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Rates of Return

Holding Period Return

Consider a stock with beginning price P_0, ending price P_1 and a dividend payment of d.

- The *holding period return* is

$$\begin{aligned} HPR &= \frac{P_1 - P_0 + d}{P_0} \\ &= \underbrace{\frac{P_1 - P_0}{P_0}}_{\text{capital gains yield}} + \underbrace{\frac{d}{P_0}}_{\text{dividend yield}} . \end{aligned}$$

This definition can be used for assets other than stocks (e.g. a bond with a coupon payment).

Holding Period Return Example

- On Nov 9th 2012, Apple stock closed at $P_0 = \$547.06$.
- On Nov 12th, Apple payed a dividend of $d = \$2.65$ per share and the price closed at $P_1 = \$542.83$.
- What was the HPR?

$$\begin{aligned} HPR &= \frac{\$542.83 - \$547.06}{\$547.06} + \frac{\$2.65}{\$547.06} \\ &= \frac{-\$4.23}{\$547.06} + \frac{\$2.65}{\$547.06} \\ &= \frac{-\$1.58}{\$547.06} \\ &= -0.00289. \end{aligned}$$

Gross and Net Returns

Forget dividends or cash payouts for a moment.

- The *capital gains yield* is

$$\underbrace{\frac{P_1 - P_0}{P_0}}_{\text{net return}} = \underbrace{\frac{P_1}{P_0}}_{\text{gross return}} - 1.$$

Gross and Net Returns

What's the difference between net and gross returns?

- The net return is the fraction of your invested money that you gain by holding the asset, excluding the original money.

- The gross return is the total gain, including your original money. It is the factor by which you multiply your original invested amount to determine the final invested amount.

Multi-period Returns

Suppose an asset has net returns $\{r_t\}_{t=0}^{T}$. Consider two forms of average returns:

$$\text{Arithmetic Average} = \frac{1}{T}\sum_{t=0}^{T} r_t$$

and

$$\text{Geometric Average} = \left(\prod_{t=0}^{T}(1 + r_t)\right)^{\frac{1}{T}}.$$

The geometric average is the *constant* return that would have to be earned each period to yield the same final value of the asset.

Annualized Returns - EAR

Suppose you enter into a contract to pay or receive a net rate of return r on an asset for each of n periods in a year.

- $n = 12$ is a monthly contract.

- $n = 4$ is a quarterly contract.

- The Effective Annual Rate (EAR) is

$$1 + \text{EAR} = (1 + r)^n.$$

Annualized Returns - APR

Suppose you enter into a contract to pay or receive a net rate of return r on an asset for each of n periods in a year.

- The Annual Percentage Rate (APR) is

$$\text{APR} = n \times r.$$

The APR ignores compounding (as seen in the following example).

Annualized Returns - Example

You invest $100 in an asset that pays 5% return each quarter for one year.

$$Q1 : \$100 \times 1.05 = \$105$$
$$Q2 : \$105 \times 1.05 = \$110.25$$
$$Q3 : \$110.25 \times 1.05 = \$115.76$$
$$Q4 : \$115.76 \times 1.05 = \$121.55$$

Annualized Returns - Example

$$EAR : (1.05)^4 - 1 = 0.2155$$
$$APR : 0.05 \times 4 = 0.2$$
$$HPR : \frac{\$121.55 - \$100}{\$100} = 0.2155.$$

EAR and APR

What is the relationship between EAR and APR?

Since $r = \frac{\text{APR}}{n}$ we have

$$1 + \text{EAR} = \left(1 + \frac{APR}{n}\right)^n.$$

We can rearrange the equation above to get

$$\text{APR} = \left[(1 + \text{EAR})^{\frac{1}{n}} - 1\right] \times n.$$

Continuous Compounding

Continuous compounding is what occurs when we allow the number of periods in the year, n, to become large.

- For daily returns, $n = 365$.
- For hourly returns, $n = 8760$.
- For returns each minute, $n = 525,000$.
- For returns each second, $n = 31,536,000$.

Continuous Compounding

Continuous compounding is the limit, when $n = \infty$. In this case

$$\lim_{n \to \infty} \left(1 + \frac{\text{APR}}{n}\right)^n = e^{\text{APR}}.$$

So, under continuous compounding

$$1 + \text{EAR} = e^{\text{APR}}$$

or

$$\text{APR} = \ln(1 + \text{EAR}).$$

Inflation

Inflation is the increase of the general price level over time.

- Inflation erodes the purchasing power of a given amount of money over time.
- In the presence of inflation, an asset that yields a return of r doesn't actually generate r units of additional real purchasing power for each dollar invested.

Nominal vs. Real Returns

In the previous slides we computed nominal returns.

- Let us momentarily change notation and refer to the nominal return of an asset as R.
- Then the real return of the asset is the nominal return discounted by inflation:

$$1 + r = \frac{1 + R}{1 + \pi}.$$

- r is the net real return and π is net inflation.

Nominal vs. Real Returns

- This relationship is approximated by

$$r \approx R - \pi.$$

See the proof on the next slide.

Nominal vs. Real Returns - Proof

The proof requires an approximation. For some small number $\varepsilon > 0$,

$$\ln(1 + \varepsilon) \approx \varepsilon.$$

Thus,

$$\begin{aligned} 1 + r &= \frac{1 + R}{1 + \pi} \\ \Rightarrow \ln(1 + r) &= \ln\left(\frac{1 + R}{1 + \pi}\right) \\ \Rightarrow \ln(1 + r) &= \ln(1 + R) - \ln(1 + \pi) \\ \Rightarrow r &\approx R - \pi. \end{aligned}$$

Nominal vs. Real Returns - Example

Suppose you can invest in a CD that pays 8% return over the next year and that inflation is 5% during the same period.

- $R = 0.08.$
- $\pi = 0.05.$
- $r \approx 0.08 - 0.05 = 0.03.$

The actual real rate of return is

$$r = \frac{1.08}{1.05} - 1 = 0.0286.$$

Expected Inflation

In practice, future inflation is not known, even though the nominal rate of return may be known with certainty.

- Think of a fixed-income asset.

- In this case

$$R = r + E[\pi].$$

- $E[\pi]$ is expected inflation.

Expected Inflation

- The returns to typical government bonds are nominal.

- In 1997, the U.S. Treasury introduced "Treasury Inflation-Protected Securities" (TIPS).

- These have coupon and principle payments that are corrected for observed inflation over time.

- The difference between these rates of return on these two instruments can be treated as a measure of expected inflation.

Binomial Trees

Random Walk

A *random walk* is a stochastic process the evolves in the following manner:

$$Y_t = Y_{t-1} + \varepsilon_t.$$

- If $\varepsilon_t \sim N(0, 1)$, then Y_t is a continuous random variable and is a referred to as a Gaussian random walk.
- If ε_t is drawn from a discrete distribution, then Y_t is also a discrete random variable.
- We will refer to ε_t as the *innovation* or *shock*.

Binomial Distribution

Suppose a random variable X can only take one of two values X_u and X_d.

- If $P(X = X_u) = p$ and $P(X = X_d) = 1 - p$, then X follows a Bernoulli distribution with parameter p.
- In notation: $X \sim Bernoulli(p)$.
- The sum of n independent Bernoulli random variables is a Binomial random variable with parameters n and p.
- In notation: if $Y_t = \sum_{i=t-n}^{t} X_i$ then $Y_t \sim Binomial(n, p)$.

Bernoulli Random Walk

A very simple model of stock prices assumes that they follow a random walk with Bernoulli innovations:

$$P_t = P_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim Bernoulli(p).$$

- Interpretation: The stock price can only move to one of two values at each time period.

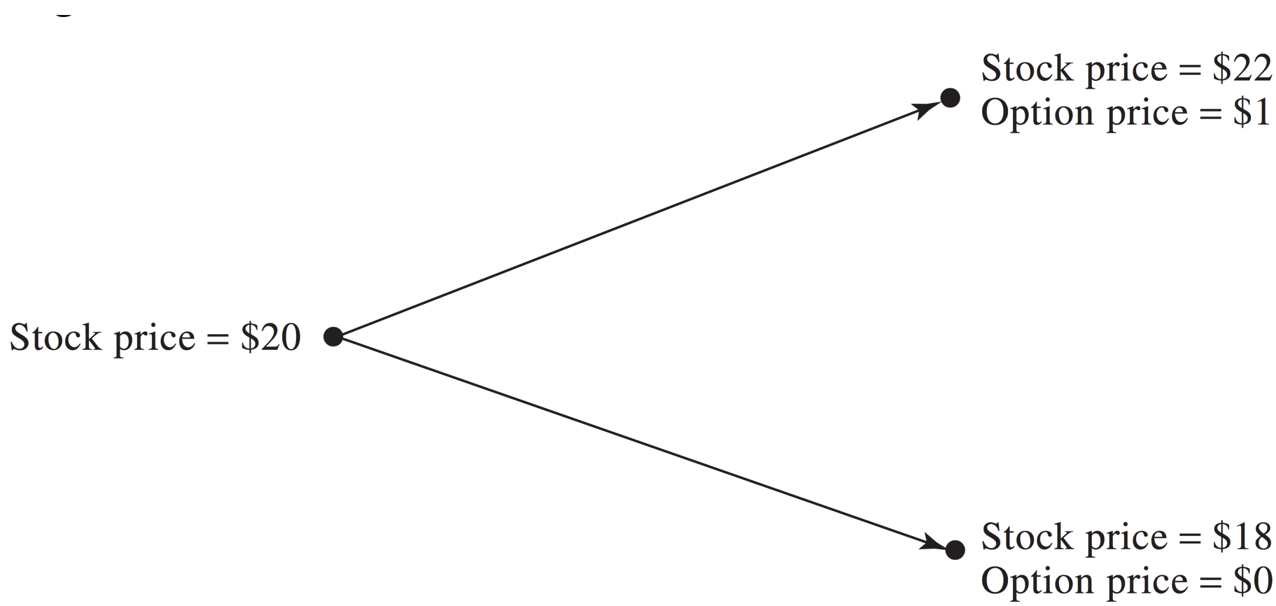
- This model is useful for valuing options on the asset.

Binomial Tree Example

Suppose a stock price is currently $\$20$ and that in three months it will either be $\$18$ or $\$22$.

- You would like to buy a call option with strike of $\$21$.
- If the price moves up to $\$22$, the option will be worth $\$1$ at expiry.
- If the price moves down to $\$18$, the option will be worthless.

Binomial Tree Example



A Riskless Portfolio

Consider the following portfolio:

- Buy Δ shares of the stock.
- Sell one call option.
- If the prices moves up, the value of the portfolio is $\$22\Delta - 1$.
- If the prices moves down, the value of the portfolio is $\$18\Delta$.
- If Δ can be chosen so that $\$22\Delta - 1 = \$18\Delta$, the portfolio is risk free.

A Riskless Portfolio

Clearly, if $\Delta = 0.25$ the payoffs are equal and the portfolio is riskless.

- Thus a portfolio which is long 0.25 shares and short 1 option is riskless.
- If the prices moves up, the value of the portfolio is $\$22\Delta - 1 = \$4.5$.
- If the prices moves down, the value of the portfolio is $\$18\Delta = \$4.5$.
- Although it's not possible to buy 0.25 shares, you can buy 100 shares and short 400 calls.

Value of the Riskless Portfolio

Since the portfolio has no risk, it must earn the risk-free rate of return.

- Suppose the continuously compounded risk-free interest rate is 12% per annum.
- The value of the portfolio today must be the present value of the riskless $\$4.5$ payoff:

$$4.5e^{-0.12\times 3/12} = 4.367.$$

Value of the Riskless Portfolio

- The cost of the portfolio is $\$20\Delta - f = 5 - f$, where f is the current price of the call.
- The cost and value of the portfolio must be equal, otherwise an arbitrage opportunity would exist:

$$\begin{aligned} 5 - f &= 4.367 \\ \Rightarrow f &= 0.633. \end{aligned}$$

One-Step Binomial Tree

We can generalize the previous example.

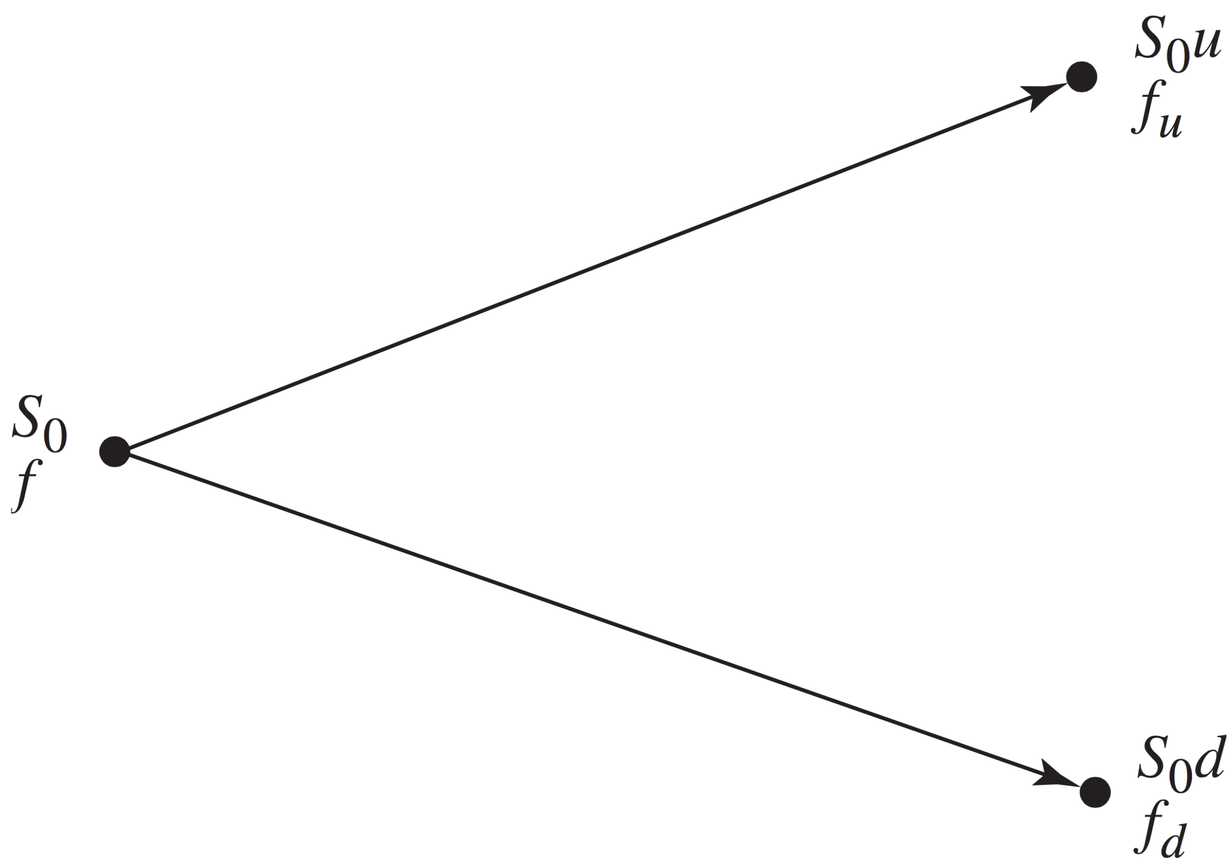
- The current stock price is S_0.
- The current option price is f.
- Time periods are T years.

- At the end of the next time period, the stock price will be either S_0u or S_0d, where $u > 1$ and $d < 1$.

- $u - 1$ represents a percentage increase and $d - 1$ represents a percentage decrease.

- The option values at the end of next time period are either f_u or f_d.

One-Step Binomial Tree



One-Step Portfolio

Consider a portfolio that is long Δ shares and short 1 call option.

- If the stock price moves up, the value is $S_0u\Delta - f_u$.

- If the stock price moves down, the value is $S_0d\Delta - f_d$.

- Find Δ such that the payoffs are equal:

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}.$$

One-Step Portfolio

Since the payoffs are equivalent in each state, the portfolio is riskless.

- Suppose the continuously compounded risk-free rate is r.
- The present value of the portfolio payoff is $\left(S_0u\Delta - f_u\right)e^{-rT}$.
- The current cost of the portfolio is $S_0\Delta - f$.

Call Option Price

Equating cost and value of the portfolio:

$$
\begin{aligned}
S_0\Delta - f &= \left(S_0u\Delta - f_u\right)e^{-rT} \\
\Rightarrow f &= S_0\Delta\left(1 - ue^{-rT}\right) + f_u e^{-rT} \\
&= S_0\frac{f_u - f_d}{S_0u - S_0d}\left(1 - ue^{-rT}\right) + f_u e^{-rT} \\
&= \frac{f_u\left(1 - de^{-rT}\right) + f_d\left(ue^{-rT} - 1\right)}{u - d} \\
&= e^{-rT}\left(pf_u + (1-p)f_d\right),
\end{aligned}
$$

where $p = \frac{e^{rT} - d}{u - d}$.

Revisiting One-Period Example

Recall the previous example:

- $u = 1.1, d = 0.9, f_u = 1, f_d = 0, r = 0.12$ and $T = 3/12$.
- Thus $p = \frac{e^{0.12 \times 3/12} - 0.9}{1.1 - 0.9} = 0.6523$.
- $f = e^{-0.12 \times 0.25}(0.6523 \times 1 + 0.3477 \times 0) = 0.633$.

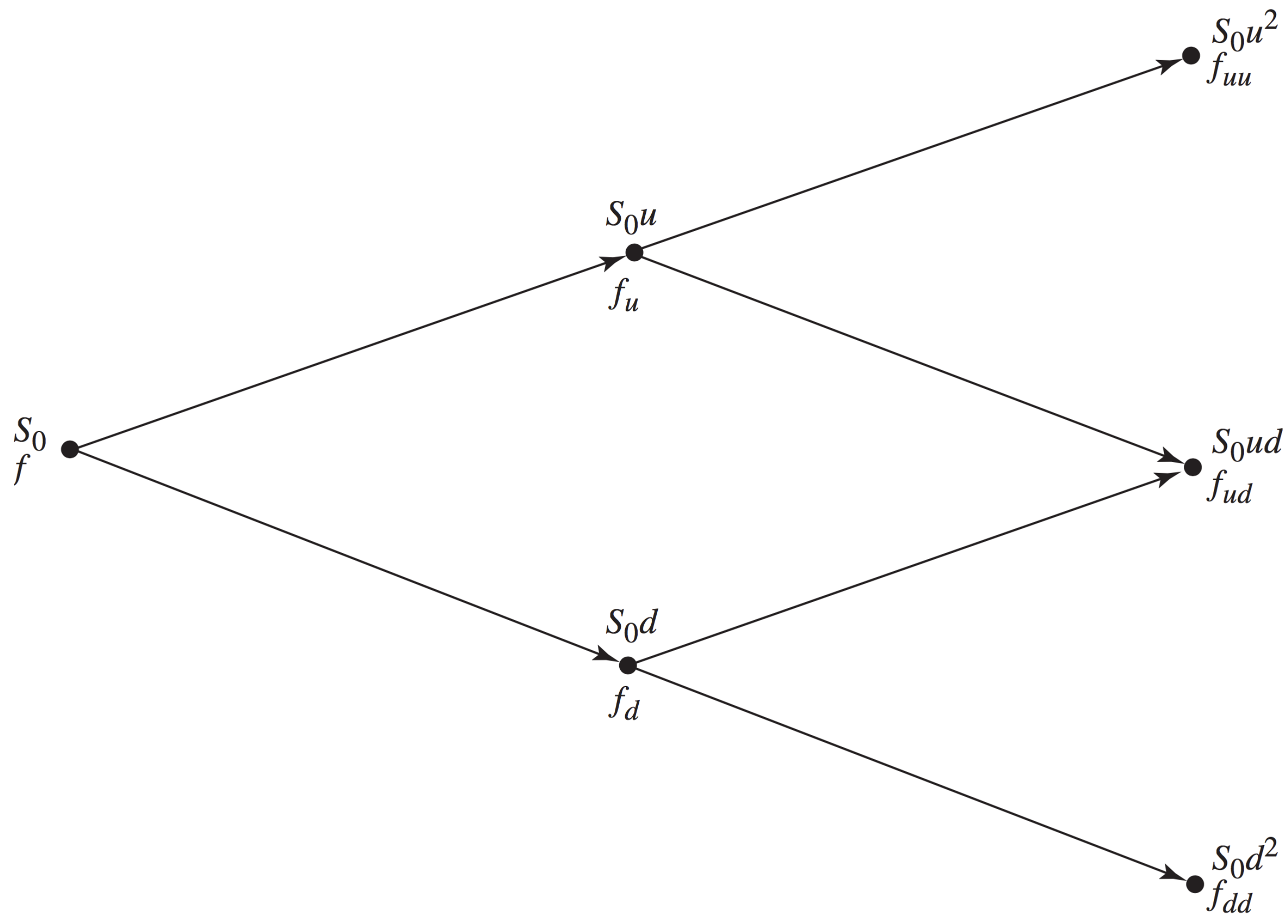
Two-Step Binomial Tree

Consider a two-period example, where the stock price can move up or down during the first period, and then can move up or down during the second period.

- There will be either three or four possible stock prices at the end of time two.

- We can solve the problem backwards:
  - First solve for the prices of the options in each state of the world at the end of period 1 (two separate problems).
  - Use those prices to solve for the price at the current time (single problem).

Two-Step Binomial Tree



Two-Step Binomial Tree

Let us now denote a time period as Δt.

- The value (price) of the call option at the end of the first period in the high state is:

$$f_u = e^{-r\Delta t}\left(pf_{uu} + (1-p)f_{ud}\right)$$

- The value (price) of the call option at the end of the first period in the low state is:

$$f_d = e^{-r\Delta t}\left(pf_{ud} + (1-p)f_{dd}\right)$$

Two-Step Binomial Tree

- The value (price) of the call option at the current time is:

$$\begin{aligned} f &= e^{-r\Delta t}\left(pf_u + (1-p)f_d\right) \\ &= e^{-2r\Delta t}\left(p^2 f_{uu} + 2 * p(1-p)f_{ud} + (1-p)^2 f_{dd}\right). \end{aligned}$$

Two-Step Binomial Tree Example

Continuing with the previous example: $u = 1.1, d = 0.9, r = 0.12$ and $\Delta t = 3/12$.

- If the stock price moves up twice $S_0 u^2 = 24.2$ and $f_{uu} = 3.2$.

- If the stock price moves down twice $S_0 d^2 = 16.2$ and $f_{dd} = 0$.

- If the stock price moves up and down $S_0 ud = 19.8$ and $f_{ud} = 0$.

Two-Step Binomial Tree Example

Given the numbers above:

$$\begin{aligned} f_u &= e^{-0.12\times 0.25}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257 \\ f_d &= e^{-0.12\times 0.25}(0.6523 \times 0 + 0.3477 \times 0) = 0 \\ f &= e^{-0.12\times 0.25}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823. \end{aligned}$$

Valuing Put Options

The foregoing treatment was not unique to call options.

- We likewise could have determine the number of shares necessary to create a riskless portfolio that is long the stock and short a put.

- This would result in a symmetric solution for Δ, but using the final values of the put in each state.

- The same formulas for the one- and two-period trees can be used to value the option.
  - The only difference is that the option payoffs at each node are different than the call.

Two-Step Put Option Example

The current price of a stock is $S_0 = \$50, u = 1.2, d = 0.8, r = 0.05$ and $\Delta t = 1$. Consider a put with strike price $\$52$.

- If the stock price moves up twice $S_0 u^2 = 72$ and $f_{uu} = 0$.
- If the stock price moves down twice $S_0 d^2 = 32$ and $f_{dd} = 20$.
- If the stock price moves up and down $S_0 ud = 48$ and $f_{ud} = 4$.
- $p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05 \times 1} - 0.8}{1.2 - 0.8} = 0.6282$.

Two-Step Put Option Example

Given the numbers above:

$$\begin{aligned} f_u &= e^{-0.05 \times 1}(0.6282 \times 0 + 0.3718 \times 4) = 1.4147 \\ f_d &= e^{-0.05 \times 1}(0.6282 \times 4 + 0.3718 \times 20) = 9.4635 \\ f &= e^{-0.05 \times 1}(0.6282 \times 1.4147 + 0.3718 \times 9.4635) = 4.1923. \end{aligned}$$

American Options

To value an American option with a binomial tree:

- At each node, compute the value of the call implied by the subsequent nodes (using the foregoing methods).
- Compute the value of immediate exercise.
- Assign the value as the maximum of these two quantities.
- Proceed in reverse fashion until the value is computed at the first node, taking into account the possibilities of early exercise.

American Option Example

Suppose that the put in the previous example is an American option.

- If the stock price moves down, the value of immediate exercise is $\$12$.
- This means $f_d = \max(9.4626, 12) = 12$.

- As a result:

$$f = e^{-0.05 \times 1}(0.6282 \times 1.4147 + 0.3718 \times 12) = 5.0894.$$

- Note that the value of the American option is greater than that of the European option.

Delta

Recall that Δ is the number of shares that must be purchased to create a riskless portfolio with a short option.

- It is the ratio of possible option values over the possible stock prices.

- It is often written and referred to as *delta*.

- Following this investment strategy is called *delta hedging*.

- Call option deltas are positive and put option deltas are negative.

Call Deltas

In the two-period call option example:

$$\begin{aligned}\Delta_0 &= \frac{2.0257 - 0}{22 - 18} = 0.5064 \\ \Delta_u &= \frac{3.2 - 0}{24.2 - 19.8} = 0.7273 \\ \Delta_d &= \frac{0 - 0}{19.8 - 16.2} = 0,\end{aligned}$$

where Δ_0 is the delta for the first time period and Δ_u and Δ_d are the deltas for the second periods if the price moves up and down.

- Notice that delta changes with time - i.e you need to rebalance your portfolio to maintain a riskless hedge.

- This is called dynamic hedging.

Put Deltas

In the two-period put option example:

$$\Delta_0 = \frac{1.4147 - 9.4636}{60 - 40} = -0.4024$$
$$\Delta_u = \frac{0 - 4}{72 - 48} = -0.1667$$
$$\Delta_d = \frac{4 - 20}{48 - 32} = -1.$$

Stock Return Volatility

Suppose σ^2 is the annualized variance of the stock returns.

- Assuming the returns are independent, $\Delta t\sigma^2$ is a good approximation of the variance for period Δt.
- This means a good approximation of the standard deviation, or volatility, for period Δt is $\sqrt{\Delta t}\sigma$.

Binomial Tree Parameters

Given the current stock price, S_0, the option strike price and the risk-free rate, r:

- The binomial tree is completely determined by three parameters: u, d and Δt.
- u and d are often set so that $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$.
- In doing so, it can be shown that the volatility of the returns in the binomial tree approximate σ .

Binomial Tree Parameters

Simple binomial tree models with long time periods are unrealistic.

- We can simply increase the number of steps in the model.
- For example, for a three-month period with daily time steps, there would be a total of roughly 66 period (22 trading days per month).
- For the recombining trees we considered above, this amounts to 66 possible terminal values and $\sum_{i=1}^{65} i = 2145$ individual options nodes to value.
- For the non-recombining trees, this amounts to 2^{65} possible terminal values and $\sum_{i=1}^{65} 2^{i-1}$ individual options nodes to value.

Options on Other Assets

We have only valued options on a stock that doesn't pay dividends.

- To value other assets, we only need to adjust the interest rate r used in the calculation of p to account for any earnings or costs associated with the underlying asset.

- Discounting to present value is always done with r.

Options on Other Assets

For stock paying dividends:

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}.$$

For currencies:

$$p = \frac{e^{(r-r_f)\Delta t} - d}{u - d}.$$

For futures:

$$p = \frac{1 - d}{u - d}.$$

Wiener Processes

Stochastic Processes

A stochastic process is a collection of random variables, indexed by time.

- More simply, it can be thought of as a single random variable that evolves dynamically with time.
- Stochastic processes can be discrete/continuous time: either they evolve discretely or continuously.
- They can be discrete/continuous value: at each time, they are drawn from either a discrete or continuous probability distribution.
- We will develop a continuous time/continuous value model for asset prices.
  - Note that we only observe discrete value and discrete time observations for actual asset prices.

Properties of Expected Value

Given random variables X and Y, define a new random variable $Z = X + Y$. Then:

$$E[Z] = E[X + Y] = E[X] + E[Y].$$

Suppose $E[X] = E[Y] = \mu$. Then:

$$E[Z] = E[X] + E[Y] = 2\mu.$$

Properties of Expected Value

Extending the prior result, given random variables $\{X_i\}_{i=1}^{n}$, define a new random variable $Z = \sum_{i=1}^{n} X_i$. Then:

$$E[Z] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i].$$

Suppose $E[X_i] = \mu, \;\; i = 1, \ldots, n$. Then:

$$E[Z] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \mu = n\mu.$$

Properties of Variance

Given random variables X and Y, define a new random variable $Z = X + Y$. Then:

$$Var(Z) = Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y).$$

Suppose $Var(X) = Var(Y) = \sigma^2$ and that they are independent $(Cov(X, Y) = 0)$. Then:

$$\begin{aligned} Var(Z) &= Var(X) + Var(Y) = 2\sigma^2 \\ \Rightarrow Sd(Z) &= \sqrt{Var(Z)} = \sqrt{2}\sigma. \end{aligned}$$

Properties of Variance

Extending the prior result, given random variables $\left\{X_i\right\}_{i=1}^{n}$, define a new random variable $Z = \sum_{i=1}^{n} X_i$. Then:

$$Var(Z) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov(X_i, X_j).$$

Suppose $Var(X_i) = \sigma^2, \;\; i = 1, \ldots, n$ and that each random variable is independent of the others $(Cov(X_i, X_j) = 0, \;\; i \neq j)$. Then:

$$\begin{aligned} Var(Z) &= \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \sigma^2 = n\sigma^2 \\ \Rightarrow Sd(Z) &= \sqrt{Var(Z)} = \sqrt{n}\sigma. \end{aligned}$$

Normal Random Variables

Suppose $X_i \sim N(\mu, \sigma), \;\; i = 1, \ldots, n.$

- Define a new random variable $Z = \sum_{i=1}^{n} X_i$.
- We already know $E[Z] = n\mu$ and $Sd(Z) = \sqrt{n}\sigma$.
- However, the sum of Normals is also Normal: $Z \sim N\left(n\mu, \sqrt{n}\sigma\right)$.

Additive Normal Example

Suppose the monthly return on an asset is Normally distributed for each month over the next year.

- We will assume the expected value is 1% and standard deviation is 5% and that returns across months are independent:

$$R_i \overset{i.i.d.}{\sim} N(0.01, 0.05), \quad i = 1, \ldots, 12.$$

- The annual return is the sum of monthly returns: $R_a = \sum_{i=1}^{12} R_i$.

- The distribution of the annual return is $R_a \sim N(0.12, 0.1732)$.

Additive Normal Example

Note that we can also work backwards.

- Suppose you know that the annual expected return is $\mu_a = 0.12$, the annual standard deviation is $\sigma_a = 0.1732$ and that monthly returns are independent.

- The monthly expected return and standard deviation are:

$$\mu_m = \frac{\mu_a}{n} = \frac{0.12}{12} = 0.01$$
$$\sigma_m = \frac{\sigma_a}{\sqrt{n}} = \frac{0.1732}{\sqrt{12}} = 0.05.$$

Wiener Process

A random variable $Z(t)$ follows a Wiener process if:

- During a short time interval Δt,

$$\Delta Z(t) = Z(t) - Z(t - \Delta t) = \varepsilon\sqrt{\Delta t}, \quad \varepsilon \sim N(0, 1).$$

- The values of $\Delta Z(t)$ for any two short intervals Δt are independent.

- As $\Delta t \to 0$, we use the notation $dZ = \varepsilon\sqrt{dt}$.

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Moments of the Wiener Process

What are the first two moments of $\Delta Z(t)$?

$$E[\Delta Z(t)] = E\left[\varepsilon\sqrt{\Delta t}\right] = \sqrt{\Delta t}E[\varepsilon] = 0$$
$$Var(\Delta Z(t)) = Var\left(\varepsilon\sqrt{\Delta t}\right) = \Delta t Var(\varepsilon) = \Delta t$$
$$Sd(\Delta Z(t)) = Sd\left(\varepsilon\sqrt{\Delta t}\right) = \sqrt{\Delta t}Sd(\varepsilon) = \sqrt{\Delta t}.$$

Wiener Process Over Long Horizon

Consider the evolution of $Z(t)$ over a longer horizon T.

- Divide the time interval $(0, T)$ into N small intervals of equal length Δt.

- Then:

$$Z(T) - Z(0) = \sum_{i=1}^{N} \varepsilon_i\sqrt{\Delta t}, \quad \varepsilon_i \overset{i.i.d.}{\sim} N(0, 1).$$

Wiener Process Over Long Horizon

- The resulting moments are:

$$E[Z(T) - Z(0)] = \sum_{i=1}^{N} \sqrt{\Delta t}E[\varepsilon_i] = 0$$
$$Var(Z(T) - Z(0)) = \sum_{i=1}^{N} \Delta t Var(\varepsilon_i) = T$$
$$Sd(Z(T) - Z(0)) = \sum_{i=1}^{N} \sqrt{\Delta t}Sd(\varepsilon_i) = \sqrt{T}.$$

Wiener Process Sample Paths

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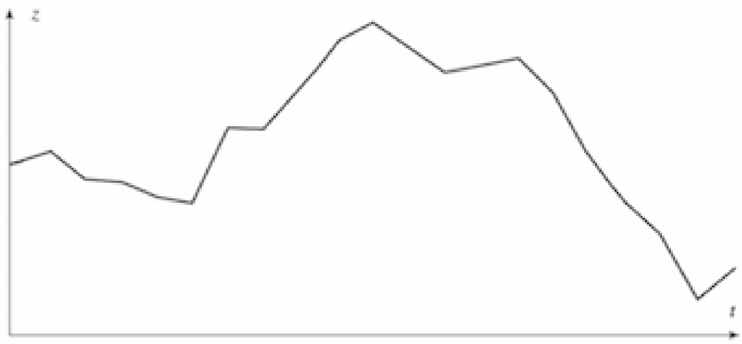
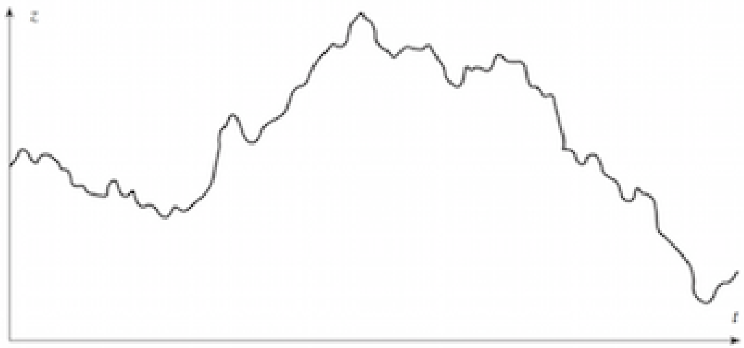
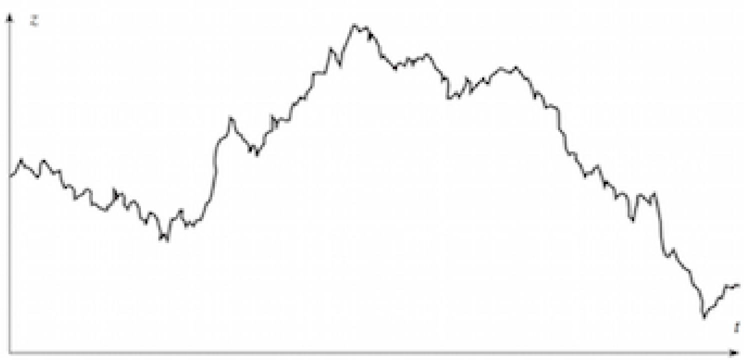


Relatively large value of Δt



Smaller value of Δt



Generalized Wiener Process

Consider the *generalized Wiener process*:

$$dX = adt + bdZ.$$

- a is the *drift rate* and b^2 is the *variance rate.*

- The basic Wiener process has drift rate zero and unit variance rate.

Generalized Wiener Process

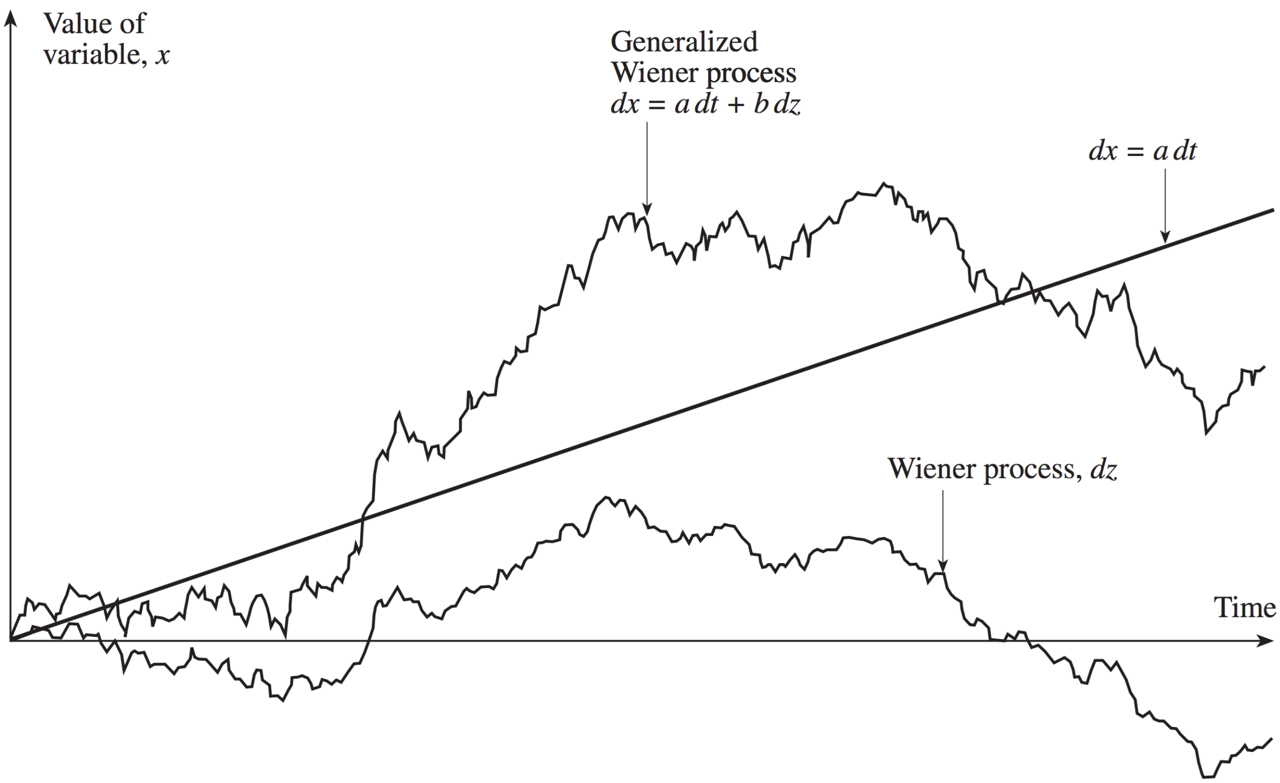
In discrete time, the generalize Wiener process is

$$\Delta X = a\Delta t + b\varepsilon\sqrt{\Delta t}.$$

The moments are:

$$E[\Delta X] = a\Delta t$$
$$Var(\Delta X) = b^2\Delta t$$
$$Sd(\Delta X) = b\sqrt{\Delta t}.$$

Wiener Process Comparison



Ito Process

An *Ito process* is a generalization of a generalized Wiener process:

$$dX = a(X,t)dt + b(X,t)dZ.$$

- The drift rate and variance rate are functions of the process value and time - i.e. they are variable.

In discrete time:

$$\Delta X = a(X,t)\Delta t + b(X,t)\varepsilon\sqrt{\Delta t}.$$

- The discrete time analog requires an approximation: the drift and variance rate are constant over small intervals Δt.

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Model for Asset Prices

We will use an Ito process as a model for asset prices.

- We think of percentage returns as being constant for a particular asset.
- This means that the size of the moves (the drift) changes with the level of the price.
- Variance also typically changes with the level of asset prices.
- This means an Ito process is a better model than a generalized Wiener process.

Model for Asset Prices

We will employ the following Ito process:

$$dS = \mu S dt + \sigma S dZ.$$

- The drift rate function takes the specific form: $a(S, t) = \mu S$.
  - The drift rate increases proportionally with the asset price and does not depend on time.
- The variance rate function takes the specific form: $b^2(S, t) = \sigma^2 S^2$.
  - The volatility, $b(S, t) = \sigma S$, increases proportionally with the asset price and does not depend on time.
- This form of an Ito process is known as *geometric Brownian motion*.

Geometric Brownian Motion

Geometric Brownian motion can also be written as:

$$\frac{dS}{S} = \mu dt + \sigma dZ.$$

- $\frac{dS}{S}$ is effectively the return.
- Note that returns evolve as a generalized Wiener process (constant drift and variance rates).
- μ and σ are the expected return and volatility of the asset.

Geometric Brownian Motion

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In discrete time, geometric Brownian motion is described as:

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}.$$

This means $\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma \sqrt{\Delta t})$.

Parameter Estimation

We can estimate the parameters μ and σ from historical data.

- Set an interval of time, Δt in years (days, weeks, months, etc).
- Collect $n + 1$ price observations for the beginning and end of each interval.
- Compute n returns for the invervals.
- Set $\hat{\mu}$ and $\hat{\sigma}$ to the sample mean and sample standard deviation of the returns.
- $\hat{\mu}$ is an estimate of $\mu \Delta t$, *not* μ.
- $\hat{\sigma}$ is an estimate of $\sigma \sqrt{\Delta t}$, *not* σ.

Simulating Geometric Brownian Motion

Let's estimate the daily mean and standard deviation of Lockheed Martin (LMT) returns using data prior to January 1, 2015:

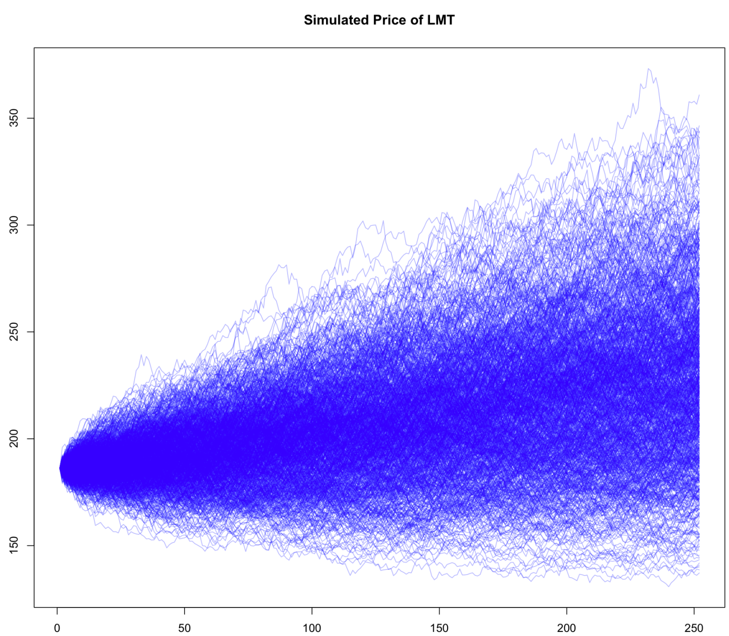
```
> library(quantmod)
> getSymbols("LMT",src="yahoo", from="2010-01-01", to="2014-12-31")
> lmtRets = dailyReturn(LMT)
> mu = mean(lmtRets)
> sigma = sd(lmtRets)
> cat(mu,sigma)
0.0008037858 0.01122918
```

We now simulate 1000 price paths of one year (250 trading days) starting on Jan 2, 2015:

```
> nSim = 1000
> nDays = 252
> S0 = 186.21 # price on Jan 2, 2015
> S = matrix(0,nrow=nDays,ncol=nSim)
> for(ix in 1:nSim){
+     SVec = rep(0,nDays)
+     SVec[1] = S0
+     for(jx in 2:nDays){
+             DeltaS = mu*SVec[jx-1] + sigma*SVec[jx-1]*rnorm(1)
+             SVec[jx] = SVec[jx-1]+DeltaS
+     }
+      S[,ix] = SVec
+ }
> matplot(S,type='l',col=rgb(0,0,1,0.3),lty=1,ylab='',main='Simulated Price of LMT')
```

Simulating Geometric Brownian Motion

The closing price on Jan 4, 2016 was $\$211.61$.



Ito's Lemma

Suppose X follows and Ito process:

$$dX = a(X,t)dt + b(X,t)dZ.$$

For some function $G(X,t)$, Ito's lemma says:

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$$dG = \left(\frac{\partial G}{\partial X}a(X,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b(X,t)^2\right)dt + \frac{\partial G}{\partial X}b(X,t)dZ.$$

Ito's Lemma

Ito's lemma says that $G(X, t)$ is an Ito process with drift and variance rates

$$\frac{\partial G}{\partial X}a(X,t) + \frac{\partial G}{\partial t} + \frac{\partial^2 G}{\partial X^2}b(X,t)$$

$$\left(\frac{\partial G}{\partial X}\right)^2 b(X,t)^2.$$

Ito's Lemma for Geometric Brownian Motion

Recall our model for stock prices:

$$dS = \mu S dt + \sigma S dZ.$$

In this case, Ito's lemma says that $G(S, t)$ follows:

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dZ.$$

Application to Forward Contracts

Recall the formula for the value of a forward contract (given current time t and horizon $T - t$):

$$F = Se^{r(T-t)}.$$

- In this case $F = G(S, t)$.

The derivatives are

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \quad \frac{\partial^2 F}{\partial S^2} = 0, \quad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}.$$

Application to Forward Contracts

Following Ito's lemma,

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$$
\begin{aligned}
dF &= \left(e^{r(T-t)}\mu S - rSe^{r(T-t)}\right)dt + e^{r(T-t)}\sigma SdZ \\
&= (\mu - r)Fdt + \sigma FdZ.
\end{aligned}
$$

Distribution of Prices

What is the distribution of S if it follows geometric Brownian motion?

- We know that returns, $\frac{dS}{S}$, are Normal.

Let's use Ito's lemma to derive the law of motion of $G(S,t) = \ln(S)$:

$$
\frac{\partial G}{\partial S} = \frac{1}{S} \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S} \quad \frac{\partial G}{\partial t} = 0
$$

$$
\Rightarrow dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ.
$$

- $\ln(S)$ follows a generalized Wiener process with drift rate $\mu - \frac{\sigma^2}{2}$ and variance rate σ^2.

Distribution of Prices

The discrete analog to the process for $\Delta\ln(S)$ is

$$
\Delta\ln(S) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}.
$$

For $\Delta t = T$, $\Delta\ln(S) = \ln(S_T) - \ln(S_0)$, which results in:

$$
\ln(S_T) - \ln(S_0) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)
$$

$$
\Rightarrow \ln(S_T) \sim N\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)
$$

- This means that prices are *lognormally* distributed.

Lognormal Distribution

Suppose $S_T \sim LN\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$.

- By defintion, this means $\ln(S_T) \sim N\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$.

- It can be shown that the mean and variance of $\smash{S_T}$ are

$$\begin{split}\begin{align} E[S_T] & = S_0 e^{\mu T} \\ Var(S_T) & = S_0^2 e^{2\mu T} \left(e^{\sigma^2 T}-1\right). \end{align}\end{split}$$

Simulation Distribution

We can check that our simulated geometric Brownian motion results in a lognormal distribution for LMT prices.

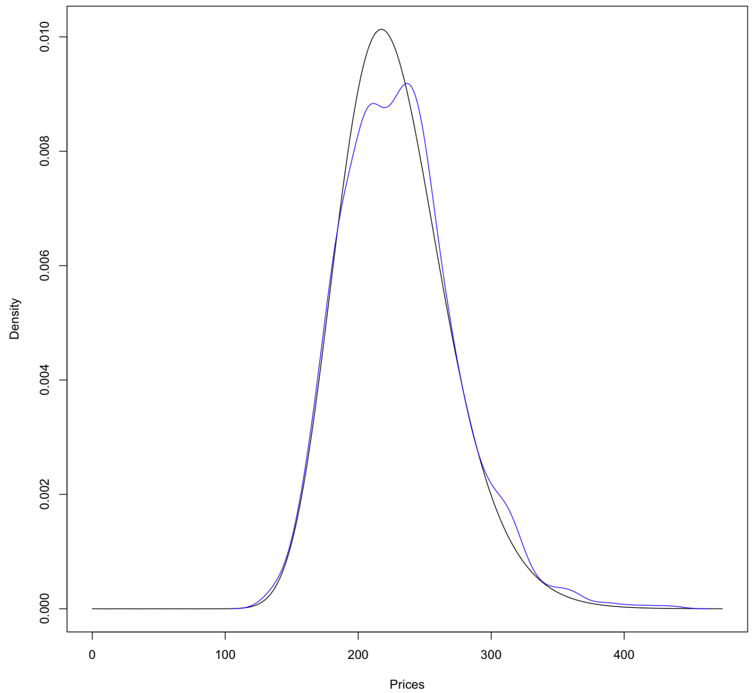
```
> lnMean = S0*exp(mu*nDays)
> lnSD = S0*exp(mu*nDays)*sqrt(exp((sigma^2)*nDays)-1)
> cat(lnMean,mean(S[nDays,]))
228.019 230.529
> cat(lnSD,sd(S[nDays,]))
40.97119 43.21174
```

Simulation Distribution

We can also plot the theoretical lognormal and the empirical distribution of LMT prices.

```
> meanOfLog = log(S0) + (mu-(sigma^2)/2)*nDays
> sdOfLog = sigma*sqrt(nDays)
> priceGrid = seq(0,lnMean+6*lnSD,length=10000)
> theoreticalDens = dlnorm(priceGrid,meanOfLog,sdOfLog)
> empiricalDens = density(S[nDays,])
> plot(priceGrid,theoreticalDens,type='l',xlab='Prices',ylab='Density')
> lines(empricalDens,col='blue')
```

Simulation Distribution



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Black-Scholes-Merton Model

Overview

The Black-Scholes-Merton (BSM) model provides a simple formula for computing the price of an option.

- It is derived in a fashion similar to the binomial option pricing model.

- A riskless portfolio is obtained by buying Δ shares of the underlying asset and shorting a single option.

- Δ represents $\frac{\partial C}{\partial S}$, where S is the price of the underlying and C is the price of the option.

- Unlike the binomial model, the BSM model Δ is only valid for an infinitesimal length of time.

BSM Assumptions

The BSM model depends on the following assumptions.

- The price of the underlying asset follows geometric Brownian motion with parameters μ and σ.
- There is no restriction on short selling.
- There are no transaction costs or taxes.
- All assets are perfectly divisible.
- The underlying doesn't pay dividends.
- There are no riskless arbitrage opportunities.
- Asset trading occurs continuously.
- The risk-free rate, r is constant and the same for all maturities.

BSM and Ito's Lemma

Suppose the price of an asset follows geometric Brownian motion:

$$dS = \mu S dt + \sigma S dZ.$$

According to Ito's lemma, the price of a derivative, f, follows:

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dZ.$$

A Riskless Portfolio

Consider a portfolio that is short one unit of the derivative and long $\frac{\partial f}{\partial S}$ units of the underlying:

$$\begin{aligned}
\Pi &= -f + \frac{\partial f}{\partial S}S \\
\Rightarrow d\Pi &= -df + \frac{\partial f}{\partial S}dS \\
&= \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt.
\end{aligned}$$

Note that the portfolio is not affected by Z, so it is riskless.

A Riskless Portfolio

- The portfolio must earn the risk-free rate of return over period dt.

$$\begin{aligned}
d\Pi &= r\Pi dt \\
\Rightarrow \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt &= r\left(f - \frac{\partial f}{\partial S}S\right)dt \\
\Rightarrow rf &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2.
\end{aligned}$$

- This is the BSM differential equation.

Boundary Conditions

The BSM differential equation is true for any derivative that depends on S.

- Boundary conditions determine the price of a particular derivative.
- For example, the boundary condition for a call option is its terminal value: $f = \max(S - X, 0)$.
- Likewise, the boundary condition for a put option is its terminal value: $f = \max(X - S, 0)$.

BSM Price of Forward

Suppose that some time ago a forward contract was entered into with delivery price K and maturity T.

- Recall that at intermediate date t the value of the forward is $f = S - Ke^{-r(T-t)}$.

- It follows:

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)} \quad \frac{\partial f}{\partial S} = 1 \quad \frac{\partial^2 f}{\partial S^2} = 0.$$

- Substituting these into the BSM equation, $rf = rS - rKe^{-r(T-t)}$, which is true.

Risk-Neutral Valuation

A basic principle of asset pricing is that investors demand higher expected return, μ, in the presence of higher volatility, σ.

- Note that μ doesn't appear in the BSM differential equation.

- This means we can treat investors *as if* they are risk neutral.

- That is, we can value assets under the assumption that investors only demand expected return r, even though they aren't really risk neutral.

- The practical implication is that once we compute future asset payoffs, we can discount them at the risk-free rate (implicitly assuming this is the rate of return that investors demand).

BSM Option Pricing Formulas

The BSM option pricing formulas are:

$$\begin{aligned}
C &= S_0\Phi(d_1) - Xe^{-rT}\Phi(d_2) \\
P &= Xe^{-rT}\Phi(-d_2) - S_0\Phi(-d_1) \\
d_1 &= \frac{\log(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\log(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.
\end{aligned}$$

- These satisfy the BSM differential equation.

- $\Phi(x)$ represents the standard Normal CDF: $P(X \leq x)$ when $X \sim N(0, 1)$.

- C and P are the prices of European call and put options on the underlying, S.

Normal CDF

Interpretation

$$
\begin{aligned}
P(S_T \geq X) &= P\left(\log(S_T) \geq \log(X)\right) \\
&= P\left(\frac{\log(S_T) - E\left[\log(S_T)\right]}{Sd\left(\log(S_T)\right)} \geq \frac{\log(X) - E\left[\log(S_T)\right]}{Sd\left(\log(S_T)\right)}\right) \\
&= 1 - \Phi\left(\frac{\log(X) - E\left[\log(S_T)\right]}{Sd\left(\log(S_T)\right)}\right) \\
&= \Phi\left(-\frac{\log(X) - E\left[\log(S_T)\right]}{Sd\left(\log(S_T)\right)}\right) \\
&= \Phi(d_2).
\end{aligned}
$$

Interpretation

The last equation above follows because, under risk-neutrality $E\left[\log(S_T)\right] = \log(S_0) + (r - \sigma^2/2)T$ and $Sd\left(\log(S_T)\right) = \sigma\sqrt{T}$:

$$
\begin{aligned}
-\frac{\log(X) - E\left[\log(S_T)\right]}{Sd\left(\log(S_T)\right)} &= \frac{\log(S_0) + (r - \sigma^2/2)T - \log(X)}{\sigma\sqrt{T}} \\
&= \frac{\log(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= d_2.
\end{aligned}
$$

Interpretation

$\Phi(d_1)$ is similar to $\Phi(d_2)$, but slightly harder to interpret.

- $S_0\Phi(d_1)e^{rT}$ is the expected asset price (under risk neutrality), conditional on the asset expiring with $S_T \geq X$.

- The expected payoff of the call option is:

$$S_0\Phi(d_1)e^{rT} - X\Phi(d_2).$$

- The present value of the call option expected payoff is:

$$S_0\Phi(d_1) - Xe^{-rT}\Phi(d_2).$$

Extreme Cases

Suppose the current stock price S_0 is very large relative to the strike X.

- In this case d_1 and d_2 are very large.

- As a result $\Phi(d_1)$ and $\Phi(d_2)$ approach 1.

- This causes the call option value to be $S_0 - Xe^{-rT}$.
  - This is identical to a forward contract, which makes sense for a deep-in-the-money call.

- Likewise $\Phi(-d_1)$ and $\Phi(-d_2)$ approach 0.

- This causes the put option value to be 0.

Extreme Cases

Suppose $\sigma \to 0$.

- If $S_0 > Xe^{-rT}$, then $d_1 \to \infty$ and $d_2 \to \infty$, causing $\Phi(d_1) \to 1$ and $\Phi(d_2) \to 1$.

- The result is a value of $S_0 - Xe^{-rT} > 0$, which is identical to the riskless payoff $\max(S_0e^{rT} - X, 0)$.

- Likewise, if $S_0 < Xe^{-rT}$, then $d_1 \to -\infty$ and $d_2 \to -\infty$, causing $\Phi(d_1) \to 0$ and $\Phi(d_2) \to 0$.

- The result is a value of 0, which is identical to the riskless payoff $\max(S_0e^{rT} - X, 0)$.

Implied Volatility

Volatility, σ, is the single parameter of the BSM model that is not directly observed.

- Typically, the BSM equation is not used to compute an option price, but to compute an *implied volatility*, given an option price.

- For example, if $C = 1.875, S_0 = 21, X = 20, r = 0.1$ and $T = 0.25$, what is the value that σ must take for the BSM equation to hold for a European call option?

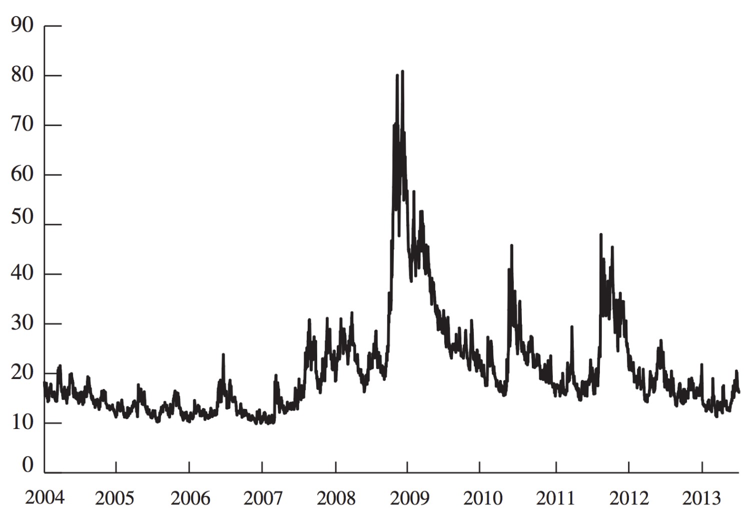
VIX Index

The VIX index is an index of implied volatilities for 30-day S&P 500 options (calls and puts).

- It is interpreted as the (annualized) one standard deviation move (in percentage) of the S&P 500 index over the next 30 days.

- Options and futures trade on the VIX itself (as well as the S&P 500).

Historical VIX



Forward Contracts

Forward Contract Definition

Definition: A forward contract is an agreement to exchange an asset at a future date at a prespecified price.

- The contract settlement date is called the *expiration date*.
- The asset that is exchanged is called the *underlying asset*.
- The buyer holds the *long* position.
- The seller holds the *short* position.
- There is no initial payment or premium.

Delivery and Settlement

There are two types of forward contract settlements.

- **Delivery**: The long position pays the prespecified price to the short position, who delivers the asset.
- **Cash settlement**: The long and short positions pay the net cash value to the other.

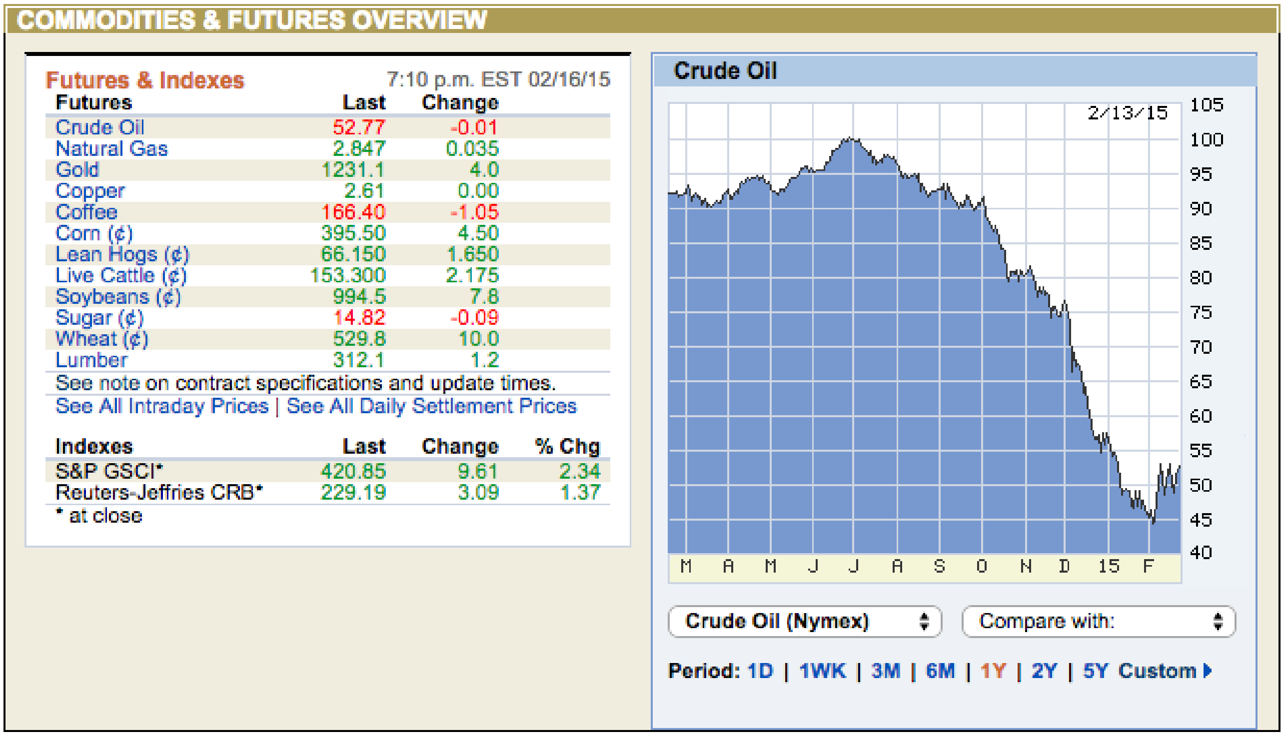
Forward Example

Two parties contract to exchange a $\$100$ bond for $\$98$ at a future date.

- If the bond is worth $\$98.25$ at expiry, the short position pays $\$0.25$ to the long position at expiry.
- If the bond is worth $\$97.50$ at expiry, the long position pays $\$0.50$ to the short position at expiry.
- Cash-settled forwards are often called NDFs, or nondeliverable forwards.

- Usually, cash settlement is used for underlying assets that are difficult to exchange (think of a stock index).

Market Prices



COMMODITIES & FUTURES OVERVIEW

Futures & Indexes 7:10 p.m. EST 02/16/15

Futures	Last	Change
Crude Oil	52.77	-0.01
Natural Gas	2.847	0.035
Gold	1231.1	4.0
Copper	2.61	0.00
Coffee	166.40	-1.05
Corn (¢)	395.50	4.50
Lean Hogs (¢)	66.150	1.650
Live Cattle (¢)	153.300	2.175
Soybeans (¢)	994.5	7.8
Sugar (¢)	14.82	-0.09
Wheat (¢)	529.8	10.0
Lumber	312.1	1.2

See note on contract specifications and update times.
See All Intraday Prices | See All Daily Settlement Prices

Indexes	Last	Change	% Chg
S&P GSCI*	420.85	9.61	2.34
Reuters-Jeffries CRB*	229.19	3.09	1.37

* at close

For current data, visit the WSJ or CME Group homepages.

Early Termination

Suppose one party in a forward contract wishes to terminate early.

- She could engage in another forward contract on the opposite side.

- Depending on market conditions, the new contract may be written at a new price.

Early Termination Example

Suppose a trader enters a long forward contract position to exchange a barrel of crude oil on 13 Feb 2015 and decides to terminate the contract on 16 Feb 2015.

- On 13 Feb, the forward price is $52.78 per barrel.

- On 16 Feb, the forward price is $52.73 per barrel.

- She can write a forward contract for $52.73 on 16 Feb.

- Note that she takes a $0.05 loss and is still exposed to risk of default on two different contracts.

- Alternatively, she can ask her original counterparty to accept the present value of $0.05 to terminate.

Notation

We will use the following notation:

- S_0: Spot price of the underlying asset today.

- F_0: Forward price of the underlying asset today.

- T: Time until delivery.

- r: Risk-free rate of interest for maturity T.

Note that any units (minutes, hours, days, weeks, months, years) may be used for T, but that the interest rate, r, must be adjusted accordingly.

Forward Valuation

The price of a forward contract with maturity T for an asset with price S_0 is:

$$F_0 = S_0 e^{rT}.$$

- r is the risk-free interest rate over period T.

- If r is constant, F_0 is a deterministic function of the spot sprice, and has nothing to do with the unknown, future price of the asset.

- e^{rT} is known as the *basis*.

- **Intuition**: the foward holder must pay the holder of the spot contract for interest that would have been earned.

Forward Valuation Example

Suppose you would like to purchase a 3-month forward contract on Coca-Cola (KO) stock on 1 Mar 2016. What is the value of the forward (assuming the stock never pays dividends)?

- Set $T = 0.25$ (i.e. time units of 1 year).

- Use Yahoo Finance to determine $S_0 = \$43.35$.

- Use Quandl to determine the (annualized) yield on the 3-month U.S. Treasury Bill: $r = 0.0033$.

Thus,

$$F_0 = S_0 e^{rT} = \$43.35 e^{0.0033 \times 0.25} = \$43.39.$$

Forward Valuation with Income

Suppose the underlying asset provides income with present value I.

- This may be a single payment or a stream of payments, all appropriately discounted:

$$I = \frac{d}{1 + \frac{r}{m}} + \frac{d}{\left(1 + \frac{r}{m}\right)^2} + \cdots + \frac{d}{\left(1 + \frac{r}{m}\right)^{mT}}.$$

- This assumes m equally spaced payments of equal size during interval T.

The value of a forward contract is now:

$$F_0 = (S_0 - I)e^{rT}.$$

Forward Valuation with Yield

Suppose the underlying asset provides income yield (continuously compounded) q. Then:

$$F_0 = S_0 e^{(r-q)T}.$$

- **Intuition**: the holder of the spot contract now pays interest (implicitly), but earns income. The foward holder must compensate the spot holder for interest, net of income earned over period T.

Forward Valuation with Yield Example

Reconsider the previous example for Coca-Cola stock.

- Now assume that KO has an annualized dividend yield of 3%.

The forward price is

$$
\begin{aligned}
F_0 &= S_0 e^{(r-q)T} \\
&= \$43.35 e^{(0.0033-0.03)\times 0.25} \\
&= \$43.06.
\end{aligned}
$$

Forward Valuation for Currency

Suppose the underlying asset is a currency, and that the risk-free interest rate in the foreign market is r_f. Then:

$$F_0 = S_0 e^{(r-r_f)T}.$$

- The foreign interest is income and the rate is the income yield.

Curreny Forward Example

What is the value of a 6-month forward contract for Canadian dollars (CAD) on 1 Mar 2016?

- Set $T = 0.5$ (i.e. time units of 1 year).
- Use Quandl to determine the spot exchange rate for USD/CAD: $S_0 = \$1.34$.
- Use Quandl to determine the (annualized) yield on the 3-month Canadian Treasury Bill: $r_f = 0.0047$. We already determined that $r = 0.0033$.

Thus,

$$F_0 = S_0 e^{(r-r_f)T} = \$1.34 e^{(0.0033-0.0047)\times 0.5} = \$1.339.$$

Forward Valuation for Commodities

Suppose that the underlying is a physical asset that must be stored. Then:

$$F_0 = (S_0 + U)e^{rT}.$$

or

$$F_0 = S_0 e^{(r+u)T}.$$

- U is the present value of storage costs.
- u is the annual storage cost expressed as a fraction of commodity value.
- Note that storage costs are like negative income.

Cost of Carry

The foregoing compounding rates are referred to as *the cost of carry*, c.

$$F_0 = S_0 e^{cT}.$$

- The cost of carry includes interest rate and storage costs, minus income.
- For a stock index that pays a dividend yield, $c = r - q$.
- For a foreign currency, $c = r - r_f$.
- For a commodity that provides income, $c = r - q + u$.

Futures Markets

Futures Contracts

Futures contracts are forward contracts with the following differences:

- They are *exchange traded*.
- The contract is standardized (by the exchange).
- They are settled daily.

Role of Exchanges

Exchanged are centralized locations with the following features:

- They standardize contracts.
- They aggregate supply and demand.
- They determine limitations on who can trade and how.
- They set limitations on borrowing and distribute risk by requiring transactions to be routed through clearing houses.

Exchanges

Major futures exchanges at present:

- CME Group.
- Intercontinental Exchange (ICE).
- Eurex.
- National Stock Exchange of India.

Contract Specifications

The futures exchange determines the following aspects of contracts:

- Price units. Dollars? Tens of dollars?
- Price increments. The Minimum price that the price can move by (ie: Penny increments) Discrete prices
- Size units. What is the minimum contract size?
- When the contract trades. Self explanatory, When it settles or expires, when you allowed to trade
- When the contract is settled.
- How the contract is settled. What is going to be delivered, on what date, what location etc
- Margin. How much you are allowed to borrow to buy a contract

Example: E-mini S&P 500 Contract

The E-mini S&P 500 futures contract is a contract on the S&P 500 index.

- Note that the S&P 500 is simply an index, not a traded asset.
- The price of the E-mini is quoted in S&P 500 index points.
- The actual size of the contract is 50x the index.
- Margin for a single contract is $5060.

E-mini S&P 500 Specifications

Contract Unit	$50 x S&P 500 Index	
Trading Hours	CME Globex:	Sunday - Friday 6:00 p.m. - 5:00 p.m. ET (5:00 p.m. - 4:00 p.m. CT) with 15-minute trading halt Monday – Friday 4:15 p.m. - 4:30 p.m. ET (3:15 p.m. - 3:30 p.m. CT). Monday - Thursday 5:00 p.m. - 6:00 p.m. ET (4:00 p.m. - 5:00 p.m. CT) daily maintenance period. For the BTIC, trading hours will be Sunday – Friday 6:00 p.m. - 4:00 p.m. ET (5:00 p.m. - 3:00 p.m. CT); Monday - Thursday 5:00 p.m. - 6:00 p.m. ET (4:00 p.m. - 5:00 p.m. CT) daily maintenance period.
	CME ClearPort:	Sunday - Friday 6:00 p.m. - 5:00 p.m. ET (5:00 p.m. - 4:00 p.m. CT). Monday - Thursday 5:00 p.m. - 6:00 p.m. ET (4:00 p.m. - 5:00 p.m. CT) daily maintenance period.
Minimum Price Fluctuation	OUTRIGHT	0.25 index points=$12.50 BTIC: 0.05 index points = $2.50
	CALENDAR SPREAD	0.05 index points=$2.50

E-mini S&P 500 Specifications

Product Code	CME Globex: ES CME ClearPort: ES Clearing: ES BTIC: EST
Listed Contracts	Five months in the March Quarterly Cycle (Mar, Jun, Sep, Dec)
Settlement Method	Financially Settled
Termination Of Trading	Trading can occur up to 8:30 a.m. on the 3rd Friday of the contract month For the BTIC, trading terminates on the Thursday before the Third Friday of the Contract Month

Futures Maturities

Note that many futures contracts with different maturities on the same underlying can exist at the same time.

- For example, E-mini contracts expire on the third Friday of each Mar/Jun/Sep/Dec.
- At any date, the next five contracts are available for trade.
- The contract closest to expiry is called the *front-month* contract and is always the most liquid.

Delivery of Commodities

For commodities, delivery is an important part of the specification.

- What type (grade) of the product can be delivered.

- Range of dates for delivery.

Terminology

- **Open interest**: Total number of contracts outstanding.

- **Trade volume**: The number of contracts traded.

- **Settlement price**: The last price before market close.

Volume vs. Open Interest

- What is the difference between open interest and trade volume on a given day?

- When are trade occurs, what are the possible effects on open interest?

- Can trade volume be greater than open interest during day?

Margin

When investors borrow money from a broker to purchase an asset, they are buying on margin.

- The *initial margin* is cash or marketable securities that an investor gives to a broker in order to purchase an asset.

- The purchased securities are maintained in an account by the broker and are monitored.
  - Gains and losses on the securities are added to the value of the account.

- The maintenance margin is a lower threshold for the value of the account.
  - When the value falls below, the investor must add cash or securities up to initial margin.

- Margin accounts are typically settled daily.

E-mini Margin

For the E-mini:

- Initial margin is $5225.
- Maintenance margin is $4750.
- The settlement price on 6 April 2016 was 2060.25, or a notional value of 50x2060.25 = $103012.5.
    - This implies a leverage ratio of $\frac{103012.5}{5225} = 19.71$

Example: Gold Futures

Suppose that an investor buys two CME Group gold futures contracts (symbol GC) for $1450 (per troy ounce).

- Initially margin is $6000 per contract.
- Maintenance margin is $4500 per contract.
- The contract size is 100 troy ounces, which implies a notional value of 1450*100 = $145,000.
- The resulting leverage ratio is $\frac{145000}{6000} = 24.17.$

Example: Gold Futures

Day	Trade Price ($)	Settle Price ($)	Daily Gain ($)	Cumul. Gain ($)	Margin Balance ($)	Margin Call ($)
1	1,450.00				12,000	
1		1,441.00	−1,800	− 1,800	10,200	
2		1,438.30	−540	−2,340	9,660	
.....		.....	.....	.....	......	
6		1,436.20	−780	−2,760	9,240	
7		1,429.90	−1,260	−4,020	7,980	4,020
8		1,430.80	180	−3,840	12,180	
.....		.....	.....	.....	......	
16	1,426.90		780	−4,620	15,180	

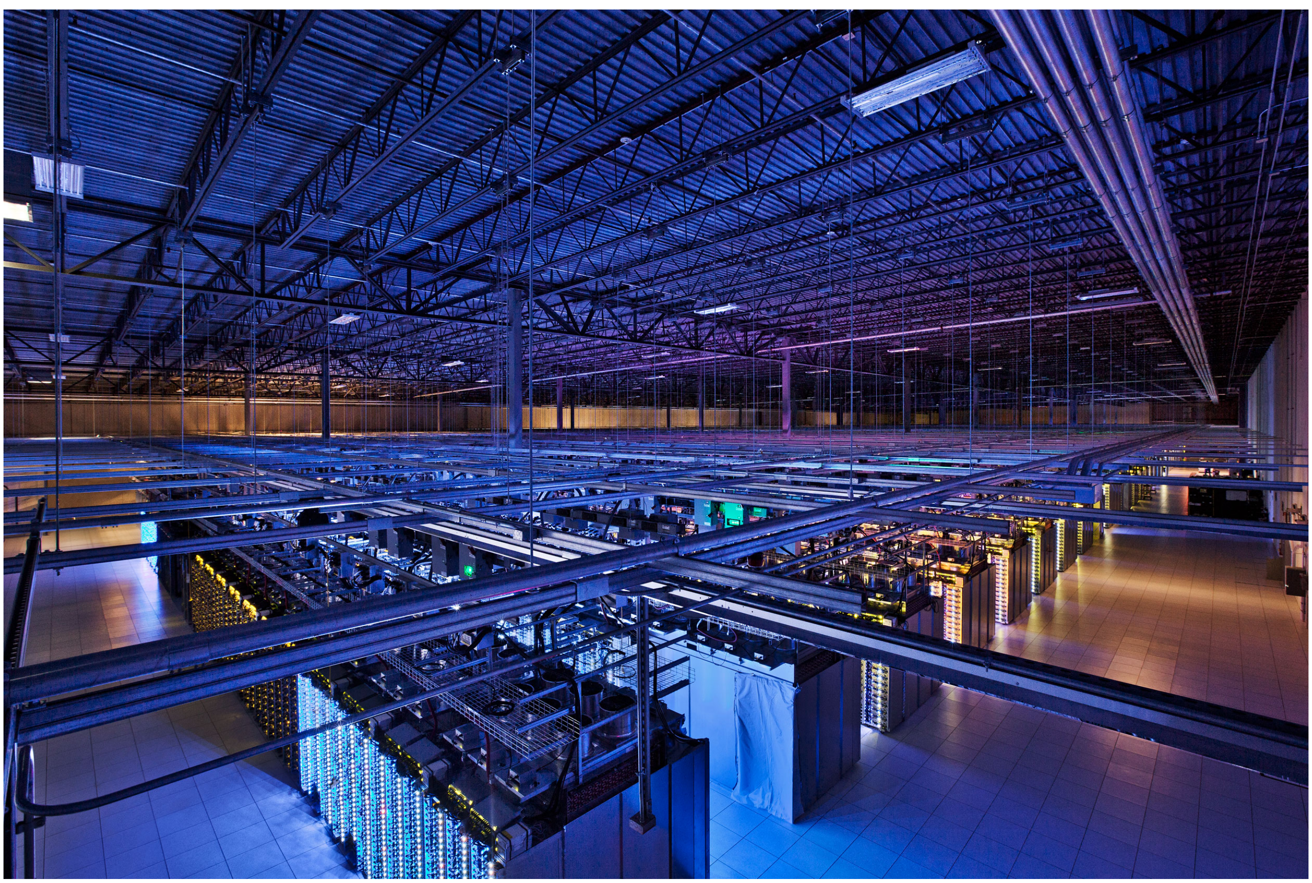
Trading

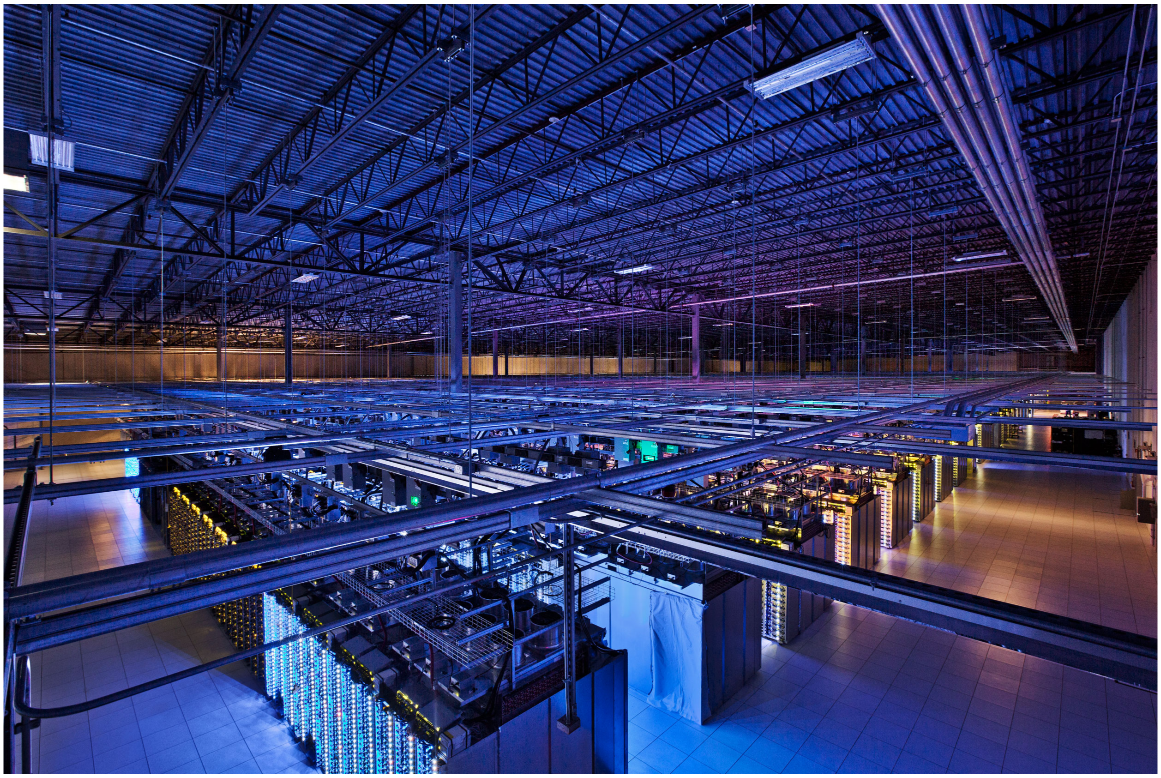
Exchanges Today

Modern financial exchanges are typically *electronic communications networks* or *ECNs*.

- They consist of large warehouses, populated by computer servers that electronically match orders.
- These are referred to as *matching engines*.
- Since order execution speed is so important, many traders co-locate their trading computers with the matching engines.

Exchanges Today

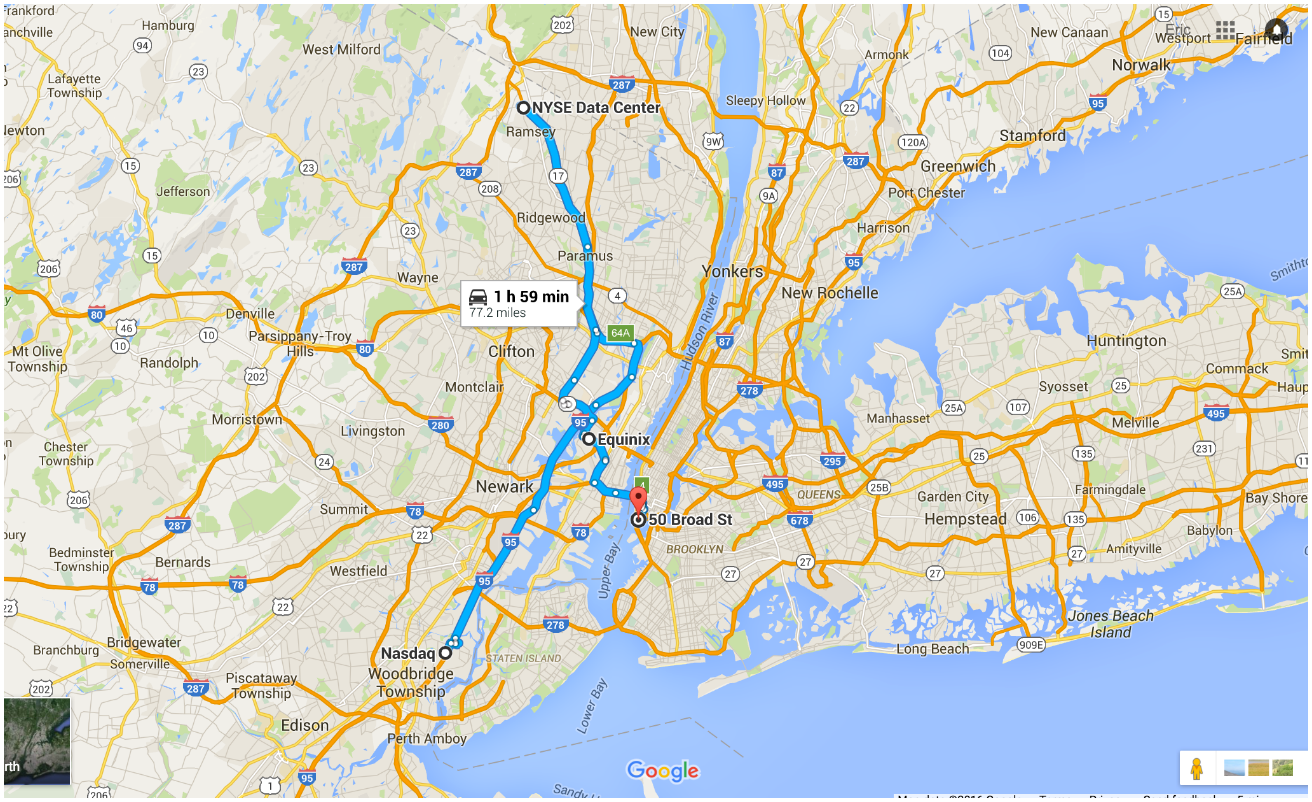




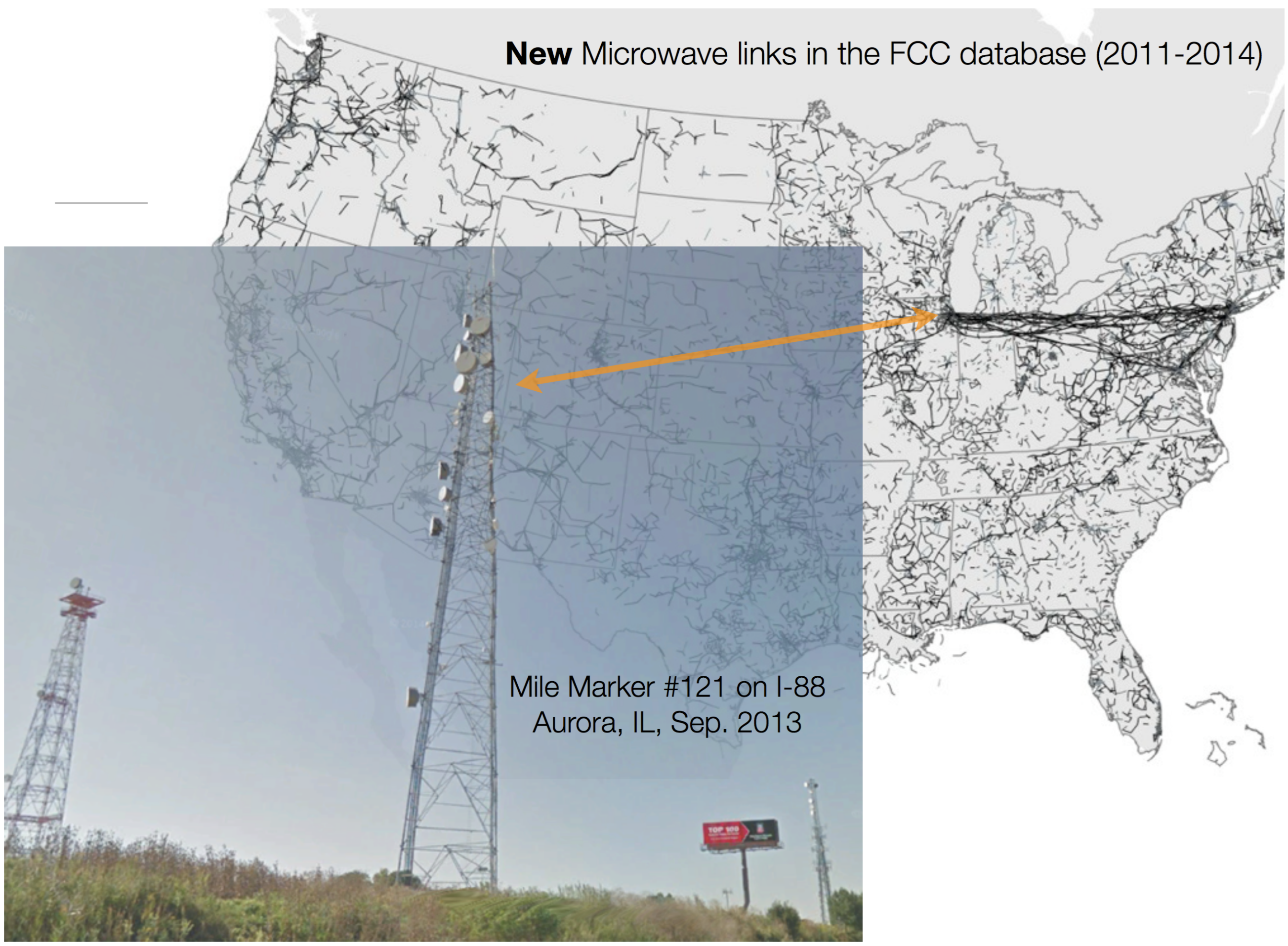
Major U.S. Exchanges

- The NYSE Arca matching engine is located in Mahwah, NJ.
- The Electronic Broking System (EBS) currency exchange is located in Secausus, NJ.
- The Chicago Board Option Exchange (CBOE) has co-located with EBS in Secaucus.
- The Nasdaq matching engine is located in Carteret, NJ.
- The CME Group matching engine is located in Aurora, IL.

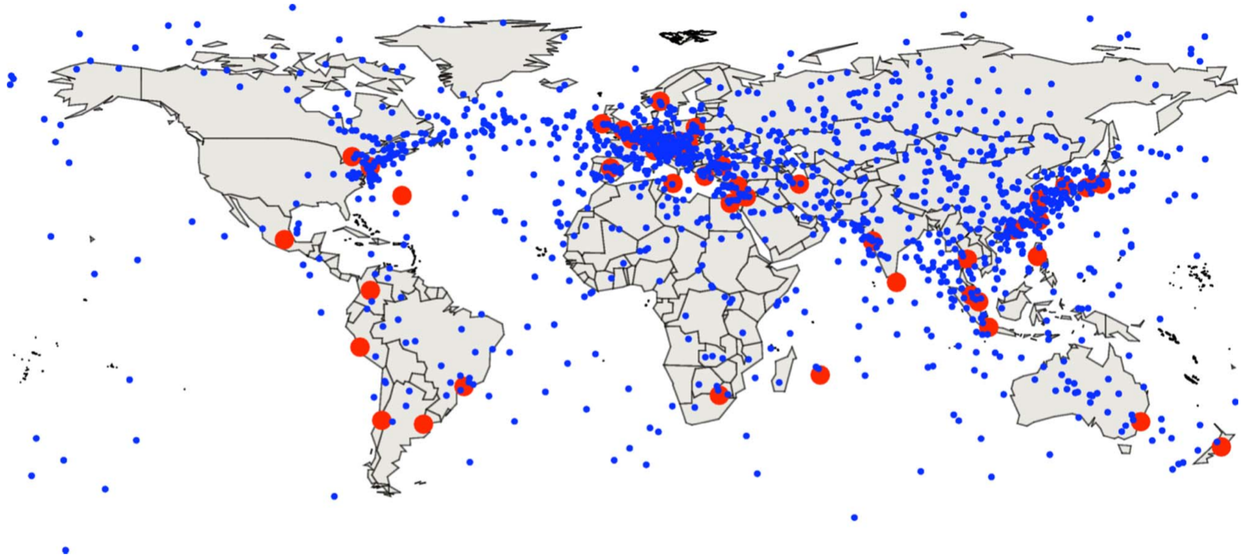
Exchanges Today



Exchanges Today



Exchanges Today



Market Makers

Participants on exchanges include makers and takers.

- Makers are required to post both quotes to buy and sell assets on the exchange.

- Quotes to buy are called *bids*.

- Quotes to sell are called *offers*.

- Typically there are many bids and offers posted on the exchange at different prices.

- Market makers are compensated for providing quotes (also referred to as *providing liquidity*).

Takers

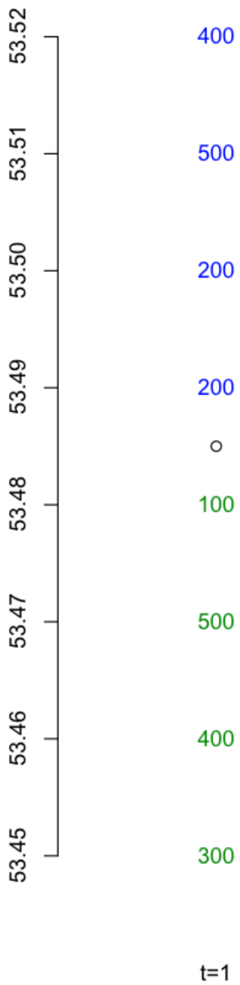
Takers actively *take* orders that have been passively posted by market makers.

- When a taker wants to buy the asset, they buy at the lowest posted offer quote.

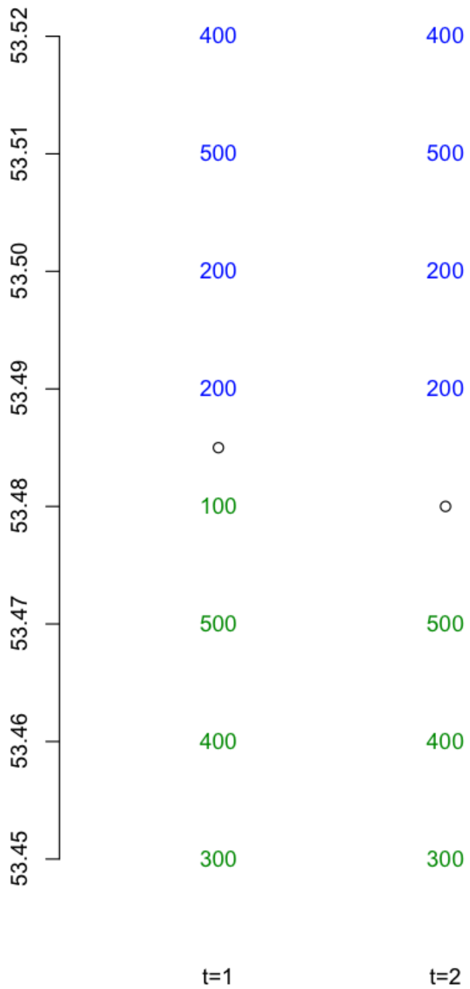
- When a taker wants to sell an asset, they sell at the highest posted bid quote.

- The difference between the highest bid quote and lowest offer quote is known as the *spread*.

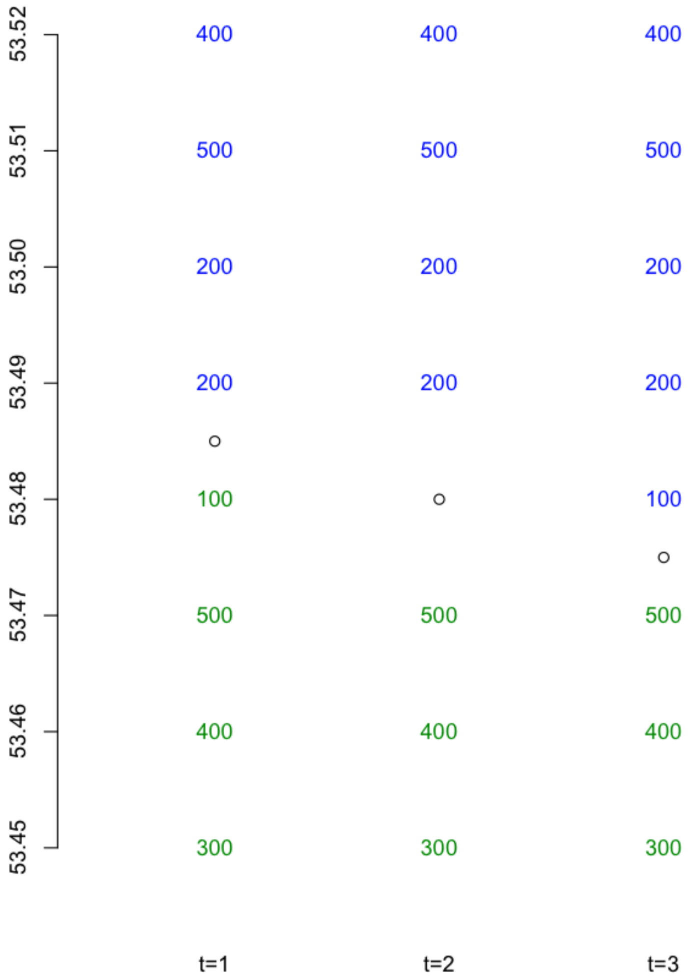
Order Book Example: time 1



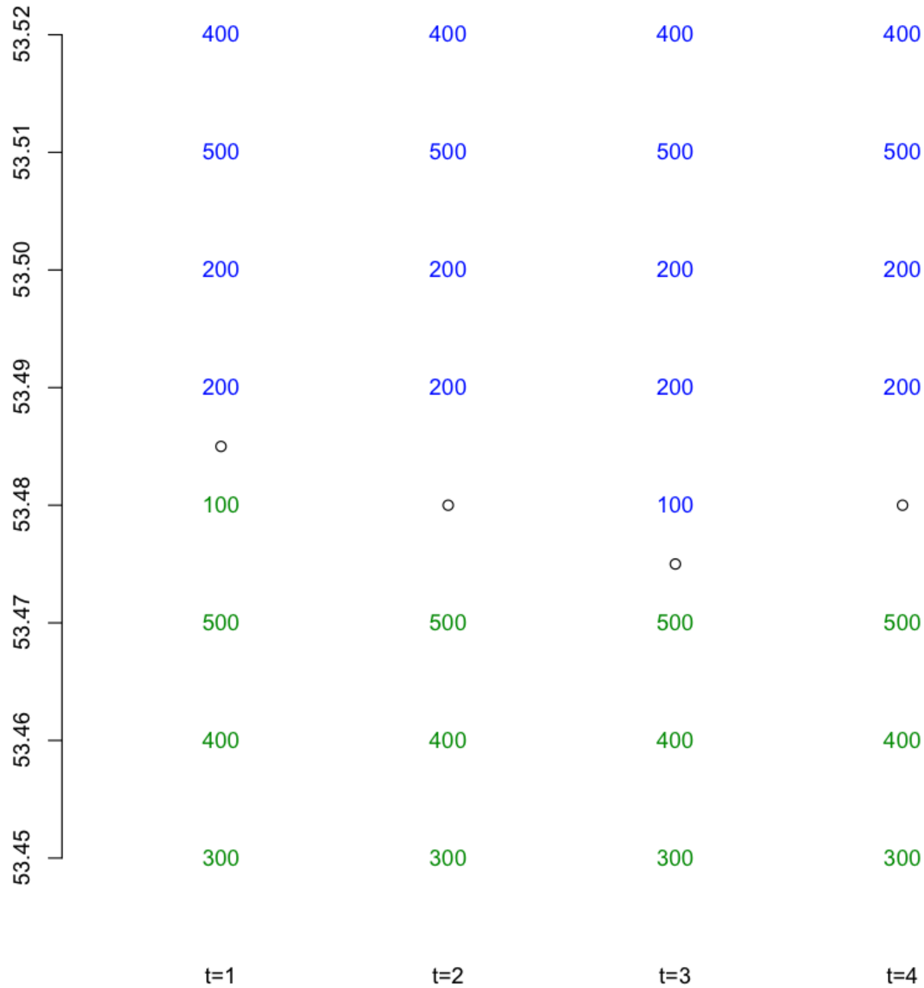
Order Book Example: time 2



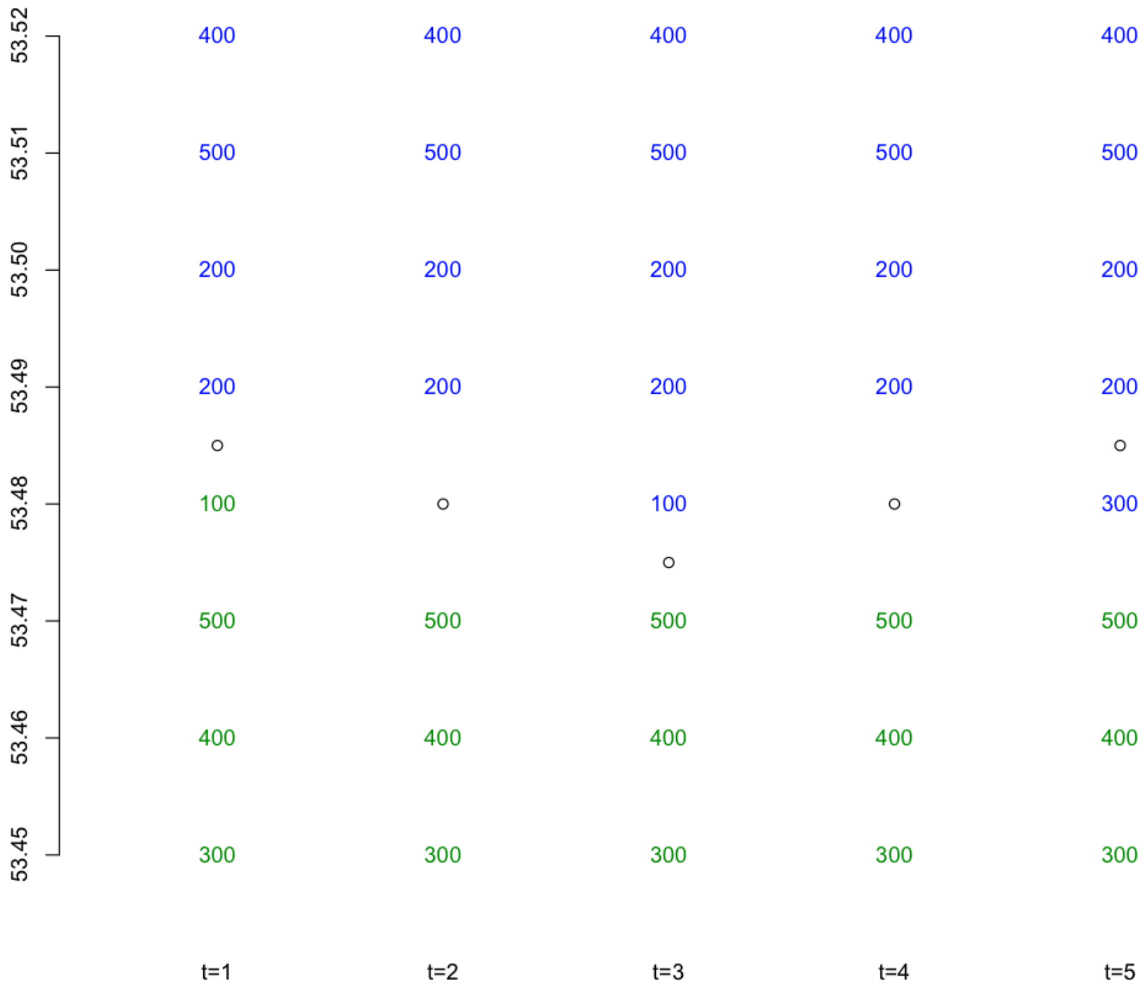
Order Book Example: time 3



Order Book Example: time 4



Order Book Example: time 5



Information Flows

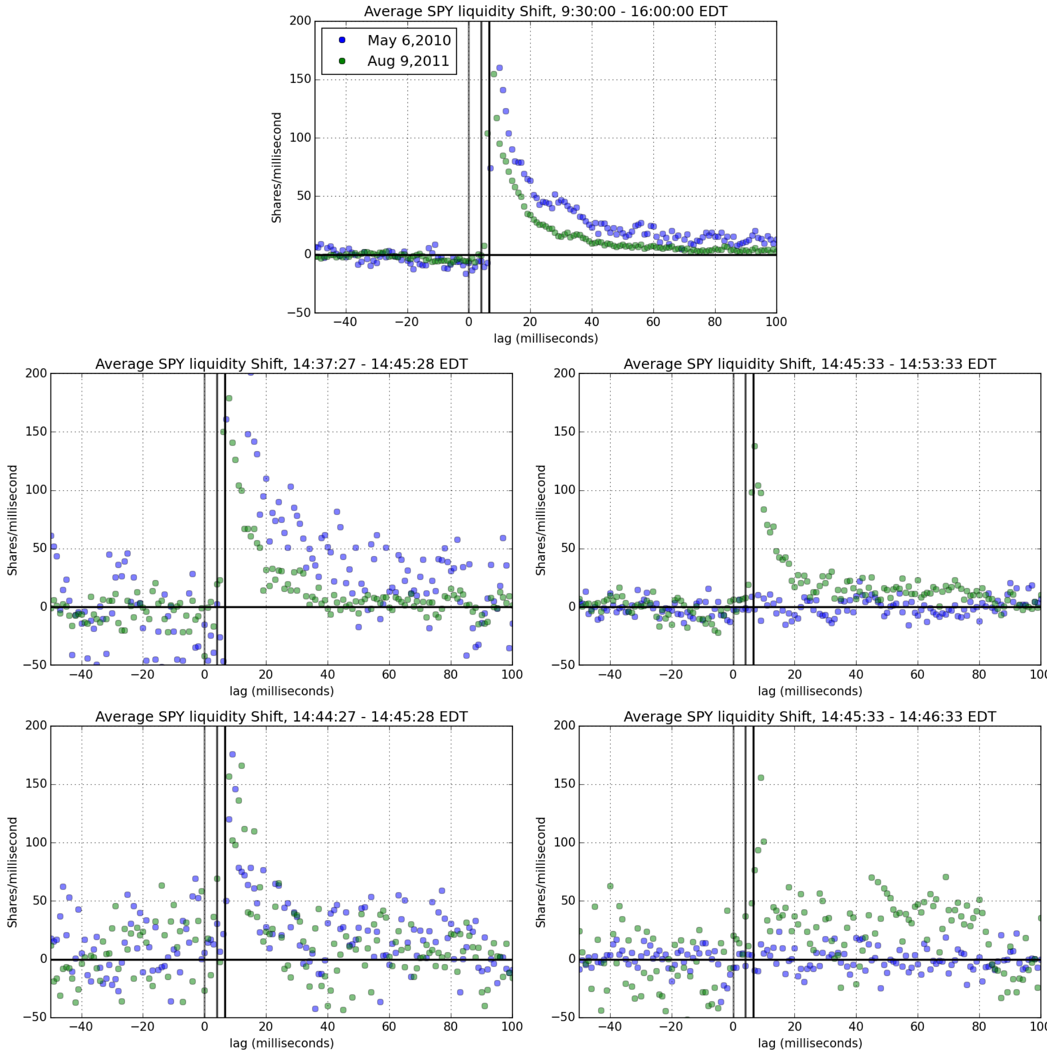
Conventional wisdom is that information flows from the futures market to the equities (cash) market.

Price Discovery and Liquidity – The primary purposes of futures markets are to provide an efficient mechanism for price discovery and risk management. The academic literature underscores the efficacy of futures markets as a tool of price discovery. According to one study, "[e]mpirical results confirm that futures market plays a price discovery role, implying that futures prices contain useful information about spot prices."[1]

As such, stock index futures frequently represent the venue in which price information is revealed first, generally followed closely by spot markets. In fact, most researchers find that "futures lead the cash index returns, by responding more rapidly to economic events than stock prices."[2]

SPY Liquidity Response to ES

The following figure shows average net SPY liquidity response (shares added to offer minus shares added to bid) in each millisecond following ES liquidity change.



Reasons for Information Flow

- The standard explanation is leverage.
- Informed traders will first trade where they can obtain highest leverage.
- The data also suggests that *relative price increment* is an important determinant.

SPY/E-mini Contract Specs

E-mini:

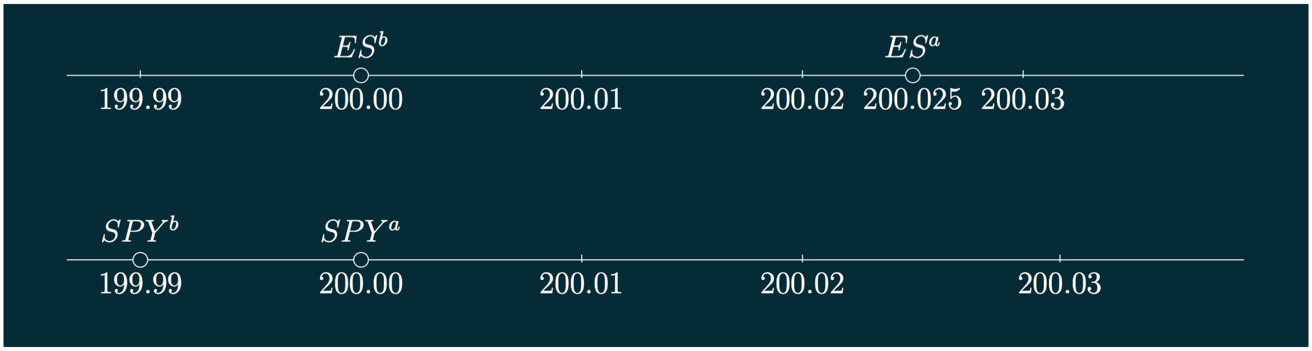
- Quoted in S&P 500 index points.
- Notional value of one contract is 50x the index.
- Minimum price increment is 0.25 index points.

SPY:

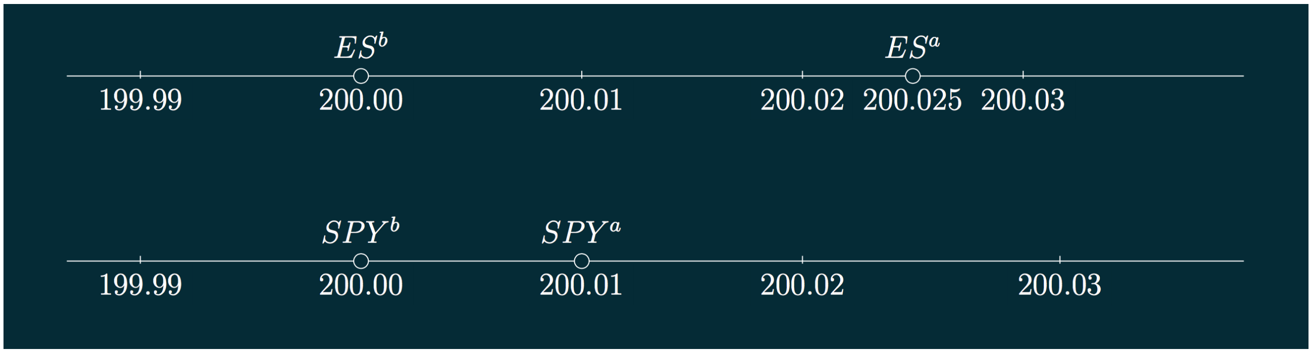
- Quoted at 1/10th the value of the S&P 500 index.
- Notional value is equal to the price quotes.
- Minimum price increment is $0.01.
  - Equivalent to 0.10 index points.

E-mini tick size is 2.5x SPY tick size.

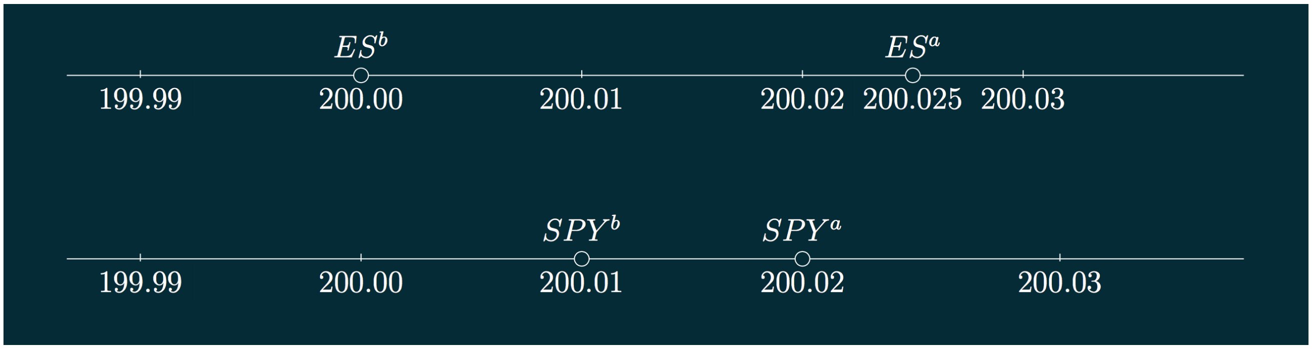
E-mini/SPY Spreads



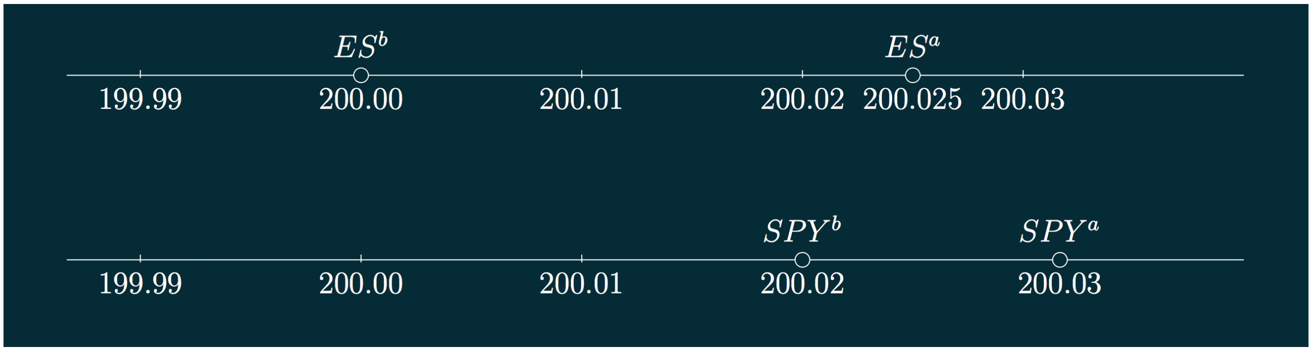
E-mini/SPY Spreads



E-mini/SPY Spreads



E-mini/SPY Spreads



Spread and Information Flow

Differences in spreads creates arbitrage opportunities.

- The size of the spread differences determines size and frequency of arbitrages.

- Large ES spread relative to SPY spread means big arbs most of the time when a market maker is filled at CME.

- Less frequently, the SPY bid or offer will allow an arb in the reverse direction.

Implications

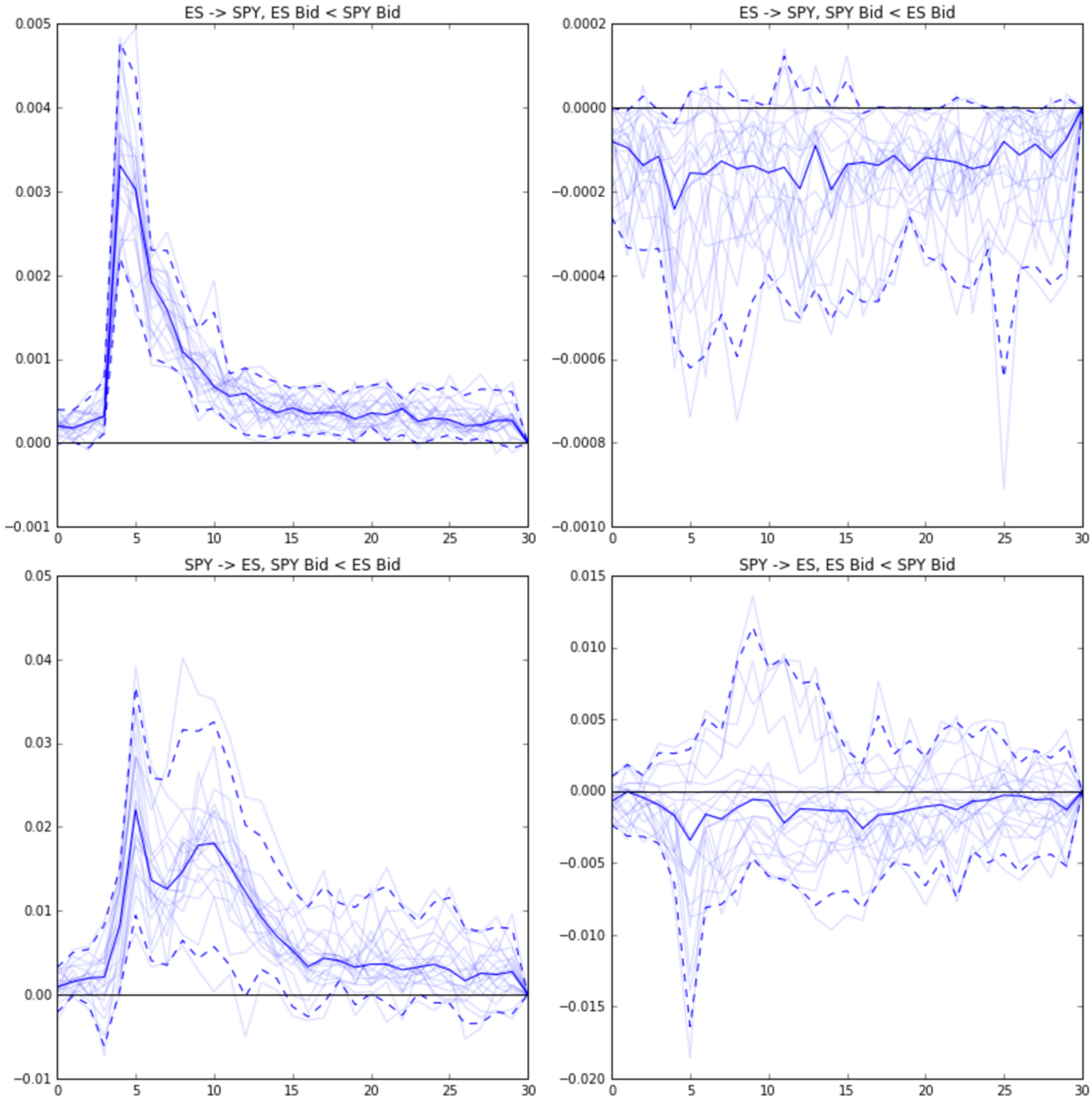
- If maker is filled at ES bid and ES bid < SPY bid: sell at SPY bid.
  - SPY market responds to ES market.

- If maker is filled at ES offer and ES offer > SPY offer: buy at SPY offer.
  - SPY market responds to ES market.

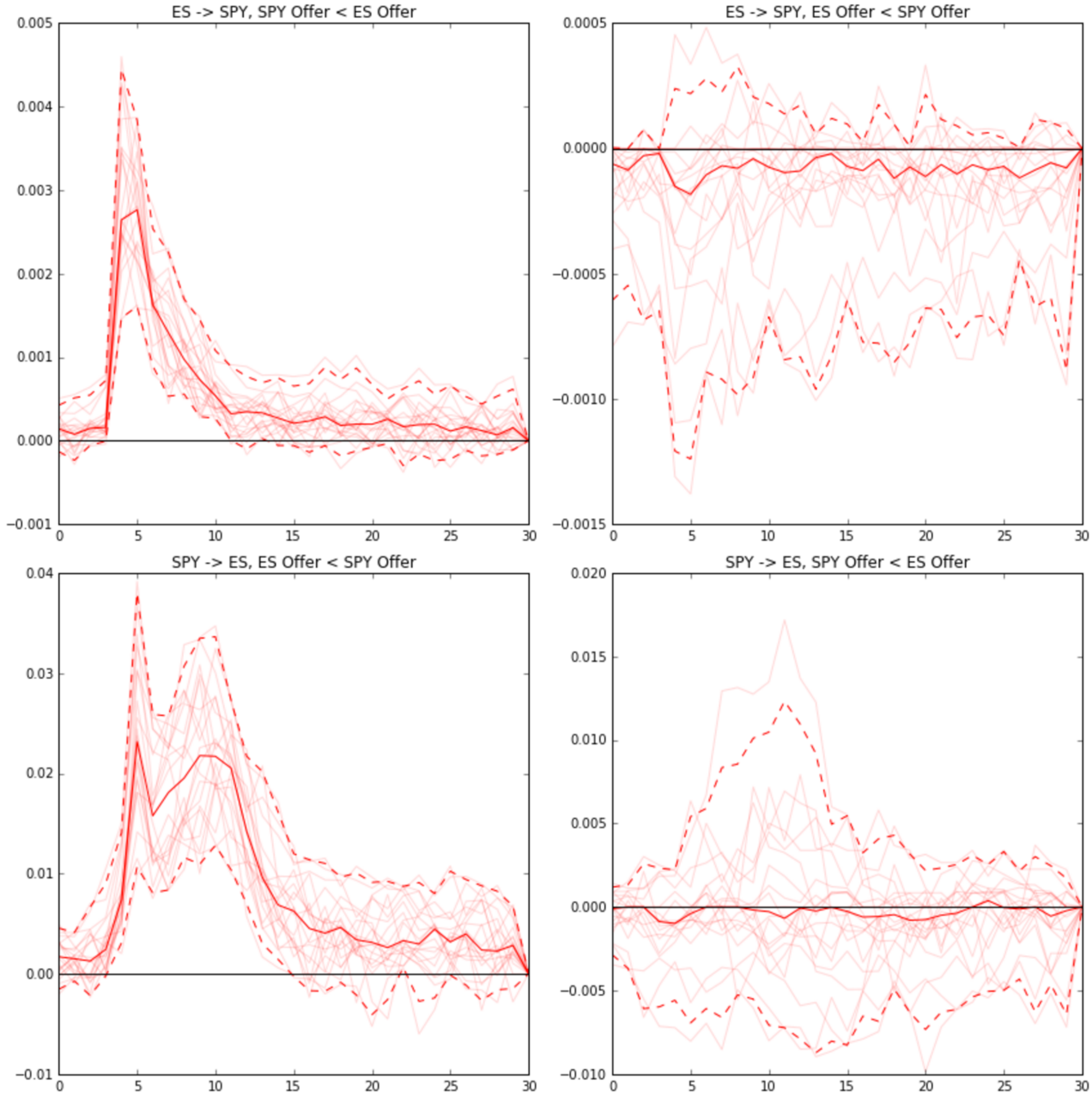
- If maker is filled at SPY bid and SPY bid < ES bid: sell at ES bid.
  - ES market responds to SPY market.

- If maker is filled at SPY offer and SPY offer > ES offer: buy at ES offer.
  - ES market responds to SPY market.

Empirical Results



Empirical Results



Notes on Results

- The previous results are base on the trading record from Aug 2015.
- The data strongly support the model predictions.
- There are typically 10 times as many ES events as SPY events.
- Each SPY event is 10 times more influential than an ES event.
- What does this mean if you are designing and exchange?

Hedging with Futures

Long and Short Hedges

A hedge is an investment that limits the risk of another risky investment. It typically consists of an offsetting position.

- Futures are used to hedge against price fluctuations of an asset that must be bought or sold at a future date.
- A long hedge is used when you will purchase an asset in the future.
- A short hedge is used when you will sell an asset in the future.

Short Hedge Example

Suppose it is May 15th and an oil producer has negotiated to sell 1 million barrels of crude oil on Aug 15. $S_0 = \$80$ and $F_0 = \$79$.

- The producer will gain/lose $10,000 for each 1 cent increase/decrease in the spot price.
- A hedge would consist of shorting 1000 futures contracts (for 1000 barrels each) with expiration as close to Aug 15th as possible.
- Suppose $S_T = \$75$ on Aug 15 - what are total profits for the producer?
- Suppose $S_T = \$85$ on Aug 15 - what are total profits for the producer?

Note that the producer will not want to deliver the barrels for the futures contract, but will close out early.

Long Hedge Example

Suppose it is Jan 15th and a copper producer has negotiated to buy 100,000 pounds of copper on May 15. $S_0 = 340$ cents and $F_0 = 320$ cents.

- The producer will gain/lose $1000 for each 1 cent decrease/increase in the spot price.

- A hedge would consist of buying 4 CME Group futures contracts (for 25,000 pounds each) with expiration as close to May 15th as possible.

- Suppose $S_T = 320$ cents on May 15 - what is the total cost for the producer?

- Suppose $S_T = 305$ cents on May 15 - what is the total cost for the producer?

Note that the producer will not want to take delivery for the futures contract, but will close out early.

Imperfect Hedging

It is typically difficult for an investor to exactly hedge a position.

1. A futures contract may not exist for the exact asset to be hedged.
    - E.g. VIX futures.

2. The expiration of the futures may not coincide with the end of the termination of the position in interest.

3. The termination date of the underlying position may be unknown.

Basis

For traders, the basis is defined as the difference between the futures and spot prices.

- Traditional definition: $b_t = S_t - F_t$.

- For financial assets: $b_t = F_t - S_t$.

- Recall

$$F_t = e^{c(T-t)} S_t$$
$$\Rightarrow b_t = S_t - F_t = \left(1 - e^{c(T-t)}\right) S_t.$$

- Note that the basis can be positive or negative.

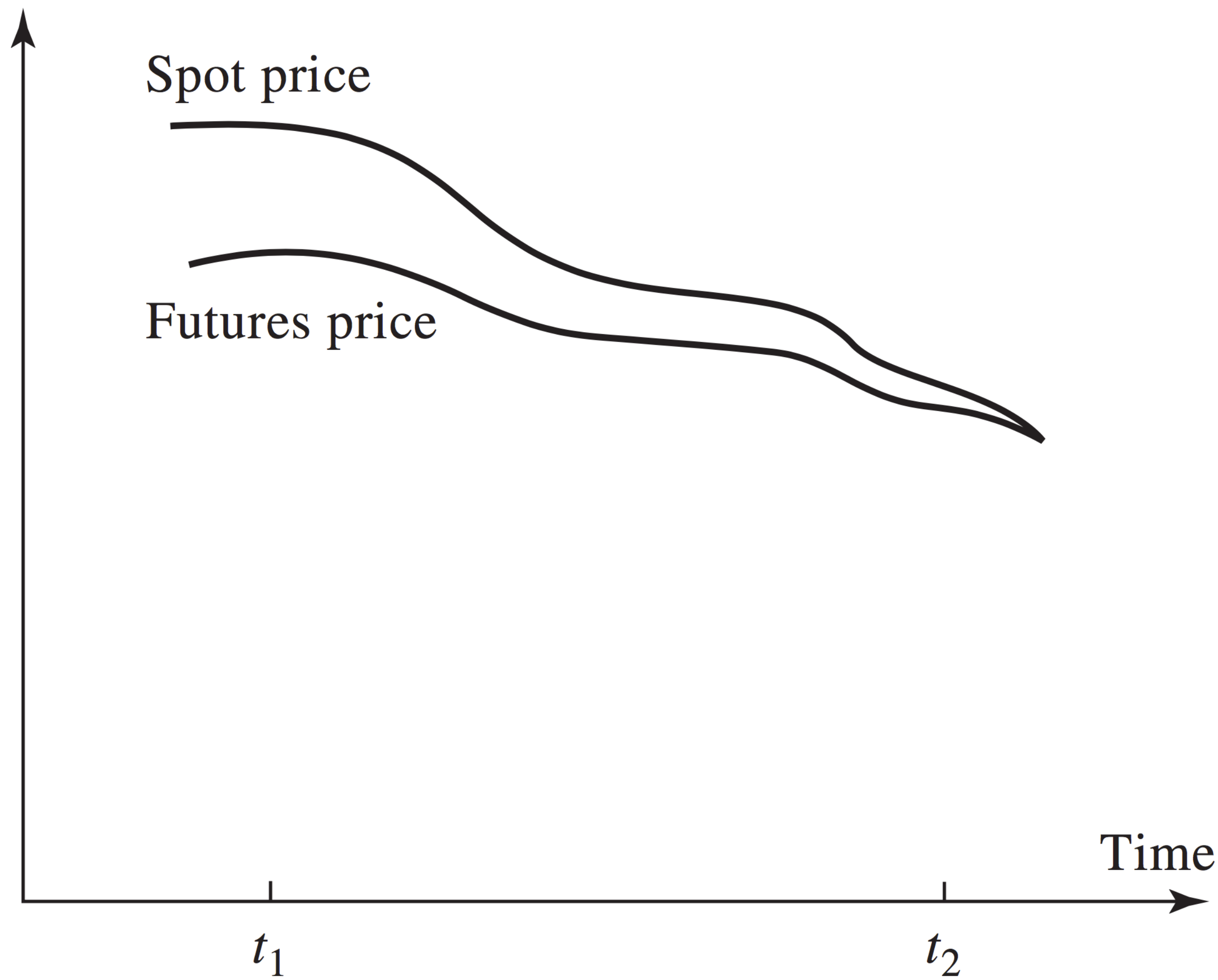
Basis Fluctuation

The basis can fluctuate through time.

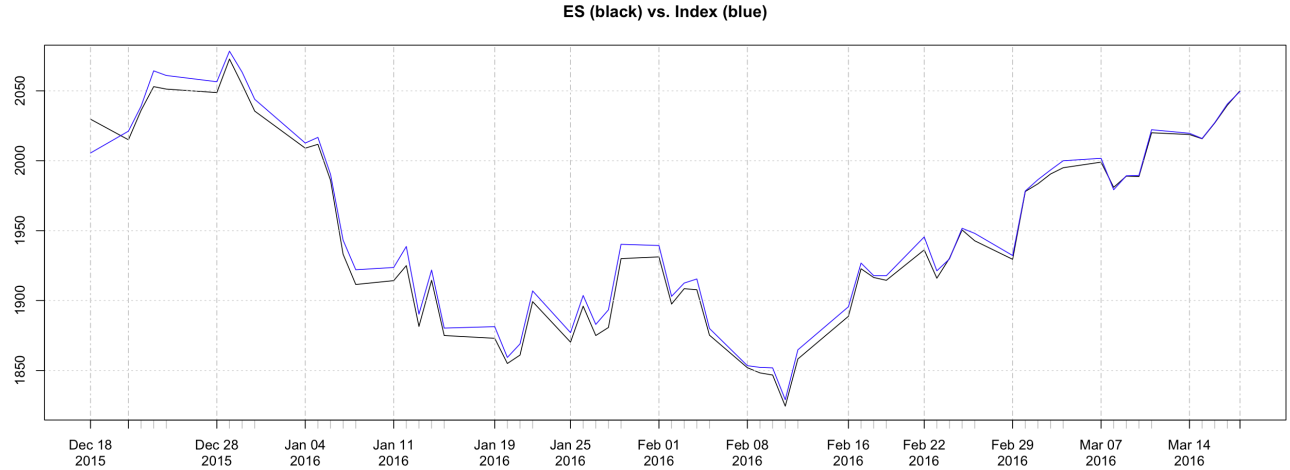
- The futures price should converge to the spot price at expiry. This is a deterministic change in the basis related to shortening time window $T - t$.

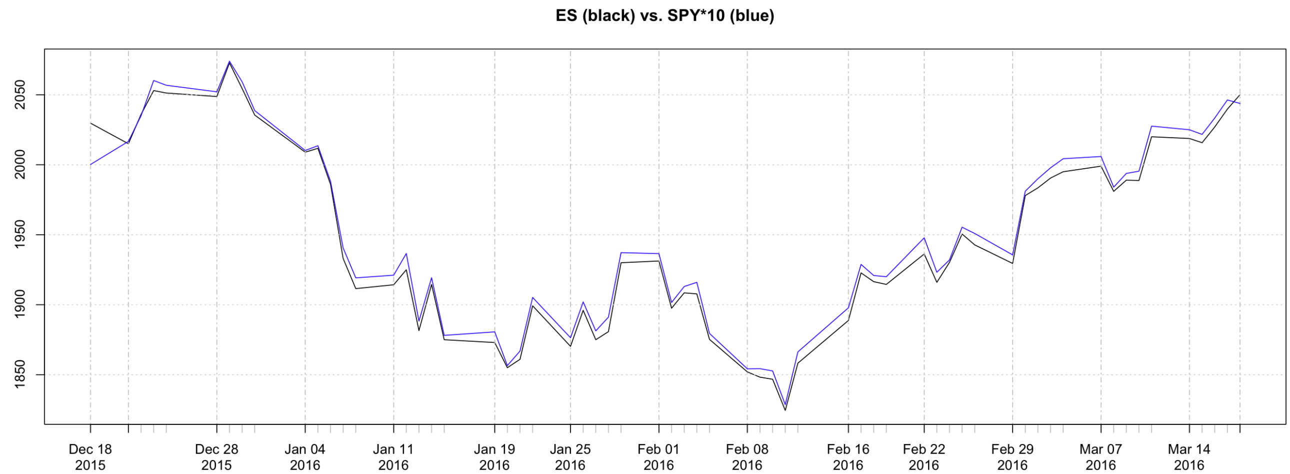
- The basis may also fluctuate due to random variation in the cost of carry, c.
    - This is caused by random fluctuations in r, r_f, dividends, storage costs, etc.

Stylized Basis Fluctuation



Actual Basis Fluctuation





Hedging and Basis

Consider an arbitrary asset with the following spot and futures prices at t_1 and t_2: $S_1 = \$2.50$, $F_1 = \$2.20$, $S_2 = \$2.00$ and $F_2 = \$1.90$.

- $b_1 = \$0.30$ and $b_2 = \$0.10$.
- If you hold the asset and plan to sell at t_2, how can you hedge?
- Profit: $S_2 + F_1 - F_2 = F_1 + b_2 = \$2.30$.
- If you need to purchase the asset at t_2, how can you hedge?
- Cost: $S_2 + F_1 - F_2 = F_1 + b_2 = \$2.30$.

Basis Risk

Note that the profit/cost of the hedging strategies above is $F_1 + b_2$.

- If b_2 is known at t_1, then a perfect hedge could be designed.
- Basis change due to $t_2 - t_1$ is perfectly foreseeable.
- Basis fluctuation due to random variations in c is not perfectly foreseeable.

Contract Choice

Perfect hedges typically don't exist. Important decisions for the hedge include:

- An asset with a futures contract that closely approximates the asset to be hedged.
- Expiry close to the necessary terminal date of the hedge.
- Typically, expiry is chosen to be a month following hedge termination so that delivery doesn't occur and rolling is unnecessary.

Example: Hedging Yen

Suppose it's March 1st and you will receive 50 million Yen at end of July. Yen futures contracts are for delivery of 12.5 million Yen in Mar/Jun/Sep/Dec.

- How can you hedge?

Assume the following spot and (Sep) futures prices (cents/Yen): $F_1 = 0.9800$, $S_2 = 0.9200$ and $F_2 = 0.9250$.

- What is the price you pay for Yen?
- $F_1 + b_2 = 0.9800 + (0.9200 - 0.9250) = 0.9750$ per Yen.
- For 50 million Yen: $50,000,000 \times 0.9750 = 48,750,000$ cents or $\$487,500$.

Cross Hedging

A *cross hedge* occurs when the asset being hedged is different from the asset underlying a futures contract.

- Since the two assets may not be perfectly correlated, an adjustment must be made to determine the optimal number of futures contracts to hold.

- The optimal number of futures is determined via the *hedge ratio*.

Hedge Ratio

The optimal hedge ratio is defined as

$$h^* = \rho\frac{\sigma_S}{\sigma_F}$$
$$\sigma_S = Sd(\Delta S)$$
$$\sigma_F = Sd(\Delta F).$$

- h^* is the slope of a regression of ΔS on ΔF.

- The hedge ratio is typically computed by estimating $\rho, \sigma_S`and:math:$ smash{sigma_F}` with historical data.

Hedge Ratio with Returns

Note that

$$\sigma_{r_S} = Sd(\Delta S/S) = Sd(\Delta S)S$$
$$\sigma_{r_F} = Sd(\Delta F/F) = Sd(\Delta F)F$$

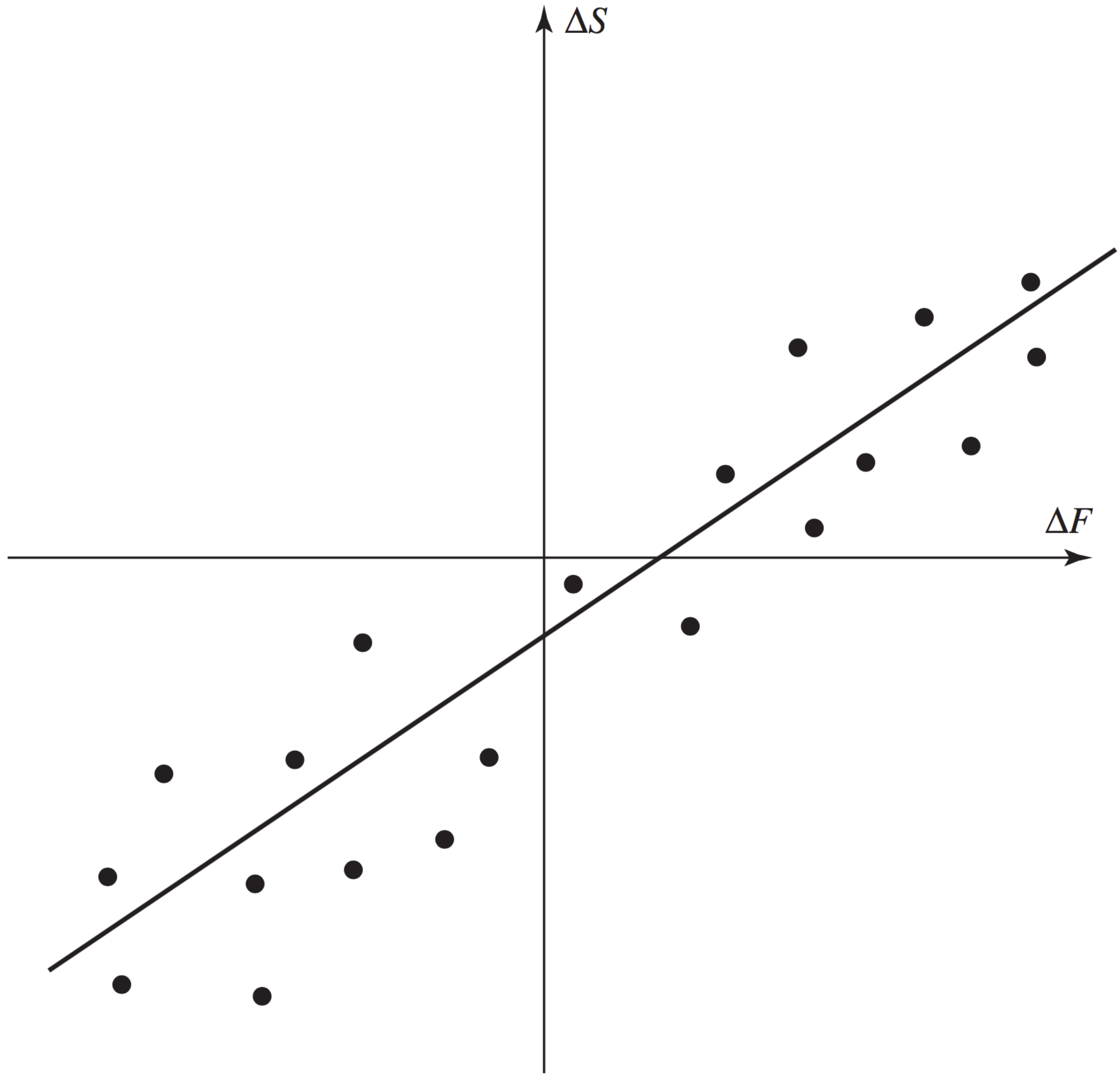
Typo: divided by S (and F); not multiplied

where r_S and r_F are returns (not price changes).

- An alternate definition of the hedge ratio is:

$$\tilde{h} = \rho\frac{\sigma_{r_S}}{\sigma_{r_F}} = \rho\frac{\sigma_S/S}{\sigma_F/F} = h^*\frac{F}{S}$$
$$\Rightarrow h^* = \tilde{h}\frac{S}{F}.$$

Hedge Ratio Regression



Hedge Ratio

Note that if the asset underlying the futures is identical to the asset being hedged:

- $\rho = 1, \sigma_S = \sigma_F$ and $h^* = 1$.

If $\rho = 1$ and $\sigma_S = 2\sigma_F$, you need to hedge with two futures.

- The assets are perfectly correlated, but price swings in the futures are only half as large as those for the asset being hedged.

Optimal Number of Contracts

The optimal number of contracts to purchase for a cross hedge is

$$N^* = h^* \frac{Q_S}{Q_F} = \tilde{h} \frac{SQ_S}{FQ_F} = \tilde{h} \frac{V_S}{V_F}.$$

where Q_S is the size of the position being hedged, Q_F is the size of a futures contract and V_S and V_F are their total valuations (shares times price).

Hedge Ratio Example

An airline company needs to purchase 2 millions gallons of jet fuel in 1 month.

- CME Group heating oil futures are the best contract to use as a hedge.

- One contract is for delivery of 42,000 gallons of heating oil.

- The table on the following slide has historical data to estimate the optimal hedge ratio.

Hedge Ratio Example

Month i	*Change in heating oil futures price per gallon* $(= \Delta F)$	*Change in jet fuel price per gallon* $(= \Delta S)$
1	0.021	0.029
2	0.035	0.020
3	–0.046	–0.044
4	0.001	0.008
5	0.044	0.026
6	–0.029	–0.019
7	–0.026	–0.010
8	–0.029	–0.007
9	0.048	0.043
10	–0.006	0.011
11	–0.036	–0.036
12	–0.011	–0.018
13	0.019	0.009
14	–0.027	–0.032
15	0.029	0.023

Hedge Ratio Example

Using the data:

- $\hat{\rho} = 0.928, \hat{\sigma}_S = 0.0263$ and $\hat{\sigma}_F = 0.0313$.

$$\hat{h}^* = \hat{\rho}\frac{\hat{\sigma}_S}{\hat{\sigma}_F} = 0.928\frac{0.0263}{0.0313} = 0.78$$

$$\hat{N}^* = \hat{h}^*\frac{Q_S}{Q_F} = 0.78\frac{2,000,000}{42,000} \approx 37.$$

Hedging an Equity Portfolio

Suppose you want to hedge an equity portfolio which has some sensitivity to the market, β:

$$r_p = r_f + \beta(r_I - r_f) + \epsilon.$$

- r_p, r_I and r_f are the portfolio, market index (S&P 500) and risk-free returns, respectively.

- Since β is the slope of the regression of excess returns, it is the hedge ratio, $\tilde{h}$.

Hedging Equity Portfolio Example

Suppose:

- $P_{SP500} = 1000$.
- $P_{ES} = 1010$.
- Portfolio Value $\$5,050,000$.
- $r_f = 0.04$ per annum.
- Index dividend yield 0.01 per annum.
- $\beta = 1.5$.

Hedging Equity Portfolio Example

Value of index in three months:	900	950	1,000	1,050	1,100
Futures price of index today:	1,010	1,010	1,010	1,010	1,010
Futures price of index in three months:	902	952	1,003	1,053	1,103
Gain on futures position ($):	810,000	435,000	52,500	−322,500	−697,500
Return on market:	−9.750%	−4.750%	0.250%	5.250%	10.250%
Expected return on portfolio:	−15.125%	−7.625%	−0.125%	7.375%	14.875%
Expected portfolio value in three months including dividends ($):	4,286,187	4,664,937	5,043,687	5,422,437	5,801,187
Total value of position in three months ($):	5,096,187	5,099,937	5,096,187	5,099,937	5,103,687

Why Hedge?

Note that the index hedge results in a portfolio that earns grows at the risk-free rate.

- So why hedge?

- Perhaps you think your portfolio will earn positive *non-market* return (alpha) but don't want to be exposed to the market.

- Perhaps you want to hold the portfolio for a long period of time, but need a brief reduction of risk exposure.

Changing Beta

A complete hedge (as above) makes the effective beta zero.

- Suppose you simply want to change the beta of your portfolio to a new value, β^*?

$$N^* = (\beta - \beta^*)\frac{V_P}{V_F}.$$

Futures on Interest Rates

Common Interest Rates

- Treasury rates.
    - U.S. Treasury rates are rates earned by on bills, notes and bonds.

- London Interbank Offer Rate (LIBOR).
    - Short-term borrowing rate between banks.
    - Published by British Bankers Association (BBA) on 10 currencies and 15 borrowing periods. Based on survey of AA-rated banks.

- Fed Funds Rate.
    - The overnight lending rate between banks which keep deposits at the Federal Reserve.

Day Counts

Different day count conventions exist for determining the period over which interest bearing instruments accrue interest.

- Actual/Actual (U.S. Treasury bonds and notes).

- 30/360 (Corporate agency, municipal bonds).

- Actual/360 (T-bills, commercial paper).

T-bill Quotations

T-bill prices are quoted using a *discount rate.*

- The discount rate is the *annualized* interest earned as a percentage of final face value.

The *cash price* of a T-bill is defined as

$$P = 100 - \frac{n}{360}Q.$$

- Q is the quoted price of the bond (the discount rate).
- n is the number of days remaining in the life of the bond.

Treasury Bond Quotations

Treasury bonds are quoted in dollars and thirty-seconds of a dollar.

- For example, on Mar 5, 2015, the 30-year semi-annual Treasury bond with 11% coupon rate maturing July 10, 2038 was quoted as 95-16, or $95.50.
- The quotations do not include accrued interest.
- Accrued interest is the prorated amount of the next coupon that must be paid to the seller, since they will not receive the coupon at a later date.
- The quoted price is the *clean price*.
- The *dirty price* includes accrued interest.

Accrued Interest

Treasury bond coupon rates are always quoted in *annual* terms.

- An 8% coupon on a semiannual bond means that a 4 dollar coupon is paid every 6 months (assuming $100 face value).
- In the example above, the 11% coupon rate means $5.50 is paid every 6 months, on Jan 10 and Jul 10.
- If you buy the bond on Mar 5:
    - 54 days have elapsed since Jan 10
    - There are a total of 181 days between Jan 10 and Jul 10.
    - Accrued interest is $\frac{54}{181}\$5.50 = \$1.64$.

The dirty price of the bond is: $\$95.50 + \$1.64 = \$97.14$.

Treasury Futures

Treasury futures typically provide a menu of instruments that can be delivered.

- Treasury bond futures: any bond with 15 to 25 years to maturity.

- Ultra T-bond futures: any bond with maturity over 25 years.

- 10-year Treasury note futures: any bond/note with 6.5 to 10 years to maturity.

- 5/2-year Treasury note futures: a note with about 5/2 years remaining and original maturity less than 5.25 years.

- Conversion factors compensate recipient for differentials in delivered instruments.

Conversion Factors

The party with the short futures position chooses the instrument to deliver.

- The cash price paid by the recipient (long position) is

$$\text{Most recent settlement price} \times \text{Conversion factor} + \text{Accrued interest.}$$

- The short party determines the *cheapest-to-deliver* bond using data on conversion factors and settlement prices.

Treasury Futures Quotations

Each futures is for delivery of $100,000 face value of bonds.

- Treasury bond and Ultra T-bond futures: quoted in thirty-seconds of a dollar per $100 face.
  - Identical to spot market.

- 10-year Treasury note futures: quoted in half of a thirty-second.

- 5/2-year Treasury note futures: quoted in quarter of a thirty-second.

Treasury Futures Quotations

	Open	*High*	*Low*	*Prior settlement*	*Last trade*	*Change*	*Volume*
Ultra T-Bond, $100,000							
June 2013	158-08	158-31	156-31	158-08	157-00	−1-08	45,040
Sept. 2013	157-12	157-15	155-16	156-24	155-18	−1-06	176
Treasury Bonds, $100,000							
June 2013	144-22	145-04	143-26	144-20	143-28	−0-24	346,878
Sept. 2013	143-28	144-08	142-30	143-24	142-31	−0-25	2,455
10-Year Treasury Notes, $100,000							
June 2013	131-315	132-050	131-205	131-310	131-210	−0-100	1,151,825
Sept. 2013	131-040	131-080	130-240	131-025	130-240	−0-105	20,564
5-Year Treasury Notes, $100,000							
June 2013	123-310	124-015	123-267	123-307	123-267	−0-040	478,993
Sept. 2013	123-177	123-192	123-122	123-165	123-122	−0-042	4,808
2-Year Treasury Notes, $200,000							
June 2013	110-080	110-085	110-075	110-080	110-075	−0-005	98,142
Sept. 2013	110-067	110-072	110-067	110-070	110-067	−0-002	13,103
30-Day Fed Funds Rate, $5,000,000							
Sept. 2013	99.875	99.880	99.875	99.875	99.875	0.000	956
July 2014	99.830	99.835	99.830	99.830	99.830	0.000	1,030
Eurodollar, $1,000,000							
June 2013	99.720	99.725	99.720	99.725	99.720	−0.005	107,167
Sept. 2013	99.700	99.710	99.700	99.705	99.700	−0.005	114,055
Dec. 2013	99.675	99.685	99.670	99.675	99.670	−0.005	144,213
Dec. 2015	99.105	99.125	99.080	99.100	99.080	−0.020	96,933
Dec. 2017	97.745	97.770	97.675	97.730	97.680	−0.050	14,040
Dec. 2019	96.710	96.775	96.690	96.760	96.690	−0.070	23

Treasury Futures Price

The value of a Treasury futures is somewhat ambiguous because the cheapest-to-deliver bond and deliver date aren't precise.

- Assuming a date and a particular bond, the price is:

$$F_0 = (S_0 - I)e^{rT}$$

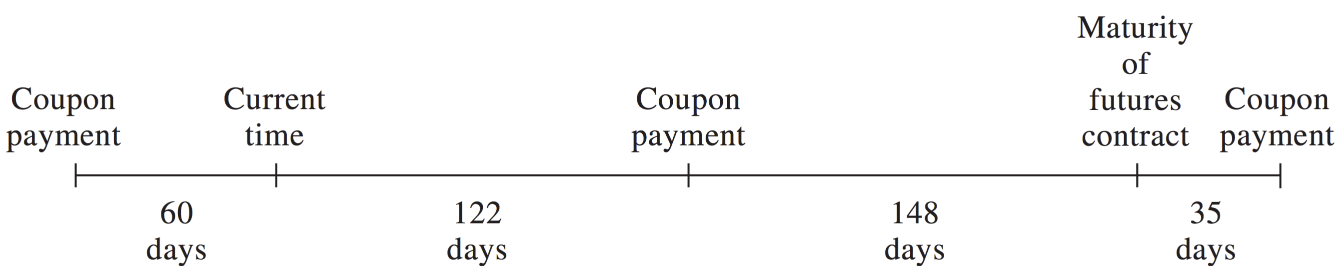
- I is the present value of coupons for the remainder of the bond's life.
- r is the risk-free interest rate.
- T is the life of the bond in years.

Treasury Futures Price Example

Suppose the following characteristics of cheapest-to-deliver bond:

- Quoted bond price is $115.
- 12% coupon rate and bond is semi-annual.
- 270 days until maturity.
- Conversion factor of 1.6000.
- Risk-free rate is 10% per annum.
- 60 since last coupon.
- 122 days until next coupon.
- Futures contract expires 35 days before bond maturity.

Treasury Futures Price Example



Treasury Futures Price Example

Cash price of bond includes accrued interest:

$$115 + \frac{60}{60 + 122} \times 6 = 116.978.$$

Present value of $6 coupon received in 122 days (0.3342 years):

$$6e^{-0.1 \times 0.3342} = 5.803.$$

Life of futures contract is 270 days (0.7397 years), so the cash price is:

$$F_0 = (116.978 - 5.803)e^{0.1 \times 0.7397} = 119.711.$$

Treasury Futures Price Example

At expiry, there are 148 days of accrued interest, so quoted price is:

$$119.771 - 6 \times \frac{148}{148 + 35} = 114.859$$
$$\Rightarrow \frac{114.859}{1.6000} = 71.79.$$

Eurodollar Futures

- A Eurodollar is a dollar deposited in a bank outside of the U.S.

- 3-month Eurodollar futures are most popular interest rate futures traded at CME Group.
  - They are futures contracts on 3-month LIBOR to be paid on $1m principle at the expiry date.

- Eurodollar futures have maturities in the four nearest months and then Mar/Jun/Sep/Dec for up to 10 years.

Eurodollar Futures Quotes

A Eurodollar futures quote is 100 minus the futures interest rate:

$$Q = 100 - R.$$

- The rate is an APR and is expressed in precent.

Since the contract is for 3 months (0.25 years), the contract price is defined as:

$$P = 10,000 \times (100 - 0.25R) = 10,000 \times (100 - 0.25(100 - Q)).$$

- The contract price is the difference between the principle ($1m) and the interest paid on the principle.

- A 0.01% change in the futures rate or futures quote causes a $25 change in paid interest or contract price.

Eurodollar Futures Table

Date	*Settlement futures price*	*Change*	*Gain per contract ($)*
May 13, 2013	99.725		
May 14, 2013	99.720	−0.005	−12.50
May 15, 2013	99.670	−0.050	−125.00
⋮	⋮	⋮	⋮
June 17, 2013	99.615	+0.010	+25.00
Total		−0.110	−275.00

Eurodollar Futures Example

In the previous table:

- The June 2013 futures price on May 13, 2013 is

$$10,000 \times (100 - 0.25(100 - 97.725)) = \$999,312.5.$$

- The June 2013 futures price on Jun 17, 2013 (expiry) is

$$10,000 \times (100 - 0.25(100 - 97.615)) = \$999,037.5.$$

- The price fell by 11 basis points or $\$25 * 11 = \$275$.

- The buyer benefits if rate (price) falls (rises).

Forward Rate Agreements

Zero Rates

A *zero rate* for maturity T is the annual rate of interest earned on a T period investment.

- There is a single payment at the end of the investment (no intermediate payments).
- It is sometimes called the *spot* rate.
- It is equivalent to the yield on a zero-coupon bond.

Zero Coupon Bond

Suppose that the price of a 3-year zero-coupon bond is 97-05.

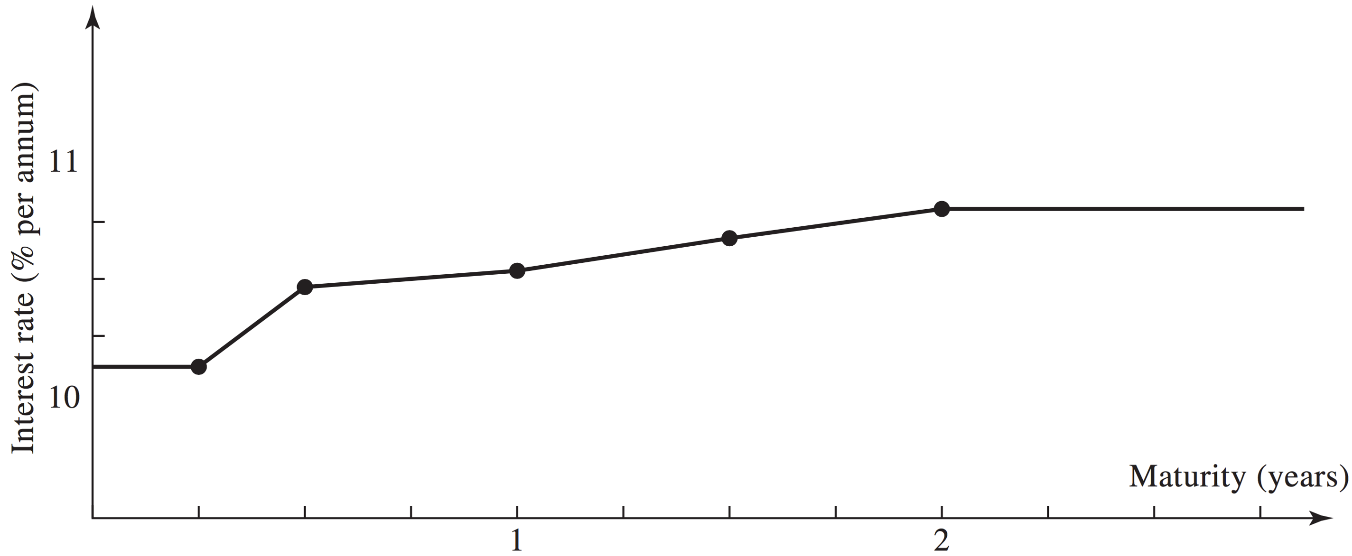
- The continuously-compounded 3-year zero rate is

$$\begin{aligned} 97.15625 e^{r_3 3} &= 100 \\ \Rightarrow r_3 &= \frac{1}{3}(\log(100) - \log(97.15625)) \\ &= 0.00961656. \end{aligned}$$

Zero Curve

Maturity (years)	*Zero rate (%) (continuously compounded)*
0.25	10.127
0.50	10.469
1.00	10.536
1.50	10.681
2.00	10.808

Zero Curve



Forward Rates

Forward rates are future spot rates implied by current spot rates. Consider two investments:

- Investment 1: At t_0 invest $100 in a t_2 year zero coupon bond, earning r_2 (continuously compounded).

- Investment 2: At t_0 invest $100 in a t_1 year zero coupon bond (where $t_2 > t_1$), earning r_1, and at t_1 roll the proceeds into a $t_2 - t_1$ year zero coupon bond.

- The forward rate is the spot rate that would have to prevail at t_1 for the investments to be equal.

$$
\begin{aligned}
100e^{r_1 \times t_1} e^{r_f \times (t_2 - t_1)} &= 100e^{r_2 \times t_2} \\
\Rightarrow r_1 t_1 + r_f (t_2 - t_1) &= r_2 t_2 \\
\Rightarrow r_f &= \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1}.
\end{aligned}
$$

Forward Rate Example

Year (n)	Zero rate for an n-year investment (% per annum)	Forward rate for nth year (% per annum)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

Forward Rates: Alternative

Rearranging the forward rate formula:

$$r_f = r_2 + (r_2 - r_1)\frac{t_1}{t_2 - t_1}.$$

- If the zero-curve is upward sloping between t_1 and t_2, $r_f > r_2 > r_1$.
- If the zero-curve is downward sloping between t_1 and t_2, $r_f < r_2 < r_1$.

Notes on Forward Rates

- Spot rates are *currently* available interest rates for investments of different maturities.
- Forward rates are *implied future* spot rates for a single maturity.
- Forward rates are not the same as future spot rates.
  - They are guaranteed rates for the future period, obtained by investing in current spot rates of differing maturities.

Forward Rate Agreements

A Forward Rate Agreement (FRA) is a contract to fix an interest rate for borrowing/lending on a specific principal amount for a specific period of time.

- The contract is over the counter.

- The benchmark interest rate is typically LIBOR.
- If LIBOR is below the contracted rate at maturity, the borrower pays the interest differential on the principal to the lender and vice versa.
- The principal and time period are variables.
- Interest is due at end of period, but present value typically paid at beginning.

FRA Example

A company enters into an FRA to receive 4% on $100m for a 3-month period starting in 3 years.

- If LIBOR is 4.5% in 3 years, cash flow to the lender at 3.25 years is:

$$100,000,000 \times (0.04 - 0.045) \times 0.25 = -\$125,000.$$

- The present value is paid at year 3:

$$-\frac{125,000}{1 + 0.045 \times 0.25} = -123,609.$$

- Note that for FRAs, interest rates are typically not quoted with continuous compounding, but with compounding frequency equal to term of agreement.

FRAs vs Eurodollar Futures

FRAs and Eurodollar futures are similar.

- Both fix an interest rate for a future period of time, tied to LIBOR.
- Eurodollar futures are exchange traded, $1m principal and 3-month LIBOR.
- FRAs are over the counter, variable principal, variable term.
- Eurodollar futures commit to pay the difference of principal and interest payment at beginning of period.
- FRAs commit to pay interest differential on principal at end of period.

FRA Notation

Suppose an FRA is set so that company X lends to Y between T_1 and T_2.

- R_K: Fixed interest rate set in FRA.
- R_F: Forward LIBOR rate (determined today) between T_1 and T_2.
- R_M: Actual LIBOR observed at T_1.
- L: Principal.

FRA Cash Flows

At T_2 the cash flow from Y to X (possibly negative):

$$L(R_K - R_M)(T_2 - T_1)$$

If settled at time T_1, the present value is:

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

FRA Valuation

An FRA is worth zero if $R_K = R_F$.

- At inception, R_K is set equal to R_F and the FRA has zero value.
- As time elapses, R_F fluctuates and the value of the FRA changes.
- Prior to T_1, the value of the FRA is computed by substituting R_F for R_M and computing the present value:

$$L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}.$$

- R_2 is the continuously compounded risk-free rate.

FRA Valuation Example

It is currently Jan 1, 2010.

- On Jan 1, 2009, a company entered into an FRA to receive 5.8% (semi-annual compounding) and pay LIBOR on $100m between July 1, 2011 and Dec 31, 2011.
- The current LIBOR forward rate for July 1 - Dec 31, 2011 is 5% (semi-annual compounding).

- The 2-year risk-free interest rate (continuous compounding) is 4%.

- The value of the FRA on Jan 1, 2010 is:

$$100,000,000 \times (0.058 - 0.050) \times 0.5 \times e^{-0.04 \times 2} = \$369,200.$$

Swaps

Swaps

A swap is an agreement to exchange cash flows in the future.

- It is an over-the-counter agreement.
- Dates, cash flows and timing of payments are variables of the contract.
- The cash flows are typically determined by an interest rate or exchange rate.

Forward/Futures as Swap

A forward or futures contract can be viewed as a simple swap:

- Suppose you buy an E-mini contract today for $F = 2000$.
- You are agreeing to a swap in which you pay $50F$ and receive $50P$, where P is the settlement price of the contract.

Swaps are typically more complex, involving a stream of swapped cash flows.

Interest Rate Swap

Basic components of an interest rate swap:

- Notional principal.
- Fixed interest rate.
- Floating interest rate.
- Predetermined time period for cash flows.

One party pays fixed interest on the principal during the predetermined period and the other pays floating interest.

LIBOR Swap

Typically LIBOR is the floating rate used in interest rate swaps.

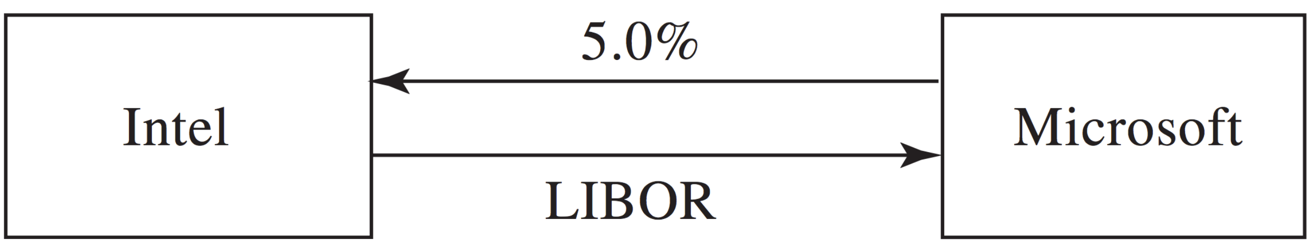
- A LIBOR swap is referred to as a ``Libor-for-fixed" swap.

LIBOR Swap Example

Suppose Microsoft and Intel engage in a 3-year swap on Mar 5, 2014.

- The notional principal is $100 million.
- Payments made semi-annually.
- Microsoft pays 5% fixed (per annum, semi-annual compounding).
- Intel pays 6-month LIBOR (per annum, semi-annual compounding).

LIBOR Swap Diagram



LIBOR Swap Cash Flows

Date	*Six-month LIBOR rate (%)*	*Floating cash flow received*	*Fixed cash flow paid*	*Net cash flow*
Mar. 5, 2014	4.20			
Sept. 5, 2014	4.80	+2.10	−2.50	−0.40
Mar. 5, 2015	5.30	+2.40	−2.50	−0.10
Sept. 5, 2015	5.50	+2.65	−2.50	+0.15
Mar. 5, 2016	5.60	+2.75	−2.50	+0.25
Sept. 5, 2016	5.90	+2.80	−2.50	+0.30
Mar. 5, 2017		+2.95	−2.50	+0.45

LIBOR Swap Cash Flows with Notional

Date	*Six-month LIBOR rate (%)*	*Floating cash flow received*	*Fixed cash flow paid*	*Net cash flow*
Mar. 5, 2014	4.20			
Sept. 5, 2014	4.80	+2.10	−2.50	−0.40
Mar. 5, 2015	5.30	+2.40	−2.50	−0.10
Sept. 5, 2015	5.50	+2.65	−2.50	+0.15
Mar. 5, 2016	5.60	+2.75	−2.50	+0.25
Sept. 5, 2016	5.90	+2.80	−2.50	+0.30
Mar. 5, 2017		+102.95	−102.50	+0.45

LIBOR Swap Example

- Microsoft pays $2.5 million to Intel on Sep 5, 2014 and each subsequent 6 months until Mar 5, 2017.

- On Sep 5, 2014, Intel pays Mar 5, 2014 LIBOR (4.2%): $2.1 million.

- Each subsequent 6 months, Intel pays the LIBOR rate from the date 6 months prior to the payment.

- The first exchange of payments is known at initiation of the swap.

LIBOR Swap Cash Flows

Typically only one party in the swap makes a payment of the difference between the two cash flows.

- In the previous example, Microsoft would pay \$0.4 million on Sep 5, 2015 and $0.1 million on Mar 5, 2015.

- Note that the principal is not exchanged at the end of the contract, which is why it is called *notional*.

- If the notional were traded, the swap could be characterized as an exchange of floating- and fixed-rate bonds.

Transforming a Liability

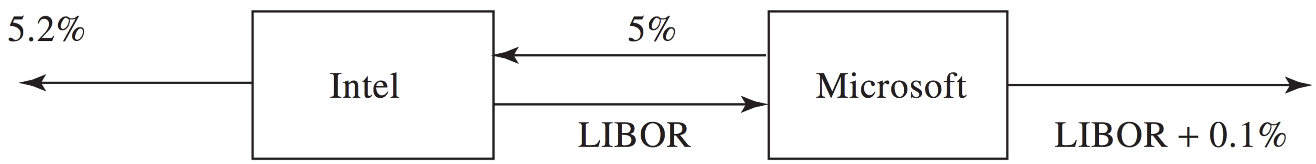
A swap can transform a floating-rate loan into a fixed-rate loan and vice versa.

- If Microsoft has a floating-rate loan on $100 million at LIBOR plus 10 basis points (1 basis point = 0.01%), the swap transforms it into a loan of 5.1% fixed.

1. Microsoft pays LIBOR plus 0.1% to outside lenders.
2. It receives LIBOR from swap.
3. It pays 5% to Intel.

- If Intel has a fixed-rate loan on $100 million at 5.2%, the swap transforms it into a loan of LIBOR plus 20 basis points:
  1. Intel pays 5.2% to outside lenders.
  2. It receives 5% from swap.
  3. It pays LIBOR to Microsoft.

Transforming a Liability



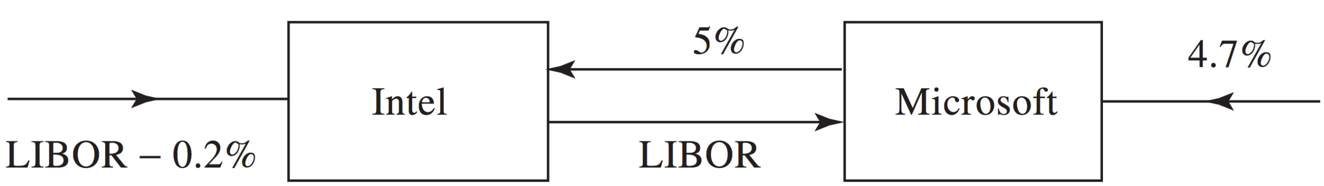
Transforming a Asset

A swap can transform a floating-rate asset into a fixed-rate asset and vice versa.

- If Microsoft holds $100 million in bonds paying 4.7% per annum, the swap transforms the bonds into asset paying LIBOR minus 30 basis points:
  1. Microsoft receives 4.7% from bonds.
  2. It receives LIBOR from swap.
  3. It pays 5% to Intel.

- If Intel holds $100 million in bonds paying LIBOR minus 20 basis points, the swap transforms the bonds into an asset paying 4.8%:
  1. Intel receives LIBOR minus 0.2% from bonds.
  2. It receives 5% from swap.
  3. It pays LIBOR to Intel.

Transforming a Liability

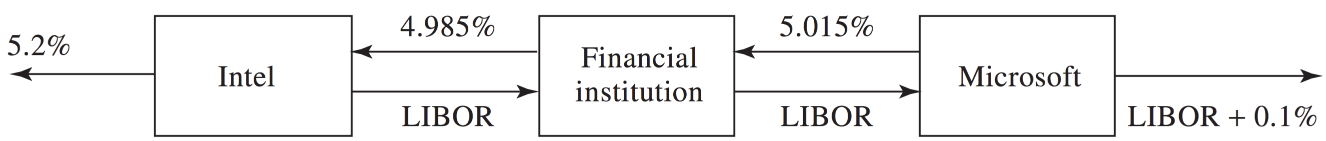


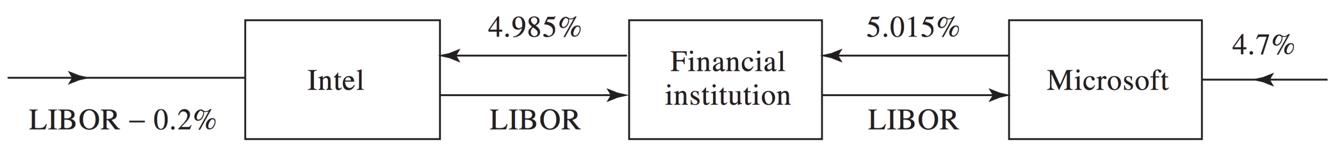
Financial Intermediaries

Typically the parties to a swap connect via a banking intermediary.

- The bank earns a commission of 3% or 4% for providing the service and assuming default risk.
- The individual swap parties don't know who is on the other side of the transaction.
- In many cases, market makers enter into swaps with one party without having an exact offsetting position.
- Market makers will post bid and offer (fixed) rates - the average of bid and offer is the *swap rate.*

Financial Intermediaries Diagram





Market Maker Quotes

Maturity (years)	*Bid*	*Offer*	*Swap rate*
2	6.03	6.06	6.045
3	6.21	6.24	6.225
4	6.35	6.39	6.370
5	6.47	6.51	6.490
7	6.65	6.68	6.665
10	6.83	6.87	6.850

An Important Fact

Recall the basic bond pricing formula:

$$P = \frac{c \times F}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{F}{(1+r)^T}.$$

- c is the coupon rate, expressed in terms of the compounding period.

- r is the discount rate.

- Note that when $c = r, P = F$.
  - This is because the income stream from the bond is equal (in percentage terms) to the rate at which you discount.

- What if $c > r$ or $c < r$?

Swap Value: Floating Side

The floating side of a swap is a bond that pays LIBOR.

- The bond is also discounted at LIBOR.

- This is a case where the coupon rate equals the discount rate.

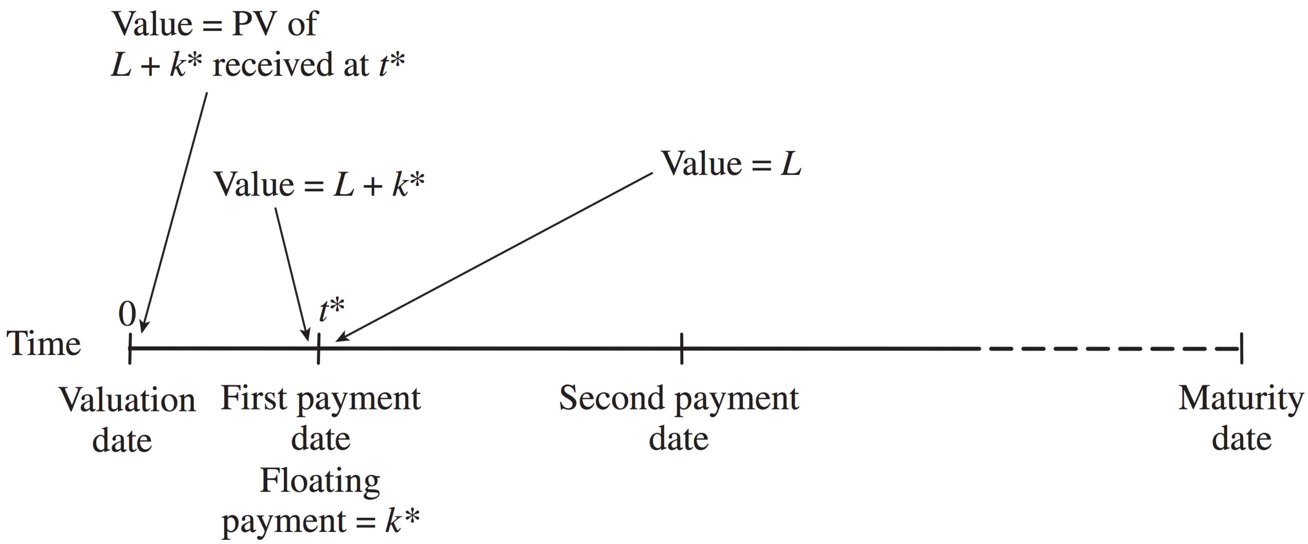
- After each interest payment, the bond is equal to the face, or notional principal, L.

- *Immediately* prior to the interest payment, k^*, which is known at the beginning of the period, the bond is worth $L + k^*$.

- If the payment is made at time t^*, we can discount back to the present using the continuously compounded risk-free, r^*:

$$B_{fl} = (L + k^*)e^{-r^* t^*}.$$

Swap Value Diagram



Swap Value: Fixed Side

The fixed side of a swap is a fixed-rate bond.

- The present value of the bond can be computed in the usual way, using spot rates:

$$B_{fix} = ce^{-r_1 t_1} + ce^{-r_2 t_2} + \ldots + (c + L)e^{-r_n t_n}.$$

- c is the fixed payment and L is the notional principal.

- n is the number of payment periods and r_i, $i = 1, \ldots, n$ are the spot rates.

Swap Value

The resulting swap value is the difference in value of the fixed and floating bonds.

- For the party holding the long position in floating:

$$V_{swap} = B_{fl} - B_{fix}.$$

- For the party holding the long position in fixed:

$$V_{swap} = B_{fix} - B_{fl}.$$

Swap Value Example

Some time ago you went long on the floating side of a swap, receiving 6-month LIBOR on $100 million principal and paying 3% per annum (semi-annual compounding).

- The swap has 1.25 years remaining.
- Continuously compounded 3-, 9- and 15-month LIBOR rates are 2.8%, 3.2% and 3.4%.
- 6-month LIBOR (compounded semi-annually) at last payment date was 2.9%.

Swap Value Example

The fixed-side bond value (in millions of dollars) is:

$$B_{fix} = 1.5e^{-0.028 \times 0.25} + 1.5e^{-0.032 \times 0.75} + 101.5e^{-0.034 \times 1.25} = 100.2306.$$

The floating-side bond value (in millions of dollars) is:

$$B_{fl} = 101.45e^{-0.028 \times 0.25} = 100.7423.$$

The swap value (in millions of dollars) is:

$$V_{swap} = B_{fl} - B_{fix} = 100.7423 - 100.2306 = 0.5117.$$

Swap Value Table

Time	B_{fix} *cash flow*	B_{fl} *cash flow*	*Discount factor*	*Present value* B_{fix} *cash flow*	*Present value* B_{fl} *cash flow*
0.25	1.5	101.4500	0.9930	1.4895	100.7423
0.75	1.5		0.9763	1.4644	
1.25	101.5		0.9584	97.2766	
Total:				100.2306	100.7423

Swap Value: FRA Portfolio

Each swap payment can be considered an individual FRA.

- In the Intel/Microsoft example, the exchange on Mar 5, 2015 is an FRA where 5% is traded for 6-month LIBOR, determined on Sep5, 2014, etc.
- Since the future LIBOR rates are unknown, each FRA is valued using the forward rates implied by the LIBOR zero curve.
- Hence, the swap can be valued as follows:
    1. Compute forward rates using the LIBOR zero curve.

2. Compute cash flows assuming future LIBOR rates are equal to the forward rates.
3. Discount the cash flows using the LIBOR zero curve.

Swap Value Example: FRA Portfolio

Continuing with the previous example:

- The LIBOR rate for the first cash receipt (at 3 months) is already determined at 2.9% per annum (semi-annual compounding).

- The forward rate (continuous compounding) for the period between 3 months and 9 months is

$$\frac{0.032 \times 0.75 + 0.028 \times 0.25}{0.5} = 0.034.$$

- With semi-annual compounding (using EAR/APR relationship):

$$2(e^{0.034/2} - 1) = 0.034291.$$

Swap Value Example: FRA Portfolio

- The forward rate for the period between 9 months and 15 months is

$$\frac{0.034 \times 1.25 + 0.032 \times 0.75}{0.5} = 0.037.$$

- With semi-annual compounding (using EAR/APR relationship):

$$2(e^{0.037/2} - 1) = 0.037344.$$

Swap Value Example: FRA Portfolio

The swap cash flows are (in millions of dollars):

$$\begin{aligned} 0.5 \times 0.029 \times 100 - 1.5 &= 1.45 - 1.5 = -0.05 \\ 0.5 \times 0.034291 \times 100 - 1.5 &= 1.7145 - 1.5 = 0.21455 \\ 0.5 \times 0.037344 \times 100 - 1.5 &= 1.867 - 1.5 = 0.3672. \end{aligned}$$

The present value is:

$$V_{swap} = -0.05e^{-0.028 \times 0.25} + 0.21455e^{-0.032 \times 0.75} + 0.3672e^{-0.034 \times 1.25} = 0.5117.$$

Swap Value Example: FRA Table

Time	*Fixed cash flow*	*Floating cash flow*	*Net cash flow*	*Discount factor*	*Present value of net cash flow*
0.25	−1.5000	+1.4500	−0.0050	0.9930	−0.0497
0.75	−1.5000	+1.7145	+0.2145	0.9763	+0.2094
1.25	−1.5000	+1.8672	+0.3672	0.9584	+0.3519
Total:					+0.5117

Currency Swaps

In a *fixed-for-fixed currency swap* two parties exchange principal and fixed interest payments on two different currencies.

- Typically the principal is exchanged at initiation, where the principal values are roughly equal, given the exchange rate.

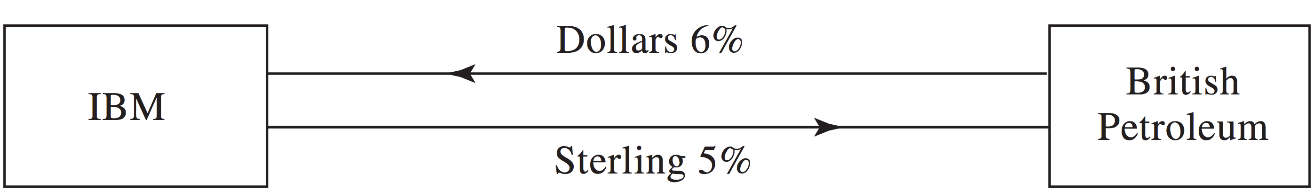
- The principal values are exchanged again at the end, but may be quite different (due to exchange rate fluctuations).

Currency Swap Example

Suppose IBM and BP enter into a 5-year currency swap on Feb 1, 2014.

- IBM pays 5% fixed in sterling and receives 6% fixed in dollars.
- Principal amounts are 15 million USD and 10 million GBP and interest payments are once a year.

Currency Swap Example



Currency Swap Example

Date	*Dollar cash flow (millions)*	*Sterling cash flow (millions)*
February 1, 2014	−15.00	+10.00
February 1, 2015	+0.90	−0.50
February 1, 2016	+0.90	−0.50
February 1, 2017	+0.90	−0.50
February 1, 2018	+0.90	−0.50
February 1, 2019	+15.90	−10.50

Currency Swap: Transforming Liabilities and Assets

Suppose IBM can issue USD bonds, but not GBP.

- The swap effectively transforms 15 million USD in debt to 10 million GBP in debt for IBM.
- Alternatively, suppose IBM has an opportunity to invest 10 million GBP at 5% interest in the UK for 5 years, but feels the dollar will strengthen.
  - The swap effectively transforms the GBP investment into a USD investment.

Currency Swap Valuation: Bonds

Let V_{swap} be the USD value of a swap where dollars are received and foreign currency paid:

$$V_{swap} = B_D - S_0 B_F.$$

- B_F is the foreign currency bond value (denominated in units of the foreign currency).
- B_D is the USD bond value (denominated in USD).
- S_0 is the spot exchange rate (as USD per unit of foreign currency).

The value of the swap where foreign currency is received and USD paid is the opposite:

$$V_{swap} = S_0 B_F - B_D.$$

Currency Swap Valuation Example: Bonds

You currently hold a swap in which you receive 5% per annum on Yen and pay 8% per annum on USD.

- The swap has 3 years remaining.
- Continuous compounded discount rates are 4% per annum in Japan and 9% per annum in the U.S.
- Principal values are 10 million USD and 1,200 million JPY.
- S_0 = 1/110 USD/JPY.

Currency Swap Valuation Example: Bonds

The present value of JPY cash flows (in millions of JPY) is:

$$60e^{-0.04\times 1} + 60e^{-0.04\times 2} + 1260e^{-0.04\times 3} = 1230.55.$$

The present value of USD cash flows (in millions of USD) is:

$$0.8e^{-0.09\times 1} + 0.8e^{-0.09\times 2} + 10.8e^{-0.09\times 3} = 9.6439.$$

The value of the swap (in millions of USD) is:

$$\frac{1230.55}{110} - 9.6439 = 1.5430.$$

Currency Swap Valuation Table: Bonds

Time	*Cash flows on dollar bond ($)*	*Present value ($)*	*Cash flows on yen bond (yen)*	*Present value (yen)*
1	0.8	0.7311	60	57.65
2	0.8	0.6682	60	55.39
3	0.8	0.6107	60	53.22
3	10.0	7.6338	1,200	1,064.30
Total:		9.6439		1,230.55

Currency Swap Valuation: Forward Portfolio

Each currency swap cash flow exchange is a single forward contract on foreign exchange.

- Recall the forward exchange rate is:

$$F_0 = S_0 e^{(r-r_f)T}.$$

- The currency forwards are valued by assuming that the forward exchange rates are realized.

Currency Swap Valuation Example

Continuing with the previous example:

- The JPY cash flows, in USD are:

$$\frac{60}{110} e^{(0.09-0.04)\times 1} = 0.57342$$
$$\frac{60}{110} e^{(0.09-0.04)\times 2} = 0.60282$$
$$\frac{1260}{110} e^{(0.09-0.04)\times 3} = 13.30828.$$

- The present value of cash flows is:

$$(0.57342 - 0.8)e^{-0.09\times 1} + (0.60282 - 0.8)e^{-0.09\times 2} + (13.30828 - 10.8)e^{-0.09\times 3} = 1.5430.$$

Currency Swap Valuation Table: Forward Portfolio

Time	*Dollar cash flow*	*Yen cash flow*	*Forward exchange rate*	*Dollar value of yen cash flow*	*Net cash flow ($)*	*Present value*
1	−0.8	60	0.009557	0.5734	−0.2266	−0.2071
2	−0.8	60	0.010047	0.6028	−0.1972	−0.1647
3	−0.8	60	0.010562	0.6337	−0.1663	−0.1269
3	−10.0	1200	0.010562	12.6746	+2.6746	2.0417
Total:						1.5430

Other Currency Swaps

Other common currency swaps include:

- Fixed-for-floating.
    - This can be regarded as portfolio of a fixed-for-fixed currency swap and a fixed-for-floating interest rate swap.

- Floating-for-floating.
    - This can be regarded as portfolio of a fixed-for-fixed currency swap and two fixed-for-floating interest rate swaps.

Other Swaps

- Many variations of interest swaps exists with differing rates, payment periods, principals, etc.

- Credit default swaps (CDS): annual premium is paid (spread) in exchange for a cash payment if a company defaults.

- Equity swaps: returns on equity index swapped for LIBOR.

- Commodity swaps: series of forward commodity contracts.

- Volatility swaps: predetermined volatility swapped with realized (historical) vol.

- Countless others: financial engineering at its finest.

Options

Options

Options are contracts that give the option to buy or sell an asset on or before a specific date at a specific price.

- A *call* option is an option to buy.
- A *put* option is an option to sell.
- A *European* option can only be exercised at maturity.
- An *American* option can be exercised any time prior to maturity.

Terminology

- Underlying asset: The asset that may be bought or sold when the option is exercised.
- Maturity (exercise) date: The date at which the contract expires.
- Strike (exercise) price: The pre-specified price at which the underlying can be bought or sold.

Underlying Assets

Common underlying assets include:

- Common stock.
- Foreign currency.
- Stock indices.
- Volatility indices.
- Futures contracts.

Options are written on many other underlying *assets*.

Options Exchanges

Many options are exchange traded.

- Chicago Board Options Exchange (CBOE).
- International Securities Exchange (ISE).
- Nasdaq PHLX.
- BATS.
- NYSE MKT (formerly American Stock Exchange, or AMEX).
- NYSE Arca.

Exchange-Traded Options

Exchanges serve to standardize contracts on popular options.

- Expiration dates.
- Strike prices.
- Class - call or put.
- American or European.
- Size of options contract.
- Size of underlying.
- Margin requirements.

VIX Options Specs

CBOE VOLATILITY INDEX® (VIX®) OPTIONS CONTRACT SPECIFICATIONS

Symbol:
VIX

Description:
The CBOE Volatility Index - more commonly referred to as "VIX" - is an up-to-the-minute market estimate of expected volatility that is calculated by using real-time S&P 500® Index (SPX) option bid/ask quotes. The VIX Index is calculated using SPX quotes generated during regular trading hours for SPX options. The VIX Index uses SPX options with more than 23 days and less than 37 days to expiration and then weights them to yield a constant, 30-day measure of the expected volatility of the S&P 500 Index.

Multiplier:
$100.

Strike (Exercise) Prices:
Generally, minimum strike price intervals are as follows: (1) $0.50 where the strike price is less than $15, (2) $1 where the strike price is less than $200, and (3) $5 where the strike price is greater than $200.

Premium Quotation:
Stated in points and fractions, one point equals $100. Minimum tick for series trading below $3 is 0.05 ($5.00); above $3 is 0.10 ($10.00).

Expiration Date:
The Expiration Date (usually a Wednesday) will be identified explicitly in the expiration date of the product. If that Wednesday or the Friday that is 30 days following that Wednesday is an Exchange holiday, the Expiration Date will be on the business day immediately preceding that Wednesday.

Contract Expirations:
Up to six 6 weekly expirations and up to 12 standard (monthly) expirations in VIX options may be listed. The six weekly expirations shall be for the nearest weekly expirations from the actual listing date and standard (monthly) expirations in VIX options are not counted as part of the maximum six weekly expirations permitted for VIX options.

Exercise Style:
European - VIX options generally may be exercised only on the Expiration Date.

Implications of Options

The buyer of a call (put) option is *not obligated* to buy (sell) the underlying asset at the strike price.

- The *buyer* has the *option* to buy (sell).

- The *seller* of the call (put) option is *obligated* to sell (buy) the underlying if the buyer wants to exercise the option.

- If the price of the underlying asset is above (below) the strike price on the maturity date, the buyer will exercise. Why?

- If the price of the underlying asset is below (above) the strike price on the maturity date, the buyer will not exercise. Why?

Options as Insurance

Options have no downside risk for the buyer.

- The buyer of a call (put) is better off if the underlying asset price rises (falls).

- If the underlying asset price falls (rises), the buyer doesn't lose anything.

Obligation of Sellers

However, the seller of an option *only* faces downside risk.

- The seller of a call (put) is worse off if the underlying asset price rises (falls).

- If the underlying asset price falls (rises), the seller doesn't gain anything.

The seller must be compensated for taking the risk of having to sell (buy) the underlying for a low (high) price.

- The buyer pays a *premium* to purchase the option contract.

Call Option Example

On March 8th 2013, stock for Chipotle Mexican Grill (CMG) sold for $\$321.84$ and an option contract with a strike price of $\$320.00$ and maturity date of March 15th 2013 cost $\$5.30$.

- If the price of Chipotle is less than $\$320.00$ on March 15th, the option will not be exercised.

- If the price is $\$325.00$ on March 15th, the option holder (buyer) will exercise the contract.

- The gain to the buyer will be $\$5.00$.

Call Option Example

- Remember that the contract cost $\$5.30$, so the buyer has a net loss of $\$0.30$.

- If the price on March 15th is greater than $\$325.30$, the buyer will have a net gain.

Put Option Example

Consider again Chipotle stock which sold for $\$321.84$ on March 8th 2013.

- A put option with a strike price of $\$320.00$ and a maturity date of March 15th 2013 costs $\$3.30$.

- If the price of the stock is above $\$320.00$ on March 15th, the option will not be exercised.

Put Option Example

- Suppose the price is below $\$320.00$ on March 15th: $\$315.00$.

- The buyer of the put will exercise the contract, buying the stock for $\$315.00$ on the market and selling to the put writer for $\$320.00$.

- The gross profit would be $\$320.00$ - $\$315.00$ = $\$5.00$.

- The net profit would be: $\$5.00$ - $\$3.30$.

Moneyness

An option is

- *In the money* when its strike price would produce profits for the holder.

- *Out of the money* when exercise would be unprofitable.

- *At the money* when the strike price is equal to the asset price.

The moneyness can be determined at any time, as if the option were exercised at that instant.

Notation

We use the following notation:

$$\begin{aligned}
T &= \text{Maturity date} \\
S_t &= \text{Underlying asset price at time } t \\
X &= \text{Strike Price} \\
C_t &= \text{Value of a call option at time } t \\
P_t &= \text{Value of a put option at time } t.
\end{aligned}$$

Call Option Payoff (Buyer)

The payoff to a call option holder (buyer) at expiration is

$$C_T = \begin{cases} S_T - X & \text{if } S_T > X \\ 0 & \text{if } S_T \leq X. \end{cases}$$

- If the asset price is above the strike, the holder can buy the underlying for X and immediately sell it for S_T, yielding a profit of $S_T - X$.

- If the asset price is below the strike, the option is worthless.

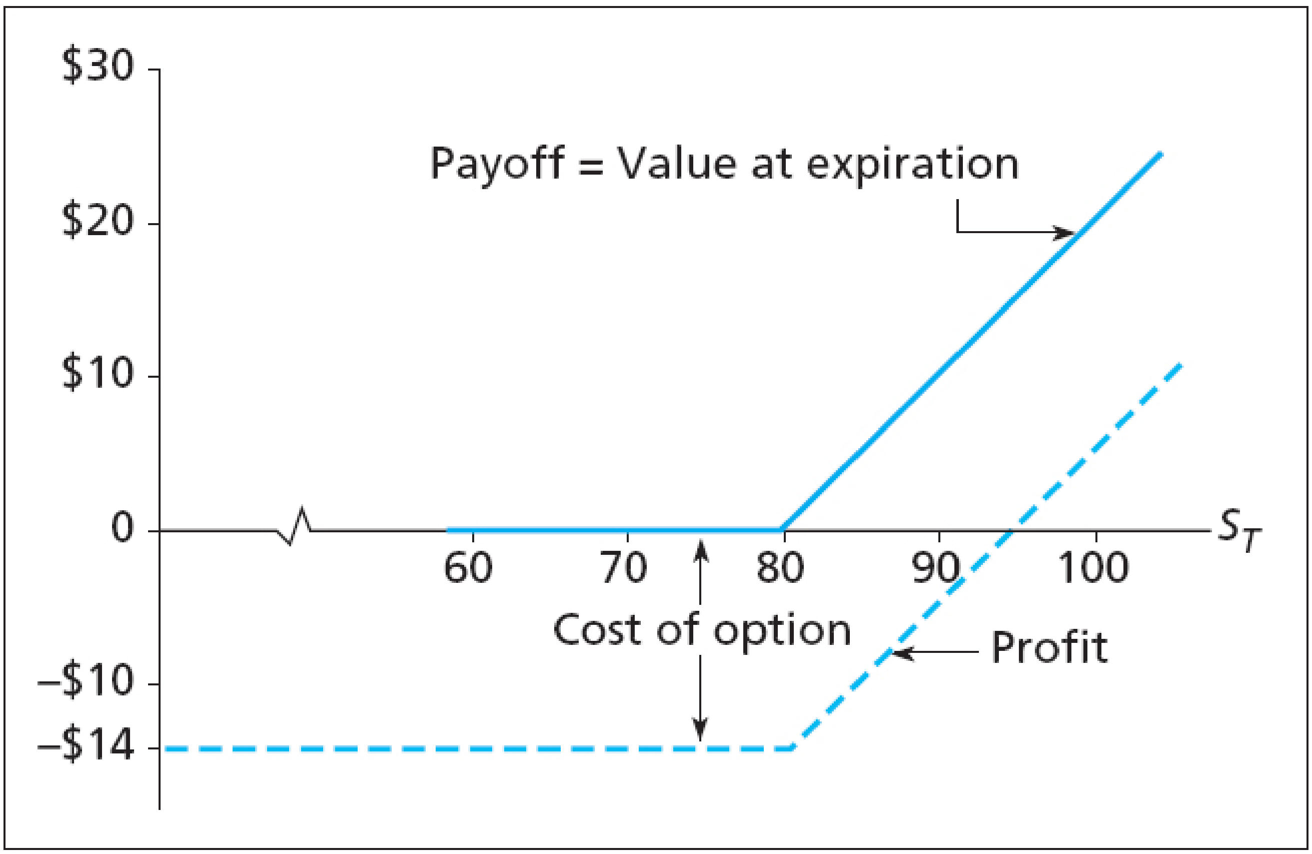
Call Option Payoff (Buyer)

The payoffs above did not account for the cost of the option.

- If the option is purchased at time t for a price of C_t, the net payoff to the holder at expiration is

$$C_T = \begin{cases} S_T - X - C_t, & \text{if } S_T > X \\ -C_t, & \text{if } S_T \leq X. \end{cases}$$

Call Option Payoff (Buyer)



Call Option Payoff (Seller)

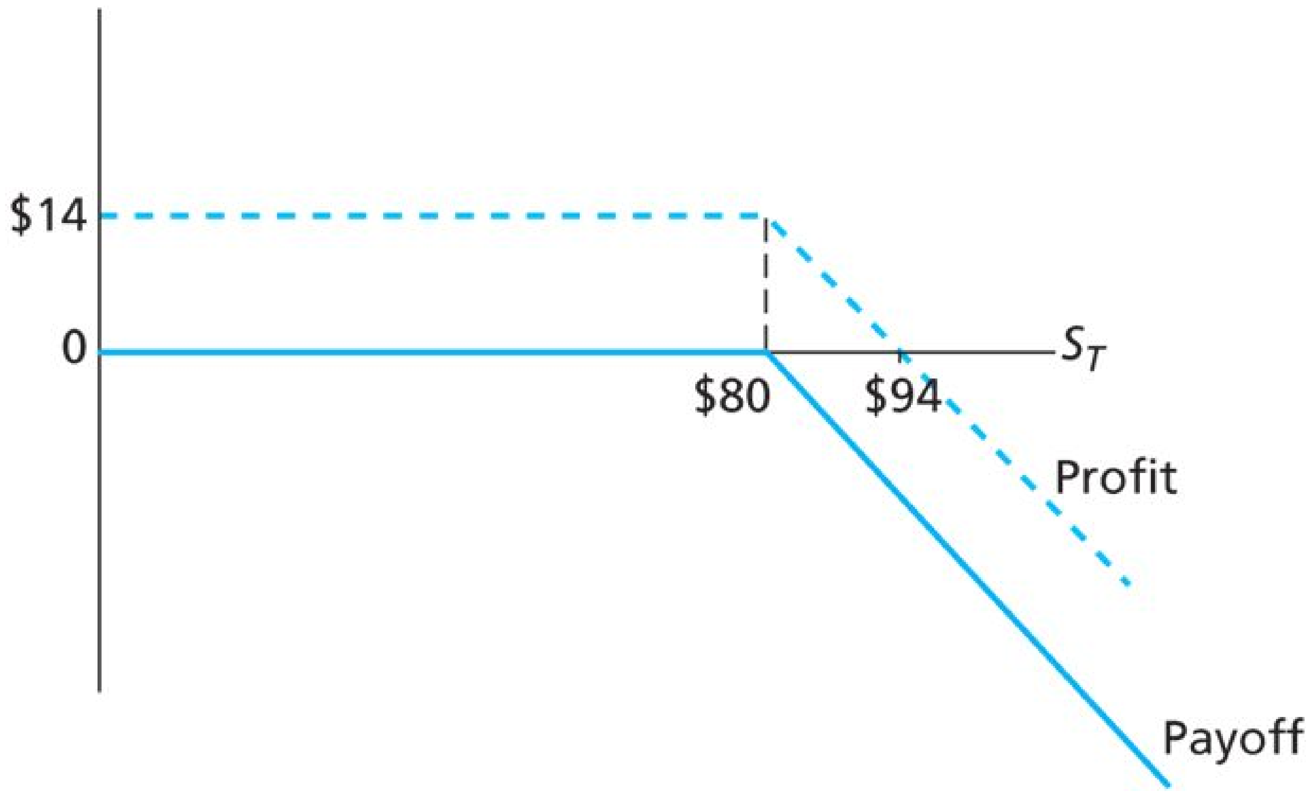
On the flip side, the gross payoff to the call option writer at expiration is

$$C_T = \begin{cases} X - S_T, & \text{if } S_T > X \\ 0, & \text{if } S_T \leq X. \end{cases}$$

The net payoff is

$$C_T = \begin{cases} X - S_T + C_t, & \text{if } S_T > X \\ C_t, & \text{if } S_T \leq X. \end{cases}$$

Call Option Payoff (Seller)



Put Option Payoff (Buyer)

The gross payoff to put option holders at expiration is

$$P_T = \begin{cases} 0, & \text{if } S_T > X \\ X - S_T, & \text{if } S_T \leq X. \end{cases}$$

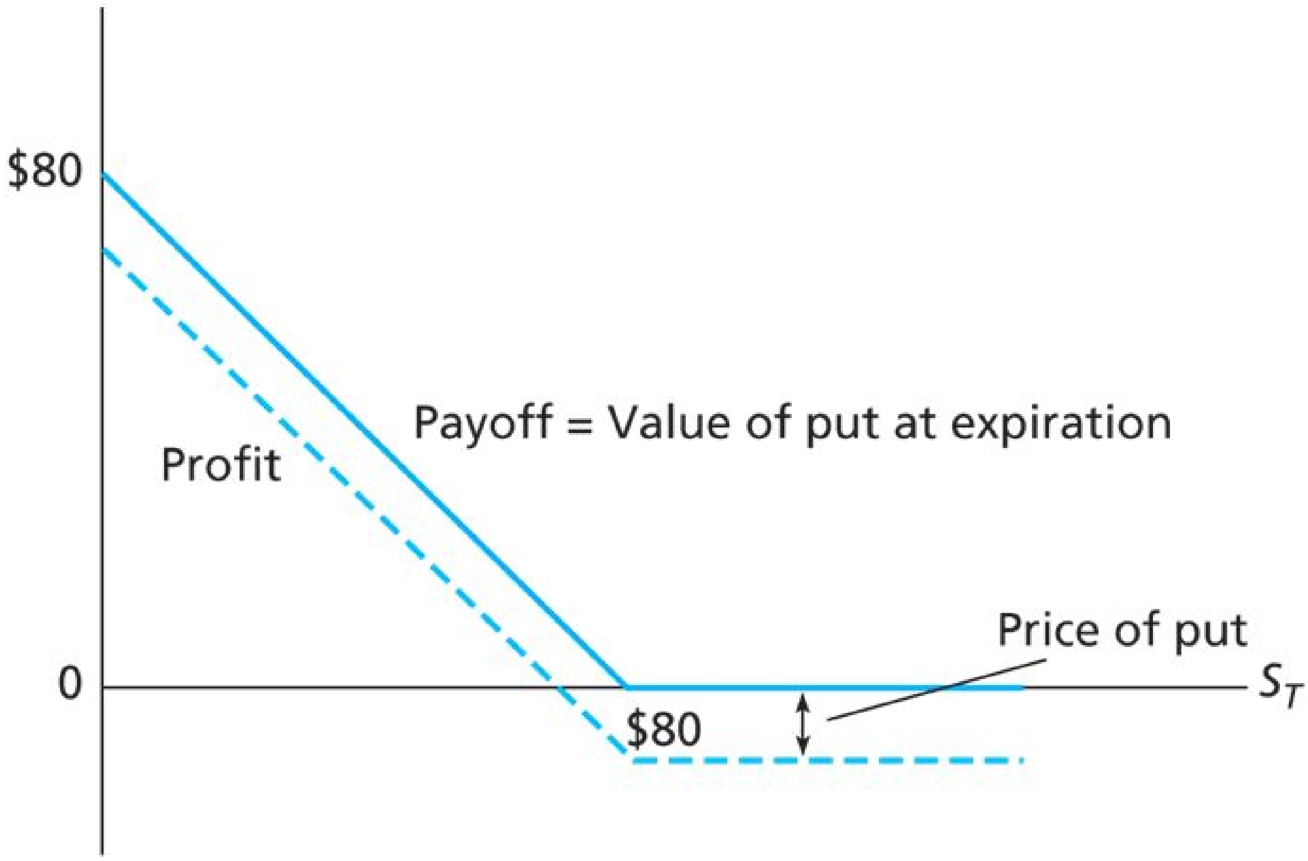
- If the underlying asset price is below the strike, the holder can purchase it for S_T and immediately resell for X, yielding a profit of $X - S_T$.
- If the asset price is above the strike at expiration, the option is worthless.

Put Option Payoff (Buyer)

The *net* payoff to put option holders is

$$P_T = \begin{cases} -P_t, & \text{if } S_T > X \\ X - S_T - P_t, & \text{if } S_T \le X. \end{cases}$$

Put Option Payoff (Buyer)



Speculation and Hedging

Options can be used for both speculation and hedging.

- Suppose you have $\$10,000$ available for investment.
- A share of stock costs $\$100$.
- An option with a strike price of $\$100$ and six-month maturity costs $\$10$.
- You can lend money (purchase the risk-free asset) at a rate of 3% for the next six months.

Speculation and Hedging

Consider three strategies.

- Strategy A: Invest entirely in stock, buying 100 shares at the current price of $\$100$.
- Strategy B: Invest entirely in at-the-money options, buying 10 call contracts (each for 100 shares) selling for $\$1000$ a piece.
- Strategy C: Purchase 100 call options (1 contract) for $\$1,000$ and invest the remaining $\$9,000$ in the risk-free asset, which will yield a total of $\$9,000 \times 1.03 = \$9,270$ at the end of the six months.

Speculation and Hedging

The values of the three strategies are:

	Stock Price					
Portfolio	\$95	\$100	\$105	\$110	\$115	\$120
A: Stock	\$9,500	\$10,000	\$10,500	\$11,000	\$11,500	\$12,000
B: Options	\$0	\$0	\$5,000	\$10,000	\$15,000	\$20,000
C: Mix	\$9,270	\$9,270	\$9,770	\$10,270	\$10,770	\$11,270

Speculation and Hedging

The returns to the three strategies are:

	Stock Price					
Portfolio	\$95	\$100	\$105	\$110	\$115	\$120
A: Stock	-5.0%	0.0%	5.0%	10.0%	15.0%	20.0%
B: Options	-100.0%	-100.0%	-50.0%	0.0%	50.0%	100.0%
C: Mix	-7.3%	-7.3%	-2.3%	2.7%	7.7%	12.7%

Speculation and Hedging

From these tables we see two features of options.

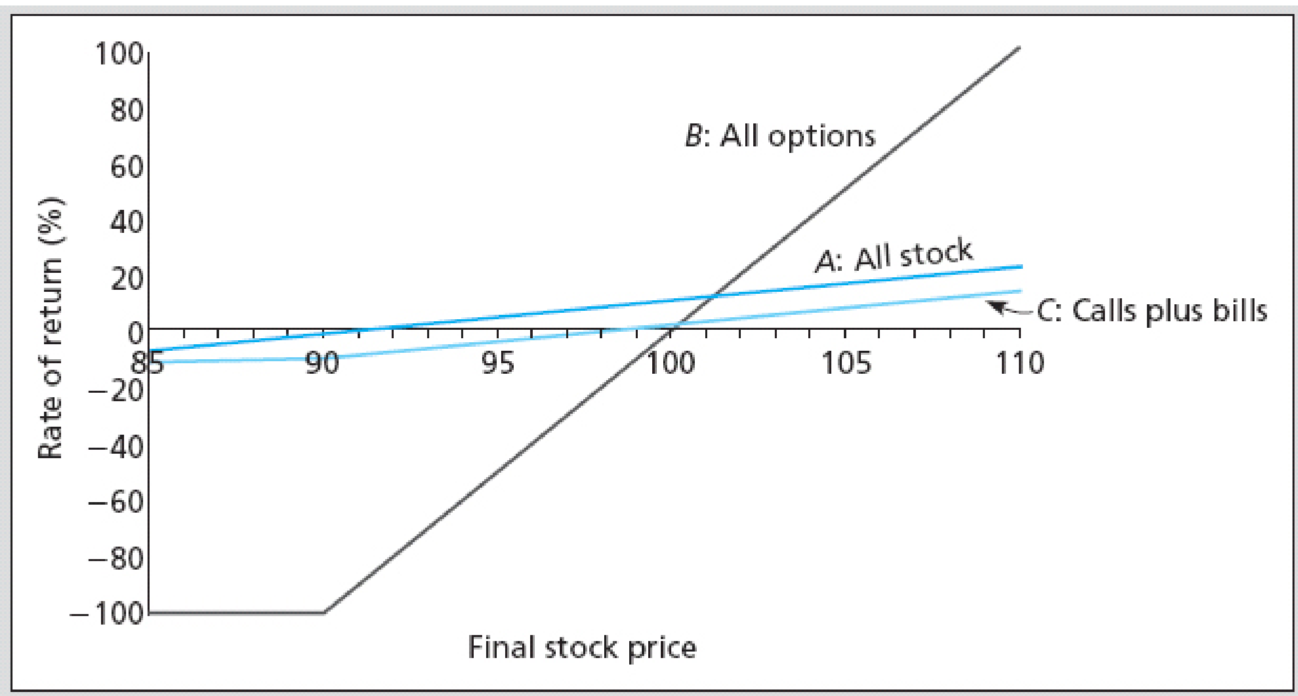
- Options offer leverage.
  - For the all-option portfolio, the return plummets to -100% when the stock price is below the strike.

- The return rockets to numbers that are much greater than simply holding the stock when the stock price increases above the strike.

Speculation and Hedging

- Options offer insurance.
  - The mixed portfolio has limited downside loss: the investor can't lose more than -7.3%.
  - It also has limited upside gains: if the stock price is above the strike, its returns are always below the portfolio comprised of only stock.

Speculation and Hedging



Protective Put

A protective put strategy consists of simultaneously purchasing a share of stock and a put option on that stock.

- This limits the potential downside loss of the stock while leaving the potential gains intact.

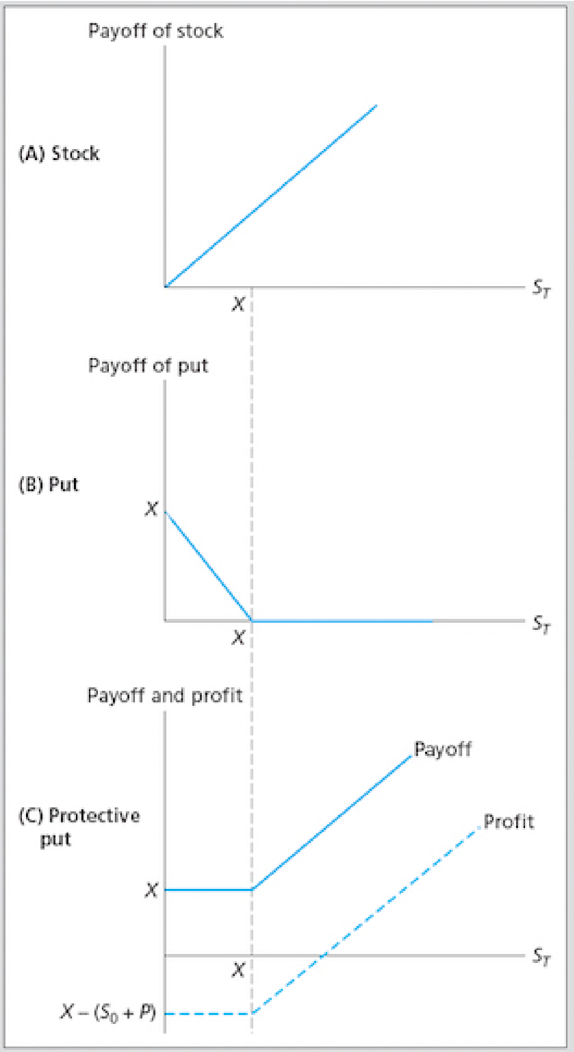
Protective Put

The put acts as insurance against loss.

- Comparing the net payoff of the protective put with the strategy of holding stock alone shows that the former comes at a cost.

- This is the insurance premium.

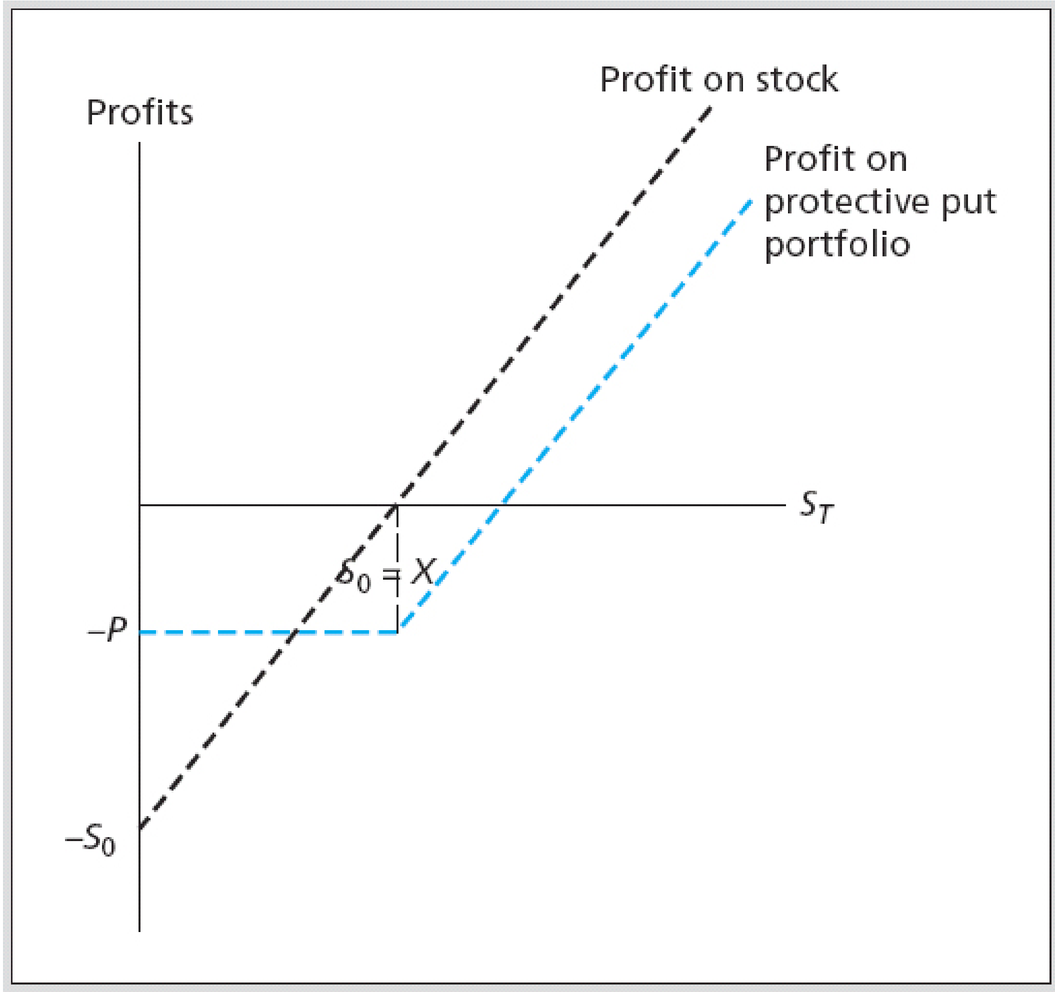
Protective Put



Protective Put

St<=X:
Stock:St
Put:X-St
Net: X

St>=X:
Stock: St
Put: 0
Net: St



Covered Call

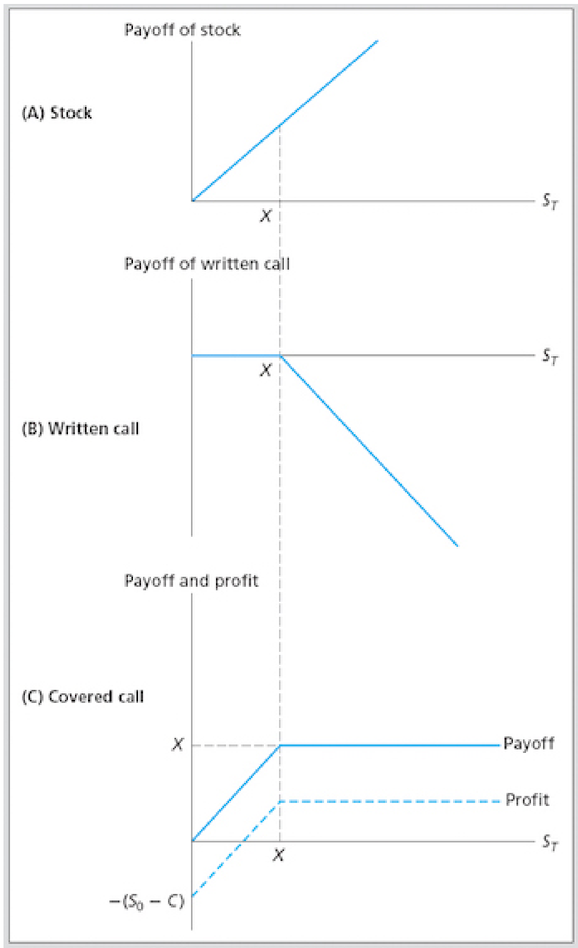
A covered call strategy consists of simultaneously purchasing a share of stock and writing a call option on that stock.

- It doesn't eliminate downside loss (like the protective put).
- It covers the obligation to deliver the stock for less than its market value if the stock price is above the strike.
- The call writer is charging a premium (the call price) in order to forsake the upside gain of holding the stock.

Covered Call

	$S_T \leq X$	$S_T > X$
Stock	S_T	S_T
Written Call	0	$-(S_T - X)$
Total	S_T	X

Covered Call



Straddle

A straddle consists of purchasing call and put options for the same asset and strike price.

- It is a bet on volatility.
- The straddle holder will earn (gross) positive returns if the stock price moves up or down, but nothing if it remains at the strike.

Straddle

	$S_T \leq X$	$S_T > X$
Call	0	$S_T - X$
Put	$X - S_T$	0
Total	$X - S_T$	$S_T - X$

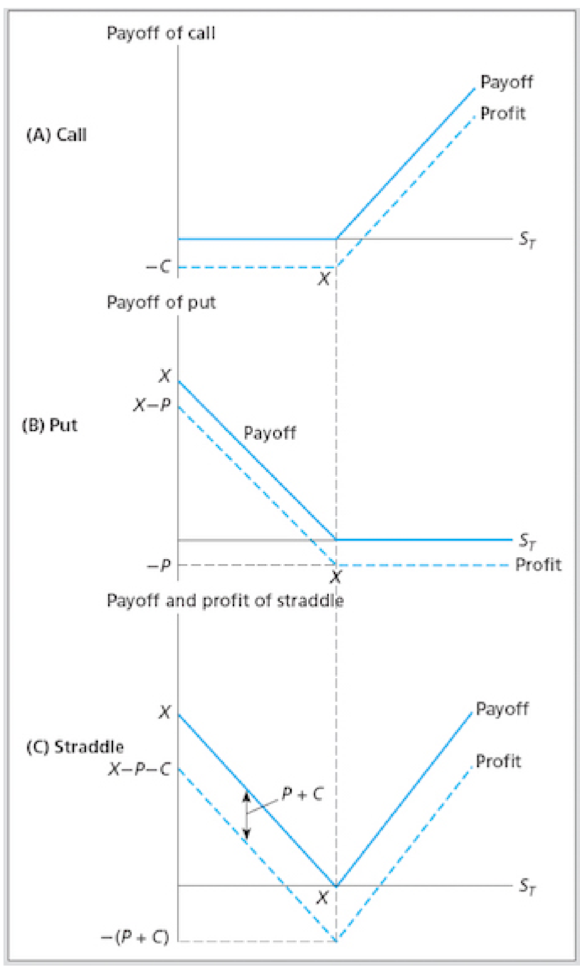
Straddle

So why doesn't everyone hold straddles?

- Because the investor must pay for both contracts.

- The investor doesn't earn a *net* return unless the stock price moves enough to compensate for the initial outlay.

Straddle



Spread

A spread is a combination of two or more options (both calls or both puts) on the same stock with different strikes.

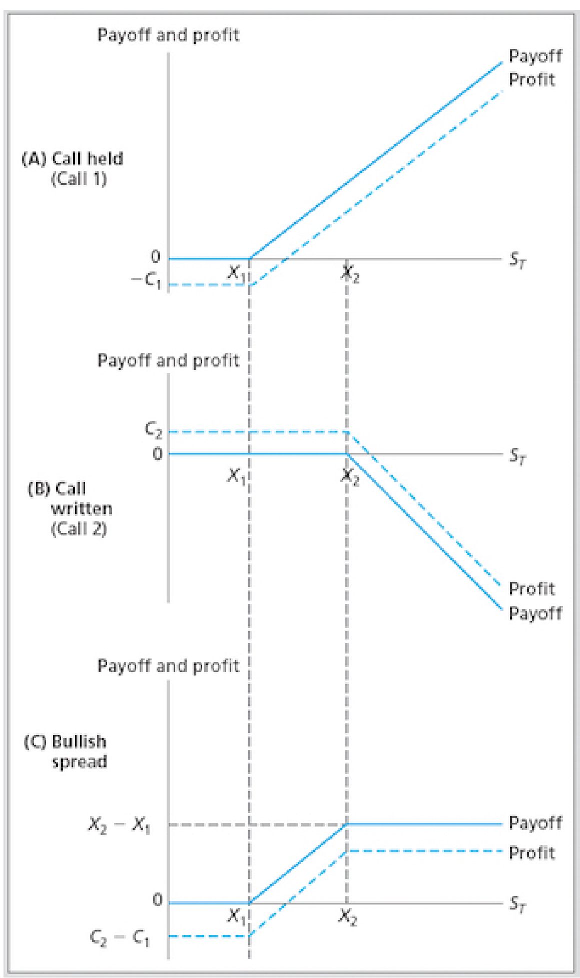
- Some of the options are purchased while others are sold.
- A money spread is the simultaneous purchase and sale of options with different strikes.
- A time spread is the simultaneous purchase and sale of options with different maturities.

Bullish Spread

A bullish spread:

	$S_T \leq X_1$	$X_1 < S_T \leq X_2$	$S_T > X_2$
Purchased Call with X_1	0	$S_T - X_1$	$S_T - X_1$
Written Call with X_2	0	0	$-(S_T - X_2)$
Total	0	$S_T - X_1$	$X_2 - X_1$

Bullish Spread



Collar

An example of a collar is the purchase of a protective put for one strike price and the sale of a call option, on the same stock, for a higher strike.

- This strategy eliminates downside losses below the strike of the put and also upside gains beyond the strike of the call.

- In this case, the investor constrains gains and losses within a region close to the current price of the stock.

Protective Put Alternative

A protective put eliminates the downside loss of holding stock. We could achieve this with an alternative strategy.

- Purchase a call option with strike price X.

- Purchase a T-bill (lend at the risk-free rate) with a face value equal to the call strike price, X, and the same maturity date as the call.

Protective Put Alternative

	$S_T \leq X$	$S_T > X$
Call	0	$S_T - X$
Bond	X	X
Total	X	S_T

Put Call Parity

The payoffs in the previous table are identical to those for the protective put.

- Hence, the cost of the protective put strategy should be equal to the cost of the call plus bonds strategy (why?!!!).

- This fact is known as the *Put-Call Parity Relationship*.

- Mathematically, it can be expressed as:

$$C_0 + Xe^{-r_f T} = S_0 + P_0.$$

- This relationship is very useful because it allows us to compute the value of a call option if we know the price of the corresponding put, and vice versa.

Put Call Parity Example

Assume

- An asset currently sells for $\$110$.

- A call option with strike $X = \$105$ and 1-year maturity sells for $\$17$.

- A put option with strike $X = \$105$ and 1-year maturity sells for $\$5$.

- The continuously-compounded risk-free interest rate is 4.879% per year.

- Does the parity relationship hold?

Put Call Parity Example

$$C_0 + Xe^{r_f} \stackrel{?}{=} S_0 + P_0.$$

$$\$117 = \$17 + \$105e^{-0.04879} \neq \$110 + \$5 = \$115.$$

- The relationship doesn't hold.

- How might an investor take advantage of the situation?

Put Call Parity Example

Position	Immediate Cash Flow	Cash Flow in 1 year	
		$S_T \leq \$105$	$S_T > \$105$
Buy Stock	-\$110	S_T	S_T
Borrow $\$105/1.05 = \$100$	\$100	-\$105	-\$105
Sell Call	\$17	\$0	$-(S_T - \$105)$
Buy Put	-\$5	$\$105 - S_T$	\$0
Total	\$2	\$0	\$0