1 Parameters Fitting To The FitzHugh-Nagumo Equations

1.1 Background

The FitzHugh-Nagumo equation is a simplification of the Hodgkin-Huxley model devised in 1952. The Hodgkin-Huxley has four variables and the FitzHugh-Nagumo equation is a reduction of that model. The reduction is from four variables to two variables where phase plane techniques may be used for the analysis of the model. The variables kept in the reduction of the model are the excitable variable and the recovery variable which are characterized as being the fast and slow variables respectively. The FitzHugh-Nagumo equations are used in computational neuroscience to model the potential in the nerve. The equations in this case are given by:

$$\frac{dv}{dt} = c(v - \frac{1}{3}v^3 + w) := f(t, v, w) \tag{1}$$

$$\frac{dw}{dt} = -\frac{1}{c}(v - a - bw) := g(t, v, w) \tag{2}$$

where v is the membrane potential and w is the recovery variable.

1.2 Synthesis Data

There are three parameters in the model, namely; a, b, and c. We are going to assume that a is fixed and fit the other parameters to the system. To get raw data, we will mimic a real life data set by first solution the equation with some parameters and perturb the solution to generate a data set that we shall take as data. Also, since the equation can not be solved exactly, we shall use the RK-4 formula to find the approximate the solution. The

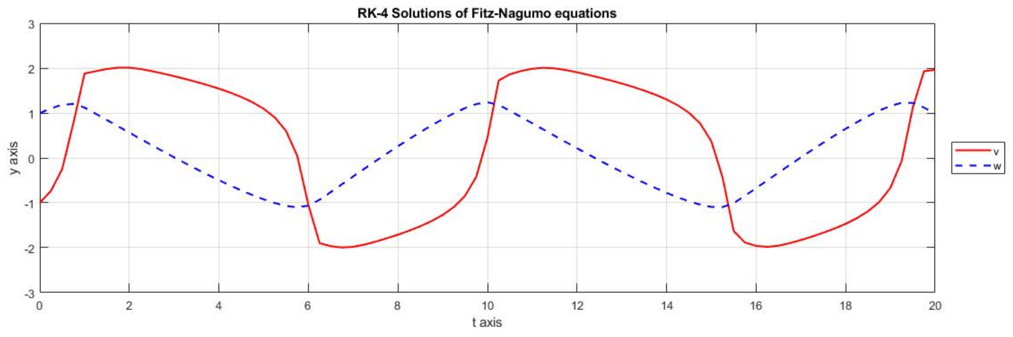


Figure 1: The numerical solution when $v(0) = -1$, $w(0) = 1$, $a = b = 0.2$, and $c = 3$.

numerical solution when $v(0) = -1$, $w(0) = 1$, $a = b = 0.2$, and $c = 3$ is given by figure (1). This is then randomize to get the data given by figure (2).

1.3 Numerical Solution by RK4

To solve the system given by (1) and (2), we use the popular Runge-Kutta formula of order 4 (RK4) given by:

$$v_{n+1} = v_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{3}$$

$$w_{n+1} = w_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{4}$$

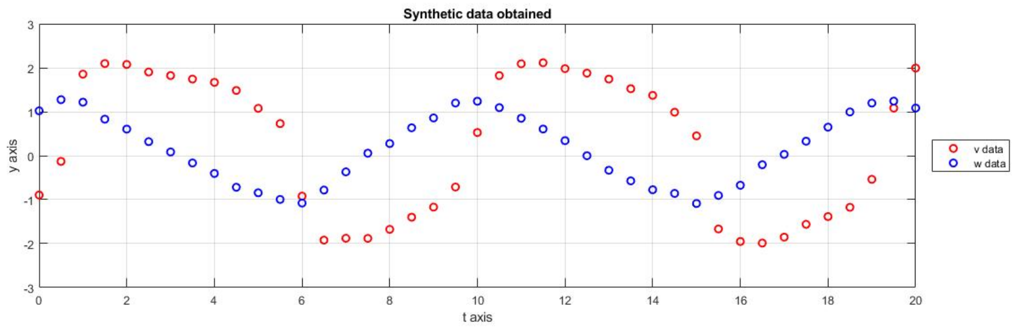


Figure 2: Data obtained by randomization

where,

$$m_1 = f(t_n, v_n, w_n); \quad k_1 = g(t_n, v_n, w_n)$$
$$m_2 = f(t_n + \frac{1}{2}h, v_n + \frac{1}{2}hm_1, w_n + \frac{1}{2}hk_1); \quad k_2 = g(t_n + \frac{1}{2}h, v_n + \frac{1}{2}hm_1, w_n + \frac{1}{2}hk_1)$$
$$m_3 = f(t_n + \frac{1}{2}h, v_n + \frac{1}{2}hm_2, w_n + \frac{1}{2}hk_2); \quad k_3 = g(t_n + \frac{1}{2}h, v_n + \frac{1}{2}hm_2, w_n + \frac{1}{2}hk_2)$$
$$m_4 = f(t_n + h, v_n + hm_3, w_n + hk_3); \quad k_4 = g(t_n + h, v_n + hm_3, w_n + hk_3)$$

1.4 Error Function

The error function to be minimized is the $l-2$ norm of the difference between the solutions of (1) and (2) and the data is denoted by $E(c, b)$ and

$$E(c, b) = \|(v(t_i), w(t_i)) - (v_i, w_i)\| \tag{5}$$

where (t_i, v_i) and (t_i, w_i) are the data, and $v(t)$ and $w(t)$ are the solutions to the system (1) and (2).

1.5 Parameters Fitting

We are going to assume that $a = 0.2$ is known, and the objective is to fit the parameter c and b to the system (1), (2). For this purpose, we are going to minimize the error function given by (5), hence we have a nonlinear regression problem to deal with.
We will use the interpolation method, that is, in order the evaluate the error function for a particular (c, b), the system (1) and (2) are first solve numerically using RK4. The solution is then used to construct a clamped cubic spline interpolation function that can be evaluated to obtain values for v and w at the various $t_i's$ in the error function. Next, we are going to use different minimization routines to find the optimum and compare the performances of these routines.

1.6 Results

We now use different optimization routine to solve the problem and the results are tabulated below.

	fminsearch	fminunc	ga
(c, b) optimal	$(3.0182, 0.3619)$	$(3.0182, 0.3619)$	
Minimum $E(c, b)$	0.587467	0.587467	
Number of iteration	41	15	

1. We used the same initial condition for both ”fminsearch” and ”fminunc” routine and as expected ”fminunc” was faster.
2. Genetic algorithm results keeps change at every run of the code. So, I could not compare it with the earlier routines.
3. I tried to use the particle swarm optimization as well, but I keep getting error. It keeps complaining about the function evaluation.

1.7 Appendices

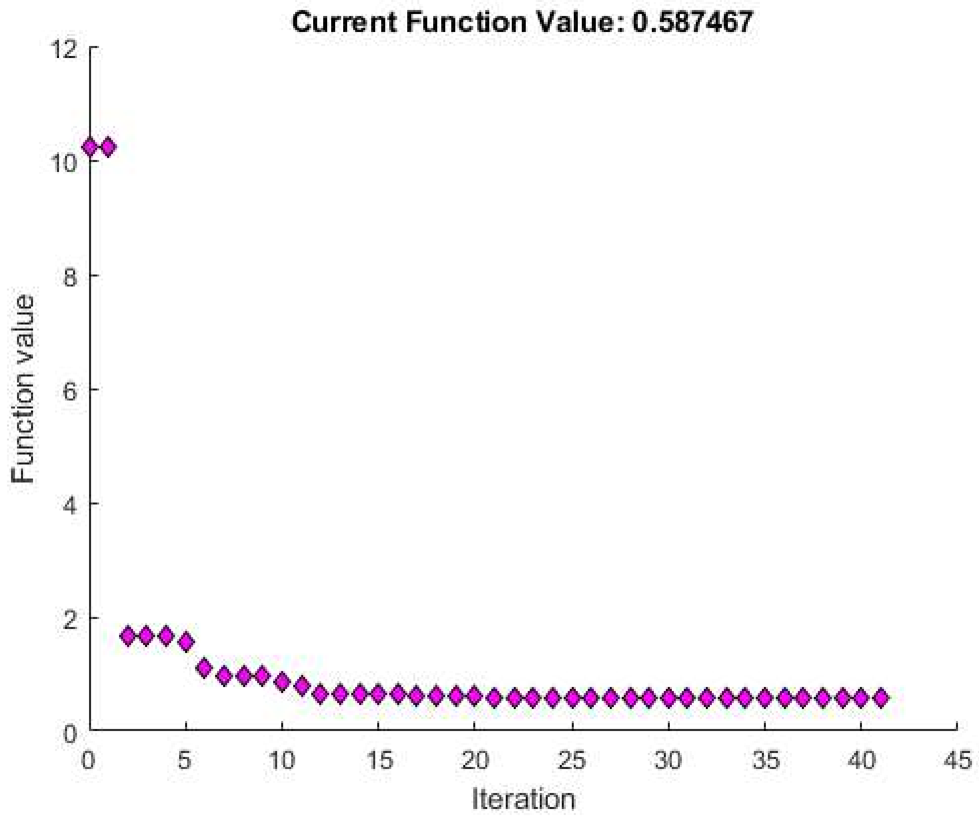
(A.) fminsearch minimization

```
% fminsearch with Nelder-Mead
% here we use fminsearch to find the minimizer of the error function E(c,b)
% and display the iteration procedure as well as the function plot
clear
clc

load('synthetic.mat')
x0 = [3 0.5];
options = optimset('Display','iter','PlotFcns',@optimplotfval);
fun = @(t)(errorFunction2(t,datav,dataw,T,M));
[x,fval,exitflag,output] = fminsearch(fun,x0,options);
```

```
 Iteration   Func-count     min f(x)         Procedure
     0            1          10.2268
     1            3          10.2268         initial simplex
     2            5          1.65302         expand
     3            6          1.65302         reflect
     4            8          1.65302         contract inside
     5           10          1.55463         contract inside
     6           12          1.08974         contract inside
     7           14         0.959423         reflect
     8           16         0.959423         contract inside
     9           18         0.959423         contract inside
    10           20         0.847515         expand
    11           22         0.803015         reflect
    12           24         0.640871         expand
    13           25         0.640871         reflect
    14           27         0.640871         contract inside
    15           29         0.640871         contract outside
    16           30         0.640871         reflect
    17           32         0.597021         expand
    18           33         0.597021         reflect
    19           35         0.597021         contract inside
    20           37         0.597021         contract inside
    21           39         0.588255         reflect
    22           41         0.588255         contract outside
    23           42         0.588255         reflect
    24           44         0.587738         contract inside
    25           46         0.587738         contract inside
    26           47         0.587738         reflect
    27           49         0.587599         contract inside
    28           51         0.587515         contract outside
    29           53         0.587515         contract inside
    30           55         0.587478         contract inside
    31           56         0.587478         reflect
    32           58         0.587478         contract inside
    33           60         0.587475         contract outside
    34           62         0.587469         contract inside
    35           64         0.587468         contract inside
    36           66         0.587468         contract outside
    37           68         0.587467         contract inside
    38           70         0.587467         contract inside
    39           72         0.587467         contract inside
    40           74         0.587467         contract outside
    41           76         0.587467         contract inside
```

```
Optimization terminated:
 the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-04
 and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-04
```



(B.) fminunc minimization

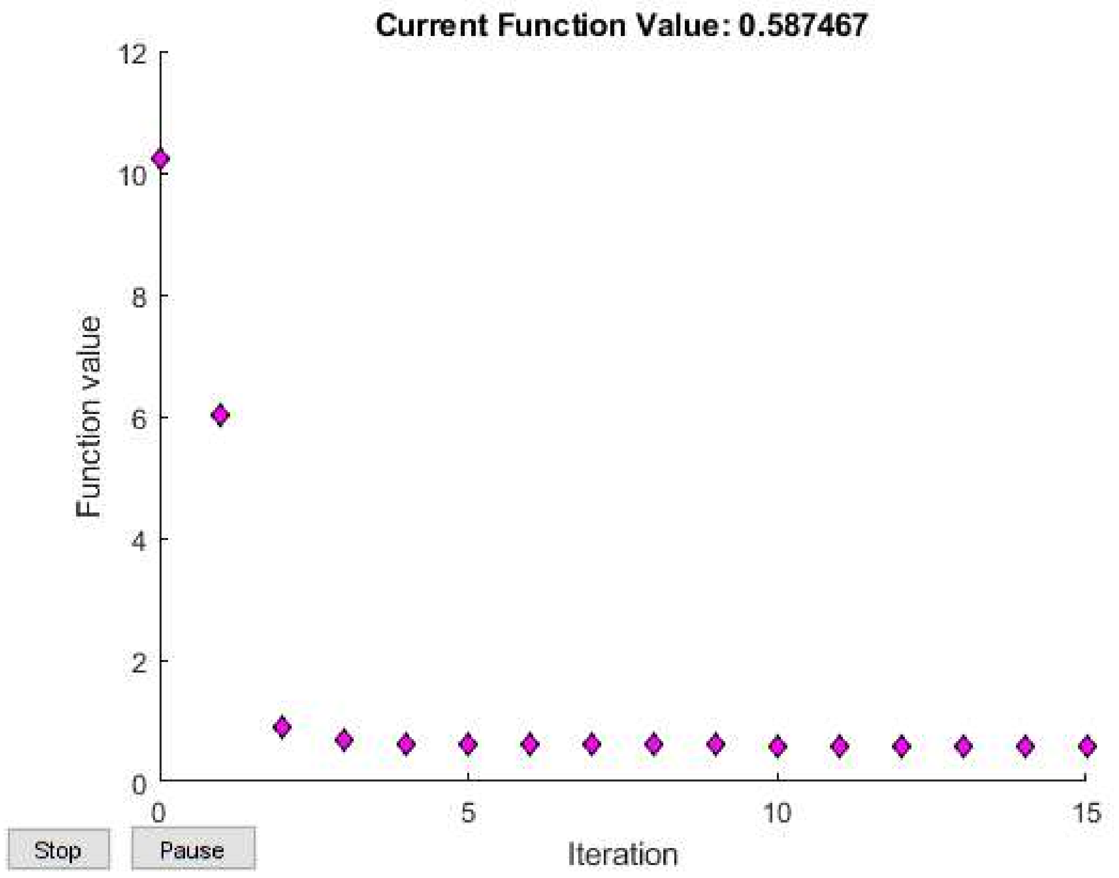
```matlab
% fminunc to minimize the error function E(c,b) for FitzHugh-Nagumo
% equations
clear
clc

load('synthetic.mat')
x0 = [3 0.5];
options = optimset('Display','iter','PlotFcns',@optimplotfval);
fun = @(t)(errorFunction2(t,datav,dataw,T,M));
[x,fval,exitflag,output] = fminunc(fun,x0,options);
```

```
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
     0           3          10.2268                           152
     1           9          6.01588      0.00112005            94
     2          12         0.895076               1          22.8  
     3          15         0.684901               1          14.9  
     4          18          0.62015               1          2.94  
     5          21         0.616523               1         0.833  
     6          24         0.616013               1         0.624  
     7          27         0.613016               1           2.1  
     8          30         0.608419               1          3.41  
     9          33         0.601601               1          4.17  
    10          36         0.594411               1          3.37  
    11          39         0.588362               1          1.07  
    12          42         0.587524               1          0.17  
    13          45         0.587468               1        0.0272  
    14          48         0.587467               1      0.000408  
    15          51         0.587467               1      2.29e-05  

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the optimality tolerance.
```

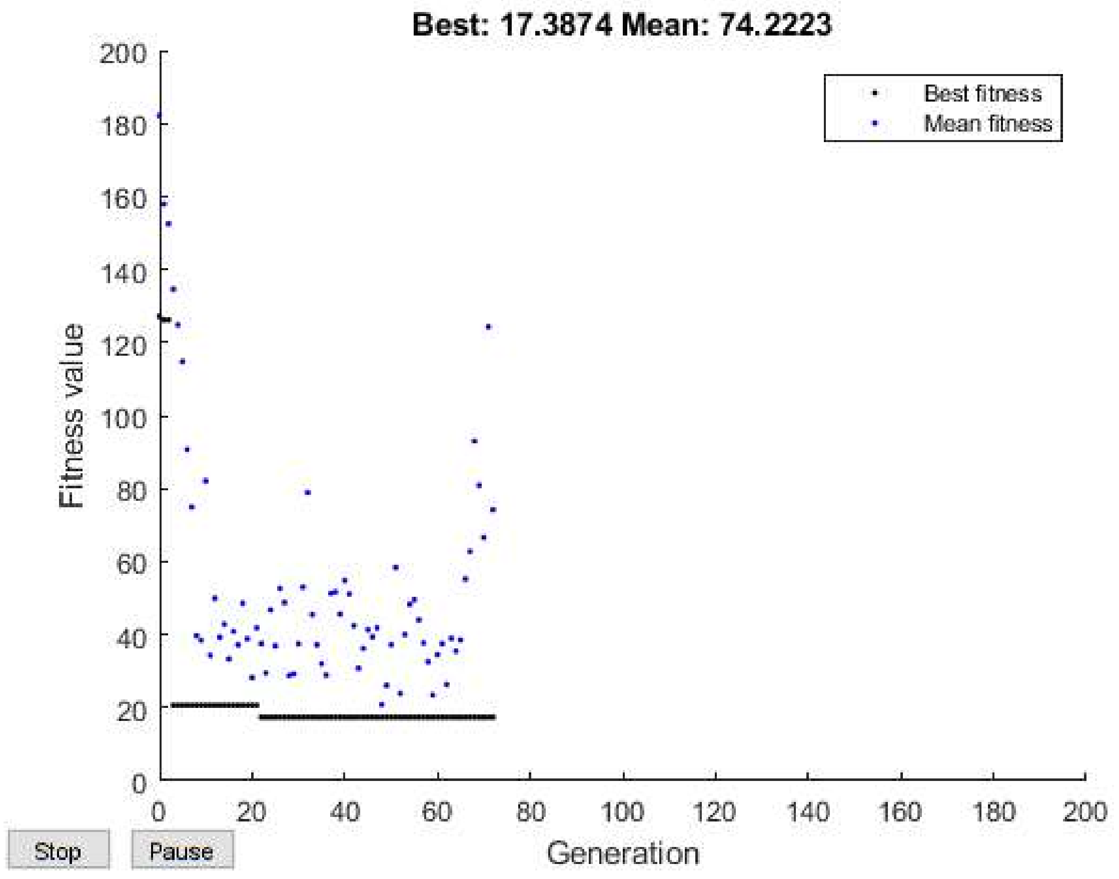


Current Function Value: 0.587467
Function value
Iteration
Stop
Pause

(C.) ga minimization

```
% ga to minimize the error function E(c,b) for FitzHugh-Nagumo
% equations
clear
clc
load('synthetic.mat')
x0 = [3 0.5];
options = optimoptions('ga','PlotFcn',@gaplotbestf);
fun = @(t)(errorFunction2(t,datav,dataw,T,M));
nvars = 2;
lb = [];
ub = [];
% [x,fval,exitflag,output] = ga(fun,nvars,lb,ub,options);
[x,fval,exitflag,output] = ga(fun,nvars,options);
```

```
Optimization terminated: average change in the fitness value less than options.FunctionTolera
nce.
```



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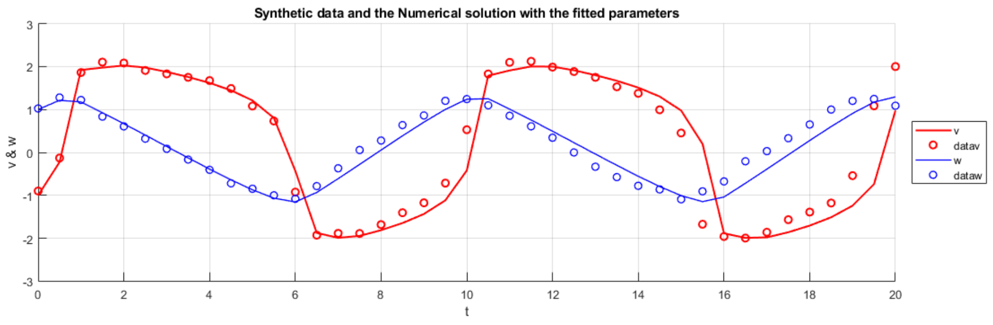

Figure 3: Data and the numerical solution with the fitted parameters