Chapter 1 *Monopoly and price discrimination*

Based on Regis Renault's Slides

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Organization

- additional material will be posted at https://andras.niedermayer.ch
- if you have any questions, write to: andras.niedermayer@cyu.fr
- please bring along your laptops for the hands-on computer exercises (starting from week 2)
- we will have a combination of lectures, hands-on exercises in class and take home work
- the grade will be based on a take home exam/term paper

Examples of Application of Empirical Industrial Organization

- car industry, environmental policy
- auctions
- price discrimination

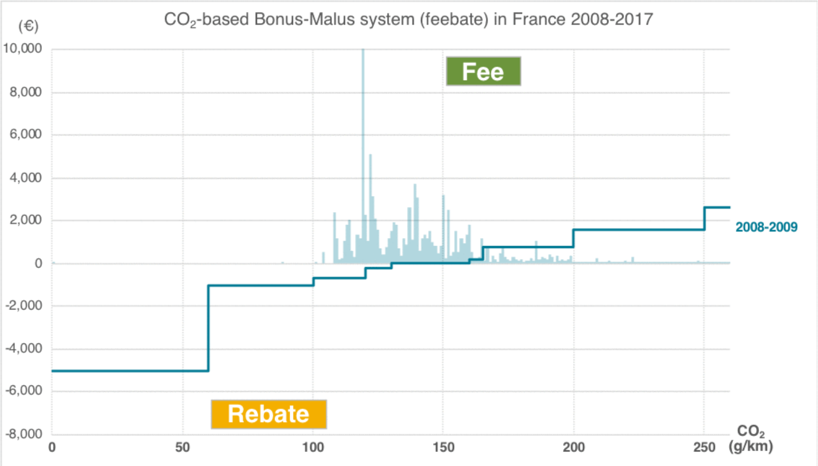
Examples: Discrete Choice

- merger control
  - for example, in 2017 the PSA Group acquired Opel and Vauxhall
  - should competition authorities have cleared the acquisition?
  - counterfactual: what is the prediction on price changes for the acquisition?

Examples: Discrete Choice

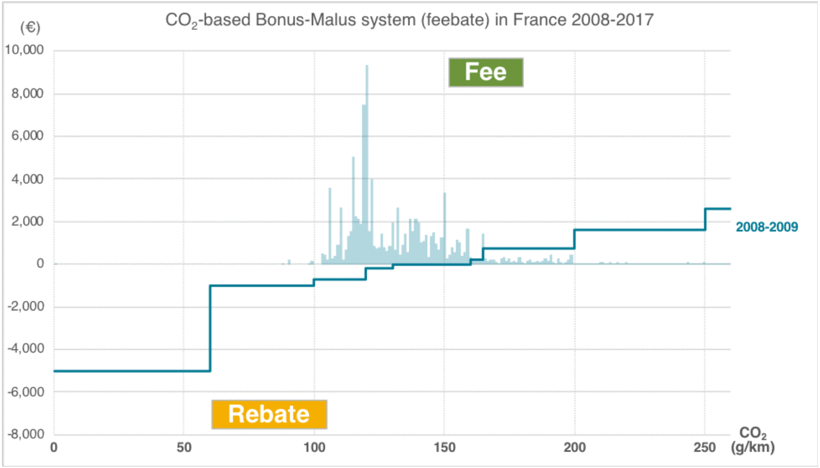
- environmental policy
  - for example, France introduced a feebate policy for cars in 2008
  - high CO2 emission cars get taxed, low CO2 emission cars get a rebate
  - the intention was to have a balanced budget

Examples: Discrete Choice



Source: International Council on Clean Transportation

Examples: Discrete Choice



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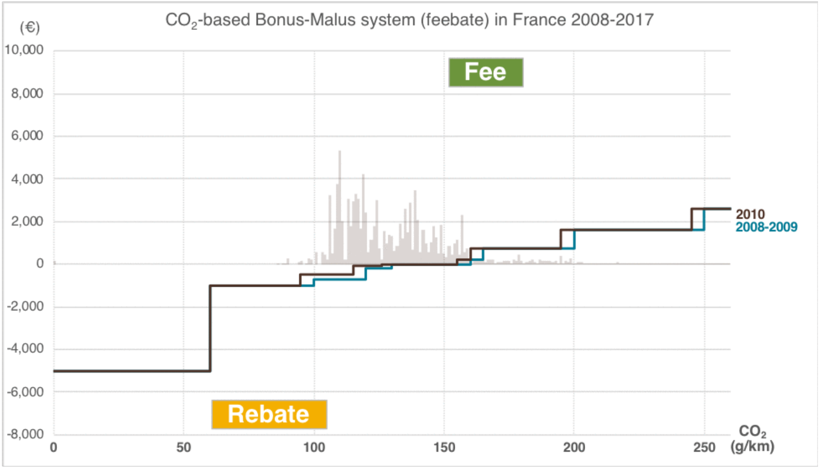

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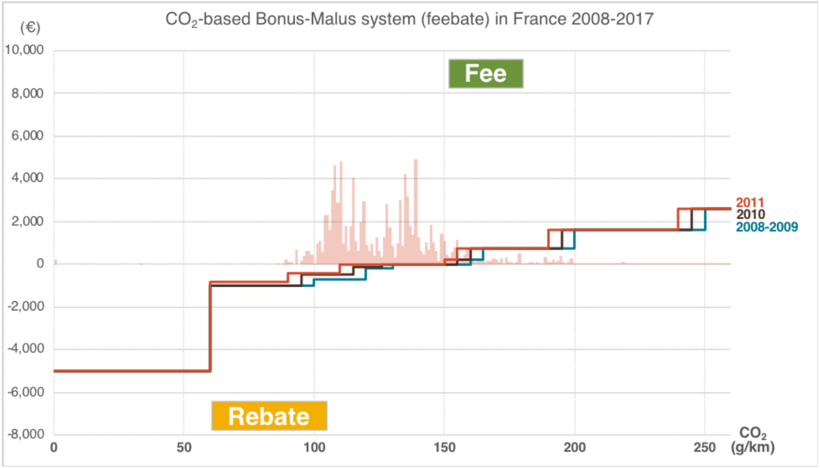

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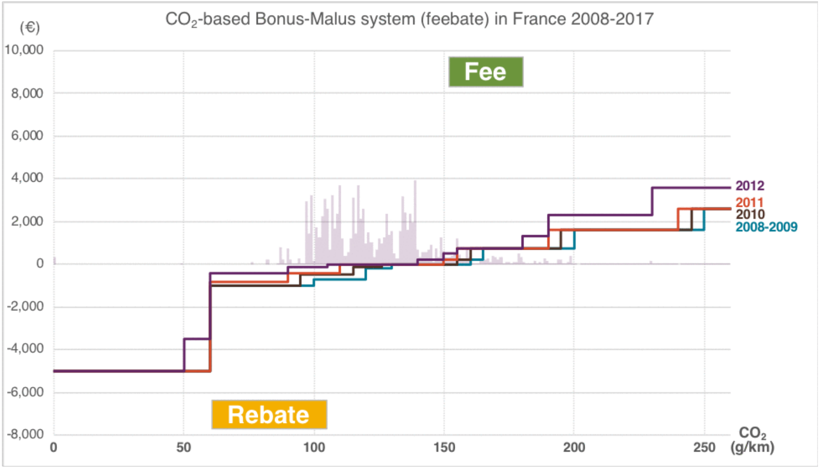

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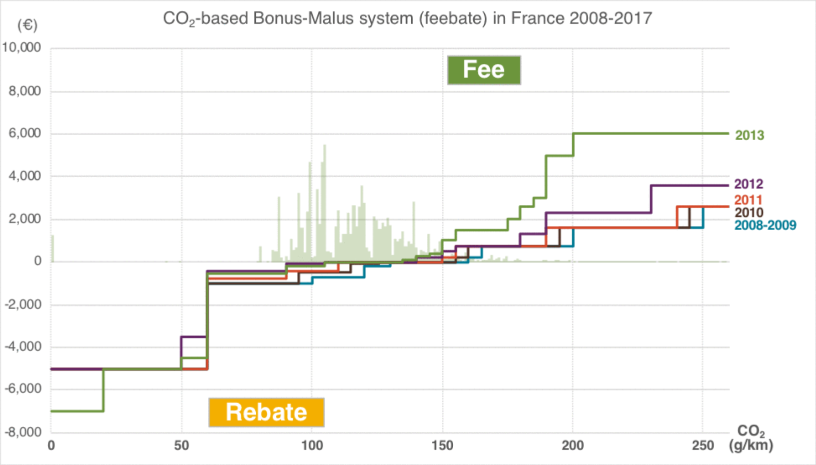
Examples: Discrete Choice



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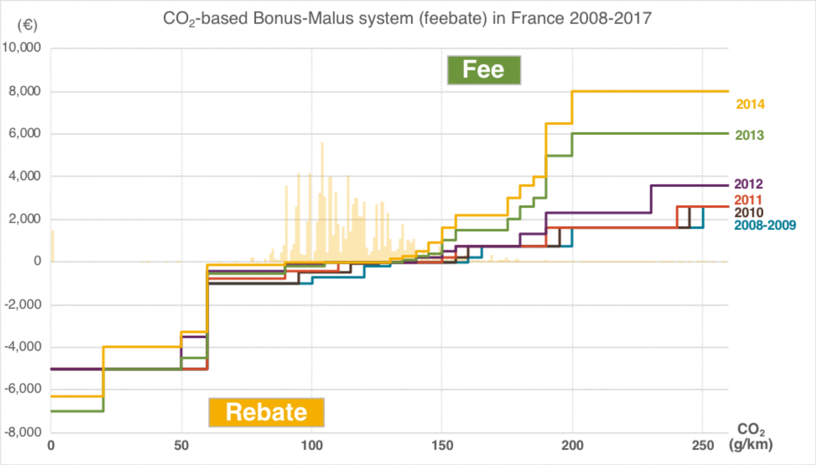
Examples: Discrete Choice



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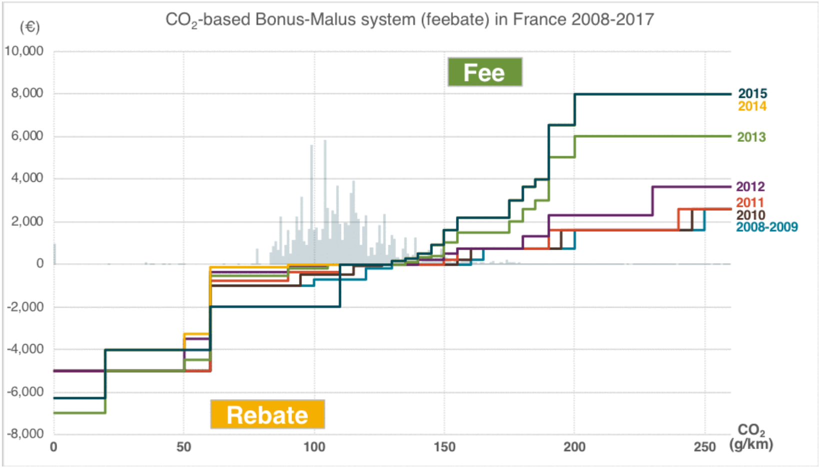

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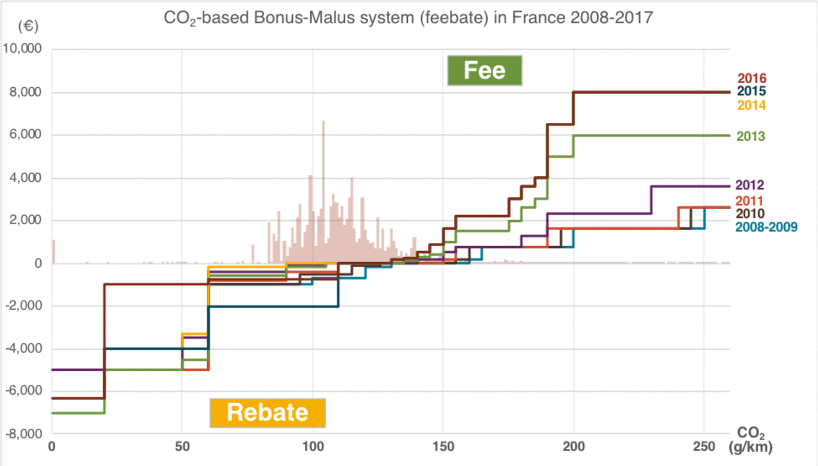

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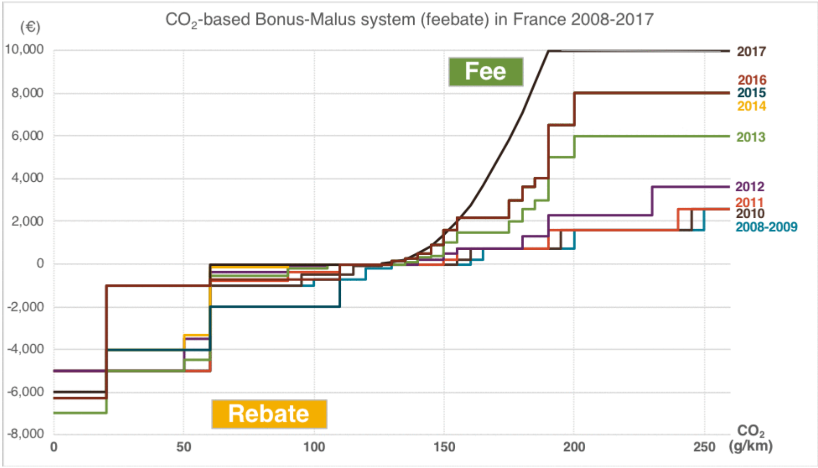
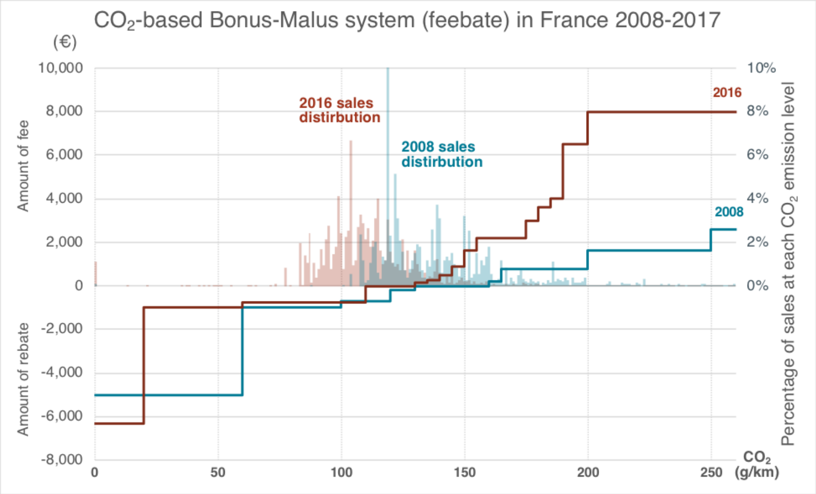


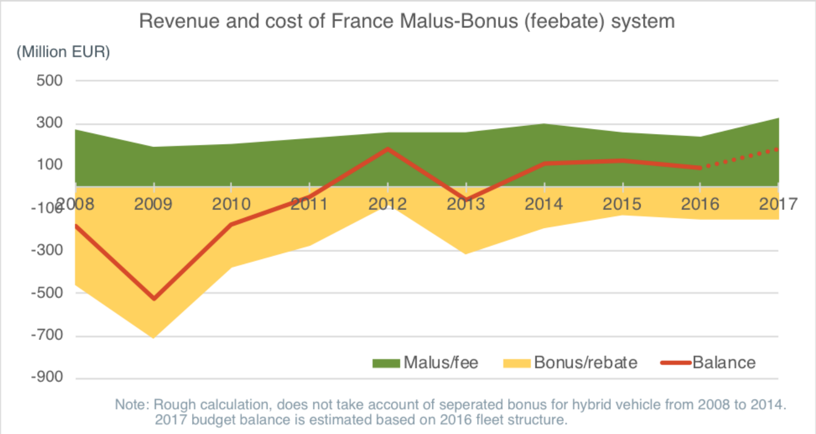
Source: International Council on Clean Transportation

2008 vs 2016



Source: International Council on Clean Transportation

Budget



Source: International Council on Clean Transportation

Examples: Auctions

Examples: Auctions

- auctions
  - for example, every year that Canadian government auctions off rights to log on government land
  - What is the optimal auction format?
  - Which minimal price should the goverment set?
- procurement auctions
  - (the government) buys from the lowest bidder on a project, e.g. the construction of roads
  - "Operation Hammer" in Quebec, started in 2009: uncovered widespread collusion in the bidding for government construction contracts
  - How do you detect collusion?
  - How do you compute damages from collusion?

Examples: Price Discrimination

Automobiles

Renault Clio

€16,600, power: 58 kW

Examples: Price Discrimination

Automobiles

€11,300
power: 43 kW

€16,600
power: 58 kW

€39,700
power: 187 kW

€11,300 €11,875 €12,375 €12,800 €12,833 €12,880 €13,025 €13,172 €13,300 €13,325 €13,433
€13,625 €13,798 €13,825 €13,875 €13,948 €13,988 €14,200 €14,300 €14,325 €14,333 €14,357
€14,358 €14,460 €14,466 €14,488 €14,525 €14,600 €14,640 €14,800 €14,820 €15,068 €15,100
€15,200 €15,298 €15,320 €15,425 €15,448 €15,450 €15,488 €15,625 €15,650 €15,680 €15,700
€15,713 €15,787 €15,858 €15,960 €15,988 €16,050 €16,070 €16,125 €16,140 €16,180 €16,400
€16,464 €16,550 €16,600 €16,620 €16,688 €16,733 €16,788 €16,900 €16,964 €17,013 €17,080
€17,100 €17,120 €17,138 €17,188 €17,212 €17,250 €17,400 €17,413 €17,430 €17,466 €17,513
€17,560 €17,580 €17,590 €17,712 €17,750 €17,788 €17,820 €17,888 €17,913 €17,930 €18,060
€18,200 €18,288 €18,320 €18,388 €18,488 €18,675 €18,760 €18,988 €19,175 €19,260 €19,488
€19,520 €19,679 €19,688 €20,020 €20,039 €20,188 €20,617 €21,200 €21,700 €23,200 €23,750
€24,367 €36,913 €38,568 €39,700

Outlook for Classes

- theoretical foundations:
  - monopoly and price discrimination
  - auction theory
  - discrete choice random utility models
- Python refresher (introduction?)
- discrete choice estimation
- auction econometrics
- econometrics of price discrimination

Standard monopoly results:

- a monopoly firm prices above marginal cost (this reflects market power);
- monopoly pricing generates a deadweight loss because the quantity produced is too small (monopoly total surplus is less than the perfect competition total surplus).
- deadweight loss exists because in order to sell more, monopolist must lower the price for ALL units (not only for the marginal unit).
  - with uniform pricing this is reflected by a marginal revenue below the inverse demand curve.

Harberger's objection (Harberger, 1954)

- Calculated that aggregate deadweight loss represented about 0.1% of US GNP.
- Hence monopoly power is not empirically relevant..

Objections to Harberger

- Methodological criticism of his empirical approach (miss specification of demand, miss specification of the perfectly competitive profit).
- His approach was partial equilibrium by taking the sum of deadweight losses over all the sectors in the economy: this may however lead to major mistakes (that may go both ways though).
- Harberger's low figure may reflect the effectiveness of US antitrust laws that go back to the late 19th century (Sherman Act and Clayton act).

Demand elasticity and monopoly pricing.

- By the inverse elasticity rule, monopoly markup is higher when demand is less elastic.
- This is why demand elasticity is a key ingredient in empirical studies on market power.

Demand elasticity and dead weight loss.

- Intuitively, a lower elasticity leads to a higher wedge between price and marg. cost but the quantity demanded is less affected by the increase in price: hence the impact on deadweight loss is ambiguous.
- More formally we may consider how a change in elasticity affects the ratio of DWL to the 1st best social surplus (sum of DWL, CS and PS).
- Consider constant elasticity demands, $D(p) = p^{\epsilon}$, p is price and $\epsilon < -1$ is price elasticity.
- It can be shown that as ϵ decreases from -1 to $-\infty$, the ratio $DWL/(DWL + PS + CS)$ increases (hence more elasticity leads to more inefficiency).

Demand curvature and dead weight loss.

- Constant elasticity is a special case of a more general class of demands: ρ-linear demands.
- Consider $D(p)$ such that $D(p)^{\rho}$ is linear in p for some real number ρ: e.g. constant elasticity demands are ρ-linear for $\rho = 1/\epsilon$.
- It can be shown that as ρ increases from -1 to $+\infty$, the ratio first increases and than decreases to zero where the turning point is for some $\rho > 0$.
- As ρ increases, the monopolist captures a larger share of the overall surplus and if that share is sufficiently high, the firm causes less inefficiency.
- The limit corresponds to a rectangular demand where there is no deadweight loss and the firm captures the entire surplus.

Outline

2 Some comparative statics
change in marginal cost

- Consider two differentiable total cost functions C_1 and C_2 such that $C_1' > C_2'$ for all positive quantities.
- q_i^m and p_i^m are monopoly quantity and price for cost function C_i.
- Because q_i^m and p_i^m maximize profit we have the two following inequalities:

$$p_1^m q_1^m - C_1(q_1^m) \geq p_2^m q_2^m - C_1(q_2^m) \tag{1}$$

and

$$p_2^m q_2^m - C_2(q_2^m) \geq p_1^m q_1^m - C_2(q_1^m). \tag{2}$$

2 Some comparative statics
change in marginal cost

- Taking the difference of the two inequalities yields:

$$[C_2(q_1^m) - C_1(q_1^m)] - [C_2(q_2^m) - C_1(q_2^m)] \geq 0, \tag{3}$$

or equivalently

$$\int_{q_2^m}^{q_1^m} C_2'(q) - C_1'(q) dq \geq 0. \tag{4}$$

- Since $C_2' - C_1' > 0$, we must have $q_1^m > q_2^m$ and hence (since demand is decreasing) $p_1^m < p_2^m$.
- This shows that an increase in marginal cost leads to an increase in the monopoly price.

The magnitude of the price increase caused by an increase in cost

- Assume a constant marginal cost $c > 0$. From the price FOC monopoly price p^m satisfies

$$p^m - c = -\frac{D(p^m)}{D'(p^m)}. \tag{5}$$

- Let $g(p^m) = D(p^m)/D'(p^m)$. Standard comparative statics shows that

$$\frac{dp^m}{dc} = \frac{1}{1 + g'(p^m)} \tag{6}$$

- The impact of a cost increase is < 1 (resp. > 1) if and only if $g' > 0$ (rep. $g' < 0$).

;

- Note that $g' > 0$ over some price range iff D is log-concave (i.e. lnD is concave) over that range.
- For instance linear demand, $D(p) = 1 - p$, is logconcave on $[0, 1]$.
- More generally, ρ-linear demand $D(p) = (1 - p)^{\frac{1}{\rho}}$, with $\rho > 0$, is logconcave on $[0, 1]$.
- This is not the case for constant elasticity demands: lnD is convex on $[0, +\infty)$.
    - then an increase in marginal cost of 1 Euro causes an increase in price of more that 1 Euro.
- More generally, all ρ-linear demands with $\rho < 0$ are logconvex.

- Consider a unit tax t. Monopolist chooses price p^m to solve:

$$\max_{p^m} p^m D(p^m + t) - C(D(p^m + t)). \tag{7}$$

- Nec. FOCs are:

$$D(p^m + t) + [p^m - C'(D(p^m + t))]D'(p^m + t) = 0. \tag{8}$$

- To restore efficiency, t must be set so that the price paid by consumers $p^m + t$ equals marg. cost $C'(D(p_t^m))$, so we have

$$t = \frac{D(p^m + t)}{D'(p^m + t)} < 0. \tag{9}$$

Taxes

- Monopoly dead weight loss may be eliminated by using a unit subsidy.
- In practice, this solution is not used much, in particular because it would require some tax revenue thus causing distortions elsewhere.
- Typically, monopolies are regulated directly or government owned.

3 Some second order conditions

- For FOCs to be not only nec. but also suf. we need restrictions on monopoly profit pi^m: for quasiconcavity in price.
- Formally, the set of prices p at which $\pi^m(p) \geq K$ for some real number k should be convex.
- So profit is quasiconcave iff it does not have an interior local minimum.
- At an interior local min., 1st derivative must be zero and 2nd derivative must be ≥ 0.
- Hence profit is quasiconcave if whenever its 1st deriv. is 0 its second deriv. is < 0.

3 Some second order conditions

- Assume constant marginal cost c.
- Profit is $(p-c)D(p)$.
- 1st deriv. being 0 implies $D(p)+(p-c)D'(p)=0$.
- 2nd deriv. is $2D'(p)+(p-c)D''(p)$ so that by substituting the zero 1st deriv. in the 2nd deriv. we have the following suf. condition for quasiconcavity:

$$2D'(p)^2 - D(p)D''(p) > 0. \tag{10}$$

ρ-concave demand functions

- A function $D > 0$ with a convex domain is said to be ρ-concave for some real number ρ if D^ρ is concave for $\rho > 0$ and $-D^\rho$ is concave for $\rho < 0$; D is zero-concave if it is logconcave.
- If D is ρ-concave for some ρ, then it is ρ'-concave for all $\rho' < \rho$.

3 Some second order conditions

- Assume D is ρ-concave for $\rho < 0$. Then the 2nd deriv. of D^ρ must be positive, which is equivalent to

$$-(\rho - 1)D'(p)^2 - D(p)D''(p) \geq 0. \tag{11}$$

- LHS strictly decreasing in ρ: so if $\rho > -1$, then the inequality is strict at $\rho = -1$, which yields the SOC (10).
- So ρ-concavity of demand, for $\rho > -1$, is sufficient for quasiconcavity of profit (in fact, (-1)-concavity is sufficient as well).
- This implies that logconcavity of demand is sufficient (this weaker assumption will be used in oligopolistic competition with product differentiation).

4 Unit demand

- A population of L consumers.
- Monopolist sells one product.
- Then individual demand is characterized by a valuation for the product such that, consumer buys iff price is weakly below.

Linear random utility model LRUM.

- Consumer ℓ has the following utility:

$$U_\ell = \epsilon_\ell - p + y_\ell, \tag{12}$$

  if she buys at price p and $u_\ell = y_\ell$ if she does not buy, where y_ℓ is her income.
- ϵ_ℓ, $\ell = 1, ..., L$, are i.i.d. random variables with support $[a, b]$, cumulative distribution function F and density f,
- Then ℓ's valuation is ϵ_ℓ, independent of her income (it is a quasilinear utility with no income effect).

4 Unit demand

- Consumer ℓ buys Iff $\epsilon_\ell \geq p$, which happens with probability $1 - F(p)$.
- Then expected demand is

$$D(p) = L[1 - F(p)]. \tag{13}$$

- $L = 1$ and a uniform distribution on $[0, 1]$ for ϵ_ℓ yields linear demand $D(p) = 1 - p$.
- If marginal cost is constant at $c \geq 0$ then price FOC is

$$p^m - c = \frac{1 - F(p^m)}{f(p^m)}. \tag{14}$$

Increasing hazard rate and logconcavity

- RHS of (14) is the inverse of the hazard rate of ϵ_ℓ (which is $h = \frac{f}{1-F}$).
- Standard assumption is h increasing.
- This is equivalent to $1 - F$ logconcave.
- Actually if f logconcave (which holds for many commonly used distributions) than $1 - F$ and F are logconcave as well (a consequence of the Prekopa-Borell theorem).

5 A durable good monopolist

- A monopolist sells over several periods a good for which each consumer needs only one unit (a durable good).
- Then consumers engage in inter-temporal substitution and can wait if they expect price to fall.
- Then the monopolist creates competition for its sales in the current period if it cannot commit to not dropping the price in the future.

5 A durable good monopolist

- A monopoly firm with 0 costs sells one unit of a durable good over 2 periods.
- one consumer with unit demand: valuation for the good ϵ uniform on $[0, 1]$.
- Product sold either in period 1 or period 2
- Consumer and firm have common discount factor $\delta \in (0, 1)$ for surplus in period 2.
- We look for an equilibrium where the consumer buys in period 1 if and only if $\epsilon \geq \tilde{\epsilon}$.
- $\tilde{\epsilon}$ is endogenously determined by price charged in period 1 and expected price in period 2.

- In period 2 demand is $D_2(p) = 1 - \frac{p}{\tilde{\epsilon}}$
- Period 2 price is $p_2 = \frac{\tilde{\epsilon}}{2}$ and this is anticipated by consumer.
- In period 1, consumer with valuation ϵ buys iff $\epsilon - p_1 > \delta(\epsilon - \frac{\tilde{\epsilon}}{2})$.
- Consumer and firm have common discount factor $\delta \in (0, 1)$ for surplus in period 2.
- Hence the consumer buys in period 1 iff

$$\epsilon \geq \frac{1}{1-\delta}\left(p_1 - \delta\frac{\tilde{\epsilon}}{2}\right) = \tilde{\epsilon}. \tag{15}$$

- Hence $p_1 = (1 - \frac{\delta}{2})\tilde{\epsilon}$.

- Consumer buys in period 1 with prob. $1-\tilde{\epsilon}$ and waits until period 2 with prob. $\tilde{\epsilon}$, in which case she buys with prob. $\frac{1}{2}$.
- Total discounted expected profit can be written as a function of $\tilde{\epsilon}$

$$(1-\tilde{\epsilon})(1-\frac{\delta}{2})\tilde{\epsilon}+\delta\frac{\tilde{\epsilon}^2}{4}. \tag{16}$$

- From necessary FOCs profit is maximized at $\tilde{\epsilon}=\frac{1-\frac{\delta}{2}}{2-\frac{3\delta}{2}}$.
- Hence $p_1=\frac{(1-\frac{\delta}{2})^2}{2-\frac{3\delta}{2}}$ and $p_2=\frac{1-\frac{\delta}{2}}{2\left(2-\frac{3\delta}{2}\right)}$.
- can be checked that because $\delta\in[0,1]$, $p_1\geq p_2$.

- Firm earns monopoly profit either when $\delta = 1$ or $\delta = 0$.
- in the former case it is because the firm is patient enough to forego any sale in period 1 so as to face the static monopoly problem in period 2 ($\tilde{\epsilon} = 1$.
- In the latter case, consumers are myopic so the firm can charge monopoly price in period 1 $p_1 = \frac{1}{2}$ without facing competition from potential future sales.
- For $\delta \in (0, 1)$, profit derivative wrt δ is $\frac{3\tilde{\epsilon}^2}{4} - \frac{\tilde{\epsilon}}{2}$ (using envelop theorem).
- It is > 0 iff $\tilde{\epsilon} \geq \frac{2}{3}$ (i.e. $\delta \geq \frac{2}{3}$) and < 0 otherwise.
- For $\delta < \frac{2}{3}$ consumer patience dominates whereas for $\delta > \frac{2}{3}$ firm patience dominates.
- IN any case, profit is less than monopoly profit.

The Coase Conjecture As the frequency of price changes becomes increasingly high, the monopoly profit tends to zero and all consumers by the product at a price close to marginal cost. This result has been proved formally.

Solutions to the Coase conjecture

1. Renting.
2. Most favored customer close whereby the firm commits to reimbursing a consumer if the price decreases.
3. Planned obsolescence.

6 Price discrimination in the unit demand setting

Strict definition Price discrimination involves selling different units of “the same” product at different prices.

- Actual price discrimination practices often involve selling different products.
- A standard form of price discrimination with only one product is *non linear pricing* (e.g. quantity discounts).
- What about price discrimination when each buyer buys one unit?

Perfect discrimination

- Assume the firm knows ϵ_ℓ for each consumer and is allowed to charge a price conditional on ϵ_ℓ.
- By charging $p(\epsilon_\ell) = \epsilon_\ell$ and selling only to consumers for whom ϵ_ℓ exceeds marginal cost, the firm captures the entire social surplus..
- If social surplus is not max. then profit can be increased either by selling to a consumer for whom $\epsilon_\ell >$ marg. cost or by not selling to some consumer for whom $\epsilon_\ell <$ marg. cost.
- Constant marg. cost case can be illustrated graphically.

- Assume now that the firm only knows the distribution of ϵ_ℓ but not its realization for each consumer.
- Then price cannot be conditional on the realization of ϵ_ℓ.
- Hence, a consumer can freely choose within the menu of prices.
- Clearly, if the product can be purchased at two different prices, all consumers pick the lowest price and there is no price discrimination.
- To prevent such *personal arbitrage* the choice of a lower price must entail some cost.

Screening

- To illustrate, assume that ϵ_ℓ can take on value θ_1 with probability $\lambda \in (0,1)$ and θ_2 with prob. $1-\lambda$, $\theta_1 < \theta_2$.
- Firm has marginal cost $c \geq 0$ and consumers are risk neutral.
- To circumvent personal arbitrage, we allow for stochastic pricing mechanisms.
- Formally, the firm offers a menu of pricing schemes (q, T), where q is the probability that the product is delivered to the consumer and T is the money transfer between the consumer and the firm..

6 Price discrimination in the unit demand setting Screening

- We have a two stage leader follower game where:
  1. in stage 1 the firm offers a menu of pricing schemes;
  2. In stage 2 each consumer selects one of the pricing schemes or does not buy.
- Let (q_i, T_i) be the pricing scheme selected in equilibrium by a type i consumer, $i = 1, 2$.
- The firm needs only offer two pricing schemes (one of them could be $(q, T) = (0, 0)$ if it is optimal not to sell to one of the consumer types).

- As a benchmark, consider the *first best* case where the firm knows the realization θ_i.
- Then it maximizes its expected profit $T_i - q_i c$ subject to the *participation constraint* that the consumer is willing to "buy", $q_i\theta_i - T_i \geq 0$.
- It can be seen graphically that the solution is $(q_i, T_i) = (1, \theta_i)$ if $\theta_i \geq c$ and $(q_i, T_i) = (0, 0)$ otherwise.
- This is the perfect discrimination solution.
- Interesting case is when $\theta_2 > \theta_1 > c$ (so both types are served in the first best).

- If the firm does not know θ_i, it maximizes expected profit

$$\lambda\,(T_1 - q_1 c) + (1-\lambda)\,(T_2 - q_2 c)\,, \tag{17}$$

subject to two participation constraints,

$$q_1\theta_1 - T_1 \geq 0 \tag{18}$$
$$q_2\theta_2 - T_2 \geq 0, \tag{19}$$

and two incentive compatibility constraints,

$$q_1\theta_1 - T_1 \geq q_2\theta_1 - T_2 \tag{20}$$
$$q_2\theta_2 - T_2 \geq q_1\theta_2 - T_1, \tag{21}$$

- Since $\theta_2 > \theta_1$, (19) is implied by (18) and (21): so (19) is not binding.
- Then IC constraint (21) must bind: else T_2 could be increased without violating (18).
- Now let us look at the solution to the problem while ignoring IC constraint (20)
- Then PC constraint (18) must bind (the low type has no rent): else, T_1 could be increased without violating the IC constraint (21)

- Substituting binding constraints (19) and (21) into the expected profit, the firm selects q_1 and q_2 so as to maximize

$$\lambda\left(\theta_1 - c\right)q_1 + (1-\lambda)\left((\theta_2 - c)q_2 - (\theta_2 - \theta_1)q_1\right). \qquad (22)$$

.

- Then the solution is $q_2 = 1$ and $q_1 = 1$ iff

$$\theta_1 - (1-\lambda)\theta_2 \geq \lambda c. \qquad (23)$$

- Corresponding transfers are $T_1 = T_2 = \theta_1$ if $q_1 = 1$ and $T_1 = 0$ and $T_2 = \theta_2$ if $q_1 = 0$.

- This is the optimal solution under uniform pricing.
- Note that this is incentive compatible for type θ_1 so (20) is satisfied and we have characterized the optimal solution.
- Hence, price posting is the optimal solution when selling one product with unit demand.
- Three ways around this:
  1. assuming demand is price sensitive.
  2. assuming different product varieties (qualities).
  3. Assuming some capacity constraint and the possibility to auction off the product.

- Assume now that the utility of consume ℓ is

$$U_\ell = \theta_\ell q - p + y_\ell, \tag{24}$$

  if she purchases the product at price p and $u_\ell = y_\ell$ if she does not purchase.

- θ_ℓ are I.i.d random variables with a support in $[0, +\infty)$ and $q > 0$ is the product's quality, where the marg. cost of producing a product of quality q is $c(q)$, where c is strictly increasing, strictly convex and twice continuously differentiable.
- The realization of θ_ℓ is unknown to the firm.

- First we consider the case where θ_ℓ is either θ_1 or θ_2, $\theta_2 > \theta_1 > 0$ and $Pr\{\theta_\ell = \theta_1\} = \lambda$.
- The firm now offers a menu of qualities sold at different prices.
- The price quality pair selected by type θ_i is denoted (q_i, T_i).

- Before deriving the profit maximizing solution let us consider the case where the firm is perfectly informed about each consumer's type and may perfectly discriminate.
- The firm would then charge $T_i = \theta_i q_i$ to type θ_i and select $q_i = q_i^*$ to maximize $\theta_i q_i - c(q_i)$.
- It is not incentive compatible because type θ_2 would pick (q_1^*, t_i^*) (see graph).
- For further reference, this (first best) quality if it is > 0, solves the FOC, $\theta_i = c'(q_i^*)$.

Firm chooses (q_i^s, T_i^s), $i = 1, 2$ to solve

$$\max_{(q_i, T_i)_{i=1}^2} \lambda(T_1 - c(q_1)) + (1 - \lambda)(T_2 - c(q_2))$$

s.t. participation constraints

$$q_1\theta_1 - T_1 \geq 0$$
$$q_2\theta_2 - T_2 \geq 0,$$

and two incentive compatibility constraints,

$$q_1\theta_1 - T_1 \geq q_2\theta_1 - T_2$$
$$q_2\theta_2 - T_2 \geq q_1\theta_2 - T_1,$$

- As before (19) is irrelevant and we first solve the problem ignoring IC (20).
- Substituting the 2 binding constraints (18) and (21) in expected profit, the optimal qualities q_1^s and q_2^s must solve

$$\max_{(q_1,q_2)} \lambda(\theta_1 q_1 - c(q_1)) + (1-\lambda)(\theta_2 q_2 - c(q_2)) - (1-\lambda)(\theta_2 - \theta_1) q_1$$

- The firm maximizes the expected total surplus minus the informational rent (which is the last term).
- If both types are served, quantities should be > 0 so that FOCs are

$$\theta_1 = c'(q_1^s) + \frac{1-\lambda}{\lambda}(\theta_2 - \theta_1) \tag{25}$$

$$\theta_2 = c'(q_2^s) \tag{26}$$

- From (26) the quality for the high valuation consumer is 1st best while from (25) the quality for the low valuation consumer is distorted downwards from the first-best (because c' is increasing by convexity of c).
- **Intuition:** the informational rent is the only source of discrepancy between expected profit and expected social surplus. Since it is unaffected by q_2 and increasing in q_1 only the latter should be distorted from its socially optimal level and it should go down to reduce the informational rent.

- Corresponding transfers are

$$T_1^s = \theta_1 q_1^s \tag{27}$$

$$T_2^s = \theta_2 q_2^s + (\theta_2 - \theta_1) q_1^s. \tag{28}$$

- To check that IC (20) is not violated first note that $q_2^s = q_2^* > q_1^* > q_1^s$.
- We can rewrite (20) as $(\theta_2 - \theta_1)(q_2^s - q_1^s) \geq 0$ which is clearly the case since $\theta_2 > \theta_1$ and $q_2^s > q_1^s$.

Takeaways

1. High valuation consumers earn an informational rent and consume a first best quality.
2. Low valuation consumers have no rent and consume a quality that is distorted downward from the first-best.

Communication

- The above pricing scheme requires no communication between the firm and consumers.
- It implements the same allocation as an optimal direct mechanism where consumers would be asked to announce their type.
- From the *revelation* principle for Bayesian implementation, a more general communication procedure (non direct mechanism) could not implement anything better.

- Now θ can take on any value in $[\underline{\theta}, \bar{\theta}]$, $\underline{\theta} > 0$ with c.d.f F and density f.
- Firm selects a pricing scheme (q, t):
    - (q, t) is a two dimensional function with domain $[\underline{\theta}, \bar{\theta}]$;
    - $(q(\theta), t(\theta))$ is the quality price pair selected by type θ in equilibrium.
- Infinitely many IC constraints:

$$\theta q(\theta) - t(\theta) \geq \theta q(\hat{\theta}) - t(\hat{\theta}), \tag{29}$$

for all θ, $\hat{\theta}$ in $[\underline{\theta}, \bar{\theta}]$, so type θ does not want to deviate and mimic type $\hat{\theta}$.

Lemma

Pricing scheme (q, t) satisfies all incentive compatibility constraints (29) if and only if q is increasing and

$$U(\theta) = \underline{U} + \int_{\underline{\theta}}^{\theta} q(s)ds, \tag{30}$$

where $U(\theta) \equiv \theta q(\theta) - t(\theta)$ is the equilibrium utility of type θ, and $\underline{U} = U(\underline{\theta})$.

Proof.

Necessary condition 1st, taking $\theta_2 > \theta_1$ the IC constraints between these two types imply that $q(\theta_2) > q(\theta_1)$. Hence q must be increasing.

Then q is differentiable almost everywhere. $\square$

Proof.

Necessary condition ctd Furthermore, form IC constraint (29), t is differentiable whenever q is.
Indeed we have

$$(\theta + h)\frac{q(\theta + h) - q(\theta)}{h} \geq \frac{t(\theta + h) - t(\theta)}{h} \geq \theta \frac{q(\theta + h) - q(\theta)}{h} \tag{31}$$

for $h > 0$, and for $h < 0$ we have the reverse inequalities.
Then $t'(\theta)$ is the limit of the middle term when h tends to zero which exists whenever $q'(\theta)$ exists (sandwich theorem). And we have

$$\theta q'(\theta) = t'(\theta). \tag{32}$$

(Note: this is also the necessary FOC for IC, which requires that announcing $\hat{\theta} = \theta$ maximizes $\theta q(\hat{\theta}) - t(\hat{\theta})$, the surplus obtained by pretending she has type $\hat{\theta}$.) □

Proof.

Necessary condition ctd 2nd,Integrating (32) between $\underline{\theta}$ and θ yields

$$\int_{\underline{\theta}}^{\theta} sq'(s)ds = t(\theta) - t(\underline{\theta}) \tag{33}$$

Integrating by parts:

$$[sq(s)]_{\underline{\theta}}^{\theta} - \int_{\underline{\theta}}^{\theta} q(s)ds = t(\theta) - t(\underline{\theta}) \tag{34}$$

or

$$\theta q(\theta) - t(\theta) = \underline{\theta} q(\underline{\theta}) - t(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)ds, \tag{35}$$

which is the desired condition (30). □

Proof.

Sufficient conditions Now assume $q(\theta)$ is increasing and (30) holds. We need to show (29) which can be rewritten as

$$U(\theta) \geq \theta q(\hat{\theta}) - t(\hat{\theta}) = U(\hat{\theta}) + (\theta - \hat{\theta}) q(\hat{\theta}). \tag{36}$$

Using (30) this simplifies to

$$\int_{\hat{\theta}}^{\theta} q(s) - q(\hat{\theta}) ds \geq 0, \tag{37}$$

which holds for q increasing. □

- Using $t(\theta) = \theta q(\theta) - U(\theta)$ and the lemma, the firm's problem can be written as

$$\max_{q,\underline{U}} \int_{\underline{\theta}}^{\bar{\theta}} \left(\theta q(\theta) - c(q(\theta)) - \underline{U} - \int_{\underline{\theta}}^{\theta} q(s)ds \right) f(\theta)d\theta \quad (38)$$

subject to q increasing and $\underline{U} \geq 0$.

- Note that because of (30) the participation constraint is relevant only for $\underline{\theta}$ and it should clearly be binding.

- Using integration by parts we have

$$\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} q(s)ds \right) f(\theta)d\theta \\
&= \left[\left(\int_{\underline{\theta}}^{\theta} q(s)ds \right) F(\theta) \right]_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)F(\theta)d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} [1 - F(\theta)]q(\theta)d\theta
\end{aligned}$$

- The firm then solves

$$\max_{q} \int_{\underline{\theta}}^{\bar{\theta}} \left(\left(\theta - \frac{1}{h(\theta)} \right) q(\theta) - c(q(\theta)) \right) f(\theta)d\theta \tag{39}$$

- The integral in (39) can be maximized point-wise and the FOC for $q(\theta)$ is

$$\theta - \frac{1}{h(\theta)} = c'(q(\theta)). \tag{40}$$

- since c' increasing, a sufficient condition for q to be increasing is that hazard rate h is increasing.
- LHS is type θ's virtual valuation for increasing the product's quality.