Chapter 14

Giant Sucking Sound: Lyapunov Functions

A system, whether it be social, biological, or physical can go to equilibrium, it can cycle, it can produce random patterns, or it can be complex. In this chapter, we're going to learn one reason why some systems go to an equilibrium. For example, the pure coordination game goes to an equilibrium. Markets go to equilibrium. Other models, such as Conway's Game of Life, do not go to equilibrium. They produce complex outcomes. A system can go to equilibrium or not go to equilibrium for a variety of reasons. To determine whether any particular system will equilibrate requires thinking carefully about the nature of the system. Once we know how the system functions, we can often predict whether or not it will equilibrate.

In this chapter, I construct a stark framework based on the Lyapunov functions. This framework will provide sufficient conditions for a system to go to an equilibrium. In brief, if we can construct a Lyapunov function for a system, then we can say with certainty that it will go to an equilibrium. We can even place an upper bound on how long that equilibration will take. For some systems, we cannot construct a Lyapunov function. This does not mean that those systems doesn't go to equilibrium. We just don't know for sure. For those systems we will need more powerful mathematics and the equilibria that we find will be less compelling. How much less compelling will be a matter of interpretation.

Before I start, I want to make a general point about round pegs and square holes. Systems that don't go to equilibria, such as international relations, cannot fit the model. If they did, they'd have equilibria. This doesn't mean that we shouldn't try to wedge them into the model. We should. By seeing what edges we have to saw off the round peg, we learn what features of those systems cause them to not equilibrate.

The Lyapunov Bucket

I introduce the Lyapunov Model with a story. Imagine that in the middle of a playing field, there sits a large wooded bucket. One day a man comes along and fills this bucket with a biscuits. During the course of the day, at five minute intervals a man named Immanuel lets his dog Alex off leash. Alex goes over to the bucket and helps himself to some biscuits. Sometimes, he takes two or three. What's important to the story is that every five minutes, he removes at least one.

Now for an easy question: Can we say for certain that at some point Alex empties the bucket of biscuits? The answer is of course yes. The bucket contains a finite number of biscuits, say one hundred and twenty four, just to pick a number. Every five minutes the number of biscuits decreases by at least one. Therefore, at some point, the bucket must be empty.

Lyapunov Functions

This bucket story can be turned it into a model. That model will have three assumptions. Assumption one will be that there exists a positive, finite resource. In the case of Alex, these were biscuits. Assumption two will be that there exists a process that occurs in discrete time steps and that if this process is not in equilibrium, then the resource decreases. And assumption three will be that amount of the decrease in resource will be by at least some fixed amount. Note that all three of those assumptions are satisfied in the bucket story. I will then show that any model that satisfies these assumptions attains an equilibrium.

To make the assumptions formal, I define the resource level at time t to be R_t. The first assumption states that R_0, the initial level of the resource is finite.

Assumption 1 $R_0 < \infty$ and $R_t \geq 0$.

I then define a function L which maps the resource level at time t, R_t into the resource level at time $t+1$, R_{t+1}. This function L will be called a *Lyaponov Function* if and only if it satisfies two additional assumptions.

Assumption 2 $R_{t+1} = L(R_t)$. If $L(R_t) \neq R_t$, then $L(R_t) < R_t$.

Assumption 3 There exists an amount X, such that if $L(R_t) \neq R_t$, then $L(R_t) < R_t - X$

I can then state the following claim

Claim 1 *A Lypaonuv Function converges to a fixed point in which $L(R_t) = R_t$, in finite time.*

[1]

[1]Written formally, the claim would state that given A1 through A3 that there exists a time t such that $L(R_t) = R_t$. The proof of this claim is easy to follow. In each period that's not an equilibrium, the amount

The only subtlety in this model concerns Assumption 3. If the amount of decrease were not at least some amount X, then the process need not stop even with the first two assumptions in place. The paradox of the never ending decreasing function was first written down by Zeno of Elea. Suppose that I am given a huge cake (a finite resource) and that every day, I eat half of the remaining cake (a function that decreases the amount of the resources). Provided I have the ability to split subatomic particles without creating a huge explosion, I'll never finish the cake. I'll always have half left.

I want to pause of a moment and take stock of what we've done. We went from a story to formal assumptions. We've then shown that when those assumptions are satisfied, our function attains an equilibrium and does so in finite time. Let's now put this model to work.

A Return to the Voter Model

I now show that the voter model, which if you recall is just our pure coordination game played on a checkerboard, can be interpreted as a Lyapunov Model. The key steps in making that interpretationwill be to define a resource and to characterize a Lyaponov Function that describes how that resource changes. I will define the resource to be the number of pairs of neighbors who take different actions. This resource is always positive and it's finite. I'll call it D_t for disagreement at time t. There's only so many neighbors who could disagree. Note that we don't have to do any calculations to determine what the maximum disagreement could be. We only need to know that the amount is finite.

In the voter model, each agent considers changing its action one at a time, and a period consists of one update by each agent. An agent can either change it's action or not. If the agent does not change its action then nothing happens to D_t. And, if no agents change their

of the resource decreases by at least X. Therefore, there exists some number K, such that if you took out X units of the resource K times, then you'd be out of the resource, i.e that $R_0 < KX$.

actions, then the system has reached an equilibrium. Suppose that an agent does change it's action. If so, it must be that at least five of the agents neighbors were taking the opposite action of the agent and that no more than three were taking the same action. Therefore, after the agent changes it's action total disagreement decreases by at least five and increases by at most three. Therefore, total disagreement falls by at least two for each agent that changes it's state. In the picture below when the agent in the center changes it's action from up (denoted by a dark square) to down (denoted by a white square), it now agrees with five of its neighbors whereas before it only agreed with three.

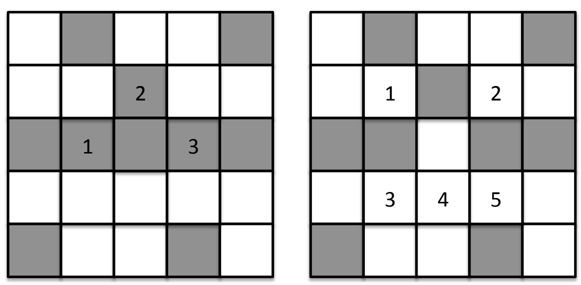

Figure 14.1: Disagreement Falls in Voter Model

Notice that we have constructed a Lyapunov Function. If the voter model is not in equilibrium, then the amount of disagreement falls by at least two (perhaps more, but by at least two). The initial amount of disagreement, just like the number of biscuits in the bucket, was finite. Therefore, disagreement can only fall for a finite time. At some point, the process has to stop.

If the analysis so far seems too easy, perhaps it is. Using disagreement as the Lyapunov Function is not the intuitive choice. That choice might instead be the number of agents in the minority. However, as the figure below shows, the number of agents in the minority won't always a Lyapunov Function. The agent in the center, which is initially in the majority, switches and joins the minority.

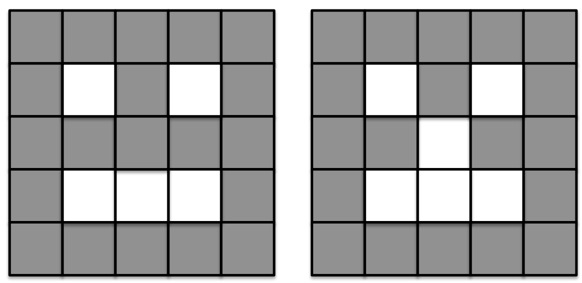

Figure 14.2: Failure of Number in Minority as a Lyapunov Function

Here then, we have our first inkling of the power of models and the connection between models. Using the Lyaponov Model, we can explain why the voter model converges. Not sometimes, not most of the time, but always. And we can also explain why it must always converge in a finite amount of time. Let's now go fry some bigger fish: models of markets and democracies.

A Pure Exchange Economy

The Lyaponov Model can also explain why a classical *pure exchange economy* reaches an equilibrium. A pure exchange economy can be described as follows: A collection of consumers shows up at a marketplace. Each brings with him or her a bundle of goods called an *endowment*. Consumers get happiness from goods that they own but may prefer to own some of the goods that others bring to the marketplace. Assume that each trade has a cost, in terms of time and effort for both parties. Call this cost c. Finally, assume that two people only trade if each benefits from the trade (in terms of happiness) by at least c.

To apply the Lyaponov Model here, it's easier to stand the logic on it's head. By that I mean that I will create a Lyaponov Function that always increases instead of creating one that always decreases. And further, to complete the model, I'll have to show that the increase is by a fixed amount and that the total amount of that resource is bounded from above.

For the pure exchange economy, that's easy. I just let the Lyapunov function equal the aggregate happiness of people with their bundles of goods. Let's think through the logic to see why this works. Each person brought a fixed endowment of goods, so the total happiness that they could enjoy in aggregate is bounded. In addition, two people only trade if they're both happier, so happiness always goes up. And finally, the cost of trading is c, so their happiness with their bundles must go up by at least c with each trade. We know then happiness is bounded (assumption 1), trade increases happiness (assumption 2), the costs of trading imply happiness increases by a fixed amount (assumption 3), so therefore trade has to stop (the claim).

One quick comments. Notice that when trade stops, the allocation doesn't have to be efficient. Of course, if it were not efficient some people in the market might then locate a trade that makes them better off, but trade could stop even though some reallocation could make everyone better off.

Lack of Equilibria

If we can write down a Lyapunov function that a system must go to equilibrium. If we cannot, that does not mean that a system doesn't go to equilibrium. It might. But the lack of a Lyapunov function means that the path to equilibrium may meander. Or, in some cases, we may not know if the system goes to equilbrium.

Consider the famous mathematical problem known as HOTPO for (*half or triple plus one*) Start with any number, say seven, if the number is odd, multiply by three and add one. If the number is even, divide it by two. If, and only if, you reach the number one, then stop. After seven, the rule gives the following sequence of numbers: 22, 11, 34, 17, 52, 26, 13, 40, 20 , 10, 5, 16, 8, 4, 2, and 1. This system has a single equilibrium, namely the number one. As the sequence goes up and down and up and down, no obvious Lyapunov function recommends itself. And, yet the sequence goes to equilibrium. Whether or not HOTPO always goes to equilibrium is not known. Every number checked so far has, but there might always be a counterexample.

To show that a system does not reach an equilibrium, it is therefore, not sufficient to say "I couldn't construct a Lyapunov Function." Instead, you have to prove that the system won't stop. Here's an example of system that won't ever reach an equilibrium point:

In this example, two pairs of people play two Rock, Paper, and Scissors. Allyson and Barry play against one another as do Camile and Dwon. The only caveat here is that the players must produce the object in order to take that action. So, to play scissors, a player needs a pair of scissors, and to play rock a person needs a rock. Initially, Allyson has a rock, Barry has paper, Camile has scissors, and Dwon also has paper. Allyson (rock) loses to Barry (paper), and Camile (scissors) defeats Dwaon (paper). The two losers, Allyson and Dwon have an incentive to trade. If they do, Allyson now ties Barry, and Dwon (rock) defeats Camile (scissors).

After the trade, Barry and Camile have an incentive to trade. If they do, Barry (scissors) defeats Allyson, and Camile (paper) defeats Dwon. A possible sequence of allocations and outcomes might look as follows (winners are in **bold**):

Players	*Allyson*	*Barry*	*Camile*	*Dwon*
Period 1	Rock	**Paper**	**Scissors**	Paper
	(Allyson trades with Dwon)			
Period 2	Paper	Paper	Scissors	**Rock**
	(Barry trades with Camile)			
Period 3	Paper	**Scissors**	**Paper**	Rock
	(Allyson trades with Dwon)			
Period 4	**Rock**	Scissors	Paper	Paper
	(Barry trades with Camile)			
Period 5	Rock	**Paper**	**Scissors**	Paper

Note that in period five, the players have returned to where they started. All that trading results in a cycle. The systems goes to an equilibrium cycle, but not an equilibrium point.

Externalities and Equilibria

In the Rock, Paper, Scissors game, the failure to reach an equilibrium point occurs because individuals care about other people's trades. When one person's action materially affects the happiness of someone not directly involved in that action, economists call the effect an *externality.* In my discussion of the Bazaar, no externalities exist. That's why trade must stop. Each trade makes the people who trade happier and has no effect on anyone else. Therefore, each trade increases happiness overall by at least the transaction cost of a trade, and we can apply the Lyapunov model. In the Rock, Paper, and Scissors game, trades materially affect others. So, we cannot apply the model.

I know want to introduce an example in which externalities exist, but we still get an equilibrium, and a Lyaponuf Function. In this example, one hundred people must each decide on a cafe': Comet Coffee or Espresso Royale. Each person has a baseline level of enjoyment from each cafe, call this H. A person's happiness equals his baseline happiness minus the number of people already at the cafe. If twenty people are at Comet Coffee then we can write Bob's happiness at comet as follows:

$$\text{Bob's Happiness} = \text{Baseline Happiness at Comet Coffee - twenty}$$

In this example, assume that initially, each person choose the cafe' that has the higher baseline for her. Then, let's assume that among those people who would like to relocate (because their favored cafe' is crowded) the person who would get the largest gain in happiness, does in fact relocate We can then ask if this process fits our Lyapunov model.

As our candidate Lyapunov Function, let's use the sum of all individual happiness scores. Initially, if anyone move, they must move from a more crowded cafe to a less crowded cafe. By assumption, that person's happiness goes up so the value of the Lyaponov Function increases. We're not finished though. This process has an externality. Everyone at the cafe' this person moves to becomes less happy. But, everyone at the old cafe is happier. And, by assumption, this person left the more crowded cafe to move to the less crowded cafe', so, in fact, total happiness does increase, even though the action of moving created a lot of externalities.

With a little effort, you can convince yourself that as long people initially went to their preferred cafe and that the person who most wants to relocate is the next to move, that no one will ever want to reverse a decision. Thus, the process has a Lyapunov Function and will stop. The lesson here is that externalities can exist but they need not prevent us from constructing a Lyapunov function.

Arms Trade

A real world example of when externalities may produce a lack of an equilibrium comes from international relations. When one country trades resources with another, those countries external to the trade may be worse off. This change in status may cause these countries to trade as well. That's not to say that there cannot be an equilibrium, but to reinforce that the externalities could be large making the construction of a Lyapunov function difficult.

In this chapter, I present two models that are represented on a checkerboard. These models both simple and abstract. One is called the *voter model* and it originated in physics. I'll show how it has implications for sociological, economic, and political processes. The second is called the *Game of Life*, and it originated on the floor of the Cambridge Mathematics Department. I'll show how it gives insights into cognition and consciousness.

These models provide are extremely simple. In each, the actors in the model, what I'll call agents, can take only two actions. In the first model, the agents will be pointing up or down. In the second model, the stakes get larger, they agents will be dead or alive. In both models, agents only care about the actions of the eight agents that surround them on the checkerboard. And yet, both models generate unexpected insights that apply in a wide variety of settings. These models fan out.