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Essay

Tipping Points*

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ABSTRACT

This paper formally defines tipping points as discontinuities between current and future states of a system and introduces candidate measures of when a system tips based on changes in the probability distribution over future states. We make two categorical distinctions between types of tips relevant in social contexts: The first differentiates between *direct tips* and *contextual tips*. A direct tip occurs when a gradual change in the value of a variable leads to a large, i.e. discontinuous, jump in that same variable in the future. A contextual tip occurs when a gradual change in the value of one variable leads to a discontinuous jump in some other variable of interest. We argue that while scholars and writers often focus on direct tips, contextual tips often make direct tips possible, such as when human rights conditions in a state deteriorate creating the potential for an uprising. The second

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differentiates tips between outcomes that belong to the same class — such as tips from one equilibrium to another — from tips that result in a change in the outcome class, such as tips that occur when an equilibrium system becomes chaotic or complex.

Introduction

References to the term tipping point have exploded in recent popular and academic writing, particularly in the context of events like the Arab Spring and the home mortgage crisis.[1] With this rapid expansion in use, the phrase is in danger of losing any scientific meaning and therefore any value for social scientists. In this paper, we formally define tipping points, correct common misconceptions about when tipping points occur, delineate substantively important categories of tipping points, and provide a candidate measure to quantify how much a system tips.

As mentioned, the media, popular writers, and many academics would have us believe that tips occur with great regularity. Elections tip (Silver, 2008), markets tip (Ellison and Fudenberg, 2003), fashions tip (Gladwell, 2000), computer applications tip (Onnela and Reed-Tsochas, 2010), public opinion tips (Liptak, 2011), societies tip (Solé, 2011), and the economy tips (Hauser, 2011). Such claims lack scientific basis given that there exists no precise definition of a tipping point. This void inhibits social scientists' ability to identify, quantify, and understand tipping behavior. In this paper, we use the mathematical formalism of dynamical systems to define tipping points as discontinuities in the relationship between present conditions and future states of the system. In other words, if gradually adjusting some variable now causes a discontinuous jump in the state of the world at some future time, then a tip occurs. As we shall show, many apparent tipping points may not be tips given our formal definition but rather inevitable upticks in adoption rates driven by positive feedbacks.

Having defined tipping points, we proceed to categorize them according to two binary distinctions that we believe will help scholars better identify tipping points empirically and to develop policies for managing tipping

[1] For example, on Sept. 1, 2011, Aljazeera ran an article entitled: *The Arab Spring: Anatomy of a Tipping Point*. In October 2011, we stopped counting at 500 the number of Google links that contained the words "arab spring" and "tipping point."

dynamics. The first distinction draws a bright line between tips that result directly from a change in the variable of interest and those that result from changes in the context surrounding that variable. We define these as *direct tips* and *contextual tips*, respectively.

As an example of a direct tip, consider the Vietcong's bombing of the United State's airbase at Pleiku on February 7, 1965, which, according to historians, caused President Johnson to escalate the war in Vietnam (Jervis, 2009). We characterize this as a direct tip. The bombing of Pleiku by the Vietcong led to escalated bombing by the United States. Of course, such an event need not be the only possible tip. As McGeorge Bundy who was then serving as national security advisor said "Pleikus are streetcars." In other words, "if you are waiting for one, it will come along" (Jervis, 2009). In contrast, when on October 18, 1987, Secretary James Baker stated on the Sunday morning talk shows that the United States would not accept Germany's interest rate increase, his actions changed the mood of investors. He did not trade any stocks, but he changed the context. The next day, Monday, October 19, a few sells resulted in more sells (direct tipping), and the Dow fell 22.6%. We characterize this a *contextual tip.*

Our second distinction separates tips that change the behavior of the system and tips that do not. Wolram (2002) classifies the states of systems, be they political, economic, or physical, as belonging to one of four classes: fixed point, periodic, random, or complex. A system can tip within — from one stable equilibrium to another — or between these classes — from random to stable or from stable to complex. We define these two types of tips as within class and across class. In many social science models, tips move the system from one equilibrium to another. For example, in social interaction models as one dials up the social influence term a stable mixed equilibrium becomes unstable creating a contextual tip. Once that system does tip, it moves to a new equilibrium configuration in which everyone takes the same action (Brock and Durlauf, 2002).

Finally, we develop a candidate measure of tippiness based on change in the expected distribution over future outcomes. The event itself does not have to be large or precipitate an immediate large response, it need only change the probability distribution over future events. So, if a small change in its virulence means that a disease will likely spread, we classify that change as a tipping point. The subsequent rise in the spread of this disease is merely the inevitable spread based on positive feedbacks and not the tip itself.

Identifying A Tip

Generally tipping points occur in some quantity (or quantities) over time, such as the number of people infected with a disease, the GDP of a nation, the approval rating of the president, or even the probabilities that actors play a particular strategy. Therefore, we will rely on the mathematical formalism of dynamical systems.[2] In dynamical systems, the variable of interest is referred to as the *state*. The state depends on time, so we denote it by x_t. Denote the initial state, sometimes called the *initial conditions*, by x_0. Let Ω_x, the *state space*, denote all possible states of the system.

The state can represent either a single entity, or it can be a statistic that aggregates characteristics of a population, or it can be a multidimensional variable that captures several quantities at once. For purposes of our classification, it will be important to distinguish between the variable of interest, x_t, and other variables, y_t, that may affect the process. For example, x_t might be a government's approval rating and y_t might be the unemployment rate. Changes in y_t can influence x_t.

Time can proceed continuously or in discrete steps. In the former case, we write the dynamical system using differential equations, such as

$$\dot{x_t} = f(x_t, y_t). \tag{1}$$

While in the latter, we use difference equations,

$$x_{t+1} = F(x_t, y_t). \tag{2}$$

Equations (1) and (2) are called *equations of motion*.

Both formalisms should be familiar to social scientists. Either can be appended to include random variables to create stochastic processes. For example, a presidential approval rating, x_t, could be described as a function of his rating in the previous period, x_{t-1}, his policy choices in the previous period, y_{t-1}, and a random shock, ϵ_t:

$$x_t = F(x_{t-1}, y_{t-1}) + \epsilon_t. \tag{3}$$

For the purpose of clarity, in what follows, we restrict attention to deterministic processes.

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2 For richer treatments of the mathematics of dynamical systems see the books by Strogatz (1994) or Hirsch *et al.* (2004).

Formal Definitions

With this basic framework in hand, it is possible to give a formal definition. For brevity, we use notation from the continuous case, but the analogous discrete time definition is straightforward. Let x_t, y_t, and z_t be dynamic processes such that future states of x_t are determined by the current states of x_t, y_t, and z_t which belong to Ω_x, Ω_y and Ω_z, respectively. We call the variable x_t the *tipping variable* and y_t the *threshold variable*, and use z_t as a place holder for anything else that might affect the function value.

At a tipping point small changes in a variable dramatically affect the state of the system at some time in the future. A tip does not imply that the state of the system undergoes some large change immediately. For example, Jackson and Yariv (2007) develop a model of how behaviors spread and prove the existence of a threshold, a tipping point, that depends on the network of connections in the population. If the initial number of adopters lies above this threshold, then the behavior spreads. If fewer people adopt initially, then the diffusion process stops. Whether or not the initial number exceeds or falls below the threshold does not have an enormous effect on the first period, but it completely changes the final distribution.

To capture this effect of present conditions on the future, we introduce a function $L_\Delta(x_t, y_t, z_t) = x_{t+\Delta}$, which gives the value of x_t exactly Δ steps after t (if the process is stochastic then L_Δ gives the expected state at $t + \Delta$).

> *A point $\tau \in \Omega_y$ is a* **tipping point** *for x_t if there exists a path $q : (-1, 1) \to \Omega_y$ with $q(0) = \tau$ such that $L_\Delta(x_t, q(s), z_t)$ is discontinuous as a function of s at $s = 0$ for some $\Delta > 0$.*

What this means is that small changes in y_t near the threshold point τ cause the future paths of x_t to rip apart in a discontinuous way.[3] Figure 1 illustrates the concept in a model of the spread of a rumor, which we detail in the section, ''contextual tipping points''. The plot depicts different paths for the number of people spreading the rumor as the probability p that the rumor is passed on in any given contact is varied. As the plot illustrates,

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[3] Note that although our definition requires a discontinuity, rather than a non-differentiable point, our definition can also capture sharp changes in the rate of change of a time series by defining the state variable x_t as the derivative or first difference of the original time series of interest.

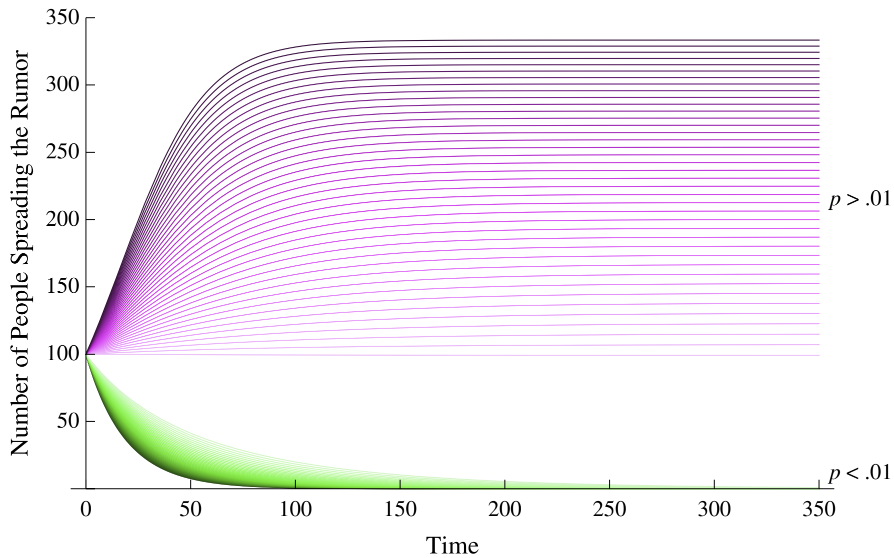

Figure 1. Different paths in the rumor spreading model (see ''contextual tipping points'' section) generated by varying the probability p that the rumor is passed on in a given contact. The paths rip away from each other when the tipping point at $p = 0.01$ is crossed.

for values of p less than 0.01 and greater than 0.01, gradually changing p gradually changes the evolution of rumor exposure, but right at $p = 0.01$ a gulf opens up between two separate groups of outcomes.

The power of the formal definition is illustrated in the following section where we develop three conditions related to tipping behavior and demonstrate how these conditions apply to common models and empirical observations.

Tipping Conditions

In this section we define three conditions that characterize potential tipping behavior. The first covers situations in which no tipping occurs. The second covers systems with initial tipping points, and the third covers cases where tipping occurs more generally.

Models Without Tips

Many social science models do not exhibiting tipping behavior. The dynamical systems framework gives a simple explanation:

> **No tipping condition:** *Whenever the equation of motion, $\dot{x}_t = f(x_t, y_t)$, intersects the line $\dot{x}_t = 0$ only once and is decreasing at that intersection, regardless of other parameter values, y_t, there is not a tipping point.*

Intuitively, the condition says that when the system is below some equilibrium point it tends to increase and when it is above that equilibrium, it tends to decrease. Thus, there is one and only one equilibrium, and that equilibrium is stable, so there is no possibility of tipping. For example, consider a common pool resource problem. Let x_t denote the number of individuals using the resource at time t. Let the payoff to each individual using the resource equal $y - 2x_t$ and normalize the payoff for those not using the resource to zero. In equilibrium, $x^* = \frac{y}{2}$, so that the payoff from using the resource equals the normalized payoff of zero. We can transform this into a dynamical system by assuming that individuals decide whether to use the resource or not based upon the relative payoffs. Then the following differential equation captures the evolution of x_t over time,

$$\dot{x}_t = y - 2x_t. \tag{4}$$

Under these dynamics, increasing the relative payoff to using the resource increases the number of individuals who use it. Figure 2 represents this graphically. The arrows along the horizontal axis represent the directional effects on x. If $x_t < x^*$, then $y - 2x_t > 0$ and x_t will increase. Alternatively, if $x_t > x^*$, then $y - 2x_t < 0$, and x_t will decrease.

We can use this formalization to determine whether this system contains a tipping point. Tipping points can occur either by changing x_t, the action, or by changing y, the parameter. First, let us consider changing x_t. Any perturbation of x_t from equilibrium will be corrected by the dynamics. Thus, we can classify this equilibrium as *stable*, and no change in x_t will cause the system to tip. Next, consider changes in y. Any small change in y shifts the line $y - 2x_t$ up or down a corresponding amount. Here again, we have no tip because as we gradually change y, the equilibrium x^* also changes gradually.

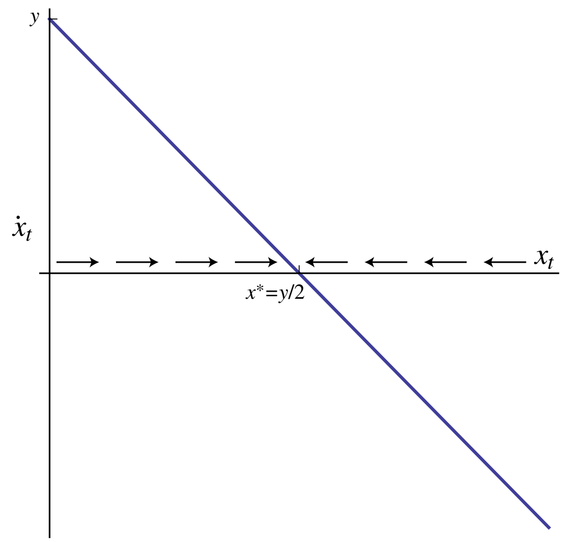

Figure 2. Dynamics in the common pool model. There is a single, stable, equilibrium at $x^* = y/2$.

This example and Figure 2 make clear why the no tipping condition guarantees a model in which gradual changes in the variable of interest or the model parameters give rise to gradual changes in outcomes. Prominent social science examples that fall in this category include the Median Voter Theorem and the equilibrium of supply and demand.

Initial Tips

In the previous section, we described a commonly satisfied condition that guarantees the non-existence of a tipping point. Now we turn to another common case in which a tipping point exists, but at a different point than typically identified. This common error in the identification of tipping points relies on graphical interpretations of data rather than formal explication of the tip.

Specifically, tipping point claims are often based on an apparent acceleration in a diffusion curve. For example, consider the graph in Figure 3, which depicts the spread of the online social network Facebook. Facebook began in a Harvard dorm room in February of 2004 and grew explosively, reaching over half a billion users by July 2010 (Facebook, 2011). From the graph, a

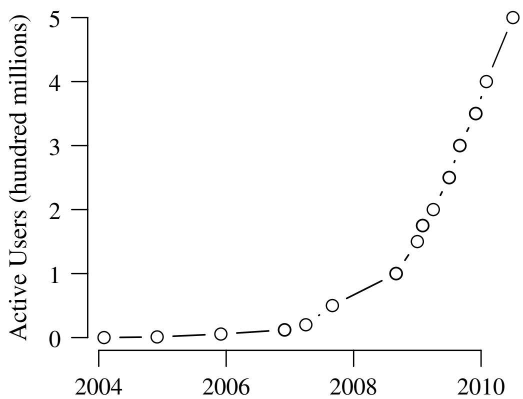

Figure 3. The growth of Facebook.

tipping point appears to occur around the beginning of 2007. But such an interpretation fails for two reasons: first to say, "Facebook reached a tipping point in early 2007," obliges some explanation as to *why* growth accelerated after 2007, not simply that it did. The cause of such an acceleration can never be determined simply by examining the diffusion curve.

Second, while a kink in the graph may appear to be compelling evidence of a tipping point, it may only be an artifact of scaling. Consider a standard exponential growth function e^{gt}, where g denotes the rate of growth per period and t denotes time. Figure 4 depicts such a function on four different time scales ($g = 4\%$). In the panel in the upper left, there appears to be a gradual increase in slope but no tipping point. As we extend the graph over 100 periods, a possible tipping point reveals itself around Period 70. But, when we extend the graph for another 150 periods, that tipping point disappears, and a new tipping point occurs around Period 200. Finally, when we plot out 500 periods, the second tipping point no longer exists, but a new one arises around Period 450. Because the vertical scale increases more rapidly than the horizontal scale, the visual tipping point shifts to the right as we extend the graph. This ambiguity, along with the fact that the curve depicts an unchanging growth process over time, leads us to reject these as tipping points.

Again, a simple model and an illustration of the equation of motion makes the intuition clear. Suppose the rate of change in industrial production of a nation is proportional to its current production, x_t. Then the equation of

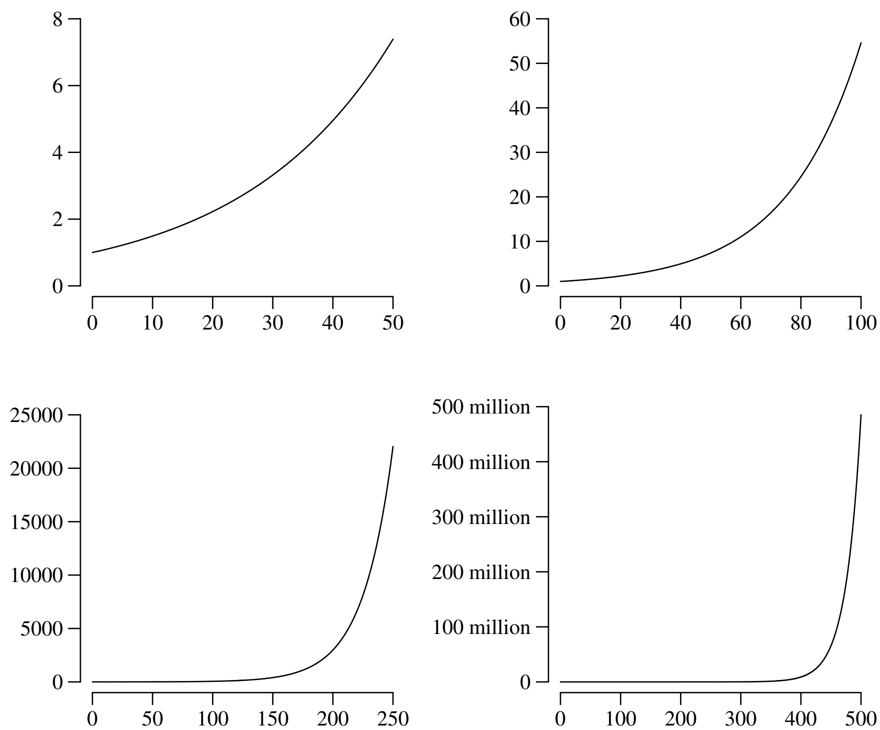

Figure 4. Exponential growth of four percent plotted on four time scales (adapted from Sterman (2000) with author's permission).

motion is

$$\dot{x}_t = rx_t, \tag{5}$$

as shown in Figure 5, and the solution is exponential growth: $x_t = x_0 e^{rt}$. As the figure illustrates, Equation (5) does not satisfy the no tipping condition. In fact there is a tipping point in this model at $x_t = 0$. There are two distinct trajectories that this model can give rise to: if $x_t = 0$ then $x_t \equiv 0$ for all time, but as soon as $x_t > 0$ for any t, then $\lim_{t\to\infty} x_t = \infty$. This observation gives rise to our second tipping condition:

> **Initial tipping condition:** *Whenever the equation of motion, $\dot{x}_t = f(x_t, y_t)$, intersects the line $\dot{x}_t = 0$ only at $x_t = x^*$ and is positive for all $x_t \neq x^*$ (and all values of y_t) then there is only one tipping point and that tipping point occurs at x^*.*

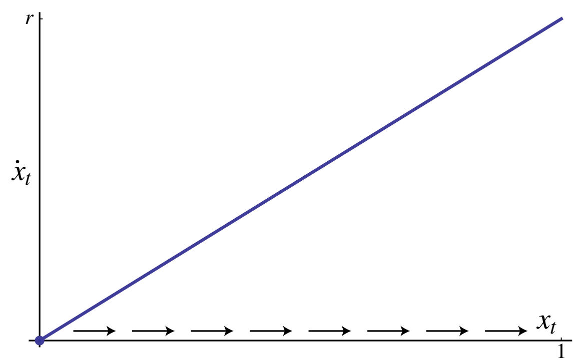

Figure 5. Dynamics in the exponential growth model.

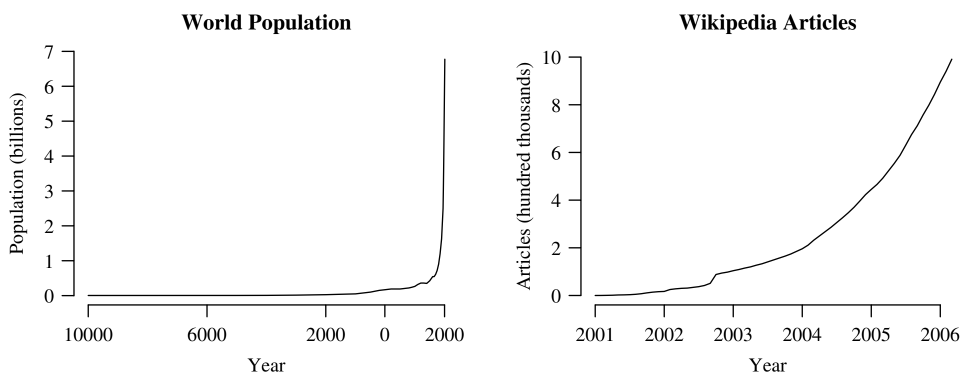

Figure 6. Exponential growth of the world population and articles on Wikipedia.

Any linear first order positive feedback loop (Sterman, 2000), like the one described in Equation (5), produces exponential growth and thus will satisfy the initial tipping condition. But exponential growth is not the only model captured by this condition. Most standard diffusion models (e.g. the Bass (1969) model) also fall into this category. Thus, it is not surprising that graphs like those in the bottom panels of Figure 4 are extremely common and that the media and others see tipping points everywhere.

For example, Figure 6 plots growth in the world population and articles on Wikipedia. While both of these graphs appear to exhibit tipping points; around 1700 for the world population and sometime in 2002 for Wikipedia, it is unlikely that either of these apparent kinks represents a tipping point by our definition. However, as described in the initial tipping point condition, these systems do contain tips — when Mark Zuckerberg put up the first "thefacebook" page and Jimmy Wales and Larry Sanger launched Wikipedia the future was forever changed — but those tipping points occurred long before the more commonly identified kinks in the diffusion curves. This distinction has important consequences. A small intervention at the initial tipping point may have prevented the creation of the world's largest social network or encyclopedia, but had something come along to impede the growth of either Facebook or Wikipedia at the times more commonly pointed to as tipping points, it most likely would have only temporarily slowed their inevitable diffusion.

Tipping Existence

The previous two sections gave conditions that eliminate or restrict the possibility of tips. Now we turn to a condition that guarantees more interesting tipping behavior. Consider the following model of a *bandwagon process*. Suppose a population of R Iowa Republicans is choosing between two presidential primary candidates, M and N. Assume that the voters' only motivation is to support the candidate most likely to win the general election, and they take the current level of support for each candidate in Iowa as a signal of the candidates' electability. As a candidate gains more support, he is viewed as more electable, leading to even more support. When the candidates have equal support, voters are indifferent between them. Specifically, let $x_t = M_t - N_t$ denote the difference in support for M and N at time t, and suppose that

$$\dot{x}_t = \begin{cases} x_t, & |x_t| \neq R \\ 0, & |x_t| = R \end{cases} \tag{6}$$

Eventually, the bandwagon process converges to either $x_t = R$ or $x_t = -R$. Notice that unlike the common pool resource model, this model has a tipping point at $x_t = 0$. Suppose first that x_t lies just to the left of 0, then the process will converge to the equilibrium in which all of the voters support candidate N. But suppose that we perturb x_t so that it exceeds 0. In that case, the process converges to all M supporters. Thus, small changes in the number of people who initially support M or N can have large effects on the final outcome. Later when we present our tipping typology we will classify

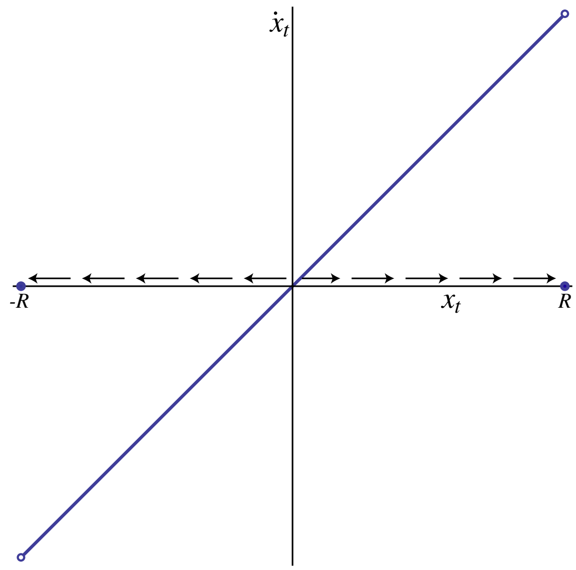

Figure 7. Dynamics in the bandwagon model. There are two stable equilibria at R and $-R$ and an unstable equilibrium at $x_t = 0$.

this as a *direct tip* as a small change in the variable itself, x_t, leads to a large change in its equilibrium value.

This example leads us to formulate the following tipping existence condition:

> **Tipping existence condition:** *Whenever the equation of motion, $\dot{x}_t = f(x_t, y_t)$, intersects the line $\dot{x}_t = 0$ at a point $x_t = x^*$ and is increasing at that intersection then x^* is a (direct) tipping point.*

The tipping existence condition describes situations in which a tipping point is guaranteed to exist. Again, many social science models fall into this category. In particular, any model with an unstable equilibrium point, such as models for the selection of a technology standard (e.g., Arthur, 1989) or game theoretic models like the one described in Table 2 below fit this condition. As we increase the dimensionality of the dynamical system and the extent of nonlinearities, tipping points become more likely as the potential for more equilibria also increases. To give a hint of why this occurs, we next construct a one dimensional nonlinear dynamical system that we will call the *resource model.* Solé (2011) shows that variations of this model have been used to characterize disease, economic collapse, communication, insect behavior, ecosystem desertification, and collective wisdom. We will describe a model of economic collapse (this is a simplified version of a model discussed by Janssen *et al.* (2003)).

Let x_t equal the amount of some renewable resource available at time t, and let C denote the maximum amount of that resource that could ever be available. Assume that the resource replenishes at a rate equal to the current availability times the excess capacity, $(C - x_t)$, and that a constant proportion α of the resource is consumed at any given time. To complete the model, assume that a fixed amount of the resource A is consumed independent of the current supply (more precisely whichever is lower of A or x_t is consumed). We can then write the change in x_t as follows:

$$\dot{x}_t = x_t(C - x_t) - \alpha x_t - \min\{A, x_t\}. \tag{7}$$

In Figure 8 we assume that $C = 1$, $\alpha = 0.25$, and $A = 0.05$. Whenever the curve is above zero, $\dot{x}_t$ is positive, so the amount of resources available will increase and the system will shift to the right along the curve. Conversely, when the curve is below zero, x_t will decrease. Values of x_t for which $\dot{x}_t = 0$ are equilibria. For these parameters, there are two interior equilibria as well as an equilibrium at $x_t = 0$. Note that this last equilibrium corresponds to the collapse of the society. Using the arrows along the horizontal axis as a guide, we can see that the equilibrium at 0.17 is unstable. Therefore, if the

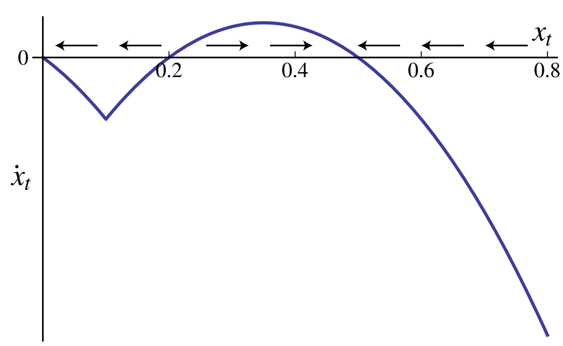

Figure 8. The differential equation governing the dynamics of the resource model. The arrows indicate whether the amount of resources available, x_t, will increase or decrease depending on the current amount available.

Table 1. The tipping point typology.

	Within class	Across class
Direct		
Contextual		

society were producing $x_t = 0.2$ it would be poised at a tipping point. One small change in behavior would lead to either collapse or a new, stable equilibrium of higher productivity. This tipping point is similar to the tipping point in the bandwagon process. We characterize it as a *direct tip* because it results from a change in the state variable.

We can contrast this with a second type of tip that results from a change in a parameter, what we call a *contextual tip* (refer again to Figure 8). Changes in any of the three parameters C, α, or A will shift the curve. For example, if we increase A, the fixed amount of resources consumed independent of resource availability, the entire curve shifts downward causing the two interior equilibria to converge. When A become sufficiently large, the entire curve lies below the horizontal axis. At that point, the interior stable and unstable equilibria disappear, and only the collapse equilibrium remains.

Note then, the two fundamental differences between the tip caused by changing A and the tip caused by changing x_t. First, the change in A is a change in a parameter, not in resource levels, x_t. Tips can therefore arise not only because of actions but because the environment shifts. The distinction between direct tips caused by changes in the state versus contextual tips caused by changes in the environment surrounding the state will form half of our typology. Second, the direct tip caused the system to move from one equilibrium to another, while the contextual tip caused the set of equilibria to change. Put differently, direct tips change where the system *will* go. Contextual tips change the set of states where the system *can* go.

A Typology of Tipping Points

Now that we have formally defined tipping points, we proceed to break the set of all tipping points up into categories that can help us to better understand when tips occur and how to prevent or encourage them. In this section we split tipping points according to the two binary distinctions that we have already discussed: tipping points are either *direct* or *contextual* and either *within class* or *across class*. This creates four categories, which are represented in Table 1. In Table 3, we present an application of the typology applied to several examples described in the text along with some other well-known tips. To determine if a particular tip is direct or contextual, identify the tipping and threshold variables. If they are the same, it is direct, if not, it is contextual. To decide if a tip is within or across class, examine the behavior before and after the tip. If they fall into the same class, the tip is within class, if not, it is across class.

Direct Tips

Our first category captures the special case when the threshold variable y_t equals the tipping variable x_t.

> *A* **direct tip** *occurs when a gradual change in x_t results in a discontinuous jump in future values of that same variable.*

That is, there is a distinguished value of our variable of interest so that any move away from this special value has big consequences for the future of that same variable. This special case covers many of the tipping points that social scientists study such as Schelling's tipping points into segregation and Jackson and Yariv's diffusion tipping points. Below we describe two subcategories of direct tipping points that arise frequently within social systems: *unstable equilibria* and *basin borders*. We also describe a type of direct tip associated with *chaotic systems* that receives less attention but which also occurs in social systems.

Unstable equilibria. Earlier we wrote a common pool resource problem and a bandwagon process as dynamical systems. In the first, the equilibrium was stable, in the latter, one of the equilibria, the interior equilibrium, was unstable. In a stable equilibrium, perturbations in the state self-correct. For example, in a perfect market if the price of a commodity randomly rises above the equilibrium price, demand falls and supply increases causing a surplus and a subsequent fall in prices. In contrast, at an unstable equilibrium, small perturbations send the system off in a new direction, typically to a new equilibrium. Many less mathematical conceptions of tipping points such as the Colliers' notion of a *critical juncture*, or moments of reorientation

Table 2. Game with two pure strategy Nash equilibria and one mixed.

	A	B
A	**2**,2	**0**,0
B	**0**,0	**1**,1

in which long term trajectories get set in place (Collier and Collier, 1991), fall into this category.

In game theory, equilibria can also be classified by their stability, with the caveat that stability depends on how one characterizes the response behavior of the players. Here we assume standard best response dynamics. Consider any 2×2 game with two pure strategy equilibria and one mixed strategy equilibrium, such as the game specified by the payoff matrix in Table 2. Both pure strategies are Nash Equilibria as is the mixed strategy in which one-third of the population chooses A and two-thirds choose B. If we assume a population of individuals playing this game against one another, and if we assume that all of the individuals best respond, then the mixed strategy will be unstable. As soon as one member of the population changes her action, she produces a direct tip.

Basin Borders. Unstable equilibria occur at specific points, but social processes can also exhibit regions of instability. This becomes inevitable in systems that involve multi-dimensional actions. If one imagines a two dimensional space with two stable equilibria, then instead of an unstable point, there will exist a curve or boundary line between the two basins of attraction for the equilibria.[4]

In addition, social systems can also produce plateaus. Standing on top of the plateau is relatively stable, but if you accidentally wander off the edge, you are shot off in a new direction. To see how this can occur, consider a variant of our bandwagon model. Suppose that each of two primary candidates, A and B, can count on 20% of the electorate but that the other 60% are independent voters who change their opinions randomly. For convenience, we can model the preferences of these other 60% as a random walk on the line. Let X_t denote the total percentage of all voters who prefer candidate A at time t. Assume that $X_0 = 50$. According to the random walk model,

[4] An equilibrium's basin of attraction characterizes the points that will lead to the equilibrium given the dynamics.

Table 3. Classification of tips.

Example	Threshold variable	Tipping variable	Class before	Class after	Type
Schelling model	Percent minorities in a neighborhood	Percent minorities in a neighborhood	Equilibrium	Equilibrium	Direct, within class
Jackson and Yariv (2007) model	Fraction of agents adopting behavior	Fraction of agents adopting behavior	Equilibrium	Equilibrium	Direct, within class
Bandwagon process (Figure 7)	Number of liberal voters	Number of liberal voters	Equilibrium	Equilibrium	Direct, within class
Game in Table 2	Fraction of agents playing mixed strategy	Fraction of agents playing mixed strategy	Equilibrium	Equilibrium	Direct, within class
HD DVD vs. BluRay	Fraction of BluRay adopters	Fraction of BluRay adopters	Stochastic	Equilibrium	Direct, across class
Arthur (1989) model	Fraction of technology 1 adopters	Fraction of technology 1 adopters	Stochastic	Equilibrium	Direct, across class
Logistic map with $r > 3.57$	x_t	x_t	Chaotic	Chaotic	Direct, within class
Basic rumor model	R_0	Fraction of population exposed	Equilibrium	Equilibrium	Contextual, within class
Game in Table 4	Payoffs	Set of equilibria	Equilibrium	Equilibrium	Contextual, within class
Spatial rumor model	Population density	Fraction of rumors that reach DC	Equilibrium	Equilibrium	Contextual, within class
Logistic map	r	x_t	Equilibrium, periodic, or chaotic	Equilibrium, periodic, or chaotic	Contextual, across class
Tunisian uprisings	Number of protesters	Number of protesters	Equilibrium	Equilibrium	Direct, within class
Egyptian uprisings	Number of protesters	Number of protesters	Equilibrium	Complex	Direct, across class
Overloaded camel	Amount of straw	Integrity of camel's back	Equilibrium	Equilibrium	Contextual, within class

Figure 9. A peak and a plateau.

with probability $\frac{1}{2}$ the proportion of the vote going to A will either increase or decrease by one each period. To create tipping points, we add one more assumption. Assume that if either candidate falls below 35% support, her core supporters will abandon her immediately. As long as the state stays in the interval $(35, 65)$, the process is random. However, once the process reaches 35 or 65, it becomes "locked-in" to a new trajectory.

In this toy example, we build in the plateau with our assumption of core supporters. Plateaus can also emerge when multiple forces are in play. Ellison and Fudenberg (2003) construct a model of agglomeration that includes increasing returns to choosing the same location as other firms, but there are price advantages from choosing the smaller market. They show that these two forces combine to form plateaus and not peaks. A similar logic may apply in the case of standards — the recent competition between HD-DVD and BluRay high definition video discs being an example. Early on consumers chose one type of player or the other based on personal preferences, price, availability, or their personal prediction of which technology will become the future standard. This is like the random walk stage of our toy example. Eventually, BluRay gained enough market share that it became clear it would be the standard, and gradually demand for HD-DVD dropped off altogether — the market crossed an absorbing barrier and was locked-in to the BluRay standard. Both the threshold variable and the tipping variable are equal to the fraction of BluRay adopters, so this is a direct tip. Arthur's (1989) well known model of standards formation shares this exact structure.

In this toy plateau model, we can identify the tipping point exactly: it is when one independent voter pushes support for one of the candidates past the threshold. We can therefore delineate the basins of attraction of the two equilibria. In the models drawn from the literature — agglomeration of industries and the lock-in to BluRay, identification of basins may be problematic. Rather than being a single point at which randomness gives way to inevitability, the transition proves more gradual. We return to this subtlety in identifying tipping points in empirical data in more detail in the conclusion.

We should also add that so far we have ignored the fact that the actors in these systems may have incentives to pull the system from one basin into another. Activists attempting to produce political change, leaders of social movements, and entrepreneurs attempting to create new markets all fit this description. Not all of these efforts succeed. In fact, most don't. They fail to reach the tipping point. That failure may have less to do with the action taken then with the context.

Sensitivity to Initial Conditions: Chaotic Tipping Points. The third type of direct tip falls under the broad category of chaos. Chaos, or extreme sensitivity to initial conditions, is more colloquially understood as the *butterfly effect* — the notion that small changes can have big consequences. We consider here deterministic chaotic systems. Because chaotic systems are so sensitive, their behavior is effectively impossible to predict: any error in measuring the initial state of the system will lead to different predicted outcomes (Oremerod, 2001).

Often we think of chaotic systems as arising from the interaction of a huge number of interacting variables, so many variables that a small change in one variable causes a cascading string of changes in many other variables, so that it is impossible to tell where things are headed. The classic example of such a system is the weather. In fact, the term "butterfly effect" came from the title of a talk given by mathematician and meteorologist Edward Lorenz, "Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?"

Focusing on the butterfly metaphor results in two common inferential errors. First, rather than producing chaos, interactions between many coupled variables often leads to stability. Think of the idealized behavior of a market in which trading and prices all work together to bring the system to equilibrium. Second, for a system to create chaos it need not have many moving parts. Minor modifications in our resource model from the "Tipping Existence" section produce a map known as the *logistic map*. In both models, the amount of the resource available next period depends on the amount this period times a constant minus that amount.

$$x_{t+1} = rx_t(1 - x_t). \tag{8}$$

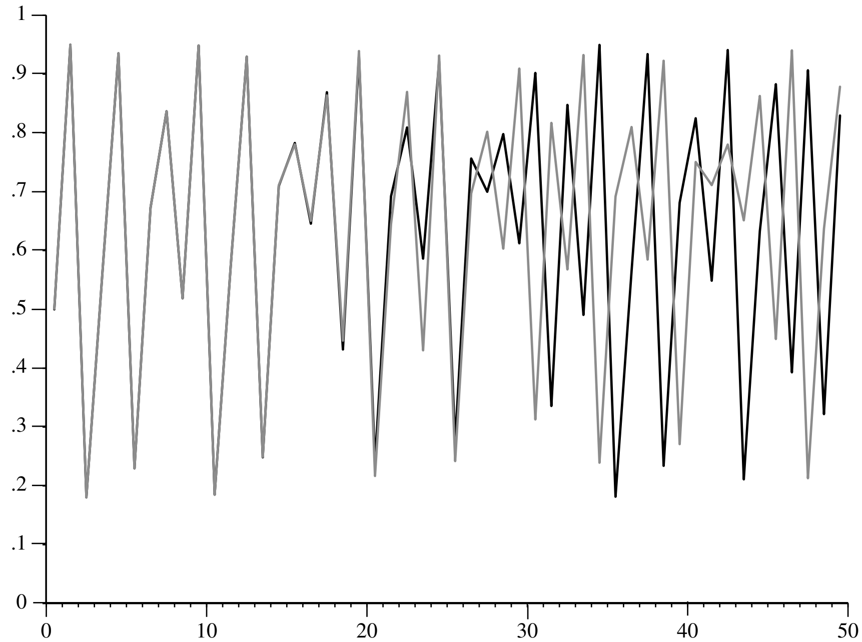

Figure 10. The logistic map ($r = 0.38$) for two nearby initial conditions.

The logistic map produces chaotic behavior for values of r above 3.57. Figure 10 shows the first 50 periods for the logistic map when $r = 3.8$ for two nearby initial conditions. The darker line shows the process for $x = 0.5$ and the lighter line shows the process for $x = 0.501$. Initially, the two paths remain close but before long the small differences multiply. It is provocative to refer to social systems as "chaotic" but to date little evidence has been found to suggest real chaos in social outcomes. However, for our purposes the notion of "chaotic tipping points" can be a mental place holder for the idea that some direct tips may steer a system from one equilibrium to another. We might think of these as tips between two equilibria or two paths. Other tipping points might well lead to more potential paths. For this to occur, the system has to be poised to move in several possible directions.

One case where tips could go in many directions is in the formation of alliances. When countries choose sides, be it in a war or an emissions agreement, the choice of a single actor could result in multiple possible final configurations. We say this without any particular case in mind and mindful of the fact that some coalitions are undoubtedly more likely than others

(Axelord *et al.* 1995). Nevertheless, based entirely on the combinatorics, 30 countries sitting on the fence create over a billion possible partitions, we believe it plausible that different actions would induce a variety of different alliances. Is this chaos? Not exactly, but it is a direct tip that could produce a lot of different outcomes.

Contextual Tipping Points

So far we have examined direct tips, in which a small change in a variable leads to large changes in that same variable. Now, we turn to contextual tips. In a contextual tip, a change in one variable, typically a variable that characterizes the environment, has a large effect on another variable. When economic conditions worsen setting the stage for a political uprising, this creates a contextual tip by creating an environment in which large social demonstrations can occur.

In our framework, contextual tipping points can be defined as the threshold values of the system parameters. We can think of those system parameters as dials that control how the system behaves — when we turn one of these dials past a tipping point, the realm of possible outcomes for the system changes significantly. Recall that this was the case in the resource model that we described previously.

> *A* **contextual tip** *occurs when a gradual change in a variable* y_t *causes a discontinuous jump in future values of some other variable* x_t.

Below we discuss two types of contextual tipping points. Physicists refer to these as bifurcations and phase transitions. To avoid confusion and to link with large existing literatures, we use the same terminology.

Bifurcations. The technical term for a contextual tipping point that results from changing a single parameter is a *bifurcation.* At a bifurcation point, a dynamical system undergoes a qualitative (topological) change in its dynamics. Consider the spread of a rumor, such as the rumor that President Obama is not a United States citizen.[5] We can describe the spread of this rumor with a simple model. In this model, at any moment in time a given person is

[5] This was (is) a real rumor. In an Internet survey Berinsky (2011) found that 27% of respondents believed that President Obama was born in another country.

either spreading the rumor or not. We assume that each person, regardless of their type, has conversations with c people per day chosen at random from the entire population. When someone spreading the rumor meets someone not spreading the rumor, the rumor is passed on with probability p. Finally, each person exposed to the rumor will spread it for D days before stopping, but a person who has stopped spreading the rumor can be reactivated if he or she is exposed to the rumor again.

Let N denote the population size, I_t the number of people who are currently spreading the rumor, and S_t the number who have yet to hear. In this model, the expected daily increase in the number of people who have heard the rumor will equal the number of conversations had by people who don't know, cS_t, times the probability that any given conversation results in someone spreading the rumor, pI_t/N.

Combining terms, we get that the rate of new people spreading the rumor equals pcS_tI_t/N. The rumor will spread if this rate exceeds the rate at which people stop spreading the rumor, which equals I_t/D. Otherwise, the rumor will die out. Thus, the equation $pcS_tI_t/N > I_t/D$ or more simply

$$pcDS_t/N > 1 \tag{9}$$

determines the long run dynamics of the system. If we assume that initially the population is large relative to the number of people actively spreading the rumor then S_t approximately equals N, so the condition simplifies further to

$$pcD > 1. \tag{10}$$

This equation contains three contextual tipping points. Increases in any of the parameters c, p, or D, can cause a rumor to shift from one that simply dies out, to one that spreads across a large fraction of the population and remains pervasive. Changes in social networking technology, the "juiciness," or the political relevance of the rumor would all be contextual changes that these variables might capture. Each of these is a contextual tipping point because small changes in c, p, or D cause a discontinuous jump in future states of I_t. The variables doing the work, c, p, or D, are the threshold variables, and the variable doing the jumping, I_t, is the tipping variable.

Figure 1 illustrates one of the three contextual tipping points. The graph plots the number of agents in a population of 1000 that are spreading the rumor over time starting from 100 initial rumor spreaders. For each of the paths, each person has 10 conversations per day and each individual actively spreads the rumor for 10 days, thus the tip in the spreading probability occurs at $p = 0.01$. As the plot illustrates, as long as $p < 0.01$ the rumor dies out, but as soon as the likelihood that a conversation results in transmission of the rumor exceeds 1% the rumor suddenly becomes endemic. Further increases in p gradually increase the eventual number of rumor spreaders, but the jump at $p = 0.01$ is discontinuous.

In many cases, the parameters c, p, and D cross the contextual tipping point before anything arrives to spread. The conditions may be ripe for an epidemic, like a gasoline soaked pile of wood waiting for a spark. In this case, the "spark" is a direct tip that occurs with the first telling, which then inevitably spreads because of the context. Watts and Dodds (2007) identified this type of behavior when trying to predict individual influence in a model of public opinion formation. They conclude, "The ability of any individual to trigger a cascade depends much more on the global structure of the influence network than on his or her personal degree of influence — that is, if the network permits global cascades, virtually anyone can start one, and if it does not permit global cascades, nobody can." In our terms, once the contextual tipping point has been crossed, any individual can trigger the direct tipping point, but before the contextual tipping point occurs, the direct tip is nearly impossible.

We now revisit game theoretic models to make an obvious, but often unrecognized point. Consider the game with payoffs given in Table 4. If the parameter r is less than 1, the game has three equilibria. Two stable pure strategies and one unstable mixed. When the parameter r becomes larger than 1, the game has a unique stable mixed strategy. Thus, as r decreases the game has a classic bifurcation: the stable equilibrium becomes unstable and two new stable equilibria appear.[6]

This phenomenon is not unique to these particular payoffs. In many game theoretic models, existence results rely on certain necessary conditions in

Table 4. Game with a contextual tipping point.

	A	B
A	**1**,1	**0**,r
B	**0**,r	**1**,1

[6] Formally, the best responses in this game will be correspondences and not functions, but the underlying logic remains the same.

the parameters. When those parameters fall below the thresholds, the equilibria being considered fail to exist or become unstable. These thresholds are contextual tipping points. This example problematizes any hope of dismissing tipping points as a fringe informal concept. Contextual tipping points are pervasive in game theoretic models. In fact, any time someone identifies a binding condition for the existence of a particular equilibrium, she characterizes a contextual tipping point.

Phase Transitions. In common usage, direct and contextual tipping points often arise in the context of something spreading — a political uprising or even a disease. In such situations, who interacts with whom, the *structure of interaction*, makes a large difference on outcomes. The second primary class of contextual tipping points that we cover, called *phase transitions*, concern the properties of the structure of interaction.[7] In a phase transition, that interaction structure undergoes a dramatic shift.

To see why interaction structures matter to such an extent, we will embed our simple rumor model in a network. We will now distribute our population of people randomly across a rectangular grid so that each cell is occupied by at most one person (some cells may be vacant), with the probability of any given cell being occupied equal to p. To keep things simple, we assume that people do not move from cell to cell. Suppose that the rumor starts down in the lower left corner of our grid — if our grid were overlaid on top of the contiguous 48 states this would be San Diego. Let us assume that as soon as a person hears the rumor they share it with any neighbors in adjacent cells to the north, east, south, or west of their cell (but that's it, no phone calls or emails). We can then ask: what are the chances the rumor will spread all the way across the country to the White House?

The answer depends on p. When p is small, the grid is mostly empty, and the rumor likely fades away. At the other extreme, if $p = 1$, then the whole grid is filled, and the rumor will spread straight across from one side of the country to the other. Our intuition tells us that as p increases, the chance the rumor makes it across the country should smoothly increase from 0 to 1, but that intuition is wrong. Instead, the chance remains near zero until p

7 In thermodynamics, phase transitions refer to changes of a thermodynamic system from one state of matter to another (e.g. solid to liquid). The term phase transition is also used in graph theory and percolation theory. A graph undergoes a phase transition if graphs of a certain size (number/density of nodes or edges) have a certain property with very low probability and graphs of slightly larger size have the property with much higher probability.

grows just larger than 0.59, then suddenly the chance jumps to near 1.[8] The critical value, $p_c \approx 0.59$, is a contextual tipping point. If we allow for a more complex interaction structure, say by including the possibility that the rumor can jump long distances through phone calls or emails, then the contextual tipping point will change (Jackson and Rogers, 2007; Lamberson, 2011).

This connection based notion of contextual tipping points resonates with accounts that the Arab Spring and other uprisings have been made possible due to new technology (Pollock, 2011). These technologies make people more connected and allow for ideas and actions to percolate across the society in ways that were precluded, often purposefully, under some regimes.

Within Class and Across Class Tips

We now turn to our second primary distinction: *within class tips* and *between class tips*. We first characterize what we mean by classes of outcomes. Wolfram (2002) distinguishes between four types of behaviors of systems: they can produce stable points, cycles, randomness, and complexity. We do not necessarily advocate Wolfram's classification for social systems. True chaos, for example, seems unlikely in the social world. We merely suggest that it provides useful classifications for thinking about how tips can change the nature of social reality. That said, we believe it to be true that some social systems do in fact approach equilibria — certainly this is true for many markets. Other social systems approximate orbits. The Democratic Party held a majority of seats in the United States Senate for over three decades. Then in 1980, the Republican Party won a majority. In the intervening years, the parties have alternated who has been in office. The business cycle provides another familiar example. And, finally, it is also true that some social science time series, such as stock market prices, appear close to random (Malkiel, 1985).

Wolfram's final class, complexity, can be characterized as having structure but being difficult to explain or predict (Mitchell, 2009; Page, 2010). Many social systems exhibit complexity and often it can make sense to think of a system in flux as complex. Countries that overthrow dictators often enter

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[8] In an infinitely large grid, the approximate value of the tipping point is 0.59274621 (Newman and Ziff, 2000). Physicists call this the *percolation threshold* of the square lattice (Stauffer and Aharony, 1994). You can try this out yourself using the "Fire" model (Wilensky, 1997) in the models library of the freely available software package NetLogo (Wilensky, 1999).

a period of confusion, after which they either transit to another dictator, to democracy, or to some other form of government. We can think of that moment of overthrow as the political system tipping from equilibrium to complex. By that we mean, that at the moment of overthrow, and this was especially true in the case of Libya, the political system can be characterized as having multiple, diverse actors interacting who produce unpredictable, hard to describe collective behavior. Often, the political system later moves to a new equilibrium. These tips can be classified as tips *between classes*. In contrast, when the United State Senate tips from a Republican to a Democratic majority, it tips within class — from one equilibrium to another.

To show how a system can tip from one type of outcome to another and to show that it can happen in the context of a very simple model, we return to the logistic equation,

$$x_{t+1} = rx_t(1 - x_t). \tag{11}$$

Here we consider a sequence of contextual tips. When $r < 1$, the state, x_t goes to zero. When r lies in the interval $[1, 3)$, x_t converges to an equilibrium value of $\frac{r-1}{r}$. Social scientists would not consider $r = 1$ to be a tipping point. Nor should they. It is merely a kink in the comparative statics curve. However, once r increases past three, but not too far, the state will enter a periodic orbit. Thus, we have a contextual tip in the class of the outcome. An equilibrium process becomes a cycle. When r increases beyond 3.45, the period of the orbit increases to 8. Further increases lead to doublings of the orbit. Eventually, when r exceeds 3.57, the equation becomes chaotic. This simple equation therefore exhibits two different tips between class: first from stable to periodic and then from periodic to chaotic.

Measuring Tips as Entropy Reduction and Entropy Divergence

We now turn to the question of how to identify tipping points using formal measures. By definition, a tipping point occurs when gradually changing a variable results in a discontinuous jump in the future value of that variable or some other related variable. Note that the large change need not and often does not happen immediately. To measure the extent of a tip, we measure the change in the expected distribution over the state of the system. To explain what we mean, we return to an early example. Recall our example of a primary election from the "Direct Tips" section. Initially, the probability of each candidate receives the nomination is 50%, but once either candidates' support falls below 35% the probability that the other candidate will be nominated jumps to 1. A small action — one more person choosing their candidate of choice — results in a substantial shift in the likely long term state of the system.

In that example, the tip reduces the amount of uncertainty. In general, it is also possible for a tip to increase the amount of uncertainty. Therefore, to measure how much a system tips, and in fact whether a tipping point exists, we need to capture changes in the distribution over possible outcomes and not just reductions in uncertainty. We do so as follows: suppose that the stochastic process $\{X_t\}$ has a discrete set of possible long-run outcomes, $A = \{a_1, \ldots, a_n\}$. These can be thought of as equilibria, but they might also be periodic orbits, complex regimes, or chaotic paths. We need only that as $t \to \infty$ the probability that $X_t \to a$ for some $a \in A$ approaches one. In words, the process always ends up converging to a member of A. For every point x in the state space and time s, we can define a probability distribution $P_{x,s}$ on A, by

$$P_{x,s}(a) = P(X_t \to a | X_s = x). \tag{12}$$

The distribution $P_{x,s}$ describes the possible long run outcomes, given that at time s the process is in state x. We would like to quantify the amount of uncertainty in this distribution: are there many possible outcomes or are we locked-in to a single path. We do this using the Shannon entropy of the random variable $(A, P_{x,s})$, or just entropy for short. The entropy of $(A, P_{x,s})$ is defined by

$$H(A, P_{x,s}) = -\sum_{i=1}^{n} P_{x,s}(a_i) \log P_{x,s}(a_i), \tag{13}$$

where $P_{x,s}(a_i) \log P_{x,s}(a_i)$ is taken to be 0 if $P_{x,s}(a_i) = 0$. At one extreme, if every a_i is equally likely so the outcome is maximally uncertain, then the entropy is $\log n$. At the other extreme, if the outcome is certain to be equal to a_i, then the entropy is zero.

A particular state is "tippy" if the system is likely to experience a significant decrease or increase in uncertainty after that state. By this measure, in the random walk with absorbing barriers discussed in "Direct Tips", a state at the edge of one of the basins of attraction would be very tippy. There is a 50% chance that in the next time step the system will become locked-in to a single outcome and therefore experience a sudden drop in uncertainty.

We can capture this intuition formally. Equation (13) gives the uncertainty over outcomes when the process is in state x_s. We can also calculate the expected uncertainty in the next time step, $s+1$, by summing up the entropy for all possible states x'_{s+1} and multiplying by the probability of those states conditional on X_s:

$$\sum_{x'} P(X_{s+1} = x'|X_s)H(A, P_{x',s+1}). \tag{14}$$

We define the *entropy ratio* τ of a state x_s to be the ratio of (13) to (14):

$$\tau(x_s) = \frac{H(A, P_{x,s})}{\sum_{x'} P(X_{s+1} = x'|x_s)H(A, P_{x',s+1})}. \tag{15}$$

If we expect a state to result in a significant decrease in uncertainty, so (14) is much smaller than (13), and therefore $\tau(x_s)$ is large, we can characterize that state as tippy.[9] The most tippy any state could be is $\log |A|$. This happens when at one time step, all of the outcomes in A are possible and equally likely, but we know with certainty that in the next time step the eventual outcome will be completely determined. Notice that this measure has the added feature that if it takes on a value close to zero, then it means that the system has tipped from uncertain to certain.

> *The parameter $\tau(x_s)$ captures the tippiness of the system, moreover, if $\tau(x_s) > 1$ then the system* **tips toward certainty***. If $\tau(x_s) < 1$, the system* **tips toward uncertainty***.*

What we call *tipping toward uncertainty* nicely captures across class tips from equilibrium to periodic, complexity, or chaos. In those cases, the uncertainty about the future state of the system would increase.

Tipping Points vs Path Dependence. Conceptually, a tip — an immediate change in the future state of affairs — differs from *path dependence* — a sequence of outcomes changing the future state. As we now show, the $\tau(x_s)$ measure of tippiness can help to distinguish the concept of path dependence from tippiness (Page, 2006). At a tipping point, probabilities change abruptly. In a path dependent process, the probabilities might change gradually along the path. To show this, we can apply our measure to the Pólya

[9] Note that we can make these same calculations relative to changes in other variables to measure contextual tips.

# Red Balls Chosen	Initial Ball = Red	Initial Ball = Blue
0	0	0.25
1	0.04	0.21
2	0.07	0.18
3	0.11	0.14
4	0.14	0.11
5	0.18	0.07
6	0.21	0.04
7	0.25	0
Entropy	2.133	2.133

Figure 11. Entropy in Pólya process after first draw.

Process, the canonical example of a path dependent process. The Pólya Process starts with one red and one blue ball in an urn. Here, we consider a seven period process. In each period, a ball is selected, noted and returned to the urn with another ball of that same color. After seven periods, there are eight possibilities: any number of red balls from 0 to 7 might be selected. With only a little effort it can be shown that each of these eight possibilities is equally likely. Therefore, using the logarithm base two for our measure, this process has an initial entropy equal to 3.[10]

The Pólya Process is most tippy at its initial state. To compute τ we need only compute the average entropy in the next state — after the first ball is chosen. That ball could be either red or blue. Figure 4 gives the probability distribution over outcomes and the entropy for each of the two possible outcomes, which equals 2.133. Taking the initial entropy of 3 and dividing it by the new expected entropy gives a tippiness equal to 1.4.

By comparison, consider a different process that has eight equally likely outcomes but that after the first period, one of those states will be the end state with probability three-fourths and another will be the end state with probability one-fourth. For that process the entropy will fall from 3 to around 0.8, so τ will equal approximately 3.7. In general then, a path dependent process could "tip" but only if one step on that path proved extremely important in determining the fate of the system. More often, and in many of the cases that we think of as path dependent, each step along the

[10] $2^3 = 8$. Therefore, $\log_2 8 = 3$.

way could have a small effect on the end state of the process. In the Pólya Process, that is certainly the case. No one draw seals the future. The future depends on an accumulation of the past.

Conclusion

In this paper, we define tipping points formally, present a typology for categorizing them, and develop a measure of tippiness. Our typology highlights the tendency to focus on direct tipping points rather than contextual ones when explaining how events unfold, despite the fact that in our view contextual tips may be more important given that contextual tips often make direct tips possible. Attaching undue importance to the direct tip seems to us an example of what social psychologists refer to as the *fundamental attribution error* (Ross, 1977). For example, recently Israel witnessed the largest protests in its history as a country in response to rising costs of living. The most popular account of the beginnings of these protests focuses on the actions of a single woman, who set up a tent on a boulevard in Tel Aviv after being evicted from her apartment. As the story goes, this woman's act of protest, along with her creation of a Facebook page encouraging others to join her, caused the resulting wave of protest (Broner, 2011). No doubt this is true to an extent. Her protest was the direct tip that set the ball in motion, but more importantly, prior to her actions we suspect there were changes in the conditions in Israel — changes in the costs of living, changes in people's attitude towards the state, and changes in the communication infrastructure — that made this explosion of public sentiment possible. In other words, there was a contextual tip that preceded the direct tip. Once this contextual tipping point was crossed, the conditions for protest were set and just waiting for a direct tip to come along. With this frame in mind the actions of the initial protester are less significant. Had she not initiated the direct tip, most likely another disgruntled citizen would have. To paraphrase McGeorge Bundy, we may be looking at streetcars.

Awareness of the distinction between direct and contextual tips changes the way we think about intervening in social systems. A leader wishing to prevent protests in a nation cannot do so effectively by focusing on the direct tips. Like so many sparks in a pile of dry leaves, if the context is set eventually one will start a fire. Similarly, marketers and social organizers hoping to cause a tip, rather than focusing on finding "influentials" to create a direct tip as is often advocated (Keller and Berry, 2003), might do better to try and create contextual changes to help their products or ideas spread, to create the conditions for a direct tip (Aral and Walker, 2011; Bakshy *et al.*, 2011).

If a tip does occur, can it be identified empirically? Yes — but doing so requires a model, and proper measurement of the appropriate variable. Because identifying a tipping point says something about why we observe a particular outcome, claims of tipping point behavior require a test of the tipping mechanism, not simply a measure of the result. The typology of tipping points described in this paper can help. Before arguing that a tipping point is responsible for some observed dynamics, the type of tipping point should be identified. To identify a direct tipping point requires a model with an unstable equilibrium or a basin border. Contextual tipping points require a causal link between a threshold variable and a tipping variable along with a qualitative change (a bifurcation or a phase transition).

To summarize, the term *tipping point* has been used frequently and means different things to different people. We show that the concept can be formally defined and can be categorized in ways that are useful to social scientists in their attempts to make sense of the world. The distinction between direct tips (actions that have lots of positive feedbacks) and contextual tips (changes in variables that set the stage for large changes to the system) is particularly important. Furthermore, to compare systems or institutions to one another, one can even apply measures of tippiness, such as the one that we propose, that capture whether uncertainty over outcomes increases or decreases. This capacity to measure tips should result in deeper understanding of how, why, and when tips occur.

Appendix

As an alternative, one could measure the change in the distribution using *Kullback–Leibler* divergence, which captures how much the entropy changes.

$$D_{\mathrm{KL}}(A, P_{x,s} \,\|\, P_{x,s+1}) = -\sum_{i=1}^{n} P_{x,s}(a_i) \log \frac{P_{x,s}(a_i)}{P_{x,s+1}(a_i)} \tag{16}$$

First, notice that if the probability distribution does not change from time period s to time period $s+1$, then the Kullback–Leibler divergence equals zero. Second, notice that the Kullback–Leibler divergence requires that $P_{x,s+1}(a_i) > 0$ if $P_{x,s}(a_i) > 0$, otherwise it takes on value infinity. Thus, this measure fails to capture tippiness if all of the uncertainty gets resolved at a tipping point. Note also that Kullback–Leibler is not symmetric. To get around these twin problems of symmetry and undefinability, Jensen and Shannon created a distribution M which equals the average of the two distributions under comparison. The *Jensen–Shannon divergence*, often called the *information radius* equals the average of the distances of the two distributions from the mean.

$$D_{\mathrm{JS}}(A, P_{x,s} \parallel P_{x,s+1}) = D_{\mathrm{KL}}(A, P_{x,s} \parallel M) + D_{\mathrm{KL}}(A, P_{x,s} \parallel M) \tag{17}$$

where $M = \frac{1}{2}(P_{x,s} + P_{x,s+1})$.

The Jensen–Shannon measure suffers from being a bit complicated. A computationally simpler alternative is to take the *total variation distance*.

$$D_V(A, P_{x,s}, P_{x,s+1}) = \frac{1}{2}\sum_{i=1}^{n} |P_{x,s}(a_i) - P_{x,s+1}(a_i)| \tag{18}$$

This measure has the disadvantage of taking values only in the interval $[0, 1]$, so it cannot distinguish the dimensionality of tips. In sum, we believe that in those cases where a tip reduces uncertainty, and this covers many of the tips of interest to social science, that our measure τ may be sufficient. However, in other cases, it will make sense to use the Jensen–Shannon measure.

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