Auction Theory

Empirical Industrial Organization

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A simple monopoly problem

- Monopoly seller sells one unit to consumers with unit demand.
- Consumer valuations are i.i.d with standard uniform distributions: $\epsilon_\ell \sim U[0, 1]$
- Zero marginal cost.
- With one consumer, the best it can do is post the monopoly price.
- Monopoly price is $\frac{1}{2}$ with corresponding monopoly profit $\frac{1}{4}$

Second price auction with two buyers

- Assume now there are two consumers.
- The seller auctions off the good using a second price/Vickrey auction.
- Highest bidder gets the good and pays the other consumer's bid.
- In equilibrium consumer ℓ bids ϵ_ℓ (weakly dominant strategy). expected revenue is $E \min\{\epsilon_1, \epsilon_2\} = \int_0^1 x(2-2x)dx = \frac{1}{3}$

First price auction with two buyers

- suppose a first price auction is used instead.
- Highest bidder gets the good and pays her own bid.
- Bidding own valuation is no more an equilibrium (it is actually weakly dominated by bidding strictly less).
- Bidders engage in shading by bidding less than their valuation.

- There is no longer a weakly dominant strategy.
- So we look for a Bayesian Nash equilibrium.
- Optimal bidding behavior now depends on the distribution of valuations for the competing bidder.
- More demanding in terms of what information bidders need.

- We look for a symmetric equilibrium.
- Bidder ℓ' s behavior characterized by a bidding function β such that if ℓ' 's valuation is ϵ_ℓ, she bids $b_\ell = \beta(\epsilon_\ell)$
- Choosing $b_\ell = \beta(\epsilon_\ell)$ must maximize ℓ' s expected surplus if she expects the other bidder is using bidding function β
- We assume β is differentiable: hence it is continuous and there is no tie in the auction.
- Further assume β strictly increasing so it admits an inverse β^{-1} which is also differentiable.
- We must have $\beta(0) = 0$ (a bidder with zero valuation bids zero).

- Bidder 1 's expected surplus if she bids b_1 is

$$\Pr\{b_2 \le b_1\}(\epsilon_1 - b_1)$$

- Now, $b_2 = \beta(\epsilon_2)$ so

$$\Pr\{b_2 \le b_1\} = \Pr\{\beta(\epsilon_2) \le b_1\} = \Pr\left\{\epsilon_2 \le \beta^{-1}(b_1)\right\} = \beta^{-1}(b_1)$$

- Then 1 's expected surplus if she bids b_1 is

$$\beta^{-1}(b_1)(\epsilon_1 - b_1)$$

- Hence b_1 satisfies the FOC

$$\beta^{-1'}(b_1)(\epsilon_1 - b_1) = \beta^{-1}(b_1)$$

- In a symmetric equilibrium we must have $\epsilon_1 = \beta^{-1}(b_1)$ which yields the differential equation for β^{-1}

$$\beta^{-1'}(b)\left(\beta^{-1}(b) - b\right) = \beta^{-1}(b)$$

- Alternatively, we must have $b_1 = \beta(\epsilon_1)$ which yields the differential equation for β

$$\frac{1}{\beta'(\epsilon)}(\epsilon - \beta(\epsilon)) = \epsilon \tag{1}$$

- (1) can be written as

$$\beta'(\epsilon)\epsilon + \beta(\epsilon) = \epsilon$$

- Because $\beta'(x)x + \beta(x)$ is the deriv. of $\beta(x)x$ and $\beta(0) = 0$ integrating (13) between 0 and ϵ yields

$$\beta(\epsilon) = \frac{1}{\epsilon}\int_0^\epsilon x dx = E\left(\epsilon_2 | \epsilon_2 \leq \epsilon\right)$$

$\left(\frac{1}{\epsilon}\right.$ is the density of ϵ_2 conditional on $\left.\epsilon_2 \leq \epsilon\right)$

- Hence $\beta(\epsilon) = \frac{\epsilon}{2}$

Revenue comparison

- In the first price auction the seller earns the highest bid but it is half of the highest valuation.
- Expected revenue is

$$\frac{1}{2}E\left[\max\left\{\epsilon_1,\epsilon_2\right\}\right]=\frac{1}{2}\int_0^1 2\epsilon^2 d\epsilon=\frac{1}{3}$$

- This is an illustration of the revenue equivalence principle.
- The strategic behavior of bidders unravels the attempt of the seller to capture more than the second highest valuation.

Reserve price

- Are these auction formats revenue maximizing?
- Clearly not
- Seller could post the monopoly price (from the one buyer case). $\frac{1}{2}$, and sell with prob. $\frac{3}{4}$
- Expected revenue of $\frac{3}{8} > \frac{1}{3}$.
- It could actually earn more by posting a higher price.

Reserve price

- In the auctions we have considered the good is sold with probability one to the highest valuation buyer: social optimum.
- Selling even when valuations are very low lowers the expected price.
- Revenue can be increased by giving up selling to low valuation buyers.
- This is achieved by using a reservation price $r > 0$ such that the product is sold only if the price exceeds r

Second price auction with reserve price

- Good is sold to the highest bidder only if she bids at least r
- She pays the max of r and the other bid.
- Bidding own valuation ϵ_ℓ is still a dominant strategy.
- Note that the expected revenue with such an auction is always $>$ than the expected revenue obtained by posting r : probability of selling is the same but there is some probability that the good is sold at a price $> r$

Second price auction with reserve price

- Revenue max reserve price is $r = \frac{1}{2}$
- Note that the hazard rate for the standard uniform is $h(\epsilon) = \frac{1}{1-\epsilon}$
- The optimal reserve price is such that the virtual value $r - \frac{1}{h(r)}$ is zero.
- Below that value, the seller is giving up too much informational rent to those with valuations above r and it is preferable to give up selling to those below r
- Also note that the good is not sold with probability $\frac{1}{4}$.
- If seller cannot commit to running an auction again, there is potential for coasian dynamics as in the durable good problem

First price auction with private values: symmetric model

- Assume now there are L buyers with i.i.d. valuations: F is the c.d.f and f the density.
- Values are private because they are drawn independently.
- Setting is symmetric because valuations are identically distributed.
- Let $Y = \max\{\epsilon_2, \ldots, \epsilon_L\}$: it has c.d.f G, where $G(x) = F(x)^{L-1}$ and density $g(x) = (L-1)f(x)F(x)^{L-2}$
- Again we look for a strictly increasing and differentiable bidding function β

First price auction with private values: symmetric model

- Bidder 1 's expected surplus if she bids b_1 is

$$\Pr\{\beta(Y) \leq b_1\}(\epsilon_1 - b_1)$$
$$= G\left(\beta^{-1}(b_1)\right)(\epsilon_1 - b_1)$$

- Hence b_1 satisfies the FOC

$$\beta^{-1'}(b_1)\, g\left(\beta^{-1}(b_1)\right)(\epsilon_1 - b_1) = G\left(\beta^{-1}(b_1)\right)$$

- In a symmetric equilibrium we must have $b_1 = \beta(\epsilon_1)$ which yields the differential equation for β

$$\frac{g(\epsilon)}{\beta'(\epsilon)}(\epsilon - \beta(\epsilon)) = G(\epsilon) \tag{2}$$

- (2) can be written as

$$\beta'(\epsilon)G(\epsilon) + \beta(\epsilon)g(\epsilon) = \epsilon g(\epsilon) \tag{3}$$

- Because $\beta'(x)G(x) + \beta(x)g(x)$ is the deriv. of $\beta(x)G(x)$ and $\beta(0) = 0$, integrating (3) between 0 and ϵ yields

$$\beta(\epsilon) = \frac{1}{G(\epsilon)} \int_0^{\epsilon} xg(x)dx = E(Y|Y \leq \epsilon)$$

($\frac{g(x)}{G(\epsilon)}$ is the density of Y at x conditional on $Y \leq \epsilon$.)

- The equilibrium bid for a buyer with valuation ϵ is the expected max of the valuations of all the other buyers conditional on ϵ being the highest valuation.

Main extensions

- Interdependent values.
- Multiple objects and sequential auctions.