Chapter 7

The S-Shaped Curve of Diffusion

The next model that we study, the diffusion model, describes processes in which something – an idea, a behavior, a disease, a joke, or even a new product - spreads across a population. Like the Markov model, this model has few moving parts. And, like the Markov model, it offers up a powerful insight. Here, we'll learn about *threshold phenomena*: outcomes that depend on some value, say the contagiousness of a disease or the hilarity of a joke, exceeding some fixed threshold. At least in the context of our model, diseases above the contagion threshold will spread, and those below will not. This means of course, that making a joke a little bit funnier (perhaps by adding a hard "k" sound in the punch line) could have a huge effect – if it pushes it over the hilarity threshold – and it could also have no effect.

I start with a basic diffusion model. We'll see some of the basic building blocks that comprise this model a little later covering chaos. Both diffusion models and models of chaos rely on *difference equations*, which describe how the process changes over time. In a diffusion model, we assume a population of people. By convention, I let N denote the number N. This can vary depending upon whether I'm considering a large city (a big N) or a small town (a small N). Initially, I assume that one of the people has got something called the "wobblies." You can think of the wobblies as a disease, as a new type of jeans, or as a behavioral affectation. These wobblies can spread to others by person to person interactions. In addition, once you've got the wobblies, you can never lose them. This makes them rather pernicious. The diffusion model allows us to figure out how fast they wobblies will spread or if they'll spread at all.

After presenting the basic model, what I call the *contact model*, I then extend it to a classic model called the SIS model. The SIS, along with a related model called the SIR model add in the fact that people can recover from the wobblies. This makes the wobblies less likely to spread, but now the model exhibits a threshold phenomenon. The contact model does not produce a threshold phenomenon. I the contact model, the wobblies alway spread.

After presenting the basics, I take a brief detour to tackle a lingering concern about unrealistic assumptions. All models contain them – even physics models. The diffusion model contains a real doozy – the random mixing assumption. So I'll take a moment to talk through when it's okay and when it's not to make an assumption that flies in the face of reality. After that, I'll present an extension of the contact model known as the *broadcast model*. It assumes that the wobblies can spread through airwaves.

The Contact Model

In the contact model, as in the Markov model, time plays out in discrete steps: hours, days, weeks, or months. The variable W_t denotes the number of people with the wobblies at time t. The contact model consists of an equation that gives the number of people who have the wobblies at time $t+1$, W_{t+1} as a function of the number who have it at time t, W_t, as well as the population size N, the frequency with which people meet, m and the likelihood that one person catches the wobblies from another β.

Let's build the equation from the bottom up. W_t people have the wobblies. This means that $(N - W_t)$ don't but could get them next period. Remember, the only way to get the wobblies is through contact with someone who has them. To make the mathematical equation more transparent, I will assume that people only meet in pairs (or that only in pairs can a wobblies uptake occur) and that when two people meet, one leads and the other follows. Wobblies can only be passed from a leader to a follower.

Okay, we're now ready for the equation. If I randomly pick someone to be a leader, the probability that person has the wobblies equals $\frac{W_t}{N}$. And, the probability that the follower doesn't have the wobblies equals $\frac{N-W_t}{N}$. Recall that β equals the probability that someone with the wobblies passes it to someone who doesn't. Given all these assumptions, the probability of a new case of the wobbles from a random meeting equals

$$\beta \left[\frac{W_t}{N}\right] \left[\frac{N - W_t}{N}\right]$$

We have one more feature to add into our model: the frequency of meetings between pairs of people. Let's assume that each of the N people acts as leader in m interactions. When people without the wobblies lead, they cannot pass anything.[1] Therefore, the total number of new cases equals the number of meetings, Nm, times the probability any one meeting leads to a new case of the wobblies.

$$W_{t+1} = W_t + Nm\beta \left[\frac{W_t}{N}\right] \left[\frac{N - W_t}{N}\right]$$

This is called a *difference equation.* If we stare at this equation, we can make three observations. First, the number of people with the wobblies always increases. And the rate

[1] The careful reader will notice that within the period someone without the wobblies could become someone with the wobblies. To make that problem go away, we'll assume that once one has to have the wobblies for a period before one can spread them.

of increase becomes larger as β and m become larger. This insight makes logical sense. The parameter β captures contagiousness, and m captures how often people meet. Increasing either should hasten the spread. Second, in light of the first observation, everyone gets the wobblies. So long as W_t is less than N the second (messy) term in the equation exceeds zero.

Third, the rate of increase will be largest when W_t equals half the population. Let's see why. We can ignore β and m because they just magnify the speed of spread. If W_t equals some small percentage of the population, say 5%, then $\left[\frac{W_t}{N}\right] = 0.05$ and $\left[\frac{N-W_t}{N}\right] = 0.95$, so their product equals 0.0475, not a very big number. But, if W_t equals half the population then $\left[\frac{W_t}{N}\right] = \left[\frac{N-W_t}{N}\right] = 0.5$, and their product equals 0.25. So the process starts slowly, builds, and then slows down again, creating the S shaped curve shown in the figure.

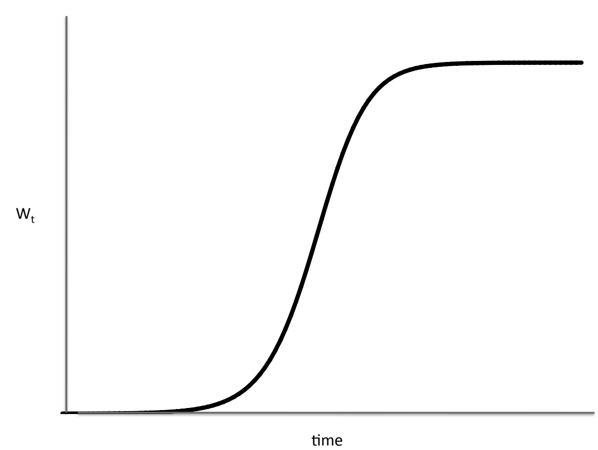

Figure 7.1: The S-Shaped Curve Produced by the Contact Model

We can think of the contact model, as a model of a disease, like the measles prior to vaccines, that will eventually infect everyone in the population As a model of the spread of an idea, like Newton's laws of gravity, it also works quite well. It even works quite well as a model of the spread of gossip. However, at first glance, the contact model seems an unrealistic as model of the spread of say a style of jeans or a new style dance. After all, not everyone will wear a particular new style or learn to Salsa dance. Fair enough. But guess what? The model handles this critique effortlessly. We just need to redefine the variable N. Instead of denoting everyone in the population, we just let N denote those people in the population who could get the wobblies. Now, if the wobblies represents a new video game, we can let N denote just the set of gamers, and our model works fine.

The SIS Model

As a model of disease, the contact model leaves out one important detail, namely that people get better. To account for that, the contact model can be extended to allow people to be in multiple states: S for susceptible, I for infected, and R for recovered (that's SIR to you). In the case we consider, those people who have recovered from the wobblies go right back into the susceptible pool (hence the name SIS). This wouldn't be true for the mumps, but it would hold for a flu virus (assuming the flu mutates sufficiently). In the cases of product, jokes, ideas, and gossip, the analog of recovery would be a loss of interest in spreading.

To account for recovery within our model, we modify the difference equation by also counting those people who recover in a given period. That number depends in turn on a recovery rate, α that captures the rate at which infected people become recovered. Our new equation for the number of the people with the wobblies as time $t+1$ now becomes

$$W_{t+1} = W_t + Nm\beta \left[\frac{W_t}{N}\right] \left[\frac{N - W_t}{N}\right] - \alpha W_t$$

which with a little rearranging becomes

$$W_{t+1} = W_t + W_t \left(m\beta \left[\frac{N - W_t}{N}\right] - \alpha\right)$$

We can use this equation to identify what in the next chapter I'll formally define as a tipping point. Suppose that the disease has just appeared, so that W_t is relatively small compared to N. We can then approximate W_{t+1} as follows:

$$W_{t+1} \approx W_t + W_t \left(m\beta - \alpha\right)$$

From this approximation, we can see that in order for the number of people with the disease to increase, the term $(m\beta - \alpha)$ must be positive. Rearranging terms a bit, we can rewrite this condition as

$$\frac{m\beta}{a} > 1$$

The term $\frac{m\beta}{a}$ is commonly referred to as the *basic reproduction number*, R_0. Diseases that have an R_0 less than one die off. We probably never hear of them. Diseases with R_0 values greater than one become matters of concern. Measles has an R_0 of around fifteen. Smallpox had an R_0 of around six. That's why they both spread so fast.

To prevent the spread of a disease, we often vaccinate a portion of the population. The percentage of people that must be vaccinated depends on R_0 in an obvious way. Let V denote the percentage of people vaccinated. In effect, vaccination reduces R_0 to $R_0(1-V)$. To prevent spread, we therefore need V to satisfy

$$R_0(1-V) < 1$$

Rearranging terms gives

$$R_0 - 1 < VR_0$$

or

$$V > \frac{R_0 - 1}{R_0}$$

Thus, to prevent the spread of measles, we need to vaccinate $\frac{14}{15}$ of the population, but to prevent the spread of smallpox we need only vaccinate $\frac{5}{6}$ of the population. The fact that our model that explains how diseases spread also tells us how many people we need to vaccinate shouldn't come as a huge surprise. In fact, it should have been expected. Once we have a model that explains some phenomenon, we can often use that same model to explore policies.

Broadcast models