7. CONSUMPTION AND SAVING

The Basic Two-Period Model

Here we consider a risk-averse expected utility maximizer who has a two-period time-horizon, labeled Period 1 (this year) and Period 2 (next year). Denote his income in the two periods by Y_1, Y_2 and his consumption amounts in the two periods by C_1, C_2; these are measured in dollars. We assume his utility-of-consequences function to be

$$u(C_1, C_2) = u_1(C_1) + \delta\ u_2(C_2),$$

where each of the functions u_1, u_2 is increasing and concave (has a positive first derivative and negative second derivative), and δ is a positive constant that discounts second-period utilities. (The $u_1(.)$ and $u_2(.)$ can have the same functional form, but they don't have to.) Denote the gross or total return he gets on each dollar of Period-1 saving by R. So if he saves S dollars,

$$C_1 = Y_1 - S, \qquad C_2 = Y_2 + R\,S\,.$$

In what follows, we consider two different kinds of uncertainty. In each case we characterize the choice of saving to maximize expected utility, and the comparative statics as various exogenous parameters change.

Before we proceed, a comment on this setup. Two different kinds of additive separability of utility are being conflated together: additivity over time, which is a common assumption in macro theories, and additivity over states, which is the basis of expected utility theory. But they jointly do entail additional restrictions. For example, suppose the function takes the form that is commonly used in macro (probably this was done in ECO 311):

$$u(C_1, C_2) = \begin{cases} \frac{1}{1-r}\,(C_1)^{1-r} + \delta\,\frac{1}{1-r}\,(C_2)^{1-r} & \text{if } r \neq 1 \\ \ln(C_1) + \delta\ \ln(C_2) & \text{if } r = 1 \end{cases}$$

We know that in the expected utility formulation where C_2 is uncertain, this makes the relative risk aversion with respect to C_2 constant and equal to r.

What are the implications in macro? The marginal rate of substitution between consumption in the two periods is

$$-\left.\frac{dC_2}{dC_1}\right|_{u=\text{constant}} = \frac{\partial u/\partial C_1}{\partial u/\partial C_2} = \frac{(C_1)^{-r}}{\delta\ (C_2)^{-r}}$$

If the consumer faces an intertemporal budget constraint

$$C_1 + \beta\ C_2 = \text{constant}\,,$$

where $\beta = 1/(1 + \text{interest rate})$ is the period-1 equivalent of a dollar in period 2, then along the budget constraint,

$$-\left.\frac{dC_2}{dC_1}\right|_{\text{budget constraint}} = \frac{1}{\beta}\,.$$

For optimality,

$$\frac{(C_1)^{-r}}{\delta\,(C_2)^{-r}} = \frac{1}{\beta}\,,$$

or

$$\frac{C_2}{C_1} = \left(\frac{\delta}{\beta}\right)^{1/r}.$$

Therefore $1/r$ is the elasticity of substitution between first-period and second-period consumption.

But r is also the coefficient of relative risk aversion, so $1/r$ is also a quantitative measure of substitution across states of nature. Thus r is playing a dual role, and in principle there seems no reason why risk aversion (substituion across states) and substitution across time should be so tied together. More general formulations can allow separate parameters for the two; you will get them in more advanced courses.

Uncertain Future Income

Suppose there is uncertainty about future income, so Y_2 is a random variable with a density function $f(Y_2)$. The rate of return R is not random. The consumer chooses S to maximize expected utility. Assume in all that follows that the solution has $S > 0$.

Expected utility, expressed as a function of the choice variable S, is given by

$$
\begin{aligned}
EU(S) &= \text{E}[\,u_1(Y_1 - S) + \delta\,u_2(Y_2 + R\,S)\,]\\
&= u_1(Y_1 - S) + \delta\int u_2(Y_2 + R\,S)\,f(Y_2)\,dY_2\,. \quad (1)
\end{aligned}
$$

Note: [1] The limits of integration (the support of Y_2) are omitted for brevity. [2] $u_1(Y_1 - S)$ is non-random and therefore its expectation is itself. [3] The multiplicative constant δ can be taken outside the expectation integral. It would not matter if you kept it inside the integration.

The first-order condition is

$$EU'(S) = -u_1'(Y_1 - S) + \delta\,R\int u_2'(Y_2 + R\,S)\,f(Y_2)\,dY_2 = 0\,, \quad (2)$$

where I have taken R outside the expectation integral becuase it is non-random here.

The second-order condition is

$$EU''(S) = u_1''(Y_1 - S) + \delta\,R^2\int u_2''(Y_2 + R\,S)\,f(Y_2)\,dY_2 < 0$$

This is satisfied because u_1'', u_2'' are both always negative.

Equation (2) implicitly defines the optimal S. We consider its comparative statics.

First suppose future income prospects become unambiguously better, in the sense that the distribution of Y_2 shifts to the right in the sense of first-order stochastic dominance (FOSD). From our general theory of comparative statics, we know that this will lead to a smaller optimal S if it lowers the whole function $EU'(S)$. And the basic property of FOSD

shifts tells us that this will happen if $u_2'(Y_2 + R S)$ is a decreasing function of the random variable Y_2 for any given S. That is true because $u_2'' < 0$. So we immediately have the result that an FOSD improvement in future income implies a less current saving. The intuition is as follows: The shift makes the person better off. Current consumption being a normal good, he wants more of it. Consuming more out of an unchanged current income lowers the saving.
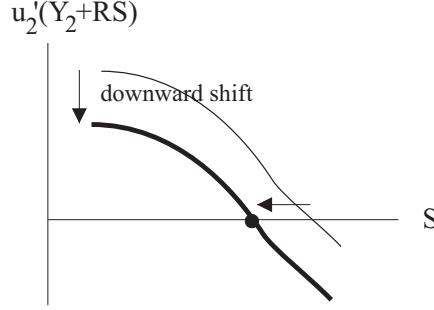
Comparative statics of FOSD improvement in Y_2

Here we have another application of the general comparative statics argument, which you need to understand and later be able to do it correctly and instinctively.

Next suppose future income becomes unambiguously riskier, that is, the distribution of Y_2 undergoes a mean-preserving spread in the sense of second-order stochastic dominance. This will raise the optimal S if it raises the expectation of $u_2'(Y_2 + R S)$ for each given R and S. We know that will happen if $u_2'(Y_2 + R S)$ is a convex function of Y_2. The condition for this is $u_2'''(Y_2 + R S) > 0$.

The textbook (p. 17) defines a measure of (absolute) *prudence:* $P(W) = -u'''(W)/u''(W)$. Since $u''(W) < 0$, we see that $P(W) > 0$ if $u'''(W) > 0$. Qualitatively, we say that the decision-maker is prudent if $P(W) > 0$. We now see the reason for using this name more clearly: if the decision-maker is prudent, he saves more in preparation for a more uncertain propsect of future income.

Uncertain Return to Saving

The situation here is exactly the same as in the previous section, except that now Y_2 is non-random, and R is random with a density function $g(R)$.

The expected utility expressed as a function of S is

$$
\begin{aligned}
EU(S) &= \mathrm{E}[\, u_1(Y_1 - S) + \delta\, u_2(Y_2 + R S)\,] \\
&= u_1(Y_1 - S) + \delta \int u_2(Y_2 + R S)\, g(R)\, dR
\end{aligned}
\tag{3}
$$

The first-order condition is

$$
EU'(S) = -u_1'(Y_1 - S) + \delta \int u_2'(Y_2 + R S)\, R\, g(R)\, dR = 0\,.
\tag{4}
$$

Note that by contast with (2), now R is random and cannot be taken outside the expectation integral.

The second-order condition is

$$EU''(S) = u_1''(Y_1 - S) + \delta \int u_2''(Y_2 + R\,S)\,R^2\,g(R)\,dR < 0$$

This is satisfied because u_1'', u_2'' are both always negative.

A rightward shift of the distribution of R in the FOSD sense will increase $EU'(S)$ at all S, and will therefore increase the optimal S, if $u_2'(Y_2 + R\,S)\,R$ is an increasing function of R for fixed Y_2, S. The condition for this is

$$u_2'(Y_2 + R\,S) + u_2''(Y_2 + R\,S)\,S\,R > 0\,.$$

The left hand side of this inequality can be written as

$$u_2'(Y_2 + R\,S)\,\left[\, 1 + S\,R\,\frac{u_2''(Y_2 + R\,S)}{u_2'(Y_2 + R\,S)}\,\right]$$

$$= u_2'(Y_2 + R\,S)\,\left[\, 1 + \frac{S\,R}{Y_2 + S\,R}\,\frac{(Y_2 + R\,S)\,u_2''(Y_2 + R\,S)}{u_2'(Y_2 + R\,S)}\,\right]$$

$$= u_2'(Y_2 + R\,S)\,\left[\, 1 - \frac{S\,R}{Y_2 + S\,R}\,\rho(C_2)\,\right]$$

Since $S\,R/(Y_2 + S\,R) < 1$, the coefficient of relative risk aversion being less than 1 is a sufficient condition to ensure that the whole expression is positive.

If ρ is too high, then an FOSD improvement in the distribution of the rate of return on the risky asset can lead to a decrease in saving. Here the interpretation of $1/\rho$ as the *intertemporal* elasticity of substitution comes useful. If ρ is large, this elasticity is small. Therefore there is only a small intertemporal substitution effect of the improvement in the rate of return, and it is overwhelmed by the income effect, which makes the consumer want to take out some of the benefit of this improved prospect in the form of today's consumption.

When the distribution of R undergoes a mean-preserving spread in the sense of second-order stochastic dominance, the expectation of $u_2'(Y_2 + R\,S)\,R$ rises for given Y_2 and S if $u_2'(Y_2 + R\,S)\,R$ is a convex function of R. If that is true, then for each S the mean-preserving spread of the distribution of R raises $EU'(S)$, and by the same argument as above, the optimal S increases. Now

$$\frac{d}{dR}[u_2'(Y_2 + R\,S)\,R] = u_2'(Y_2 + R\,S) + u_2''(Y_2 + R\,S)\,S\,R\,,$$

and

$$\frac{d^2}{dR^2}[u_2'(Y_2 + R\,S)\,R] = u_2''(Y_2 + R\,S)\,S + u_2''(Y_2 + R\,S)\,S + u_2'''(Y_2 + R\,S)\,S^2\,R$$

$$= S\,[\, 2\,u_2''(Y_2 + R\,S) + u_2'''(Y_2 + R\,S)\,S\,R\,]$$

We are told that $S > 0$, so the sign of the second derivative is the same as the sign of the expression in the bracket. The condition can be interpreted in terms of a measure of relative prudence, but it is not particularly useful.