Chapter 3 *Auction theory*

Based on Regis Renault's Slides

Andras Niedermayer[1]

[1]Department of Economics (THEMA), Cergy Paris Université

Winter 2025

1 A simple monopoly problem

- Monopoly seller sells one unit to consumers with unit demand.
- Consumer valuations are i.i.d with standard uniform distributions: $\epsilon_\ell \sim U[0,1]$.
- Zero marginal cost.
- With one consumer, the best it can do is post the monopoly price.
- Monopoly price is $\frac{1}{2}$ with corresponding monopoly profit $\frac{1}{4}$.

Second price auction with two buyers

- Assume now there are two consumers.
- The seller auctions off the good using a *second price/Vickrey* auction.
- Highest bidder gets the good and pays the other consumer's bid.
- In equilibrium consumer ℓ bids ϵ_ℓ (weakly dominant strategy).
- Expected revenue is $E\min\{\epsilon_1, \epsilon_2\} = \int_0^1 x(2-2x)dx = \frac{1}{3}$.

1 A simple monopoly problem
First price auction with two buyers

- suppose a *first price* auction is used instead.
- Highest bidder gets the good and pays her own bid.
- Bidding own valuation is no more an equilibrium (it is actually weakly dominated by bidding strictly less).
- Bidders engage in *shading* by bidding less than their valuation.

First price auction with two buyers

- There is no longer a weakly dominant strategy.
- So we look for a *Bayes-Nash* equilibrium.
- Optimal bidding behavior now depends on the distribution of valuations for the competing bidder.
- More demanding in terms of what information bidders need.

First price auction with two buyers

- We look for a symmetric equilibrium.
- Bidder ℓ's behavior characterized by a bidding function β such that if ℓ's valuation is ϵ_ℓ, she bids $b_\ell = \beta(\epsilon_\ell)$.
- Choosing $b_\ell = \beta(\epsilon_\ell)$ must maximize ℓ's expected surplus if she expects the other bidder is using bidding function β
- We assume β is differentiable: hence it is continuous and there is no tie in the auction.
- Further assume β strictly increasing so it admits an inverse β^{-1} which is also differentiable.
- We must have $\beta(0) = 0$ (a bidder with zero valuation bids zero).

- Bidder 1's expected surplus if she bids b_1 is

$$Pr\{b_2 \leq b_1\}(\epsilon_1 - b_1). \tag{1}$$

- Now, $b_2 = \beta(\epsilon_2)$ so

$$Pr\{b_2 \leq b_1\} = Pr\{\beta(\epsilon_2) \leq b_1\} = Pr\{\epsilon_2 \leq \beta^{-1}(b_1)\} = \beta^{-1}(b_1)$$

.

- Then 1's expected surplus if she bids b_1 is

$$\beta^{-1}(b_1)(\epsilon_1 - b_1). \tag{2}$$

First price auction with two buyers

- Hence b_1 satisfies the FOC

$$\beta^{-1'}(b_1)(\epsilon_1 - b_1) = \beta^{-1}(b_1) \tag{3}$$

- In a symmetric equilibrium we must have $\epsilon_1 = \beta^{-1}(b_1)$ which yields the differential equation for β^{-1},

$$\beta^{-1'}(b)(\beta^{-1}(b) - b) = \beta^{-1}(b). \tag{4}$$

- Alternatively, we must have $b_1 = \beta(\epsilon_1)$ which yields the differential equation for β

$$\frac{1}{\beta'(\epsilon)}(\epsilon - \beta(\epsilon)) = \epsilon. \tag{5}$$

- (5) can be written as

$$\beta^{'}(\epsilon)\epsilon + \beta(\epsilon) = \epsilon. \tag{6}$$

- Because $\beta^{'}(x)x + \beta(x)$ is the derivative of $\beta(x)x$ and $\beta(0) = 0$, integrating (6) between 0 and ϵ yields

$$\beta(\epsilon) = \frac{1}{\epsilon}\int_0^{\epsilon} x dx = E[\epsilon_2 | \epsilon_2 \leq \epsilon], \tag{7}$$

  ($\frac{1}{\epsilon}$ is the density of ϵ_2 conditional on $\epsilon_2 \leq \epsilon$).
- Hence $\beta(\epsilon) = \frac{\epsilon}{2}$.

Revenue comparison

- In the first price auction the seller earns the highest bid but it is half of the highest valuation.
- Expected revenue is

$$\frac{1}{2}E(\max\{\epsilon_1, \epsilon_2\} = \frac{1}{2}\int_0^1 2\epsilon^2 d\epsilon = \frac{1}{3}. \tag{8}$$

- This is an illustration of the *revenue equivalence* principle.
- The strategic behavior of bidders unravels the attempt of the seller to capture more than the second highest valuation.

- Are these auction formats revenue maximizing?
- Clearly not
  - Seller could post the monop. price (from the one buyer case). $\frac{1}{2}$, and sell with prob. $\frac{3}{4}$.
  - Expected revenue of $\frac{3}{8} > \frac{1}{3}$.
- It could actually earn more by posting a higher price.

Reserve price

- In the auctions we have considered the good is sold with probability one to the highest valuation buyer: social optimum.
- Selling even when valuations are very low lowers the expected price.
- Revenue can be increased by giving up selling to low valuation buyers.
- This is achieved by using a reservation price $r > 0$ such that the product is sold only if the price exceeds r.

Second price auction with reserve price

- Good is sold to the highest bidder only if she bids at least r.
- She pays the max of r and the other bid.
- Bidding own valuation ϵ_ℓ is still a dominant strategy.
- Note that the expected revenue with such an auction is always $>$ than the expected revenue obtained by posting r: probability of selling is the same but there is some probability that the good is sold at a price $> r$.

Second price auction with reserve price

- Revenue max reserve price is $r = \frac{1}{2}$.
- Note that the hazard rate for the standard uniform is $h(\epsilon) = \frac{1}{1-\epsilon}$.
- The optimal reserve price is such that the *virtual value* $r - \frac{1}{h(\epsilon)}$ is zero.
- Below that value, the seller is giving up too much informational rent to those with valuations above r and it is preferable to give up selling to those below r.
- Also note that the good is not sold with probability $\frac{1}{4}$.
  - If seller cannot commit to running an auction again, there is potential for *Coasian dynamics* as in the durable good problem.

2 First price auction with private values: symmetric model

- Assume now there are L buyers with i.i.d. valuations: F is the c.d.f and f the density.
- Values are private because they are drawn independently.
- Setting is symmetric because valuations are identically distributed.
- Let $Y = \max\{\epsilon_2, ..., \epsilon_L\}$: it has c.d.f G, where $G(x) = F(x)^{L-1}$ and density $g(x) = (L-1)f(x)F(x)^{L-2}$.
- Again we look for a strictly increasing and differentiable bidding function β.

- Bidder 1's expected surplus if she bids b_1 is

$$Pr\{\beta(Y) \leq b_1\}(\epsilon_1 - b_1) \tag{9}$$
$$= G(\beta^{-1}(b_1))(\epsilon_1 - b_1). \tag{10}$$

- Hence b_1 satisfies the FOC

$$\beta^{-1'}(b_1)g(\beta^{-1}(b_1))(\epsilon_1 - b_1) = G(\beta^{-1}(b_1)). \tag{11}$$

- In a symmetric equilibrium we must have $b_1 = \beta(\epsilon_1)$ which yields the differential equation for β

$$\frac{g(\epsilon)}{\beta'(\epsilon)}(\epsilon - \beta(\epsilon)) = G(\epsilon). \tag{12}$$

- (12) can be written as

$$\beta^{'}(\epsilon)G(\epsilon)+\beta(\epsilon)g(\epsilon)=\epsilon g(\epsilon). \tag{13}$$

- Because $\beta^{'}(x)G(x)+\beta(x)g(x)$ is the derivative of $\beta(x)G(x)$ and $\beta(0)=0$, integrating (13) between 0 and ϵ yields

$$\beta(\epsilon)=\frac{1}{G(\epsilon)}\int_0^{\epsilon}g(s)xdx=E[Y|Y\leq\epsilon], \tag{14}$$

  ($\frac{g(x)}{G(x)}$ is the density of Y conditional on $Y\leq\epsilon$).
- The equilibrium bid for a buyer with valuation ϵ is the expected max of the valuations of all the other buyers conditional on ϵ being the highest valuation.

3 Main extensions

- Interdependent values.
- Multiple objects and sequential auctions.