General Blotto: Games of Allocative Strategic Mismatch

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Abstract

The Colonel Blotto game captures strategic situations in which players attempt to mismatch the actions of their opponents. In the early history of game theory, Colonel Blotto commanded substantial attention, but in recent decades has faded from view. In this paper, we promote Colonel Blotto, literally, to a class of General Blotto games that allow for more general payoffs and for externalities between fronts. We show that General Blotto games are relevant to problems that arise in fields ranging from international politics, electoral politics, business, the law, biology, and sports but that they rarely have pure strategy equilibria.

Background

Two games propelled game theory into the real politick of international relations: the Prisoners' Dilemma and Colonel Blotto. Each game made a different contribution. The Prisoners' Dilemma game clarified our understanding of the incentives to accumulate weapons and of the importance of repeated interactions in inducing cooperation, the so-called shadow of the future (Schelling 1960). The Prisoner's Dilemma game has a preferred outcome, mutual cooperation. The challenge is in how to achieve it. Colonel Blotto made a different contribution. It revealed the intricacies of allocating limited resources across domains in competitive environments. Unlike the Prisoners' Dilemma game, Colonel Blotto has no preferred outcome. It is a zero sum game. When one player wins, the other player loses. Nor does Colonel Blotto have a straightforward solution. It has multiple mixed strategy equilibria, which except in rare cases, are difficult to characterize.

Over the past half century, game theoretic reasoning has continued to provide insights into international relations. In addition, game theoretic models and reasoning

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have become central to the study of economics, policy, business, politics, law, and even biology and philosophy. The Prisoners' Dilemma and the strategic problem it identifies can be found in each of these contexts. Academic and applied interest in the Prisoners' Dilemma continue unabated, and for good reason. Whether modeled as a two person game or in its N person incarnation, the Prisoners' Dilemma continues to provide insights into how collections of entities, be they ants, people, political parties, corporations, or nations, achieve cooperation either through evolution or rational choice (Axlerod 1984).

Our concern here is not the Prisoners' Dilemma but its long lost brother: Colonel Blotto. The Colonel Blotto game is easy to describe: two players must allocate resources to a finite number of contests. Each contest is won by the player who allocates the greater number resources to it, and a player's payoff equals the number of contests won. Thus, a player's goal is to strategically mismatch the actions of it's opponent. Colonel Blotto can be thought of as an elaborate version of Rock, Paper, and Scissors, an analogy we return to later. In the standard version of Blotto, each player has the same number of resources. In extensions, one player can have greater resources than the other. Sometimes the resources are infinitely divisible, and sometimes they are integer valued.

The list of original contributors to the Colonel Blotto literature reads like an excerpt from who's who in mathematics. It includes such luminaries as Borel (1921), Savage (1953), Tukey (1949), and Bellman (1969). Yet, over the past twenty five years, only a handful of papers have been written on Colonel Blotto (Roberson 2006). Blotto's shrinking popularity partly stems from its complexity. The clean, precise comparative statics results that game theorists love are not accessible for Blotto. And yet, as we discuss in the conclusion, this complexity makes Blotto all the more compelling in its interpretations.

A second reason that Colonel Blotto receives little attention has to do with changing circumstances. In its original form, Colonel Blotto captures the strategic problem of how to allocate troops or regiments fighting along fronts or in trenches.[1] The conventional military application of the Colonel Blotto game assumes that battles are always won by the side with the most troops and that both sides care only about the number of fronts won. This may be a fine model of traditional warfare, but a modern military is more agile, more flexible to defend against asymmetric attacks. Resources can be deployed to multiple fronts. And often, fronts are not independent. Resources may contribute to multiple fronts at once. And finally, victory on a front need not be decisive. The benefit to an army could depend on the margin of victory. A generalized version of Colonel Blotto not only better aligns with modern military allocation games, it also captures situations in economics, politics, law, biology, and sports.

Generalizing Colonel Blotto opens up the possibility of cleaner results. Perhaps

[1]We retain the language of the original problem and often refer to the contests as fronts, without regard to what we are actually modeling.

some generalization of Colonel Blotto can capture incentives for strategic mismatch, yet admit a more tractable equilibrium. Most existing results apply only to the specific game of Colonel Blotto and not to a general class of games of strategic mismatch. As a result, we cannot be sure whether the intuitions that we derive from Blotto hold more generally or if they are peculiar to the specific assumptions. For example, in a context where competition on a front is not a winner-take-all battle, but the rest of the game resembles Colonel Blotto, do players have an optimal pure strategy? Are Colonel Blotto's complicated mixed strategy equilibria an artifact of the winner-take-all assumption or an inevitable result in situations of strategic mismatch? In this paper, we follow an approach outlined by de Marchi (2005) and explore a feature space for a class of models. Within that feature space, we vary assumptions and check the generality of results.

In what follows, we generalize Colonel Blotto in three ways. First, we allow for a more general payoff structure. Whereas contests in Colonel Blotto are winner-take-all, we allow the payoff at a front to depend on the size of the resource advantage at that front. The standard assumption from economics is that the benefits diminish in the extent of the victory on a front, but in other contexts, perhaps law and sports, we might find marginally increasing benefits from advantages. Our second extension allows for interactions between the fronts. It allows competition on combinations of fronts. The outcome of a military maneuver may depend on the combined efforts of ground forces and air forces. The third extension considers an N player version of Blotto in which players pick their strategies first and those fixed strategies play in pairwise games against the other players. In making these extensions, we transform Colonel Blotto into a class of General Blotto games.[2] Though we might expect smooth payoff functions to create a more uniform equilibrium and externalities to create a pure strategy equilibrium, we find that only in rare cases do we get clean results that allow for comparative statics.

The remainder of this paper is organized as follows. In the next section, we describe the Colonel Blotto game. We then define a class of General Blotto games that generalize Colonel Blotto and show that General Blotto games only have pure strategy equilibria in a few cases. We then discuss why Colonel Blotto needs to be generalized by describing several contexts that create incentives for strategic mismatches. We conclude with a discussion of how populations of agents might learn to play equilibria in General Blotto games.

Colonel Blotto

We begin by defining the Colonel Blotto game. We also characterize a simple equilibrium for the case of indivisible resources. Suppose two players each have ten troops to allocate across five fronts. A player's payoff is the number of fronts he wins minus

[2]Formally, making a Colonel a General is a promotion. In the U.S. Army, Colonels belong to officer class six and Generals to classes seven and above.

the number he loses. This game has no pure strategy equilibrium: any allocation of troops across the five fronts can be defeated. Suppose that Player 1 allocates two troops to every front. Then Player 2 can allocate three troops to each of the first three fronts and one to the fourth and win the battle, as shown in the table below.

Colonel Blotto With Five Fronts

Player	*Front 1*	*Front 2*	*Front 3*	*Front 4*	*Front 5*
Player 1 (P1)	2	2	2	2	2
Player 2 (P2)	3	3	3	1	0
Winner	P2	P2	P2	P1	P1

Player 2's strategy is also easily defeated. All that Player 1 would have to do is shift two troops from the second front to the third front. While this game does not have an optimal pure strategy, it does have many mixed strategy equilibria. One set of those equilibria requires troop allocations to each front to be uniformly distributed across a proportion of the total number of troops. One such equilibrium for the case of five fronts and ten troops is given by the following five strategies, denoted $S1$ through $S5$.

Colonel Blotto Mixed Strategy

Player	*Front 1*	*Front 2*	*Front 3*	*Front 4*	*Front 5*
S1	4	3	2	1	0
S2	0	4	3	2	1
S3	1	0	4	3	2
S4	2	1	0	4	3
S5	3	2	1	0	4

If both players play each of these strategies with probability one-fifth, then this is a mixed strategy equilibrium. The proof follows from Proposition 2, below.

The formal version of the Colonel Blotto game can be written as follows: It is a two-player, zero-sum game. A strategy for player X can be written as a real vector $(x_1, x_2, ..., x_m)$ with

$$\sum_{i=1}^{m} x_i = 1,\ x_i \in [0,1]$$

where x_i represents the fraction of the budget allocated to front i, and m is the total number of fronts. Here, both players have the same available budget. The payoff to $\mathbf{x}$ against $\mathbf{y}$ is

$$\sum_{i=1}^{m} \mathrm{sgn}(x_i - y_i)$$

where the function

$$\text{sgn}(\chi) = \begin{cases} 1 & \text{if } \chi > 0; \\ 0 & \text{if } \chi = 0; \\ -1 & \text{if } \chi < 0. \end{cases}$$

We assume $m > 2$. Otherwise, the game always ends in a tie. It can be shown that there is no pure strategy Nash Equilibrium. A pure strategy $\mathbf{x}$ that allocates resources to the i'th front, $x_i > 0$, will lose to the strategy $\mathbf{y}$ that allocates no resources to the i'th front and more resources to all other fronts,

$$y_i = 0,\ y_j = x_j + \frac{x_i}{m-1} \text{ for all } j \neq i.$$

Borel provides the first solution of the Colonel Blotto game with $m = 3$ (Borel and Ville (1938)), and Gross and Wagner (1950) extend this result for $m > 3$. We restate their characterization of a mixed strategy Nash Equilibrium for Colonel Blotto here.

Proposition 1 *(Gross and Wagner (1950)) The Colonel Blotto game has a mixed strategy equilibrium in which the marginal distributions are uniform on* $[0, \frac{2}{m}]$ *along all fronts.*

Proof Let $\mathbf{x}$ be an allocation picked randomly according to a joint distribution of strategies $(x_1, ..., x_m)$ such that each variable x_i is uniformly distributed on $[0, \frac{2}{m}]$. The expected payoff to a pure strategy $\mathbf{y}$ against $\mathbf{x}$ is at most 0. The probability that $y_i > x_i$ is $\min\left[1, \frac{my_i}{2}\right]$. The expected payoff at front i is $\min[1, my_i - 1]$, so the game's total expected payoff is

$$\sum_{i=1}^{m} \min[1, my_i - 1] \leq \sum_{i=1}^{m} (my_i - 1) = 0. \quad \blacksquare$$

Equilibrium play dictates that the allocation to a given front is equally likely to be anything between 0 and $\frac{2}{m}$. This unpredictability leaves the opponent with no preference for one strategy or another as long as no front is allocated more than $\frac{2}{m}$ resources. Gross and Wagner (1950) show a geometrical construction of a joint distribution with such marginals (See also Laslier and Picard (2002)). Roberson (2006) constructs other mixed strategy equilibria and proves that all equilibria must have these marginal distributions.

Following the argument of Gross and Wagner (1950) in the case of infinitely divisible goods, we can find mixed strategy equilibria when resources are discrete indivisible units. This proves that our example above is an equilibrium to the game.

Proposition 2 *Let n be the number of troops to be allocated between m fronts. Assume m divides n. Then there is a Nash Equilibrium when both players play a mixed strategy such that the distribution of troops along any particular front is uniform between* 0 *and* $\frac{2n}{m}$.

Proof Let player X be playing such a mixed strategy. For all i, x_i is uniformly distributed on $[0, \frac{2n}{m}]$. For a given strategy $\mathbf{y}$, the probability that $y_i > x_i$ is $\min[1, \frac{y_i}{\frac{2n}{m}+1}]$. The probability that $y_i < x_i$ is $\max[0, \frac{\frac{2n}{m}-y_i}{\frac{2n}{m}+1}]$. The expected payoff from front i is at most

$$\frac{my_i}{2n+m} - \frac{2n-my_i}{2n+m} = \frac{2my_i - 2n}{2n+m}.$$

The maximum total expected payoff to $\mathbf{y}$ subject to the condition $\sum_{i=1}^{m} y_i = n$ is

$$\sum_{i=1}^{m} \frac{2my_i - 2n}{2n+m} = \frac{2mn - 2nm}{2n+m} = 0.$$

Player X cannot lose. His strategy must be a best response to itself. ∎

In the example above, $n = 10$ and $m = 5$, and the strategy that mixes $S1$ through $S5$, each with probability one-fifth, allocates troops uniformly on $[0, 4]$ on each front. In general, for a suitable choice of n, we can construct such mixed strategies.

Proposition 3 *Let $n = \frac{m^2-m}{2}$. Then we can mix permutations of $(0, 1, \cdots, m-1)$ to allocate troops uniformly on $[0, \frac{2n}{m}]$ on each front.*

Proof The permutations are legal strategies for our choice of n. That is, they satisfy the budget constraint,

$$\sum_{i=1}^{m} x_i = \sum_{j=0}^{m-1} j = \frac{1}{2}(m-1)(m) = n.$$

Observe that $m-1 = \frac{2n}{m}$, so if we mix evenly between all $m!$ permutations, we will be uniform on $[0, \frac{2n}{m}]$ on each front. ∎

We can also mix evenly on just m different permutations to achieve uniform distributions on each front. For example, if $(x_1, \cdots, x_m)$ is one such permutation, $\mathbf{x_1}$, then define $\mathbf{x_i}$ to be $(x_i, \cdots, x_m, x_1, \cdots, x_{i-1})$. We can mix evenly between $\mathbf{x_1}$ through $\mathbf{x_m}$.

General Blotto

We now formally extend the Colonel Blotto game to a class of General Blotto games. First we generalize the payoff function on each front to allow linear payoffs, diminishing marginal payoffs, increasing marginal payoffs, and winner-take-all payoffs as special cases. Baye, Kovenock, and De Vries (1998) consider a generalized class of payoff functions for single dimension contests. We also consider a payoff function that values sets of fronts. In making this second extension, we copy Shubik and Weber

(1981) who also considered a more general game form to make Blotto more applicable to real world military defense situations, such as the defense of networks. In this framework, winning certain combinations of fronts may be more valuable than winning the greatest number of fronts. Shubik and Weber (1981) used a general functional form to compare the value of a combination of fronts with and without the particular front in question. Their framework was not specific enough to find an equilibrium, but they could state that an equilibrium would exist. Finally, we consider games with N players who play pairwise games of Blotto against all other players. In this setting, we can get pure strategy equilibria.

Thus, in a General Blotto game, we allow competitions on individual fronts and combinations of fronts and we have a scoring function f for each of these fronts and front combinations. The payoff to a player equals the sum of the scores across all fronts and combinations of fronts. Given a subset s of the fronts, we define the function $v_{|s|}$ to be a single valued function defined over a set of $|s|$ variables. In a slight abuse of notation, we write $v_{|s|}(\{x_i\}_{i\in s})$. In the case of $|s|=1$, $v(x)=x$, the number of troops on a front. In cases with more than one front, $v_{|s|}()$ gives the value of the resources on that combination of fronts. Depending on what is being modeled, $v_{|s|}(s)$ could take any of several forms. It could be the sum or the product of the resources on the included fronts, or it could equal the minimum or the maximum number of resources.

As before, a strategy for player X can be written as a real vector $(x_1, x_2, ..., x_m)$ with

$$\sum_{i=1}^{m} x_i = 1,\ x_i \in [0,1].$$

We still assume that both players have equal budgets. However, now the payoff to $\mathbf{x}$ against $\mathbf{y}$ equals

$$\sum_{s\in S} f(v_{|s|}(\{x_i\}_{i\in s}) - v_{|s|}(\{y_i\}_{i\in s}))$$

where S is the set of combinations of fronts (including the individual fronts) along which competition takes place. The function $v_{|s|}$ is different in each context, depending on what is being modeled.

Now we look at a few suggestive examples, with an eye for situations where the game becomes more predictable. Rather than trying to model a specific situation, we address the question of whether General Blotto has a pure strategy equilibrium. Frequently, it does not. However, in some special cases, General Blotto does have a dominant strategy.

In the analysis that follows, we consider a specific functional form for the scoring function,

$$f(\chi) = \mathrm{sgn}(\chi) \cdot |\chi|^p.$$

Fronts can be valued in isolation, just as in Colonel Blotto, but we also consider valuing some specific combinations of fronts that retain the symmetry that no front is special. In one case, all pairs of fronts are valued as highly as the individual fronts. In another, every set of fronts has equal value. We assume that the function v is the product of the resources on the contributing fronts.

Definition When only *isolated fronts* are valued, $S = \{s : |s| = 1\}$.

The payoff function becomes

$$\sum_{i=1}^{m} f(x_i - y_i).$$

Definition When *all pairs* of fronts are valued, $S = \{s : |s| \leq 2\}$.

The payoff becomes

$$\sum_{j=1}^{m-1} \sum_{k=j+1}^{m} f(x_j x_k - y_j y_k) \; + \; \sum_{i=1}^{m} f(x_i - y_i).$$

Definition When *all sets* of fronts have equal value, $S = \{s\}$.

The payoff is

$$\sum_{\mu=1}^{m} \; \sum_{i_1 > i_2 > \cdots > i_\mu}^{m} f\left(\prod_{j=1}^{\mu} x_{i_j} - \prod_{j=1}^{\mu} y_{i_j} \right).$$

We consider separately the cases $p = 0$, $0 < p < 1$, $p = 1$, and $p > 1$. The margin of victory is irrelevant in the case $p = 0$, just like Colonel Blotto. The concave case, $0 < p < 1$, is a similar game, but more complicated. The linear case, $p = 1$, allows us to consider a strategy independently of the opponent's strategy, and thus makes it easier to find pure equilibria. The convex case, $p > 1$, is a very different game, in which margin of victory becomes more important than the number of fronts won.

Results

We now prove the existence or non-existence of pure strategy Nash Equilibria for specific classes of General Blotto games. When only isolated fronts are valued and $p = 0$, we have a Colonel Blotto game. Recall from Section 2 that this game has no pure Nash Equilibrium. The next claim extends that result. It states that a pure strategy equilibrium does not exist for the winner-take-all scoring function when all pairs of fronts are valued as well, if there are more than 3 fronts.

Proposition 4 *There is no pure strategy Nash Equilibrium in the case of $p = 0$ in General Blotto valuing all pairs of fronts as well as the individual fronts, if $m > 3$.*

Proof Any strategy $\mathbf{x}$ with $x_i > 0$ for some i will lose to the strategy

$$\left(y_i = 0,\, y_j = x_j + \frac{x_i}{m-1} \text{ for all } j \neq i\right).$$

Strategy $\mathbf{y}$ will win $m-1$ of m single fronts and $\left(\begin{array}{c} m-1 \\ 2 \end{array}\right)$ of $\left(\begin{array}{c} m \\ 2 \end{array}\right)$ pairs of fronts. As $m > 3$, $\mathbf{x}$ loses more than it wins. ∎

We encounter a special case when $m = 3$ due to symmetry between individual fronts and pairs of fronts. Specifically, each individual front is the complement of a set of a pair of fronts. When $m = 3$, $p = 0$, and all pairs of fronts are valued, there is a Nash Equilibrium when both players play $\frac{1}{3}$ on every front. We treat the general case in the Corollary to Proposition 5, which follows.

This next result shows that pure strategy equilibria exist for General Blotto, when all subsets of fronts are valued with a winner-take-all scoring rule.

Proposition 5 *For $p = 0$, there is a Nash Equilibrium in pure strategies to the General Blotto game valuing all sets of fronts, in which both players allocate $\frac{1}{m}$ to every front.*

Proof Playing the strategy $(\frac{1}{m}, \cdots, \frac{1}{m})$ guarantees a player of winning at least half of the sets of fronts regardless of the opponent's play. Observe that if this strategy loses a particular set of fronts, it must win the complement of those fronts.

Let $\{i_1, \cdots, i_k\}$ be a set of fronts on which this strategy loses to $\mathbf{x}$.

$$\prod_{j=1}^{k} x_{i_j} > \frac{1}{m^k}.$$

Then

$$\frac{\sum_{j=1}^{k} x_{i_j}}{k} > \frac{1}{m}.$$

So

$$\frac{\sum_{\mu \neq i_j \forall j \in [1,k]} x_\mu}{m-k} < \frac{1}{m}.$$

And thus

$$\prod_{\mu \neq i_j \forall j \in [1,k]} x_\mu < \frac{1}{m^{m-k}}.$$

Playing $\frac{1}{m}$ on all fronts beats $\mathbf{x}$ on the complement of $\{i_1, \cdots, i_k\}$. Thus, its payoff against any opponent is non-negative. In a zero-sum game, a strategy with non-negative payoff must be a best response to itself. ∎

Corollary 1 *For $p = 0$, there is a Nash Equilibrium in which both players allocate $\frac{1}{m}$ to every front in the General Blotto game defined over any collection of subsets of fronts and their complements.*

The corollary is a straightforward consequence of the previous proof.

The existence of pure strategy equilibria in one extension of General Blotto suggests that making the rules of the game more complex can make the strategy of the game simpler. Surprisingly, such simple optimal strategies are lost in the extension of General Blotto to a concave scoring function. The concavity of the scoring function, which one might think would lead to more regularity, creates incentives away from equal division.

Proposition 6 *For $0 < p < 1$, there is no pure strategy Nash Equilibrium in the General Blotto games valuing isolated fronts only, all pairs of fronts as well, or all sets of fronts.*

Proof If fronts are valued in isolation, any strategy $\mathbf{x}$ with $x_i > 0$ for some i will lose to the strategy

$$\left(y_i = 0,\ y_j = x_j + \frac{x_i}{m-1} \text{ for all } j \neq i \right).$$

The payoff to $\mathbf{x}$ is

$$x_i^p - (m-1)\left(\frac{x_i}{m-1}\right)^p = x_i^p\left(1 - (m-1)^{1-p}\right).$$

Note

$$(m-1)^{1-p} > 1 \text{ because } p < 1 \text{ so the payoff to } \mathbf{x} \text{ is negative.}$$

When all pairs of fronts are valued too, every pure strategy still loses to some other pure strategy. For any strategy $\mathbf{x}$ we can re-number the fronts without loss of generality so that $x_1 \geq x_2 \geq x_3$ and $x_1 > 0$. Then $\mathbf{x}$ loses to $\mathbf{y}$ where

$$(y_1, y_2, y_3) = \left(x_1 - \epsilon, x_2 + \frac{\epsilon}{2}, x_3 + \frac{\epsilon}{2} \right) \text{ and } (y_i = x_i\ \forall\, i > 3) \text{ for sufficiently small } \epsilon.$$

The payoff to $\mathbf{y}$ is

$$2\left(\frac{\epsilon}{2}\right)^p - \epsilon^p + \left(\frac{\epsilon}{2}\left(1 - x_1 + \frac{\epsilon}{2}\right)\right)^p - \left(\frac{\epsilon}{2}\left(2x_2 - x_1 + \epsilon\right)\right)^p - \left(\frac{\epsilon}{2}\left(2x_3 - x_1 + \epsilon\right)\right)^p + \sum_{i=4}^{m} 2^{1-p}(\epsilon x_i)^p$$

$$\begin{aligned} &= \left(\tfrac{\epsilon}{2}\right)^p \left(2 - 2^p + (1-x_1)^p - (2x_2 - x_1)^p - (2x_3 - x_1)^p + \textstyle\sum_{i=4}^m 2x_i^p\right) \text{ to first order in } \epsilon \\ &\geq \left(\tfrac{\epsilon}{2}\right)^p \left(2 - 2^p + (1-x_1)^p - x_2^p - x_3^p + \textstyle\sum_{i=4}^m 2x_i^p\right) \end{aligned}$$

which has a minimum value under the restriction $\sum_{i=1}^m x_i = 1$, $x_i \geq 0$, $x_2 \leq \frac{1}{2}$, $x_3 \leq \frac{1}{2}$ of

$$\left(\frac{\epsilon}{2}\right)^p \left(2 - 2^p + 1 - 2\left(\frac{1}{2}\right)^p\right) > 0.$$

These restrictions are more than satisfied by the assumptions of the game. This minimum value is positive because

$$2-2^p = 2^p\left(2\left(\frac{1}{2}\right)^p - 1\right) \text{ and } 2^p > 1 \text{ and } 2-2^p > 0.$$

The payoff to strategy $\mathbf{x}$ is negative.

For the case valuing all sets of fronts, we will use induction on the number of fronts. Consider $m=3$. This case follows from the proof above, when we considered just pairs of fronts. Here we have the same sets as before, all single fronts and all pairs of fronts, and one additional set, the set of all three fronts. We have to add just one term to the payoff function. The payoff to $\mathbf{y}$ is that same positive quantity as before, plus $\left(\frac{\epsilon}{2}\right)^p (x_1x_3 + x_1x_2 - 2x_2x_3)$ to first order in ϵ. This new term can only increase $\mathbf{y}$'s payoff because $(x_1x_3 + x_1x_2 - 2x_2x_3) \geq 0$.

Furthermore, in showing that $\mathbf{y}$'s payoff is positive, we could just as well have assumed that $\sum_{i=1}^m x_i$ was at most 1 instead of exactly 1.

Assume that any $\mathbf{x}$ with

$$\sum_{i=1}^m x_i \leq 1,\ x_1 \geq x_2 \geq x_3, \text{ and } x_1 > 0$$

loses to $\mathbf{y}$ where

$$(y_1, y_2, y_3) = \left(x_1 - \epsilon, x_2 + \frac{\epsilon}{2}, x_3 + \frac{\epsilon}{2}\right) \text{ and } y_i = x_i\ \forall\, i > 3$$

for sufficiently small ϵ when all sets of fronts $1 \cdots m$ are valued. Let $\mathbf{y}$'s payoff be $\pi > 0$. Now add front $m+1$. An arbitrary pure strategy $\mathbf{x}'$ can be formed by taking $x'_i = x_i\ \forall\, i < m+1$ and choosing any x'_{m+1} such that $\sum_{i=1}^{m+1} x'_i \leq 1$. We define $\mathbf{y}'$ just as we defined $\mathbf{y}$. Strategy $\mathbf{y}'$ plays against $\mathbf{x}'$ on all sets of fronts where $\mathbf{x}$ played $\mathbf{y}$ plus on each of those sets with front m+1 added, as well as on front m+1 in isolation. The payoff to $\mathbf{y}'$ on the sets of fronts excluding front m+1 is π. The payoff to $\mathbf{y}'$ on the sets of fronts including m+1 is $(x_{m+1})^p\,\pi$. The payoff to $\mathbf{y}'$ on front m+1 in isolation is 0. So the total payoff to $\mathbf{y}'$ is positive. ∎

Next, we let $p=1$. This scoring function fosters pure equilibria. In fact, every pair of pure strategies is a Nash Equilibrium when $p=1$ and fronts are valued in isolation, because the game must end in a tie. The payoff function is

$$\sum_{i=1}^m (x_i - y_i) = \sum_{i=1}^m x_i - \sum_{i=1}^m y_i = 0$$

as both players must use up their entire budgets.

The next claim extends to the cases when all pairs of fronts are valued and when all sets of fronts are valued. In these cases, the linearity of the scoring function is sufficient to create a Nash Equilibrium in pure strategies.

Proposition 7 *There is a Nash Equilibrium when both players allocate $\frac{1}{m}$ to every front, $p = 1$, and all pairs of fronts or all sets of fronts are valued.*

Proof Because margin of victory is valued linearly, a player's optimal strategy is to maximize his own score on the sets of fronts being valued without regard to his opponent's strategy. The symmetry between fronts guarantees that the maximum score occurs when all fronts are allocated equally, $\mathbf{x} = (\frac{1}{m}, \cdots, \frac{1}{m})$. Consider an arbitrary strategy $(y_1, \cdots, y_m)$. The payoff to $\mathbf{y}$ will always improve by replacing (y_i, y_j) with $(\frac{y_i+y_j}{2}, \frac{y_i+y_j}{2})$, unless this changes nothing at all as in the case $y_i = y_j$. The sets without fronts i and j are unaffected. The sets with just front i are affected equally and oppositely from the sets with just front j. And the payoffs on the sets with both fronts i and j increase because $\left(\frac{x_i+x_j}{2}\right)^2 > x_i x_j$. By repeating this substitution for all i and j, $\mathbf{y}$ approaches the optimal strategy, $\frac{1}{m}$ on all fronts. ∎

When $p > 1$, the scoring function becomes convex. For this class of games, a large margin of victory on one front is more valuable than a small victory on many fronts, and the General Blotto game defined on the isolated fronts has a unique class of equilibria.

Proposition 8 *When $p > 1$ and only the isolated fronts are valued, there is a Nash Equilibrium when both players allocate all their resources to any one front. The front that player X picks need not be the same front player Y picks.*

Proof Let $\mathbf{x}$ be a strategy that allocates the entire budget to one front, and without loss of generality assume it's the first front.

$$\mathbf{x} = (1, 0, \cdots, 0).$$

Then the payoff to strategy $\mathbf{y}$ is

$$\begin{aligned} & \sum_{i=2}^{m} y_i^2 \; - \; (1 - y_1)^2 \\ = \; & \sum_{i=2}^{m} y_i^2 \; - \; \left(\sum_{i=2}^{m} y_i\right)^2 \\ = \; & - \sum_{j>i=2}^{m} 2 y_i y_j. \end{aligned}$$

This payoff has a maximum of 0 when all terms are 0. If for some k, $y_k = 1$, then for all $\mu \neq k$, $y_\mu = 0$, so every term in the payoff is 0 because at least one factor, y_i or y_j, must be 0. So a best response to player X allocating all resources to a single front is for player Y also to allocate all resources to a single front. ∎

Once again, an extension to a General Blotto game can simplify optimal play, but the simple solution disappears when further extensions are made. When other sets of fronts are valued with the convex scoring rule, players want big margins of victory, but also want to be competitive on all fronts. A given strategy can always be exploited by one of these approaches.

Proposition 9 *There is no pure strategy Nash Equilibrium for the case $p > 1$ valuing all pairs of fronts or all sets of fronts.*

Proof Let $\mathbf{x}$ be an arbitrary pure strategy. First consider the case that $\mathbf{x} \neq \left(\frac{1}{m}, \cdots, \frac{1}{m}\right)$. Without loss of generality we can assume $x_1 \neq x_2$ because not all x_i are equal. The strategy $\mathbf{y}$ where

$$y_1 = y_2 = \frac{x_1 + x_2}{2} \text{ and } y_i = x_i \;\forall\, i > 2$$

then beats $\mathbf{x}$ with payoff $\left(\frac{x_1 - x_2}{2}\right)^{2p}$ in the case of all pairs, and with payoff

$$\left(\prod_{i=3}^{m}(1 + x_i^p)\right)\left(\frac{x_1 - x_2}{2}\right)^{2p}$$

in the case of all sets.

Now consider the case that $\mathbf{x} = \left(\frac{1}{m}, \cdots, \frac{1}{m}\right)$. Then the strategy $\mathbf{y}$ where

$$y_1 = y_2 = \frac{1}{m} - \epsilon,\; y_3 = \frac{1}{m} + 2\epsilon, \text{ and } y_i = \frac{1}{m} \;\forall\, i > 3$$

with sufficiently small ϵ beats $\mathbf{x}$. The payoff to $\mathbf{y}$ in the case of all pairs is

$$\begin{aligned}
& (2\epsilon)^p - 2(\epsilon)^p + 2\left(\frac{1}{m}\epsilon - 2\epsilon^2\right)^p - \left(\frac{2}{m}\epsilon - \epsilon^2\right)^p + (m-3)(2^p - 2)\left(\frac{\epsilon}{m}\right)^p \\
= \;& \epsilon^p(2^p - 2)\left[1 + (m-4)\left(\frac{1}{m}\right)^p\right] \text{ to first order in } \epsilon \\
> \;& 0.
\end{aligned}$$

In the case of all sets, the payoff is

$$\begin{aligned}
& \left(1 + \left(\frac{1}{m}\right)^p\right)^{m-3}\left[(2\epsilon)^p - 2(\epsilon)^p + 2\left(\frac{1}{m}\epsilon - 2\epsilon^2\right)^p - \left(\frac{2}{m}\epsilon - \epsilon^2\right)^p - \epsilon^{2p}\right] \\
= \;& \left(1 + \left(\frac{1}{m}\right)^p\right)^{m-3}\left[\epsilon^p(2^p - 2)\left(1 - \left(\frac{1}{m}\right)^p\right)\right] \text{ to first order in } \epsilon \\
> \;& 0. \quad \blacksquare
\end{aligned}$$

We can summarize our results in the following table.

Pure Nash Equilibria in Blotto Games

Payoff Structure	Isolated Fronts	All Pairs	All Sets
$\mathbf{p = 0}$	none	none ($m > 3$)	$\frac{1}{m}$
$\mathbf{0 < p < 1}$	none	none	none
$\mathbf{p = 1}$	anything	$\frac{1}{m}$	$\frac{1}{m}$
$\mathbf{p > 1}$	$(1, 0, \cdots, 0)$	none	none

The entry $\frac{1}{m}$ refers to the strategy that allocates this amount to every front. The entry $(1, 0, \cdots, 0)$ refers to all strategies that allocate the entire budget to a single front.

Our third extension considers a finite population of Blotto players who all play pairwise. This N player version of Blotto has a pure strategy equilibrium whenever the mixed strategies that form a two-player equilibrium have finite support with rational probabilities over that support. Recall Propositions 2 and 3, which discuss mixed equilibria of the Colonel Blotto game.

Applications of General Blotto

The original Colonel Blotto describes contexts of *allocative strategic mismatch.* It captures situations in which a player allocates resources hoping to mismatch what the other player does. The preferred mismatch is not symmetric. Each player seeks to win by a little on as many fronts as possible and to lose by a lot on only a few fronts. Colonel Blotto also captures *competitive games with budgets:* players allocate fixed budgets and their payoffs depend on the allocation of the other player. Allocative strategic mismatch and competitive games with budgets arise in a variety of contexts. Unfortunately, most real world games usually do not fit into the rigid structure of Colonel Blotto. As we now show, General Blotto, which allows for non-winner-take-all payoffs, externalities between fronts, and pairwise competition among a population of players, more accurately captures many of these situations.

Military Allocations

Modern wars no longer take place in trenches and on fields. Even though fighter planes can tear through the sky at remarkable speeds, moving aircraft carriers and tens of thousands of troops still takes time as does establishing command and control centers and bases of operations. Thus, as in the past, a military power must decide where to allocate its resources: its troops, carriers, planes, and intelligence personnel. A military prefers having more resources than its opponent on any front. However, given the ability make quick local movements of troops and the potential for escalation, militaries also care about resources at combinations of fronts. The former emphasis on the number of troops along a border has been replaced with a concern over the number of troops in a theater of operation, which might include several borders.

A military also allocates resources to different departments and projects. Here the fronts are strategic goals rather than geographic areas. For example, a military must decide how many troops to train for peacekeeping and how many to train for covert operations. At the same time, terrorist organizations, like al-Qaeda, decide whether to join insurgent movements or to help coordinate communication and recruitment at a broader scale. The need for peacekeepers depends on al-Qaeda's focus on the insurgency, while covert operatives would be better equipped to disrupt communications and infiltrate the network. The military has more resources overall, but needs a large resource advantage to succeed on a given front. Furthermore, the fronts are related. Infiltration of the terrorist network might hurt al-Qaeda's ability to support the insurgency, and putting down the insurgency might hinder al-Qaeda's recruitment efforts.

Politics

General Blotto applies to domestic politics as well as international relations. Consider a political party's problem of where to send party luminaries to help with congressional campaigns. The fight for control of the house is a game of allocative strategic mismatch with 435 fronts, one for each congressional district, and the fight for control of the senate is a strategic mismatch game on combinations of these fronts. Sending a party luminary to a congressional district necessarily sends that person to a senate district as well. Thus, the congressional campaign game is General Blotto in which combinations of fronts matter.

Or consider the game in which candidates make campaign promises. In effect, politicians partition the electorate and make promises to groups within that partition (Myerson 1993). If we assume that whichever candidate offers the most to a group wins the group's vote, we have a Colonel Blotto game. However, that assumption seems like a bit of a stretch. More likely, we should expect that the better promise, the larger the share of the vote. The expected payoff at each front would not be winner-take-all but a smoothly increasing function of the size of the resource advantage.

Sports

Many sports can be modeled as games of strategic mismatch. We begin by considering boxing. Boxers develop different skills, either through training or natural ability, and then compete pairwise against each other. We might characterize the relevant skills for a boxer along five dimensions: reach, speed, stamina, power, and chin (ability to take a punch). We could compare two boxers by comparing their reach, speed, and stamina and aligning the power of each boxer with the chin of his opponent. We then have a Colonel Blotto game. Consider the following approximations of the skills for three famous boxers from the 1970's. The ratings are meant to establish which fighter had an advantage in which category, and should not be interpreted as measurements on some absolute scale.

Boxer	**Reach**	**Speed**	**Stamina**	**Power**	**Chin**
Ali	2	3	2	1	3
Frazier	1	1	3	2	2
Foreman	2	2	1	3	2

Given these assumptions, Ali has a 3-2 advantage over Frazier and a 2-1 advantage over Foreman (with two ties), and Foreman has a 3-1 advantage over Frazier (with one tie). In actual contests, Ali beat Frazier in two of three grueling fights, and Ali beat Foreman using a "rope-a-dope" strategy. In two fights, Foreman dominated Frazier. The skills of these boxers cannot be captured on a single dimension. Frazier and Ali were evenly matched. Ali had at best a slight advantage over Foreman. But Foreman dominated Frazier. Colonel Blotto helps us makes sense of how this could be. We need to know where strengths and weaknesses lie, and how they match up in a particular competition. This attempt to model boxing as a Colonel Blotto game organizes our thinking and shows the possibility of cycles, but it ignores the fact that combinations of skill matter. Having an advantage in both speed and power is especially valuable. This extension can be handled by General Blotto.

General Blotto also captures competitions between teams in a league with a salary cap. In the NBA, for example, teams organize their roster by position: point guard, shooting guard, small forward, power forward, and center (positions 1,2,3,4, and 5). It's a game of allocative strategic mismatch, and the benefits marginally increase with advantage at a position. A team with Kobe Bryant may exploit his advantage at the 2 position to the tune of 81 points. Moreover, combinations of positions matter. A strong back court (positions 1 and 2) can mean good ball handling, a front court (positions 3, 4, and 5) rebounding, and a high-low duo (positions 1 and 4, or 1 and 5) can exploit the pick and roll. Ball movement and help defense depend on the combination of all five positions. Alternatively, a team could organize its roster by skill set: passers, shooters, defenders, energy guys, and post-up players. The particular form of the payoff function would change, but the game would still be one of allocative strategic mismatch.

In the NBA, as in all leagues, teams play pairwise games, and they are – for the most part – stuck with the same attributes (or strategy) in each of those games. Thus, we can think of each team as picking a pure strategy that they then have to use in games against a set of opponents. To make the model most applicable to sports, the set of strategies available to each player would be further restricted because the talent pool is fixed.

Business

The business world is full of competitive games with budgets. Firms often allocate resources across several markets. The markets could be geographic or demographic, while the resources might be sales forces or advertising budgets. Outspending the competition in a market gets a firm a larger share of that market. The payoffs increase in resource difference, often with diminishing marginal returns. Firms must also allocate resources to different areas of operation. Consider two pharmaceutical companies racing to patent drugs to treat the common cold. Drug A is a better cough drop, drug B a new fever reducer. The companies have fixed budgets to divide between research and development on each drug as well as marketing. Competition on the research and development fronts is winner-take-all; only one company can get the patent on a drug. If competition on the marketing front is winner-take-all (perhaps a national drug store chain has room for only one new product on its shelves), then this is a Colonel Blotto game. On the other hand, it's possible that research and development for one drug can prove beneficial in development of the other drug, too. General Blotto would be relevant in this case.

School Rankings

Colleges and business schools compete for higher rankings in publications such as U.S. News and World Report, Business Week, and the Wall Street Journal. Given limited budgets, they have to decide how much money to spend on faculty, classrooms, computers, dormitories, and more. This creates a game of General Blotto. The payoff is a smooth mapping from advantage in each comparison. The school rankings game is played by multiple schools at once, but each school cares how it matches up against each other school as a pair. And, as in team sports, each school must play the same strategy – i.e. have the same attributes – in each pairwise comparison.

College Admissions

Just as universities compete for students, students compete for admission into college. College admissions depend on grade point average, SAT scores, and extracurricular activities. A high school student has a limited amount of time and effort to devote to studying for classes, taking an SAT prep course, or taking a leadership position in a student group. The college admissions game is multiplayer, but we could assume that colleges compare all applicants pairwise, and accept the applicants who compare favorably most frequently. An applicant's profile is, in essence, a pure strategy for this game. Competition on each front is clearly not winner-take-all. The more outstanding an applicant is in one respect, the better he looks in total. Also, natural ability combines with time and effort exerted to determine an applicant's profile, so some strategies are unattainable. It is unlikely that admissions officers look at grades, SAT scores, and extracurriculars in isolation. Taken together, they are more indicative of academic ability and personal development than viewed alone. The students compete in a game where combinations of fronts matter and payoffs are not winner-take-all, a General Blotto game.

Legal Competition

Legal cases create games of allocative strategic mismatch in their preparation stage. Technical subject matter may require both sides to call expert witnesses. If the law firms have fixed budgets to pay experts, they must choose the fields in which professional judgment will be most relevant. At the same time, they must predict their opponent's choice to avoid being faced with expert testimony that cannot be rebutted. Lawyers also must allocate their time and effort when a lawsuit provides multiple possible lines of accusation. While the defense has some time to react to the plaintiff's arguments, they benefit greatly from anticipating the plaintiff's line of accusation. In this situation, the plaintiff is better off picking one line of accusation and running with it. The plaintiff's goal is to win one front by as much as possible, rather than to win as many fronts as possible. General Blotto can handle this sort of allocative strategic mismatch too.

Organized Crime & the FBI

Consider the game played by the FBI and a criminal network to persuade suspects to rat or to hush. Each side makes an offer to each suspect (the offers are private knowledge), and each suspect takes the best offer. The suspects are fronts in a Blotto game, and the offers are allocations of limited resources. This game is winner-take-all on each front, and if we think of the payoffs as depending solely on the number of informants, then it's Colonel Blotto. Alternatively, certain pairs of suspects may be able to offer corroborative testimony. In this case, the payoffs depend on persuading pairs of suspects to rat in addition to persuading individuals. So again, we care about combinations of fronts.

Co-evolution of Predators and Prey

General Blotto may seem far removed from biology, but it is a good model of co-evolutionary adaptation. Predator and prey compete along several dimensions. For example, frogs and flies compete on speed. Flies want to fly quickly. Frogs want fast tongues. They compete on adherence; flies want to be slippery, and frogs want sticky tongues. We can think of each gene as a front, and competition takes place on the collections of genes that determine physical traits in a game of General Blotto. As most physical traits are determined by large numbers of genes, the function that compares two players at a particular collection of fronts may be quite complicated. The available genetic code serves as a resource constraint. At each front there may be only a few possible allocations. Structural limitations specific to the biological system may also play the role of a resource constraint. Having a hard shell and being light and quick are contradictory design specifications.[3] Again, we have a multiplayer game where each player must pick one strategy and stick with it forever. We assume interactions are pairwise (which seems safe in the case of frogs and flies, less so in the case of wolves and sheep). The society as a whole can exhibit a variety of different genotypes, a mixed strategy.

Applying General Blotto

As these examples suggest, General Blotto could be applied in a variety of circumstances. To put General Blotto to work in practice would require knowledge of the payoff function. Let's suppose, in the case of team sports, that one team could discern the payoff function accurately. If so, that team would be at an advantage relative to the other teams. However, as we've shown in this paper, it's unlikely that knowing the payoff function would provide them with an optimal pure strategy (unless of course the opposing teams were stuck with their current rosters). Just how much of an advantage accrues from knowing the payoff function is an open question, but it would seem to be as important as having more resources.

Discussion

In this paper, we have extended the Colonel Blotto game to a class of General Blotto games. General Blotto is a more realistic model of games of allocative strategic mismatch in the real world than the stylized Colonel Blotto game. Versions of General Blotto appear relevant to national security, politics, economics, social competition, and biology. Unfortunately, like Colonel Blotto, most General Blotto games fail to have pure strategy equilibria. Smoothing the payoff function and creating externalities did not simplify the strategic context. If anything, the generalizations have made the game more difficult to analyze.

The lack of pure strategy can be seen in two ways. We can see the results in a negative light, as a reason to ignore General Blotto as well as Colonel Blotto. Or, we can see these results positively. Our results show that the complexity of Blotto is not an artifact of particular assumptions. Not all games have nice equilibria. Lack of a pure strategy equilibrium, though not convenient, makes for more interesting problem domains. So, we can now say with greater certainty that games of allocative mismatch should produce interesting sequences of outcomes. Certainly, international relations remains complex (Jervis 1997) as do political competition and sports.

The complexity of the Blotto games comes not just from the lack of a pure strategy equilibrium. Matching pennies has no pure strategy equilibrium, yet we don't think it too complex to analyze. The complexity arises from the complicated mixed strategy equilibria. In General Blotto, those equilibria may be even less intuitive. Therefore, a natural question to ask is whether players could learn these equilibria.

[3] The armadillo is the exception that proves this rule.

The answers appears to be that it depends on how the players learn. Colonel Blotto and some versions of General Blotto are zero sum games, so we know that fictitious play converges to an equilibrium on those games (Brown 1951). However, discrete time replicator dynamics and best response functions do not converge for Colonel Blotto. In numerical experiments applying replicator dynamics to the divisible resources version of Colonel Blotto, we found that the system located combinations of strategies that resembled a Rock, Paper, Scissors game with many more strategies in the support. The learning dynamic cycles through these strategies without reaching a stable point. The fact that the two learning rules lead to different conclusions leaves open the question of whether the mixed strategy of Colonel Blotto is practically attainable. Also left open is whether the equilibria of some versions of General Blotto could be easier to attain.

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