GrowthRates.pdf

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Econ 202: Notes on growth rates in discrete time

Winter 2017

GrowthRates.tex, 1/12/2017

A key approximation During Econ 202, we will generally use models expressed in discrete time. Because Romer develops growth theory in continuous time, these notes are may be useful as you relate the text's treatment to the treatment I will give in the lectures.

In continuous time, a variable X_t that grows at the rate g from some initial level X_0 can be expressed as

$$X_t = X_0 e^{gt}.$$

Taking natural logs,

$$\ln X_t = \ln X_0 + gt$$

and the log change is equal to the growth rate:

$$\ln X_t - \ln X_{t-1} = g.$$

Remark 1 *In discrete time, the growth rate of a variable X_t from $t-1$ to t is*

$$\frac{X_t - X_{t-1}}{X_{t-1}} = \frac{X_t}{X_{t-1}} - 1 = g_t,$$

or

$$\frac{X_t}{X_{t-1}} = 1 + g_t.$$

Tanking logs,

$$\ln X_t - \ln X_{t-1} = \ln (1 + g_t).$$

Remark 2 *The natural log of* 1 *plus a small number is approximately equal to the small number. Thus, we will often assume*

$$\ln X_t = \ln X_{t-1} \approx g_t.$$

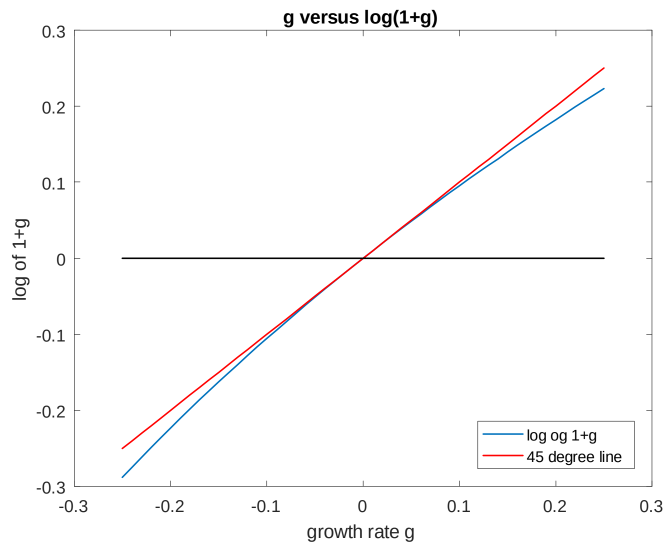
We can write this as

$$\Delta \ln X_t = g_t,$$

where Δ denotes the first difference operator.

Remark 3 *A growth rate of* 3% *means $g_X = 0.03$. Let g_X vary from -0.2 to* 0.2 *(i.e., from a* -20% *growth rate to a positive* 20% *growth rate).*

The figures illustrates that g_X is a good approximation to $\ln(1 + g_X)$ for growth rates in the range of -0.1 to 0.1 (i.e., -10% to 10%).



Data frequency The most common data frequency employed in macro is quarterly, as this is the frequency with which data on GDP is normally available. Suppose X_t grows at the rate g per quarter. The growth rate over a year when the data are quarterly is

$$\frac{X_t}{X_{t-4}} = 1 + g^a,$$

where g^a denotes the annual growth rate. Thus,

$$\frac{X_t}{X_{t-4}} = \frac{X_t}{X_{t-1}} \frac{X_{t-1}}{X_{t-2}} \frac{X_{t-2}}{X_{t-3}} \frac{X_{t-3}}{X_{t-4}} = (1+g)^4 = 1 + g^a,$$

so

$$4 \ln (1+g) = \ln (1 + g^a)$$

and

$$g^a \approx 4g.$$

So an annual growth rate of 4% translates into a 1% growth rate when expressed at a quarterly rate.

Remark 4 *In addition to growth rates, interest rates are measured at annual rates and so their value has a time dimension and depends on whether we express things at annual rates or at quarterly rates. Interest rates and inflation rates are, by convention, always expressed at annual rates. So an interest rate of* 6% *means the return is* 1.25% *per quarter. Our models will generally assume* *the time period is a quarter. So if P_t is the price level, the inflation rate is $\ln P_t - \ln P_{t-1}$, but that would give us inflation at a quarterly rate. So to convert it to annual rates, we would need to multiply by* 4.

Growth rates of two variables If the growth rates of X_t and Y_t are $g_{X,t}$ and $g_{Y,t}$ respectively, than the growth rate of $X_t Y_t$ is $g_{X,t} + g_{Y,t}$. Because the growth rate equals the change in the log,

$$\Delta \ln (X_t Y_t) = \Delta [\ln X_t + \ln Y_t] = \Delta \ln X_t + \Delta \ln Y_t = g_{X,t} + g_{Y,t}.$$

If the growth rates of X_t and Y_t are $g_{X,t}$ and $g_{Y,t}$ respectively, than the growth rate of X_t divided by Y_t is $g_{X,t} - g_{Y,t}$. Because the growth rate equals the change in the log,

$$\Delta \ln \left(\frac{X_t}{Y_t} \right) = \Delta [\ln X_t - \ln Y_t] = \Delta \ln X_t - \Delta \ln Y_t = g_{X,t} - g_{Y,t}.$$

The Solow neoclassical growth model

Carl E. Walsh

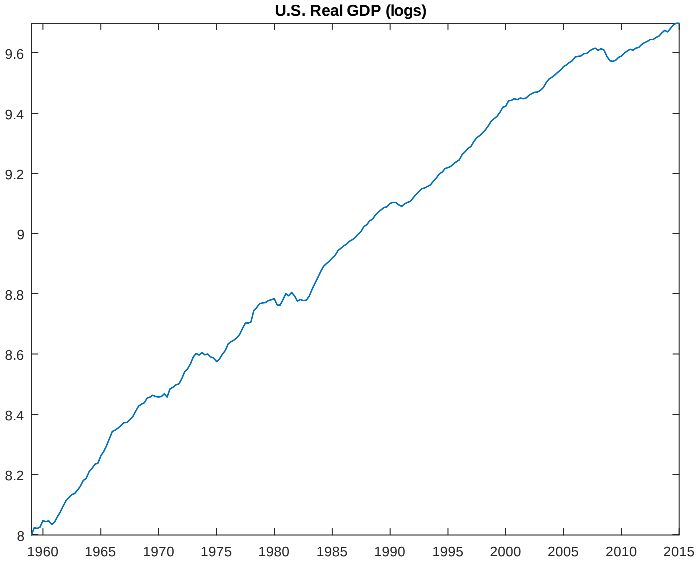
UC Santa Cruz

Winter 2017

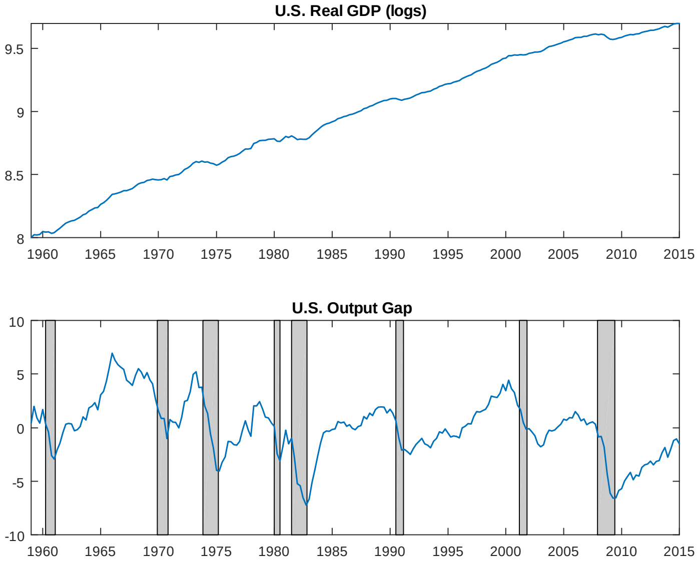
Economics 202: Macroeconomics

1. Web site: http://people.ucsc.edu/~walshc/UCSC/202_w17/
2. Dynare and matlab
3. Slides_SolowModel_W2017.pdf

U.S. real GDP: 1959-2014



US business cycles



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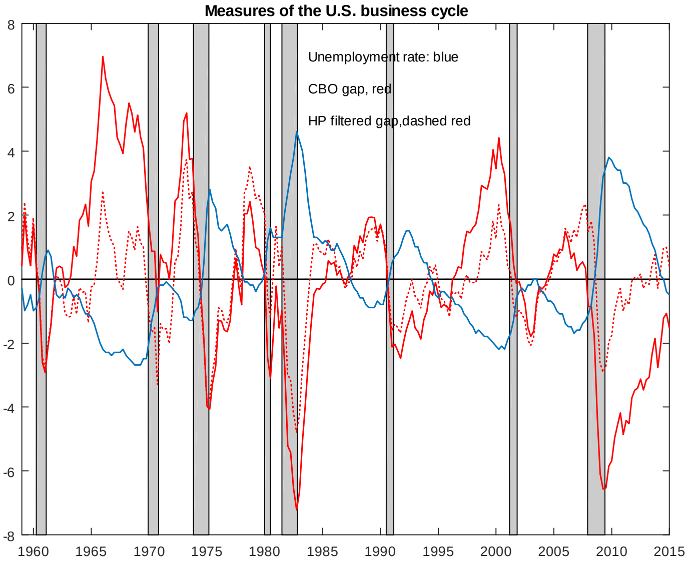

Trend growth and business cycles

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Trend growth and business cycles

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$$y_t = y_t^{Trend} + y_t^{Cycle}.$$

- We observe y_t. There are various methods of decomposing it into trend and cycle.
- Since trend dominates over time, we start by asking what factors account for long-run growth.

A note on representation

- Two representations of a rose:

A note on representation

- Three representations of the same economics:

$$\frac{dk(t)}{dt} = y(t) - c(t) - \delta k(t)$$

$$K_{t+1} - K_t = Y_t - C_t - \delta K_t$$

$$K_t - K_{t-1} = Y_t - C_t - \delta K_{t-1}$$

The Solow neoclassical growth model

Key components

1. An aggregate production function describing the economy's technology.

$$Y_t = F(K_t, L_t, A_t)$$

   1. Neoclassical in allowing substitutability between capital and labor.
   2. Note – I will use a discrete time setup to map more easily into business cycle models calibrated using quarterly data.

2. A description of the evolution of the inputs to production – i.e., a description of capital accumulation, labor force growth, and changes in technology.

The Solow neoclassical growth model: the production function

$$Y_t = F\left(K_t, L_t, A_t\right)$$

1. Assumptions about technology:
    1. Labor augmenting (Harrod neutral): $Y_t = F\left(K_t, A_t L_t\right)$;
    2. Capital augmenting: $Y_t = F\left(A_t K_t, L_t\right)$;
    3. Disembodied technological change: $Y_t = A_t F\left(K_t, L_t\right)$.
2. We will assume labor augmenting technological change.
3. Also will assume constant returns to scale:

$$F\left(\lambda K_t, \lambda A_t L_t\right) = \lambda F\left(K_t, A_t L_t\right).$$

The Solow neoclassical growth model: the production function

$$Y_t = F\left(K_t, A_t L_t\right)$$

- With constant returns to scale, let $\lambda = 1/AL$:

$$\frac{1}{A_t L_t} F\left(K_t, A_t L_t\right) = F\left(\frac{K_t}{A_t L_t}, 1\right),$$

  or

$$y_t = f\left(k_t\right)$$

  where y_t is output per effective unit of labor and k_t is capital per effective unit of labor.
- The function $f\left(k_t\right)$ is called the intensive-form production function.

The Solow neoclassical growth model: the production function

$$y_t = f(k_t)$$

- We will assume positive but diminishing marginal productivity so that $f'(k) \geq 0$ and $f''(k) < 0$.
- We will also assume the Inda conditions:

$$\lim_{k \to 0} f'(k) = \infty$$

$$\lim_{k \to \infty} f'(k) = 0$$

The Solow neoclassical growth model: the production function

- A common function form used for the production function is Cobb-Douglas:

$$Y_t = K_t^{\alpha} \left(A_t L_t \right)^{1-\alpha}.$$

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$$y_t \equiv \frac{Y_t}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \left(\frac{A_t L_t}{A_t L_t}\right)^{1-\alpha} = k_t^{\alpha}.$$

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$$\ln y_t = \alpha \ln k_t$$

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- In log form,

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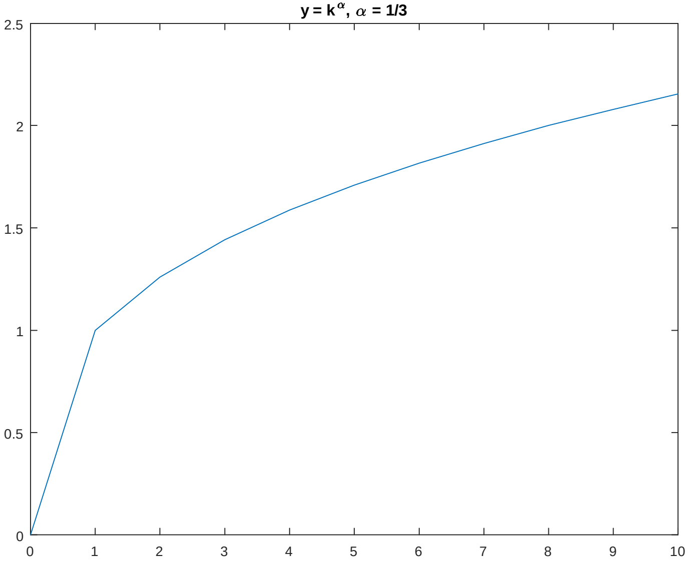
- So growth rate of y is equal to α times the growth rate of k.
- Growth rate of Y is growth rate of y plus growth rate of effective units of labor.
  - Growth rate of effective units of labor is growth rate of technology plus growth rate of labor force:

$$g_Y = g_y + g_{AL} = \alpha g_k + g_A + g_L.$$

Example of a Cobb-Douglas production function



The Solow neoclassical growth model: evolution of the inputs

- Exogenous in basic model – technology and labor force.
- Assume these grow at constant rates:

$$L_t = (1+n)^t L_0$$

$$A_t = (1+g)^t A_0$$

- Hence,

$$A_t L_t = (1+g)^t (1+n)^t A_0 L_0 \approx (1+g+n)^t A_0 L_0.$$

- This uses the approximation that, to first order,

$$(1+g)(1+n) = 1+g+n+gn \approx 1+g+n.$$

The Solow neoclassical growth model: evolution of capital

- The evolution of capital is the key endogenous source of growth in the Solow model.
- From the economy's resource constraint:

$$Y_t = C_t + K_{t+1} - (1 - \delta) K_t,$$

where C is consumption and δ is the depreciation rate.

- Assume savings is a fixed fraction s of income: $S_t = sY_t$, so $C_t = Y_t - sY_t = (1 - s) Y_t$.
- This means

$$K_{t+1} - (1 - \delta) K_t = I_t = Y_t - C_t = S_t = sY_t.$$

- Start with $A_0 L_0$ and K_0. These determine Y_0. This determines S_0 and K_1. Given K_1 and $A_1 L_1$, these determine Y_1, ...

The Solow growth model: review of last lecture

- Aggregate production function describes the economy's technology:

$$Y_t = F\left(K_t, L_t, A_t\right)$$

  - Positive but diminishing marginal products:

$$F_K \geq 0,\ F_{KK} \leq 0$$

$$F_L \geq 0,\ F_{LL} \leq 0$$

- Standard assumption functional form of the production function – Cobb-Douglas:

$$Y_t = K_t^{\alpha}\left(A_t L_t\right)^{\beta},\ 0 < a, \beta < 1$$

  - Constant returns to scale:

$$\alpha + \beta = 1$$

  - Intensive form:

$$Y_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha} \Rightarrow y_t \equiv \left(\frac{Y_t}{A_t L_t}\right) = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = k_t^{\alpha}$$

The Solow growth model: evolution of the inputs

$$y_t = k_t^{\alpha}$$

- Assumptions about labor force growth and technology:

$$L_t = (1+n)^t L_0$$

$$A_t = (1+g)^t A_0$$

- Hence,

$$A_t L_t = (1+g)^t (1+n)^t A_0 L_0 \approx (1+g+n)^t A_0 L_0.$$

- Growth accounting:

$$\begin{aligned}
\Delta \ln Y_t &= \Delta \ln y_t + \Delta \ln A_t L_t = \alpha \Delta \ln k_t + \Delta \ln A_t + \Delta \ln L_t \\
&= \alpha \left(\Delta \ln K_t - \Delta \ln A_t - \Delta \ln L_t \right) + \Delta \ln A_t + \Delta \ln L_t \\
&= \alpha \Delta \ln K_t + (1-\alpha) \Delta \ln L_t + (1-\alpha) \Delta \ln A_t
\end{aligned}$$

The Solow growth model: evolution of the inputs

- Assumptions about growth in the capital stock – constant saving rate s:

$$K_{t+1} = (1-\delta) K_t + I_t = (1-\delta) K_t + sY_t.$$

- Evolution of capital per effective unit of labor k:

$$\frac{K_{t+1}}{A_t L_t} = (1-\delta) \frac{K_t}{A_t L_t} + s \frac{Y_t}{A_t L_t}.$$

  - Write left side in terms of k_{t+1}:

$$\frac{K_{t+1}}{A_t L_t} = \frac{A_{t+1} L_{t+1}}{A_t L_t} \frac{K_{t+1}}{A_{t+1} L_{t+1}} = (1+g+n) k_{t+1}.$$

- So we have

$$(1+g+n) k_{t+1} = (1-\delta) k_t + sy_t.$$

The steady state

- Consider the situation in which k is constant (so y will also be constant).
- From the capital accumulation equation, $k_{t+1} = k_t = \bar{k}$ implies

$$(1 + g + n)\,\bar{k} = (1 - \delta)\,\bar{k} + sy\left(\bar{k}\right)$$

or

$$(g + n + \delta)\,\bar{k} = sy\left(\bar{k}\right) = s\bar{k}^{\alpha}.$$

- This is a single equation to determine $\bar{k}$.

The steady state

- From the capital accumulation equation

$$(1+g+n)\,k_{t+1} = (1-\delta)\,k_t + sy_t.$$

$$(1+g+n)\,(k_{t+1}-k_t) = sy_t - (g+n+\delta)\,k_t.$$

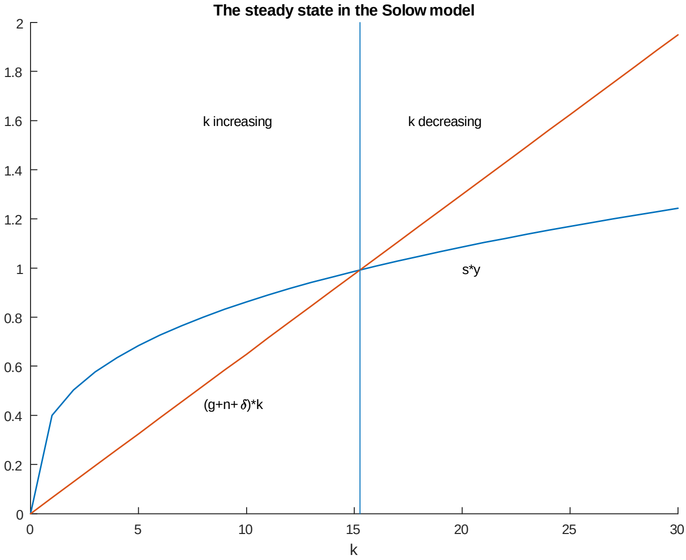
- So

$$k_{t+1} > k_t \Leftrightarrow sy > (g+n+\delta)\,k_t$$

$$k_{t+1} < k_t \Leftrightarrow sy < (g+n+\delta)\,k_t$$

$$k_{t+1} = k_t \Leftrightarrow sy = (g+n+\delta)\,k_t$$

The steady state



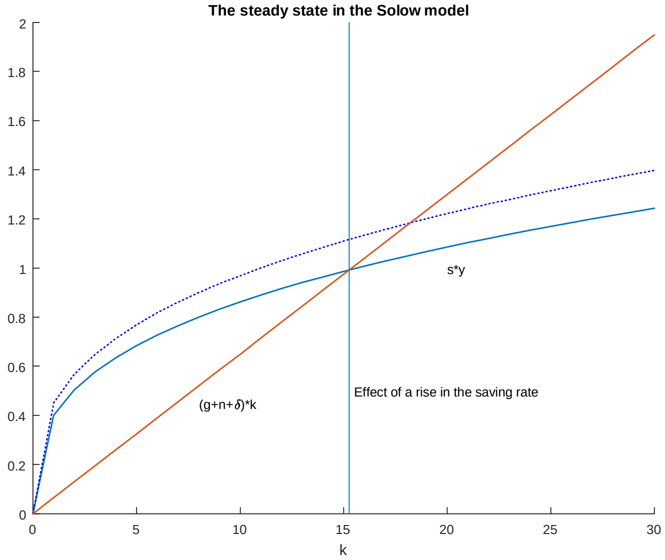
Properties of the steady state

- Since $k = K/AL$, k constant implies K and AL must be growing at the same rate.
- Since $y = k^{\alpha}$ is constant, Y must be growing at the same rate as AL.
- In the steady state,
  - A grows at rate g;
  - L grows at rate n;
  - Y, K, and C grow at the rate $g + n$.
    - $C = Y - \delta K \Rightarrow (C/AL) = (Y/AL) - \delta (K/AL) = y - \delta k$ and so is constant.
    - And consumption per capital C/L grows at the rate g.

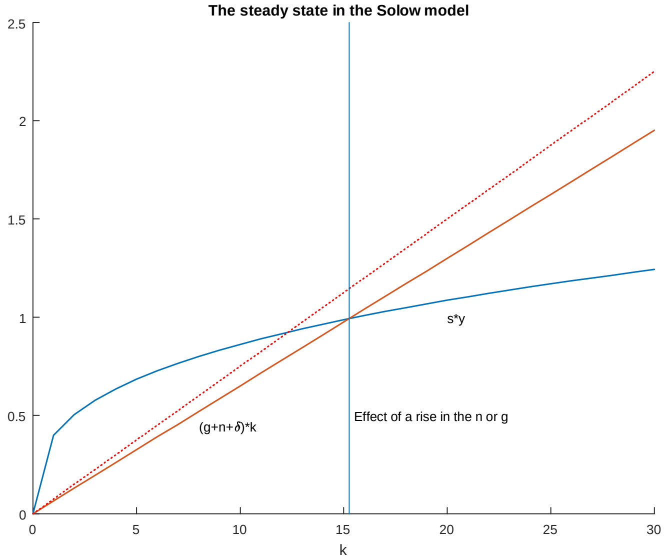
Properties of the steady state

- Steady-state growth rate independent of the savings rate s.
  - Increase in s leads to capital deepening (a rise in $\bar{k}$) but not a long-run increase in the growth rate.
- Increase in g or n increases steady-state growth rate.
  - But they reduce the level of income and capital per effective unit of labor.

Effects of an increasing in the saving rate



Effects of an increasing in labor growth rate



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Properties of the model

- Growth convergence
  - Growth accounting:

$$g_Y = \alpha g_K + (1-\alpha)\, g_L + (1-\alpha)\, g_A$$

  - Empirical support weak even after attempting to deal with measurement issues (e.g., quality of L).
  - Early empirical tests suffered from a selection bias by looking at countries with high incomes at end of sample.
  - Solow residual, $(1-\alpha)\, g_A$ is large.

The Solow growth model: review of last lecture

- Output determined by capital, labor and technology.
- Capital accumulation endogenous, labor and technology exogenous in basic model.
- Key relationships:
  - Production function: $y_t = k_t^{\alpha}$, $0 < \alpha < 1$.
  - Capital accumulation:
    $(1 + g + n)\, k_{t+1} = y_t - c_t + (1 - \delta)\, k_t = s k_t^{\alpha} + (1 - \delta)\, k_t$.
- Steady-state defined by constant $\bar{k}$, given as the solution to

$$(\delta + g + n)\, \bar{k} = s \bar{k}^{\alpha} \Rightarrow \bar{k} = \left(\frac{s}{\delta + g + n} \right)^{\frac{1}{1-\alpha}}$$

The Solow growth model: review of last lecture

- If $sy_t > (g+n+\delta)\,k_t$, k grows. If $sy_t < (g+n+\delta)\,k_t$, k declines.
- Hence, k converges to a steady state k^* where k and y are constant.
- In the steady state,
  - K and Y grow at rate $g+n$.
  - Y/L grows at rate $(g+n)-n=g$.
  - Consumption $C=Y-S=(1-s)\,Y$ so C grows at rate $g+n$.
  - C/L grows at rate g.
- Changes in s and δ affect steady state levels of k, y, and c but not steady state growth rate.
- Changes in g and n affect steady state levels of k, y, and c *and* steady state growth rate.

How does steady state consumption depend on the saving rate?

- Steady-state consumption is output minus investment required to maintain constant $\bar{k}$:

$$\begin{aligned}\bar{c} &= \bar{y} - (\delta + g + n)\,\bar{k} \\ &= \bar{k}^{\alpha} - (\delta + g + n)\,\bar{k}.\end{aligned}$$

- Two questions:
  - What $\bar{k}$ maximizes $\bar{c}$? Call it $k^{\max}$
  - What saving rate would make $k^{\max}$ the economy's steady state k?

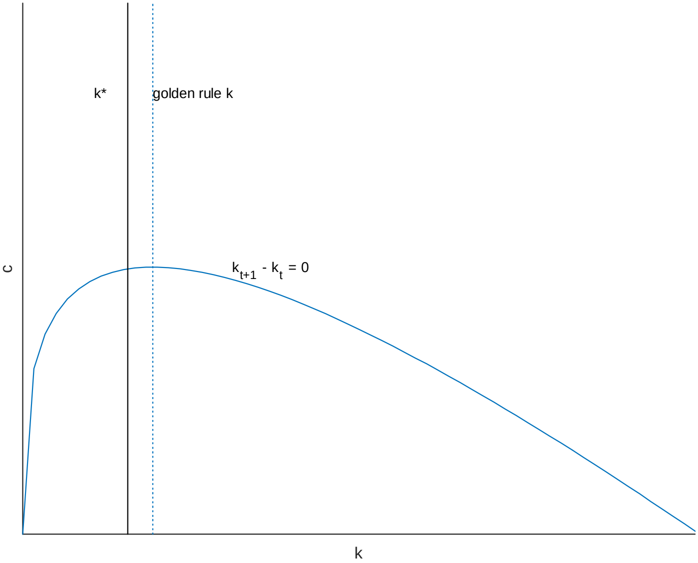
How does steady state consumption depend on the saving rate?

- What $\bar{k}$ maximizes $\bar{c}$?
  - Answer. Since $\bar{c} = \bar{k}^{\alpha} - (\delta + g + n)\,\bar{k}$, maximize this with respect to $\bar{k}$.
  - First-order condition is

$$\alpha \left(\bar{k}^{\max}\right)^{\alpha-1} - \delta = g + n \Rightarrow \bar{k}^{\max} = \left(\frac{\alpha}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}.$$

  - The left side is the marginal product of k net of depreciation, the right side is the growth rate.
- What saving rate would make $k^{\max}$ the economy's steady state k?
  - $s^{\max} = \alpha$.

Maximizing household consumption: the golden rule



Key questions:

- What level of savings would rational, optimizing households choose?
- What is the optimal savings rate?
  - Frank Ramsey (1927)

Utility maximizing households

- What level of savings would rational, optimizing households choose?
- Suppose representative household consists of L_t members and household wishes to maximize utility given by

$$\sum_{i=0}^{\infty} \beta^i L_{t+i} U\left(\frac{C_{t+i}}{L_{t+i}}\right) = \sum_{i=0}^{\infty} \beta^i L_{t+i} U\left(A_{t+i}\frac{C_{t+i}}{A_{t+i}L_{t+i}}\right)$$
$$= \sum_{i=0} \beta^i L_{t+i} U\left(A_{t+i}c_{t+i}\right)$$

for a standard utility function defined over consumption ($U' \geq 0$, $U'' \leq 0$ and $0 < \beta < 1$).

- The notation is $c_t = C_t / A_t L_t$.

Utility maximizing households

- Assume the following form for the utility function:

$$U(X) = \frac{X^{1-\sigma}}{1-\sigma},\ \sigma > 0 \Rightarrow U(A_{t+i}c_{t+i}) = A_{t+i}^{1-\sigma}\frac{c_{t+i}^{1-\sigma}}{1-\sigma}.$$

- Marginal utility of consumption is

$$U' = A_t^{1-\sigma}c_t^{-\sigma},\ U'' = -\sigma A_t^{1-\sigma}c_t^{-(1+\sigma)}$$

Utility maximizing households

- Household problem:

$$\max_{c_{t+i},k_{t+1+i}} \sum_{i=0}^{\infty} \beta^i L_{t+i} A_{t+i}^{1-\sigma} \frac{c_{t+i}^{1-\sigma}}{1-\sigma}$$

subject to

$$k_t^{\alpha} + (1-\delta) k_t = c_t + (1+g+n) k_{t+1}$$

Optimal consumption choice

- Suppose the household has chosen a path for c_t, c_{t+1}, c_{t+2},.... and k_{t+1}, k_{t+1}, k_{t+2}, ...
- Consider a small reallocation of consumption:
  - At time t, reduce c_t, use the resources made available to invest in more k_{t+1}.
  - At time $t+1$, consume the extra income generated by the extra k_{t+1} plus what is left over of the extra capital after depreciation.
  - This leaves c_{t+2},.... and k_{t+2+1}, ... unchanged from the original path.
- If the original path was optimal, then the marginal cost of this reallocation (due to the reduction in c_t) must just equal the marginal benefit of the higher c_{t+1}.
- If it doesn't, for example if the marginal benefit exceeds the marginal cost, then the original path wasn't optimal and the household should reduce c_t.

Marginal costs and benefits

- The marginal cost of reducing c_t a small amount is the loss in utility per household member times the number of household members:

$$\left(L_t A_t^{1-\sigma} c_t^{-\sigma}\right) \Delta c_t.$$

- What is the marginal benefit?
  - Because k_t is fixed, budget constraint

$$k_t^{\alpha} + (1-\delta)\, k_t = c_t + (1+g+n)\, k_{t+1}$$

implies

$$\Delta k_{t+1} = -\left(\frac{1}{1+g+n}\right) \Delta c_t.$$

Marginal costs and benefits

- What is the marginal benefit in terms of higher consumption at $t+1$ from having Δk_{t+1} more capital?
- The household has the extra income this produces at $t+1$, which is the marginal product of k_{t+1} and the household has the left over extra capital $(1-\delta)\,\Delta k_{t+1}$.
  - The marginal produce of capital is
$$\frac{\partial y_{t+1}}{\partial k_{t+1}} = \frac{\partial k_{t+1}^{\alpha}}{\partial k_{t+1}} = \alpha k_{t+1}^{\alpha-1}.$$
  - So extra resources available for consumption at $t+1$ are
$$\Delta c_{t+1} = \left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right) \Delta k_{t+1}.$$
- Using relationship between Δk_{t+1} and Δc_t,
$$\Delta c_{t+1} = \left(\frac{\alpha k_{t+1}^{\alpha-1} + 1 - \delta}{1+g+n}\right) \Delta c_t.$$

Marginal costs and benefits

- What is the marginal benefit of Δc_{t+1}?
- The marginal benefit of increasing c_{t+1} by Δc_{t+1} is the gain in utility per household member times the number of household members:

$$\left(L_{t+1}A_{t+1}^{1-\sigma}c_{t+1}^{-\sigma}\right)\Delta c_{t+1} = \beta\left(L_{t+1}A_{t+1}^{1-\sigma}c_{t+1}^{-\sigma}\right)\left(\frac{\alpha k_{t+1}^{\alpha-1}+1-\delta}{1+g+n}\right)\Delta c_t.$$

- Marginal cost equal marginal benefit (i.e., the original consumption plan was optimal) when

$$\left(L_t A_t^{1-\sigma}c_t^{-\sigma}\right)\Delta c_t = \beta\left(L_{t+1}A_{t+1}^{1-\sigma}c_{t+1}^{-\sigma}\right)\left(\frac{\alpha k_{t+1}^{\alpha-1}+1-\delta}{1+g+n}\right)\Delta c_t$$

or

$$L_t A_t^{1-\sigma}c_t^{-\sigma} = \beta\left(L_{t+1}A_{t+1}^{1-\sigma}c_{t+1}^{-\sigma}\right)\left(\frac{\alpha k_{t+1}^{\alpha-1}+1-\delta}{1+g+n}\right).$$

Marginal costs and benefits

$$L_t A_t^{1-\sigma} c_t^{-\sigma} = \beta \left(L_{t+1} A_{t+1}^{1-\sigma} c_{t+1}^{-\sigma} \right) \left(\frac{\alpha k_{t+1}^{\alpha-1} + 1 - \delta}{1 + g + n} \right).$$

- Simplify this by rearranging and recalling that L grows at rate n and A grows at rate g.
- Divide both sides by L_t and $A_t^{1-\sigma}$:

$$c_t^{-\sigma} = \beta \left(\frac{L_{t+1}}{L_t} \right) \left(\frac{A_{t+1}}{A_t} \right)^{1-\sigma} \left(\frac{\alpha k_{t+1}^{\alpha-1} + 1 - \delta}{1 + g + n} \right) c_{t+1}^{-\sigma}.$$

- This can be written as

$$c_t^{-\sigma} = \beta (1+n)(1+g)^{1-\sigma} \left(\frac{\alpha k_{t+1}^{\alpha-1} + 1 - \delta}{1 + g + n} \right) c_{t+1}^{-\sigma}.$$

Marginal costs and benefits: simplifying

- Define

  Marginal product of capital net of depreciation: $r_{k,t+1} \equiv \alpha k_{t+1}^{\alpha-1} - \delta$

$$\rho \equiv \frac{1-\beta}{\beta} > 0 \Rightarrow \beta = \frac{1}{1+\rho}$$

- Now simplify by using our results on first-order approximations:

$$(1+\rho)^{-1} \approx 1-\rho$$

$$(1+g)^{1-\sigma} \approx 1+(1-\sigma)g$$

$$(1+n)(1+g)^{1-\sigma} \approx 1+n+(1-\sigma)g$$

$$\frac{1}{1+g+n} \approx 1-g-n.$$

- Thus,

$$\frac{(1+n)(1+g)^{1-\sigma}}{1+\rho}\left(\frac{1+r_{k,t+1}}{1+g+n}\right) \approx 1+r_{k,t+1}-\rho-\sigma g$$

Marginal costs and benefits: simplifying

- Putting that all together, along the optimal path,

$$c_t^{-\sigma} \approx (1 + r_{k,t+1} - \rho - \sigma g)\, c_{t+1}^{-\sigma}.$$

- In the steady state, c is constant, so $c_t^{-\sigma} = c_{t+1}^{-\sigma}$ and

$$r_k^* = \rho + \sigma g$$

- Steady state k^* such that

$$\alpha \left(k^*\right)^{\alpha-1} = \rho + \delta + \sigma g \Rightarrow k^* = \left(\frac{\alpha}{\rho + \delta + \sigma g}\right)^{\frac{1}{1-\alpha}}$$

Utility maximizing households: steady state

- So Solow model was

$$(1+g+n)\,k_{t+1} = (1-\delta)\,k_t + \left(k_t^{\alpha} - c_t\right)$$

$$c_t = (1-s)\,k_t^{\alpha}$$

- With optimizing households, it becomes

$$(1+g+n)\,k_{t+1} = (1-\delta)\,k_t + \left(k_t^{\alpha} - c_t\right)$$

$$c_t^{-\sigma} = \left(1 + r_{k,t+1} - r_k^*\right) c_{t+1}^{-\sigma}$$

$$r_{k,t} = \alpha k_t^{\alpha-1} - \delta$$

$$r_k^* = \rho + \sigma g$$

Convergence to the steady state

- Optimal condition for consumption was

$$c_t^{-\sigma} = (1 + r_{k,t+1} - \rho - \sigma g)\, c_{t+1}^{-\sigma}.$$

- Because $r_k^* = \rho + \sigma g$, we can write this as

$$c_t^{-\sigma} = (1 + r_{k,t+1} - r_k^*)\, c_{t+1}^{-\sigma}.$$

- Rearranging,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = (1 + r_{k,t+1} - r_k^*)\,.$$

- If $k < k^*$, then $r_k > r_k^*$ and c is growing. If $k > k^*$, then $r_k < r_k^*$ and c is shrinking.
- c constant when $r_k = r_k^*$.

Convergence to the steady state

- Optimal condition for consumption was

$$c_t^{-\sigma} = (1 + r_{k,t+1} - \rho - \sigma g)\, c_{t+1}^{-\sigma}.$$

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- If $k < k^*$, then $r_k > r_k^*$ and c is growing. If $k > k^*$, then $r_k < r_k^*$ and c is shrinking.
- c constant when $r_k = r_k^*$.

Convergence to the steady state

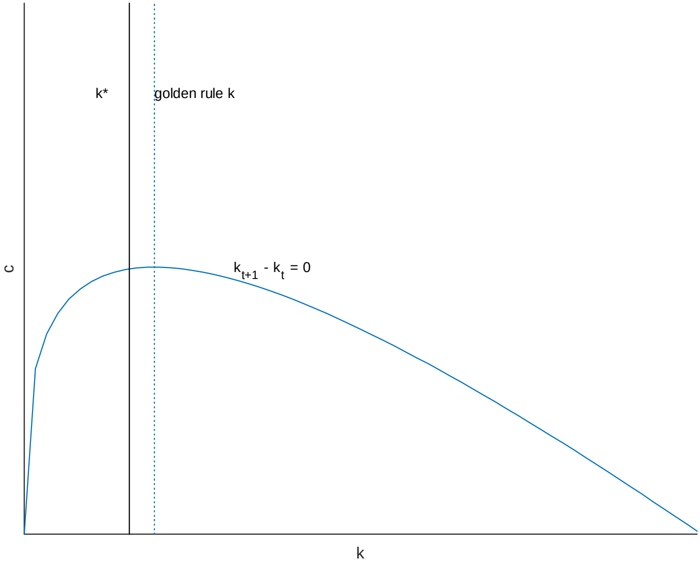
- Capital evolves as

$$(1+g+n)\left(k_{t+1}-k_t\right)=\left(k_t^{\alpha}-c_t\right)-(\delta+g+n)\,k_t.$$

- So k constant if

$$c_t=k_t^{\alpha}-(\delta+g+n)\,k_t.$$

Utility maximizing household: steady state



Utility maximizing household: steady state

- Capital evolves as

$$(1+g+n)(k_{t+1}-k_t) = (k_t^{\alpha} - c_t) - (\delta+g+n)\, k_t.$$

- So k constant if

$$c_t = k_t^{\alpha} - (\delta+g+n)\, k_t.$$

- Consumption c is maximized (in steady-state) when

$$\alpha k^{\alpha-1} = \delta+g+n \Rightarrow k^m = \left(\frac{\delta+g+n}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

or

$$r_k^m = \alpha k^{\alpha-1} - \delta = g+n.$$

Solow/Ramsey model

- Level of consumption jumps to put economy on path that converges to $(c^* \ k^*)$ where $c^* = (k^*)^\alpha - (\delta + g + n) k^*$.
- Equilibrium is efficient.
- Faster population growth shifts $\Delta k = 0$ line down but does not affect k^*.
- Faster productivity growth shifts $\Delta k = 0$ line down but also shifts $\Delta c = 0$ line to left.

What's next

- Think of the Solow/Ramsey as representing a market economy with households maximizing utility and firms renting capital that is used to product output.
- All markets are perfectly competitive, so equilibrium is efficient.
- Now suppose technology A_t, rather than growing smoothly over time, is subject to variation over time.
  - Sometime A grow faster than the rate g, sometimes it grows more slowly.
- Would the economy display behavior that looks like real world business cycles?

From Solow/Ramsey to RBC: Lectures 4 and 5

Carl E. Walsh

UC Santa Cruz

Winter 2017

The Solow-Ramsey model: review of last lecture

- Basic Solow model was

$$(1+g+n)\,k_{t+1} = (1-\delta)\,k_t + \left(k_t^{\alpha} - c_t\right)$$

$$c_t = (1-s)\,k_t^{\alpha}$$

- With optimizing households, model becomes

$$(1+g+n)\,k_{t+1} = (1-\delta)\,k_t + \left(k_t^{\alpha} - c_t\right)$$

$$c_t^{-\sigma} = \left(1 + r_{k,t+1} - r_k^*\right) c_{t+1}^{-\sigma}$$

$$r_{k,t} = \alpha k_t^{\alpha-1} - \delta$$

$$r_k^* = \rho + \sigma g$$

A basic real business cycle (RBC) model

- Now suppose technology A_t, rather than growing smoothly over time, is subject to random variation over time.
  - Sometimes A grow faster than the rate g, sometimes it grows more slowly.
- Would the economy display behavior that looks like real world business cycles?
- Kydland and Prescott (1982) argued the answer was yes.

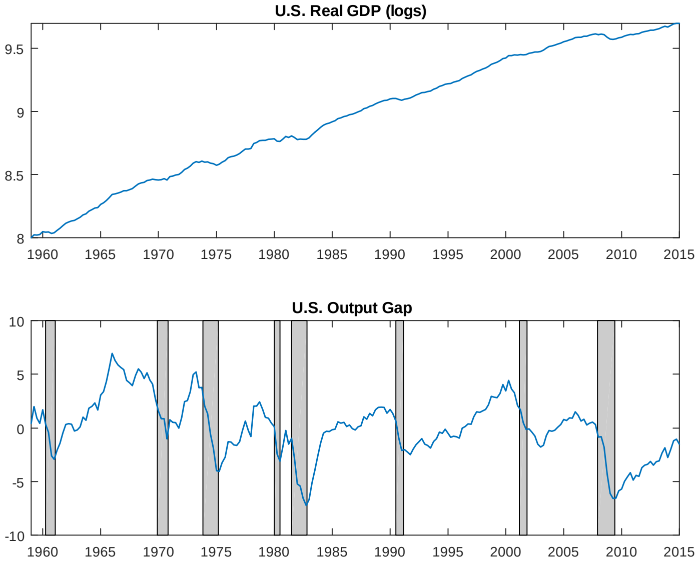
A basic real business cycle (RBC) model

- Key components

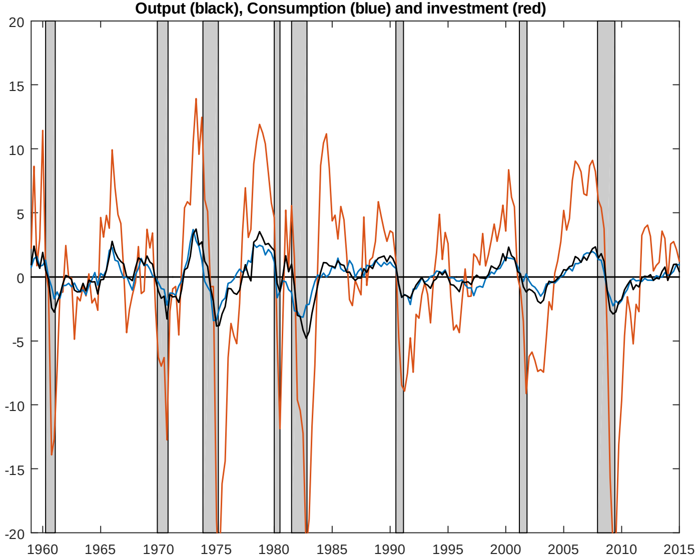
1. Optimizing agents – well defined decision problems faced by households and firms.
2. Consistent with Ramsey-Solow neoclassical growth model (but we will now ignore growth to focus on business cycles).
3. General equilibrium perspective.

U.S. business cycles: 1956-2016

USBusinessCycles.m



U.S. business cycles: 1959-2014



U.S. business cycles: 1959-2014

Quarterly, HP filtered data

Business cycle volatility: 1959-2014

	Std.	Std. relative to output
output	1.50	1.0
consumption	1.22	0.82
investment	6.67	4.45

A basic real business cycle (RBC) model

- Why might a model with perfect markets, rational, optimizing households and firms display business cycle like behavior if it experiences stochastic shocks to productivity?
- Key idea:
  - Consumption smoothing.

From growth to business cycles

- Households hold capital, make decisions about consumption and amount of labor to supply.
  - They maximize utility subject to budget constraint.
- Firms rent capital and hire labor, product to maximize profits.
- Firms and households interact in competitive goods and factor (capital and labor) markets.
- All markets clear (i.e., prices adjust so that demand equals supply in all markets:
  - Output produced by firms equals consumption chosen by households.
  - Labor supplied by households equals labor demand by firms.
  - Capital rented by firms equals capital supplied by households.

A basic real business cycle (RBC) model

Equilibrium

- Given an exogenous process for the productivity shock A, equilibrium consists of time paths for output, consumption, capital, employment, wages and rental rates that satisfy

$$Y_t = F(K_t, L_t, A_t)$$

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t$$

$$F_N(K_t, L_t, A_t) = \omega_t$$

$$F_K(K_t, L_t, A_t) = r_t$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}$$

$$\frac{\chi L_t^{\eta}}{C_t^{-\sigma}} = \omega_t$$

- Where do these equation come from?
- Does the equilibrium display business cycle behavior?

Aggregate production function

$$Y_t = F(K_t, L_t, A_t)$$

- Assume

$$Y_t = F(K_t, L_t, A_t) = A_t K_t^{\alpha} L_t^{1-\alpha}.$$

- Constant returns to scale.
- Remove growth but assume

$$A_t = e^{z_t} A_0,\ z_t \text{ random, persistent}$$

$$z_t = \rho z_{t-1} + e_t$$

where e_t is stochastic white noise (zero mean, constant variance, serially uncorrelated).

Firms

- Firm takes wages and rental rate of capital as given and maximizes profits.
- Firm problem is

$$\max_{N,K} \left\{ A_t K_t^{\alpha} L_t^{1-\alpha} - \omega_t L_t - r_t K_t \right\}$$

where ω is the real wage and r_k is the rental rate on capital.

- First-order conditions:

$$F_L = (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) \frac{Y_t}{L_t} = \omega_t$$

$$F_K = \alpha K_t^{\alpha-1} L_t^{1-\alpha} = \alpha \frac{Y_t}{K_t} = r_t$$

Firms

- Conditions to be satisfied in equilibrium:
  - Production function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

  - Marginal product of labor equals the real wage:

$$F_L = (1-\alpha)\frac{Y_t}{L_t} = \omega_t$$

  - Marginal product of capital equals the rental rate on capital:

$$F_K = \alpha \frac{Y_t}{K_t} = r_t$$

From growth to business cycles: households

- Households decide on how much to consume and how much to work.
- Optimal condition for consumption was

$$MUC_t = \beta (1 + r_{t+1} - \delta) MUC_{t+1}$$

  - Note – previously used r_k to denote $r - \delta$.

- With random fluctuations in productivity, future outcomes are uncertain, so we need to replace right side with its expectations:

$$MUC_t = \beta \mathrm{E}_t (1 + r_{t+1} - \delta) MUC_{t+1}.$$

- Expectations because now future productivity (which affects output and consumption) is unknown.

From growth to business cycles: households

- Assume utility at time t is

$$\frac{C_t^{1-\sigma}}{1-\sigma} - \chi \frac{N_t^{1+\eta}}{1+\eta}.$$

- Note – I am changing notation and using N for employment instead of L.
- Optimal consumption path satisfies

$$MUC_t = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) MUC_{t+1}.$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}.$$

From growth to business cycles: households

- Households also decide on how much labor to supply.
- Normalize amount of time available to equal 1.
- Assume household can allocate time to market work (N_t) and leisure ($1 - N_t$).
- If household optimizes, then marginal cost of working an extra hour will equal the marginal benefit:

$$MUL_t = \omega_t MUC_t,$$

where MUL is the marginal utility of leisure (the cost of working more), ω_t is the real wage, and MUC is the marginal utility of consumption, so ωMUC is the marginal benefit of working more.

From growth to business cycles: households

- With utility at time t given by

$$\frac{C_t^{1-\sigma}}{1-\sigma} - \chi \frac{N_t^{1+\eta}}{1+\eta}.$$

the optimal choice of hours of work must satisfy

$$\chi N_t^{\eta} = \omega_t C_t^{-\sigma}.$$

- Marginal rate of substitution between leisure and consumption must equal the real wage:

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \omega_t.$$

Households

- Conditions to be satisfied in equilibrium:

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \omega_t.$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}.$$

Market clearing

- Economy's resource constraint:

$$Y_t = C_t + K_{t+1} - (1 - \delta) K_t$$

- Labor market clears: equilibrium real wage such that labor demand by firms equals labor supply of households.
- Capital market clears: equilibrium rental rate on capital such that demand for capital by firms equals supply of capital.

A basic real business cycle (RBC) model

- Given an exogenous process for the productivity shock A, equilibrium consists of time paths for output, consumption, capital, employment, wages and rental rates that satisfy

$$Y_t = e^{z_t} K_t^{\alpha} N_t^{1-\alpha}$$

$$Y_t = C_t + K_t - (1-\delta) K_{t-1}$$

$$(1-\alpha)\frac{Y_t}{N_t} = \omega_t$$

$$\alpha \frac{Y_t}{K_t} = r_t$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}$$

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \omega_t$$

$$z_t = \rho z_{t-1} + e_t$$

- Seven equations in Y_t, C_t, N_t, K_{t+1}, ω_t, r_t and z_t. Solve for initial K_t and z_t with exogenous e_t.

Solving and analyzing the model

- We have a set of dynamic nonlinear equations involving expectations of the future.
  - Equilibrium today depends on what agents expect the equilibrium to be in the future.
- Solution:
  - Linearize model to obtain first-order approximation to the equilibrium conditions.
  - Assume rational expectations – agent's expectations about the future are consistent with the model.
  - Obtain numerical solutions.
- Will need to assign values to the parameters.
  - What are sensible values for the model's parameters?

Rules for taking approximations

- Use the simple rules employed previously.
  - $(1+x)^a \approx 1 + ax$ for x small.
  - $(1+x)(1+y) \approx 1 + x + y$
  - $(1+x)/(1+y) \approx 1 + x - y$
- Let $\hat{x}_t$ denote the percent deviation of x_t around its steady-state value.
- Exception – returns (r_t) are already in percent terms so $\hat{r}_t \equiv r_t - r^*$.
- See ApproximatingRBC.pdf:

First-order linear approximation

- For production function:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

$$\hat{y}_t = z_t + \alpha \hat{k}_t + (1-\alpha)\,\hat{n}_t$$

where ˆ denotes percent deviation around the steady state.

- For labor demand,

$$(1-\alpha)\,\frac{Y_t}{N_t} \approx (1-\alpha)\,\frac{Y}{N}\,(1+\hat{y}_t-\hat{n}_t) = \omega\,(1+\hat{\omega}_t)$$

so

$$\hat{n}_t^d = \hat{y}_t - \hat{\omega}_t$$

First-order linear approximation

- For capital,

$$\alpha \frac{Y}{K} \left(1 + \hat{y}_t - \hat{k}_t\right) = r_t$$

- Subtract $\sigma Y/K = r^*$ from both sides:

$$\alpha \frac{Y}{K} \left(\hat{y}_t - \hat{k}_t\right) = r_t - r^* = \hat{r}_t$$

First-order linear approximation

- Production function:

$$\hat{y}_t = z_t + \alpha \hat{k}_t + (1 - \alpha)\, \hat{n}_t$$

- Labor demand:

$$\hat{n}_t^d = \hat{y}_t - \hat{\omega}_t$$

- Capital demand:

$$\hat{k}_t^d = \hat{y}_t - \left(\alpha \frac{Y}{K}\right)^{-1} (r_t - r^*)$$

First-order linear approximation

- Optimal condition for consumption was

$$c_t^{-\sigma} = (1 + r_{t+1} - r^* - \delta)\, c_{t+1}^{-\sigma}.$$

- Now let c denote the steady state value of c and $\hat{c}_t$ the percent deviation of c_t around c:

$$c_t = c\,(1 + \hat{c}_t)$$

- Then the optimality condition for consumption is

$$c^{-\sigma}\,(1 + \hat{c}_t)^{-\sigma} = \mathrm{E}_t\,(1 + r_{t+1} - r^* - \delta)\, c^{-\sigma}\,(1 + \hat{c}_{t+1})^{-\sigma}.$$

- Expectations because now future productivity (which affects output and consumption) is unknown.

First-order linear approximation

$$c^{-\sigma}\left(1+\hat{c}_t\right)^{-\sigma} = \mathrm{E}_t\left(1+r_{t+1}-r^*-\delta\right)c^{-\sigma}\left(1+\hat{c}_{t+1}\right)^{-\sigma}.$$

- Approximate this as

$$\left(1+\hat{c}_t\right)^{-\sigma} = \mathrm{E}_t\left(1+r_{t+1}-r^*-\delta\right)\left(1+\hat{c}_{t+1}\right)^{-\sigma}$$

$$1-\sigma\hat{c}_t = \mathrm{E}_t\left[1+r_{t+1}-r^*-\delta-\sigma\hat{c}_{t+1}\right]$$

$$\hat{c}_t = \mathrm{E}_t\hat{c}_{t+1}-\left(\frac{1}{\sigma}\right)\mathrm{E}_t\left(r_{t+1}-r^*-\delta\right)$$

First-order linear approximation

- Then optimal choice of hours of work must satisfy

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \omega_t.$$

- Marginal rate of substitution between leisure and consumption must equal the real wage.
- Linear approximation is

$$\eta \hat{n}_t + \sigma \hat{c}_t = \hat{\omega}_t.$$

Summarizing households

- Consumption choice:

$$\hat{c}_t = \mathrm{E}_t \hat{c}_{t+1} - \left(\frac{1}{\sigma}\right) \mathrm{E}_t \left(r_{k,t+1} - r_k^*\right)$$

- Labor supply choice:

$$\eta \hat{n}_t + \sigma \hat{c}_t = \hat{\omega}_t$$

or

$$\hat{n}_t^s = \left(\frac{1}{\eta}\right) \left(\hat{\omega}_t - \sigma \hat{c}_t\right)$$

Market equilibrium

- Goods market:

$$Y_t = C_t + K_{t+1} - (1-\delta) K_t$$

or

$$Y(1+\hat{y}_t) = C(1+\hat{c}_t) + K\left(1+\hat{k}_{t+1}\right) - (1-\delta) K\left(1+\hat{k}_t\right)$$

$$\hat{y}_t = \frac{C}{Y}\hat{c}_t + \frac{K}{Y}\hat{k}_{t+1} - (1-\delta)\frac{K}{Y}\hat{k}_t$$

- Labor market – labor demand equals labor supply:

$$\hat{n}_t = \hat{n}_t^s = \left(\frac{1}{\eta}\right)(\hat{\omega}_t - \delta \hat{c}_t) = \hat{n}_t^d = \hat{y}_t - \hat{\omega}_t$$

- Capital market – capital demand equals supply:

$$\hat{k}_t = \hat{k}_t^d = \hat{y}_t - \left(\alpha \frac{Y}{K}\right)^{-1} \left(r_{k,t} - r_k^*\right)$$

A basic real business cycle (RBC) model: equilibrium conditions

- Endogenous variables: $\hat{y}_t$, $\hat{c}_t$, $\hat{k}_{t+1}$, $\hat{n}_t$, $r_{k,t}$, $\hat{\omega}_t$ and z_t. Seven equations:

$$\hat{y}_t = z_t + \alpha \hat{k}_t + (1-\alpha)\,\hat{n}_t$$

$$\hat{y}_t = \left(\frac{C}{Y}\right)\hat{c}_t + \left(\frac{K}{Y}\right)\left[\hat{k}_{t+1} - (1-\delta)\,\hat{k}_t\right]$$

$$\hat{c}_t = \mathrm{E}_t \hat{c}_{t+1} - \left(\frac{1}{\sigma}\right)\mathrm{E}_t \hat{r}_{t+1}$$

$$\hat{n}_t = \left(\frac{1}{\eta}\right)(\hat{\omega}_t - \sigma \hat{c}_t)$$

$$\hat{n}_t = \hat{y}_t - \hat{\omega}_t$$

$$\hat{r}_t = \left(\alpha \frac{Y}{K}\right)(\hat{y}_t - \hat{k}_t)$$

$$z_t = \rho z_{t-1} + e_t$$

A basic real business cycle (RBC) model in dynare

- Note: Dynare would interpret $\hat{k}_{t+1} as$ really meaning $\mathrm{E}_t \hat{k}_{t+1}$.
- So to solve the model in dynare, equations with capital need to be written as

$$\hat{y}_t = z_t + \alpha \hat{k}_{t-1} + (1-\alpha)\,\hat{n}_t$$

$$\hat{y}_t = \left(\frac{C}{Y}\right)\hat{c}_t + \left(\frac{K}{Y}\right)\left[\hat{k}_t - (1-\delta)\,\hat{k}_{t-1}\right]$$

$$\hat{r}_t = \left(\alpha \frac{Y}{K}\right)\left(\hat{y}_t - \hat{k}_{t-1}\right)$$

- That is, $\hat{k}_{t-1}$ is predetermined at time t and $\hat{k}_t$ is an endogenous choice variable at time t.

Calibrating a basic real business cycle (RBC)

- Preferences: η, σ, β.
- Technology: α, δ.
- Productivity: ρ, σ_e.
- Steady state values: K/Y, C/Y

Calibrating a basic real business cycle (RBC)

- Preferences: η, σ, β:
- Average real return from consumption Euler equation:

$$1 = \beta\left(1 + r - \delta\right) \Rightarrow 1 + r - \delta = \frac{1}{\beta}$$

- If average real return on capital (net of depreciation) is 0.04 at annual rate, then $\beta \approx 1/\left(1 + 0.01\right) = 0.99$.
- Standard case: log utility, so $\sigma = 1$.
- Labor supply choice by households implies

$$n_t^s = \left(\frac{1}{\eta}\right)\hat{\omega}_t - \left(\frac{\sigma}{\eta}\right)\hat{c}_t.$$

  - So η is inverse wage elasticity of labor supply. Lots of controversy over correct value for η. Will start with $\eta = 2$ case.

Calibrating a basic real business cycle (RBC)

- Technology: α, δ:
  - Depreciations – common value is 10% at annual rate (0.10). This implies $\delta = 0.025$ at quarterly frequency.
  - From labor demand condition, $MPL = (1-\alpha)\, Y/N = \omega$. So

$$1 - \alpha = \frac{\omega N}{Y} = \text{ labor's share of total income}$$

  - Common value for α is $1/3$.
- Productivity: ρ, σ_e:
  - Pick σ_e so that model matches standard deviation of observed output. Set $\sigma_e = 0.0033$.
  - Will experiment with different values of ρ.
  - Typical value is $\rho = 0.95$.

Calibrating a basic real business cycle (RBC)

- Steady state values: K/Y, C/Y.
- In steady state, Y, K, and C constant (recall, we are ignoring growth):

$$Y = C + I = C + [K - (1-\delta)K] = C + \delta K$$

- So in steady state,

$$1 = \left(\frac{C}{Y}\right) + \delta\left(\frac{K}{Y}\right) \Rightarrow \left(\frac{C}{Y}\right) = 1 - \delta\left(\frac{K}{Y}\right).$$

- What is K/Y?

$$\frac{1}{\beta} = 1 + r - \delta = 1 + \alpha\frac{Y}{K} - \delta$$

so

$$\frac{K}{Y} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right).$$

A basic real business cycle (RBC) model

- Solution will be of form

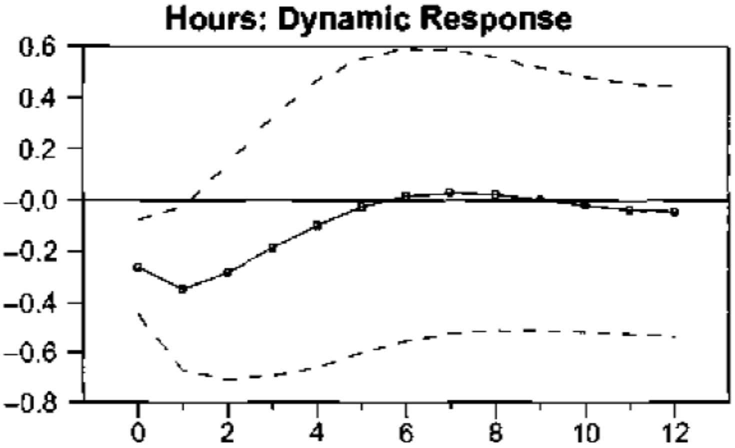
$$\begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{k}_{t+1} \\ \hat{n}_t \\ \hat{r}_t \\ \hat{\omega}_t \\ z_t \end{bmatrix} = M \begin{bmatrix} \hat{k}_t \\ z_{t-1} \\ e_t \end{bmatrix}$$

A basic real business cycle (RBC) model

- Evaluation
- Persistence
- Real wages and employment
- Ability to match historical episodes – e.g., 1979-1982

A basic real business cycle (RBC) model

Galí and Rabanal (2004): Productivity shocks and hours



The RBC model: Lecture 6

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Winter 2017

A basic real business cycle (RBC) model

- Given an exogenous process for the productivity shock A, equilibrium consists of time paths for output, consumption, capital, employment, wages and rental rates that satisfy

$$Y_t = F(K_t, L_t, A_t)$$

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t$$

$$F_N(K_t, L_t, A_t) = \omega_t$$

$$F_K(K_t, L_t, A_t) = r_t$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}$$

$$\frac{\chi L_t^{\eta}}{C_t^{-\sigma}} = \omega_t$$

- Where do these equation come from?
- Does the equilibrium display business cycle behavior?

A basic real business cycle (RBC) model

- Given an exogenous process for the productivity shock A, equilibrium consists of time paths for output, consumption, capital, employment, wages and rental rates that satisfy

$$Y_t = e^{z_t} K_t^{\alpha} N_t^{1-\alpha}$$

$$Y_t = C_t + K_t - (1-\delta) K_{t-1}$$

$$(1-\alpha)\frac{Y_t}{N_t} = \omega_t$$

$$\alpha \frac{Y_t}{K_t} = r_t$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}$$

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \omega_t$$

$$z_t = \rho_z z_{t-1} + e_t$$

- Seven equations in Y_t, C_t, N_t, K_{t+1}, ω_t, r_t and z_t. Solve for initial K_t and z_t with exogenous e_t.

Solving and analyzing the model

- We have a set of dynamic nonlinear equations involving expectations of the future.
  - Equilibrium today depends on what agents expect the equilibrium to be in the future.
- Solution:
  - Linearize model to obtain first-order approximation to the equilibrium conditions.
  - Assume rational expectations – agent's expectations about the future are consistent with the model.
  - Obtain numerical solutions.
- Will need to assign values to the parameters.
  - What are sensible values for the model's parameters?

A basic real business cycle (RBC) model: equilibrium conditions

- Endogenous variables: $\hat{y}_t$, $\hat{c}_t$, $\hat{k}_{t+1}$, $\hat{n}_t$, $r_{k,t}$, $\hat{\omega}_t$ and z_t. Seven equations:

$$Y_t = e^{z_t} K_{t-1}^{\alpha} N_t^{1-\alpha} \Rightarrow \hat{y}_t = z_t + \alpha \hat{k}_{t-1} + (1-\alpha)\,\hat{n}_t$$

$$\begin{aligned} Y_t &= C_t + K_t - (1-\delta) K_{t-1} \Rightarrow \\ \hat{y}_t &= \left(\frac{C}{Y}\right)\hat{c}_t + \left(\frac{K}{Y}\right)\left[\hat{k}_t - (1-\delta)\,\hat{k}_{t-1}\right] \end{aligned}$$

$$(1-\alpha)\,\frac{Y_t}{N_t} = \omega_t \Rightarrow \hat{\omega}_t = \hat{y}_t - \hat{n}_t$$

A basic real business cycle (RBC) model

$$\alpha \frac{Y_t}{K_{t-1}} = r_t \Rightarrow \hat{r}_t = \left(\alpha \frac{Y}{K}\right)\left(\hat{y}_t - \hat{k}_{t-1}\right)$$

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma} \Rightarrow \hat{c}_t = \mathrm{E}_t \hat{c}_{t+1} - \left(\frac{1}{\sigma}\right) \mathrm{E}_t \hat{r}_{t+1}$$

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \omega_t \Rightarrow \hat{\omega}_t = \eta \hat{n}_t + \sigma \hat{c}_t$$

$$z_t = \rho_z z_{t-1} + e_t$$

A basic real business cycle (RBC) model in dynare

- Note: Dynare would interpret $\hat{k}_{t+1}$as meaning $\mathrm{E}_t \hat{k}_{t+1}$ so denote time t capital available for production by k_{t-1}.
- So to solve the model in dynare, equations with capital need to be written as

$$\hat{y}_t = z_t + \alpha \hat{k}_{t-1} + (1-\alpha)\,\hat{n}_t$$

$$\hat{y}_t = \left(\frac{C}{Y}\right)\hat{c}_t + \left(\frac{K}{Y}\right)\left[\hat{k}_t - (1-\delta)\,\hat{k}_{t-1}\right]$$

$$\hat{r}_t = \left(\alpha \frac{Y}{K}\right)\left(\hat{y}_t - \hat{k}_{t-1}\right)$$

- That is, $\hat{k}_{t-1}$ is predetermined at time t and $\hat{k}_t$ is an endogenous choice variable at time t.

Calibrating a basic real business cycle (RBC)

- Preferences: η, σ, β.
- Technology: α, δ.
- Productivity: ρ_z, σ_e.
- Steady state values: K/Y, C/Y

Calibrating a basic real business cycle (RBC)

- Preferences: η, σ, β:
- Average real return from consumption Euler equation:

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-\sigma}$$

- In steady state, this becomes

$$C^{-\sigma} = \beta \left(1 + r - \delta\right) C^{-\sigma}$$

$$1 = \beta \left(1 + r - \delta\right) \Rightarrow 1 + r - \delta = \frac{1}{\beta}$$

- If average real return on capital (net of depreciation) is 0.04 at annual rate (0.01 at quarterly rate), then $\beta \approx 1/\left(1 + 0.01\right) = 0.99$.

Calibrating a basic real business cycle (RBC)

- Standard case: log utility, so $\sigma = 1$.
- Labor supply choice by households implies

$$n_t^s = \left(\frac{1}{\eta}\right)\hat{\omega}_t - \left(\frac{\sigma}{\eta}\right)\hat{c}_t.$$

  - So η is inverse wage elasticity of labor supply. Lots of controversy over correct value for η. Will start with $\eta = 2$ case.

Calibrating a basic real business cycle (RBC)

- Technology: α, δ:
  - Depreciation – common value is 10% at annual rate (0.10). This implies $\delta = 0.025$ at quarterly frequency.
  - From labor demand condition, $MPL = (1-\alpha)\,Y/N = \omega$. So

$$1 - \alpha = \frac{\omega N}{Y} = \text{ labor's share of total income}$$

  - Common value for α is $1/3$.
- Productivity: ρ, σ_e:
  - Pick σ_e so that model matches standard deviation of observed output. Set $\sigma_e = 0.0033$.
  - Will experiment with different values of ρ.
  - Typical value is $\rho = 0.95$.

Calibrating a basic real business cycle (RBC)

- Steady state values: K/Y, C/Y.
- In steady state, Y, K, and C constant (recall, we are ignoring growth):

$$Y = C + I = C + [K - (1-\delta) K] = C + \delta K$$

- So in steady state,

$$1 = \left(\frac{C}{Y}\right) + \delta \left(\frac{K}{Y}\right) \Rightarrow \left(\frac{C}{Y}\right) = 1 - \delta \left(\frac{K}{Y}\right).$$

- What is K/Y?

$$\frac{1}{\beta} = 1 + r - \delta = 1 + \alpha \frac{Y}{K} - \delta$$

so

$$\frac{K}{Y} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right).$$

A basic real business cycle (RBC) model

- Solution will be of form

$$\begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{k}_t \\ \hat{n}_t \\ \hat{r}_t \\ \hat{\omega}_t \\ z_t \end{bmatrix} = M \begin{bmatrix} \hat{k}_{t-1} \\ z_{t-1} \\ e_t \end{bmatrix}$$

A basic real business cycle (RBC) model

- Adding back in growth
- Evaluating the RBC model
  - Real wages and employment
  - Persistence
  - Ability to match historical episodes – e.g., 1979-1982
  - The response of hours to productivity

A digression to show how to add growth back in

- Suppose $A_t = (1+g)^t A_0$. Model equations become

$$Y_t = e^{z_t} K_{t-1}^{\alpha} (A_t N_t)^{1-\alpha} \Rightarrow \frac{Y_t}{A_t} = e^{z_t} \left(\frac{K_{t-1}}{A_t} \right)^{\alpha} N_t^{1-\alpha}$$
$$\Rightarrow \quad y_t = e^{z_t} k_{t-1}^{\alpha} N_t^{1-\alpha}$$

$$\frac{Y}{A_{t\,t}} = \frac{C_t}{A_t} + \frac{K_t}{A_t} - (1-\delta)\frac{K_{t-1}}{A_t}$$
$$\Rightarrow \quad y_t = c_t + \frac{K_t}{A_{t+1}} \frac{A_{t+1}}{A_t} - (1-\delta) k_{t-1}$$
$$\Rightarrow \quad y_t = c_t + (1+g) k_t - (1-\delta) k_{t-1}$$

$$(1-\alpha) \frac{Y_t}{N_t} \frac{1}{A_t} = \frac{\omega_t}{A_t} \Rightarrow (1-\alpha) \frac{y_t}{N_t} = \tilde{\omega}_t$$

$$\alpha \frac{Y_t}{K_{t-1}} \frac{A_t}{A_t} = r_t \Rightarrow \alpha \frac{y_t}{k_{t-1}} = r_t$$

A digression to show how to add growth back in

$$
\begin{aligned}
C_t^{-1} &= \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) C_{t+1}^{-1} \Rightarrow \\
\left(\frac{C_t}{A_t}\right)^{-1} &= \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) \left(\frac{C_{t+1}}{A_{t+1}} \frac{A_{t+1}}{A_t}\right)^{-1} \Rightarrow \\
c_t^{-1} &= \beta \mathrm{E}_t \left(1 + r_{t+1} - \delta\right) c_{t+1}^{-1} \left(1 + g\right)^{-1} \Rightarrow \\
c_t^{-1} &= \beta \mathrm{E}_t \left(\frac{1 + r_{t+1} - \delta}{1 + g}\right) c_{t+1}^{-1}
\end{aligned}
$$

$$
\frac{\chi N_t^{\eta}}{C_t^{-1}} = \chi N_t^{\eta} \left(\frac{C_t}{A_t}\right) = \frac{\omega_t}{A_t} \Rightarrow \chi N_t^{\eta} C_t = \tilde{\omega}_t
$$

A digression to show how to add growth back in

$$y_t = e^{z_t} k_{t-1}^{\alpha} N_t^{1-\alpha}$$

$$y_t = c_t + (1+g)\, k_t - (1-\delta)\, k_{t-1}$$

$$(1-\alpha)\frac{y_t}{N_t} = \tilde{\omega}_t$$

$$\alpha \frac{y_t}{k_{t-1}} = r_t$$

$$c_t^{-1} = \beta \mathrm{E}_t \left(\frac{1 + r_{t+1} - \delta}{1+g} \right) c_{t+1}^{-1}$$

$$\chi N_t^{\eta} c_t = \tilde{\omega}_t$$

$$z_t = \rho_z z_{t-1} + e_t$$

- Seven equations in y_t, c_t, N_t, k_t, $\tilde{\omega}_t$, r_t and z_t. Solve for initial k_{t-1} and z_t with exogenous e_t.

A digression to show how to add growth back in

- Economy converges to a steady state in which y_t, c_t, N_t, k_t, $\tilde{\omega}_t$ and r_t are constant.
- Actual output grows on average at the rate g:

$$\begin{aligned} Y_t &= A_t y_t \\ &\Rightarrow \ln Y_t = \ln A_t + \ln y_t \\ &= \ln A_t + z_t + \alpha \ln k_{t-1} + (1-\alpha) \ln N_t \end{aligned}$$

- Consumption grows on average at the rate g.
- Real wages growth on average at the rate g.
- Average hours per worker and the rental rate on capital converge to constants.

A digression to show how to add growth back in

- From

$$c_t^{-1} = \beta \mathrm{E}_t \left(\frac{1 + r_{t+1} - \delta}{1+g} \right) c_{t+1}^{-1},$$

in the steady state, c is constant so

$$1 = \beta \left(\frac{1 + r - \delta}{1+g} \right)$$

- We can approximate this by

$$\begin{aligned} 1 &= \frac{1 + r - \delta}{(1+\rho)(1+g)} \approx 1 + r - (\delta + \rho + g) \\ \Rightarrow & \quad r = \delta + \rho + g \end{aligned}$$

A basic real business cycle (RBC) model

- Adding back in growth
- Evaluating the RBC model
  - Real wages and employment
  - Persistence
  - Ability to match historical episodes – e.g., 1979-1982
  - The response of hours to productivity

A basic real business cycle: real wages and employment

- The model assumes a competitive labor market.
- Observed real wages and employment are equilibrium outcomes – labor demand equals labor supply.
- Problem: in the data, real wages do not move a lot over the business cycle, but employment does.
- Can account for this if
  - labor supply curve is relatively flat.
  - or if there are additional shocks that shift labor supply.
- But micro evidence suggests labor supply relatively steep.
- Will see later that fiscal policy might shift labor supply.

A basic real business cycle: persistence

- Calibrated model assumed $\rho = 0.95$.
- Persistence due to
  - the persistence of the shocks because $\rho_z > 0$
  - the capital accumulation channel.
- Problem: the capital accumulation channel is weak:

$$\hat{k}_t = (1 - \delta)\,\hat{k}_{t-1} + \delta inv_t$$

  where inv_t is investment (expressed as percent deviation around steady state).
- But δ is small, around 0.025. So even large increases in investment have little effect on the total stock of capital.

Dropping capital to simplify the model

$$y_t = e^{z_t} N_t^{1-\alpha}$$

$$y_t = c_t$$

$$(1-\alpha)\frac{y_t}{N_t} = \omega_t$$

$$c_t^{-1} = \beta \mathrm{E}_t \left(\frac{1 + r_{t+1} - \delta}{1+g} \right) c_{t+1}^{-1}$$

$$\chi N_t^{\eta} c_t^{\sigma} = \omega_t$$

$$z_t = \rho_z z_{t-1} + e_t$$

Dropping capital: the labor market

- Labor demand and labor supply conditions imply

$$(1-\alpha)\frac{y_t}{N_t} = \chi N_t^{\eta} c_t^{\sigma}.$$

- Goods market equaility implies $y_t = c_t$, so

$$\begin{aligned} (1-\alpha)\frac{y_t}{N_t} &= \chi N_t^{\eta} y_t^{\sigma} \\ &\Rightarrow (1-\alpha) y_t^{1-\sigma} = \chi N_t^{1+\eta} \end{aligned}$$

- Using the production function, we have two equations to solve for y_t and N_t:

$$(1-\alpha) y_t^{1-\sigma} = \chi N_t^{1+\eta}$$

$$y_t = e^{z_t} N_t^{1-\alpha}$$

Solving for equilibrium output

- Two equations to solve for y_t and N_t:

$$(1-\alpha)\, y_t^{1-\sigma} = \chi N_t^{1+\eta}$$

$$y_t = e^{z_t} N_t^{1-\alpha}$$

- Simplify by assuming constant returns to scale ($\alpha = 1$). Then

$$y_t = e^{z_t} N_t \Rightarrow N_t = e^{-z_t} y_t$$

- Use this in the labor demand equals labor supply condition:

$$(1-\alpha)\, y_t^{1-\sigma} = \chi N_t^{1+\eta} = \chi e^{-(1+\eta)z_t} y_t^{1+\eta}$$

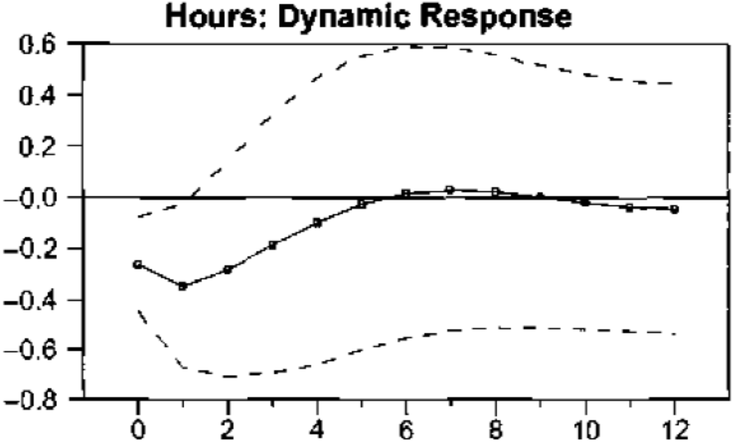
- Solving for outout,

$$y_t = \left[\left(\frac{1-\alpha}{\chi} \right) e^{(1+\eta)z_t} \right]^{\frac{1}{\sigma+\eta}}$$

A basic real business cycle: historical episodes

A basic real business cycle (RBC) model

Galí and Rabanal (2004): Productivity shocks and hours



From RBC to NK: Lectures 7 and 8

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Winter 2017

A basic real business cycle (RBC) model

- Evaluation
  - Real wages and employment
  - Persistence
  - Ability to match historical episodes – e.g., 1979-1982
  - The response of hours to productivity

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- Problem: the capital accumulation channel is weak:

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Dropping capital to simplify the model

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$$y_t = c_t$$

$$(1-\alpha)\frac{y_t}{N_t} = \omega_t$$

$$c_t^{-1} = \beta \mathrm{E}_t \left(\frac{1 + r_{t+1} - \delta}{1+g} \right) c_{t+1}^{-1}$$

$$\chi N_t^{\eta} c_t^{\sigma} = \omega_t$$

$$z_t = \rho_z z_{t-1} + e_t$$

Dropping capital: the labor market

- Labor demand and labor supply conditions imply

$$(1-\alpha)\frac{y_t}{N_t} = \chi N_t^{\eta} c_t^{\sigma}.$$

- Goods market equaility implies $y_t = c_t$, so

$$\begin{aligned}(1-\alpha)\frac{y_t}{N_t} &= \chi N_t^{\eta} y_t^{\sigma} \\ &\Rightarrow (1-\alpha) y_t^{1-\sigma} = \chi N_t^{1+\eta}\end{aligned}$$

- Using the production function, we have two equations to solve for y_t and N_t:

$$(1-\alpha) y_t^{1-\sigma} = \chi N_t^{1+\eta}$$

$$y_t = e^{z_t} N_t^{1-\alpha}$$

Solving for equilibrium output

- Two equations to solve for y_t and N_t:

$$(1-\alpha)\, y_t^{1-\sigma} = \chi N_t^{1+\eta}$$

$$y_t = e^{z_t} N_t^{1-\alpha}$$

- Simplify by assuming constant returns to scale ($\alpha = 1$). Then

$$y_t = e^{z_t} N_t \Rightarrow N_t = e^{-z_t} y_t$$

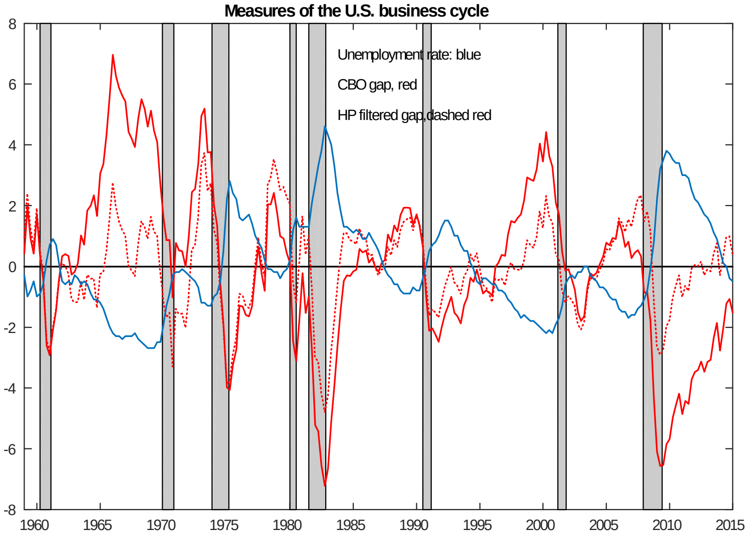
- Use this in the labor demand equals labor supply condition:

$$(1-\alpha)\, y_t^{1-\sigma} = \chi N_t^{1+\eta} = \chi e^{-(1+\eta) z_t} y_t^{1+\eta}$$

- Solving for outout,

$$y_t = \left[\left(\frac{1-\alpha}{\chi}\right) e^{(1+\eta) z_t}\right]^{\frac{1}{\sigma+\eta}}$$

The 1979-1982 period



The 1979-1982 period: interest rates

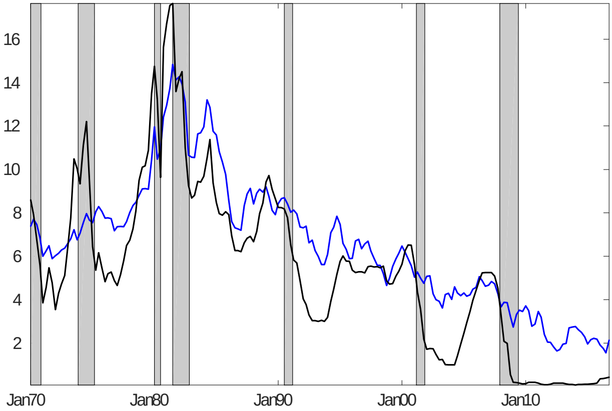
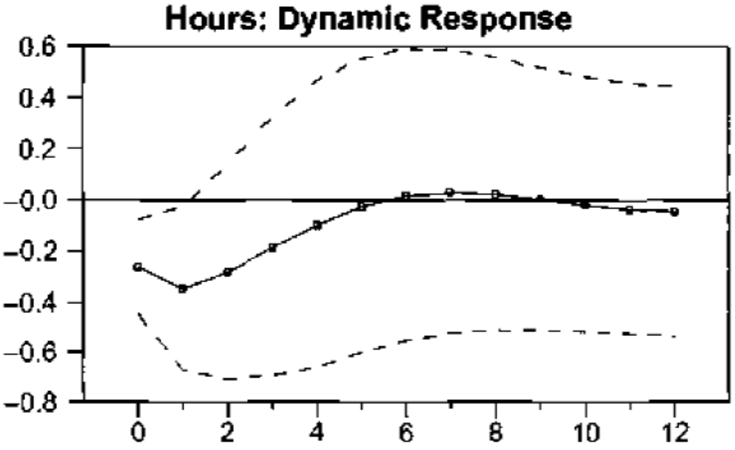


Figure: Federal Funds Rate (black) and 10-yr Treasury Rate (blue)

A basic real business cycle (RBC) model

Galí and Rabanal (2004): Productivity shocks and hours



From the real business cycle model to the new Keynesian model

- Add money:
  - to incorporate monetary policy and investigate the effects of monetary shocks on the economy.
- Add sticky prices (will add sticky wages later).
  - macro models assume households and firms care about relative prices and real quantities, including real quantity of money: $\frac{M}{P}$
  - changes in the nominal quantity of money together with a proportional change in all prices (including the price of labor) leave $\frac{M}{P}$ and all relative prices unchanged.
  - so there would be no effect on any real decisions.
  - if prices are slow to adjust, monetary policy can have real effects in the short-run.
- Add imperfect competition.
  - if prices are sticky, someone must be making decisions about when to change prices and by how much to change them.

From the real business cycle model to the new Keynesian model

- Because we are adding complications to model, let's simplify RBC model in other ways.
- Drop the capital stock – didn't play much role for analysis of cycles.
- Simplify production to be constant returns to scale in labor alone:

$$Y_t = A_t N_t.$$

A basic real business cycle (RBC) model without capital

- Endogenous variables: $\hat{y}_t$, $\hat{c}_t$, $\hat{n}_t$, $r_{k,t}$, $\hat{\omega}_t$ and z_t. Six equations:

$$\hat{y}_t = z_t + \hat{n}_t$$

$$\hat{y}_t = \hat{c}_t$$

$$\hat{c}_t = \mathrm{E}_t \hat{c}_{t+1} - \left(\frac{1}{\sigma}\right) \hat{r}_t$$

$$\hat{n}_t = \left(\frac{1}{\eta}\right) (\hat{\omega}_t - \sigma \hat{c}_t)$$

$$\hat{n}_t = \hat{y}_t - \hat{\omega}_t$$

$$z_t = \rho z_{t-1} + e_t$$

A basic real business cycle (RBC) model without capital

- This is the equilibrium when prices are flexible so it will be denoted with a superscript f.
- Use $\hat{y}_t^f = \hat{c}_t^f$ to eliminate $\hat{c}_t^f$, use $\hat{n}_t^f = \hat{y}_t^f - z_t$ to eliminate $\hat{n}_t^f$ and $\hat{\omega}_t^f$.

$$\eta \left(\hat{y}_t^f - z_t \right) = \left(z_t - \sigma \hat{y}_t^f \right) \Rightarrow \hat{y}_t^f = \left(\frac{1+\eta}{\eta+\sigma} \right) z_t$$

$$\hat{y}_t^f = \mathrm{E}_t \hat{y}_{t+1}^f - \left(\frac{1}{\sigma} \right) \hat{r}_t^f \Rightarrow r_t^f = \sigma \left(\mathrm{E}_t \hat{y}_{t+1}^f - \hat{y}_t^f \right)$$

Dynamic Stochastic General Equilibrium new Keynesian (DSGE) models

- Provides frameworks for analyzing macro policy issues.
- Builds on real business cycle model foundations (which built on Solow/Ramsey foundations)
- Model consists of households who supply labor, purchase goods for consumption, and hold money and bonds, and firms who hire labor and produce and sell differentiated products in monopolisticly competitive goods markets.
- Households and firms behave optimally: households maximize the expected present value of utility and firms maximize profits.
- Each firm set the price of the good it produces, but not all firms reset their price each period.
- Model fits the macro date and can be used for policy analysis.

The core of DSGE – a new Keynesian model

- What is the core structure of the modern DSGE models?:
- Core consists of six equations (variables expressed as deviations around steady state):

$$y_t^f = \left(\frac{1+\eta}{\eta+\sigma}\right) z_t$$

$$r_t^f = \sigma \mathrm{E}_t \left(y_{t+1}^f - y_t^f\right)$$

$$x_t \equiv y_t - y_t^f$$

$$x_t = \mathrm{E}_t x_{t+1} - \sigma^{-1} \left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa \left(\eta + \sigma\right) x_t + \varepsilon_t$$

$$i_t = a_\pi \pi_t + a_x x_t.$$

- These determine the flex-price output, the natural real interest rate, the output gap, inflation, and the nominal interest rate.

Where do these critical components come from?

- Agents in the model: households, firms and the central bank.
- Households choose consumption and labor supply;
- Firms hire labor and set prices in an environment of monopolistic competition;
- Central bank determines the nominal interest rate.

Households

- Household preferences:

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left[\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} \right]$$

- Households face budget constraint. Can hold one-period bonds that pay gross nominal return of $R_t = 1 + i_t$.
- The following optimality conditions must also hold in equilibrium

$$C_t^{-\sigma} = \beta R_t \mathrm{E}_t \left(\frac{P_t}{P_{t+1}} \right) C_{t+1}^{-\sigma};$$

$$\frac{\chi_t N_t^{\eta}}{C_t^{-\sigma}} = \frac{W_t}{P_t} = \omega_t.$$

Households

- Euler condition for intertemporal consumption allocation:

$$C_t^{-\sigma} = \beta R_t \mathrm{E}_t \left(\frac{P_t}{P_{t+1}} \right) C_{t+1}^{-\sigma;}$$

- Linearize around the steady state to obtain

$$\hat{c}_t = \mathrm{E}_t \hat{c}_{t+1} - \left(\frac{1}{\sigma} \right) \left(i_t - \mathrm{E}_t \pi_{t+1} \right)$$

where $\hat{c}_t$ is the log deviation of consumption around the steady state and i_t and π_{t+1} are deviations from steady state.

Households

- Define $\hat{y}_t$ and $\hat{y}_t^f$ as log deviations of output and output with flexible prices each expressed as deviations around steady state, and define $x_t \equiv \hat{y}_t - \hat{y}_t^f$.
- Without investment or government spending, goods market equilibrium requires $C_t = Y_t$.
- Then since $\hat{c}_t = \hat{y}_t$,

$$\hat{c}_t = \mathrm{E}_t \hat{c}_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - \mathrm{E}_t \pi_{t+1})$$

becomes

$$\hat{y}_t = \mathrm{E}_t \hat{y}_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - \mathrm{E}_t \pi_{t+1})$$

becomes

$$\hat{y}_t - \hat{y}_t^f = x_t = \mathrm{E}_t \left(y_{t+1} - \hat{y}_{t+1}^f\right) - \left(\frac{1}{\sigma}\right)(i_t - \mathrm{E}_t \pi_{t+1}) + \mathrm{E}_t \hat{y}_{t+1}^f - \hat{y}_t^f$$

or

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

Back to the basic model structure: sticky prices and inflation

- Inflation adjustment:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa \left(\eta + \sigma \right) x_t + \varepsilon_t^p$$

- Where does this come from?
- Need to model firms' pricing decisions.

Micro evidence on price setting

- Micro level data on price adjustments suggest prices remain unchanged for appreciable periods of time and not all prices adjust every period.
  - True for online prices as well (Gorodnichenko and Talavera 2015).

Price adjustment: micro facts – prices are sticky

- Bils and Klenow (JPE, 2004)
  - median duration between price changes is 4.3 months for items in the U.S. CPI with wide variation in frequency across different categories of goods and services.
- Klenow and Kryvtsov (QJE 2008)
  - price changes large on average, but a significant fraction of price changes are small.
  - variations in the size of price changes, rather than variation in the fraction of prices that change, can account for most of the variance of aggregate inflation.
- Nakamura and Steinsson (QJE 2008)
  - excluding sales increases median duration between changes from 4.5 months to 10 months.
  - the probability the price of an item changes (the hazard function) declines during the first few months after a change in price.

Models of price adjustment

- Time dependent pricing models
  - Calvo – fixed probability of adjusting each period.
  - Taylor's model of fixed-length contracts
- State dependent price models
  - Firms' decision depends on the "state" of the economy.

Time dependent pricing models: Calvo

- Each period, the firms that adjust their price are randomly selected: a fraction $1-\omega$ of all firms adjust while the remaining ω fraction do not adjust.
  - The parameter ω is a measure of the degree of nominal rigidity; a larger ω implies fewer firms adjust each period and the expected time between price changes is longer.
- For those firms who do adjust their price at time t, they do so to maximize the expected discounted value of current and future profits.
  - Profits at some future date $t+s$ are affected by the choice of price at time t only if the firm has not received another opportunity to adjust between t and $t+s$. The probability of this is ω^s.

Firms' pricing decisions – start with household demand

- Household purchase differentiated goods that are imperfect substitutes.
- The composite consumption good that enters the household's utility function is

$$C_t = \left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \qquad \theta > 1.$$

- Implies demand functions and price level defined by

$$c_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\theta} C_t; \; P_t = \left[\int_0^1 p_{jt}^{1-\theta} dj \right]^{\frac{1}{1-\theta}}.$$

- The parameter θ governs the price elasticity of demand for the individual goods.
- This price elasticity subject to stochastic variation.

Firms: start with case of flexible prices

- Simple CRS production technology for each firm j: $y_{j,t} = Z_t N_{j,t}$.
  - Real marginal cost is $mc_t = \omega_t / Z_t$. To produce one more unit of output requires $1/Z_t$ units of labor at a cost of $\omega_t (1/Z_t)$.
- Firm's maximize profit given demand curve and technology:

$$
\begin{aligned}
\text{profits}_{j,t} &= \left(\frac{p_{j,t}}{P_t}\right) y_{j,t} - \omega_t N_{j,t} = \left(\frac{p_{j,t}}{P_t}\right) y_{j,t} - \left(\frac{\omega_t}{Z_t}\right) y_{j,t} \\
&= \left[\left(\frac{p_{j,t}}{P_t}\right) - mc_t\right] \left(\frac{p_{j,t}}{P_t}\right)^{-\theta} C_t
\end{aligned}
$$

Firms: the case of flexible prices

- First order condition for choice of p_{jt}:

$$(1-\theta_t)p_{j,t}^{-\theta}\left(\frac{1}{P_t}\right)^{1-\theta}C_t + \theta mc_t p_{j,t}^{-\theta-1}\left(\frac{1}{P_t}\right)^{-\theta}C_t = 0$$

$$(1-\theta)\left(\frac{p_{j,t}}{P_t}\right) + \theta mc_t = 0$$

- or

$$\left(\frac{p_{j,t}}{P_t}\right) = \left(\frac{\theta}{\theta-1}\right)mc_t = \mu \times mc_t,\ \mu \equiv \left(\frac{\theta}{\theta-1}\right) \geq 1$$

Firms: the case of flexible prices

- Firms set price equal to a markup over nominal marginal cost: $p_{j,t} = \mu \times mc_t P_t$.
- With all firms doing this, $p_{j,t} = P_t$ and $1 = \mu \times mc_t \Rightarrow \mu = Z_t/\omega_t > 1$ or $\omega_t = Z_t/\mu \leq Z_t$.
- However, the real wage must also equal the marginal rate of substitution between leisure and consumption to be consistent with household optimization:

$$mrs_t = \left(\frac{\chi N_t^{\eta}}{C_t^{-\sigma}}\right) = \omega_t = \frac{Z_t}{\mu} \leq mpl_t.$$

  - Workers paid less than their marginal product – *source of inefficiency.*

Price adjustment: the case of sticky prices

The firm's decision problem

- The firm's profits at time $t+i$ are

$$\begin{aligned}\Pi_{j,t} &= \left[\left(\frac{p_{jt}}{P_{t+i}}\right) c_{jt+i} - mc_{t+i} c_{jt+i}\right] \\ &= \left[\left(\frac{p_{jt}}{P_{t+i}}\right)^{1-\theta} C_{t+i} - mc_{t+i}\left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta} C_{t+i}\right].\end{aligned}$$

- The optimal pricing decision then involves picking p_{jt} to maximize

$$\mathrm{E}_t \sum_{i=0}^{\infty} \omega^i \Omega_{i,t+i} \left[\left(\frac{p_{jt}}{P_{t+i}}\right)^{1-\theta} C_{t+i} - mc_{t+i}\left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta} C_{t+i}\right],$$

where the discount factor $\Omega_{i,t+i}$ is given by $\beta^i (C_{t+i}/C_t)^{-\sigma}$.

Price adjustment

- All firms adjusting in period t face the same problem, so all adjusting firms will set the same price.
- Let p_t^* be the optimal price chosen by all firms adjusting at time t.
- The first order condition for the optimal choice of p_t^* yields

$$\left(\frac{p_t^*}{P_t}\right) = \mu \frac{\mathrm{E}_t \sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} mc_{t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta}}{\mathrm{E}_t \sum_{i=0}^{\infty} \omega^i \beta^i C_{t+i}^{1-\sigma} \left(\frac{P_{t+i}}{P_t}\right)^{\theta-1}}. \tag{1}$$

- Combine the first order condition with

$$P_t^{1-\theta} = (1-\omega)(p_t^*)^{1-\theta} + \omega P_{t-1}^{1-\theta}.$$

Price adjustment: the new Keynesian inflation equation

- Linearize around a zero steady state inflation rate yields

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa mc_t$$

where

$$\kappa = \frac{(1-\omega)(1-\omega\beta)}{\omega}.$$

Forward-looking inflation adjustment

- The New Keynesian Phillips curve is forward-looking; when a firm sets its price, it must be concerned with inflation in the future because it may be unable to adjust its price for several periods.
- Solving forward,

$$\pi_t = \sum_{i=0}^{\infty} \beta^i \mathrm{E}_t \kappa mc_{t+i},$$

- Inflation is a function of the present discounted value of current and future real marginal cost.
- Inflation depends on real marginal cost and not directly on a measure of the gap between actual output and some measure of potential output or on a measure of unemployment relative to the natural rate, as is typical in traditional Phillips curves.

Assessing models of nominal price rigidities

- Klenow and Kryvtsov (QJE 2008)
  - No model fits all the micro facts
  - Calvo does surprisingly well except for the declining hazard rate
    - ★ Variations in the size of price changes, rather than variation in the fraction of prices that change, can account for most of the variance of aggregate inflation.
- Carlsson and Nordström Skans (AEJ Macro 2012)
  - Uses Swedish firm-level data to compare models of staggered price adjustment with models of flexible prices but imperfect information (sticky information, rational inattention to macro factors)
  - Concludes Calvo explains data best.

Real marginal cost and the output gap

- The firm's real marginal cost is equal to the real wage it faces divided by the marginal product of labor: $s_t = \omega_t / Z_t$.
- Because nominal wages have been assumed to be completely flexible, the real wage must equal the marginal rate of substitution between leisure and consumption plus any wage markup: $\hat{\omega}_t = \hat{w}_t - \hat{p}_t = \eta \hat{n}_t + \sigma \hat{y}_t$.
- Recalling that $\hat{c}_t = \hat{y}_t$, $\hat{y}_t = \hat{n}_t + z_t$,

$$
\begin{aligned}
mc_t &= (\eta \hat{n}_t + \sigma \hat{y}_t) - \hat{z}_t = [\eta (\hat{y}_t - z_t) + \sigma \hat{y}_t] - z_t \\
&= (\sigma + \eta) \left[\hat{y}_t - \left(\frac{1+\eta}{\eta+\sigma} \right) z_t \right] \\
&= (\sigma + \eta) \left(\hat{y}_t - \hat{y}_t^f \right).
\end{aligned}
$$

Real marginal cost and the output gap

- Using these results, the inflation adjustment equation becomes

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa \left(\sigma + \eta\right) x_t$$

  where $\kappa = \left(\eta + \sigma\right) \tilde{\kappa} = \left(\eta + \sigma\right) \left(1 - \omega\right) \left(1 - \beta\omega\right) / \omega$ and $x_t \equiv \hat{y}_t - \hat{y}_t^f$ is the gap between actual output and the flexible-price equilibrium output.
- This inflation adjustment or forward-looking Phillips curve relates output, in the form of the deviation around the level of output that would occur in the absence of nominal price rigidity, to inflation.

New Keynesian Part 2

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The core of DSGE – a new Keynesian model

- What is the core structure of the modern DSGE models?:
- Core consists of six equations (variables expressed as deviations around zero-inflation steady state):

$$y_t^f = \left(\frac{1+\eta}{\eta+\sigma}\right) z_t$$

$$r_t^f = \sigma \mathrm{E}_t \left(y_{t+1}^f - y_t^f\right)$$

$$x_t \equiv y_t - y_t^f$$

$$x_t = \mathrm{E}_t x_{t+1} - \sigma^{-1}\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa\left(\eta+\sigma\right) x_t + \varepsilon_t$$

$$i_t = a_\pi \pi_t + a_x x_t.$$

- These determine the flex-price output, the natural real interest rate, the output gap, inflation, and the nominal interest rate.

The NK model: Does only the short-rate matter?

- NK expectational IS curve relates output gap (essentially consumption) to the policy rate $i_{t.}$.
- Central banks typically control a short-term interest rate.
  - In U.S., Federal Reserve sets target for the federal funds rate (currently 0.25 percent).
  - Fed funds rate is an overnight rate.
- Does this mean long-term interest rates do not matter? No.

Solving expectational equations forward

- Consider an equation of the form

$$p_t = d_t + \beta \mathrm{E}_t p_{t+1}.$$

- If this holds for all $t + i$, $i \geq 0$, then

$$\mathrm{E}_t p_{t+1} = \mathrm{E}_t d_{t+1} + \beta \mathrm{E}_t p_{t+2}.$$

- Therefore,

$$\begin{aligned} p_t &= d_t + \beta \mathrm{E}_t p_{t+1} \\ &= d_t + \beta \mathrm{E}_t d_{t+1} + \beta^2 \mathrm{E}_t p_{t+2} \end{aligned}$$

Solving expectational equations forward

- Keep recursively substituting for the future p_{t+i}:

$$\begin{aligned} p_t &= d_t + \beta \mathrm{E}_t p_{t+1} \\ &= d_t + \beta \mathrm{E}_t d_{t+1} + \beta^2 \mathrm{E}_t p_{t+2} \\ &= d_t + \beta \mathrm{E}_t d_{t+1} + \beta^2 \mathrm{E}_t d_{t+2} + \beta^3 \mathrm{E}_t p_{t+3} \\ & \quad .... \\ &= \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i d_{t+i} + \lim_{T \to \infty} \beta^T p_T \\ &= \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i d_{t+i}. \end{aligned}$$

- p_t depends on the present discounted value of the current and future values of d_{t+i}.

Solving the expectational IS curve forward

$$
\begin{aligned}
x_t &= \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right) \\
&= \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(r_t - r_t^f\right).
\end{aligned}
$$

- This implies

$$
\begin{aligned}
x_t &= -\left(\frac{1}{\sigma}\right)\left(r_t - r_t^f\right) + \mathrm{E}_t x_{t+1} \\
&= -\left(\frac{1}{\sigma}\right)\left(r_t - r_t^f\right) - \left(\frac{1}{\sigma}\right)\left(r_{t+1} - r_{t+1}^f\right) + \mathrm{E}_t x_{t+2} \\
&\cdots \\
&= -\left(\frac{1}{\sigma}\right)\mathrm{E}_t \sum_{i=0}^{\infty}\left(r_{t+i} - r_{t+i}^f\right).
\end{aligned}
$$

  - Changes in the one-period rate that are persistent, so that they also influence expectations of future interest rates, will have stronger effects on x_t then more temporary changes in r.

The real interest rate gap

- Basic model:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t$$

$$i_t = a_\pi \pi_t + a_x x_t .$$

- The policy rule cannot be arbitrary;
- Suppose $a_x = 0$. Then there is a unique equilibrium if and only if $a_\pi > 1$.
  - This known as the Taylor Principle.
  - Central bank needs to respond more than one-for-one to inflation.

The Taylor Principle

- Basic model:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t$$

- The impact of monetary policy on output and inflation operates through the real rate of interest;
- The real interest rate gap $i_t - \mathrm{E}_t \pi_{t+1} - r_t^n$ summaries the impact of monetary policy.

Other channels of monetary transmission

The role of money

- So far, monetary policy only works via the real interest rate gap.
- No direct role for money.
- Central bank has to allow the money supply to adjust to clear teh money market at its targeted interest rate.

Other channels of monetary transmission (to be discussed later)

Money can also affect the economy via

- Credit channels
- Exchange rate channels

Special case – serially uncorrelated shock

- When shocks are i.i.d., expected future output gap and inflation equal zero.
- Model becomes

$$x_t = -\left(\frac{1}{\sigma}\right)\left(i_t - r_t^n\right)$$

$$\pi_t = \kappa x_t + u_t$$

$$i_t = a_\pi \pi_t + v_t.$$

- Substituting the policy rule into the IS equation gives a two equation model:

$$x_t = -\left(\frac{1}{\sigma}\right)\left(a_\pi \pi_t + v_t - r_t^n\right)$$

$$\pi_t = \kappa x_t + u_t$$

Special case – serially uncorrelated shock

$$x_t = -\left(\frac{1}{\sigma}\right)\left(a_\pi \pi_t + v_t - r_t^n\right)$$

$$\pi_t = \kappa x_t + u_t$$

- First equation has a negative slope in $(x\ \pi)$ space: $\pi_t = -\left(\sigma x_t + r^n - v_t\right)/a_\pi$
  - Slope depends in $\sigma/\alpha_\pi - -$ policy affects the slope.
- Second equation has a policy slope in in $(x\ \pi)$ space: $\pi_t = \kappa x_t + u_t$.

Simulations

- Basic model with shocks:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + u_t$$

$$i_t = a_\pi \pi_t + a_x x_t + v_t.$$

- Parameter values: $\sigma = 1$, $\beta = 0.99$, $\kappa = 0.25$, $a_\pi = 1.5$, $a_x = 0$.
- Assume r_t^n, u_t and v_t are independent AR(1) processes.
- See dynare mod file: NKM.dyn

Summary

- Can derive simple model consistent with optimizing agents and nominal rigidities;
- Interest rate rules cannot be arbitrary – Taylor principle is important;
- Transmission via real interest rate gap;

New Keynesian Part 3

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Real marginal cost and the output gap

- The firm's real marginal cost is equal to the real wage it faces divided by the marginal product of labor: $mc_t = \omega_t / Z_t$.
- Because nominal wages have been assumed to be completely flexible, the real wage must equal the marginal rate of substitution between leisure and consumption plus any wage markup: $\hat{\omega}_t = \hat{w}_t - \hat{p}_t = \eta \hat{n}_t + \sigma \hat{c}_t$.
- Recalling that $\hat{c}_t = \hat{y}_t$, $\hat{y}_t = \hat{n}_t + z_t$,

$$
\begin{aligned}
mc_t &= (\eta \hat{n}_t + \sigma \hat{y}_t) - \hat{z}_t = [\eta (\hat{y}_t - z_t) + \sigma \hat{y}_t] - z_t \\
&= (\sigma + \eta) \left[\hat{y}_t - \left(\frac{1+\eta}{\eta+\sigma} \right) z_t \right] \\
&= (\sigma + \eta) \left(\hat{y}_t - \hat{y}_t^f \right) \\
&= (\sigma + \eta) x_t .
\end{aligned}
$$

Real marginal cost and the output gap

- Using these results, the inflation adjustment equation becomes

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t$$

  where $\kappa = (\eta + \sigma)(1 - \omega)(1 - \beta\omega)/\omega$ and $x_t \equiv \hat{y}_t - \hat{y}_t^f$ is the gap between actual output and the flexible-price equilibrium output.
- This inflation adjustment or forward-looking Phillips curve relates output, in the form of the deviation around the level of output that would occur in the absence of nominal price rigidity, to inflation.

The basic new Keynesian model

- Three equation system (treating r_t^f and y_t^f as exogenous):

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(\hat{\imath}_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t$$

$$\hat{y}_t = \hat{y}_t^f + x_t$$

- Consistent with
  - optimizing behavior by households and firms
  - budget constraints
  - market equilibrium
- Three equations but three unknowns: x_t, π_t,and i_t – need to specify monetary policy

Closing the model by specifying monetary policy

- Could add money demand condition from household's optimization problem and treat nominal quantity of money as the policy instrument.
- But, major central banks use the nominal interest rate as their chief policy instrument (ignoring ZLB).
- Can we close model with any rule for setting i_t? No – Taylor Principle must be satisfied.

$$i_t = rr_t^f + \phi\pi_t,\ \phi > 1.$$

Basic NK model with interest rate rule

- Case of serially uncorrelated shocks r_t^f and e_t.
- Use policy rule to eliminate i_t:

$$x_t = -\left(\frac{1}{\sigma}\right)\phi\pi_t$$

$$\pi_t = \kappa x_t + e_t$$

- Two equations in two unknowns, x_t and π_t.
- Modern day equivalent to textbook IS-LM-AS model.
  - No LM as policy determines i and lets the money supply adjust endogenously.

Basic NK model with interest rate rule

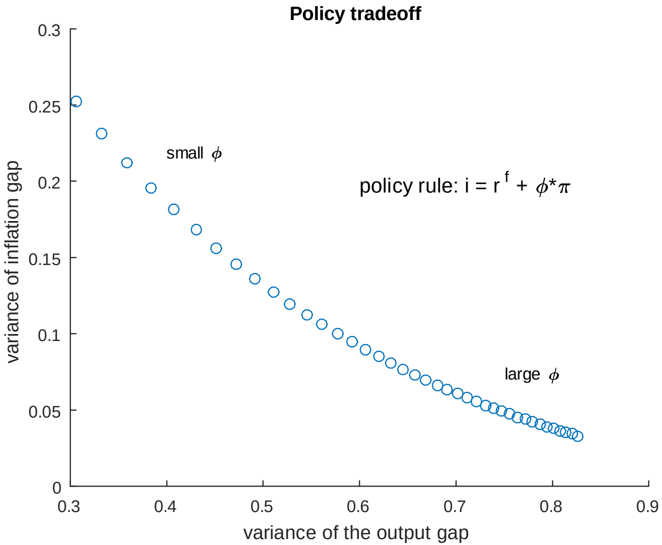
- Case of serially uncorrelated shocks r_t^f and e_t.
- Solving the model yield

$$x_t = -\left(\frac{\phi}{\sigma + \phi\kappa}\right) e_t$$

$$\pi_t = \left(\frac{\sigma}{\sigma + \phi\kappa}\right) e_t$$

- Trade-off in volatilities.

Policy trade-off frontier



Basic NK model with interest rate rule

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t$$

$$i_t = \phi_\pi \pi_t + \phi_x x_t + v_t$$

- With serially correlated shocks, easier to solve numerically.
  - See NKM.dyn

Optimal monetary policy in the NK model

The basic new Keynesian model

Three core elements:

1. Expectational IS curve, expressed in terms of the output gap:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

2. An expectational Phillips Curve:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t$$

3. A describtion of monetary policy.

Optimal policy in the basic new Keynesian model

- Policy objectives – assume central bank wants to minimizeAssume objective is to minimize

$$\frac{1}{2}\mathrm{E}_t\sum_{i=0}^{\infty}\beta^i\left(\pi_{t+i}^2+\lambda x_{t+i}^2\right).$$

- Inflation creates relative price dispersion that is inefficient.
  - Consumers respond to relative prices, so inflation leads to an inefficicent combination of goods being bought to produce the consumption composite that yields utility.
  - Essentially, more labor hours needed to achieve the same C_t.
- Risk averse households do not like output gap volatility.

Policy implications of price stickiness

- When the price level fluctuates, and not all firms are able to adjust, price dispersion results. This causes the relative prices of the different goods to vary. If the price level rises, for example, two things happen.
  1. The relative price of firms who have not set their prices for a while falls. They experience in increase in demand and raise output, while firms who have just reset their prices reduce output. This production dispersion is inefficient.
  2. Consumers increase their consumption of the goods whose relative price has fallen and reduce consumption of those goods whose relative price has risen. This dispersion in consumption reduces welfare.
- The solution is to prevent price dispersion by stabilizing the price level by keeping inflation equal to zero.

Key distortion: goods market clearing

- Output is

$$Y_t = \int c_{jt} dj = Z_t \int N_{jt} dj = Z_t N_t$$

- But

$$Y_t = \int c_{jt} dj = C_t \int \left(\frac{p_{jt}}{P_t} \right)^{-\theta} dj = C_t \Delta_t \,,$$

where

$$\Delta_t = \int \left(\frac{p_{jt}}{P_t} \right)^{-\theta} di \geq 1$$

is a measure of price dispersion.

- Therefore,

$$\Delta_t^{-1} Y_t = C_t \leq Y_t.$$

- Since $\Delta_t \geq 1$, price dispersion means more has to be produced to achieve a given level of C_t.

Optimal policy under discretion

- Central bank treats expectations as given. Reduces to a sequence of one-period problems.
- Central bank wants to minimize

$$\frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right)$$

subject to

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t$$

and

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

- Expectations IS curve not a constraint – always move i_t to offset r_t^f.
  - Only a problem when a big negative r_t^f shock pushes i_t to zero.
  - Will discuss this case later.

Optimal policy under discretion

- Central bank wants to

$$\min_{\pi_t, x_t} \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right) + \psi\left(\pi_t - \beta \mathrm{E}_t \pi_{t+1} - \kappa x_t - e_t\right).$$

- Solve this problem then use expectational IS relationship to solve for i_t that is consistent with the desired values of inflation and the output gap.

Optimal policy under discretion

$$\min_{\pi_t, x_t} \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right) + \psi_t\left(\pi_t - \beta \mathrm{E}_t \pi_{t+1} - \kappa x_t - e_t\right).$$

- First order conditions – differentiate with respect to π and x and set equal to zero:

$$\pi_t + \psi_t = 0$$

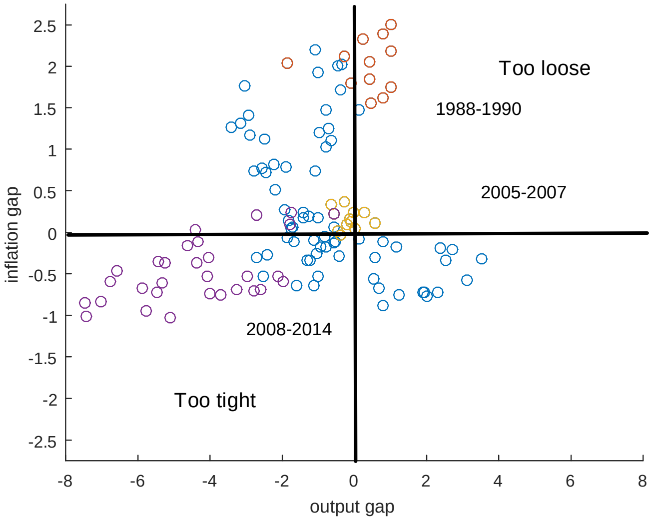
$$\lambda x_t - \psi_t \kappa = 0$$

- Eliminate ψ_t to obtain a *targeting criterion* or *targeting rule*:

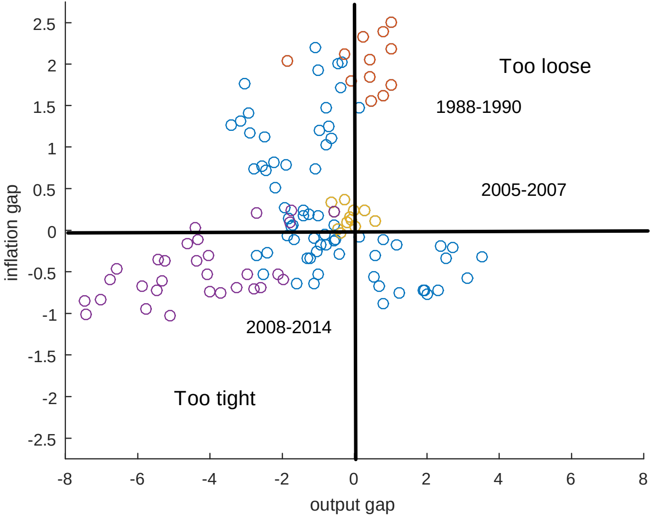
$$\lambda x_t + \kappa \pi_t = 0$$

- Keep this relationship between inflation and the output gap equal to zero.
  - If $\pi_t > 0$ (can think of this as inflation is above target), the output gap should be negative; if $\pi_t < 0$ (inflation is below target), the output gap should be positive.
  - If $\pi_t > 0$ and $x_t > 0$, policy is too loose; if $\pi_t < 0$ and $x_t < 0$, policy is too tight.

The Qvigstad graph



Assessing policy: the Qvigstad rule



The equilibrium under optimal discretion

- Three equations for inflation, the output gap, and the nominal interest rate are

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t$$

$$\lambda x_t + \kappa \pi_t = 0$$

Optimal precommitment

- The central bank's problem is to pick π_{t+i} and x_{t+i} to minimize

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left[\frac{1}{2} \left(\pi_{t+i}^2 + \lambda x_{t+i}^2 \right) + \psi_{t+i} \left(\pi_{t+i} - \beta \pi_{t+i+1} - \kappa x_{t+i} - e_{t+i} \right) \right].$$

- The first order conditions can be written as

$$\pi_t + \psi_t = 0 \tag{1}$$

$$\mathrm{E}_t \left(\pi_{t+i} + \psi_{t+i} - \psi_{t+i-1} \right) = 0 \qquad i \geq 1 \tag{2}$$

$$\mathrm{E}_t \left(\lambda x_{t+i} - \kappa \psi_{t+i} \right) = 0 \qquad i \geq 0. \tag{3}$$

- Dynamic inconsistency – at time t, the central bank sets $\pi_t = -\psi_t$ and promises to set $\pi_{t+1} = -\left(\mathrm{E}_t \psi_{t+1} - \psi_t \right)$. When $t+1$ arrives, a central bank that reoptimizes will again obtains $\pi_{t+1} = -\psi_{t+1}$ – the first order condition (1) updated to $t+1$ will reappear.

Timeless precommitment

- An alternative definition of an optimal precommitment policy requires the central bank to implement conditions (2) and (3) for all periods, including the current period so that

$$\pi_{t+i} + \psi_{t+i} - \psi_{t+i-1} = 0 \qquad i \geq 0$$

$$\lambda x_{t+i} - \kappa \psi_{t+i} = 0 \qquad i \geq 0.$$

- Woodford (2003) has labeled this the "timeless perspective" approach to precommitment.

Timeless precommitment

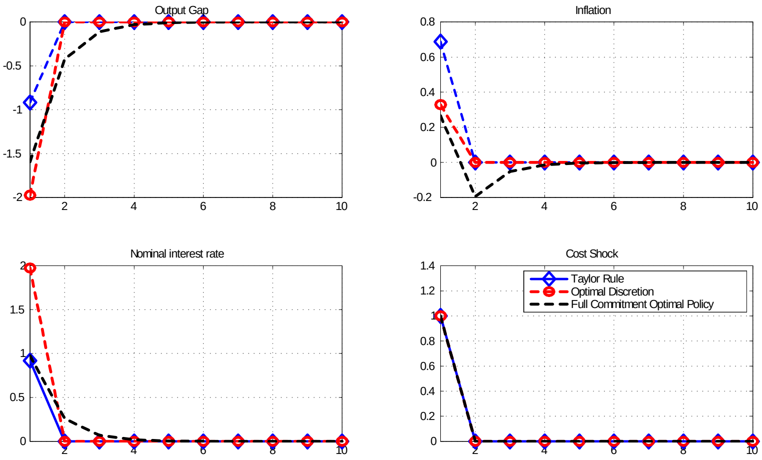
- Under the timeless perspective optimal commitment policy, inflation and the output gap satisfy

$$\pi_{t+i} = -\left(\frac{\lambda}{\kappa}\right)(x_{t+i} - x_{t+i-1}) \tag{4}$$

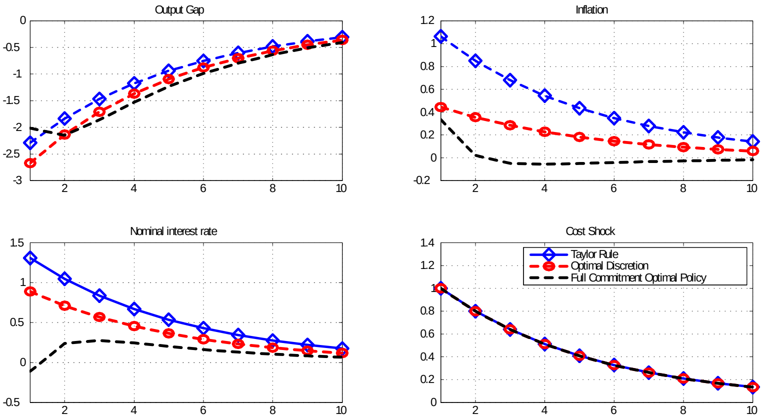
for all $i \geq 0$.

- Even if e is serially uncorrelated so that there is no natural source of persistence, the optimal commitment policy introduces inertia into the output gap and inflation processes.
- This commitment to inertia implies that the central bank's actions at date t allow it to influence expected future inflation. Doing so leads to a better trade-off between gap and inflation variability than would arise if policy did not react to the lagged gap.

Illustrating commitment versus discretion in the simple NK model



The case of a persistent cost shock



Improved trade-off under commitment

- The difference in the stabilization response under commitment and discretion is the stabilization bias due to discretion.
- Consider a positive inflation shock, $e > 0$.
- A given change in current inflation can be achieved with a smaller fall in x if expected future inflation can be reduced:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t$$

- Requires a commitment to future deflation (i.e., inflation below target if we interpret π as inflation relative to target).
- By keeping output below potential (a negative output gap) for several periods into the future after a positive cost shock, the central bank is able to lower expectations of future inflation. A fall in $\mathrm{E}_t \pi_{t+1}$ at the time of the positive inflation shock improves the trade-off between inflation and output gap stabilization faced by the central bank.

Summing up – issues to keep in mind

- Forward looking aspects of private sector behavior implies optimal commitment policies react to past states.
- Have so far have ignored the ZLB.
- Have so far have ignored sticky wages.

Policy weights

- Theory says something about the weights in the loss function:

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i V_{t+i} \approx -\Omega \mathrm{E}_t \sum_{i=0}^{\infty} \frac{1}{2} \beta^i \left[\pi_{t+i}^2 + \lambda \left(x_{t+i}^f - x_t^* \right)^2 \right],$$

where

$$\Omega = \frac{1}{2} \bar{Y} U_c \left[\frac{\omega}{(1-\omega)(1-\omega\beta)} \right] \left(\theta^{-1} + \eta \right) \theta^2$$

and

$$\lambda = \left[\frac{(1-\omega)(1-\omega\beta)}{\omega} \right] \frac{(\sigma + \eta)}{(1+\eta\theta)\,\theta}.$$

- Greater nominal rigidity (larger ω) reduces λ.
- Greater elasticity of demand (θ) reduces weight on output gap.
- Calvo specification implies λ is small – Taylor specification leads to larger weight on output gap.

Optimal policy

- The solution is to prevent price dispersion by stabilizing the price level.
- What is critical for this result is that nominal wages are assumed to be completely flexible.
- But the same argument would apply if wages are sticky and prices flexible. With sticky wages and flexible prices, monetary policy should stabilizes the nominal wage.

Sticky prices and wages

- When both wages and prices display stickiness, life becomes more complicated for the policy authority.
- Welfare costs arise from price dispersion *and* from wage dispersion.
- Approximation to the welfare of representative agent is now equal to the expected present discounted value of

$$L_t = \lambda_p \pi_t^2 + \lambda_w \pi_{wt}^2 + \lambda_x \left(x_t^f - x_t^* \right)^2$$

where π_w is wage inflation, $\lambda_p + \lambda_w = 1$, and

$$\frac{\lambda_w}{\lambda_p} \propto \frac{\kappa_p}{\kappa_w}.$$

- Place more weight on stabilizing wage inflation if wages are stickier than prices.

Sticky prices and wages

- Stabilizing the price level is no longer a sufficient guideline for monetary policy.
- The reason is that if the price level is stabilized, sticky wages will prevent the real wage from adjusting in response to shocks. So inefficient output variability would be introduced. Similarly, stabilizing the wage level still leaves prices sticky so real wages cannot jump.

The new Keynesian Model: part 4

ELB and conventional policies

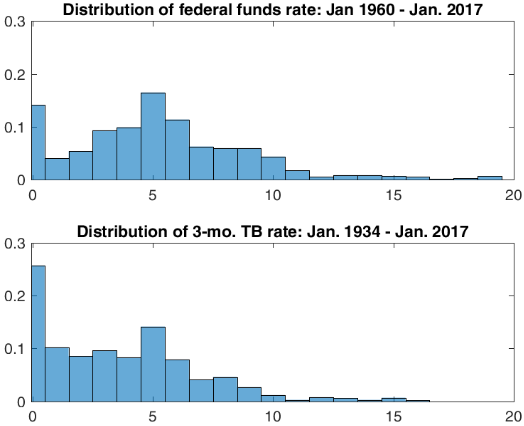
Carl E. Walsh

UC Santa Cruz

Winter 2017

The Effective Lower Bound and Conventional Policies

The effective lower bound (ELB)



Back to basic model

- Two equation system

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t$$

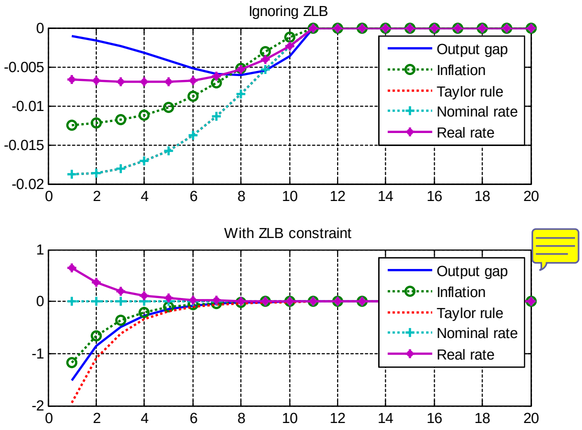
- Suppose nominal rate is at its ELB (assumed to be zero). Then system becomes

$$x_t = \mathrm{E}_t x_{t+1} + \left(\frac{1}{\sigma}\right)\left(\mathrm{E}_t \pi_{t+1} + r_t^f\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t.$$

- Negative demand shock r_t^f is not neutralized.

Demand shock in the face of the ZLB



Is the ELB a constraint?

- Is the ELB a constraint on monetary policy?
- Conventional model based on an expectational IS relationship:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right)$$

- Interest rates – both current and expected future matter:

$$\begin{aligned} x_t \quad = \quad & -\left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1}\right) - \left(\frac{1}{\sigma}\right)\mathrm{E}_t \sum_{i=1}^{\infty}\left(i_{t+i} - \pi_{t+1+i}\right) \\ & + \left(\frac{1}{\sigma}\right)\mathrm{E}_t \sum_{i=0}^{\infty} r_{t+i}^f, \end{aligned}$$

- Reflects a narrow view of the transmission mechanism – no role for quantitative easing or credit easing policies.

Conventional instruments at the ELB

- Even at the ELB, policy has the potential to influence real spending if it can affect expectations of future real interest rates.
- If $i_t = 0$ and is expected to remain at zero until $t + T$, then

$$x_t = \left(\frac{1}{\sigma}\right)\sum_{i=0}^{T}\mathrm{E}_t\pi_{t+1+i} - \left(\frac{1}{\sigma}\right)\mathrm{E}_t\sum_{i=T+1}^{\infty}\left(i_{t+i} - \pi_{t+1+i}\right) + \left(\frac{1}{\sigma}\right)\mathrm{E}_t\sum_{i=0}^{\infty} r^f_{t+i}.$$

Future path of interest expectation

Exogenous shock

- Raising expected future inflation or committing to lower future nominal rates can stimulate current spending.

Simple example: Discretionary equilibrium

- Suppose economy hit by $r_t^n = r^{bad} < 0$.
  - With probability q, economy exits the ZLB in following period. With probability $1 - q$, $r_{t+1}^n = r^{bad}$.
  - Once economy exits, optimal policy under discretion sets $i = r^n$ so that $\pi = x = 0$.
- Equilibrium given by solution to

$$\pi^{zlb} = \beta(1-q)\pi^{zlb} + \kappa x^{zlb}$$

$$x^{zlb} = (1-q)x^{zlb} + \left(\frac{1}{\sigma}\right)\left[(1-q)\pi^{zlb} + r^{bad}\right]$$

Example

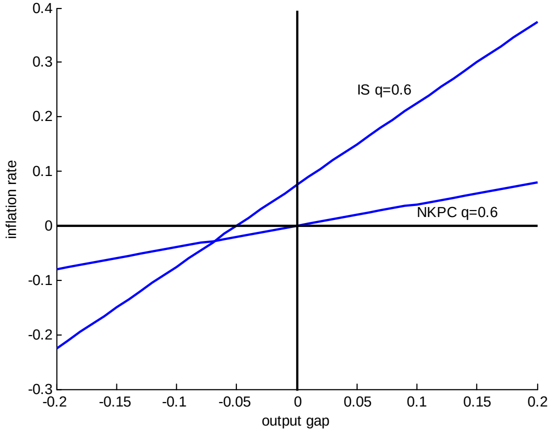


Figure: Equilibrium at the ZLB; q is the probability of exiting in following period.

Example

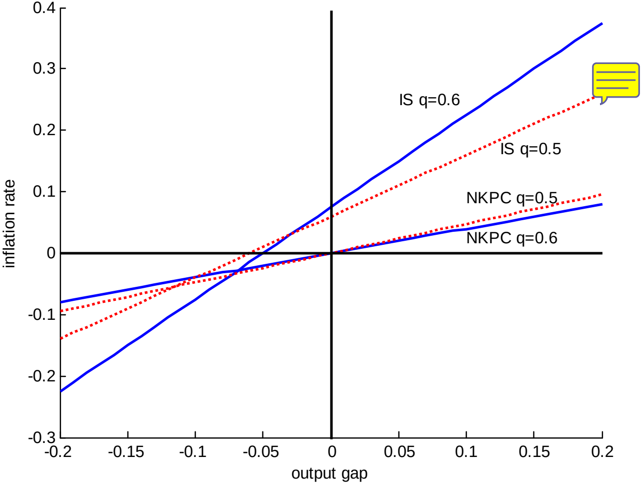


Figure: Equilibrium at the ZLB; effects of a fall in q (a rise in probability of remaining at the ZLB).

Promising future inflation and forward guidance

- Credibly promising to keep future interest rates at zero is a powerful tool.
- Consider simple model at ZLB:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1}\right)$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t.$$

$$i_{t+i} = \begin{cases} i^* \text{ for } i = 1, , , T \\ \phi_\pi \pi_t + \phi_x x_t \text{ for } i = T+1, ... \end{cases}$$

- With no inflation shocks, $\pi_{t+i} = x_{t+i} = 0$ for $i = T+1, ...$

Promising future inflation

- Equilibrium for $i = 1, ..., T$ is solution to

$$\begin{aligned}\left(\frac{1}{\kappa}\right)(\pi_t - \beta \mathrm{E}_t \pi_{t+1}) &= \left(\frac{1}{\kappa}\right)(\mathrm{E}_t \pi_{t+1} - \beta \mathrm{E}_t \pi_{t+2}) \\ &\quad - \left(\frac{1}{\sigma}\right)(i^* - \mathrm{E}_t \pi_{t+1})\end{aligned}$$

- Or

$$\pi_t = -\left(\frac{\kappa}{\sigma}\right) i^* + \left(1 + \beta + \frac{\kappa}{\sigma}\right) \mathrm{E}_t \pi_{t+1} - \beta \mathrm{E}_t \pi_{t+2}$$

with terminal condition $\pi_{t+T+1} = 0$.

Promising future inflation

- Consider a perfect foresight equilibrium.
- Solve backwards:

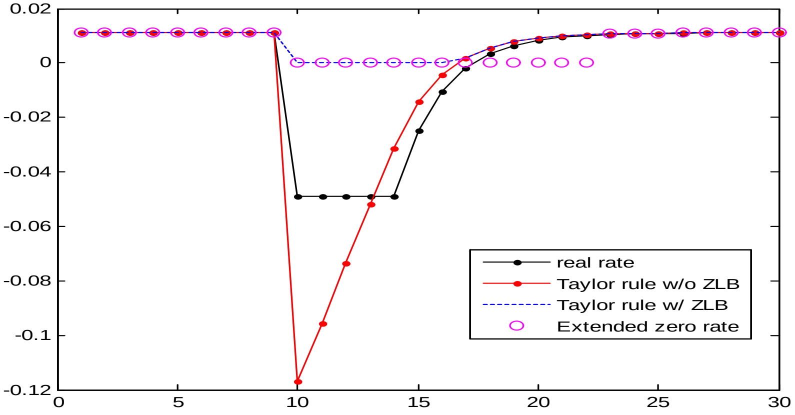
$$\pi_{t+T} = -\left(\frac{\kappa}{\sigma}\right) i^* + \left(1+\beta+\frac{\kappa}{\sigma}\right)\pi_{t+T+1} - \beta\pi_{t+T+2} = -\left(\frac{\kappa}{\sigma}\right) i^*$$

$$\pi_{t+T-1} = -\left(\frac{\kappa}{\sigma}\right) i^* + \left(1+\beta+\frac{\kappa}{\sigma}\right)\pi_{t+T} = -\left(2+\beta+\frac{\kappa}{\sigma}\right)\left(\frac{\kappa}{\sigma}\right) i^*$$

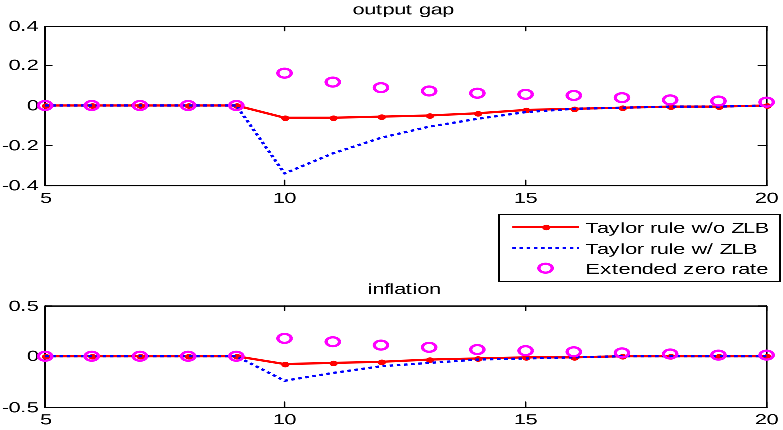
$$\pi_{t+T-2} = -\left(\frac{\kappa}{\sigma}\right) i^* + \left(1+\beta+\frac{\kappa}{\sigma}\right)\pi_{t+T-1} - \beta\pi_{t+T}$$

- For π_{t+T-s}, process is explosive as s gets larger.

Promising future inflation



Promising future inflation



Can forward guidance really be that powerful?

McKay, Nakamura, and Steinsson (NBER 20882, 2015).

- McKay, Nakamura, and Steinsson argue more realistic responses occur with idiosyncratic risk and incomplete markets that lead to precautionary savings.
- Approximate incomplete markets model by replacing standard Euler condition with a "discounted Euler equation":

$$x_t = \alpha \mathrm{E}_t x_{t+1} - \frac{\zeta}{\sigma}\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right),$$

with $0 < \alpha < 1$, $0 < \zeta < 1$.

- Combine with

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t$$

$$i_t = \max\left(0,\ r_t^n + \phi \pi_t\right),\ \phi > 1$$

Can forward guidance really be that powerful?

Table 3: How Severe a Constraint Is the ZLB?

	Output	Inflation
Standard Model	-14.3%	-10.5%
($\alpha = 1$, $\zeta = 1$)		
Discounted Euler Equation Model	-2.9%	-2.1%
($\alpha = 0.97$, $\zeta = 0.75$)		

Response of output and inflation when the natural rate falls to -2% (annualized) with a 10% per quarter probability of returning to normal.

Figure: From McKay, Nakamura, and Steinsson (2015)

Solutions to the ELB: regime change

- Replacing inflation targeting.
- Optimal policy at the ZLB can be implemented via a time-varying price-level target (Eggertsson and Woodford 2003).
- Intuition – under commitment, central bank has many tools even if current policy rate at zero.
  - Can promise future low interest rates and a boom.
  - This generates expectations of future inflation which raises current inflation and lowers current real interest rate.
- Promise to generate inflation to achieve price-level target.

Price level targeting

- Distorting objectives in a discretionary environment can improve outcomes (Walsh AER 1995).
- Central banks may find it easier to commit to objectives than to future policy actions.
- Vestin (JME 2006) shows price level targeting can replicate the timeless precommitment solution if the central bank is assigned the loss function $p_t^2 + \lambda_{PL} x_t^2$ in an environment of discretion.
  - Svensson (JMCB 1999), Vestin (JME 2006), work at the Bank of Canada (Cateau, et. al (2008), Dib, et. al. (2008), Kryvstov, et. al. (2008)).
- Under timeless precommitment,

$$\pi_t = (1 - L)p_t = \left(\frac{\lambda}{\kappa}\right)(x_t - x_{t-1}) \Rightarrow p_t = \left(\frac{\lambda}{\kappa}\right) x_t$$

- Price level targeting makes inflation expectations act as an automatic stabilizer.

Price level targeting

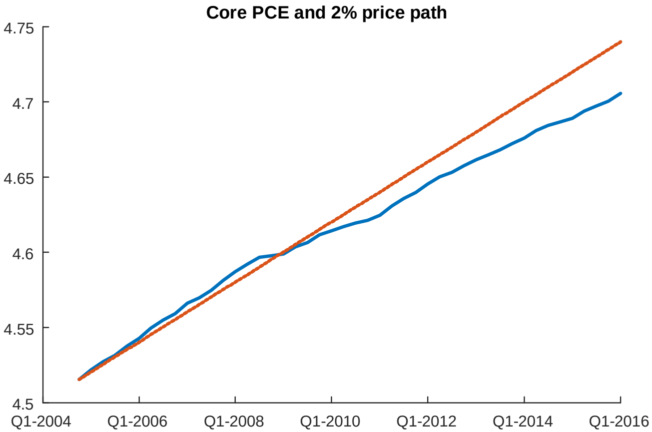


Figure: Price level target path and actual core PCE for the U.S (logs).

PLT: other considerations

- Advantages of PLT require that expectations act as automatic stabilizers.
  - Raises issues of credibility and learning
- Switching policy regimes in a crisis risks gains in credibility achieved by inflation targeters.
  - When adopted, the choice of price index, the underlying trend inflation rate, and the speed with which deviations from target path are expected to be reversed are all important.
- Walsh (AER 2003) adds lagged inflation to the inflation adjustment equation and shows that the advantages of price level targeting over inflation targeting decline as the weight on lagged inflation increases.

Other considerations

- Advantages of PLT or nominal income targeting require that expectations act as automatic stabilizers.
  - Raises issues of credibility and learning
  - Is it easier to commit to a policy framework such as PLT/NIT than it is to a future contingent path for policy?
- Switching policy regimes in a crisis risks gains in credibility achieved by inflation targeters.
  - Optimal commitment means doing what you had previously promised to do, even if it is not the optimal thing to do at the moment.

Nominal income targeting

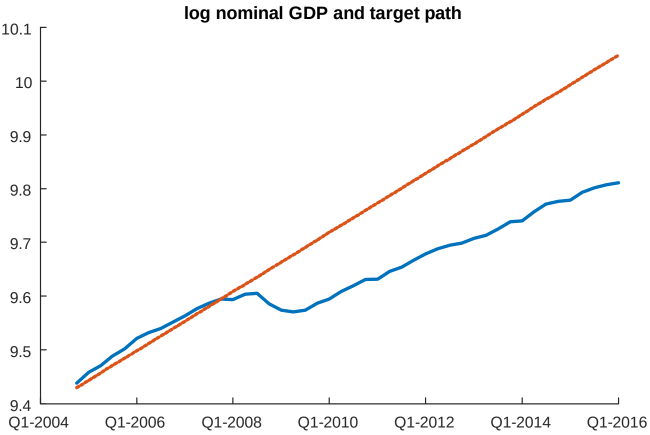


Figure: Nominal income in the U.S. and target path

The zero lower bound: other solutions for getting out

- Svensson's "foolproof way"
  - Depreciation as visible means of committing to a higher price level
- Fiscal policy:
  - "fiscal policy must be seen *not* to be committed to... conventional prescriptions for good fiscal policy...". (Sims 2000, p. 969, italics in original) – more on this later.
  - unconventional fiscal policies: Correia, et. al. (AER 2011) – time varying consumption taxes at the ZLB:

$$\zeta_t C_t^{-\sigma} = \beta \mathrm{E}_t \left(\frac{1+\tau_t^c}{1+\tau_{t+1}^c} \right) \zeta_{t+1} C_{t+1}^{-\sigma}.$$

    A negative demand shock (ζ_t low relative to ζ_{t+1} calls for rise in future consumption taxes.
- Quantitative easing/credit easing policies.

Reforming inflation targeting

Raising the inflation target

- Raising the inflation target raises average nominal interest rates and makes hitting the ELB less likely.
- Trade-off – steady-state loss of higher inflation against ability to improve stabilization.
- Schmitt-Grohe and Uribe (Handbook 2010), Billi (AEJ: Macro 2011), Coibion, Gorodnichenko, and Wieland (REStud 2012)
  - optimal π still small when ZLB taken into account.
- Will return to this issue later.

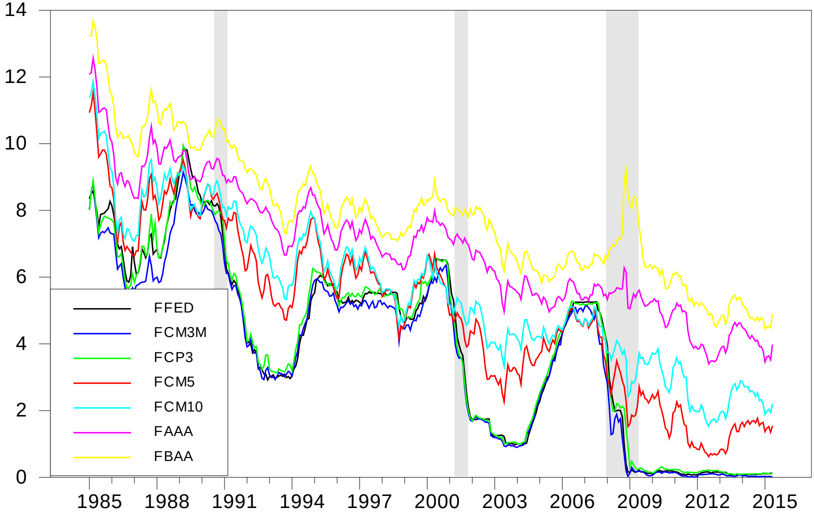
Econ 202: Slides on Spreads and Frictions

Carl Walsh

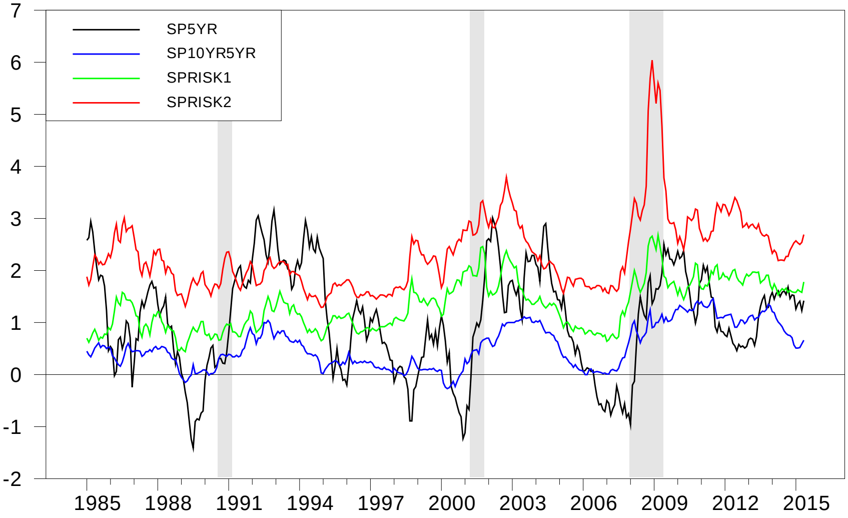
Winter 2017

Multiple interest rates and interest rate spreads

Short-term and long-term interest rates



Interest rate spreads



The new Keynesian framework with a single interest rate

- The expectational IS relationship:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right)$$

- The Phillips curve:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + u_t$$

- The policy rule:

$$i_t = r_t^n + \phi_\pi \pi_t$$

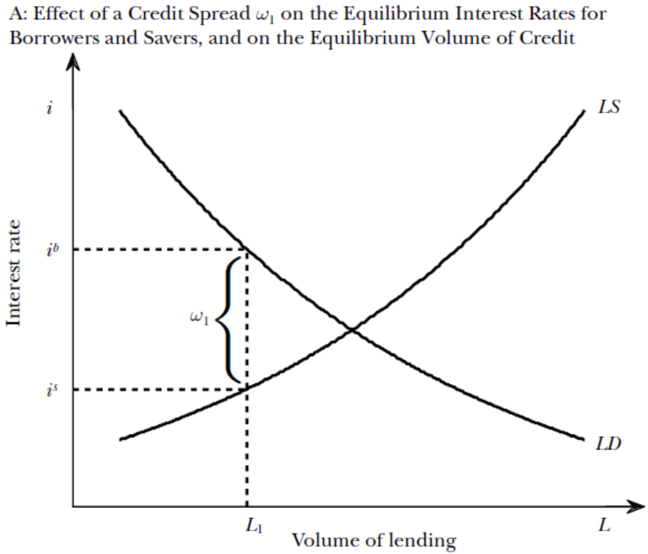
The new Keynesian framework with a single interest rate

- Implicit financial structure – savers lend directly to borrowers.
- Single interest rate adjusts to equilibriate market for borrowing and lending.
- This is the same rate the central bank affects.

Adding financial intermediaries

- Assume savers channel funds to firms through financial interemediaries.
  - Call intermediaries banks for short, but they can include all institutions that borrow directly from households and lend to firms and (other) households.
- Supply of funds to intermediaries depends on interest rate i_t^s available to savers.
- Demand for loans from intermedaries depends on interest rate i_t^b charged to borrowers.

The credit market and the credit spread



What determines the spread?

- The intermediary's marginal profit on a loan is a function of the spread.
  - The intermediary borrows at i^s and lends at i^b.
- In a perfectly competitive financial system, if $i^b > i^s + c$, where c represents the costs of intermediaries, intermediaries would have an incentive to expand lending.
  - This would push up i^s as intermediaries try to attract more funds and lower i^b as they try to lend more.

What determines the spread?

- The intermediary's marginal profit on a loan is a function of the spread.
  - The intermediary borrows at i^s and lends at i^b.
- In a perfectly competitive financial system, if $i^b > i^s + c$, where c represents the costs of intermediaries, intermediaries would have an incentive to expand lending.
  - This would push up i^s as intermediaries try to attract more funds and lower i^b as they try to lend more.
  - Equilibrium would have $i^b = i^s + c$.
- What determines c?

What determines the spread?

- Intermediary costs include cost of labor, capital, etc.
- But what else?
  - Time – assets maturity at differnt times in the future
  - Risk – assets have different risk characteristics
  - Financial frictions such as
    - Asymmetric information leading to adverse selection, agency costs, moral hazard.

The term structure of interest rates

Is it only the short-term policy rate that matters?

- No.
- Conventional model based on an expectational IS relationship:

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n\right)$$

- Interest rates – both current and expected future matter:

$$x_t = -\left(\frac{1}{\sigma}\right)\mathrm{E}_t \sum_{i=0}^{\infty}\left(i_{t+i} - \pi_{t+1+i}\right) + \left(\frac{1}{\sigma}\right)\mathrm{E}_t \sum_{i=0}^{\infty} r_{t+i}^n.$$

- How does the path of future policy rates relate to long term interest rates?

The expectations theory

- Under the expectations theory of the term structure, the n-period interest rate equals an average of the current short-term rate and the future short-term rates expected to hold over the n-period horizon.
- For example, if $i_{n,t}$ is the nominal yield to maturity at time t on an n-period discount bond, while i_t is the one-period rate, the pure expectations hypothesis in the absence of uncertainty would imply that

$$(1 + i_{n,t})^n = \prod_{i=0}^{n-1} (1 + i_{t+i}).$$

- This condition ensures that the holding period yield on the n-period bond is equal to the yield from holding a sequence of one-period bonds.

The expectations theory

$$(1+i_{n,t})^n = \prod_{i=0}^{n-1}(1+i_{t+i}).$$

- Taking logs of both sides and recalling that $\ln(1+x) \approx x$ for small x yields a common approximation:

$$i_{n,t} \approx \frac{1}{n}\sum_{i=0}^{n-1} i_{t+i}.$$

- Since an n-period bond becomes an $n-1$ period bond after one period, these two relationships can also be written as

$$(1+i_{n,t})^n = (1+i_t)\prod_{i=0}^{n-2}(1+i_{t+1+i}) = (1+i_t)\left(1+i_{n-1,t+1}\right)^{n-1}$$

and

$$i_{n,t} \approx \left(\frac{1}{n}\right) i_t + \left(\frac{n-1}{n}\right) i_{n-1,t+1}.$$

The expectations theory

- These condition will not hold exactly under conditions of uncertainty for two reasons.
  - First, if risk-neutral investors equate expected one-period returns, then the one-period rate $1 + i_t$ will equal $\mathrm{E}_t(1 + i_{n,t})^n / (1 + i_{n-1,t+1})^{n-1}$, which, from Jensen's inequality, is not the same as $(1 + i_{n,t})^n = (1 + i_t)\mathrm{E}_t(1 + i_{n-1,t+1})^{n-1}$.
    - Suppose $P_{n,t}$ is the time-t price of an n period discount bond. An n-period discount bond yields \$1 in n periods and sells today at a price $P_{n,t}$.
    - Then the return if held for n periods is $i_{n,t}$ such that

$$(1 + i_{n,t})^n P_{n,t} = 1.$$

    - Since at time $t + 1$ this becomes an $n - 1$ period bond, the one-period gross return is

$$1 + i_t = \mathrm{E}_t P_{n-1,t+1} / P_{n,t} = \mathrm{E}_t(1 + i_{n,t})^n / (1 + i_{n-1,t})^{n-1}.$$

  - Second, Jensen's inequality implies that $\ln \mathrm{E}_t(1 + i_{n-1,t+1})$ is not the same as $\mathrm{E}_t \ln(1 + i_{n-1,t+1})$.

The expectations theory

- We will ignore these two issues, however, to illustrate the basic linkages between the term structure of interest rates and monetary policy.
- It will be sufficient to simplify further by dealing only with one- and two-period interest rates.
- Letting $I_t \equiv i_{2,t}$ be the two-period rate (the long-term interest rate), the term structure equation becomes

$$(1 + I_t)^2 = (1 + i_t)(1 + \mathrm{E}_t i_{t+1}), \tag{1}$$

  and this will be approximated as

$$I_t = \frac{1}{2}\left(i_t + \mathrm{E}_t i_{t+1}\right). \tag{2}$$

- The critical implication of this relationship for monetary policy is that the current term structure of interest rates will depend on current short-term rates *and* on market expectations of future short rates.
- Since the short rate is set by monetary policy, I_t will depend on expectations about future policy.

Expected inflation and the term structure

- Since market interest rates are the sum of an expected real return and an expected inflation premium, the nominal interest rate on an n-period bond can be expressed as

$$i_t^n = \frac{1}{n}\sum_{i=0}^{n} \mathrm{E}_t r_{t+i} + \frac{1}{n}\mathrm{E}_t \bar{\pi}_{t+n}$$

  where $\mathrm{E}_t r_{t+i}$ is the one-period real rate expected at time t to prevail at $t+i$ and $\mathrm{E}_t \bar{\pi}_{t+n} \equiv \mathrm{E}_t p_{t+n} - p_t$ is the expected change in log price from t to $t+n$.

Expected inflation and the term structure

$$i_t^n = \frac{1}{n}\sum_{i=0}^{n} \mathrm{E}_t r_{t+i} + \frac{1}{n}\mathrm{E}_t \bar{\pi}_{t+n}$$

- If real rates are stationary around a constant value $\bar{r}$, then $\frac{1}{n}\sum_{i=0}^{n} \mathrm{E}_t r_{t+i} \approx \bar{r}$ and

$$i_t^n \approx \bar{r} + \frac{1}{n}\mathrm{E}_t \bar{\pi}_{t+n}.$$

- In this case, fluctuations in the long rate will be caused mainly by variations in expected inflation.

Macro finance and affine models

Micro finance

- The finance literature uses the idea of latent (unobserved) factors to model the term strurure.
- In recent years, these latent factors have been integrated into macro models, with the latent factors identified with macro factors.
- The term structure is represented using affine, no-arbitrage models. The unobserved latent variables that determine bond prices in these models are linked to macroeconomic variables, either through nonstructural statistical models or by using a new Keynesian model to represent macroeconomic and monetary policy outcomes.

Affine models (affine = linear)

- Suppose there are two latent (unobserved) factors that determine bond prices.
- Denote these factors by L_t and S_t and assume they follow a joint time series process given by

$$F_t \equiv \begin{bmatrix} L_t \\ S_t \end{bmatrix} = \rho \begin{bmatrix} L_{t-1} \\ S_{t-1} \end{bmatrix} + \Sigma e_t = \rho F_{t-1} + \Sigma e_t, \quad (3)$$

where e_t is independently and identically distributed as a normal, mean zero, unit variance process, and Σ is a 2×2 nonsingular matrix.

Affine models

- Assume further that we can write the short-term interest rate i_t as a function of the two factors. Specifically,

$$i_t = \delta_0 + \delta_1 F_t. \tag{4}$$

Finally, assume the prices of risk associated with each factor are linear functions of the two factors, so that if $\Lambda_{i,t}$ is the price of risk associated with conditional volatility of factor i,

$$\Lambda_t = \begin{bmatrix} \Lambda_{L,t} \\ \Lambda_{S,t} \end{bmatrix} = \lambda_0 + \lambda_1 F_t. \tag{5}$$

Affine models

- If i_t is the return on a one-period bond, than the structure given by (3) - (5), together with the assumption that no arbitrage opportunities exist, allows one to price longer term bonds.
- In particular, if $b_{j,t}$ is the log price of a j-period nominal bond, one can show that

$$b_{j,t} = \bar{A}_j + \bar{B}_j F_t,$$

where

$$\bar{A}_1 = -\delta_0; \ \bar{B}_1 = -\delta_1$$

and for $j = 2, ...J$,

$$\bar{A}_{j+1} - \bar{A}_j = \bar{B}_j \left(-\Sigma\lambda_0\right) + \frac{1}{2}\bar{B}_j \Sigma\Sigma' \bar{B}_j + \bar{A}_1;$$

$$\bar{B}_{j+1} = \bar{B}_j \left(\rho - \Sigma\lambda_1\right) + \bar{B}_1.$$

Affine models: the macro factors

- Empirical research aimed at estimating this type of no-arbitrage model generally finds that one factor affects yields at all maturities and so is called the level factor, while the other factor affects short and long rates differently and so is called the slope factor.
- The macro-finance literature has attempted to identify the level and slope factors with macroeconomic factors.

Affine models: the macro factors

- For example, in new Keynesian models, the short-term interest rate is often represented in terms of a Taylor rule of the form

$$i_t = r^* + \pi_t^T + a_\pi \left(\pi_t - \pi_t^T \right) + a_x x_t,$$

where π_t^T is the central bank's inflation target and x_t is the output gap.

  - In this case, changes in the inflation target should affect nominal interest rates at all maturities by altering inflation expectations. It is a prime candidate for the level factor.
  - The slope factor might then be capturing the central bank's policy actions intended to stabilize the economy in the short-run.

- Thus, one could model the factors explicitly in terms of the policy behavior of the central bank.

Adverse selection and the market for lemons

The market for lemons

- Akerlof (QJE 1970) showed how markets can fail if problems of asymmetric information become too severe.
- Example of used car market
  - Assume there are good used cars and bad used cars (lemons).
  - Assume everyone agrees a good used car is worth \$10,000 and a lemon is worth \$2,000.
- If buyers and sellers can tell which used car is good and which is a lemon, then two markets will coexist. One for good used cars with an average price of \$10,000 and one for lemons with an average price of \$2,000.

The market for lemons

- Now assume buyers cannot tell a good used car from a bad one.
- If the proportion of good used cars in the population of all used cars is 50%, then buyers would be willing to pay approximately $6,000 for a used car.
  - This is its expected value – if buyers are risk averse, they would be willing to pay somewhat less than $6,000.
  - Because our focus is on asymmetric information, not risk aversion, assume everyone is risk neutral.

The market for lemons

- Now assume the owners of used cars know whether their car is a lemon or not, but buyers cannot tell the difference.
  - This is the asymmetric information.
- If the market price were \$6,000, no owner of a good used car would want to sell (why accept \$6,000 for a good used car worth \$10,000), and only owners of lemons would want to sell their cars (since they much prefer \$6,000 to a lemon worth only \$2,000).
- The lemons drive the good used cars off the market – the price falls to \$2,000 and only lemons are available on the market.
  - Example of adverse selection.
- Markets can fail to function if asymmetric information becomes too big of a problem.

The market for lemons: financial market example

- Suppose a bank holds some mortgage backed securities – some are good, some are toxic.
- A potential buyer can't tell which are good and which are toxic.
- If the market price reflects some sort of average value of these securities, then the only ones a bank will want to sell are the toxic ones.
- Knowing (or fearing) that the only mortgage backed securities being offered for sale are the toxic ones, no one is willing to buy them.

Simple loan market model of adverse selection

- Consider a simple loan ar an interest rate r.
  - L is the loan amount.
  - C is the borrower's collateral.
- Risky project: return is R:

$$R = \begin{cases} R' - x \text{ with probability } \frac{1}{2} \\ R' + x \text{ with probability } \frac{1}{2} \end{cases}$$

- An increase in x is an increase in the project's riskiness. (Changes variance but does not change expected return.)

Return to the lender

- Lender is repaid, i.e., receives $(1 + r)\,L$ is

$$(1 + r)L \leq R + C.$$

- If

$$(1 + r)L > R + C,$$

  assume the lender gets $R + C$.

Return to the lender

- Assume
$$(1 + r)L > R' - x + C,$$
so borrower defaults in bad state (when $R = R' - x$).
- Expected profit to lender is
$$\mathrm{E}\pi^L = \frac{1}{2}(1 + r)L + \frac{1}{2}\left(R' - x + C\right)$$
which is decreasing in x (i.e., expected return to lender is lower for riskier projects).

Return to the borrower

- In the good state, the borrow keeps

$$R' + x - (1 + r)L.$$

- In the bad state, borrower defaults and loses

$$-C.$$

- Expected profit to the borrower is

$$\mathrm{E}\pi^B = \frac{1}{2}\left[R' + x - (1 + r)L\right] - \frac{1}{2}C.$$

  Notice that the expected return to the borrow is increasing in x – i.e., expected return to the borrower is higher for risky projects because the downside loss to the borrower is limited to $-C$ while the borrower gets to keep the good outcomes.

Cutoff level of riskiness

- Define x^* such that

$$\mathrm{E}\pi^B(x^*) = \frac{1}{2}\left[R' + x^* - (1+r)L\right] - \frac{1}{2}C = 0.$$

  Or

$$x^*(r, L, C, R') = (1+r)L + C - R'.$$

- Expected profit to the borrower is positive for all $x > x^*$.

Adverse selection

$$x^*(r, L, C, R') = (1 + r)L + C - R'.$$

- x^* is increasing in r. Some borrowers with less risky projects (lower x) will find it unprofitable to borrow when r increases.
- A rise in r changes the mix of borrowers – fewer safe borrowers take out loans as r rises.

Adverse selection

- Suppose there are two types of borrower: safe borrowers with x_g and risky borrowers with $x_b > x_g$.
- If the loan rate is such that

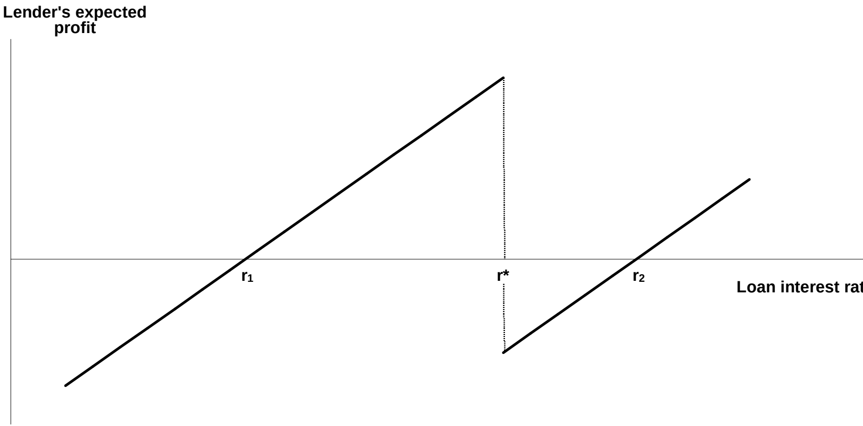
$$x^*(r,\, L,\, C,\, R') < x_g < x_b,$$

  then both types borrow and the lender's profits are increasing in r.
- But once r rises such that

$$x_g < x^*(r,\, L,\, C,\, R') < x_b,$$

  the safe borrowers drop out and the lender's profits fall.

Adverse selection



Agency costs and borrowing constraints

Why are there limitations of how much one can borrow?

- Suppose the opportunity cost of funds is r.
- If lenders are risk neutral, then any firm with an investment project whose expected return is $R > r$ should be able to borrow to undertake the project.
- But often how much a firm can borrow depends on how much of the project they can finance with their own funds.
  - Cost of external financing is greater than the cost of internal financing.
- Often the amount that can be borrowed is limited even if $R > r$.
- Why?
- Useful to read section 9.9 of Romer (2012).

Financial accelerator effects: cash flow

- Financial frictions may amplify the effects of shocks to the economy.
- Positive shock that boosts firms' cash flow makes it easier to borrow.
  - This further increases aggregate demand, output and asset prices.
- A negative shock that leads to a fall in cash flow makes it hard to borrow.
  - This decline in borrowing further decreases aggregate demand, output and asset prices.
- Evidence suggests investment depends on things like a firm's cash flow. Why?

Cash flow, costly state verification and agency costs

- In the model of Bernanke and Gertler (1989), firms are assumed to be able to costlessly observe the outcome of their own investment projects; others must incur a monitoring cost to observe project outcomes.
- The optimal lending contract must maximize the expected payoff to the firm, subject to several constraint.
  - First, the lender's expected return must be at least as great as her opportunity cost rB.
  - Second, the firm must have no incentive to report the bad state when if fact the good state occurred.
- Since auditing is costly, the optimal auditing probability is just high enough to ensure the firm truthfully reports the good state when it occurs.

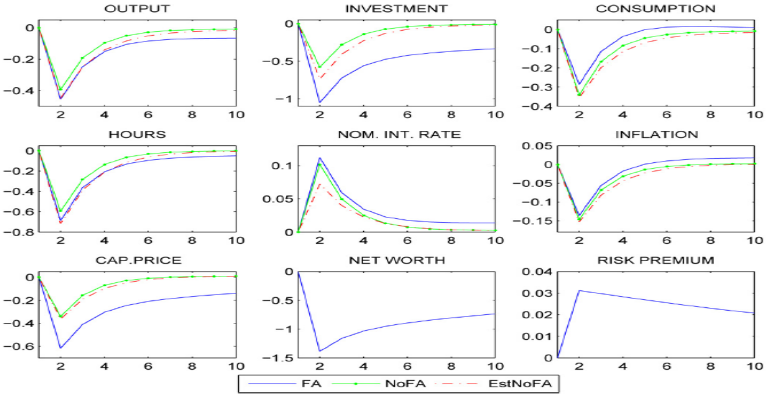
Agency costs

- Costly state verification model suggested agency costs a source of the spread.
- Expected loan cost to firm is the intermediaries opportunity cost plus expected auditing costs.
- The probability of auditing is decreasing with the amount of funds the borrower can put into the project.
- So a fall in firms' cash flow increases agency costs and widens the spread. Lending falls.

Does the financial accelerator matter?

Christensen-Dib (RED 2008)

166 *I. Christensen, A. Dib / Review of Economic Dynamics 11 (2008) 155–178*

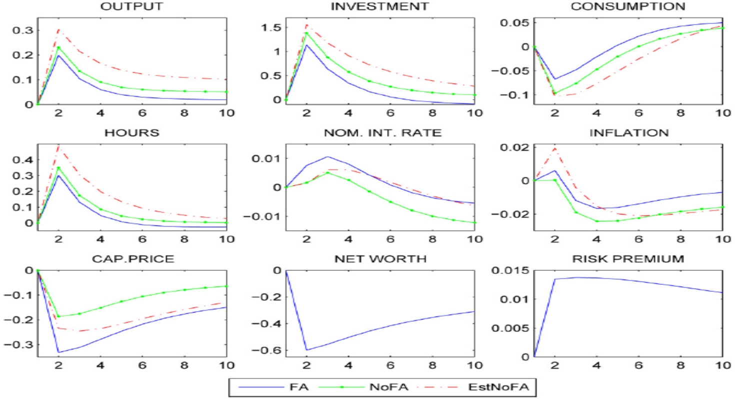


Note: The responses are percentage deviations of a variable from its steady-state value.

Fig. 1. The economy's responses to a tightening monetary policy shock.

Does the financial accelerator matter?

Christensen-Dib (RED 2008)



Note: The responses are percentage deviations of a variable from its steady-state value.

Fig. 5. The economy's responses to a positive investment-efficiency shock.

Moral hazard and borrowing constraints

Banking model with moral hazard

- Introduce agency problem between borrowers and lenders – creates a wedge between external finance cost and opportunity cost of internal finance.
- Assume financial intermediaries have skills in evaluating borrowers – makes it efficient for credit to flow through intermediaries.
- Households deposit funds with intermediaries, intermediaries lend to non-financial firms.
- Agency problem limits ability of intermediary to obtain funds from depositors. This accounts for wedge between deposit and loan rates. Spread widens in a crisis which raises cost of funds to non-financial firms.

The model: banks

- Banks raise funds in a national financial market.
  - Consists of a retail market in which banks raise funds from households and a wholesale market in which they raise funds from other banks.
- Each period, bank raises d_t in deposits at rate R_{t+1}.
- After retail market closes, investment opportunities are determined (i.e., it is like a limited participation model).

The agency problem

- After receiving deposit funds, banker invests (buys assets, makes loans).
- Balance sheet is

$$n_t + d_t + b_t = s_t$$

where d are deposits, n the bank's own capital, b are funds raised in wholesale credit markets, and s_t is it's asset holdings.

The agency problem

- The banker is able to divert a fraction θ of its assets.
  - Divertible assets are

$$s_t - \omega_d d_t - \omega_b b_t,\ 0 < \omega_d, \omega_b < 1$$

  - If $\omega_d = \omega_b = 1$, divertible assets are $s_t - d_t - b_t = n_t$, i.e., it can only divert the capital it already owns.
  - Assume $\omega_d < \omega_b$ – banks can more easily divert depositors' funds.

The agency problem

- If bank diverts funds, it fails and creditors obtain $1-\theta$ of the funds, i.e., they get what's left.
- Because of bank's incentive to divert funds, creditors will limit the amount they lend. This means banks will face a borrowing constraint.
- Let $V(s_t, b_t, d_t)$ be value of the bank – i.e., the value to the owners of continuing in business.
- Then to ensure bank does not divert funds, the *incentive constraint*

$$V(s_t, b_t, d_t) \geq \theta\,(s_t - \omega_d d_t - \omega_b b_t)$$

  must hold.
- No creditor would lend to bank if this condition isn't satisfied.

Bank balance sheet and valuation

- Guess the value function for a bank is linear:

$$V(s_t, b_t, d_t) = v_{s,t}s_t - v_{b,t}b_t - v_{d,t}d_t \tag{6}$$

  - v_{st} is the value to bank at end of period of additional unit of assets;
  - v_{bt} is marginal cost of wholesale (interbank or private credit market) debt;
  - v_t is marginal cost of deposits.
- In frictionless financial markets, we would have $v_{s,t} = v_{b,t} = v_{d,t}$.

Bank's decision problem

- Let λ_t be Lagrangian multiplier on the incentive constraint.
- The bank's problem (since $n_t = s_t - d_t - b_t$):

$$
\begin{aligned}
& Max_{d_t, s_t, b_t} \{ [v_{s,t} s_t - v_{b,t} b_t - v_{d,t} d_t] \\
& + \lambda_t [v_{s,t} s_t - v_{b,t} b_t - v_{d,t} d_t - \theta (s_t - \omega_d d_t - \omega_b b_t)] \}
\end{aligned}
$$

- Use balance sheet to eliminate s_t:

$$
Max_{d_t, b_t} \{ [v_{s,t} (n_t + d_t + b_t) - v_{b,t} b_t - v_{d,t} d_t] \\
+ \lambda_t \left[\begin{array}{c} v_{s,t} (n_t + d_t + b_t) - v_{b,t} b_t - v_{d,t} d_t \\ -\theta (s_t - \omega_d d_t - \omega_b b_t) \end{array} \right] \Bigg\}
$$

Bank's decision problem

$$Max_{d_t,b_t}\left\{\left[v_{s,t}\left(n_t+d_t+b_t\right)-v_{b,t}b_t-v_{d,t}d_t\right]\right.$$
$$\left.+\lambda_t\left[v_{s,t}\left(n_t+d_t+b_t\right)-v_{b,t}b_t-v_{d,t}d_t-\theta\left(s_t-\omega_d d_t-\omega_b b_t\right)\right]\right\}$$

- FOC with respect to d_t:

$$\left(v_{s,t}-v_{d,t}\right)\left(1+\lambda_t\right)=\lambda_t\theta\left(1-\omega_d\right) \tag{7}$$

  - If $\theta = 0$ – no moral hazard problem, then

$$v_{s,t}=v_{d,t}.$$

  - Bank raises deposit funds until marginal cost of a dollar of deposits equals the marginal return on a dollar of assets.
  - This would be the outcome with perfect capital markets.
  - $v_{s,t}$ also equals $v_{d,t}$ if the incentive constraint is not binding ($\lambda_t = 0$) or $\omega_d = 1$ – can't divert depositors' funds.

Bank's decision problem

- FOC with respect to b_t:

$$\left(v_{s,t} - v_{b,t}\right)\left(1+\lambda_t\right) = \lambda_t \theta \left(1-\omega_b\right).$$

  - If $\theta = 0$ – no moral hazard problem, then

$$v_{s,t} = v_{b,t}.$$

  - Bank raises wholesale funds until marginal cost of a dollar of borrowing equals the marginal return on a dollar of assets.
  - This would be the outcome with perfect capital markets.
  - $v_{s,t}$ also equals $v_{b,t}$ if the incentive constraint is not binding ($\lambda_t = 0$) or $\omega_b = 1$ – can't divert lenders' funds.

Interest rate spreads

- How do the marginal costs of raising funds from different sources compare?
- Subtracting FOC for b from FOC for d:

$$\left(v_{b,t} - v_{d,t}\right)\left(1 + \lambda_t\right) = \lambda_t \theta \left(\omega_b - \omega_d\right) > 0$$

- Marginal cost of fund sources are equal, $v_{b,t} = v_{d,t}$, if $\lambda_t = 0$, $\theta = 0$, or $\omega_b = \omega_d$. i.e.,
  - if incentive constraint non-binding;
  - if there is no moral hazard problem;
  - if frictions are same for both sources of funding.

Incentive constraint

- The incentive constraint was

$$v_{s,t}s_t - v_{b,t}b_t - v_{d,t}d_t \geq \theta\left(s_t - \omega_d d_t - \omega_b b_t\right)$$

- Since $d_t = s_t - b_t - n_t$, the incentive constraint becomes

$$v_{s,t}s_t - v_{b,t}b_t - v_{d,t}\left(s_t - b_t - n_t\right) \geq \theta\left[\begin{array}{c} s_t - \omega_d\left(s_t - b_t - n_t\right) \\ -\omega_b b_t \end{array}\right]$$

$$\begin{aligned} n_t \quad &\geq \quad \left[\frac{\theta\left(1-\omega_d\right)-\left(v_{s,t}-v_{d,t}\right)}{v_{d,t}+\theta\omega_d}\right]s_t \\ &\quad - \left[\frac{\theta\left(\omega_b-\omega_d\right)-\left(v_{b,t}-v_{d,t}\right)}{v_{d,t}+\theta\omega_d}\right]b_t \end{aligned}$$

- Value of net worth must be at least as large as a weighted measure of the bank's asset holdings and debt (s_t and b_t).
  - Weights are related to the value of diverting the asset.

Incentive constraint: special cases

$$n_t \geq \left[\frac{\theta\left(1-\omega_d\right)-\left(v_{s,t}-v_{d,t}\right)}{v_{d,t}+\theta\omega_d}\right] s_t - \left[\frac{\theta\left(\omega_b-\omega_d\right)-\left(v_{b,t}-v_{d,t}\right)}{v_{d,t}+\theta\omega_d}\right] b_t$$

- Suppose only depositors' funds can be diverted so $\omega_b = 1$ and assume $\omega_d = 0$.
- Then
$$\left[\frac{\theta-\left(v_{s,t}-v_{d,t}\right)}{v_{d,t}}\right]\left(s_t-b_t\right) \leq n_t$$
since $v_{s,t} = v_{b,t}$ when $\omega_b = 1$.

Incentive constraint: special cases

- Rewrite
$$\left[\frac{\theta - (v_{s,t} - v_{d,t})}{v_{d,t}}\right](s_t - b_t) \leq n_t$$
as
$$s_t - b_t \leq \left[\frac{v_{d,t}}{\theta - (v_{s,t} - v_{d,t})}\right] n_t \equiv \phi_t n_t .$$
- Using (7),
$$s_t - b_t \leq (1+\lambda)\left[\frac{v_{d,t}}{\theta}\right] n_t \equiv \phi_t n_t .$$

Incentive constraint: special cases

- Recalling that $s_t - b_t = n_t + d_t$,

$$n_t + d_t \leq (1+\lambda)\left[\frac{v_{d,t}}{\theta}\right] n_t$$

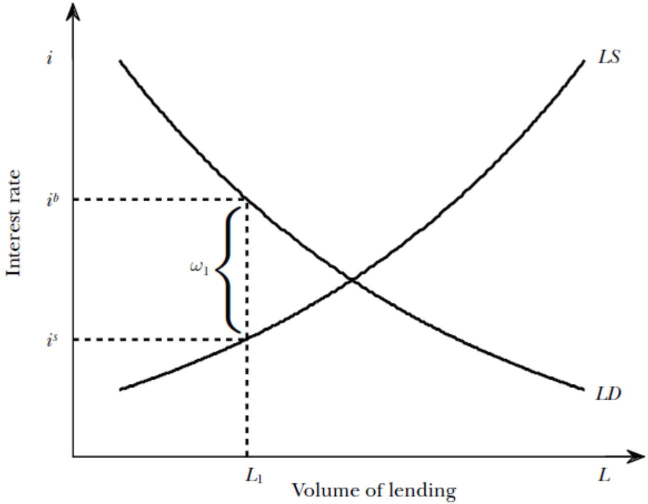
or

$$d_t \leq \left[\frac{(1+\lambda)\, v_{d,t} - \theta}{\theta}\right] n_t \equiv \phi_t n_t$$

- Financial frictions limit leverage: $\partial \phi_t / \partial \theta < 0$.
- Borrowing constraint arises as moral hazard problem limits ability to borrow.
- Tightness of this constraint depends on the extent to which bank can divert assets (θ) and on excess value of bank assets ($v_{s,t} - v_{d,t}$). The greater this is, the greater is the franchise value of the bank and the less likely it is to divert funds.

The credit market and the credit spread

A: Effect of a Credit Spread ω_1 on the Equilibrium Interest Rates for Borrowers and Savers, and on the Equilibrium Volume of Credit



Limits to leverage

- Model of moral hazard implied a limit to how much the financial intermediary can borrow.
- Treating all the bank's debt as equivalent,

$$i^b - i^s = \frac{\lambda}{1+\lambda}\theta \geq 0$$

  where θ was the fraction of assets that could be diverted and λ was the Lagrangian multiplier on the incentive constraint.
- Limits to leverage were a function of intermediaries capital.
  - Fall in bank equity would increase spread and reduce lending.

Econ 202: Slides on Bank Runs, QE, and Fiscal Policy

Carl Walsh

March 2017

Diamond-Dybvig (1983)

Defining an illiquid asset

- Invest at time $T = 0$.
- Value at $T = 1$ is r_1.
- Value at $T = 2$ is r_2.
- $r_1 < r_2$.
- $\frac{r_1}{r_2}$ measures liquidity – the lower this is, the less liquid the asset.

Investors

- Investors do not know how long they will want to hold the asset – uncertain horizon.
- At $t = 0$, investor does not know whether he/she will want to consume (need to liquidate the asset) at $T = 1$ or $T = 2$.
- This need to consume is private information – otherwise insurance markets could develop.
- Type 1 investors consume at time $T = 1$; type 2 consume at $T = 2$.
- At time $T = 0$, investors do not know which type they will be, but they know they have a probability t of being type 1 and $1 - t$ of being type 2.

Investors

- Suppose there are 100 investors, each has 1 unit to invest. There are 25 type 1's so $t = 1/4$.
- An investor holding asset (r_1, r_2) consumes r_1 if type 1 and r_2 if type 2 for expected utility of

$$tU(r_1) + (1-t)U(r_2).$$

- Assume

$$U(c) = 1 - \left(\frac{1}{c}\right).$$

So marginal utility is $1/c^2 > 0$ but diminishing in c.

Assets

- Asset a (the illiquid asset) has $r_1^a = 1$, $r_2^a = R$. Asset b (the more liquid asset) has $r_1^b > 1$ and $r_2^b < R$.
- Suppose $R = 2$, $r_1^b = 1.28$, and $r_2^b = 1.813$. (reason for these values explained later.)
- Expected utility from holding asset a:

$$\frac{1}{4}U(1) + \frac{3}{4}U(2) = 0.375.$$

Assets

- Expected utility from holding asset b:

$$\frac{1}{4}U(1.28) + \frac{3}{4}U(1.813) = 0.391 > 0.375.$$

- All investors prefer to hold the liquid asset even though its expected payoff is lower:

$$\text{expected payoff to } a\text{: } \frac{1}{4}(1) + \frac{3}{4}(2) = 1.75;$$

$$\text{expected payoff to } b\text{: } \frac{1}{4}(1.28) + \frac{3}{4}(1.813) = 1.68 < 1.75.$$

- Because investors do not know when they will need to liquidate their investment, and they are risk averse, they prefer asset b even though its expected return is lower. The lower return is compensated for by the greater liquidity.

Bank created liquidity

- Banks can create more liquidity by holding the less liquid asset and issuing more liquid liabilities.
- Bank invests in asset a (with $r_1^a = 1$ and $r_2^a = 2$).
- Bank offers deposits at $T = 0$ that offers a payout of $r_1^d = 1.28$ to investors who withdraw at $T = 1$ and a payout $r_2^d = 1.813$ to those who withdraw at $T = 2$.
- Suppose bank receives \$100 in deposits from the 100 investors. Bank buys 100 units of asset a.

Bank created liquidity

- Suppose at $T = 1$ the 25 type 1 investors withdraw 1.28 each, for total withdrawals of $25 \times 1.28 = 32$.
- The bank has to liquidate 32% of its portfolio. It still has 68% left in asset a.
- At $T = 2$, bank receives $68 \times 2 = 136$ which allows it to pay each of the remaining 75 depositors

$$\frac{138}{75} = 1.813$$

as promised.

Bank created liquidity

- Investors prefer the more liquid asset. The bank is able to offer a new asset (the deposit) that is as liquid but pays a higher return at $T = 1$.
- The bank is only holding the illiquid asset – the bank has created the liquid asset.
- Liquidity transformation – transforming the illiquid asset into a liquid asset – is an important service provided by banks.
- This is an equilibrium – because if all investors expect 25 to withdraw at $T = 1$, it is optimal for the remaining 75 not to withdraw since the type 2 investors prefer 1.813 available at $T = 2$ to the 1.28 available at $T = 1$.

Bank created liquidity

- An individual needs all or none of her liquidity. A bank knows it will only need a fraction t of its liquidity.
- What's in it for banks? Note that investors would also prefer a deposit contract that paid out $(1.28, 1.8)$ since

$$\frac{1}{4}U(1.28) + \frac{3}{4}U(1.8) = 0.388 > 0.375.$$

  This means the bank could keep $0.13 \times 68 = 0.884$ as profit.
- But competition and free entry in banking would drive r_2^b up to 1.813.

Bank runs

- Banks can have liquidity problems – if too may depositors want their money out at $T = 1$, the bank will not have enough assets to payoff the remaining depositors at $T = 2$.
- Suppose at $T = 1$ the 30 investors (25 type 1 investors and 5 type 2 investors) withdraw 1.28 each, for total withdrawals of $30 \times 1.28 = 38.4$. The bank has to liquidate 38.4% of its portfolio. It still has 61.6% left in asset b. At $T = 2$, bank receives $61.6 \times 2 = 123.2$ which allows it to pay each of the remaining 70 depositors

  $$\frac{123.2}{75} = 1.76 < 1.813$$

  So this is less than the bank had promised.
- If the depositor type were verifiable, it could be written into the contract that only type 1 investors can withdraw at $T = 1$. But type is assumed to be private information.

Multiple equilibria

1. The good equilibrium – only type 1 investors withdraw at $T = 1$.
2. The bad equilibrium – all investors withdraw at $T = 1$ – this is the bank run.

- Consider the example above in which 30 investors withdraw. Generalize this and assume a fraction f of all depositors withdraw, where $f \geq 1/4$ (since type 1 investors always withdraw).
- Each investor must form a forecast of f since the return at $T = 2$ will depend on f.
- Let $\hat{f}$ be the forecast.
- In a Nash equilibrium, there is a self-fulfilling equilibrium with $f = \hat{f}$, i.e., based on a forecast that $\hat{f}$ will withdraw, a fraction $f = \hat{f}$ actually do withdraw. One Nash equilibrium is $f = \hat{f} = 1/4$; the good equilibrium.

Multiple equilibria

- If you are one of the other $1 - f$ type 2 investors, what is your best strategy if you think $\hat{f}$ investors will withdraw their funds?
- Your expected return at $T = 2$ if you do not withdraw will be

$$\frac{2(1 - \hat{f} r_1^b)}{1 - \hat{f}}.$$

- This has to be greater than 1.28 – otherwise you would withdraw at $T = 1$.
- The bank will fail if it has to pay out more than 100 at $T = 1$, so that is the most it can get if it liquidates its entire portfolio. So if $\hat{f} > 78$, the bank will fail since $79 \times 1.28 = 101.12 > 100$.
- Note that $\hat{f} = 79$ is not an equilibrium. If you are one of the 21 investors not withdrawing, and you believe $\hat{f} = 79$, then the best strategy is for you to also withdraw. So $f = \hat{f}$ is an equilibrium. This is a bank run.

Multiple equilibria

- In an earlier example, we saw that if $\hat{f} = 30$, it make sense for the remaining 70 investors to leave their funds in the bank, because they expect to receive 1.76, which is less than they were promised (i.e., 1.813) but greater than 1.28, which is what they would get by withdrawing.
- Let $\hat{f}^*$ be the critical tipping point. That is, $\hat{f}^*$ is the point such that

$$\frac{2(1 - \hat{f}^* r_1^b)}{1 - \hat{f}^*} = 1.28 \Rightarrow \hat{f}^* = 0.5625.$$

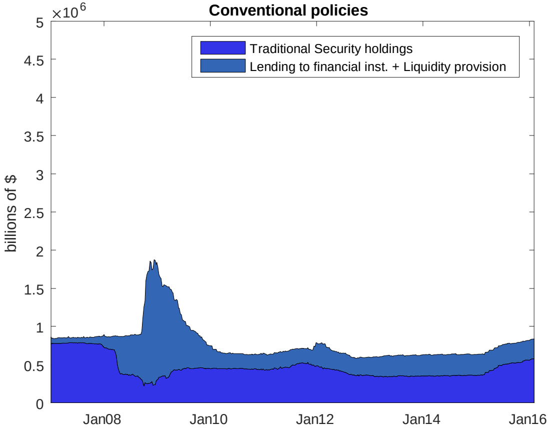
  So if $\hat{f} \geq 0.5625$, there is a run on the bank.

Multiple equilibria

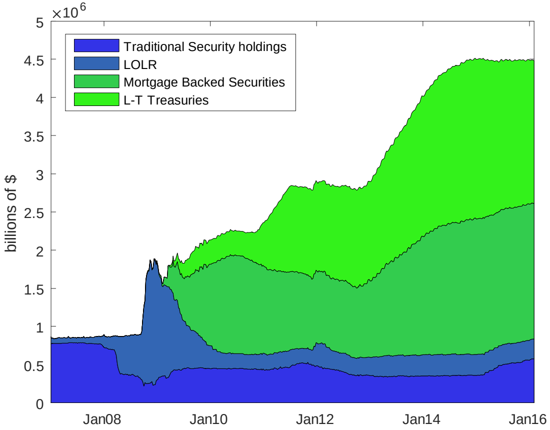
- What could shift $\hat{f}$ from 0.25 to 0.56? A bad news story. Even if most people knew the story were untrue, it is in their interests to try to withdraw.
  - Banks use to have big lobbies so teller lines wouldn't spill out the door – it they did, it might cause a run if everyone though others were withdrawing.
  - Suspension of convertibility – banks could announce they will only allow 25 depositors to withdraw and shut the doors to the 26th. This supports the good equilibrium as no type 2 wants to withdraw (only the 25 type 1 investors want to and they are able to). But if the number of type 1 investors fluctuates, suspending convertibility might prevent some type 1 investors with withdrawing. This is socially costly.
- Deposit insurance can stop runs. Need an outside agent like the government to fund it.
- Central bank lending could avoid need for banks to liquidate the illiquid asset (which is always a good asset – i.e., the banks are solvent).

QE and balance sheet policies

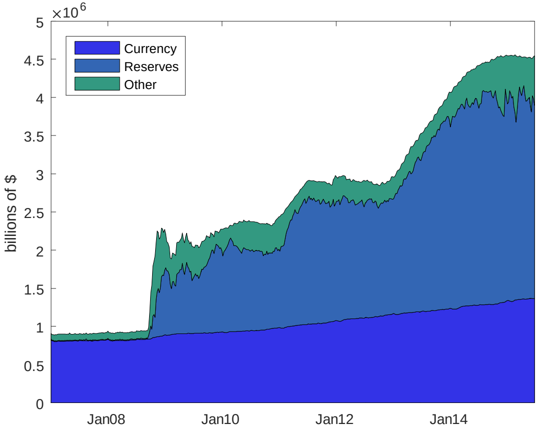
The Fed's balance sheet: liquidity provision



The Fed's balance sheet: total asset holdings



The Fed's balance sheet: total liabilities



Balance sheet of the central bank

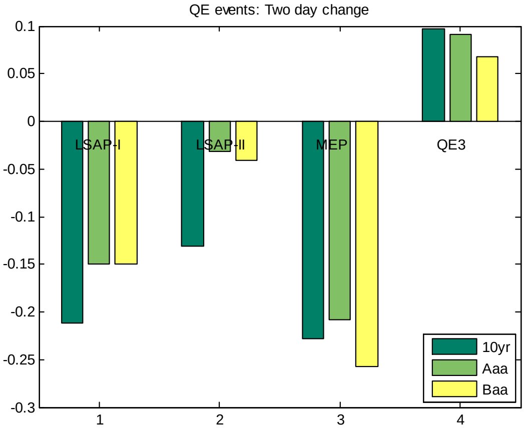
Assets	Liabilities and net worth
S-T Treasury securities	Currency
L-T Treasury securities	Reserves
Loans	Treasury deposits
Other assets	Capital

Balance sheet policies by the Fed

Table 4: Key Federal Reserve QE dates

QE1	Nov. 2008	Purchase of L-T Treasuries, MBS, agency debt
	March 2009	QE1 Expanded
	March 2010	QE1 ended
QE2	Nov. 2010-June 2011	Purchase of US Treasuries
MEP	Sept 2011-Dec. 2012	Extended ave. maturity of Treasury holdings.
QE3	Sept 2012-ongoing	Purchase of \$85 b/mo. of L-T Treasuries and MBS.
Tapering	6/6/2013	Bernanke suggests tapering to start in Sept.
	9/18/2013	FOMC does not begin taper
	12/18/2014	FOMC begins taper

Two day effects of QE announcements



Modigliani-Miller and open market operations

- Wallace (AER 1981) demonstrated a Modigliani-Miller result for open market operations.
- Price of asset j with payoffs $x_j(s)$ in future state s is

$$p_j = \sum_{s=1}^{S} \pi(s) m(s) x_j(s) = \sum_{s=1}^{S} \pi(s) \frac{\beta U_c\left[c(s)\right]}{U_c(c_t)} x_j(s)$$

  where $\pi(s)$ is the probability of state s.
- Asset prices are independent of the central bank's balance sheet – open market operations and the form they take (short-term gov't debt, long-term gov't debt, private assets) are irrelevant.
- Conditions needed: (1) assets are valued only for the pecuniary returns and (2) all investors can purchase arbitrary quantities of the same assets at the same prices.

Translating to the NK framework

- Suppose household utility depends on consumption, leisure, and money holdings.
- Household can hold one-period bonds, two-period bonds, and money:

$$y_t + T_t + W_t \geq c_t + p_{1,t} b_{1,t} + p_{2,t} b_{2,t} + m_t$$

  where

$$W_t \equiv \left(\frac{1}{1+\pi_t}\right)\left[b_{1,t-1} + p_{1,t} b_{2,t-1} + m_{t-1}\right].$$

- Let $Y_t \equiv y_t + T_t$.

Translating to the NK framework

- FOC for household maximizing expected discounted utility includes (where λ_t is MUC)

$$p_{1,t} \geq \beta \mathrm{E}_t \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left(\frac{1}{1+\pi_{t+1}} \right), \text{ with equality if } b_1 > 0$$

$$p_{2,t} \geq \beta \mathrm{E}_t \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left(\frac{p_{1,t+1}}{1+\pi_{t+1}} \right), \text{ with equality if } b_2 > 0$$

$$1 \geq \frac{U_{m,t}}{\lambda_t} + \beta \mathrm{E}_t \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left(\frac{1}{1+\pi_{t+1}} \right), \text{ with equality if } m > 0$$

Translating to the NK framework

- If all assets held, budget constraint can be solved forward to yield

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) Y_{t+i} + W_t = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) c_{t+i} + \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) \left(1 - p_{1,t+i}\right) m$$

$$1 + i_t = \frac{1}{p_{1,t}} \Rightarrow 1 - p_{1,t} = \frac{i_t}{1 + i_t}$$

$$U_{c,t} = \beta \mathrm{E}_t \left(\frac{1 + i_t}{1 + \pi_{t+1}} \right) U_{c,t+1}$$

$$\frac{U_{m,t}}{U_{c,t}} = 1 - p_{1,t} = \left(\frac{i_{1,t}}{1 + i_{1,t}} \right)$$

Budget identities

- In general, one can show that the budget constraint can be written as

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) (y_{t+i} + T_{t+i}) + W_t = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) c_{t+i} + \Omega_t ,$$

  where

$$\Omega_t \equiv \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) \left(\Delta_{1,t+i} b_{1,t+i} + \Delta_{2,t+i} b_{2,t+i} + \Delta_{m,t+i} m_{t+i} \right) .$$

- The $\Delta' s$ are wedges beteween price and discounted pecuniary return.

Pricing wedges

$$\Omega_t \equiv \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) \left(\Delta_{1,t+i} b_{1,t+i} + \Delta_{2,t+i} b_{2,t+i} + \Delta_{m,t+i} m_{t+i} \right).$$

- $\Delta_{m,t+i} = 1 - p_{1,t+i}$.
- If short-term debt held only for precuniary return, $\Delta_1 = 0$. In this case, b_1 drops out.
- Under the pure expectations hypothesis of the term structure, $\Delta_{2,t+i} = 0$ for all i.
  - In this case, $b_{2,t+i}$ drops out.
  - It is irrelevant whether changes in m are engineered via open market operations involving one-period or two-period debt.
- As long as $\Delta_{m,t+i} \neq 0$ for all i, changes in m may matter.

Pricing wedges

$$\Omega_t \equiv \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left(\frac{\lambda_{t+i}}{\lambda_t} \right) \left(\Delta_{1,t+i} b_{1,t+i} + \Delta_{2,t+i} b_{2,t+i} + \Delta_{m,t+i} m_{t+i} \right).$$

- The spirit of a preferred habitat model such as Vayanos and Vila (NBER 2009) can be captured by assuming the household's holdings of the two-period bond directly affects utility.
- In this case, $\Delta_{2,t} = U_{b,t} / U_{C,t}$ where U_b is the marginal utility derived from long-term bonds.
- If $\Delta_{2,t} \neq 0$, open market operations (OMO) involving two-period bonds can have effects that are distinct from OMO in one-period bonds.

Pricing wedges

- If monetary services enter the utility function, then $\Delta_{m,t} = U_{m,t} / U_{C,t}$, where U_m is the marginal utility of these services.
- The household, in optimally choosing its money holdings will ensure
$$\Delta_{m,t} = \frac{U_{m,t}}{U_{C,t}} = 1 - p_{1,t}.$$
- If i_t is the nominal interest rate is zero, $p_{1,t} = 1$ and $\Delta_{m,t} = 0$. In this case, while OMO involving money and one-period bonds would not affect Ω_t, OMO involving two-period bonds would if $\Delta_{b,t} \neq 0$.

Budget identities: Different cases

1. Absence of preferred habitat effects or other nonpecuniary returns to bonds, Then b_2 does not appear.
   1.1 Open market operations affecting the time path of m and b_1 are equivalent to ones involving m and b_2.
   1.2 And, selling short-term government bonds to purchase long-term government bonds is neutral. Issuing money to purchase housing is not.
2. If there are preferred habitat effects, then it matters whether m is used to purchase short-term or long-term government debt.
   2.1 Also, changing the maturity composition of the government debt held by the private sector matters because the time path of b_2 enters the houshold's budget constraint.

Budget identities: Different cases

1. If the nominal interest rate is zero, $U_m = 0$ and m does not appear. If $\Delta_{2,t} \neq 0$, debt maturity changes can still matter.
2. Changing future m can still matter in nominal interest rate expected to eventually be positive.

Credit channel: Imperfect credit markets

- What are the sources of possible credit market imperfections? – where do the wedges come from?
  - Imperfect access/market segmentation/limits to arbitrage
  - Imperfect information, imperfect monitoring, transactions costs.
- What are the consequences of these imperfections? The *credit view* stresses the distinct role played by financial assets and liabilities and argues that macroeconomic models need to distinguish between different nonmonetary assets:
  - bank versus nonbank sources of funds, internal versus external financing;
  - a rise in interest rates may have a much stronger contractionary impact on the economy if balance sheets are already weak, introducing the possibility that nonlinearities in the impact of monetary policy may be important.

Portfolio balance models

- Assumes imperfect asset substitutability, asset market segmentation
- Monetarists vs. Keynesians debates
  - Meltzer 1995, Tobin 1969;
  - Debate about quantitative significance
  - Andrés, López-Salido, and Nelson (JME 2004), Goodfriend (2000, 2010).
- Empirical evidence
  - Modigliani and Sutch (AER 1967): Found little evidence that Operation Twist mattered in the 1960s;
  - Clouse, et. al. (2003). Bernanke, Reinhart, and Sack (BPEA 2004): It would require extremely large open market operation in non-standard assets to have a significant impact on yields;
  - Gagnon et al (IJCB 2010), Joyce, Lasaosa, Stevens, and Tong (IJCB 2010), Krishnamurthy, Vissing-Jorgensen (BPEA 2011, Jackson Hole 2013).

Fiscal Issues

Fiscal and monetary interactions

- Several alternative assumptions possible about the relationship between monetary and fiscal policies.
  - Fiscal policy assumed to adjust to ensure the government's intertemporal budget is always in balance, while monetary policy is free to set the nominal money stock or the nominal rate of interest – described as a Ricardian regime (Sargent 1982), monetary dominance, or one with fiscal policy passive and monetary policy active (Leeper JME 1991).
  - The fiscal authority sets its expenditure and taxes without regard to intertemporal budget balance. Seigniorage must adjust to ensure intertemporal budget constraint is satisfied. Case of fiscal dominance (or active fiscal policy) and passive monetary policy.

Intertemporal budget balance and seigniorage

- The intertemporal budget constraint implies that any government with a current outstanding debt must run, in present value terms, future surpluses.
- One way to generate a surplus is to increase revenues from seigniorage.
- Let $s_t^f \equiv t_t - g_t$ be the primary fiscal surplus excluding seigniorage revenue.
- The government's budget constraint can be written as

$$b_{t-1} = R^{-1} \sum_{i=0}^{\infty} R^{-i} s_{t+i}^f + R^{-1} \sum_{i=0}^{\infty} R^{-i} s_{t+i}. \tag{1}$$

- The current real liabilities of the government must be financed by, in present value terms, either a fiscal primary surplus $R^{-1} \sum_{i=0}^{\infty} R^{-i} s_{t+i}^f$ or by seigniorage.

Intertempora budget balance and seigniorage

Unpleasant arithmetic

- Sargent and Wallace (FRB Minn QR 1981) – "unpleasant monetarist arithmetic" in a regime of fiscal dominance:
  - If the present value of the fiscal primary surplus is reduced, the present value of seigniorage must rise to maintain intertemporal budget balance.
  - Reducing inflation now can mean higher inflation in the future.

Fiscal policy and flex-price output

- Suppose government imposes lump-sum taxes to finance exogenous stream of expenditures.
- A rise in government expenditures has a negative wealth effect on the private sector because taxes must be raised.
- This increases labor supply (so output increases) and reduces private consumption.
- With distorting taxes, steady-state equilibrium output is reduced by taxes.

The fiscal policy and flex-price output

- If $Y_t = A_t N_t^{\alpha}$, $H_t = f^{-1}(Y_t) \Rightarrow N_t = \left(A_t^{-1} Y_t\right)^{\frac{1}{a}}$.
- If $U = C_t^{1-\sigma}/(1-\sigma) - N_t^{1+\eta}/(1+\eta)$. labor supply condition is $MRS = MPL$ or

$$\left(A_t^{-1} Y_t\right)^{\frac{\eta}{a}} C_t^{-\sigma} = \alpha A_t^{\frac{1}{\alpha}} Y_t^{1-\frac{1}{\alpha}} \Rightarrow C_t^{\sigma} = \alpha A_t^{\frac{1+\eta}{\alpha}} Y_t^{1-\frac{1+\eta}{\alpha}}$$

- Goods market clearing condition is

$$Y_t = C_t + G_t.$$

- These imply

$$(Y_t - G_t)^{-\sigma} = \alpha A_t^{\frac{1+\eta}{\alpha}} Y_t^{1-\frac{1+\eta}{\alpha}} \Rightarrow \frac{dY_t}{dG_t} = \frac{1}{1 + \left(\frac{C}{Y}\right)\left(\frac{1+\alpha\eta}{\sigma\alpha}\right)} < 1.$$

Linearized flex-price equilibrium and fiscal policy

- Goods clearing: $Y_t = C_t + G_t$, or

$$y_t \approx \left(\frac{C}{Y}\right) c_t + \left(\frac{G}{Y}\right) g_t .$$

- Labor market: $MRS = \omega = MPL/\mu$:

$$\sigma c_t^f + \eta n_t^f = z_t - \mu_t$$

- Using hte production function, $n_t^f = y_t^f - z_t$, and $c_t^f = (Y/C)\, y_t^f - (G/C)\, g_t$,

$$\sigma \left[(Y/C)\, y_t^f - (G/C)\, g_t \right] + \eta \left(y_t^f - z_t \right) = z_t - \mu_t$$

or

$$y_t^f = \left(\frac{1}{\sigma (Y/C) + \eta} \right) \left[(1+\eta)\, z_t + \sigma \left(\frac{Y}{C}\right) \left(\frac{G}{Y}\right) g_t - \mu_t \right]$$

Fiscal policy in a new Keynesian model

- With government spending, $Y_t = C_t + G_t$, or

$$c_t = \left(\frac{Y}{C}\right) y_t - \left(\frac{G}{C}\right) g_t.$$

- So Euler equation becomes

$$\frac{Y}{C}\left[y_t - \frac{G}{Y} g_t\right] = \frac{Y}{C}\left[\mathrm{E}_t y_{t+1} - \frac{G}{Y}\mathrm{E}_t g_{t+1}\right] - \left(\frac{1}{\sigma}\right)\left(i_t - E_t \pi_{t+1}\right)$$

or

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\tilde{\sigma}}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right),$$

where

$$\tilde{\sigma} = \sigma \left(\frac{Y}{C}\right)$$

$$r_t^f = -\tilde{\sigma}\left(\frac{G}{C}\right)\left(\mathrm{E}_t g_{t+1} - g_t\right) + \tilde{\sigma}\left(\mathrm{E}_t y_{t+1}^f - y_t^f\right).$$

- So fiscal shocks affect the *IS* curve.

Fiscal policy in a new Keynesian model

- Fiscal shocks can also affect inflation via marginal cost.
- If $Y_t = e^{z_t} N_t$, marginal cost is $mc_t = \sigma c_t + \eta n_t - z_t$, or

$$mc_t = \tilde{\sigma}\left[y_t - \left(\frac{G}{Y}\right) g_t\right] + \eta(y_t - z_t) - z_t,$$

- or

$$mc_t = (\tilde{\sigma} + \eta)y_t - \tilde{\sigma}\left(\frac{G}{Y}\right) g_t - (1+\eta)z_t + \mu_t = (\tilde{\sigma}+\eta)\left(y_t - y_t^f\right)$$

where

$$y_t^f = \frac{\tilde{\sigma}\left(\frac{G}{Y}\right) g_t + (1+\eta)z_t - \mu_t}{\tilde{\sigma} + \eta}.$$

Fiscal policy in a new Keynesian model

- IS equations becomes

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\tilde{\sigma}}\right)\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t^f\right),$$

  which is the same as before, but r^n depends on $\mathrm{E}_t g_{t+1} - g_t$.
- Inflation equation becomes

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t,$$

  where $\kappa = \bar{\kappa}(\sigma + \eta)$, which is also the same as before.

Fiscal policy at the ZLB

- Suppose monetary policy cannot neutralize fiscal policy, for example because interest rates are at the ZLB.
- Argument is that at ZLB, fiscal policy is more powerful..
- At ZLB, old fashion Keynesian multiplier returns.
  - No crowding out since real interest rate does not rise.
  - Output expansion raises marginal cost and inflation, lowering the real interest rate, so a form of crowding in occurs.

Output at the ZLB

- Assume $i = 0$ and with probability μ economy is still at ZLB in following period. If economy exits ZLB, output and inflation are zero.
- Equilibrium given by solutions to

$$\begin{aligned} y^Z &= \mu y^Z - \left(\frac{1}{\sigma}\right)\left(-\mu\pi^Z - r^n\right) \\ &\Rightarrow \quad y^Z = \left(\frac{1}{1-\mu}\right)\left(\frac{1}{\sigma}\right)\left(\mu\pi^Z + r^n\right) \end{aligned}$$

$$\pi^Z = \beta\pi^Z + \kappa y^Z$$

where μy^Z and $\mu\pi^Z$ are expected future output and inflation.

Adding taxes

- Eggertsson (NBER Macro Annual 2010) – adds wage and sales taxes (plus other taxes)
- Marginal cost:

$$mc_t = (w_t - p_t) - mpl_t$$

$$\begin{aligned} w_t^{aftertax} - p_t^{cpi} &\approx (w_t - \tau_t^w) - (p_t + \tau_t^s) \\ &= \eta n_t + \sigma \left[\left(\frac{Y}{C} \right) y_t - \left(\frac{G}{C} \right) g_t \right] \end{aligned}$$

so

$$\begin{aligned} mc_t &= \eta(y_t - z_t) + \sigma \left[\left(\frac{Y}{C} \right) y_t - \left(\frac{G}{C} \right) g_t \right] + \tau_t^s + \tau_t^w - z_t \\ &= \left[\eta + \sigma \left(\frac{Y}{C} \right) \right] y_t - \sigma \left(\frac{G}{C} \right) g_t + \tau_t^s + \tau_t^w - (1+\eta) z_t \end{aligned}$$

Adding taxes:

- The NKPC becomes

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa \left\{ \left[\eta + \sigma \left(\frac{Y}{C} \right) \right] y_t - \sigma \left(\frac{G}{Y} \right) g_t + \tau_t^s + \tau_t^w - (1+\eta) z_t \right\}$$

  - Note that a rise in the wage tax or sales tax increases marginal costs.
  - So wage tax cuts are deflationary

Adding taxes:

- Aggregate demand:

$$
\begin{aligned}
y_t &= \mathrm{E}_t y_t - \left(\frac{1}{\sigma}\right)\left(i_t - \mathrm{E}_t \pi^{cpi}_{t+1} - r^n_t\right) - \left(\frac{G}{Y}\right)(\mathrm{E}_t g_{t+1} - g_t) \\
&= \mathrm{E}_t y_t - \left(\frac{1}{\sigma}\right)(i_t - \mathrm{E}_t \pi_{t+1} - r^n_t) - \left(\frac{G}{Y}\right)(\mathrm{E}_t g_{t+1} - g_t) \\
&\quad + \left(\frac{1}{\sigma}\right)(\mathrm{E}_t \tau^s_{t+1} - \tau^s_t)
\end{aligned}
$$

Adding taxes: fiscal policy at the ZLB

- Now consider the case of the ZLB. Equilibrium is given by

$$
\begin{aligned}
\pi_t^Z &= \beta \mathrm{E}_t \pi_{t+1} + \kappa \left[\eta + \sigma \left(\frac{Y}{C} \right) \right] y_t^Z - \kappa\sigma \left(\frac{G}{Y} \right) g_t \\
&\quad + \kappa \left(\tau^s + \tau^w \right) - \kappa (1+\eta) z_t
\end{aligned}
$$

$$
\begin{aligned}
y_t^Z &= \mathrm{E}_t y_{t+1} + \left(\frac{1}{\sigma} \right) \left(\mu \pi_{t+1}^Z + r_t^n \right) - \left(\frac{G}{Y} \right) \left(\mathrm{E}_t g_{t+1} - g_t \right) \\
&\quad + \left(\frac{1}{\sigma} \right) \left(\mathrm{E}_t \tau_{t+1}^s - \tau^s \right)
\end{aligned}
$$

- Expansionary policies:
  - temporary sales tax cut (boosts demand and inflation)
  - temporary rise in government spending (boosts demand but lowers inflation) (possibly)
  - Expansionary policies: increase in wage tax (raises inflation and demand).

Optimal taxation with distortionary taxes

Tax smoothing

- Optimal intertemporal taxation
- Equalize marginal distortionary costs per dollar of revenue across
  - tax instruments
  - time

Distortionary taxes

- If there are two tax rates τ_1 and τ_2, and distortions per dollar raises are $D(\tau_1, \tau_2)$,

$$\frac{\partial D(\tau_{1t}, \tau_{2t})}{\partial \tau_{1t}} = \frac{\partial D(\tau_{1t}, \tau_{2t})}{\partial \tau_{2t}}$$

  and

$$\frac{\partial D(\tau_{1t}, \tau_{2t})}{\partial \tau_{1t}} = \mathrm{E}_t \left[\frac{\partial D(\tau_{1t+1}, \tau_{2t+1})}{\partial \tau_{1t+1}} \right].$$

- If D is quadratic, this second condition implies

$$\tau_{jt} = \mathrm{E}_t \tau_{jt+1} \Rightarrow \tau_{jt+1} = \tau_{jt} + \xi_{jt+1}$$

Distortionary taxes

Tax smoothing

- Tax rates follow random walks.
- Level is set to finance expected present discounted value of government expenditures.
- This is like a permanent income model of taxes
  - Taxes are like consumption – smooth them and use debt to finance transitory fluctuations in spending.