



WORKING PAPERS

N° 1648

August 2025

## “The Theory of Economic Complexity”

César Hidalgo and Viktor Stojkoski

# The Theory of Economic Complexity

César A. Hidalgo<sup>\*1,2,3</sup> and Viktor Stojkoski<sup>1,2,4</sup>

<sup>1</sup>Center for Collective Learning, IAST, Toulouse School of Economics

<sup>2</sup>Center for Collective Learning, CIAS, Corvinus University of Budapest

<sup>3</sup>Alliance Manchester Business School, University of Manchester

<sup>4</sup>Ss. Cyril and Methodius University in Skopje

August 28, 2025

## Abstract

Economic complexity methods have become popular tools in economic development, economic geography, and innovation. Yet, despite their widespread adoption, we lack a mechanistic model that provides these methods with a solid mathematical foundation. Here, we analytically derive the economic complexity eigenvector associated with a mechanistic model where the output of an economy in an activity (e.g. of a country in a product) depends on the combined presence of the factors required by the activity. Using analytical and numerical derivations we show that the economic complexity index (or *ECI*) is a monotonic function of the probability that an economy is endowed with many factors, validating the idea that it is an agnostic estimate of the presence of multiple factors of production. We then generalize this result to other production functions and to a short-run equilibrium framework with prices, wages, and consumption, showing that the derived wage function is consistent with economies converging to an income that is compatible with their complexity. Finally, we use this model to explain differences in the networks of related activities, such as the product space and the research space, showing that the shape of these networks can be explained by different factor distributions. These findings solve long standing puzzles in the economic complexity literature and validate commonly used metrics of economic complexity as estimates of the combined presence of multiple factors.

## 1 Introduction

A key tenet of the economic complexity literature is the idea that the combined presence of factors of production can be estimated without having to define them. This notion is central to the two key contributions that jump-started the study of economic complexity in the late 2000s.

The first example is the product space [1], a network of related products based on the idea that “if two goods are related because they require similar institutions, infrastructure, physical factors, technology, or some combination thereof, they will tend to be produced in tandem.” By using an outcomes based measure, the product space can be used to create estimates of economic potential that do not rely on defining specific factors of production, but that leverage instead implicit information about unknown factors present in patterns of co-specialization. Networks of related activities, such as the product space [1, 2], industry space [3–5], research space [6, 7], and technology space [8, 9], have become important tools in economic geography, innovation studies, and international development, as they provide a means to formalize notions of path dependency and

---

\*Corresponding author: cesar.hidalgo@tse-fr.eu

estimate the chances that an economy is endowed with the factors needed to engage in an activity.

The second example is the development of metrics of economic complexity, which attempt to estimate the combined presence of factors available in an economy. In [10], economic complexity metrics were introduced as a mean to estimate capabilities that are not directly observed or named. These metrics of complexity have also become useful tools in economic geography, international development, and innovation, because of their ability to explain international and regional variations in economic growth [2, 10–28], inequality [29–37], and sustainability outcomes [38–47].<sup>1</sup> Yet, despite several attempts to develop a mathematical theory of economic complexity [2, 10, 20, 21, 56–65], we still lack an analytical connection between the metrics used in the empirical literature and a production function based model from where we derive these metrics from first principles.<sup>2</sup> Here, we connect the empirical economic complexity literature with a theoretical model to provide four contributions.

First, we provide an analytical derivation of the eigenvector known as the economic complexity index (*ECI*) for a model where economies (such as countries or cities) are differently endowed with the capabilities required by each activity (such as products or industries). In this model, the output of an economy is constrained by the capabilities it has while the geography of an activity is limited to the places endowed with the capabilities the activity requires. We solve the one capability instance of this model analytically and show that the economic complexity index, or *ECI*, is a vector separating economies with an above average capability endowment from those with a below average capability endowment. Interestingly, this property is independent of how capabilities are distributed and can be generalized to other production functions (such as a shifted Cobb-Douglas factor intensity function).

Second, we extend this result numerically to models involving many capabilities assigned idiosyncratically to each economy. We show that in this case *ECI* is a monotonic function of the average capability endowment of an economy and recovers the first singular vector of the matrix of capability endowments. By exploring models combining correlated and uncorrelated capabilities, we show this result to be robust to substantial levels of noise, holding even when more than 50 percent of the capability endowments are assigned at random. This helps show that *ECI* is a measure of economic complexity, as it can captures whether an economy is endowed with multiple capabilities without having to make assumptions about their nature.

Third, we extend the single capability model to a short-run equilibrium framework where we calculate wages, prices, and consumption. We show analytically that under these assumptions *ECI* still separates economies among those with high and low capability endowments. We also determine an equilibrium wage to help interpret the known empirical relationship between economic complexity and growth, and show that the prices of goods in this model follows a concave function of their capability requirements, indicating a high premium for the production of complex goods.

Finally, we use the multi-capability model to explain known variations in the shapes of networks of related activities, such as the product space (based on product co-exports) and the research space (based on co-publication patterns). We show that the core-periphery structure observed in the product space [1], comes from correlated capability endowments and that the ring structures observed in networks of related research fields [6, 68] can be explained by capability endowments following a circulant matrix.

---

<sup>1</sup>Among other outcomes [48–55].

<sup>2</sup>For a review of the field see [66, 67].

There are a few reasons why these results should be of interest to those working on economic complexity, economic growth, and international development.

First, while the economic complexity index or *ECI* enjoys wide adoption in policy circles<sup>3</sup>, the lack of a theoretical foundation has left it open to criticism of being an ad-hoc or uninterpretable measure [21, 63, 80, 81]. Our findings provide a clear interpretation for *ECI* in the context of multi-factor model of production. We show that *ECI* provides a monotonic estimate of the factor endowment of an economy derived from a multi-product specialization matrix. This provides an interpretation in terms of a model’s parameters that is consistent with previous work exploring the interpretability of the *ECI* as a clustering method [58, 61, 62] and connecting *ECI* with the notion of log-supermodularity [57].

Second, these findings dispel the notion that economic complexity is a measure of diversity, as it was originally suggested [10]. The analytical solutions show that economies specialized in the largest number of activities (the more diverse economies) are not necessarily the ones with the highest probability of having a capability<sup>4</sup>. In fact, the model predicts that economic development is a process of diversification only until a certain point, since economies with the highest capability endowments are expected to specialize in complex activities—and are therefore—less diverse than slightly less complex economies. This provides a theoretical foundation for the finding that countries at high-level of development tend to specialize (e.g., Imbs and Warciag [82]) and is consistent with the notion that *ECI* is higher for “small” yet sophisticated and somewhat specialized economies, such as those of Singapore, Switzerland, and Finland<sup>5</sup>. Still, the model predicts a positive correlation between capability endowments and diversity, but through a non-monotonic function, explaining why measures of diversity or concentration are non-ideal estimates of the complexity of an economy.

Third, these results also provide a mean to interpret the structure of the networks of related activities, such as the product space [1], industry space [3], or research space [6]. These networks have been used extensively to model path dependencies and generate measures of export or employment potential [3–6, 9, 66, 83–91]. Yet, the structure of these networks differs depending on the data used to generate them. For instance, networks derived from co-export data, are known to have a core composed of densely interconnected activities that are high in complexity surrounded by a periphery of low complexity activities [1]. Research spaces, connecting academic fields based on citations [68] or co-authorships [6], follow a ring structure, with fields connected with a few neighbors and without a clear center<sup>6</sup>. While these differences in structure are self-evident, we hitherto lacked a method to explain them based on a mechanistic model. Here, we show how to generate network structures that resemble those observed in the empirical literature by changing the shape of the capability endowment matrices.

Finally, we present a short-run equilibrium version of the model showing that our main result is robust to these additional assumptions.

---

<sup>3</sup>For example, it is the number one mission of Malaysia’s New Industrial Master Plan [69], it was used in the recent European competitiveness report by Mario Draghi [70], and it is a key development target for rich resource intensive economies, such as Saudi Arabia and the United Arab Emirates. It has also motivated the creation of regional reports for Australia [71], Turkey [72], Uruguay [73], Russia [74, 75], Mexico [76, 77], Quebec [78], and Italy [79], among other places.

<sup>4</sup>The notion that economic complexity is different from diversity was noted theoretically by [58] and has been in the literature from early on, since the work introducing *ECI* showed that measures of diversity or concentration, such as entropy or the Herfindahl-Hirschmann index (HHI) failed to explain future economic growth as *ECI* did [10].

<sup>5</sup>While larger and more diverse economies, like those of Spain and Italy, are not necessarily as complex.

<sup>6</sup>In simple, the ring is: medicine, biology, chemistry, physics, computer science and math, economics, cognitive science, neuroscience, and back to medicine.

Together, these findings help solve some long-standing puzzles in the economic complexity and international development literature by providing a theoretical foundation for economic complexity.

## 1.1 Empirical and Theoretical Work in Economic Complexity

Empirical work in economic complexity usually starts with matrices summarizing the geography of many economic activities (e.g., exports by country and product, payroll by city and industry, patents by city and technology, etc.). These rectangular matrices (or bipartite networks) are then used for two things. The first one is to estimate networks of similar activities [1, 3, 6–9, 66, 83, 91–93] which are used to estimate the diversification potential of an economy. These measures of “relatedness” have been used to establish the principle that economies are more likely to enter (and less likely to exit) activities that share capabilities with each other<sup>7</sup>.

The second one is to create measures of the value of the portfolio of activities an economy specializes in, known as measures of economic complexity [2, 10, 19–21, 66, 94, 95]. These measures were also motivated as agnostic estimates of the capabilities available in an economy [10] and are often based on the assumption that high-complexity economies specialize in high-complexity activities. In fact, the economic complexity index or *ECI*, defines the complexity of an economy as the average complexity of the activities it specializes in, and the complexity of an activity as the average complexity of the economies specialized in that activity.<sup>8</sup>. These measures of complexity, in particular *ECI*, enjoy wide adoption in international and regional development circles, as they have been shown to be robust estimators of future economic growth [2, 10–21], and of international variations in inequality [29–33, 37, 44], and emissions [38–43, 45, 46, 96].

These two strands of literature support the notion that an economy’s pattern of specialization matters for subsequent economic development [97, 98], which has been a key intuition motivating these efforts<sup>9</sup>. Yet, despite copious empirical work, we still lack an understanding of why the eigenvectors used as measures of economic complexity are good predictors of an economy’s subsequent growth and development.

Theoretical work on economic complexity has focused instead on the construction of models of development and innovation that follow a combinatorial tradition [10, 56, 108–117]. This tradition builds on the notion that economies are endowed with capabilities [2, 118–122], or factors, which activities may or may not require. Since these capabilities are complementary, producing an activity requires the simultaneous presence of many of them. That’s why these theories have been dubbed as the “Lego” or “Scrabble” theory of development. In these models, the ability of economies to produce a product depends on having the right combination of capabilities, like in a proverbial game of scrabble where products are “words” and economies are endowed with “letters.”

---

<sup>7</sup>What is known in the specialized literature as The “Principle of Relatedness” [83].

<sup>8</sup>A similar definition was proposed over a decade later by [21]. In their words: “If a country is known to be more capable than another, say the United States (US) versus Bangladesh (BG), then one can identify any good  $k$  as more complex than another reference good  $k_0$  if, relative to the reference good, it is more likely to be exported by the United States than Bangladesh. [...] Conversely, if a good is known to be more complex than another, say medicines (ME) versus men’s underwear (UW), then one can identify any country  $i_1$  as more capable than another reference country  $i_0$  if, relative to the reference country, it is more likely to export medicines than underwear.”

<sup>9</sup>This policy intuition is connected to an old debate in development economics, going back to at least Alexander Hamilton’s Report on Manufactures [99], which advocated for the industrialization of the United States, and has been central to the works of scholars such as Rosenstein-Rodan [100, 101], Rostow [102], Hirschman [103], Prebisch [104], Gerschenkron [105], and Balassa [106]. For a discussion on how these different development theories related to economic complexity see [107].

Here we build on the combinatorial model introduced by [10], which is a generalization of Kremer’s O-Ring model of development [110], or more precisely the Kremer-Shockley model of productivity, since the same multiplicative formula was introduced in 1957 by William Shockley in a paper explaining differences in productivity among researchers [123].

The Kremer-Shockley model assumes a multi-step production process where the output of an economy is the product of the probabilities that it succeeds at each step. In other words, producing an item in this model requires a sequence of tasks, each of which has a probability of failing. This implies that the output of an economy decays exponentially with the length of the production chain at a rate determined by the probability of succeeding at a task. The key outcome is that economies with higher probabilities of completing a task should specialize in activities requiring multiple steps<sup>10</sup>. Here we focus on a generalized version of this model, where economies are endowed with probabilities of having a capability (similar to the probability of succeeding at a task in the Kremer-Shockley model) and where activities also differ in the probability of requiring a capability. This allows us to model matrices involving an arbitrary number of economies, activities, and capabilities, while also making the capabilities specific to activities and economies. The resulting matrices, which can be made as large as the ones used in the empirical literature, can be used to create theoretical estimates of the economic complexity eigenvector  $ECI$  that we can interpret in connection with the key parameter of the model: the unobservable matrix of capability endowments.

We find that, for a wide variety of model specifications,  $ECI$  recovers the probability that an economy is endowed with multiple capabilities, even when these are highly idiosyncratic. We then generalize this finding to a Cobb-Douglas type factor intensity production function and find that the ability of  $ECI$  to separate among better and worse endowed economies can be generalized to any shifted production function of the form  $Y_{cp} = B + f_c g_p$  where  $f_c$  is a general function characterizing an economy  $c$  and  $g_p$  is a general function characterizing an activity  $p$ .

As in Kremer-Shockley [110, 123], we start from a supply-side model that assumes prices are exogenous and do not provide an explicit model of wages or demand. So, we then embed the single capability model in a short-run equilibrium framework to estimate functions for the implied wages, consumption, and prices. We show that wages increase with the capability endowment of an economy, consumption grows with income, and prices are higher for more complex products (products that are more demanding of the capability). This also allows us to show that our main result—that the economic complexity eigenvector separates high- from low-capability economies—holds after introducing these additional assumptions.

Finally, we use this model to explain the structures of networks of related activities, such as the product space and research space, and show that it is possible to generate networks with a similar structure than the ones observed in the empirical literature by manipulating the capability endowment and capability requirement matrices.

The remainder of the paper is organized as follows. The next section (Section 2) introduces the single-capability model and presents an analytical derivation of the economic complexity index ( $ECI$ ). Section 3 generalizes these results numerically to several versions of a multi-capability model. Section 4 explores additional production functions and Section 5 embeds the model in a short-run equilibrium framework. Section 6 uses the model to explain the structure of networks of related activities, and Section 7 concludes.

---

<sup>10</sup>See also [124] for an extension of the O-Ring model to trade.

## 2 The Single Capability Model

We start with the basic model of economic complexity introduced numerically in [10]. This model assumes that an economy  $c$  is endowed with capability  $b$  with probability  $r_{c,b}$  and that activity  $p$  requires a capability  $b$  with probability  $q_{p,b}$ .<sup>11</sup>

For pedagogical reasons we start with the a single capability or factor and an arbitrary number of economies and activities (that is  $r_{c,b} \rightarrow r_c$  and  $q_{p,b} \rightarrow q_p$ ). This case will allow us to get a basic intuition that we will then generalize to more complex versions of the model. The advantage of starting with the single capability model is that we can derive its economic complexity index (*ECI*) analytically.

Let the output  $Y_{cp}$  of economy  $c$  in activity  $p$  be given by the matrix<sup>12</sup>:

$$Y_{cp} = A(1 - q_p(1 - r_c)) \quad (1)$$

where  $A$  is a constant or scale factor and  $1 - q_p(1 - r_c)$  is the probability that economy  $c$  has the capability that product  $p$  requires. This probability is written as a complement. That is, one minus the probability that the activity requires a capability ( $q_p$ ) that the economy does not have ( $1 - r_c$ ). In matrix form, the output matrix implied by this model is given by:

$$Y_{cp} = A \begin{bmatrix} 1 - q_1(1 - r_1) & 1 - q_2(1 - r_1) & \dots & \dots \\ 1 - q_1(1 - r_2) & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 - q_N(1 - r_N) \end{bmatrix} \quad (2)$$

Going forward, we sort rows in descending order of  $r$  and columns in ascending order of  $q$ . That is, the first cell of the matrix ( $Y_{11}$ ) is the output of the economy with the highest probability of having the capability in the activity with the lowest probability of requiring it. This sorting convention will greatly facilitate the visual inspection of these matrices.

A key difference between this implementation of the model and previous work[10, 56] is that here we use the model to simulate an output matrix ( $Y_{cp}$ ), whereas previous work used it to simulate a specialization matrix (what we will later call  $M_{cp}$ ). Specialization matrices have already been through important manipulations and normalizations. Our results show that these steps are essential for connecting *ECI* with the model parameters. So, the remainder of this section will focus on performing the manipulations used in the empirical literature to this theoretical matrix. These manipulations are:

- (i) Estimating the matrix of revealed comparative advantage or RCA  $R_{cp}$  according to Balassa's (1965) definition[125]. This matrix normalizes the output matrix  $Y_{cp}$  by the sum of its rows and columns and it is equivalent to a matrix comparing the observed output ( $Y_{cp}$ ) with the expected output in a probabilistic model (see eqn. (3)). RCA is also known as the location quotient (LQ) in economic geography and innovation studies.
- (ii) Estimating the binary specialization matrix  $M_{cp}$ . This is a matrix that is 1 if  $R_{cp} \geq 1$  and 0 otherwise. This binary matrix is motivated in the empirical literature as a means to

<sup>11</sup>This deviates from previous work [10, 56] which tended to assume a distribution of  $rs$  and  $qs$  for all economies and activities instead of endowing each country and activity with an individual parameter.

<sup>12</sup>We note that for  $q_p = 1$  this is the rectifier or ReLU function, which is a key activation function in neural networks.

remove the tails of the  $R_{cp}$  matrix, since the ratio definition of  $R_{cp}$  results in larger variance for economies with low levels of output (small  $Y_c = \sum_p Y_{cp}$ ) and activities with small markets (small  $Y_p = \sum_c Y_{cp}$ ).

(iii) Estimating the complexity matrix  $M_{cc'}$ . This is a square matrix connecting economies with similar specialization patterns and is the one used to derive the economic complexity index. This matrix is defined using the reciprocal averaging method known as the method of reflections [10], but it can also be defined as the product of a four matrices (we will introduce the exact formula at that point).

We begin with the standard definition of the RCA matrix or  $R_{cp}$  which is:

$$R_{cp} = \frac{Y_{cp} \sum_{c,p} Y_{cp}}{\sum_c Y_{cp} \sum_p Y_{cp}} \quad (3)$$

Also, since it will simplify the math going forward, we use Einstein's notation, where summed indices are "suppressed" or "muted" (e.g.  $Y_c = \sum_p Y_{cp}$ ). In this notation  $R_{cp}$  takes the more compact form:

$$R_{cp} = \frac{Y_{cp} Y}{Y_c Y_p} \quad (4)$$

To estimate  $R_{cp}$  for the single capability model we need to notice a couple of things. First, since the scale factor  $A$  is common to all terms, it cancels out of  $R_{cp}$  (so we can ignore it). Second, we should notice that applying the sum operator to the terms in  $R_{cp}$  transforms variables into averages. We can illustrate this by using the sum over  $p$  as an example (the derivation is analogous for the other terms):

$$\begin{aligned} Y_p &= \sum_p (1 - q_p(1 - r_c)) \\ Y_p &= N_p - (1 - r_c) \sum_p q_p \\ Y_p &= N_p (1 - (1 - r_c) \langle q \rangle) \end{aligned} \quad (5)$$

where  $N_p$  is the number of activities or products and  $\langle q \rangle$  is the average of  $q_p$  over all activities. Using this property, we can now rewrite  $R_{cp}$  as:

$$R_{cp} = \frac{(1 - q_p(1 - r_c))(1 - \langle q \rangle(1 - \langle r \rangle))}{(1 - q_p(1 - \langle r \rangle))(1 - \langle q \rangle(1 - r_c))}. \quad (6)$$

To derive  $M_{cp}$  we need to identify when  $R_{cp}$  is larger or smaller than one. We can do this by manipulating the inequality.

$$(1 - q_p(1 - r_c))(1 - \langle q \rangle(1 - \langle r \rangle)) \geq (1 - q_p(1 - \langle r \rangle))(1 - \langle q \rangle(1 - r_c)). \quad (7)$$

Which simplifies to:

$$q_p(1 - r_c) + \langle q \rangle(1 - \langle r \rangle) \leq q_p(1 - \langle r \rangle) + \langle q \rangle(1 - r_c) \quad (8)$$

leading to the condition:

$$(r_c - \langle r \rangle)(q_p - \langle q \rangle) \geq 0 \quad (9)$$

Since this is an inequality, we need to be careful about the signs of  $(q_p - \langle q \rangle)$  and  $(r_c - \langle r \rangle)$ . Changes in sign flip the inequality operator. So what this condition means is that  $R_{cp} \geq 1$  when  $r_c \geq \langle r \rangle$  and  $q_p - \langle q \rangle \geq 0$  or when  $r_c < \langle r \rangle$  for  $q_p - \langle q \rangle < 0$ . We can also get this condition intuitively by considering the case when  $q_p = \langle q \rangle$  or  $r_c = \langle r \rangle$ . In these two cases  $R_{cp} = 1$ , meaning that these lines divide the matrix into regions where the values of  $R_{cp}$  are higher or smaller than one. In sum, from the condition above  $M_{cp}$  is a matrix divided into four quadrants:

$$\begin{aligned} M_{cp} &= 1 & \text{if } r_c \geq \langle r \rangle \quad \& \quad q_p \geq \langle q \rangle \\ M_{cp} &= 1 & \text{if } r_c < \langle r \rangle \quad \& \quad q_p < \langle q \rangle \\ M_{cp} &= 0 & \text{if } r_c < \langle r \rangle \quad \& \quad q_p \geq \langle q \rangle \\ M_{cp} &= 0 & \text{if } r_c \geq \langle r \rangle \quad \& \quad q_p < \langle q \rangle \end{aligned} \quad (10)$$

This matrix represents a world where countries with a high probability of having the capability ( $r_c$  higher than average), specialize in products with high probability of requiring the capability ( $q_p$  higher than average), and countries with low probability of having the capability specialize in products with low probability of requiring it. This is related to the idea of log super-modularity in trade theory [57].

As an example, consider a world with four countries and six products, where two countries have above average  $r_c$  and three products have above average  $q_p$ . In this example, the binary specialization matrix  $M_{cp}$  takes the form:

$$M_{cp} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (11)$$

Finally, we use  $M_{cp}$  to derive  $M_{cc'}$ . Here we use the standard reciprocal average method or method reflections. This method proposes that the complexity of an economy is the average complexity of the activities that economy is specialized in, and that the complexity of an activity is the average complexity of the economies specialized in that activity. Using the economic complexity index ( $ECI$ ) and the product complexity index ( $PCI$ ) to indicate the complexity of economies and activities we obtain:

$$\begin{aligned} ECI_c &= \frac{1}{M_c} \sum_p M_{cp} PCI_p \\ PCI_p &= \frac{1}{M_p} \sum_c M_{cp} ECI_c \end{aligned} \quad (12)$$

putting the second equation into the first one can show that  $ECI_c$  is the solution to the following self-consistent equation:

$$ECI_c = \sum_{c'} M_{cc'} ECI_{c'} \quad (13)$$

with

$$M_{cc'} = \frac{1}{M_c} \sum_p \frac{M_{cp} M_{c'p}}{M_p} \quad (14)$$

Meaning that the economic complexity vector  $ECI_c$  must be an eigenvector of the  $M_{cc'}$  matrix representing the steady state of the mapping defined by the system in eqns. (12) (the same derivation can be used to define the  $M_{pp'}$  matrix used to estimate  $PCI$ ).<sup>13</sup>

Estimating the first eigenvector of  $M_{cc'}$  is trivial because  $M_{cc'}$  is a stochastic matrix (each row adds to one). That means its first eigenvector will always be the vector  $\mathbf{1}$ . This is easy to prove by summing  $M_{cc'}$  over  $c'$ .

$$\begin{aligned} M_{cc'} \mathbf{1} &= \sum_{c'} \frac{1}{M_c} \sum_p \frac{M_{cp} M_{c'p}}{M_p} \\ M_{cc'} \mathbf{1} &= \frac{1}{M_c} \sum_p \frac{M_{cp} M_p}{M_p} \\ M_{cc'} \mathbf{1} &= \frac{1}{M_c} \sum_p M_{cp} = \mathbf{1} \end{aligned} \quad (15)$$

Since the first eigenvector is  $\mathbf{1}$ , the steady state of the system represented by eqns (12) is given by the second eigenvector. To estimate that eigenvector, we need to calculate  $M_{cc'}$ . Here, we consider three cases. When the number of economies and activities is even, when the number of economies is odd and the number of activities is even, and when both the number of economies and activities are odd. The need to consider these cases separately will become self-evident once they are introduced.

We begin with the simplest case, that of an even number of economies and activities. We let also  $\langle r \rangle$  and  $\langle q \rangle$  be the medians of their distributions. In that case,  $M_{cc'}$  reduces to a block diagonal matrix with two blocks with values of  $1/M_p$  (all economies have the same diversity and all activities the same ubiquity). That is:

$$\begin{aligned} M_{cc'} &= \frac{1}{M_p} \quad \text{if } r_c \& r_{c'} \geq \langle r \rangle \quad \text{or} \quad r_c \& r_{c'} < \langle r \rangle \\ M_{cc'} &= 0 \quad \text{otherwise} \end{aligned} \quad (16)$$

For the example above, with four economies and six activities,  $M_{cc'}$  takes the form:

$$M_{cc'} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \quad (17)$$

Since we know the first eigenvector of this matrix is the vector  $e_c^1 = \mathbf{1}$ , and since this matrix is symmetric, and has therefore orthogonal eigenvectors, we can use these properties to find the second eigenvector, which is:

---

<sup>13</sup> $M_{cc'}$  can also be defined as the product of four matrices  $M_{cc'} = D_c M_{cp} D_p M_{pc'}$  where  $D_c$  is a diagonal matrix of  $1/M_c$  and  $D_p$  is a diagonal matrix of  $1/M_p$ .

$$e_c^2 = ECI = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad (18)$$

In this case, this eigenvector is also associated with the eigenvalue of one (this matrix is degenerate, meaning that it has more than one eigenvector associated with the same eigenvalue).<sup>14</sup> This eigenvector is easy to verify through multiplication.

What it is important for us is that this eigenvector separates economies with above and below average  $r$ , that is:

$$\begin{aligned} e_c^2 = ECI_c &= 1 & \text{if } r_c \geq \langle r \rangle \\ e_c^2 = ECI_c &= -1 & \text{if } r_c < \langle r \rangle \end{aligned} \quad (19)$$

showing that in this example the second eigenvector of the  $M_{cc'}$  matrix or  $ECI$  separate economies that are above or below average in their probability of having the only capability in the model.<sup>15</sup>

Figure 1 visualizes the matrices in the single capability model for a case involving an even number of economies and activities (10 economies and 20 activities). These graphical representations will help us develop our intuition when interpreting more complex models later.

From top left to bottom right, we start with the output matrix ( $Y_{cp}$ ), the specialization or RCA matrix ( $R_{cp}$ ), the binary specialization matrix  $M_{cp}$ , and the complexity matrix  $M_{cc'}$  from which we derive  $ECI$ . The output matrix  $Y_{cp}$  shows a nested pattern, which is a tendency for the rows that are less filled to be subsets of the rows that are more filled. Nestedness is a well-known feature of matrices summarizing the geography of fine-grained economic activities, such as exports by country and product, employment by city and industry, or patents by city and technology [126]. It is also a common feature of bipartite networks in ecology (e.g. pollinator networks or geographic specialization networks[127, 128]). This example shows how these transformations simplify  $Y_{cp}$ , reducing it to a couple of clusters with above and below average probability of having a capability. Yet, the symmetry of this example limits our ability to explore key properties of the method, such as the ability to separate capability endowments from simple measures of diversity. For that, we need to consider other cases.

Next, we focus on the case where the number of economies is odd and the number of activities is even (and where the averages of  $r$  and  $q$  are still their medians). For example,  $N_c = 5$  and  $N_p = 6$ . This example is interesting, because unlike in the previous case where

---

<sup>14</sup>In this case, all linear combinations of these eigenvectors are eigenvectors themselves. For example the vector  $[a, a, b, b]$  is also an eigenvector, since we can construct it as a linear combination of  $[1, 1, 1, 1]$  and  $[1, 1, -1, -1]$ .

<sup>15</sup>At this point it is worth noting that a standard property of eigenvectors is that they have a freedom of sign. That is, if  $e_c$  is an eigenvector of a matrix  $M$  so is  $-e_c$ . This is trivial from the fact that if  $Me_c = \lambda e_c$  then  $M(-e_c) = \lambda(-e_c)$ . This means that the eigenvector derivation of  $ECI$  separates among economies based on their capability endowments, but is agnostic about which of the two clusters is the high-capability cluster. In the empirical literature, this is solved by iterating the system of eqns. 12 to estimate  $ECI$  starting from an initial condition that is correlated with the high-capability cluster (e.g. initializing the system with diversity  $M_c$ ) and stopping at an even iteration. Other methods to estimate complexity empirical (e.g. [21] also rely on an initialization guess).

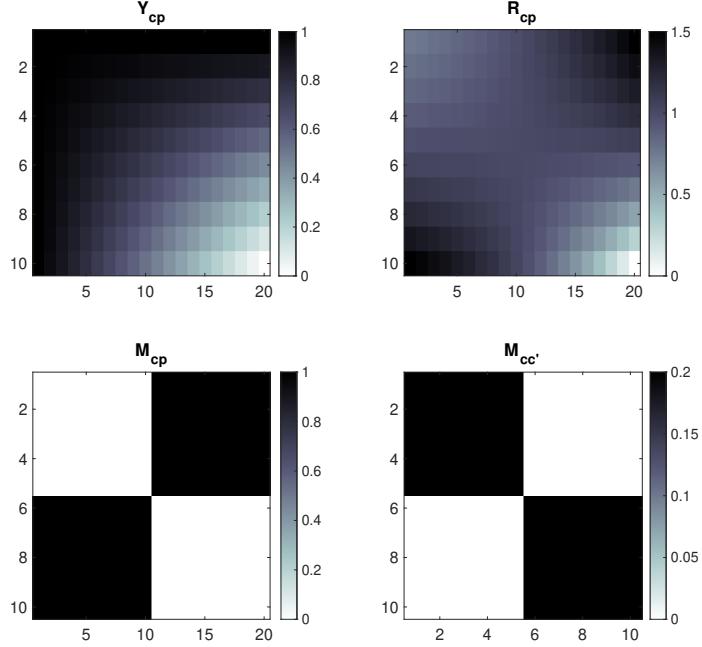


Figure 1: Graphical description of the four matrices involved in the single capability model for 10 countries and 20 products. In  $cp$  matrices row represents economies (countries) and columns represent activities (products). Rows are sorted from highest  $r_c$  to lowest  $r_c$  and columns are sorted from lowest  $q_p$  to highest  $q_p$ . That is, cell  $(1, 1)$  is the output of the country with the highest probability of having the capability on the product with the lowest probability of requiring it, and cell  $(10, 20)$  is the output of the country with the lowest probability of having a capability in the product with the highest probability of requiring it.

the diversity of economies and the ubiquity of activities was constant, here only the ubiquity of activities remains fixed. This example is important because it will teach us about the ability of  $ECI$  to recover  $r_c$ , even when the most diverse economy is the one that has a probability of having a capability equal to the average ( $r_c = \langle r \rangle$ ) (it is actually specialized in all activities).

In this *odd-even* case,  $M_{cp}$  is given by:

$$\begin{aligned}
 M_{cp} &= 1 \quad \text{if } r_c > \langle r \rangle \quad \& \quad q_p > \langle q \rangle \\
 M_{cp} &= 1 \quad \text{if } r_c < \langle r \rangle \quad \& \quad q_p < \langle q \rangle \\
 M_{cp} &= 1 \quad \text{if } r_c = \langle r \rangle \\
 M_{cp} &= 0 \quad \text{otherwise}
 \end{aligned} \tag{20}$$

which for  $N_c = 5$  and  $N_p = 6$  results in the binary specialization matrix that is completely filled on the third row (so the matrix is no longer symmetric):

$$M_{cp} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Clearly the most diverse economy is the one in the third row, which is specialized in all activities.

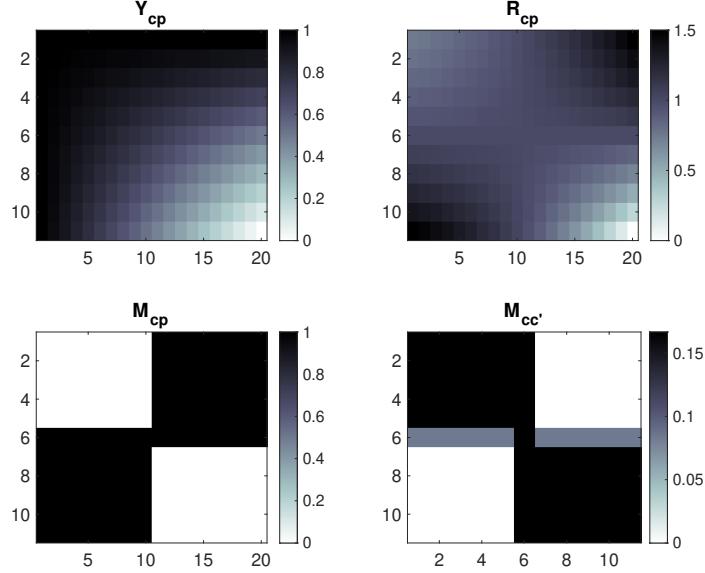


Figure 2: Graphical description of the four matrices involved in the single capability model for 11 economies (e.g. countries) and 20 activities (e.g products). In  $cp$  matrices row represents economies (countries) and columns represent activities (products). Rows are sorted from highest  $r_c$  to lowest  $r_c$  and columns are sorted from lowest  $q_p$  to highest  $q_p$ . That is, cell  $(1, 1)$  is the output of the country with the highest probability of having the capability on the product with the lowest probability of requiring it, and cell  $(11, 20)$  is the output of the country with the lowest probability of having a capability in the product with the highest probability of requiring it.

Moving to  $M_{cc'}$  gives us:

$$\begin{aligned}
 M_{cc'} &= \frac{1}{M_p} \quad \text{if } c = c' \\
 M_{cc'} &= \frac{1}{M_p} \quad \text{if } r_c \& r_{c'} > \langle r \rangle \quad \text{or} \quad r_c \& r_{c'} < \langle r \rangle \\
 M_{cc'} &= \frac{1}{M_c} \sum_p \frac{M_{cp} M_{c'p}}{M_p} \quad \text{if } r_c \& r_{c'} = \langle r \rangle \quad \& \quad c \neq c' \\
 M_{cc'} &= 0 \quad \text{otherwise}
 \end{aligned} \tag{22}$$

which for five economies and six activities results in the matrix:

$$M_{cc'} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/6 & 1/6 & 1/3 & 1/6 & 1/6 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix} \tag{23}$$

This matrix is also quite regular, and has the following second eigenvector which can be verified simply using matrix multiplication:

$$e_c^2 = ECI_c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \tag{24}$$

In more general terms it is given by:

$$\begin{aligned} e_c^2 &= ECI_c = 1 & \text{if } r_c > \langle r \rangle \\ e_c^2 &= ECI_c = -1 & \text{if } r_c < \langle r \rangle \\ e_c^2 &= ECI_c = 0 & \text{if } r_c = \langle r \rangle \end{aligned} \quad (25)$$

This is an interesting result, since it shows that the second eigenvector or  $ECI$  is not “fooled by diversity.” On the contrary, it is able to recover the fact that the economy that is specialized in all activities has a probability of having a capability that is in between that of the high probability and low probability clusters.

Figure 2 summarizes the matrices in the single capability model for a case involving an odd number of economies and an even number of activities (11 economies and 20 activities). In this case, the key difference is that center row of  $M_{cp}$  which extends through all columns of the matrix and results in a small overlap between the two clusters in  $M_{cc'}$ .

Next, we consider the case in which the number of economies and activities are odd. In this case, the diversity of economies and the ubiquity of activities is no longer constant. Now the  $M_{cp}$  matrix has both, one row and one column that are completely filled, which correspond respectively to the economy and activity with  $r_c = \langle r \rangle$  and  $q_c = \langle q \rangle$ . That is:

$$\begin{aligned} M_{cp} &= 1 & \text{if } r_c > \langle r \rangle \quad \& \quad q_p > \langle q \rangle \\ M_{cp} &= 1 & \text{if } r_c < \langle r \rangle \quad \& \quad q_p < \langle q \rangle \\ M_{cp} &= 1 & \text{if } r_c = \langle r \rangle \\ M_{cp} &= 1 & \text{if } q_c = \langle q \rangle \\ M_{cp} &= 0 & \text{otherwise} \end{aligned} \quad (26)$$

Which we can bring to an example with five economies and seven activities:

$$M_{cp} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (27)$$

In this case  $M_{cc'}$  will have a more complex form which we can express by noticing that the diversity and ubiquity of the economy and activity in the middle row and column of  $M_{cp}$  is the number of economies  $N_c$  and the number of activities  $N_p$ . Since all other economies and activities have the same diversity and ubiquity, which we will denote by  $M_c$  and  $M_p$ , we obtain:

$$\begin{aligned}
M_{cc'} &= \frac{1}{M_c} \left( 1 + \frac{1}{N_p} \right) \quad \text{if } r_c \& r_{c'} > \langle r \rangle \quad \text{or} \quad r_c \& r_{c'} < \langle r \rangle \\
M_{cc'} &= \frac{1}{M_c N_p} \quad \text{if } r_c > \langle r \rangle \quad \& \quad r_{c'} < \langle r \rangle \text{ and vice versa} \\
M_{cc'} &= \frac{1}{N_c} \left( \frac{1}{N_p} + \frac{N_c - 1}{M_p} \right) \quad \text{if } c = c' \quad \& \quad r_c = \langle r \rangle \\
M_{cc'} &= \frac{1}{N_c} \left( 1 + \frac{1}{N_p} \right) \quad \text{if } c \neq c' \quad \& \quad r_c = \langle r \rangle \\
M_{cc'} &= \frac{1}{M_c} \left( 1 + \frac{1}{N_p} \right) \quad \text{if } r_{c'} = \langle r \rangle
\end{aligned} \tag{28}$$

Which might be easier to parse when presented in matrix form:

$$M_{cc'} = \begin{bmatrix} \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) & \dots & \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) & \dots & \frac{1}{M_c N_p} \\ \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) & \dots & \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) & \dots & \frac{1}{M_c N_p} \\ \frac{1}{N_c} \left( 1 + \frac{1}{N_p} \right) & \dots & \frac{1}{N_c} \left( \frac{1}{N_p} + \frac{N_c-1}{M_p} \right) & \dots & \frac{1}{N_c} \left( 1 + \frac{1}{N_p} \right) \\ \frac{1}{M_c} \left( \frac{1}{N_p} \right) & \dots & \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) & \dots & \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) \\ \frac{1}{M_c} \left( \frac{1}{N_p} \right) & \dots & \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) & \dots & \frac{1}{M_c} \left( \frac{N_p+1}{N_p} \right) \end{bmatrix} \tag{29}$$

Bringing this the five economies and seven activities example gives us:

$$M_{cc'} = \begin{bmatrix} 3/10 & 3/10 & 3/10 & 1/20 & 1/20 \\ 3/10 & 3/10 & 3/10 & 1/20 & 1/20 \\ 6/35 & 6/35 & 11/35 & 6/35 & 6/35 \\ 1/20 & 1/20 & 3/10 & 3/10 & 3/10 \\ 1/20 & 1/20 & 3/10 & 3/10 & 3/10 \end{bmatrix} \tag{30}$$

Which again has a second eigenvector of the form:

$$e_c^2 = ECI_c = \begin{bmatrix} a \\ a \\ 0 \\ -a \\ -a \end{bmatrix} \tag{31}$$

This is easy to verify through multiplication. Since the vector adds all of the elements up to the center column and then subtracts all of the elements after the central column, and since the number of elements before and after the central column are the same, we can simply subtract the first and last element of the first row of matrix (eqn. 29) to obtain:

$$M_{cp} v_c = \frac{a}{M_c} \left( 1 + \frac{1}{M_p} \right) - \frac{a}{M_c} \frac{1}{M_p} = \frac{a}{M_c} \tag{32}$$

Doing the same operation on the last row we get:

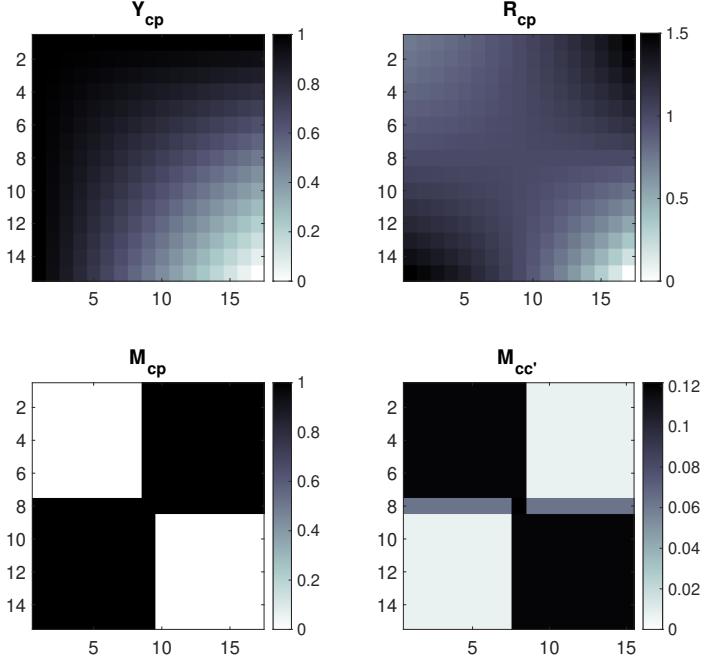


Figure 3: Graphical description of the four matrices involved in the single capability model for 15 economies (e.g. countries) and 17 activities (e.g products). In  $cp$  matrices row represents economies (countries) and columns represent activities (products). Rows are sorted from highest  $r_c$  to lowest  $r_c$  and columns are sorted from lowest  $q_p$  to highest  $q_p$ . That is, cell (1, 1) is the output of the economy or country with the highest probability of having the capability on the product or activity with the lowest probability of requiring it, and cell (11, 20) is the output of the economy with the lowest probability of having a capability in the activity or product with the highest probability of requiring it.

$$M_{cp}v_c = \frac{a}{M_c} \left( \frac{1}{M_p} \right) - \frac{a}{M_c} \left( 1 + \frac{1}{M_p} \right) = -\frac{a}{M_c} \quad (33)$$

Since in the central row of the matrix all elements, except the one in the diagonal, are the same, this vector sends that row to zero. Thus, up to a normalization constant, the second eigenvector of  $M_{cc'}$  is given by:

$$\begin{aligned} e_c^2 &= ECI_c = a & \text{if } r_c > \langle r \rangle \\ e_c^2 &= ECI_c = -a & \text{if } r_c < \langle r \rangle \\ e_c^2 &= ECI_c = 0 & \text{if } r_c = \langle r \rangle \end{aligned} \quad (34)$$

Figure 3 presents these matrices in graphical form. We would like to notice two things about this version of the single capability model. The first one is that in this case  $M_{cc'}$  no longer has blocks of 0s. The second one is that this is also an example in which the highest diversity economy (the 3<sup>rd</sup> row in  $M_{cp}$  is correctly identified as not being the economy with the highest probability of having the capability.

Thus, we have shown that, in the context of the single capability or single factor model, the second eigenvector of the  $M_{cc'}$  matrix, known as the economic complexity index or  $ECI$ , separates economies among those that have a higher and lower than average probability of having the single capability in the model.

In the next section we will use similar figures to explore more complex forms of this model, involving multiple and heterogeneous capabilities. We will then move to different production functions to explore the generalizability of this result.

### 3 The Multi Capability Model

The multi capability version of the combinatorial model can be defined by letting the probability that a country has capability  $b$  be  $r_{c,b}$  and the probability that a product requires a capability  $b$  be  $q_{p,b}$ . For a country to produce a product it needs to have all of the capabilities that the product requires. That is, the product of these probabilities for all of the capabilities in the model. Mathematically, that translates into an output matrix of the form:<sup>16</sup>

$$Y_{cp} = A \prod_{b=1}^{N_b} (1 - q_{p,b}(1 - r_{c,b})) \quad (36)$$

To avoid over-parameterizing the model too early, and to simplify our exploration, we will begin with the case in which these probabilities are independent of the capability index and of each other, and where the pre-factor  $A_{cp}$  is constant. That is:

$$Y_{cp} = A \prod_{b=1}^{N_b} (1 - q_p(1 - r_c)) \quad (37)$$

Which reduces to a well-known binomial form<sup>17</sup>:

$$Y_{cp} = (1 - q_p(1 - r_c))^{N_b} \quad (39)$$

This form assumes that a country has the same probability of having each of the different capabilities required by a product. The need for multiple capabilities, therefore, enters only in the probability of missing one of them, making this similar in spirit to Kremer's O-Ring model [110]. In fact, Kremer's O-Ring production function can be recovered from eqn.(37) by setting  $q_{p,b} = 1$  for all activities and  $r_{c,b} = r_c$  for each economy.<sup>18</sup>

---

<sup>16</sup>This model assumes capabilities are not substitutable. A model with substitutable capabilities would take the form

$$Y_{cp} = A_{cp} \prod_{b=1}^{N_b} (1 - q_{p,b}(1 - r_{c,b} - \sum_{b' \neq b} S_{bb'} r_{cb'})) \quad (35)$$

where  $S_{bb'}$  is a matrix describing the level of substitutability between capabilities  $b$  and  $b'$ .

<sup>17</sup>While this form looks relatively simple, even the solution for  $N_b = 2$  can result in a mathematical form that is substantially more complicated than the one for the single-capability model. In fact, after some algebra one can show that the condition for  $R_{cp} \geq 1$  in the  $N_b = 2$  case is:

$$\begin{aligned} & [q_p^2 - \langle q^2 \rangle][r_c^2 - \langle r^2 \rangle + 2(r_c - \langle r \rangle)] + 2[q_p - \langle q \rangle][r_c - \langle r \rangle] + \\ & 2[q_p \langle q^2 \rangle - \langle q \rangle q_p^2][r_c - \langle r \rangle r_c^2 \langle r \rangle - \langle r^2 \rangle r_c + \langle r^2 \rangle - r_c^2] \geq 0. \end{aligned} \quad (38)$$

<sup>18</sup>In that case, the production function reduces to the Kremer-Shockley function:

$$Y_c = A \prod_{b=1}^{N_b} r_c = r_c^{N_b}. \quad (40)$$

Figure 4 shows the same matrices we derived analytically for the single capability model but for a model involving ten capabilities, one hundred economies, and one thousand activities. This gets to a granularity that is similar to the one used in empirical economic complexity studies.

In this example, economies and activities are modeled using evenly spaced probabilities in the  $[0, 1]$  interval. That is, for an eleven economy model the probabilities are given by  $0, 0.1, 0.2, \dots, 0.9, 1$ . The result is a highly nested output matrix  $Y_{cp}$  and strongly off-diagonal specialization matrices ( $R_{cp}$  and  $M_{cp}$ ).

It is worth noting that the more diverse economies in this model are not the ones with the highest  $r_c$ , but the ones with an  $r_c$  that is below the largest (around 0.8). This is because the reduced output of these economies in the most demanding activities (the ones with highest  $q_p$ ) means they are relatively more specialized in products with lower  $q_p$ s compared to the highest  $r_c$  economies. This effect is analogous to what we saw in the one capability model when we considered an odd number of economies.

Figure 4 also shows that  $M_{cc'}$  follows a similar block diagonal structure than before, but much smoother than in the single capability model.

While it would certainly be substantially more difficult to estimate the eigenvectors of this model analytically, we can still explore them numerically. Figure 5 compares the  $r_c$  of each economy with its second eigenvector of the  $M_{cc'}$  matrix (the non-normalized  $ECI$ ), diversity ( $M_c$ ), and the ranking of economies according to  $ECI$ . Unlike in the single capability model, where  $ECI$  told us only if an economy was above or below average, in this example we get a less discrete second eigenvector that increases monotonically with  $r$ . This results in a perfect correlation between the ranked values of  $r$  and  $ECI$ . Diversity, however, peaks for economies with  $r_c$  less than the maximum, meaning that it is a non-ideal way to estimate the capability endowment of economies in this model. That is, we recover the fact that the second eigenvector of  $M_{cc'}$ —the economic complexity index ( $ECI$ )—is a good method to estimate the key parameter for the economies in the model ( $r_c$ ), extending our previous result to a multi-capability setting.

Figure 6 illustrates the behavior of this model for different number of capabilities (from 2 to 60). Overall, the behavior observed is consistent with the one observed for the ten capability example. Across the board,  $ECI$  behaves as a perfect estimator of the probability that a country is endowed with a capability. We can observe, however, that diversity improves as an indicator for the models with the highest number of capabilities (60), becoming almost perfectly monotonic in that case.

To continue our exploration we relax our assumptions about the distributions of  $r_c$  and  $q_p$ . So far, our simulations have involved evenly spaced  $r_c$ s and  $q_p$ s in the  $[0, 1]$  interval, which are an idealized uniform distribution. So we replace these uniform distributions for Gaussians by drawing a random number from a normal distribution for each  $r_c$  and  $q_p$  and min-max normalizing these random numbers to ensure they fall in the  $[0, 1]$  interval.

Figures 7 and 8 show the results of this exercise. Unlike in the previous example, the specialization matrix  $M_{cp}$  exhibits a bit more “roughness”, with non-perfectly smooth edges. That said, the behavior of this model is otherwise similar to the previous one.  $M_{cc'}$  is roughly block diagonal and the second eigenvector of  $M_{cc'}$  or  $ECI$  almost perfectly captures  $r_c$ , as seen in its monotonic relationship with  $r$  and in their rank correlation (Figure 8), whereas diversity peaks for economies with an  $r_c$  of around 3/4, making it a non-ideal estimator of  $r_c$ .

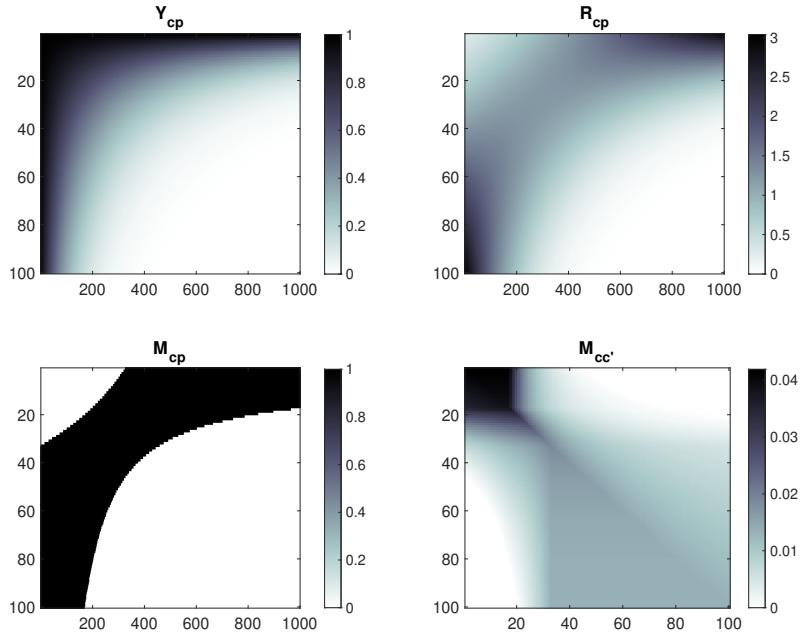


Figure 4: The four matrices involved in economic complexity calculations using a ten capability model for 100 countries and 1,000 products. In  $cp$  matrices row represents economies (countries) and columns represent activities (products). Rows are sorted from highest  $r_c$  to lowest  $r_c$  and columns are sorted from lowest  $q_p$  to highest  $q_p$ .

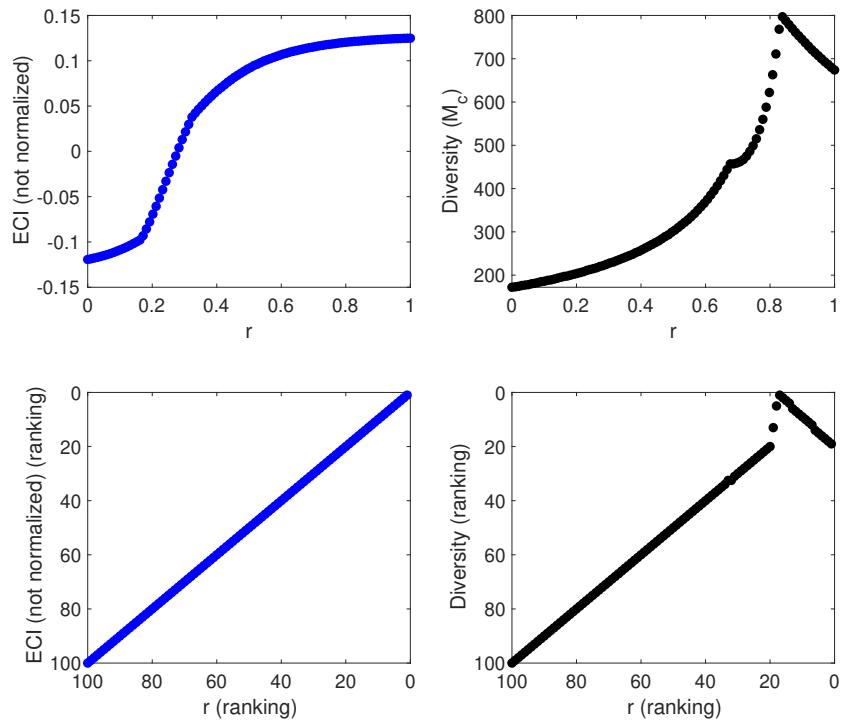


Figure 5: Comparison between the key parameters representing economies in the model ( $r$ ), the second eigenvector of the  $M_{cc'}$  matrix ( $ECI$ ), and the diversity ( $M_c$ ) of economies in the model. The top two panels show the raw relationship between the variables while the bottom two compare their rankings.

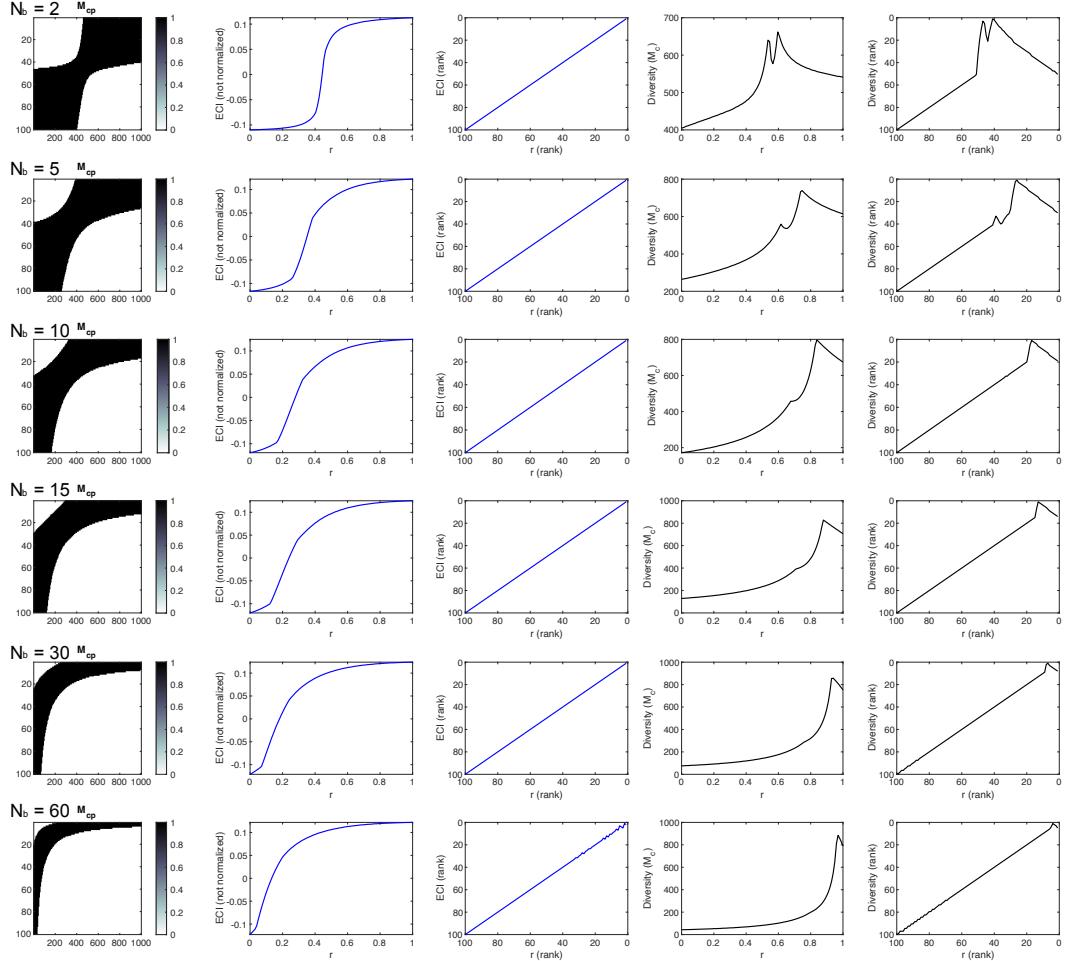


Figure 6: Comparison for the binary specialization matrix and the correlation between complexity, diversity, and the probability that a country is endowed with a capability  $r$  for models using 2, 5, 10, 15, 30, and 60 capabilities.

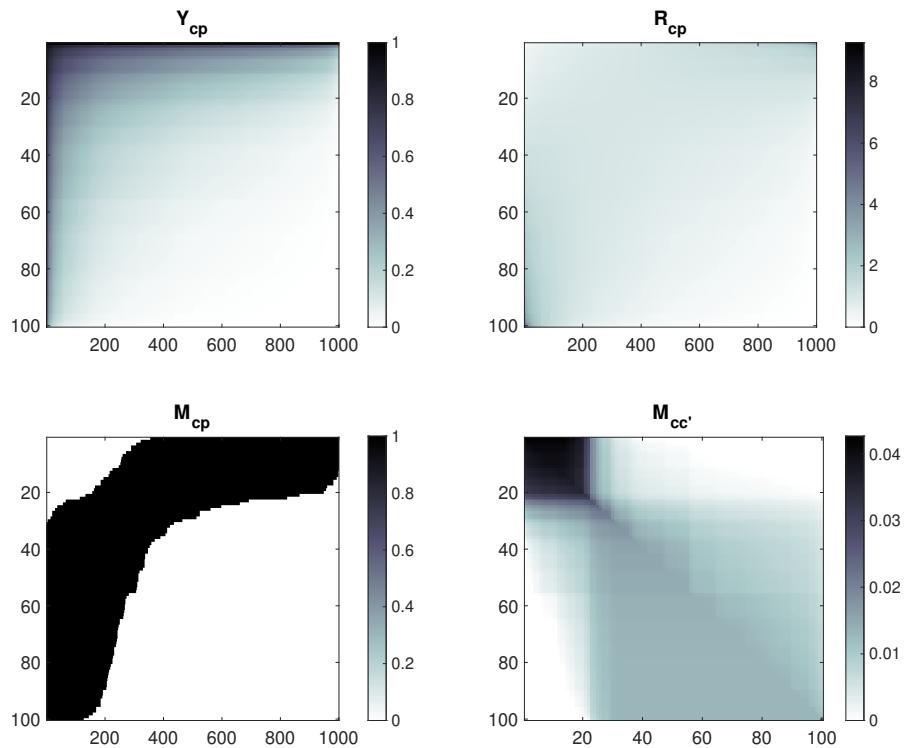


Figure 7: The four matrices involved in economic complexity calculations using a ten capability model for 100 economies and 1000 activities using randomly assigned  $r_{cs}$  and  $q_{ps}$  according to a normal distribution. In  $cp$  matrices row represents economies (countries) and columns represent activities (products). Rows are sorted from highest  $r_c$  to lowest  $r_c$  and columns are sorted from lowest  $q_p$  to highest  $q_p$ .

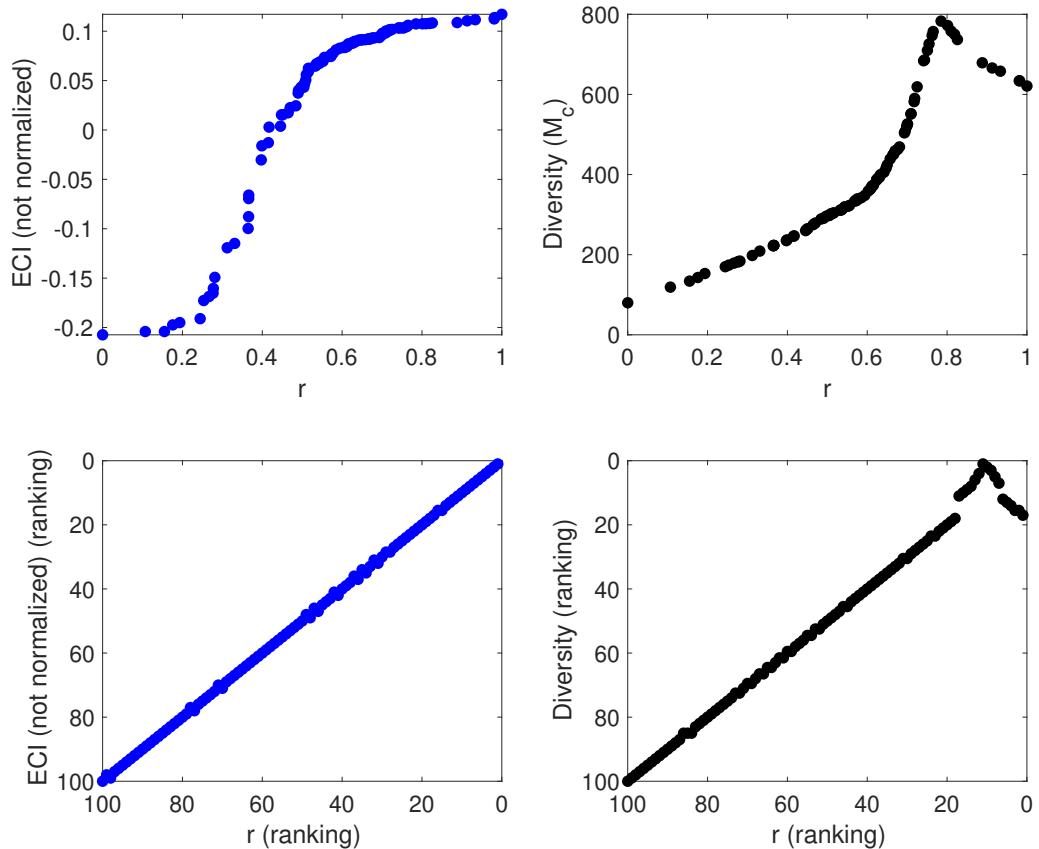


Figure 8: Comparison between the key parameter representing economies in the model ( $r$ ), the second eigenvector of the  $M_{cc'}$  matrix ( $ECI$ ), and the diversity ( $M_c$ ) of economies in a model involving 100 economies, 1000 activities, and 10 capabilities. The top two figures show the relationship between the raw variables and the bottom two show that relationship in rankings.

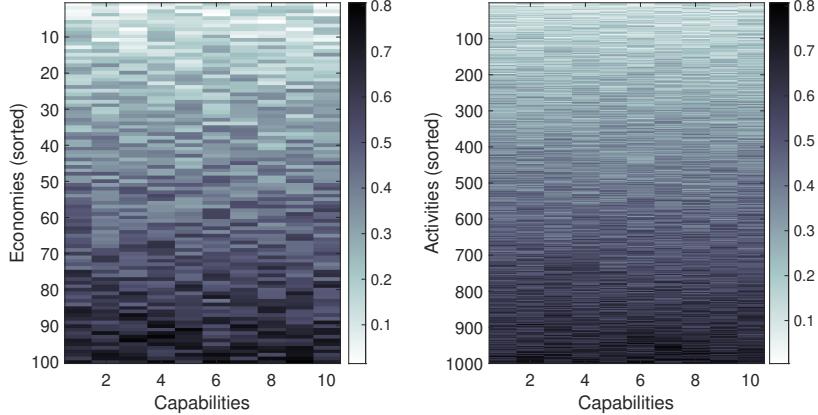


Figure 9: Parametrization of  $r_{c,b}$  and  $q_{p,b}$  for ten capabilities in a model where the probability that an economy is endowed with a capability, or that an activity requires it, is  $3/4$  of a linearly spaced baseline in the  $[0, 1]$  interval and  $1/4$  random.

Now we consider the case in which the probability that an economy is endowed with a capability, and that an activity requires one, is not equal across all capabilities. That is, we consider heterogeneous capabilities of the form:

$$r_c \rightarrow r_{c,b} \quad (41)$$

$$q_p \rightarrow q_{p,b} \quad (42)$$

Using the following parametrization:

$$\begin{cases} r_{c,b} = \alpha r_c + (1 - \alpha) \text{random}(0, 1), \\ q_{p,b} = \alpha q_p + (1 - \alpha) \text{random}(0, 1). \end{cases} \quad (43)$$

That is, we set a baseline level for the probability that an economy has a capability or that an activity requires one (which we do through a linear function), and mix that with a random number according to the proportions  $\alpha$  and  $1 - \alpha$ . When  $\alpha = 1$  the probability that an economy is endowed with a capability is the same for all economies and we recover our previous case. When  $\alpha = 0$  the capability endowments are fully random.

In this case, our goal is to explore whether  $ECI$  is able to recover the underlying structure of capability endowments. So, we compare  $ECI$  with both the average capability endowment  $\langle r_c \rangle = \sum_b r_{cb}/N_b$  and the leading singular vector of the capability matrix  $r_{cb}$ . The average captures the overall level of capabilities in each country, whereas the leading singular vector identifies the dominant mode of variation—the main direction along which countries differ in their capability profiles.

Figure 9 provides an illustration of this parametrization for the case when the probability that an economy is endowed with a capability, or that an activity requires it, is  $3/4$  of a linearly spaced baseline in the  $[0, 1]$  interval and  $1/4$  random. The matrices resulting from this model are shown in Figure 10.

We can see that despite introducing substantial variation in the capability endowments, the matrices retain a similar shape. In fact, we find that  $ECI$  continues to perform well as an estimator of both the average capability endowment  $\langle r \rangle_c$  and the leading singular vector of the capability matrix, as shown in Figure 11. This means that in the context of a model with

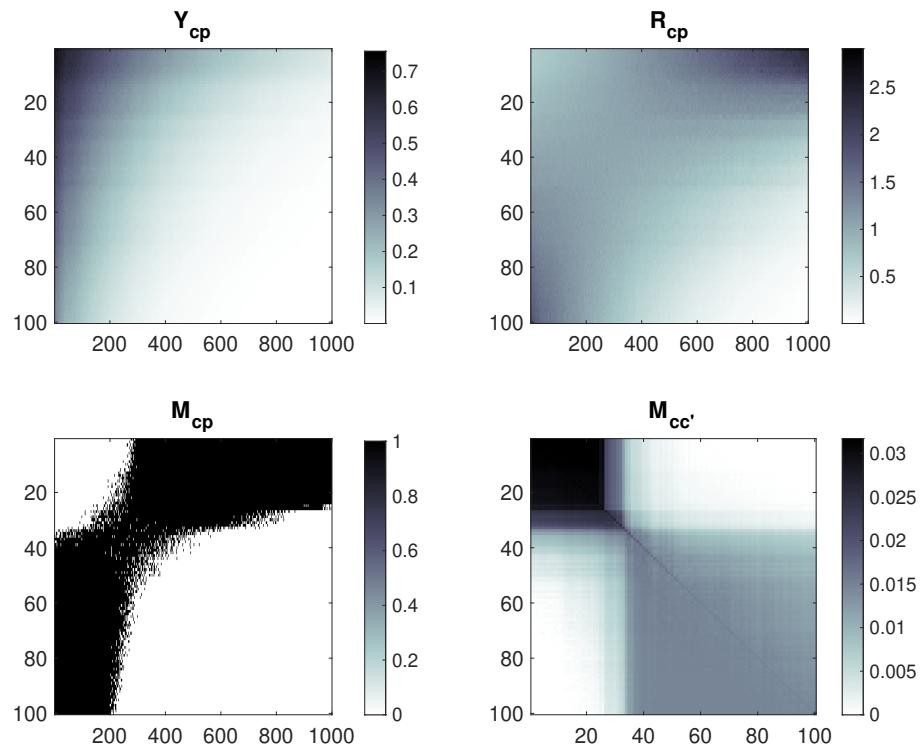


Figure 10: Matrices for a 10 capability, 100 economies, and 1000 activities model, where the probability that an economy is endowed with a capability, or that an activity requires it, is  $3/4$  of a linearly spaced baseline in the  $[0, 1]$  interval and  $1/4$  random.

multiple capabilities we can interpret  $ECI$  as an estimate of both the average capability endowment of an economy and of the dominant direction of variation in capabilities across locations.

But how far can we take this intuition? Does this method work for completely random capability endowments? Or does it require an adequate level of correlation between the different capabilities?

We can explore this by using the parametrization introduced in equation (43) to vary the level of randomness in capability endowments. Figure 12 performs this exploration, by showing the capability endowment matrices,  $M_{cp}$ , and the correlation between the ranks of  $ECI$  and 1) the average capability endowment of an economy  $\langle r \rangle_c$  and 2) the leading singular vector, for  $\alpha = [0.9, 0.75, 0.6, 0.45, 0.3, 0.15]$ . This exercise reveals that  $ECI$  is rather robust, and is able to capture both the average capability endowment and the dominant pattern of variation for an economy even when the endowment is 60 percent random and 40 percent based on a baseline. This exercise also shows that the relationship between  $ECI$  and both  $\langle r \rangle_c$  and the first singular vector breaks somewhere between  $\alpha = 0.45$  and  $\alpha = 0.3$ , suggesting a potential phase transition in this behavior.

Figure 13 explores this phase transition by presenting the average correlation between  $ECI$  and  $\langle r \rangle_c$  (left panel) and  $ECI$  and the first singular vector (right panel) observed after sweeping through the parametrization parameter  $\alpha$  250 times using a linearly spaced grid of 50 points for the interval  $\alpha = [0.01, 1]$ . We can see that there is a phase transition around 0.35, meaning that the ability of  $ECI$  to recover the average capability endowment and the pattern of variation in  $r_{cb}$  of an economy in this model ( $\langle r \rangle_c$ ) is valid as long as there is a strong enough correlation among the probabilities that an economy is endowed with different capabilities.

Overall, despite the added complexity of the multi-capability model, and the added variation of using randomly drawn probabilities for  $r_{cb}$  and  $q_{pb}$ , the behavior of the second eigenvector of  $M_{cc'}$  or  $ECI$ , and the shapes of the matrices leading to its calculation, are largely consistent with the intuition we developed in the single capability model. That extends the robustness of this idea to models with a wide range of capabilities, including models with heterogeneous capabilities and substantial levels of noise on how those capabilities are assigned.

But are these observations particular to models based on capabilities and probabilities? Or can we use the second eigenvector method of economic complexity to recover factors in models based on other production functions?

In the next section we explore extensions of this method to other production functions to delineate the effective boundaries of this theory.

## 4 Other Production Functions

You may now be wondering if the ability of  $ECI$  to recover the key parameters characterizing an economy is a property that applies to a wider family of production functions. Is this an idea that applies only to stochastic models of capabilities? Or is it a more general idea that works also for other production functions? In the case of the latter, what are the characteristics that a production function needs to satisfy to fall within the scope of this theory?

We begin by considering a production function that won't work and that will teach us a valuable lesson about those that do. This is a relative factor intensity Cobb-Douglas type

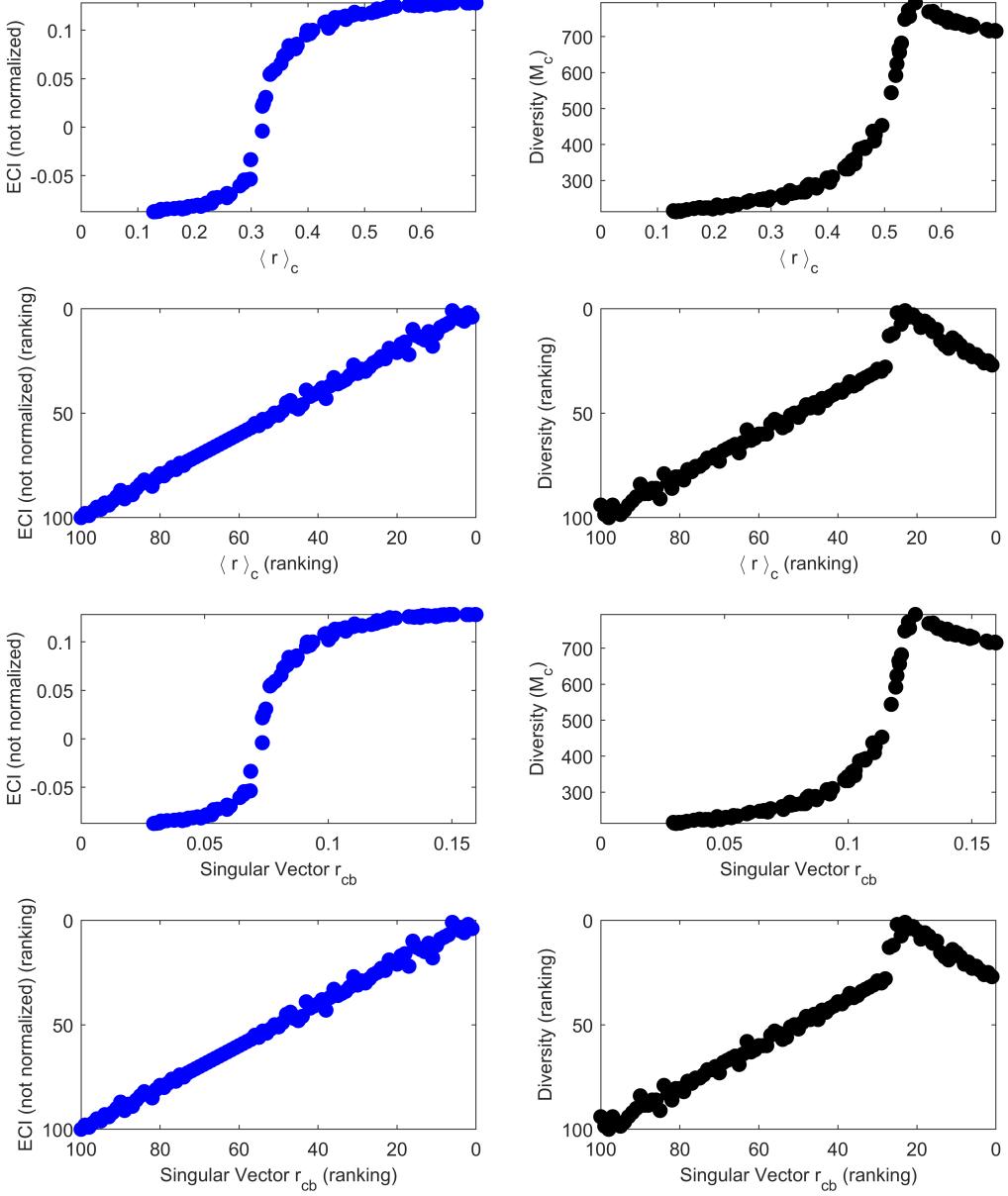


Figure 11: Relationship between the average capability endowment of an economy  $\langle r \rangle_c$ , the leading singular vector of the capability matrix  $r_{cb}$ ,  $ECI$ , and diversity, including their rankings. The model assumes that the probability an economy is endowed with a capability or that an activity requires it is composed of a structured component (3/4 linearly spaced in  $[0, 1]$ ) and a random component (1/4). The top two rows show the relationship between  $\langle r \rangle_c$ ,  $ECI$ , and diversity, while the bottom two rows show the same relationships using the leading singular vector of  $r_{cb}$  instead of the average.

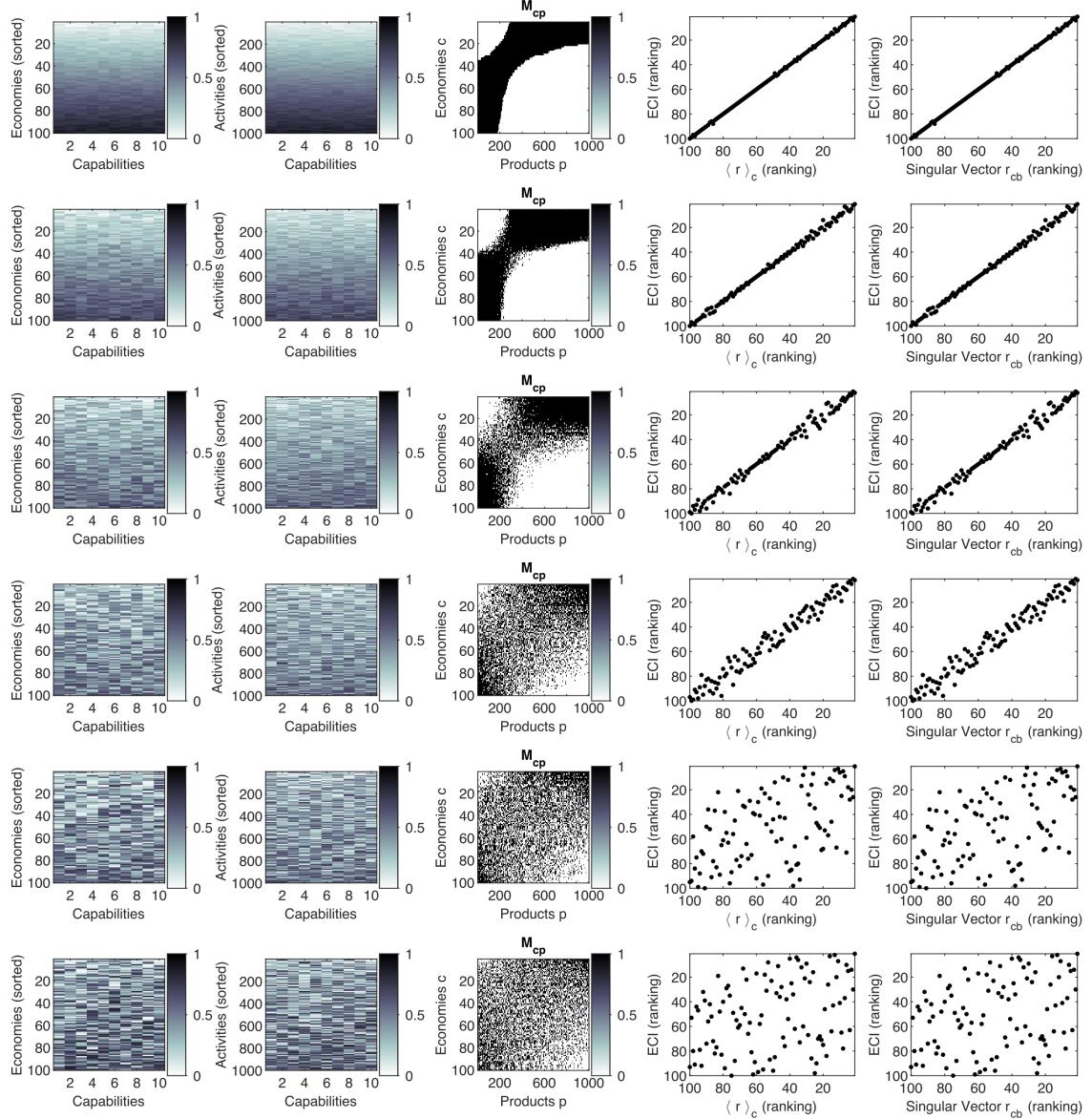


Figure 12: Numerical implementation of the multi-capability model for 100 economies, 1000 activities, and 10 capabilities. The probabilities that economies are endowed with a capability, or that activities require them, follows the parametrization in equation (43). Each row of this figure represents a different level of mixing between a baseline probability and a uniform random number. From top to bottom, the weight of the baseline  $\alpha$  are 0.9, 0.75, 0.6, 0.45, 0.3, and 0.15.

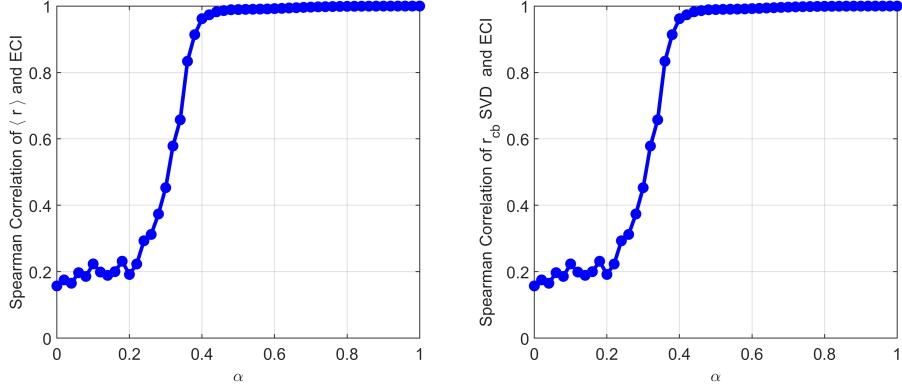


Figure 13: Average correlation between  $ECI$  and 1) the average capability endowment of an economy  $\langle r \rangle_c$  and 2) the leading singular vector or  $r_{cb}$  as a function of the mixing probability  $\alpha$ . We can see that the correlation remains close to one for mixing probabilities above 0.4.

production function of the form:

$$Y_{cp} = A(K_c/K_p)^\gamma \quad (44)$$

The problem with eqn. (44) is that in this model all economies have a comparative advantage equal to one in all activities. This is because, in this model, the output of an economy is perfectly proportional to a power of its factor endowment, meaning that there cannot be any visible specialization (at least not visible using Balassa's (1965) revealed comparative advantage indicator). This is easy to prove using the formula for  $R_{cp}$ .

$$R_{cp} = \frac{(K_c/K_p)^\gamma \sum_{c,p} (K_c/K_p)^\gamma}{\sum_c (K_c/K_p)^\gamma \sum_p (K_c/K_p)^\gamma} \quad (45)$$

which after some manipulation becomes

$$R_{cp} = \frac{(K_c/K_p)^\gamma \sum_c K_c^\gamma \sum_p (1/K_p)^\gamma}{(K_c/K_p)^\gamma \sum_c K_c^\gamma \sum_p (1/K_p)^\gamma} = 1. \quad (46)$$

In fact, we can extend this property to all separable functions of the form:

$$Y_{cp} = A f(K_c) g(K_p) \quad (47)$$

using the same calculation.

This gives us a hint of what was special of about the capability model. What made the capability model work was not that we were working with probabilities and the concept of capabilities, but that we were working with a non-multiplicative-separable function of the form  $B + f_c g_p$ <sup>19</sup>. So next, we explore a shifted version of the Cobb-Douglas factor intensity production function which breaks the separability symmetry by adding a term  $B$  which we can interpret as a baseline cost when it is negative and a baseline level of production when positive. That is, we let:

$$Y_{cp} = B + f_c g_p \quad (48)$$

<sup>19</sup>With  $B = 1$ ,  $f_c = 1 - r_c$ , and  $g_p = q_p$

Where  $f_c$  is a function describing the factor endowment of an economy and  $g_p$  is a function describing the factor intensity requirements of an activity. Applying the revealed comparative advantage formula to this shifted production function we get:

$$R_{cp} = \frac{(B + f_c g_p) \sum_{c,p} (B + f_c g_p)}{\sum_c (B + f_c g_p) \sum_p (B + f_c g_p)} \quad (49)$$

Bringing this to an inequality in which  $R_{cp} \geq 1$  and doing some algebra will lead us to the condition:

$$(f_c - \langle f \rangle)(g_p - \langle g \rangle) \geq 0 \quad (50)$$

which means:

$$\begin{aligned} M_{cp} &= 1 & \text{if } f_c \geq \langle f \rangle \quad \& \quad g_p \geq \langle g \rangle \\ M_{cp} &= 1 & \text{if } f_c < \langle f \rangle \quad \& \quad g_p < \langle g \rangle \\ M_{cp} &= 0 & \text{otherwise} \end{aligned} \quad (51)$$

In words, we recovered the binary specialization matrix of the single capability model, and the results that come downstream from that.

At this point, it is important to note one more peculiarity of the Cobb-Douglas factor intensity function that can teach us a lesson. Note that equation (50) is expressed in terms of the functions, not the factors. This is important because it means that the slopes of these function come into play. The Cobb-Douglas factor intensity model in equation (44) has opposite derivatives for the factor related to economies and the factor related to activities. That is:

$$\frac{dY_{cp}}{dK_c} > 0 \quad \& \quad \frac{dY_{cp}}{dK_p} < 0 \quad (52)$$

when  $\gamma > 0$ .

This means that when  $K_p$  is large  $g_p$  will be smaller than average  $g_p < \langle g \rangle$  and economies with a high factor endowment will specialize in activities with low factor intensity requirements. That will make  $M_{cp}$  block-diagonal. Yet, even though this is economically unreasonable (it means high complexity economies specialize in low complexity activities), it does not change the ability of the second eigenvector to separate among these two clusters.

Zooming out, there are three reasons that make the condition in equation (50) interesting.

First, it tells us that the single capability model results are valid for any production functions of the form  $Y_{cp} = B + f_c g_p$ . Second, working through the algebra tells us that this comes from the symmetry break introduced by adding the shifting term ( $B$  in this case), which makes the function non multiplicative-separable, and hence, the specialization of economies in activities not perfectly proportional to their factor endowments. And third, since the single capability model divides the world into two clusters, the more continuous eigenvectors we observe in the empirical literature, as well as the specialization matrices (e.g.  $M_{cp}$ ) can be taken as evidence of

a more complex model, or at least, a model with multiple factors.

## 5 Prices, Wages, and Consumption

We conclude our theoretical exploration by considering an extension of the single-capability model to a short-run equilibrium framework, with variable prices, wages, and consumption. We let the output of an economy in an activity depend explicitly on the price of each activity  $\pi_p$  by generalizing our output function to:

$$Y_{cp} = \pi_p(1 - q_p(1 - r_c)) = \pi_p y_{cp} \quad (53)$$

We use this function to explore a few things. First, we derive a simple relationship between capability endowments and wages. Then, we derive a new condition from the specialization matrix  $R_{cp}$ , which is the key condition connecting the empirical economic complexity estimate  $ECI$  with the model's capability endowment. Finally, we estimate product prices by exploring an extension of the model where economies maximize their utility of consumption constrained by their income and the global supply of goods.

First, we focus on wages.

In a perfectly competitive market where labor is the only factor, and all income goes into wages, then the total income of an economy  $Y_c$  must equal the wages  $w_c$  it pays times the amount of labor  $L_c$  it employs. That is:

$$Y_c = w_c L_c \quad (54)$$

Which implies:

$$w_c = \frac{\sum_p \pi_p(1 - q_p(1 - r_c))}{L_c} \quad (55)$$

dividing the numerator and denominator by  $1/N_p$  (one over the total number of activities) we can transform the sums into averages to obtain an expected wage  $w_c^*$ :

$$w_c^* = \frac{N_p (\langle \pi \rangle + \langle q\pi \rangle (r_c - 1))}{L_c} \quad (56)$$

which means that wages are proportional to the probability an economy is endowed with a capability  $r_c$ —which we can interpret as a measure of human capital, knowledge, or skill in that capability. In fact, wages grow in proportion to the product of prices times the probability an activity requires a capability and are inversely proportional to population:

$$\frac{dw_c^*}{dr_c} = \frac{N_p \langle q\pi \rangle}{L_c} \quad (57)$$

This finding is consistent with the notion that economic complexity, which we now understand as an estimate of  $r_c$ , implies an equilibrium level of wages for an economy, and thus, explains future economic growth. In this model, economies must have a wage given by eqn. 56 in equilibrium. When out of equilibrium, economies should adjust (to first order) according to:

$$\frac{dw_c}{dt} \propto -\eta(w_c - w_c^*) \quad (58)$$

where  $\eta$  is some proportionality constant (e.g. a speed or rate of adjustment). Economies with wages larger than equilibrium experience a downward pressure, whereas those with wages lower than equilibrium experience an upward pressure on their incomes.

Next, we calculate  $R_{cp}$  to determine the condition separating the two specialization clusters that are key to determining economic complexity. Going back to the definition of  $R_{cp}$  implies the condition :

$$R_{cp} = \frac{\pi_p(1 - q_p(1 - r_c)) \sum_{cp} \pi_p(1 - q_p(1 - r_c))}{\sum_c \pi_p(1 - q_p(1 - r_c)) \sum_p \pi_p(1 - q_p(1 - r_c))} \geq 1 \quad (59)$$

which after some algebra results in the inequality:

$$(r_c - \langle r \rangle)(q_p \langle \pi \rangle - \langle q \pi \rangle) \geq 0 \quad (60)$$

That brings us again to a specialization condition based on two clusters where economies with an above average probability of being endowed with the capability ( $r_c > \langle r \rangle$ ) are specialized in products with a higher probability of requiring the capability, and where those with a below average probability of being endowed with the capability ( $r_c < \langle r \rangle$ ) specialize in less demanding products. Yet, the threshold for activities is now:

$$q_p \geq \frac{\langle q \pi \rangle}{\langle \pi \rangle} \quad (61)$$

which using the standard covariance identity

$$\langle q \pi \rangle = \langle q \rangle \langle \pi \rangle + \text{cov}(q, \pi) \quad (62)$$

yields:

$$q_p \geq \langle q \rangle + \frac{\text{cov}(q, \pi)}{\langle \pi \rangle}, \quad (63)$$

this means that we recover the naked single-capability model when prices are uncorrelated with the probability that an activity requires a capability (when  $\text{cov}(q, \pi) = 0$ ). This equation also tells us that the specialization of high complexity economies in demanding activities is more pronounced when there is a positive correlation between the price of an activity and the probability it requires the capability in the model (which is a reasonable assumption). That is, in a world where prices are higher for more demanding activities, high complexity economies will specialize in a more narrow set of complex activities. Yet, for the purposes of this paper, what is important is that the specialization matrix is still divided into two clusters, just like in the single-capability model with no prices, and that these clusters separate among economies with high and low capability endowments.

Finally, we explore an extension of this model including a demand side, by assuming a logarithmic utility function. That is, we let the utility of economy  $c$  be given by:

$$U_c = \sum_p B_{cp} \log(C_{cp}) \quad (64)$$

We also assume that consumption is limited by the budget constraint:

$$\sum_p \pi_p C_{cp} \leq Y_c \quad (65)$$

which means that economies consumption is limited by the revenue generated by their total output. We also assume that the global production of goods is limited by the availability of capabilities, thus:

$$\sum_p C_{cp} = y_c \quad (66)$$

This means that in this model production capacity is fixed, and what adjusts is the price of an activity based on how demanding it is and how preferences for that activity are distributed across economies.

We start by maximizing utility following the Lagrangian:

$$\mathcal{L} = \sum_p B_{cp} \log(C_{cp}) - \lambda \left( \sum_p \pi_p C_{cp} - Y_c \right) \quad (67)$$

differentiating against consumption  $C_{cp}$  and equating to zero we obtain the condition:

$$C_{cp} = \frac{B_{cp}}{\lambda \pi_p} \quad (68)$$

And using the budget constraint equation (which we use here as an equality) we can solve for  $\lambda$ :

$$\sum_p \pi_p \frac{B_{cp}}{\lambda \pi_p} = Y_c \rightarrow \lambda = \frac{\sum_p B_{cp}}{Y_c} \quad (69)$$

meaning that consumption is given by:

$$C_{cp} = \frac{B_{cp} Y_c}{\pi_p \sum_{p'} B_{cp'}} \quad (70)$$

Finally, since:

$$Y_c = N_p (\langle \pi \rangle - (1 - r_c) \langle q\pi \rangle) \quad (71)$$

then, consumption is given by:

$$C_{cp} = \frac{B_{cp} N_p (\langle \pi \rangle - (1 - r_c) \langle q\pi \rangle)}{\pi_p \sum_p B_{cp}} \quad (72)$$

moving the  $N_p$  to the denominator allows us to transform the remaining sum into an average:

$$C_{cp} = \frac{B_{cp} (\langle \pi \rangle - (1 - r_c) \langle q\pi \rangle)}{\pi_p \langle B_c \rangle} \quad (73)$$

which means that consumption is downward sloping with the price of a good ( $\pi_p$  appears only in the denominator <sup>20</sup>) and grows with an economy's preference for a specific activity ( $B_{cp}$ ) and

<sup>20</sup> $\pi_p$  also appears implicitly in the average  $\langle \pi \rangle$  and  $\langle q\pi \rangle$  but its contribution is much smaller (divided by  $1/N_p$ ). Also, the average can be thought of as a common price level, since it is the same for all products  $p$

its capability endowment ( $r_c$ ).

Now, we estimate prices by using the market clearing condition:

$$\sum_c C_{cp} = y_p \quad (74)$$

and since:

$$y_p = N_c(1 - q_p(1 - \langle r \rangle)) \quad (75)$$

then:

$$\sum_c \frac{B_{cp}(\langle \pi \rangle - (1 - r_c)\langle q\pi \rangle)}{\pi_p \langle B_c \rangle} = N_c(1 - q_p(1 - \langle r \rangle)) \quad (76)$$

which after some algebra can be brought to the form<sup>21</sup>:

$$\pi_p = \frac{\sum_c \frac{B_{cp}}{\langle B_c \rangle} (\langle \pi \rangle - \langle q\pi \rangle (1 - r_c))}{N_c(1 - q_p(1 - \langle r \rangle))} \quad (77)$$

which means the price of activity  $p$  grows with the probability it requires the capability ( $q_p$ ), since the denominator is the smallest it can be when  $q_p = 1$  and it is the maximized for  $q_p = 0$ . Prices also grow when high capability economies (high  $r_c$  and hence high-income  $Y_c$  and high-wage economies) have a stronger preference ( $B_{cp}$ ) for an activity.

## 6 Relatedness and The Product Space

Another observable used frequently in the economic complexity literature is the network of related activities [1–3, 5–7, 83, 87, 89–91, 129–136]. When these activities are products, this network goes by the name of "product space." From an application perspective, the product space is used to estimate the potential of economy in an activity (e.g. the probability that a city specializes in an industry [3–5, 89], a country starts exporting a product [1, 2, 83], or a university starts producing papers in a given field [6, 7]. These estimates of potential are known as measures of relatedness, and are akin to traditional recommender system methods in computer science [137]. Yet, in the economic complexity literature, they are used to explain economic development trajectories (e.g. countries entering new products) instead of individual consumption patterns (e.g. customers choosing to purchase a products at an online retailer) and to explore strategies to optimize industrial promotion efforts[138–140]<sup>22</sup>

Product space type networks are important in empirical work since they help capture information about an economy's productive structure that is specific to an economy and activity. Thus, they can be used to either model path dependencies, or to control for them in work looking at the impact of other factors in economic diversification[149–152].

Here we begin by focusing on a particular characteristics of the product space that was emphasized when it was introduced nearly twenty years ago: the fact that the core of the product

<sup>21</sup>here we used the notion that averages are constants to arrive to an expression where  $\pi_p$  is expressed as a function of its ensemble averages.

<sup>22</sup>In recent years there have also been multiple efforts to look at relatedness in the context of sustainability, starting from the idea of a green product space, [141–148]

space, its most densely connected part, is composed of high-complexity activities[1].

This is a characteristic that is particular to networks derived from trade data, since networks derived from other sources can have different forms. For example, networks connecting research fields based on citation patterns or co-authorships tend to follow a "ring" structure [6, 68]. Networks connecting skills based on the occupations that require them tend to follow a "dumbbell" structure (two big clusters connected by a bridge) [91].

We begin our exploration of the structure of the product space by estimating a measure of proximity, which is an estimate of the similarity between products. Unlike in the case of *ECI*, where we have a more strict definition based on a second eigenvector, measures of proximity, in both the economic complexity and recommender systems literature, tend to be more free. In [1] for instance, proximity was introduced using the minimum of the conditional probability that two products are exported in tandem. In our notation, this translates to:

$$\phi_{pp'} = \frac{\sum_c M_{cp} M_{cp'}}{\max(M_p, M_{p'})} \quad (78)$$

In [3] proximity is simply the number of activities that are common to two economies.

$$\phi_{pp'} = \sum_c M_{cp} M_{cp'} \quad (79)$$

In general, it is not uncommon to find proximity matrices and recommender systems based on variations of  $\sum_c M_{cp} M_{cp'}$  (usually with a normalization), so we will focus our exploration in this basic form.

The product space implied by the single-capability model can be derived easily for the case in which the number of economies and activities is even. In that case, the proximity matrices are:

$$\phi_{pp'} = \sum_c M_{cp} M_{cp'} = M_p \quad \text{if } q_p > \langle q \rangle \quad \& \quad q_{p'} > \langle q \rangle \quad (80)$$

$$\phi_{pp'} = \frac{\sum_c M_{cp} M_{cp'}}{\max(M_p, M_{p'})} = 1 \quad \text{if } q_p > \langle q \rangle \quad \& \quad q_{p'} > \langle q \rangle \quad (81)$$

Which means a network composed of two clusters, one connecting the activities that are produced in high-complexity economies, and one connecting the activities produced in low complexity economies.

A more interesting exercise is to consider the networks implied by the heterogeneous multi-capability model. Here we present three examples in which we estimate networks for different model parameters that we visualize by estimating their minimum spanning tree and adding on top of that all of the links that are one standard deviation above the mean. This is a similar visualization exercise than the one used in the paper that introduced the product space network and will ensure the same visualization method is used for all examples.

Figure 14 presents this exercise for a model involving 200 activities, 100 economies, and 10 capabilities. Darker nodes indicate higher complexity. The number on top of each network

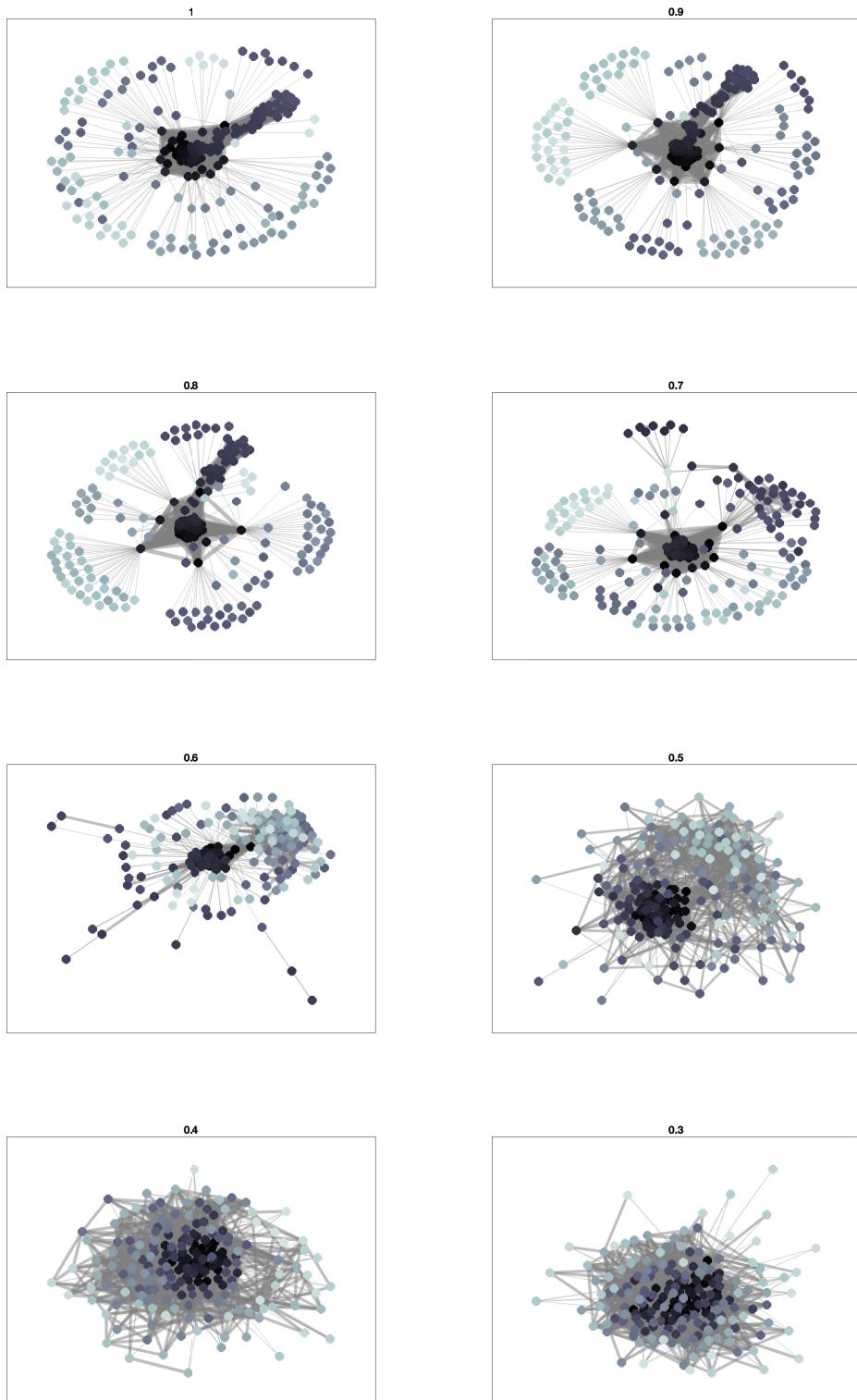


Figure 14: Networks of related activities generated for models with 200 activities, 100 economies, and 10 capabilities. Node shading corresponds to activity complexity (darker nodes denote greater complexity). Each panel is labeled with the value of the mixing parameter  $\pi$ , which specifies the degree to which random and non-random capabilities are combined (Equation (43)).

visualization shows the mixing parameter  $\alpha$  used to combine random and non-random capabilities.

We can see clearly in this example that all of the networks that are above the phase transition threshold are centered around a core of high-complexity activities, with lower complexity activities being peripheral. This reproduces the empirical observation that the core of the product space is composed of more sophisticated activities.

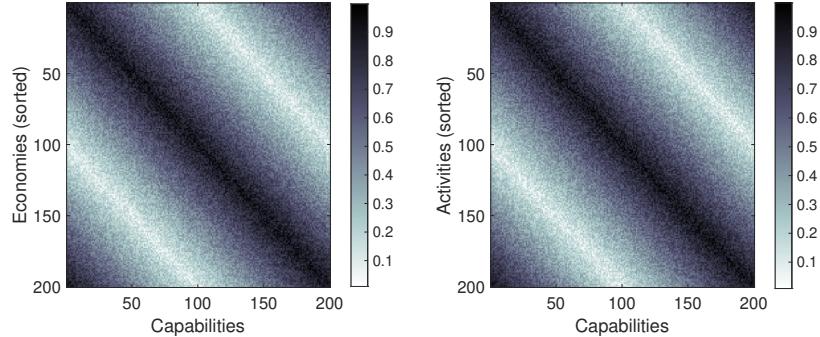


Figure 15: Parametrization of capability endowments and requirements using a symmetric Toeplitz circulant matrix combined with a random matrix in 80 percent and 20 percent proportions.

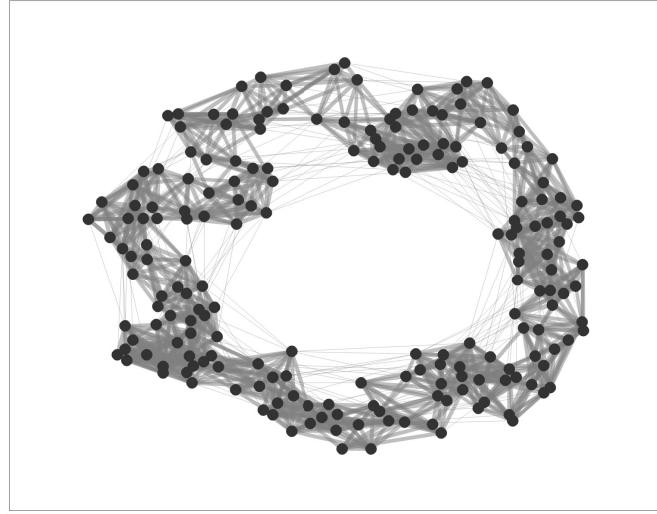


Figure 16: Network of related activities derived from the parametrization presented in figure 15.

But can we use this model to generate the network structure of the research space, which follows a ring instead of a core-periphery structure? Or do we need to radically change our assumptions to obtain that shape?

To generate a ring type network we can use Toeplitz-like matrices for the capability endowments. A Toeplitz matrix is constant along each diagonal. By setting diagonals with decreasing values or  $r_{c,b}$  and  $q_{p,b}$  we can define correlations among subsets of related activities.

Here, we use a parametrization where we combine a symmetric Toeplitz circulant matrix and a random matrix by using proportions of  $(\alpha)$  and  $(1 - \alpha)$ . A circulant matrix is a particular

type of Toeplitz matrix that has periodic boundary conditions. A symmetric circulant matrix can be constructed by starting from a row that is symmetric with respect to the center. Here, we generate these circulant matrices using linearly spaced probabilities for  $r_c$  and  $q_p$  that grow symmetrically from the center column of the first row. Figure 15 shows an example of this parametrization for a model with 200 economies, 200 activities, and 200 capabilities. We note that in this model talking about higher and lower complexity economies is not a useful construct, since economies do not differ on their average capability endowment (they are all equal on average), but in which subset of capabilities they are specialized in.

Figure 16 visualizes the network derived from this parametrization using the same method than before (minimum spanning tree, plus links that are one standard deviation above the average weight). The visualization shows a clear ring structure mimicking the one observed in networks involving research fields. The connectivity pattern of this network can be interpreted as research fields having a few related activities that share capabilities among them (e.g. capabilities are more re-deployable between molecular biology and biochemistry, than between polymer sciences and experimental psychology). This results in a network structure where each field is connected to a few neighbors.

Finally, we use the same approach to model a “dumbbell” network, which is a network with two well-defined clusters, such as the one observed when connecting skills and occupations [91]. Figures 17 and 18 show an example with 100 economies, 500 activities, and 20 capabilities. We note that obtaining this dumbbell structure requires a good level of mixing between the clusters, which can be achieved by setting the noise levels to be high enough so that some of the between cluster links are comparable in strength to the within cluster links.

What is exciting about this general idea is that it provides us with an intuitive way to read capability endowments from network structures. For instance, the core periphery-structure of the product space tells us that the capabilities associated with exporting products are correlated among economies, with high complexity economies like that of Singapore, Japan, or the United States, being well endowed with a wide set of capabilities. The ring structure of the research space tells a different story. It is a story of specialization in a world of fine grained capabilities. Similarly, we can use this intuition to think about dumbbell structures, which can be modeled by assuming capability endowments made of slightly overlapping blocks.

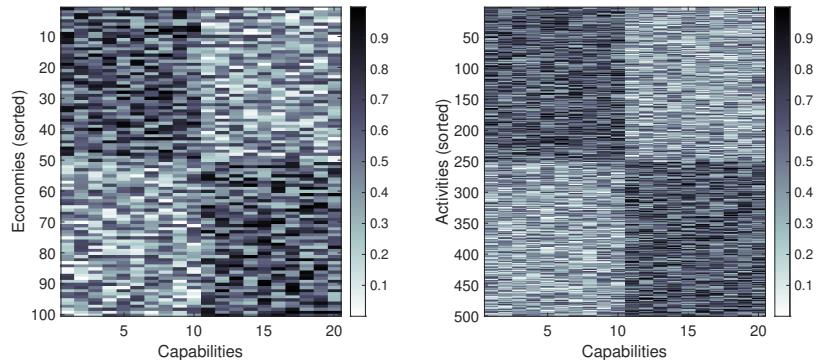


Figure 17: Parametrization of capability endowments and requirements using a 25 percent of a block diagonal matrix and 75 percent of a random matrix.

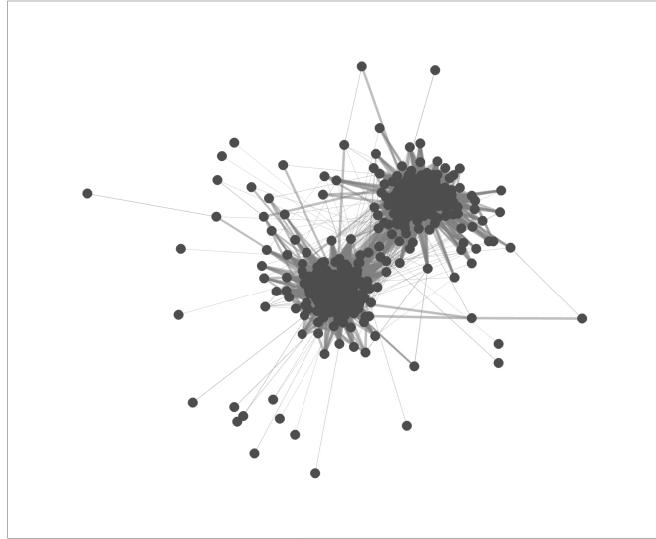


Figure 18: Network of related activities derived from the parametrization presented in figure 17.

## 7 Conclusion

Economic complexity has for long attempted to study economic growth and development using methods that are agnostic about the exact nature of factors of production. In this paper, we contribute to this goal by providing an analytical foundation for the economic complexity index (*ECI*) and showing that it can be indeed consider an estimate of the combined presence of undefined or unknown factors of production. For the single-capability model, we could derive the key eigenvector analytically and show that *ECI* separates economies among those with an above- and below-average probability of having the capability. We then extended this result numerically to a multi-capability setting to show that *ECI* is a monotonic estimator of an economy's average capability endowment—even when a substantial share of the capabilities are randomly assigned. In the multi-capability model, *ECI* is no longer a discrete measure separating low from high capability economies, but a monotonic transformation of the average capability endowment of an economy that recovers the first singular vector of the capability endowment matrix  $r_{cb}$ . These findings differentiate *ECI* from measures of diversity, which peak for capability endowments below the maximum (they are non-monotonic functions of  $r_c$ ), and thus are non-ideal estimates of the complexity of an economy. These results validate *ECI* as a measure of composition or complexity, since they show the eigenvector captures information about an economy being endowed with multiple capabilities, regardless of how these capabilities are defined.

Interestingly, our main result does not depend on assuming a stochastic model or a theory based on capabilities, since this idea can be easily generalized to models including factors that are specific to economies and activities. The key condition for the measure of complexity to work is for output to not be perfectly proportional to factor endowments. This condition can be achieved by simply shifting the production function by a constant to make it non-multiplicatively separable.<sup>23</sup>

What is also interesting is that the condition needed for our main result to hold comes from

<sup>23</sup>This mechanism is akin to the idea of symmetry-breaking in physics, since the shift removes symmetries of the function. For example  $K_c^\gamma$  satisfies the scale-invariance symmetry  $f(\lambda K) = \lambda^\gamma f(K)$ , whereas  $B + K_c^\gamma$  lacks this symmetry.

calculating the matrix of specialization  $R_{cp}$ . This is a key difference with previous attempts to connect economic complexity theory and empirics [10, 56] which jumped directly to the binary specialization matrix  $M_{cp}$ . That assumption results in a monotonic relationship between the number of activities an economy is specialized (its diversity) and its capability endowments<sup>24</sup>, which is an uncomfortable result since we have known for a long time that measures of diversity fail to explain future economic growth like measures of complexity do [10]. We now understand that calculating these specialization matrices is a key step, and that skipping this step in theoretical work results in a flawed connection between complexity and capability endowments. This change not only uncovers a tight connection between the economic complexity index and the capability, but explains other findings, like that of Imbs and Warcziag [82], which says that economies diversify only until a certain point.

Our results also open questions about alternative measures of complexity. During the last fifteen years, many alternatives to the economic complexity index have been proposed, such as the Fitness index [19], the Ability index [20], and many others [21, 63, 80, 81, 153–156]. Since these indexes tend to exhibit strong correlations with  $ECI$ , our results provide a way to theoretically explore whether they are also monotonic functions of an economy’s capability endowment. If they are, this opens up the question about the importance of the functional form connecting measures of complexity and capabilities.

Our work also speaks to the literature attempting to explain the economic complexity index. A key result in this literature is the idea that  $ECI$  is a clustering algorithm [58, 62], separating economies into different groups. Our work is consistent with this idea and provides a theoretical interpretation for the clusters, as it shows that what  $ECI$  is doing is providing a sigmoid function sorting economies into a high- and low-capability cluster. This sigmoid behavior is a well known feature of the second eigenvector or eigenfunction of diffusion maps, Fokker-Planck operators, and spectral clustering methods (e.g. see [157]). This provides an interesting link to the general idea of diffusion albeit in the context of a model of economic development, opening the door to the notion that these methods could be capturing a generalizable property of economic systems subject to spillovers.

We also embedded this model in a short-run equilibrium framework to estimate wages, consumption, and prices. In this framework, wages increase with capability endowments and prices are higher for products that demand more capabilities. The latter result is highly concave ( $\sim 1/(1-q)$ ), meaning that there is an important premium for producing high complexity products. Interestingly, prices do not strongly affect the specialization condition, meaning that they leave the connection between capability endowments and economic complexity mostly unchanged.<sup>25</sup>

Finally, we showed that the model can explain structural differences in networks of related activities, such as the product space and research space. By controlling the shape of the capability endowment matrices, we were able to reproduce the core-periphery structure observed in the product space [1], the ring structure observed for scientific publications [6, 68], and the dumbbell structure observed for networks of occupations and skills [91].

Together, these findings help resolve a few long-standing tensions in the economic complexity literature. First, and most importantly, the disconnect between its empirical metrics and their theoretical underpinnings. Our findings show that  $ECI$  is not an arbitrary or ad-hoc measure, but can be thought of as an estimator of an economy’s capability endowments derived from its

---

<sup>24</sup>That equation is provided in [56].

<sup>25</sup>We assume prices are the same across economies.

pattern of specialization. This is an interesting finding, since it provides a mean to estimate the combined presence of factors or capabilities even when these cannot be identified.

Second, we use standard macroeconomic assumptions to estimate the wages and prices associated with this model, which help support the well known empirical fact that economies tend to converge to a level of income that is related to their economic complexity [2, 10–27].

And third, we provide theoretical underpinnings for the structure of the networks of related activities. Our findings show that the structure of these networks is a reflection of how capabilities are distributed across economies and activities.

More broadly, our work helps clarify a field that had grown rapidly in its empirical scope while lacking a shared theoretical core. By grounding complexity metrics in production functions, and explaining the structure of networks of relatedness using a capability-based model, we offer a framework that not only explains the empirical robustness of *ECI*, but should also open new paths for integrating economic complexity ideas further into development economics and trade theory.

## Acknowledgments

This work owes a very special acknowledgment to Cristian Jara-Figueroa. In 2014, Cristian joined my (César’s) group at the MIT Media Lab. During the first year of his Master’s he worked on the mathematical theory of economic complexity producing an impressive internal manuscript with many results. Those results were never published, but stayed in my group. In 2025, while looking at Cristian’s work, I realized we had made an important and simple mistake at the very beginning, which was to assume that the capability model was a model of  $M_{cp}$  instead of a model of the output matrix  $Y_{c,p}$ . This required reworking everything, and motivated me to estimate the intermediate matrices in the model (such as  $R_{cp}$ ). In my mind this work owes enormously to that effort by Cristian many years ago. We would also like to acknowledge comments by Johanness Wachs, Harasiddh Ganesh, and other members of the Center for Collective Learning. The section on prices and wages was motivated by a inspiring conversation with Jean Tirole.

We acknowledge the support of the European Union LearnData, GA no. 101086712 a.k.a. 101086712-LearnDataHORIZON-WIDERA-2022-TALENTS-01 (<https://cordis.europa.eu/project/id/101086712>), IAST funding from the French National Research Agency (ANR) under grant ANR-17-EURE-0010 (Investissements d’Avenir program), and the European Lighthouse of AI for Sustainability grant number 101120237-HORIZON-CL4-2022-HUMAN-02.

## References

- [1] C.A. Hidalgo, B. Klinger, A.-L. Barabási, and R. Hausmann. The Product Space Conditions the Development of Nations. *Science*, 317(5837):482–487, July 2007.
- [2] Ricardo Hausmann, César A. Hidalgo, Sebastián Bustos, Michele Coscia, Alexander Simoes, and Muhammed A. Yildirim. *The atlas of economic complexity: Mapping paths to prosperity*. MIT Press, 2014.
- [3] Frank Neffke, Martin Henning, and Ron Boschma. How Do Regions Diversify over Time? Industry Relatedness and the Development of New Growth Paths in Regions. *Economic Geography*, 87:237–265, 2011.

- [4] Frank Neffke and Martin Henning. Skill relatedness and firm diversification. *Strategic Management Journal*, 34(3):297–316, 2013.
- [5] C. Jara-Figueroa, Bogang Jun, Edward L. Glaeser, and Cesar A. Hidalgo. The role of industry-specific, occupation-specific, and location-specific knowledge in the growth and survival of new firms. *Proceedings of the National Academy of Sciences*, 115(50):12646–12653, December 2018.
- [6] Miguel R. Guevara, Dominik Hartmann, Manuel Aristarán, Marcelo Mendoza, and César A. Hidalgo. The research space: using career paths to predict the evolution of the research output of individuals, institutions, and nations. *Scientometrics*, 109(3):1695–1709, December 2016.
- [7] Matteo Chinazzi, Bruno Gonçalves, Qian Zhang, and Alessandro Vespignani. Mapping the physics research space: a machine learning approach. *EPJ Data Science*, 8(1):33, 2019. ISBN: 2193-1127 Publisher: Springer Berlin Heidelberg.
- [8] Dieter F. Kogler, David L. Rigby, and Isaac Tucker. Mapping Knowledge Space and Technological Relatedness in US Cities. *European Planning Studies*, 21(9):1374–1391, September 2013.
- [9] Dieter F. Kogler, David L. Rigby, and Isaac Tucker. Mapping knowledge space and technological relatedness in US cities. In *Global and Regional Dynamics in Knowledge Flows and Innovation*, pages 58–75. Routledge, 2015.
- [10] César A. Hidalgo and Ricardo Hausmann. The building blocks of economic complexity. *Proceedings of the National Academy of Sciences*, 106(26):10570–10575, June 2009.
- [11] Juan Carlos Chávez, Marco T. Mosqueda, and Manuel Gómez-Zaldívar. Economic complexity and regional growth performance: Evidence from the Mexican Economy. *Review of Regional Studies*, 47(2):201–219, 2017.
- [12] Giacomo Domini. Patterns of specialization and economic complexity through the lens of universal exhibitions, 1855-1900. *Explorations in Economic History*, 83:101421, 2022. ISBN: 0014-4983 Publisher: Elsevier.
- [13] Philipp Koch. Economic Complexity and Growth: Can value-added exports better explain the link? *Economics Letters*, 198:109682, 2021. ISBN: 0165-1765 Publisher: Elsevier.
- [14] Viktor Stojkoski, Zoran Utkovski, and Ljupco Kocarev. The Impact of Services on Economic Complexity: Service Sophistication as Route for Economic Growth. *PLOS ONE*, 11(8):e0161633, August 2016.
- [15] Viktor Stojkoski and Ljupco Kocarev. The relationship between growth and economic complexity: evidence from Southeastern and Central Europe. 2017.
- [16] Guzmán Ourens. Can the Method of Reflections help predict future growth? *Documento de Trabajo/FCS-DE*; 17/12, 2012. Publisher: UR. FCS-DE.
- [17] Sandra Poncet and Felipe Starosta de Waldemar. Economic Complexity and Growth. *Revue économique*, 64(3):495–503, 2013. ISBN: 0035-2764 Publisher: Presses de Sciences Po.
- [18] Viktor Stojkoski, Philipp Koch, and César A. Hidalgo. Multidimensional Economic Complexity: How the Geography of Trade, Technology, and Research Explain Inclusive Green Growth, September 2022. arXiv:2209.08382 [cond-mat, q-fin].

- [19] Andrea Tacchella, Matthieu Cristelli, Guido Caldarelli, Andrea Gabrielli, and Luciano Pietronero. A New Metrics for Countries' Fitness and Products' Complexity. *Scientific Reports*, 2:srep00723, October 2012.
- [20] Sebastian Bustos and Muhammed A. Yıldırım. Production ability and economic growth. *Research Policy*, 51(8):104153, 2022. ISBN: 0048-7333 Publisher: Elsevier.
- [21] David Atkin, Arnaud Costinot, and Masao Fukui. Globalization and the ladder of development: Pushed to the top or held at the bottom? Technical report, National Bureau of Economic Research, 2021.
- [22] Felipe Orsolini Teixeira, Fabricio José Missio, and Ricardo Dathein. Economic complexity, structural transformation and economic growth in a regional context: Evidence for brazil. *PSL Quarterly Review*, 75(300):63–79, 2022.
- [23] Lilis Hoeriyah, Nunung Nuryartono, and Syamsul Hidayat Pasaribu. Economic complexity and sustainable growth in developing countries. *Economics Development Analysis Journal*, 11(1):23–33, 2022.
- [24] Zhuqing Mao and Qinrui An. Economic complexity index and economic development level under globalization: An empirical study. *Journal of Korea Trade*, 25(7):41–55, 2021.
- [25] Roberto Basile and Gloria Cicerone. Economic complexity and productivity polarization: Evidence from Italian provinces. *German Economic Review*, March 2022. Publisher: De Gruyter.
- [26] Ben-Hur Francisco Cardoso, Eva Yamila da Silva Catela, Guilherme Viegas, Flávio L Pinheiro, and Dominik Hartmann. Export complexity, industrial complexity and regional economic growth in brazil. *arXiv preprint arXiv:2312.07469*, 2023.
- [27] J Romero, E Freitas, F Silveira, G Britto, F Cimini, and G Jayme. Economic complexity and regional economic development: evidence from brazil. *EAEP, Online Proceedings*, pages 1–22, 2021.
- [28] Santiago Pérez-Balsalobre, Carlos Llano Verduras, and Jorge Díaz-Lanchas. Measuring subnational economic complexity: An application with spanish data. Technical report, JRC Working Papers on Territorial Modelling and Analysis, 2019.
- [29] Dominik Hartmann, Miguel R. Guevara, Cristian Jara-Figueroa, Manuel Aristarán, and César A. Hidalgo. Linking Economic Complexity, Institutions, and Income Inequality. *World Development*, 93:75–93, May 2017.
- [30] Margarida Bandeira Morais, J. Swart, and J. A. Jordaan. Economic Complexity and Inequality: Does Productive Structure Affect Regional Wage Differentials in Brazil? *USE Working Paper series*, 18(11), 2018. Publisher: USE Research Institute.
- [31] Angelica Sbardella, Emanuele Pugliese, and Luciano Pietronero. Economic development and wage inequality: A complex system analysis. *PloS one*, 12(9), 2017. Publisher: Public Library of Science.
- [32] Emilie Le Caous and Fenghueih Huarng. Economic Complexity and the Mediating Effects of Income Inequality: Reaching Sustainable Development in Developing Countries. *Sustainability*, 12(5):2089, January 2020. Number: 5 Publisher: Multidisciplinary Digital Publishing Institute.
- [33] Myriam Ben Saâd and Giscard Assoumou-Ella. Economic Complexity and Gender Inequality in Education: An Empirical Study. *Economics Bulletin*, 39(1):321–334, 2019.

- [34] Fadi Fawaz and Masha Rahnama-Moghadamm. Spatial dependence of global income inequality: The role of economic complexity. *The International Trade Journal*, 33(6):542–554, 2019. ISBN: 0885-3908 Publisher: Taylor & Francis.
- [35] Shengjun Zhu, Changda Yu, and Canfei He. Export structures, income inequality and urban-rural divide in China. *Applied Geography*, 115:102150, February 2020.
- [36] Radu Barza, Cristian Jara-Figueroa, CÃ Hidalgo, and Martina Viarengo. Knowledge Intensity and Gender Wage Gaps: Evidence from Linked Employer-Employee Data. 2020. Publisher: CESifo Working Paper.
- [37] Chien-Chiang Lee and En-Ze Wang. Economic complexity and income inequality: Does country risk matter? *Social Indicators Research*, 154(1):35–60, 2021.
- [38] Muhlis Can and Buhari Doğan. The effects of economic structural transformation on employment: an evaluation in the context of economic complexity and product space theory. In *Handbook of research on unemployment and labor market sustainability in the era of globalization*, pages 275–306. IGI Global, 2017.
- [39] Olimpia Neagu. The Link between Economic Complexity and Carbon Emissions in the European Union Countries: A Model Based on the Environmental Kuznets Curve (EKC) Approach. *Sustainability*, 11(17):4753, 2019.
- [40] Olimpia Neagu and Mircea Constantin Teodoru. The relationship between economic complexity, energy consumption structure and greenhouse gas emission: Heterogeneous panel evidence from the eu countries. *Sustainability*, 11(2):497, 2019.
- [41] João P. Romero and Camila Gramkow. Economic complexity and greenhouse gas emissions. *World Development*, 139:105317, March 2021.
- [42] Athanasios Lapatinas, Antonios Garas, Eirini Boleti, and Alexandra Kyriakou. Economic complexity and environmental performance: Evidence from a world sample, March 2019. Library Catalog: mpra.ub.uni-muenchen.de.
- [43] Penny Mealy and Alexander Teytelboym. Economic complexity and the green economy. *Research Policy*, page 103948, 2020. ISBN: 0048-7333 Publisher: Elsevier.
- [44] Manuel Gómez-Zaldívar, María Isabel Osorio-Caballero, and Edgar Juan Saucedo-Acosta. Income inequality and economic complexity: Evidence from mexican states. *Regional Science Policy & Practice*, 14(6):344–364, 2022.
- [45] Daniel Balsalobre-Lorente, Clara Contente dos Santos Parente, Nuno Carlos Leitão, and José María Cantos-Cantos. The influence of economic complexity processes and renewable energy on co2 emissions of brics. what about industry 4.0? *Resources Policy*, 82:103547, 2023.
- [46] Fabricio Silveira, João P Romero, Arthur Queiroz, Elton Freitas, and Alexandre Stein. Economic complexity and deforestation in the brazilian amazon. *World Development*, 185:106804, 2025.
- [47] Gertjan Dordmond, Heder Carlos de Oliveira, Ivair Ramos Silva, and Julia Swart. The complexity of green job creation: An analysis of green job development in Brazil. *Environment, Development and Sustainability*, pages 1–24, 2020. ISBN: 1573-2975 Publisher: Springer.

- [48] Barbaros Güneri and A. Yasemin Yalta. Does economic complexity reduce output volatility in developing countries? *Bulletin of Economic Research*. ISBN: 0307-3378 Publisher: Wiley Online Library.
- [49] Trung V. Vu. Does LGBT Inclusion Promote National Innovative Capacity? SSRN Scholarly Paper ID 3523553, Social Science Research Network, Rochester, NY, January 2020.
- [50] Diogo Ferraz, Herick Fernando Morales, Jessica Suárez Campoli, Fabíola Cristina Ribeiro de Oliveira, and Daisy Aparecida do Nascimento Rebelatto. Economic complexity and human development: Dea performance measurement in asia and latin america. *Gestão & Produção*, 25(4):839–853, 2018.
- [51] Athanasios Lapatinas and Marina-Selini Katsaiti. EU MECI: A Network-Structured Indicator for a Union of Equality. *Social Indicators Research*, February 2023.
- [52] Radu Barza, Edward L. Glaeser, César A. Hidalgo, and Martina Viarengo. Cities as Engines of Opportunities: Evidence from Brazil, May 2024.
- [53] Taylan Yenilmez. Understanding complexity in the author-journal space. *Scientometrics*, pages 1–28, 2025.
- [54] Ronald Djeunankan, Sosson Tadadjeu, Henri Njangang, and Ummad Mazhar. The hidden cost of sophistication: economic complexity and obesity. *The European Journal of Health Economics*, 26(2):243–265, 2025.
- [55] Omar Lizardo. The mutual specification of genres and audiences: Reflective two-mode centralities in person-to-culture data. *Poetics*, 68:52–71, 2018. ISBN: 0304-422X Publisher: Elsevier.
- [56] Ricardo Hausmann and César A. Hidalgo. The network structure of economic output. *Journal of Economic Growth*, pages 1–34, 2011.
- [57] Ulrich Schetter. Comparative advantages with product complexity and product quality. 2016. Publisher: Kiel und Hamburg: ZBW-Deutsche Zentralbibliothek für ....
- [58] Penny Mealy, J. Doyne Farmer, and Alexander Teytelboym. Interpreting economic complexity. *Science Advances*, 5(1):eaau1705, January 2019.
- [59] Muhammed A. Yildirim. Sorting, Matching and Economic Complexity. *CID Working Paper Series*, 2021. Publisher: Center for International Development at Harvard University.
- [60] Benjamin Cakir, Isabelle Schluep, Philipp Aerni, and Isa Cakir. Amalgamation of export with import information: The economic complexity index as a coherent driver of sustainability. *Sustainability*, 13(4):2049, 2021.
- [61] Vito DP Servedio, Alessandro Bellina, Emanuele Calò, and Giordano De Marzo. Economic Complexity in Mono-Partite Networks. *arXiv preprint arXiv:2405.04158*, 2024.
- [62] Carlo Bottai, Jacopo Di Iorio, and Martina Iori. Reinterpreting Economic Complexity: A co-clustering approach, June 2024. arXiv:2406.16199 [econ, q-fin, stat].
- [63] James McNerney, Yang Li, Andres Gomez-Lievano, and Frank Neffke. Bridging the short-term and long-term dynamics of economic structural change, March 2023. arXiv:2110.09673 [physics, q-fin].

- [64] Zoran Utkovski, Melanie F. Pradier, Viktor Stojkoski, Fernando Perez-Cruz, and Ljupco Kocarev. Economic complexity unfolded: Interpretable model for the productive structure of economies. *PLoS one*, 13(8):e0200822, 2018.
- [65] Benoît Desmarchelier, Paulo José Regis, and Nimesh Salike. Product space and the development of nations: A model of product diversification. *Journal of Economic Behavior & Organization*, 145:34–51, 2018.
- [66] César A. Hidalgo. Economic complexity theory and applications. *Nature Reviews Physics*, pages 1–22, 2021. ISBN: 2522-5820 Publisher: Nature Publishing Group.
- [67] Pierre-Alexandre Baland, Tom Broekel, Dario Diodato, Elisa Giuliani, Ricardo Hausmann, Neave O’Clery, and David Rigby. The new paradigm of economic complexity. *Research Policy*, 51(3):104450, 2022. ISBN: 0048-7333 Publisher: Elsevier.
- [68] Katy Börner, Richard Klavans, Michael Patek, Angela M. Zoss, Joseph R. Biberstine, Robert P. Light, Vincent Larivière, and Kevin W. Boyack. Design and update of a classification system: The UCSD map of science. *PLoS one*, 7(7):e39464, 2012. ISBN: 1932-6203 Publisher: Public Library of Science.
- [69] Ministry of Investment Trade and Industry of Malaysia. New industrial master plan. <https://www.nimp2030.gov.my/>. [Accessed 16-06-2025].
- [70] Mario Draghi. The future of european competitiveness part a: A competitiveness strategy for europe. 2024.
- [71] Christian Reynolds, Manju Agrawal, Ivan Lee, Chen Zhan, Jiuyong Li, Phillip Taylor, Tim Mares, Julian Morison, Nicholas Angelakis, and Göran Roos. A sub-national economic complexity analysis of Australia’s states and territories. *Regional Studies*, 52(5):715–726, 2018. ISBN: 0034-3404 Publisher: Taylor & Francis.
- [72] Birol Erkan and Elif Yildirimci. Economic Complexity and Export Competitiveness: The Case of Turkey. *Procedia-Social and Behavioral Sciences*, 195:524–533, 2015. ISBN: 1877-0428 Publisher: Elsevier.
- [73] Natalia Ferreira-Coimbra and Marcel Vaillant. Evolución del espacio de productos exportados: ¿está Uruguay en el lugar equivocado? *Revista de economía*, 16(2):97–146, 2009. ISBN: 0797-5546 Publisher: Banco Central del Uruguay.
- [74] Ivan L. Lyubimov, Maria V. Lysyuk, and Margarita A. Gvozdeva. Atlas of economic complexity, Russian regional pages. *VOPROSY ECONOMIKI*, 6, 2018.
- [75] I. Lyubimov, M. Gvozdeva, M. Kazakova, and K. Nesterova. Economic Complexity of Russian Regions and their Potential to Diversify. *Journal of the New Economic Association*, 34(2):94–122, 2017.
- [76] Fernando Gómez Zaldívar, Edmundo Molina, Miguel Flores, and Manuel de Jesús Gómez Zaldívar. Economic Complexity of the Special Economic Zones in Mexico: Opportunities for Diversification and Industrial Sophistication. *Ensayos Revista de Economía (Ensayos Journal of Economics)*, 38(1):1–40, 2019. ISBN: 2448-8402.
- [77] Carla Carolina Pérez Hernández, Blanca Cecilia Salazar Hernández, and Jessica Mendoza Moheno. Diagnóstico de la complejidad económica del estado de Hidalgo: de las capacidades a las oportunidades. *Revista mexicana de economía y finanzas*, 14(2):261–277, 2019. ISBN: 1665-5346 Publisher: Instituto Mexicano de Ejecutivos de Finanzas, AC.

- [78] Yihan Wang and Ekaterina Turkina. Economic complexity, product space network and Quebec's global competitiveness. *Canadian Journal of Administrative Sciences/Revue Canadienne des Sciences de l'Administration*, 37(3):334–349, 2020. ISBN: 0825-0383 Publisher: Wiley Online Library.
- [79] Roberto Basile, Gloria Cicerone, and Lelio Iapadre. Economic complexity and regional labor productivity distribution: evidence from Italy. *Economic complexity and regional labor productivity distribution: evidence from Italy*, 2019.
- [80] Jorge Valverde-Carbonell. Rethinking the Literature on Economic Complexity Indexes. *Economic Analysis and Policy*, May 2025.
- [81] Carla Sciarra, Guido Chiarotti, Luca Ridolfi, and Francesco Laio. Reconciling contrasting views on economic complexity. *Nature communications*, 11(1):1–10, 2020. ISBN: 2041-1723 Publisher: Nature Publishing Group.
- [82] Jean Imbs and Romain Wacziarg. Stages of diversification. *American economic review*, 93(1):63–86, 2003. ISBN: 0002-8282.
- [83] Hidalgo, Pierre-Alexandre Balland, Ron Boschma, Mercedes Delgado, Maryann Feldman, Koen Frenken, Edward Glaeser, Canfei He, Dieter F. Kogler, Andrea Morrison, Frank Neffke, David Rigby, Scott Stern, Siqi Zheng, and Shengjun Zhu. The Principle of Relatedness. In Alfredo J. Morales, Carlos Gershenson, Dan Braha, Ali A. Minai, and Yaneer Bar-Yam, editors, *Unifying Themes in Complex Systems IX*, Springer Proceedings in Complexity, pages 451–457. Springer International Publishing, 2018.
- [84] Bogang Jun, Aamena Alshamsi, Jian Gao, and César A. Hidalgo. Bilateral relatedness: knowledge diffusion and the evolution of bilateral trade. *Journal of Evolutionary Economics*, pages 1–31, 2019. ISBN: 0936-9937 Publisher: Springer.
- [85] Teresa Farinha, Pierre-Alexandre Balland, Andrea Morrison, and Ron Boschma. What drives the geography of jobs in the us? unpacking relatedness. *Industry and Innovation*, 26(9):988–1022, 2019. ISBN: 1366-2716 Publisher: Taylor & Francis.
- [86] Pierre-Alexandre Balland, José Antonio Belso-Martínez, and Andrea Morrison. The dynamics of technical and business knowledge networks in industrial clusters: Embeddedness, status, or proximity? *Economic geography*, 92(1):35–60, 2016.
- [87] Ron Boschma, Asier Minondo, and Mikel Navarro. The Emergence of New Industries at the Regional Level in Spain: A Proximity Approach Based on Product Relatedness. *Economic Geography*, 89(1):29–51, January 2013.
- [88] Teresa Farinha, Pierre-Alexandre Balland, Andrea Morrison, and Ron Boschma. What drives the geography of jobs in the us? unpacking relatedness. *Industry and Innovation*, 26(9):988–1022, 2019. ISBN: 1366-2716 Publisher: Taylor & Francis.
- [89] Zhao Chen, Sandra Poncet, and Ruixiang Xiong. Inter-industry relatedness and industrial-policy efficiency: Evidence from China's export processing zones. *Journal of Comparative Economics*, 45(4):809–826, December 2017.
- [90] Benno Ferrarini and Pasquale Scaramozzino. The product space revisited: China's trade profile. *The World Economy*, 38(9):1368–1386, 2015. ISBN: 0378-5920 Publisher: Wiley Online Library.
- [91] Ahmad Alabdulkareem, Morgan R. Frank, Lijun Sun, Bedoor AlShebli, César Hidalgo, and Iyad Rahwan. Unpacking the polarization of workplace skills. *Science Advances*, 4(7):eaao6030, July 2018.

- [92] Fengmei Ma, Heming Wang, Asaf Tzachor, César A Hidalgo, Heinz Schandl, Yue Zhang, Jingling Zhang, Wei-Qiang Chen, Yanzhi Zhao, Yong-Guan Zhu, et al. The disparities and development trajectories of nations in achieving the sustainable development goals. *Nature Communications*, 16(1):1107, 2025.
- [93] Rachata Muneepeerakul, José Lobo, Shade T Shutters, Andrés Goméz-Liévano, and Murad R Qubbaj. Urban economies and occupation space: Can they get “there” from “here”? *PloS one*, 8(9):e73676, 2013.
- [94] Matthieu Cristelli, Andrea Gabrielli, Andrea Tacchella, Guido Caldarelli, and Luciano Pietronero. Measuring the intangibles: A metrics for the economic complexity of countries and products. *PloS one*, 8(8):e70726, 2013.
- [95] Carla Sciarra, Guido Chiarotti, Luca Ridolfi, and Francesco Laio. Reconciling contrasting views on economic complexity. *Nature communications*, 11(1):3352, 2020. ISBN: 2041-1723 Publisher: Nature Publishing Group UK London.
- [96] Bernardo Caldarola, Dario Mazzilli, Lorenzo Napolitano, Aurelio Patelli, and Angelica Sbardella. Economic complexity and the sustainability transition: A review of data, methods, and literature. *Journal of Physics: Complexity*, 2024.
- [97] Ricardo Hausmann, Jason Hwang, and Dani Rodrik. What you export matters. *Journal of Economic Growth*, 12(1):1–25, March 2007.
- [98] Dani Rodrik. What’s so special about China’s exports? *China & World Economy*, 14(5):1–19, 2006. ISBN: 1671-2234 Publisher: Wiley Online Library.
- [99] Alexander Hamilton. Report on manufactures. 1791. Publisher: Washington, DC: United States.
- [100] Paul N. Rosenstein-Rodan. Notes on the theory of the ‘big push’. In *Economic Development for Latin America*, pages 57–81. Springer, 1961.
- [101] Paul N. Rosenstein-Rodan. Problems of industrialisation of eastern and south-eastern Europe. *The economic journal*, 53(210/211):202–211, 1943. ISBN: 0013-0133 Publisher: JSTOR.
- [102] Walt Whitman Rostow. The stages of economic growth. *The economic history review*, 12(1):1–16, 1959. ISBN: 0013-0117 Publisher: JSTOR.
- [103] Albert O. Hirschman. A generalized linkage approach to development, with special reference to staples. *Economic development and cultural change*, 25:67, 1977. ISBN: 0013-0079 Publisher: University of Chicago Press.
- [104] Raul Prebisch. The economic development of Latin America and its principal problems. *Economic Bulletin for Latin America*, 1962.
- [105] Alexander Gerschenkron. The early phases of industrialization in Russia: afterthoughts and counterthoughts. In *The economics of take-off into sustained growth*, pages 151–169. Springer, 1963.
- [106] Bela Balassa. Exports, policy choices, and economic growth in developing countries after the 1973 oil shock. *Journal of Development Economics*, 18(1):23–35, May 1985.
- [107] César A. Hidalgo. The policy implications of economic complexity. *Research Policy*, 52(9):104863, 2023. ISBN: 0048-7333 Publisher: Elsevier.

- [108] Martin L. Weitzman. Recombinant Growth. *The Quarterly Journal of Economics*, 113(2):331–360, May 1998.
- [109] Stuart A. Kauffman. *The origins of order: Self-organization and selection in evolution*. Oxford University Press, USA, 1993.
- [110] Michael Kremer. The O-ring theory of economic development. *The Quarterly Journal of Economics*, 108(3):551–575, 1993. ISBN: 1531-4650 Publisher: MIT Press.
- [111] Francesca Tria, Vittorio Loreto, Vito Domenico Pietro Servedio, and Steven H. Strogatz. The dynamics of correlated novelties. *Scientific reports*, 4:5890, 2014. ISBN: 2045-2322 Publisher: Nature Publishing Group.
- [112] T. M. A. Fink, M. Reeves, R. Palma, and R. S. Farr. Serendipity and strategy in rapid innovation. *Nature Communications*, 8(1):2002, December 2017.
- [113] T. M. A. Fink and M. Reeves. How much can we influence the rate of innovation? *Science Advances*, 5(1):eaat6107, January 2019.
- [114] Anton Pichler, François Lafond, and J Doyne Farmer. Technological interdependencies predict innovation dynamics. *arXiv preprint arXiv:2003.00580*, 2020.
- [115] James McNerney, J Doyne Farmer, Sidney Redner, and Jessika E Trancik. Role of design complexity in technology improvement. *Proceedings of the National Academy of Sciences*, 108(22):9008–9013, 2011.
- [116] Lee Fleming. Recombinant uncertainty in technological search. *Management science*, 47(1):117–132, 2001.
- [117] Alje Van Dam and Koen Frenken. Variety, complexity and economic development. *Research Policy*, 51(8):103949, 2022.
- [118] Giovanni Dosi. Technological paradigms and technological trajectories: a suggested interpretation of the determinants and directions of technical change. *Research policy*, 11(3):147–162, 1982.
- [119] David J Teece, Gary Pisano, and Amy Shuen. Dynamic capabilities and strategic management. *Strategic management journal*, 18(7):509–533, 1997.
- [120] Sanjaya Lall. Technological capabilities and industrialization. *World development*, 20(2):165–186, 1992.
- [121] Richard R. Nelson and Sidney G. Winter. *An Evolutionary Theory of Economic Change*. Belknap Press of Harvard University Press, 1982. Google-Books-ID: uRm5AAAAIAAJ.
- [122] Jorge M Uribe. Investment in intangible assets and economic complexity. *Research Policy*, 54(1):105133, 2025.
- [123] William Shockley. On the statistics of individual variations of productivity in research laboratories. *Proceedings of the IRE*, 45(3):279–290, 1957.
- [124] Marc J Melitz and Stephen J Redding. Missing gains from trade? *American Economic Review*, 104(5):317–321, 2014.
- [125] Bela Balassa. Trade liberalisation and “revealed” comparative advantage 1. *The manchester school*, 33(2):99–123, 1965. ISBN: 1463-6786 Publisher: Wiley Online Library.

- [126] Sebastián Bustos, Charles Gomez, Ricardo Hausmann, and César A. Hidalgo. The Dynamics of Nestedness Predicts the Evolution of Industrial Ecosystems. *PLOS ONE*, 7(11):e49393, November 2012.
- [127] Manuel Sebastian Mariani, Zhuo-Ming Ren, Jordi Bascompte, and Claudio Juan Tessone. Nestedness in complex networks: Observation, emergence, and implications. *Physics Reports*, 2019. ISBN: 0370-1573 Publisher: Elsevier.
- [128] Mário Almeida-Neto, Paulo Guimaraes, Paulo R Guimaraes Jr, Rafael D Loyola, and Werner Ulrich. A consistent metric for nestedness analysis in ecological systems: reconciling concept and measurement. *Oikos*, 117(8):1227–1239, 2008.
- [129] Louis Knuepling and Tom Broekel. Does relatedness drive the diversification of countries' success in sports? *European Sport Management Quarterly*, 22(2):182–204, 2022.
- [130] Benjamin Klement and Simone Strambach. Innovation in creative industries: Does (related) variety matter for the creativity of urban music scenes? *Economic Geography*, 95(4):385–417, 2019.
- [131] Fabian Stephany and Ole Teutloff. What is the price of a skill? the value of complementarity. *Research Policy*, 53(1):104898, 2024.
- [132] Gloria Cicerone, Philip McCann, and Viktor A. Venhorst. Promoting regional growth and innovation: relatedness, revealed comparative advantage and the product space. *Journal of Economic Geography*, 20(1):293–316, 2020. ISBN: 1468-2702 Publisher: Oxford University Press.
- [133] Sándor Juhász, Tom Broekel, and Ron Boschma. Explaining the dynamics of relatedness: the role of co-location and complexity. *Papers in Regional Science*, 2020.
- [134] César A. Hidalgo, Elisa Castañer, and Andres Sevtsuk. The amenity mix of urban neighborhoods. *Habitat International*, page 102205, 2020.
- [135] Jonathan Borggren, Rikard H. Eriksson, and Urban Lindgren. Knowledge flows in high-impact firms: How does relatedness influence survival, acquisition and exit? *Journal of Economic Geography*, 16(3):637–665, 2016. ISBN: 1468-2710 Publisher: Oxford University Press.
- [136] Rachata Muneepakul, José Lobo, Shade T. Shutters, Andrés Goméz-Liévano, and Murad R. Qubbaj. Urban Economies and Occupation Space: Can They Get “There” from “Here”? *PloS one*, 8(9):e73676, 2013.
- [137] Paul Resnick and Hal R. Varian. Recommender systems, March 1997.
- [138] Aamena Alshamsi, Flávio L. Pinheiro, and Cesar A. Hidalgo. Optimal diversification strategies in the networks of related products and of related research areas. *Nature Communications*, 9(1):1328, April 2018.
- [139] Marcin Waniek, Khaled Elbassioni, Flávio L. Pinheiro, César A. Hidalgo, and Aamena Alshamsi. Computational aspects of optimal strategic network diffusion. *Theoretical Computer Science*, 2020. ISBN: 0304-3975 Publisher: Elsevier.
- [140] Viktor Stojkoski and César A Hidalgo. Optimizing economic complexity. *arXiv preprint arXiv:2503.04476*, 2025.
- [141] Robert Hamwey, Henrique Pacini, and Lucas Assunção. Mapping green product spaces of nations. *The Journal of Environment & Development*, 22(2):155–168, 2013.

- [142] Nicola Daniele Coniglio, Raffaele Lagravinese, and Davide Vurchio. Production sophistication and growth: evidence from Italian provinces before and during the crisis, 1997–2013. *Cambridge Journal of Regions, Economy and Society*, 9(2):423–442, July 2016. Publisher: Oxford Academic.
- [143] Sandro Montresor and Francesco Quatraro. Green technologies and Smart Specialisation Strategies: a European patent-based analysis of the intertwining of technological relatedness and key enabling technologies. *Regional Studies*, pages 1–12, 2019.
- [144] Penny Mealy and Alexander Teytelboym. Economic complexity and the green economy. *Available at SSRN 3111644*, 2017.
- [145] Mark Huberty and Georg Zachmann. Green exports and the global product space: prospects for EU industrial policy. Technical report, Bruegel working paper, 2011.
- [146] Luca Fraccascia, Ilaria Giannoccaro, and Vito Albino. Green product development: What does the country product space imply? *Journal of cleaner production*, 170:1076–1088, 2018.
- [147] François Perruchas, Davide Consoli, and Nicolò Barbieri. Specialisation, diversification and the ladder of green technology development. *Research Policy*, 49(3):103922, 2020.
- [148] Artur Santoalha and Ron Boschma. Diversifying in green technologies in European regions: does political support matter? *Regional Studies*, pages 1–14, 2020. ISBN: 0034-3404 Publisher: Routledge.
- [149] Ron Boschma and Gianluca Capone. Institutions and diversification: Related versus unrelated diversification in a varieties of capitalism framework. *Research Policy*, 44(10):1902–1914, December 2015.
- [150] Shengjun Zhu, Canfei He, and Yi Zhou. How to jump further and catch up? Path-breaking in an uneven industry space. *Journal of Economic Geography*, 17(3):521–545, May 2017.
- [151] Yongyuan Huang and Shengjun Zhu. Regional industrial dynamics under the environmental pressures in China. *Journal of Cleaner Production*, page 121917, 2020. ISBN: 0959-6526 Publisher: Elsevier.
- [152] Nicola Cortinovis, Jing Xiao, Ron Boschma, and Frank G. van Oort. Quality of government and social capital as drivers of regional diversification in Europe. *Journal of Economic Geography*, 17(6):1179–1208, 2017. ISBN: 1468-2702 Publisher: Oxford University Press.
- [153] Giorgio Gnecco, Federico Nutarelli, and Massimo Riccaboni. A machine learning approach to economic complexity based on matrix completion. *Scientific Reports*, 12(1):9639, 2022. ISBN: 2045-2322 Publisher: Nature Publishing Group UK London.
- [154] Abdulrahman M. AlQurtas. A New Indicator of Economic Complexity to Guide Industrial Policies, 2018.
- [155] Inga Ivanova, Øivind Strand, Duncan Kushnir, and Loet Leydesdorff. Economic and technological complexity: A model study of indicators of knowledge-based innovation systems. *Technological Forecasting and Social Change*, 120:77–89, 2017.
- [156] Inga Ivanova, Nataliya Smorodinskaya, and Loet Leydesdorff. On measuring complexity in a post-industrial economy: The ecosystem’s approach. *Quality & Quantity*, 54(1):197–212, 2020. ISBN: 1573-7845 Publisher: Springer.

- [157] Boaz Nadler, Stephane Lafon, Ioannis Kevrekidis, and Ronald Coifman. Diffusion maps, spectral clustering and eigenfunctions of fokker-planck operators. *Advances in neural information processing systems*, 18, 2005.