

# How to Refer to Abstract Objects

By

James Davies

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Graduate Department of Philosophy

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Consider the statement “ $7 + 5 = 12$ ”. It appears to be true. Moreover, it appears to entail that something is identical to  $7 + 5$ . But this thing that is identical to  $7 + 5$  does not appear to be located in space or time – where and when would we find it? Nor does it appear to enter into any causal interactions – what causes can influence it, and what events has it produced?

*Mathematical platonism* is the view that all these intuitive appearances are correct: many of our mathematical statements are true statements about non-spatiotemporal and non-causal objects, or as philosophers say, *abstract* objects. Mathematical platonism stands in direct tension with many contemporary theories about how our thoughts and utterances can be about mind-independent objects, as most of those theories explain how we can think and speak about such objects in causal terms. This observation gives rise to the following argument, which I call the *semantic argument* against mathematical platonism. For our any one of our expressions to refer to an object  $o$ , it must be possible for  $o$  to stand in at least one causal relation. It is impossible for abstract mathematical objects to stand in causal relations. Therefore reference to abstract mathematical objects is impossible. But if reference to abstract mathematical objects is impossible, then hardly any of our mathematical statements are true – at least when read at *face value*. This substantially undermines mathematical platonism.

This dissertation defends mathematical platonism against the semantic argument. It has four chapters and a brief conclusion. I ultimately describe a package of views that may allow the

platonist to resist the semantic argument. In particular this package of views allows the platonist to claim that not only can we refer to abstract mathematical objects, but we can refer *singularly* to them. Hence not only are apparent mathematical truths like “ $7 + 5 = 12$ ” indeed true, but they say what they appear to say.

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I thank my mother for always believing that I can do anything I set my mind to, and that whatever I set my mind to must be worth doing. I hope this dissertation proves her right on both counts. I lovingly dedicate it to her.

*Contents*

Chapter One: Introduction	1
Chapter Two: Mathematical Fictionalism and the Semantic Argument	48
Chapter Three: Semantic Indeterminacy and Abstract Objects	84
Chapter Four: Singular Reference and Abstract Objects	114
Conclusion	164
Bibliography	171

## *Chapter One*

### **Introduction**

#### *Abstract*

In this introduction I describe the main subject of this dissertation. This is the *semantic argument against mathematical platonism*. Its conclusion is that we cannot refer to abstract objects. I begin with a brief preamble roughly describing the semantic argument and why it is important. Next I give a more detailed description of the semantic argument, in particular the notion of *reference* operating therein. After describing the logical geography formed by the various possible responses to the semantic argument I distinguish the semantic argument from its better-known sibling, the *epistemological* argument against mathematical platonism. Lastly I outline two framework observations about the terms of the debate and give a more detailed outline of the subsequent three chapters.

## Contents

### 1: Preamble

### 2: The Semantic Argument Against Mathematical Platonism

### 3: Logical Geography: Responses to the Semantic Argument

### 4: Comparison with the Epistemological Argument Against Mathematical Platonism

### 5: Framework Observations

### 6: Outline of the Coming Chapters

#### *1: Preamble*

In this dissertation I will defend a thesis about abstract mathematical objects: *if* abstract mathematical objects were to exist, *then* we would be able to refer to them. An abstract mathematical object is a mathematical object incapable of standing in any causal or spatiotemporal relations. Hence natural and real numbers appear to be abstract, tables and chairs are not, but I take no stand on whether impure sets like the set of my table and my chair, or geometrical objects like shapes, are abstract. I will defend this thesis about abstract mathematical objects against an argument I call the *semantic argument against mathematical platonism*, or “semantic argument” for short. Here is a preliminary sketch of the semantic argument (I’ll give a more detailed exposition shortly):

P1: For an expression  $e$  to refer to an object  $o$ ,  $o$  must stand in a causal relation to a use of

*e*.

P2: It is impossible for any abstract object to stand in any causal relations.

But,

P3. The natural numbers and real numbers are abstract (if they exist).

Therefore:

C: It is impossible for any expression to refer to any abstract mathematical object (like a natural or real number).

Let us say that an expression *e* *refers to* an object *o* if and only if the job of *e* is to introduce a single object as relevant to the truth or falsity of statements in which *e* occurs, such that  $\lceil e \text{ if } F \rceil$  is true if and only if *o* is *F*. In such cases we say that *e* is a *singular term*. (I say something further about this terminological decision very soon.) The proper name “James” is a singular term, so if “James” refers to me, then “James is English” is true if and only if I am English. (In order to simplify the coming discussions I will speak of expressions themselves referring, rather than speakers using expressions to refer. But because the features of reference that turn on maintaining this distinction – such as context sensitivity, opacity, and the like – will not affect any of the points I will make in this dissertation, this simplification should be harmless enough.)

In this dissertation I take *mathematical platonism*<sup>1</sup> to be the conjunction of the following semantic, alethic, epistemological, and metaphysical theses. Let A *pure* mathematical statement be a statement containing mathematical expressions only, e.g. “ $7 + 5 = 12$ ” and “ $e^{i\pi} = -1$ ”. (Pure mathematical *truths* are true pure mathematical statements.) *Impure* mathematical statements thus

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<sup>1</sup> Throughout this dissertation I follow the convention of using “platonism” with a lower-case “p” to indicate that nothing I attribute to the platonist is to taken as being attributed to the historical Plato.



contain both mathematical and non-mathematical expressions, e.g. “the number of planets is 8” and “1 is the loneliest number”. A *pure mathematical singular term* is a singular term appearing in a pure mathematical statement. Then the semantic component thesis of mathematical platonism is:

(S): Pure mathematical statements should be read at *face-value*: the surface syntactical form of pure mathematical statements like “ $7 + 5 = 12$ ” is a good guide to their *logical form*.

By *logical form* I mean the structural features of a statement relevant to the entailment relations that statement stands in. This notion of logical form can be found in Davidson:

[T]o give the logical form of a sentence is to give its logical location in the totality of sentences, to describe it in a way that explicitly determines what sentences it entails and what sentences it is entailed by. (Davidson 1967/2006, p.64.)

Hence if “ $7 > 5$ ” entails “ $\exists x(x > 5)$ ”, but “he left us in the lurch” does not entail “ $\exists x(\text{he left us in } x)$ ”, this is because “ $7 > 5$ ” and “he left us in the lurch” have different logical forms, even though they both having the same surface syntactical form, namely  $\lceil \Phi(\alpha, \beta) \rceil$  where  $\Phi$  ranges over binary predicates and  $\alpha$  and  $\beta$  range over singular terms. Thus claim (S) – that we should read pure mathematical statements at face-value – entails that the logical form of “ $7 + 5 = 12$ ” is  $\lceil \S(\alpha, \beta) = \gamma \rceil$  where  $\alpha$ ,  $\beta$  and  $\gamma$  range over singular terms. So a face-value reading of “ $7 + 5 = 12$ ” takes “7”, “5” and “12” to be singular terms. Seeing as “ $7 + 5 = 12$ ” is a pure mathematical statement, “7”, “5” and “12” are pure mathematical singular terms. Returning now to characterizing mathematical platonism, we have the alethic thesis:

(A): Some pure mathematical statements (that are not negative existentials) are (non-vacuously) true.

The parenthetical remarks appear in (A) because there are some pure mathematical statements that would be true even if there were no mathematical objects. These statements fall into two categories: negative existentials, like “there is no even prime”, and material conditionals, like “every prime number is greater than zero”. The truth of “there is no even prime” obviously does not require the existence of any mathematical objects. As for “every prime number is greater than zero”, note that if we write that statement slightly more formally as “ $\forall x[Px \rightarrow (x > 0)]$ ”, we see that because it is a universally quantified material conditional it is *vacuously* true if there are no prime numbers, or indeed, no mathematical objects at all. Thesis (A) is intended as a *stronger* claim than that some pure mathematical statements are vacuously true, or that there are true pure mathematical negative existentials. The truth of (A) is intended to require the truth of claims like “there is an even prime number” which definitely *do* require the existence of mathematical objects (when read at face-value). Throughout this dissertation when I speak of ‘mathematical statements’ in the general, I mean mathematical statements that are neither negative existentials nor vacuously true material conditionals.

Note also that the conjunction of (S) and (A) entails that at least some mathematical singular terms like “7” and “the even prime” successfully refer. The metaphysical component thesis of platonism is a thesis about the natures of the referents of “7”, “the even prime” and the like:

(M): The referent of a pure mathematical singular term, if it has one, is an abstract object, incapable of standing in spatiotemporal or causal relations.

Note that (M), (A) and (S) together entail that abstract mathematical objects exist. Now, thesis (M) might look too obvious to require being stated, but some philosophers have denied it and claimed that the referents of “7”, “the even prime” and the like can stand in spatiotemporal

relations and/or have causal powers. In section 3 I give a detailed investigation of how plausible it is to deny (M). Lastly we have the epistemological platonist thesis:

(E): We *know* (or are at least capable of knowing) statements containing expressions referring to and/or quantifying over abstract mathematical objects which, if true, are not vacuously true.

Note that because knowledge is factive – if  $P$  is known, then  $P$  is true – the denial of thesis (A) entails the denial of (E).

The semantic argument spells trouble for mathematical platonism because if we cannot refer to abstract mathematical objects, then (S), (A) and (M) are not all true. If the job of “7”, “5” and “12” is to introduce objects relevant to the truth or falsity of “ $7 + 5 = 12$ ”, then the truth of “ $7 + 5 = 12$ ” requires that “7”, “5” and “12” have referents. If (M) holds then those referents can only be abstract objects. But if reference to abstract objects is not possible, then “7”, “5” and “12” cannot refer to abstract objects. Thus “ $7 + 5 = 12$ ” cannot be true. This is the denial of component thesis (A) (and the denial of (A) entails the denial of (E)). Hence if reference to abstract objects is not possible then mathematical platonism is not true. My goal in this dissertation is to describe a package of views that allows the platonist to resist the semantic argument.

This dissertation has four chapters. In this first introductory chapter I lay the necessary background and state what I take to be the terms of the debate. In the second chapter I investigate the status of the semantic argument with respect to mathematical *fictionalism*. Fictionalism combines component thesis (S) of platonism with the denial of (A): pure mathematical statements are to be read at face-value, but they are not true. Fictionalism is thus consistent with *nominalism*, the view that there are no abstract objects. The semantic argument is often touted by fictionalists

as a reason to prefer fictionalism over platonism. But it turns out, surprisingly, that if the fictionalist uses the semantic argument in this way then they are committed to taking stands on issues that are at first glance far removed from the concerns that motivate fictionalism. In the third chapter I articulate one particular argument for P1 of the semantic argument. I explore the possibility of using a *reference-magnet* style view to deny P1. In the fourth chapter I consider an argument for a more restricted version of P1 of the semantic argument, to the effect that *singular* reference to abstract mathematical objects is impossible. After showing that this restricted version of P1 nevertheless makes trouble for platonism, I show that one non-causal theory of singular thought – a hybrid view combining *cognitivism* about non-perceptual singular thought with (something like) acquaintance theory about perceptual demonstrative singular thought – both allows for singular reference to abstract mathematical objects and avoids some of the pitfalls associated with other non-acquaintance-theoretic views of singular reference.

Here is how the rest of this introduction goes. In the next section I explain the semantic argument in greater detail. This involves two main tasks. The first is explaining in more detail the notion of *reference* that I am working with. After that I describe the relationships between the semantic argument and two of the best-known arguments for mathematical platonism: the *Fregean* argument and the *indispensability* argument. This leads to a taxonomy of the possible responses to the semantic argument in terms of whether one rejects P1 or P2 of the semantic argument or component thesis (S), (A) or (M) of mathematical platonism.

In the third section of this introduction I compare the semantic argument to its better-known sibling, the *epistemological* argument against mathematical platonism (“epistemological argument” for short). This argument attacks mathematical platonism by arguing that thesis (M)

entails the denial of (E), rather than (S). I explain several extant responses to the epistemological argument. I then argue that for all these responses say, even if they go through against the epistemological argument, the semantic argument still stands. But because knowledge is factive – if  $P$  is known then  $P$  is true, whence if  $P$  is not true  $P$  cannot be known – the denial of the alethic thesis (A) entails the denial of the epistemic thesis (E). Thus no response to the epistemological argument can succeed if the semantic argument is allowed to stand. Thus the semantic argument is a pressing concern for the mathematical platonist.

In the fourth section I will make two important claims about the framework in which the semantic argument operates. The first such framework claim is that in the course of defending their view against the semantic argument, the mathematical platonist is entitled to assume that abstract mathematical objects exist. The second framework claim is that if a specific argument for P1 of the semantic argument can be shown to generate the result that we cannot even refer to ordinary concrete objects, then the mathematical platonist can take this as constituting a *reductio* of that argument for P1 of the semantic argument.

## *2: The Semantic Argument Against Mathematical Platonism*

### *2.1: The Argument*

The core idea behind the semantic argument is that because abstract mathematical objects cannot stand in causal or spatiotemporal relations, they pose special problems for the theory of reference. While this kind of argument against platonism is not as commonly made as the parallel argument that the abstractness of mathematical objects poses special problems for the theory of *knowledge* (see section 4 of this chapter), the semantic argument can be found in one form or another in the

writings of several anti-platonists.<sup>2</sup> Here is a representative instance:

Probably the main ground for suspicion about mathematical entities is the difficulty that these entities raise for the theory of knowledge and for the theory of reference or theory of belief content. According to the platonist picture [...] There are no causal connections between the entities in the platonic realm and ourselves; how then can we have any knowledge of what is going on in that realm? And perhaps more fundamentally, what could make a particular word like ‘two’, or a particular belief state of our brains, *stand for* or *be about* a particular one of the absolute infinity of objects in that realm? (Field 1989, p.67/8; emphasis original.)

However the sentiment expressed in this passage – bewilderment at the notion that an expression (or belief state) could ‘stand for or be about’ an abstract mathematical object – needs to be made more definite before a committed platonist can mount a response. The purpose of this section is turn such sentiments into a definite argument.

We begin with the reference relation. In the preamble I said that an expression  $e$  refers to  $o$  if and only if the job of  $e$  is to introduce  $o$  as relevant to the truth or falsity of statements in which  $e$  occurs. (Recall also that for simplicity’s sake I will mostly speak of expression types referring to objects, as opposed to agents or their token uses referring.) I also said that I will call an expression that has this job a *singular term*. One important thing to note here is that this counts *definite descriptions* like “the largest star” and “the even prime” as singular terms. Now, there is a fairly old philosophical debate over whether definite descriptions should be counted as singular terms or not. (The debate goes back at least as far as Frege and Russell; Frege held that definite descriptions are singular terms, Russell argued that they are quantifiers.<sup>3</sup>) However my use of the

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<sup>2</sup> Here is a non-exhaustive list of places the semantic argument can be found: Michael Jubien (1977) ‘Ontology and Mathematical Truth’, Jonathan Lear (1977) ‘Sets and Semantics’, Hodes (1984) ‘Logicism and the Ontological Commitments of Arithmetic’. The semantic argument has its roots in the epistemological argument of Benacerraf (1973) ‘Mathematical Truth’.

<sup>3</sup> Frege (1892/1952) ‘On Sense and Reference’; Russell (1905) ‘On Denoting’.

expression “singular term” to cover definite descriptions is not intended to signal that I take a stand on this issue. Rather, I need a convenient term for the category of expressions whose job is to introduce an object as relevant to the truth or falsity of statements in which they occur. And, importantly, I need this term to cover expressions that *try*, but *fail* at this job – which is what the likes of “7” and “the even prime” do if both (S) and the semantic argument are correct. “Singular term” fits this bill, so I will use it as such, on the understanding that I’m not intending any specific stand on the philosophical question mentioned above.

The core idea behind the semantic argument against mathematical platonism is that the causal isolation of abstract mathematical objects prevents our expressions from referring to them. This is an instance of a more general claim, the *causal constraint on reference*. Here is a first pass at articulating the causal constraint on reference (we’ll improve upon this first pass very soon):

For an expression  $e$  to refer to an object  $o$ , there must be at least one causal relation between  $o$  and (at least one use of)  $e$ .

The idea that causal relations can play a role in determining reference has been popular since the advent of the “new” theory of reference in the late 1960’s and early 1970’s. The causal constraint on reference results from generalizing the idea that causal relations *can* play a role in determining reference to the much stronger thesis that they *must* play such a role, and that in the absence of causal relations, reference cannot be determined.

Now, my characterization of singular terms counts definite descriptions like “the largest star” and “the even prime” as singular terms. But the causal constraint on reference, as I’ve just stated it, is not very plausible as claim about definite descriptions. The claim that for a definite description  $d$  to refer to an object  $o$  there must be a causal relation between  $o$  and (uses of)  $d$  is

subject to obvious counterexamples. General Relativity Theory entails that if the largest star is outside of our light-cones, then there are no causal relations between ourselves and that star. But it is very implausible that this lack of causal contact should result in my uses of the definite description “the largest star” failing to refer to that star.

We can avoid such counterexamples by relaxing the causal constraint on reference as follows. We say that for *e* to refer to *o* requires that it be metaphysically *possible* for *o* to stand in causal relations. (Nothing I will say depends on specifics about what metaphysical possibility is, so I will not attempt to precisify that notion here.) Note that while this version of the causal constraint on reference is no starting point for building any specific causal theory of reference, it is nevertheless a *consequence* of any theory that takes reference to require causal contact. Weakening the causal constraint on reference in this way avoids the problem with “the largest star” because the largest star does stand in causal relations; moreover, it is metaphysically possible for the largest star to stand in causal relations *to us*.

Nevertheless so long as abstract mathematical objects are abstract by metaphysical necessity, even this extremely weak causal constraint on reference will rule out reference to abstract mathematical objects. Moreover the claim that the abstractness of abstract mathematical objects is a matter of metaphysical necessity appears to be accepted by the majority of mathematical platonists. Even those who claim that there are *contingently* abstract objects (perhaps my possible twin sister is abstract in the actual world but concrete in some other possible worlds) tend to hold that actually abstract *mathematical* objects are necessarily abstract.<sup>4</sup> And so we have our second pass at the causal constraint on reference – hereon abbreviated “CCR” – in

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<sup>4</sup> “The familiar abstract particulars, such as numbers and sets, have the property of being abstract in every possible world.” (Linsky & Zalta 1994 ‘In Defense of the Simplest Quantified Modal Logic’, p.26.)



terms of possible causal relations:

(CCR): in order for an expression  $e$  to refer to an object  $o$ , it must be (metaphysically) possible for  $o$  to stand in at least one causal relation.

Let me emphasize again that while the CCR is no starting point on which to build a positive theory of reference, it is a *consequence* of any causal theory of reference. We can now run the semantic argument against mathematical platonism in terms of the CCR:

P1: In order for an (human) expression  $e$  to refer to an object  $o$ , it must be (metaphysically) possible for  $o$  to stand in at least one causal relation (to at least one use of  $e$ ). (This is the CCR.)

P2: It is not metaphysically possible for abstract mathematical objects (i.e. the referents, if any, of pure mathematical singular terms) to stand in causal relations. (This is a consequence of component thesis (M) of mathematical platonism.)

Therefore,

C: It is not metaphysically possible for any expression to refer to any abstract mathematical object.

Recall that mathematical platonism is the conjunction of (M), (S), (A) and (E). If the above conclusion (C) holds, then we cannot have all of (M), (S), and (A). (And if we can't have (A), then we cannot have (E) either.) Thus if the CCR holds then mathematical platonism is not true.

The semantic argument is valid as stated. Hence rejecting its conclusion requires rejecting at least one of its premises. Later in this introduction I sketch some reasons against rejecting P2, and in chapter 3 I give a more principled argument for P2. Chapter 3 also contains an argument

for P1 (i.e. the CCR), and in chapter 4 I give an argument for a restricted version of the CCR, the causal constraint on *singular* reference. My ultimate goal in this dissertation is to describe a combination of views allowing the platonist to reject P1 (the CCR) but keep P2.

Before mapping out various possible kinds of response to the semantic argument, let's first look at the status of the semantic argument with respect to two of the best-known arguments *for* mathematical platonism: the Fregean argument and the indispensability argument.

## 2.2: The Fregean Argument

The Fregean argument for mathematical platonism – so-called because it appears in Frege's *Foundations of Arithmetic* (cf. Hale (1987), p.11-14) – goes like this:

P1: “ $7 + 5 = 12$ ” is true.

P2: “7” is a singular term.

P3: If “ $t$ ” is a singular term and  $\lceil \Phi[x/t] \rceil^5$  is true, then there is an object that is the referent of “ $t$ ”.

Therefore,

C: There is an object that is the referent of “7”.

The supposed abstractness of this referent of “7” is then derived by noting the apparent absurdity of assigning mathematical objects spatiotemporal locations or causal powers.<sup>6</sup> (I will very soon

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<sup>5</sup>  $\lceil \Phi[x/t] \rceil$  is the result of taking an open sentence  $\Phi$  with one free variable  $x$  and substituting the term  $t$  into every place in  $\Phi$  occupied by  $x$ .

<sup>6</sup> E.g. Hale (1997/2001), p.169: “Since it appears to make no sense to ask where numbers or sets are located, or when they came into existence and how long they will last, the platonist concludes that they are abstract objects, lying outside space and time.”

give some reasons for supposing that mathematical objects, natural numbers in particular, should be regarded as abstract in the sense of lacking spatiotemporal locations or causal powers.)

It is not hard to see that the Fregean argument begs the question against the semantic argument. For P1 and P2 of the Fregean argument are the platonist's alethic thesis (A) and semantic thesis (S). The semantic argument says exactly that we cannot have both (A) and (S); at least, not so long as the platonist's metaphysical thesis (M) and the CCR are both in play. Thus the only way to maintain both the semantic argument and the Fregean argument is to reject component thesis (M) of mathematical platonism and hold that the referent of "7" is (possibly) a causally related object.

Before moving on to the indispensability argument I will briefly note that there is another way of resisting the Fregean argument, by denying P3. To do this is to move from *classical* logic to *free* logic which allows empty singular terms to feature in true atomic identity statements (like " $a = b$ "). I will not say anything about this move except to note its existence and to refer the reader to Sainsbury (2005) and MacFarlane (2009)<sup>7</sup> for discussion. Throughout this dissertation I will assume that the presuppositions of classical logic, such as that " $a = b$ " cannot be true unless both " $a$ " and " $b$ " refer, are not up for grabs. Let us now turn to the indispensability argument.

### 2.3: *The Indispensability Argument*

The indispensability argument for mathematical platonism runs as follows:

P1: We are committed to the truth of our best scientific theories.

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<sup>7</sup> Sainsbury (2005) *Reference Without Referents* Clarendon Press: Oxford; MacFarlane (2009) 'Double Vision: Two Questions about the Neo-Fregean Program'.

P2: Some pure mathematical theories are indispensable parts of our best scientific theories.

Therefore,

C: We are committed to the truth of some of our pure mathematical theories.

Thus we have component (A) of platonism. Now, if we're committed to the truth of a mathematical theory, then we are committed to the existence of the entities postulated by that theory. So given (S), we are committed to mathematical objects. The abstractness of these objects follows from the usual considerations – it is absurd to take mathematical objects as being spatiotemporally located or as having causal powers – thus we have the metaphysical component (M) of platonism.

(Note as an interesting historical aside that the indispensability argument can be seen to have roots in a claim of Frege's: "it is applicability alone which elevates arithmetic from a game to the rank of a science."<sup>8</sup> Nevertheless it was first made explicit in some writings of Quine and Putnam.)

The situation here is slightly more complicated than with the Fregean argument. For not only is it possible to maintain both the semantic argument and the indispensability argument by rejecting (M); one could reject (S) as well, seeing as (S) is not part of the indispensability argument. We are coming close to requiring a taxonomy of responses to the semantic argument against platonism; hence let us now provide one.

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<sup>8</sup> Frege (1952), p.187.

### *3: Logical Geography: Responses to the Semantic Argument*

The semantic argument says that if the CCR holds – if reference is constrained by metaphysically possible causal contact – then the component theses of mathematical platonism (S), (A) and (M) are not all true. (We can leave out mentioning the epistemological thesis (E) because if (S), (A) and (M) are not all true, then (S), (A), (M) and (E) are not all true. So we need only speak of the former.) The semantic argument is valid. So it looks like our responses are limited to rejecting either one of the argument's premises, or one of the component theses of platonism. Hence the various responses to the semantic argument can be categorized in terms of what exactly they reject. Here are the categories:

- (i) If CCR & (A) & (M), then not-(S). Pure mathematical statements should not be read at face-value.
- (ii) If CCR & (M) & (S), then not-(A). Pure mathematical statements are not true.
- (iii) If CCR & (S) & (A), then not-(M). Pure mathematical objects – the referents of “7”, “the even prime” and the like – are not abstract.
- (iv) If (M) & (S) & (A), then not-CCR. Reference does not require causal contact.

In the following sub-sections I discuss each of (i)-(iv) in turn.

#### *3.1: Reconstrual Responses*

Option (i) retains components (A) and (M) of platonism, and the CCR, at the cost of rejecting (S): pure mathematical statements like “ $7 + 5 = 12$ ” should not be read at face-value. For by (A), “ $7 + 5 = 12$ ” is true, and by (M), the only possible referents of mathematical singular terms are abstract; but the CCR entails that no expression can refer to an abstract object. So there had better not *be*

any mathematical singular terms; we had better not read “ $7 + 5 = 12$ ” as containing any singular terms. But the surface syntax of “ $7 + 5 = 12$ ” certainly *suggests* that “ $7 + 5 = 12$ ” contains singular terms. Thus rejecting (S) requires admitting that the surface syntactical form of “ $7 + 5 = 12$ ” and the like is *not* a good guide to the *logical form* of those statements.<sup>9</sup> One example of a view according to which the surface syntactical form of “ $7 + 5 = 12$ ” is not a good guide to its logical form is the positive view described in Hodes (1984).<sup>10</sup> According to Hodes’ view, the logical form of “ $7 + 5 = 12$ ” is not  $\ulcorner \S(\alpha, \beta) = \gamma \urcorner$  (with  $\S$  ranging over binary functional expressions and  $\alpha, \beta$  and  $\gamma$  over singular terms), but:

$$(H): \forall X \forall Y [((\exists_5 x)Xx \ \& \ (\exists_7 x)Yx \ \& \ \sim(\exists x)(Xx \ \& \ Yx)) \rightarrow (\exists_{12} x)(Xx \vee Yx)].^{11}$$

(H) basically says that if you have five things, and seven other things, then you have twelve things in total. (Here “ $\exists_5 x$ ”, “ $\exists_7 x$ ”, and “ $\exists_{12} x$ ” are *numerical quantifiers*, defined thus:  $\exists_n x(Fx) \Leftrightarrow_{def} \exists x_1, \dots, \exists x_n [Fx_1 \ \& \ \dots \ \& \ Fx_n \ \& \ x_1 \neq \dots \neq x_n \ \& \ \forall y (Gy \rightarrow y = x_1 \vee \dots \vee y = x_n)]$ .) Note that if the logical form of “ $7 + 5 = 12$ ” is as described by (H), then “ $7 + 5 = 12$ ” does not entail “ $\exists x(x + 5 = 12)$ ”. Hence the truth of “ $7 + 5 = 12$ ” does not mean that there is some *thing* that is the number 7. Thus Hodes does not accept component thesis (S) of mathematical platonism, and he is able to accept both the CCR and (A): reference to abstract mathematical objects is not possible, but pure mathematical statements – or at least, pure *arithmetical* statements – are nevertheless true. (Note that because Hodes accepts (A) he has no problems with the indispensability argument.) Call any response to the semantic argument that like Hodes’ view denies (S) a *reconstruction* response – because it *reconstructs* the logical forms of mathematical statements as something other than what

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<sup>9</sup> Recall that by ‘logical form’ I mean the Davidsonian notion: to give the logical form of a statement is to specify its inferential properties.

<sup>10</sup> ‘Logicism and the Ontological Commitments of Arithmetic’ (1984).

<sup>11</sup> Ibid., p.140.

their surface syntax suggests.<sup>12</sup> Suffice it to say, the project of finding workable reconstructions of pure mathematical statements is not trivial, and becomes exceedingly difficult once we move beyond elementary arithmetic.<sup>13</sup> At any rate, reconstruction responses are not the subject of this dissertation. Throughout this dissertation I will consider it a necessary condition on the success of a purported defence of mathematical *platonism* against the semantic argument that it preserve claim (S).

### 3.2: *Error-Theoretic Responses*

The second broad category of response to the semantic argument retains (M), (S) and the CCR at the cost of denying (A): pure mathematical statements are not literally true (unless they are negative existentials or vacuously true material conditionals). Such responses are *error-theoretic* responses, because they need to supply an explanation of why pure mathematical statements are so useful to our scientific and everyday practices – perhaps even indispensably so – if they are not true. One of the best-known mathematical error theories, Hartry Field’s mathematical fictionalism, is the main topic of the next chapter. Thus I defer further discussion of mathematical error-theories until chapter 2.

### 3.3: *Metaphysical Responses*

The third broad category of response to the semantic argument retains (S) and (A) and the CCR,

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<sup>12</sup> Another variety of reconstruction response, with certain affinities to Hodes’, is Hofweber’s view in ‘Number Determiners, Numbers, and Arithmetic’ (2005). According Hofweber phrases like “twelve trees” are determiner phrases like “most trees”, rather than shorthand for statements containing singular terms for numbers.

<sup>13</sup> “It is worth noting that outside the arithmetic of natural numbers, it is much harder to find non-face value interpretations with any plausibility at all.” (Field 1993, p.288, f/n 5.)

at the cost of rejecting (M) (i.e. rejecting P2 of the semantic argument): mathematical objects – the referents of pure mathematical singular terms – are not abstract, and thus meet the constraint on reference imposed by the CCR. In particular, it *is* metaphysically possible for the referents of expressions like “7” and “the even prime” to stand in causal relations. Such lines of thought have been pursued in the past. One such line was pursued by Penelope Maddy in a series of papers in the 1980’s (1980, 1981, 1983) and a book in 1990.<sup>14</sup> In short, Maddy held that natural numbers are properties of sets, which themselves may be spatiotemporally located and causally efficacious. Thus numbers are no less causal than properties of ordinary concrete objects like colours and masses. Another, related line of reasoning can be found in work by David Armstrong and his followers.<sup>15</sup> According to this line, numbers are immanent or ‘Aristotelian’ universals that are wholly present in their instances.

I defer my strongest defense of (M) until chapter 3, where I argue that the only “genuine” properties<sup>16</sup> that it is (metaphysically) possible for the natural numbers to have are arithmetical properties, i.e. the relational properties numbers bear to each other, like being less than 17. So because numbers cannot have the kinds of properties an object needs to have to stand in causal relations, numbers cannot be causally related. Nevertheless I will now briefly sketch two proposals for denying (M) in the particular case of natural numbers – the first holding that natural numbers are spatiotemporally located, the second that they are causally related – and show that both proposals have implausible consequences. My hope is that this will give pause to the idea of

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<sup>14</sup> ‘Perception and Mathematical Intuition’, ‘Sets and Numbers’, ‘Proper Classes’ and *Realism in Mathematics* respectively.

<sup>15</sup> Armstrong (1989) *A Combinatorial Theory of Possibility*, chapter 9, and Armstrong & Forrest (1987) ‘The Nature of Number’ *Philosophical Papers* 16 (3): 165-186.

<sup>16</sup> “Genuine” properties like the property of being human contrast with “mere-Cambridge” properties like the property of being such that Obama is president, the gaining or losing of which does not result in a ‘real change’ in the object. It is notoriously difficult to make this distinction precise, and I don’t have the space to try to do so here. Thus we will content ourselves with an intuitive grasp of the distinction.



escaping the semantic argument by denying (M) (i.e. P2), at least until chapter 3. Given that it is relatively intuitive (though perhaps not mandatory) to take being spatiotemporally located as a necessary condition on being causally related,<sup>17</sup> I begin with spatiotemporal location.

One fairly intuitive way of taking an entity *o* to be spatiotemporally located is as having a (possibly improper) part that occupies a spatiotemporal region.<sup>18</sup> (This way of doing things may not return a satisfying verdict for every kind of entity. Frege held that the equator is abstract (to put it a little anachronistically).<sup>19</sup> The equator is spatiotemporal in the sense that it has a definite *location*, but it is not completely intuitive to regard it as *occupying* a region. But let us press on anyway.) Ordinary concrete objects like boats and birds are thus spatiotemporal, as are fusions of such things. We can turn Maddy's claim that numbers are properties of sets into a view according to which numbers are spatiotemporally located in two steps as follows. The first step is to take sets as co-located with the fusions of their (located) members. For example, if Big Ben occupies a spatiotemporal region  $l_i$ , and The Eiffel Tower occupies a region  $l_j$ , then the fusion of Big Ben and the Eiffel Tower occupies the fusion of  $l_i$  and  $l_j$ . Hence the set {Big Ben, The Eiffel Tower} is co-located with the fusion of Big Ben and the Eiffel Tower, i.e. it occupies the fusion of  $l_i$  and  $l_j$ . (If a set has no located members, then it is not located. If a set has some located members and some non-located members, then it is co-located with the fusion of its located members.) The second step is to take a property as co-located with everything that instantiates it. (I am here officially neutral with respect to whether a property is wholly co-located with its instances or

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<sup>17</sup> "If abstract objects are not even spatial, they presumably cannot cause anything to happen." (Heck 2011, p.200.)

<sup>18</sup> See Parsons (2007) 'Theories of Location' for a precise system in which to discuss location and occupancy. I will nevertheless continue to speak somewhat informally about spatial location.

<sup>19</sup> Frege (1884/1950) *The Foundations of Arithmetic*, section 26. I say this is slightly anachronistic because Frege used the expression "actual" (*wirklich*) rather than "concrete". Nevertheless Frege regarded planets as "actual" and numbers as non-"actual", so he probably had something along the same lines as our contemporary abstract/concrete distinction (whatever that is) in mind.

divides its location among them.) For example the property of being red is co-located with every (red) firetruck, with every stop sign, and so forth. In this case, properties of (located) sets will be co-located with those sets. Thus the number 2, which is a property of {Big Ben, the Eiffel Tower}, will according to this proposal occupy the spatiotemporal region occupied by that set – which is the fusion of  $l_i$  and  $l_j$ . Similarly the number 3 is a property of {Big Ben, the Eiffel Tower, the CN Tower} and thus occupies the fusion of the regions occupied by each of those objects.

So far, so good – we have a way of taking numbers to be spatiotemporally located. Note also that the claim that numbers are *properties* of sets is not essential to this proposal. We could instead take numbers to be classes of equinumerous sets – i.e. the number 2 is the class of all two-membered sets – and then numbers can be spatiotemporally located objects (provided classes are located in the same way that sets are, i.e. by being co-located with the fusion of their members).

The problem with this proposal is that it entails, to put it crudely, that almost every number is almost everywhere. The number 2 is co-located with every two-membered set. But two-membered sets are all over the place. For every arbitrary pair of objects in my current visual field – and there are very many such pairs, my desk is quite cluttered – there is a two-membered set co-located with the fusion of those objects. Thus the number 2 is in my visual field many, many times over. Likewise for lots of other numbers: in general, if there are  $n$  distinct objects in your visual field right now, then every number equal to or less than  $n$  is in your field of vision. Things get worse when we consider subatomic particles. The number of subatomic particles making up the room I am currently working in is presumably very, very large. For each number  $n$ , that number is co-located with every set of  $n$  subatomic particles. Thus the number of subatomic particles making up the room I am in – which is presumably a very *big* number – is in this room with me, along with every one of its predecessors. This is, to put it mildly, very implausible.

Things get even worse for those who like Field (1989, chapter 6) hold that each non-empty region contains continuum-many self-subsisting spacetime points. For then the number that is the cardinality of the continuum is in this room with me too.

Now, this problem may not be strictly *fatal* to the idea that numbers are spatiotemporally located. But anyone holding that view has some pretty hefty bullets to bite. Having put the located-numbers-theorist on the defensive, we now turn to the question of whether mathematical objects – natural numbers in particular – can stand in causal relations.

Now, philosophers usually take *events* to be the primary terms of causal relations.<sup>20</sup> It was not Brutus himself that, in the first instance, caused Caesar's death, but Brutus's stabbing of Caesar. Brutus himself caused Caesar's death only insofar as he *participated* in that stabbing of Caesar. If that's right then objects are causal only in a secondary sense: an object is causal in virtue of participating in a causally related event. Suppose we follow Jaegwon Kim<sup>21</sup> in taking an event to be a 'structure' (basically an ordered triple) of objects, properties (or universals), and a time. Then Brutus participates in the stabbing of Caesar by being an element in the structure  $\langle \text{Brutus}, \text{stabs Caesar}, \text{the Ides of March 44 BCE} \rangle$ . And if that structure is a causally related event, then Brutus is a causally related object.

According to this way of taking objects as standing in causal relations, the claim that natural numbers can stand in causal relations is officially the claim that natural numbers can be among the elements of causally related events. Assessing this latter claim requires knowing what it is for a given event to be causally related. Now, there are as many ways to answer this question

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<sup>20</sup> "It is events, rather than objects or properties, that are usually taken by philosophers to be the terms of the causal relationship." Shoemaker (2003), p.206. Cf. also Rosen (2012), §3.2.

<sup>21</sup> This is the view in 'Events as Property Exemplifications' Kim (1976), p.312: "an event (or state) is a structure consisting of a substance (an  $n$ -tuple of substances), a property (an  $n$ -adic relational attribute), and a time."

as there are theories of causation, which is to say, *a lot*. Let us focus on one particular kind of theory of causation: a counterfactual theory. At a first approximation – which while extremely naive, nevertheless suffices for our purposes<sup>22</sup> – it is sufficient for an event  $E_i$  to cause another event  $E_j$  that  $E_i$  and  $E_j$  both occur, and that the following two counterfactual conditionals are true:

(1) If  $E_i$  were to occur, then  $E_j$  would occur.

(2) If  $E_i$  were to not occur, then  $E_j$  would not occur.

If (1) and (2) are both true, and  $E_i$  and  $E_j$  both occur, then  $E_i$  and  $E_j$  are both causally related events – they are causally related to each other. Suppose that we already know that we are causally related to  $E_j$ . Then we can show that we are causally related to  $E_i$  – and thus also to the objects participating in  $E_i$  – by showing that  $E_i$  occurs and that (1) and (2) are true.

Suppose now that, as many platonists believe, “2 is prime” is a necessary truth. Then given the Kim-style conception of an event as a structure consisting of some objects, some properties (possibly relational), and a time, we can view 2’s being prime as an event that is necessarily always ‘occurring’: 2 is prime at every time in every possible world. Consider now the counterfactual conditional ( $t_i$  is some fixed arbitrary time and  $E_j$  some arbitrary event):

(3) If  $\langle 2, \text{is prime}, t_i \rangle$  were to occur,  $E_j$  would occur.

On a standard semantics for counterfactual conditionals,<sup>23</sup> (3) is true if and only if in every

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<sup>22</sup> In particular I am here ignoring complications arising from pre-emption and epiphenomena. One way to begin accommodating those complications is to adopt David Lewis’s (1973/1986a) two-step proposal: an event  $E_j$  *causally depends* on another event  $E_i$  if and only if both occur and the counterfactuals “if  $E_i$  were to occur,  $E_j$  would occur” and “if  $E_i$  were to fail to occur, then  $E_j$  would fail to occur” are both true. An event  $E_h$  *causes* an event  $E_k$  if and only if there is a sequence of events, each element of which causally depends on the previous event, leading from  $E_h$  to  $E_k$ . Note that because causal dependence counts as *one kind* of causation – if  $E_j$  causally depends on  $E_i$ , then  $E_i$  causes  $E_j$  – these complications shouldn’t affect the argument I’m about to describe.

<sup>23</sup> Cf. Lewis (1973a) *Counterfactuals*, (1973/1986a) ‘Causation’, reprinted in his *Philosophical Papers, Volume II*.

nearby<sup>24</sup> possible world where  $\langle 2, \text{is prime}, t_i \rangle$  occurs,  $E_j$  also occurs. Suppose that  $E_j$  is an event that occurs in every nearby world, such as the birth of the universe. (Assuming that the universe was born. I also set aside any worries arising from the idea that no event can occur ‘before’ the birth of the universe. There are also more pressing worries to do with whether the birth of the universe does in fact occur in every *nearby* world where 2 is prime (see footnote 24). But I will grant this for the moment.) Then every nearby world is a world in which both  $\langle 2, \text{is prime}, t_i \rangle$  and  $E_j$  occur. Thus the truth-conditions of (3) are satisfied, and (3) is true (on that standard semantics for counterfactual conditionals).

Consider now, on the other hand, the following ‘negative’ counterfactual conditional:

(4) If  $\langle 2, \text{is prime}, t_i \rangle$  were to fail to occur, then  $E_j$  would fail to occur.

On our standard semantics for counterfactual conditions, (4) is true if and only if there are no nearby worlds in which  $\langle 2, \text{is prime}, t_i \rangle$  fails to occur but  $E_j$  still occurs. But seeing as  $\langle 2, \text{is prime}, t_i \rangle$  is a ‘necessary occurrence’, there is no world where it fails to occur. *A fortiori* there are no nearby worlds where  $\langle 2, \text{is prime}, t_i \rangle$  fails to occur but  $E_j$  does occur. Thus the truth-conditions for (4) are satisfied (vacuously), and (4) is true.

But if both (3) and (4) are true, then according to the naïve view that if  $E_i$  and  $E_j$  both occur and (1) and (2) are true then  $E_i$  causes  $E_j$ , 2’s being prime causes the birth of the universe (a surprising result). (Note that this argument is vulnerable to the objection that vacuously true counterfactual conditionals cannot give rise to causal relations; but I set that worry aside.) But then, given that we are causally related to the birth of the universe, we are causally related to 2’s

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<sup>24</sup> A possible world  $w$  is “near” to the actual world if and only if  $w$  is similar to the actual world in the relevant salient respects. This notion of nearness is context-dependent and probably vague, but I hope the point I am making is nevertheless clear enough.

being prime. Thus 2 participates in an event to which we are causally related; we are causally related to 2. And so, there is at least *logical* space for the claim that we can be causally related to natural numbers.

The problem with this proposal comes into view when we observe that standard mathematical platonism regards not only “2 is prime” but *every* pure mathematical truth as a necessary truth.<sup>25</sup> All pure mathematical truths are true in every possible world (and at every time). We just saw an argument that  $\langle 2, \text{is prime}, t_i \rangle$  is an event to which we are causally related because it occurs in every nearby world where the birth of the universe (to which we are causally related) occurs, and there are no worlds where the birth of the universe occurs but  $\langle 2, \text{is prime}, t_i \rangle$  fails to occur. The same holds for any pure mathematical truth. Let  $M$  be a pure mathematical truth like “7 is prime”, “the even prime is a Fibonacci number”, or whatever, and  $\langle M, t_x \rangle$  the event of  $M$ ’s holding at some time  $t_x$ . Then we have:

(5) If  $\langle M, t_x \rangle$  were to occur, the birth of the universe would occur.

(6) If  $\langle M, t_x \rangle$  were to fail to occur, the birth of the universe would not occur.

The truth of (5) and (6) can be established by reasoning directly parallel to that concerning (3) and (4).  $M$  is true at every time in every possible world, so for every nearby world  $w$  and every time  $t_x$ ,  $\langle M, t_x \rangle$  occurs at  $w$ . The birth of the universe occurs in every nearby possible world. Thus (5) is true and (6) is vacuously true. So if we are causally related to 2 for the reasons given in the above paragraph, then we are causally related to every natural number (of which there are countably infinitely many) and perhaps to every mathematical object (of which there are at least

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<sup>25</sup> Hale (1997/2001), p.178: “The platonist – or at least, the kind of platonist whose position I am anxious to defend – will hold that true mathematical statements are, like true statements of logical consequence, necessarily so.” I won’t attempt to answer the question of what *kind* of necessity is operating here, but I will assume – uncontroversially, I hope – that it is *metaphysical* necessity (if not stronger).

uncountably many).

Now, this is not yet necessarily fatal to the claim that we are causally related to mathematical objects like the natural numbers. But I do hope to have shown, at least, that that claim has some unforeseen implausible consequences.

(Note that the above argument may be a specific instance of the more general problem for philosophical analyses of causation of distinguishing between causes and background conditions. Necessary truths are background conditions for every possible event, so any theory of causation that has trouble distinguishing causes from background conditions may be capable of counting pure mathematical objects as causally related.)

We might wonder at the prospects of taking ourselves as causally related to natural numbers under a successor to Lewis's two-step counterfactual view of causation (see footnote 22): Lewis's (2000) 'causation as influence' view.<sup>26</sup> The intuitive idea behind the causation-as-influence view is that for one event  $E_i$  to cause another event  $E_j$ , not only must whether  $E_j$  occurs depend counterfactually on whether  $E_i$  occurs, but the details of how, where and when  $E_j$  occurs must depend counterfactually on the details of how, where, and when  $E_i$  occurs. So for Brutus's stabbing Caesar to cause Caesar's death not only must it be the case that if the former were to occur, the latter would occur, but also that if the former were to occur at a different time or place – if Brutus were to stab Caesar in the Forum on the summer solstice, instead of in the Senate on the Ides of March – then the latter would also occur in a different time and place; Caesar would die in the Forum on the summer solstice, rather than in the Senate on the Ides of March.

Here is the view more precisely. Let an *alteration* of an event  $E_j$  be a possible event that

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<sup>26</sup> Lewis (2000) 'Causation as Influence' *Journal of Philosophy* 97 (4): 182-197.

is either  $E_j$  itself or a very similar<sup>27</sup> event that occurs in a slightly different time (or place; but for ease of exposition I will concentrate on alteration along the temporal dimension only). Say that event  $E_i$  *influences*  $E_j$  if and only if there is a ('substantial'<sup>28</sup>) range of alterations  $E_{i1}, \dots, E_{in}$  and a ('substantial') range of alterations  $E_{j1}, \dots, E_{jn}$  such that the counterfactual conditionals "if  $E_{i1}$  were to occur,  $E_{j1}$  would occur", "if  $E_{i2}$  were to occur,  $E_{j2}$  would occur", ..., "if  $E_{in}$  were to occur,  $E_{jn}$  would occur", are all true. (Note that the correlation need not be one-one; one alteration of the effect may be influenced by many alterations of the cause.<sup>29</sup>) Then  $E_h$  *causes*  $E_k$  if and only if there is a chain of events leading from  $E_h$  to  $E_k$ , such that each member of the chain is influenced by its predecessor.<sup>30</sup>

In order for us to be causally related to an abstract mathematical object like 2 under a causation-as-influence view, 2 must participate in an event that influences (or is influenced by) some other event to which we are causally related. As before let's take the 'event' of 2's being prime at some time  $t_i$ . Then the alterations of  $\langle 2, \text{is prime}, t_i \rangle$  will be 2's being prime at other times: we'll write them as  $\langle 2, \text{is prime}, t_{i-m} \rangle, \dots, \langle 2, \text{is prime}, t_{i+m} \rangle$ . We now ask whether there is an event  $E$  to which we are causally related, such that there is a significant range of alterations of  $E$ ,  $E_1, E_2, \dots, E_j$  such that the following counterfactual conditionals are all true:

(A<sub>1</sub>) If  $\langle 2, \text{is prime}, t_{i-m} \rangle$  were to occur,  $E_1$  would occur.

(A<sub>2</sub>) If  $\langle 2, \text{is prime}, t_{i-(m-1)} \rangle$  were to occur,  $E_2$  would occur.

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<sup>27</sup> The view will need an account of how similar two events need to be, and in what respects, for one to count as an alteration of the other; but I won't pursue such questions here. (Cf. *ibid.*, p.188.)

<sup>28</sup> Just as with similarity, I leave aside questions of what it is for a range of alterations to count as 'substantial' in the required sense.

<sup>29</sup> Cf. the diagrams at *ibid.*, pp.192-193.

<sup>30</sup> *Ibid.*, p.190.



(A<sub>j</sub>) If  $\langle 2, \text{is prime}, t_{i+m} \rangle$  were to occur,  $E_j$  would occur.

Now, under the standard semantics for counterfactual conditionals, (A<sub>1</sub>)-(A<sub>j</sub>) are true if and only if every nearby possible world where the relevant alteration  $\langle 2, \text{is prime}, t_x \rangle$  occurs is also a world where the relevant alteration  $E_y$  occurs. According to the standard platonist, “2 is prime” is a necessary truth: it is true at every time in every possible world. Thus not only does  $\langle 2, \text{is prime}, t_i \rangle$  actually occur: every alteration of it occurs in every possible world. Thus the counterfactual conditionals (A<sub>1</sub>)-(A<sub>j</sub>) above will be true if and only if every alteration of  $E$  occurs in every nearby possible world. For suppose that there is an alteration of  $E$ ,  $E_k$ , that does not occur in some nearby world  $w$ . Seeing as every alteration of  $\langle 2, \text{is prime}, t_i \rangle$  occurs in every possible world, every alteration of that event occurs in  $w$ . Now, if  $E_k$  is an alteration of  $E$ , then at least one of (A<sub>1</sub>)-(A<sub>j</sub>) features  $E_k$  in its consequent. Any counterfactual conditional featuring any alteration of  $\langle 2, \text{is prime}, t_i \rangle$  in its antecedent and  $E_k$  in its consequent is falsified by  $w$ . Hence not all of (A<sub>1</sub>)-(A<sub>j</sub>) are true. So  $\langle 2, \text{is prime}, t_i \rangle$  does not influence  $E$ .

Thus  $\langle 2, \text{is prime}, t_i \rangle$  can only influence events for which every alteration occurs in every nearby world. But now we have a problem for the claim that we are causally related to 2 via 2’s being prime (at some time) influencing an event we already know we are causally related to. The problem is that it is not very plausible that *any* of the events we already know we are causally related to *are* such that every one of their alterations occurs in every nearby world. Most of the everyday events we know we are causally related to are contingent enough that there are nearby worlds where no alteration of that event occurs. There are nearby worlds where I did not have breakfast this morning, and so no alteration of the event of my morning meal occurs there. And for events that *do* occur in every nearby world it still seems the case that only *some* alterations of those events occur in each nearby world. Consider the invasion of England by William the

Conqueror in 1066. For the sake of the example, let's assume that there are no nearby worlds where no such invasion happened. Suppose there is an alteration of that event where it occurs in 1067 instead of 1066, and that there is a nearby world where William invades England in 1067. In that world some alteration of the invasion of 1066 occurs, but others do not; in particular, the actual invasion of 1066 does not occur. Thus even events that occur in every nearby world, but which occur at different times in different worlds, do not fit the bill for being influenced by 2's being prime. Indeed, the same goes for the birth of the universe. If the time of the birth of the universe varies across nearby worlds, then 2's being prime does not influence it. On the other hand, if the time of the birth of the universe does *not* vary across nearby worlds – maybe it's necessary that the moment of the birth of the universe is the first moment in time – then the birth of the universe does not have alterations. (At least, it does not admit of alterations along the temporal dimension, which is all we're concentrating on right now. But the extension to spatial alteration should be obvious). Now, it is a prerequisite for an event's being subject to influence that it admit of a 'significant' range of alterations. So if the birth of the universe occurs at the same time in every nearby world, then it does not admit of (temporal) alterations, and thus cannot be influenced by anything.

I will not pursue further the question of whether there is any event  $E$  both such that we are causally related to  $E$  and every alteration of  $E$  (of which there is a 'significant' range) occurs in every nearby world. Suffice it to say the prospects appear dim. However, there is a response available to anyone intent on 'causifying' numbers in a causation-as-influence framework. The move is to point out that while 2's being prime may be unable to influence any event to which we are causally related, 2's being prime is influenced *by* every such event. Take 2's being prime and an event  $E$  to which we are causally related, like the invasion of 1066. Let  $E_1$ - $E_i$  be the alterations

of  $E$ . If  $E$  is to influence 2's being prime, then we need that all the counterfactual conditionals of the form "if  $E_x$  were to occur, then  $\langle 2, \text{is prime}, t_y \rangle$  would occur" are true. Such conditionals are true (on the standard semantics) if and only if every nearby world where  $E_x$  occurs is also a world where  $\langle 2, \text{is prime}, t_y \rangle$  occurs. "2 is prime" is a necessary truth; 2 is prime at every time in every possible world. So every nearby possible world is a world where, for every time  $t_y$ ,  $\langle 2, \text{is prime}, t_y \rangle$  occurs. And so *a fortiori* every nearby world where  $E_x$  occurs is a world where  $\langle 2, \text{is prime}, t_y \rangle$  occurs. Thus every counterfactual conditional of the form "if  $E_x$  were to occur, then  $\langle 2, \text{is prime}, t_y \rangle$  would occur" is true. Hence  $E$  influences 2's being prime. And so 2 participates in an event influenced by an event that we are causally related to.

Therefore it looks like someone truly intent on regarding numbers as causally related to us in the context of a causation-as-influence view (combined with Kim's analysis of events) has just enough logical space to do so. However even if the above proposal works, it faces exactly the same problem as was raised against using a naïve counterfactual analysis of causation. If 2 is causally related in virtue of the event of its being prime being influenced by some event we are causally related to, then given that every pure mathematical truth is metaphysically necessary, it follows that every abstract mathematical object participates in an event influenced by the invasion of 1066. Take any pure mathematical truth  $M$ , and let  $\langle M, t_x \rangle$  be the event of  $M$ 's obtaining at some time  $t_x$ . Then for every event  $E$ , the counterfactual conditional "if  $E$  were to occur,  $\langle M, t_x \rangle$  would occur" is true. Thus every pure mathematical truth is influenced by every event. And so, if we are causally related to any abstract mathematical objects in virtue of being causally related to events that influence events corresponding to pure mathematical truths, then we are causally related to *every* abstract mathematical object (of which there are uncountably many).

Now once again this is not completely fatal to the view that numbers stand in causal

relations to us. But while I have not shown that moving away from the claim that mathematical objects are abstract objects is a definitive non-starter, it does seem to come at a pretty high cost. We will, however, see a more definitive argument against regarding pure mathematical objects as capable of standing in causal relations in chapter 3.

### *3.4: Semantic Responses*

There is one final broad category of response to the semantic argument: retaining (M), (S) and (A) at the cost of rejecting the CCR. This is the project of showing that we *do* possess the ability to refer to abstract mathematical entities (provided such things exist). The aim of this dissertation is to pursue this kind of response. More specifically, I will seek to arrive at a package of views that gives the platonist the best chance of defending the combination of (M), (S), and (A).

Having gotten clear on what exactly the semantic argument amounts to and seen some reasons for pursuing the rejection of CCR over the rejection of (M), (A), or (S), let us now compare the semantic argument to its more famous sibling: the *epistemological* argument against mathematical platonism.

## *4: Comparison with the Epistemological Argument Against Mathematical Platonism*

This section is about the relationship between the semantic argument and its more-discussed sibling, the *epistemological* argument against mathematical platonism. The epistemological argument can be run as follows:

P1: In order for an agent *S* to know that *P*, there must be a causal connection between *S*

and the features of the world that make  $P$  true. (Call this the *causal constraint on knowledge*, or “CCK” for short.)

P2: It is not possible for any abstract mathematical object to stand in any causal relations; thus there can be no causal relations between any agent and the features of the world making any pure mathematical statement true (when read at face-value).

Therefore,

C: It is not possible for any agent to know any pure mathematical truth.

This conclusion is the denial of the epistemological component thesis of mathematical platonism (E). So the epistemological argument says that in the presence of the CCK, (M) and (S) entail the denial of (E). Hence platonism is self-undermining.

The epistemological argument is better known than the semantic argument, and as a result, more has been written about the former than the latter. In particular, at least two responses to the epistemological argument have been proposed: Mark Balaguer’s ‘full-blooded platonism’ (hereon “FBP”) and a ‘trivialist’ move partially developed by Bob Hale from a remark of David Lewis’s. I will now argue that neither of these moves, as it stands, furnishes us with an adequate response to the semantic argument. But then given that we cannot *know* pure mathematical statements unless those statements are *true*, and if the semantic argument goes through those statements cannot be true (on a face-value reading), a defence of platonism (taken as including (S), the claim that we should read pure mathematical statements at face-value) against the epistemological argument is all but useless unless we also have a defence against the semantic argument. Thus the semantic argument is an important concern for the platonist in its own right, and is not a mere spin-off of the epistemological argument.

#### 4.1: Benacerraf's Epistemological Argument

The best-known instance of the epistemological argument against mathematical platonism occurs in Paul Benacerraf's (1973) article "Mathematical Truth". Benacerraf argued that if statements like "there are three perfect numbers greater than 17" are interpreted at face-value as statements whose truth is a matter of what some abstract mathematical objects are like, then we can never *know* that such statements are true. This is because the theories of knowledge that were popular around the time of Benacerraf's article (for example that described in Goldman "A Causal Theory of Knowing"<sup>31</sup>) had it that in order for *S* to know that *P*, *S* needs to stand in some causal relation to the features of the world that make *P* true; this is the CCK. Consider the following statement:

(7) There are three perfect numbers greater than 17.

If (7) is read at face-value as about abstract objects, then standing in causal relations to the features of the world that make (7) true is exactly what we cannot do. Thus (M), (S), (E) and CCK cannot all be true. Hence if component thesis (M), (S) and the CCK all hold then (E) does not: we do not know any pure mathematical truths. (Except perhaps negative existentials and vacuously true material conditionals; but this will hardly comfort the platonist.)

Note right off the bat the following difference between the semantic and epistemic arguments against mathematical platonism. The epistemic argument has it that the conjunction of (S) and (E) is not true. The semantic argument has it that the conjunction of (S) and (A) is not true. Because knowledge is factive (if *P* is known then *P* is true),  $\sim[(S) \& (A)]$  entails  $\sim[(S)$

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<sup>31</sup> *Journal of Philosophy* 64 (12): 357-372.

& (E)]. But  $\sim[(S) \& (E)]$  does not entail  $\sim[(S) \& (A)]$ . Hence the conclusion of the semantic argument entails the conclusion of the epistemic argument: if pure mathematical statements can't be true (on a face-value reading), then we cannot *know* those statements (on a face-value reading). But on the other hand, the conclusion of the epistemic argument is consistent with the claim that our pure mathematical statements are in fact true (though perhaps unknowable) on a face-value reading. Thus the semantic argument is strictly stronger than the epistemic argument. Therefore any successful platonist response to the epistemic argument (that preserves claim (S)) *requires* a successful response to the semantic argument. However I will now argue that neither of two contemporary responses to the epistemological argument – Balaguer's FBP and Hale's triviality response – show us how to respond to the semantic argument.

We can find our way into FBP and the triviality response by beginning with one of the first responses to Benacerraf's original epistemological argument: Mark Steiner's<sup>32</sup> claim that any satisfactory full articulation of the causal theory of knowledge presupposed by the epistemological argument will itself presuppose the truth of some pure mathematical theory.<sup>33</sup> Steiner interprets the CCK as follows:

*S* cannot know that a sentence *P* is true, unless *P* must be used in a causal explanation of *S*'s knowing (or believing) that *P* is true. (Steiner 1973, p.60; I have very slightly re-phrased the requirement.<sup>34</sup>)

Steiner then reasons as follows. Suppose *S* believes that “ $7 + 5 = 12$ ” is true. Something is causally

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<sup>32</sup> ‘Platonism and the Causal Theory of Knowledge’ (1973). Fun fact: Steiner's article actually appeared around 9 months *before* Benacerraf's. The latter appeared in the November 1973 issue of the *Journal of Philosophy*, the former in that journal's *February* 1973 issue.

<sup>33</sup> Steiner (1973), p.63: “the most plausible version of the causal theory of knowledge admits platonism; and the version most antagonistic to platonism is implausible.”

<sup>34</sup> The original reads: “One cannot know that a sentence *S* is true, unless *S* must be used in a causal explanation of one's knowing (or believing) that *S* is true.” (Ibid., p.60.)

responsible for *S* having this belief. Steiner argues that any ‘satisfactory’ causal explanation of *S*’s believing that “ $7 + 5 = 12$ ” will appeal to some theory (we might guess that such a theory contains claims about photons impacting on retinas and the like). Steiner claims that this theory, capable of giving a ‘satisfactory’ causal explanation of *S*’s belief that “ $7 + 5 = 12$ ” is true, will contain the axioms of analysis and number theory.<sup>35</sup> Hence that theory will contain (or more accurately, entail) “ $7 + 5 = 12$ ”. Thus our ‘satisfactory’ causal explanation of how *S* came to believe that “ $7 + 5 = 12$ ” appeals to “ $7 + 5 = 12$ ”. Nevertheless Steiner claims that *S*’s belief that “ $7 + 5 = 12$ ” is true satisfies his version of the CCK: that very sentence must be used in a causal explanation of how *S* came to believe that sentence. Or so Steiner argues.

Here are two responses to Steiner’s move. The first is that the success of that move is contingent on our inability to formulate the true causal theory of knowledge without using any abstract mathematical language. But it is hardly *a priori* impossible that we could do this. Steiner merely asserts, and does not argue, that the correct causal theory of knowledge will contain the axioms of number theory and analysis (see footnote 35). And if we ever do hit upon a physicalistic formulation of the true causal theory of knowledge then Steiner’s response to the epistemological argument loses any force it might have had.

The second response is perhaps more important. Steiner’s move is often regarded as having missed the point of Benacerraf’s epistemological argument.<sup>36</sup> If we regard the epistemological argument as demanding an account of how causally bound beings like ourselves

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<sup>35</sup> “We can assume that *something* is causally responsible for our belief, and that there exists a theory – actual or possible, known or unknown – which can satisfactorily explain our belief in a causal style. This theory, like all others, WILL CONTAIN THE AXIOMS OF NUMBER THEORY AND OF ANALYSIS.” (Ibid., p.61, capital emphasis original.)

<sup>36</sup> E.g. by W. D. Hart in this widely-cited passage: “Superficial worries about the intellectual hygiene of causal theories of knowledge are irrelevant to and misleading from this problem, for the problem is not so much about causality as about the very possibility of natural knowledge of abstract objects.” (Hart 1977, p.125/6)



could gain knowledge of abstract mathematical objects, then it is plain that Steiner's response provides no such account. If anything Steiner's response re-iterates the *need* for an account of how we could gain mathematical knowledge. For showing that the true causal theory of knowledge contains pure mathematical statements does not provide an account of how we can come to know those statements. It only shows that the possibility of such knowledge is required if we are to be able to know the true causal theory of knowledge. And while this is a desirable philosophical goal – epistemologists may not be pleased at the prospect that the true theory of knowledge is unknowable – Steiner has not given us an *account* of how we could come to know truths about abstract mathematical objects. (Note as we're going past that for *externalist* theories of knowledge – theories denying that in order for *S* to know that *P*, *S* must know that they know that *P* – the notion that we cannot know the true theory of knowledge does not entail that we cannot know anything at all. So the prospect that the true theory of knowledge might be unknowable because it is a causal theory with mathematical parts threatens the non-externalist more than the externalist.)

Thus the mathematical platonist should not rest content with Steiner's response to the epistemological argument. But luckily for the platonist, causal theories of knowledge of the kind sufficient for the CCK are no longer regarded as being as plausible as they were when Benacerraf and Steiner were addressing these issues.<sup>37</sup> Moreover, the idea that there is a totally *global* constraint on knowledge may have implausible consequences, for instance that we can have no knowledge of the future.<sup>38</sup> But without the CCK Benacerraf's epistemological argument has no

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<sup>37</sup> "Goldman's theory, however promising it seemed initially, increasingly encountered difficulties. It has by now long since come to be considered less satisfactory than several rival, *non-causal* theories." (Burgess & Rosen 1997, p.36; my emphasis.)

<sup>38</sup> Lewis (1986), p.110. Wright (1983) *Frege's Conception of Numbers as Objects*, pp.95-6 discusses *a priori* knowledge of necessary truths like "no book in my house is both blue and green all over".

force.

The proponent of the epistemological argument has a response. The response is to re-cast the argument without talking about *knowledge* at all, but rather *reliably true belief*. This avoids any commitment to any particular theory of knowledge altogether, thus avoiding Steiner's objection. This move was first made by Hartry Field.<sup>39</sup>

#### 4.2: Field's Reliable-True-Belief Challenge

Field frames his version of the epistemological argument not in terms of a causal constraint on knowledge, but in terms of when it is plausible to accept a strong correlation between an agent's beliefs and what those beliefs are about. The mathematical platonist believes that expert mathematicians are reliably good at having true beliefs about abstract mathematical objects and refraining from having false beliefs about such objects. Thus the mathematical platonist believes in a strong correlation between the beliefs of mathematicians and the mathematical facts. But a view positing such a strong correlation needs to provide some means for *explaining* that correlation: if no such explanation is forthcoming, then that correlation can only be a brute coincidence, and the plausibility of that view will suffer accordingly. Field claims that the complete causal isolation of abstract mathematical objects makes it very hard to see how such a correlation could be any more than a brute coincidence:<sup>40</sup>

[A] theory tends to be undermined if it needs to postulate massive coincidence. [...] But if this is so, won't platonism also be highly suspect, unless it postulates some explanation of the correlation between our mathematical beliefs and the mathematical facts [...] And it is hard to see how to explain such a correlation without postulating something extremely mysterious: a causal influence of mathematical objects on our

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<sup>39</sup> Field "Realism, Mathematics, and Modality", reprinted in his (1989) book *Realism, Mathematics and Modality*.

<sup>40</sup> Cf. Field (1989), p.230.

belief states, a god who predisposes us to recognize the basic truths, or whatever. (Field 1998/2001, p.324/5.)

Thus platonism incurs an explanatory obligation that it cannot meet; and to the extent that platonism incurs that obligation, its plausibility suffers. But unlike Benacerraf's version of the epistemological argument, Field's challenge makes no mention of *knowledge*, and hence does not presuppose any particular theory of knowledge susceptible to Steiner-type objections.

Let's get a bit clearer on what exactly the demand to 'explain the correlation' between mathematician's beliefs and mathematical facts amounts to. Burgess & Rosen articulate Field's explanandum as the following *reliability thesis*:<sup>41</sup>

(R) When mathematicians believe a claim about mathematical objects, that claim is true.

Note that (R) is intended to hold as a general rule that allows for occasional exceptions. Mathematicians are reliably good at believing mathematical truths, but not infallibly so. Field's explanatory demand then is a demand for an explanation of how (R) could be true without taking it as a brute fact. Balaguer's FBP and the Hale's triviality move are both attempts at showing that (R) is true, but it is not a *brute* truth.

According to FBP every consistent mathematical theory (read at face-value) has its own abstract model.<sup>42</sup> Hence every consistent pure mathematical theory is true. So for instance, there are sets that make ZF+CH – the axioms of Zermelo-Fraenkel set theory, plus the continuum hypothesis – true; but there are also different sets that make ZF+~CH true. There are other sets that make von Neumann-Bernays-Gödel set theory true; yet other sets that make Quine's New

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<sup>41</sup> Burgess & Rosen (1997), p.41.

<sup>42</sup> See Balaguer (1995) 'A Platonist Epistemology' and (1998) *Platonism and Anti-Platonism in Mathematics*.

Foundations set theory true; and so on. (Though there are no ‘naïve’ sets, because naïve set theory is inconsistent.) The same goes for every other (consistent) mathematical theory.

FBP entails (R) via the claim that mathematicians are reliably good at having *consistent* pure mathematical beliefs: if mathematicians believe  $P$ , then  $P$  is consistent. Seeing as my purpose is not to assess the merits of FBP but to get it on the table in order to compare its response to Field’s version of the epistemic argument with its prospects for responding to the semantic argument, I won’t take the time to quibble with the claim that mathematicians are reliably good at having consistent beliefs. (But see section 5 of Balaguer (1995) for a defence of it.) Once that claim is in place, the FBPIst can make the following argument:

P1: If mathematicians believe  $P$ , then  $P$  is consistent.

P2: If  $P$  is consistent,  $P$  is true.

Therefore,

(R): If mathematicians believe  $P$ , then  $P$  is true.

Thus the conjunction of the core claim of FBP with the claim that mathematicians are reliably good at believing mathematical consistencies entails that mathematicians are reliably good at believing mathematical truths.

The second proposal for showing that (R) is a non-brute truth aims to be consistent with ‘sparse’ platonism, where the consistency of a mathematical theory alone is not sufficient for its truth. The move starts with the platonist assumption that pure mathematical truths are necessary truths. Here is Bob Hale compactly articulating the proposal:

[W]here beliefs of a certain sort are, when true, necessarily so, there is simply no purchase for the idea that our tendency to get things right is, at bottom, a tendency for

our beliefs to vary (counterfactually) with the facts. For in such a case, the relevant facts could not have been otherwise, so that it is merely vacuously true that, had things been otherwise, we would not have held the beliefs that we do. (Hale 1997/2001, p.178.)

The proposal has obvious roots in Lewis's claim that "nothing can depend counterfactually on non-contingent matters. [...] Nothing sensible can be said about how our opinions would be different if there were no number seventeen."<sup>43</sup> Here is one way to put some meat on the bones of Hale's appropriation of Lewis's move.

First we grant that pure mathematical truths like " $7 + 5 = 12$ " are necessary in the sense required for them to be true in every possible world. Second we assume that explaining a reliable correlation between our beliefs and the facts requires no more than showing that our beliefs counterfactually co-vary with the facts; that is, that the following counterfactual conditionals are true:

(8): If  $P$  were to be true, then  $S$  would believe that  $P$ .

(9): If  $P$  were to not be true, then  $S$  would not believe that  $P$ .

(Note that (8) and (9) comprise the counterfactual components of Robert Nozick's truth tracking account of knowledge.<sup>44</sup>)

We can now pursue a line of reasoning directly parallel to the proposal we considered in section 3 above that pure mathematical objects participate in causally related events. Suppose that  $M$  is a pure mathematical statement like " $7 + 5 = 12$ ", and that this  $M$  is true in every

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<sup>43</sup> Lewis (1986) *On the Plurality of Worlds* p.111.

<sup>44</sup> Nozick (1981) *Philosophical Explanations*, chapter 3 pp.172-178. According to Nozick  $S$  knows that  $P$  if and only if (i)  $P$  is true; (ii)  $S$  believes that  $P$ ; (iii) if  $P$  were true,  $S$  would believe that  $P$ ; and (iv) if  $P$  were not true,  $S$  would not believe that  $P$ . (I'm leaving out some of Nozick's 'refinement and epicycles'; but what I've given is the central mechanism of the view.)

(accessible) possible world. Suppose  $S$  is a mathematician  $S_M$ . Then we have:

(8\*): If  $M$  were to be true, then  $S_M$  would believe that  $M$ .

(9\*): If  $M$  were to not be true, then  $S_M$  would not believe that  $M$ .

(8\*) is true on a standard Lewis-style semantics if and only if every nearby possible world where  $M$  is true,  $S_M$  believes that  $M$ .  $M$  is true in every possible world. And in every nearby possible world,  $S_M$  believes  $M$ : worlds where  $S$  does not believe “ $7 + 5 = 12$ ” are too distant from the actual world to be relevant to whether the truth-conditions of (8\*) are satisfied. (They are worlds very different from the actual world; worlds where  $S_M$  does not exist, or has a severe cognitive defect, etc.) So in every nearby possible world where  $M$  is true,  $S_M$  believes  $M$ ; hence (8\*) is true. (9\*) is true if and only if there are no nearby possible worlds where  $M$  is not true but  $S_M$  nevertheless believes  $M$ . But if  $M$  is a necessary truth then there are no possible worlds where  $M$  is not true. So it’s trivial that there no worlds where  $M$  is not true but  $S_M$  believes  $M$ . Thus (9\*) is true. And so for the platonist who believes that pure mathematical truths are necessarily true, (R) follows from at least one standard semantics for counterfactual conditionals.

This ‘trivialist’ proposal faces several very difficult challenges. For instance, consider the following alternative instances of (8\*) and (9\*):

(8\*\*): If “ $7 + 5 = 12$ ” were to be the case and  $S_M$  were to have the requisite understanding, then  $S_M$  would believe that  $e^{i\pi} = -1$ .

(9\*\*): If “ $7 + 5 = 12$ ” were to not be the case, then  $S_M$  would not believe that  $e^{i\pi} = -1$ .

The statements “ $7 + 5 = 12$ ” and “ $e^{i\pi} = -1$ ” are both necessary truths for the platonist; they are both true in every possible world. Hence if (8\*) and (9\*) are true, so are (8\*\*) and (9\*\*) (provided that  $S_M$  understand complex analysis in every nearby world; presumably there will be

at least *some* mathematicians for which this is so). But it seems wrong to take the truth of (8\*\*) and (9\*\*) as showing that mathematician's belief that  $e^{i\pi} = -1$  reliably correlates with the mathematical facts.

Nevertheless my business here is not to speculate on the merits of the FBP or the trivialist responses to Field's challenge. Rather, I want to get these responses on the table in order to show that *even if* they constitute successful responses to Field's challenge on its own terms, it is not obvious that either furnishes us with a response to the *semantic* argument against mathematical platonism.

#### 4.3: The Responses to the Epistemic Argument and the Semantic Argument

First up is FBP. The problem here is that it is not clear what positive effect the move from sparse platonism to FBP would have on whether we can refer to any abstract mathematical objects. If the CCR holds, then the causal isolation of an object  $o$  prevents us from using any expression to refer to  $o$ . It is not clear how or why adding *more* causally isolated objects to the universe should change this. It seems that merely adding all the objects described by every consistent mathematical theory would have no effect on whether those objects are non-spatiotemporal and non-causal, hence on whether the CCR allows reference to any such object. Now, it may turn out that going full-blooded may furnish the platonist with the resources to *refute* the CCR. But nothing that has been said *so far* hints at how this could be done. There is no *obvious* incompatibility between the claim that every pure mathematical theory has a model (or at least one model) and the CCR. And so the full-blooded mathematical platonist needs to say *something* more.

Let us turn to the trivialist move: every pure mathematical truth holds in every possible world. There is no obvious inconsistency between this claim and the CCR; and thus if pure mathematical objects are abstract, then by the CCR we cannot refer to them, and none of our statements containing singular terms purportedly referring to mathematical abstracta are true. Though, to be sure, it seems superfluous to postulate a realm of necessarily existing mathematical abstracta if we cannot refer to them and hence cannot think or utter truths about them; but predilections for desert landscapes aside, this is not an *inconsistent* view. Nevertheless the point is that the trivialist line does not obviously contain the resources required for a response to the semantic argument.

Thus neither FBP nor the trivialist move appears to provide us with the resources required for resisting the semantic argument against mathematical platonism. But as we've seen, in order for any response to the epistemic argument – that is, any attempt to shore up the conjunction  $[(M) \& (E) \& (S)]$  – to hold any water, we must already have disposed of the semantic argument – that is, shown that we can have  $[(M) \& (A) \& (S)]$ . For in order for  $S$  to know a statement  $P$ ,  $P$  needs to be true. And if  $P$  is to be true, then the singular terms occurring in  $P$  need to be able to refer. So it does nothing to advance the platonist's cause to show how we could come to know that  $7 + 5 = 12$  if we cannot also account for how we can use “7” (or any other expression) to refer to the number seven in the first place. Thus it behooves the platonist to find a response to the semantic argument that is consistent with their platonism, i.e., a way of rejecting CCR.

Before concluding by briefly outlining the coming chapters I will make some important observations about the framework of the debate over the CCR and platonism.



### 5: Framework Observations

Here's my first observation about the larger framework of the debate over reference to abstract mathematical objects. The conclusion of the semantic argument entails that even if 7, 5 and 12 exist, " $7 + 5 = 12$ " is not true on a face-value reading. Hence even if (M) is true, the conjunction [(S) & (A)] is not. Given that this conditional claim is entailed by the conclusion of the semantic argument, refuting that conditional suffices for overturning the semantic argument. And so, to resist the semantic argument it suffices to prove the conditional claim that *if* abstract mathematical objects were to exist, then we *would* be able to refer to them. This means that when defending themselves from the semantic argument, the platonist is entitled to assume that abstract mathematical objects *do* exist. (Though it would of course be illicit for the platonist to assume that we can refer to them, or that we can say or think anything true about them.) Hence this dissertation is not concerned with proving that abstract mathematical objects exist. I will help myself to the assumption that abstract mathematical objects exist, i.e. that (M) is true, throughout this dissertation.

The second framework observation that I will make is about the kind of opponent the platonist can take as mounting the semantic argument. Mathematical platonism's main historical opponents have been philosophers who take abstract mathematical objects to pose *special* problems for the theories of reference and knowledge – problems that ordinary concrete objects like boats and birds, and theoretical scientific objects like bosons, are not supposed to pose. Thus it suffices to refute the semantic argument, as mounted by such a philosopher, to show that abstract mathematical objects pose no *special* problem for the theory of reference. Hence one strategy available to the platonist for resisting the semantic argument, or a specific argument for the CCR, is to show that it overgenerates its conclusion: if reference to abstract objects is not possible, then

reference to *any* external object is not possible. If the platonist can prove this conditional claim, then abstract mathematical objects pose no *special* problem for the theory of reference. Thus the platonist can consider the semantic argument refuted if they can show that it overgenerates its conclusion in this way. (This includes showing that the semantic argument entails that there is no reference relation at all.) We will see this strategy get put to use in chapters 2 and 3.

Having made two framework observations about the terms of the debate, I'll now briefly describe the contents of the remaining three chapters.

#### *6: Outline of the Coming Chapters*

The next chapter of this dissertation is about one well-known error-theoretic response, Hartry Field's mathematical fictionalism. Field accepts component thesis (S) of platonism – pure mathematical statements should be read at face-value – but he denies (A); those statements are not true. Nevertheless such statements can occur in scientifically useful theories because (briefly) a theory *T* containing (e.g.) Peano Arithmetic as a sub-theory cannot prove anything about its concrete subject-matter that cannot also be proven, in a perhaps more long-winded fashion, from the fragment of *T* that talks only about concrete objects. (This imprecise formulation will be improved upon in the next chapter.) Field also claims that the semantic argument against mathematical platonism is a reason to adopt his fictionalism. I will argue that rather than lending support to fictionalism, the semantic argument commits the fictionalist to taking stands in debates whose relevance to fictionalism is not immediately obvious.

The third chapter is about one argument for the CCR: causal contact is required for singular terms like “7” and “the even prime” to have *unique* referents, and if a singular term does

not have a unique referent then it has *no* referent. The argument that I will discuss is an adaptation of Benacerraf's argument in "What Numbers Could Not Be" that because there is no fact of the matter about which objects (e.g.) the natural numbers are, they are not objects at all. First I will adapt this into an argument for premise 2 of the semantic argument: because we are causally isolated from abstract mathematical objects, there is no fact of matter about which objects our uses of mathematical expressions like "7" or "the even prime" refer to – even if there *is* a fact of the matter about which objects are numbers (*pace* Benacerraf's original argument). After locating this argument in the wider context of Quinean worries about inscrutability of reference and showing how it can be used to argue that natural numbers cannot stand in causal relations, I will tentatively investigate the possibility of using David Lewis's notion of a *reference magnet*<sup>45</sup> to resist this argument.

The fourth and final chapter is about the distinction between descriptive and singular reference. I will give a restricted version of the semantic argument, that even if we can refer descriptively to abstract mathematical objects, we cannot refer *singularly* to them. After showing that this restricted semantic argument still spells trouble for platonism I will try to resist it by arguing that an account of singular reference to abstract objects can be built around a new theory of singular thought, Robin Jeshion's *cognitivism*.<sup>46</sup> I will consider three objections; in response to one such objection I will move to a hybrid theory combining cognitivism about non-perceptual reference with a causal theory of perceptual demonstrative reference.

I will ultimately conclude by briefly describing the package of views that, if everything I say is correct, give the mathematical platonist (and if chapter two is correct, the mathematical

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<sup>45</sup> 'Putnam's Paradox' (1984), 'New Work for a Theory of Universals' (1983).

<sup>46</sup> 'The Significance of Names' (2009), 'Acquaintance, Semantic Instrumentalism, and Cognitivism' (2010).

fictionalist who either does not want to take a stand on surprising debates or does not want to become embroiled in such debates in the first place) a chance of effectively responding to the semantic argument against mathematical platonism. That package is some form of reference magnet theory conjoined with hybrid cognitivism about singular thought.

## *Chapter Two*

### **Mathematical Fictionalism and the Semantic Argument**

#### *Abstract*

This chapter is about the relationship between the semantic argument against mathematical platonism and *mathematical fictionalism*: the view that the scientific utility of pure mathematical theories can be explained without claiming that those theories are true. Hence fictionalists can maintain thesis (S) of platonism – the thesis that we should read pure mathematical statements at face-value, that is, so that their surface syntax is a good guide to their inferential properties – whilst rejecting (A), the alethic thesis that pure mathematical statements are true. And so because fictionalists do not accept both (S) and (A), they do not have to believe in abstract mathematical objects. It is widely held that the semantic argument against platonism provides a good reason for being a fictionalist. In this chapter I argue that this is not so. Rather than motivating fictionalism, the idea that we cannot refer to abstract mathematical objects raises problems for fictionalism. In particular, claiming that we cannot refer to abstract mathematical objects forces the fictionalist to take stands in debates that are at first glance far removed from the initial goal of explaining the scientific utility of pure mathematical theories without appealing to their truth.

## Contents

### 1: Introduction

### 2: Fictionalism and Mathematical Contingency

### 3: Should Fictionalists Believe We Can Refer to Abstract Objects?

### 4: Logical Possibility and Conceptual Impossibility

### 5: Fictionalism and Primitivism about Logical Possibility

### 6: Inferentialism, Intuitionism, and Anti-Realism

### 7: Conclusion

#### *1: Introduction*

Mathematical fictionalism holds that explaining the utility of pure mathematical statements in scientific and everyday contexts does not require taking those statements to be true. Fictionalism is like platonism in that it retains the platonistic thesis (S), that we should read pure mathematical statements at face-value. But fictionalism is unlike platonism in that it rejects thesis (A): according to fictionalism, pure mathematical statements are not true. (Except perhaps negative existentials like “there are no even prime numbers greater than 2” and universally quantified conditionals like “if there are no even primes greater than 2 then there are no even primes greater than 4”. But I’ll elide over this subtlety in what follows.) Hence fictionalists do not have to believe in abstract mathematical objects.

Here is the semantic argument against platonism once again:

P1: CCR: it is a necessary condition on an expression  $e$ 's referring to an object  $o$  that it be metaphysically possible for  $o$  to stand in at least one causal relation.

P2: It is metaphysically impossible for any pure mathematical object to stand in any causal relations.

Therefore,

C: It is metaphysically impossible for any of our expressions to refer to any pure mathematical object.

C, plus the platonist's metaphysical thesis (M) (that the referents of pure mathematical singular terms can only be abstract objects), entails the denial of the conjunction of the alethic thesis (A) and the semantic thesis (S). This is because according to (S), " $7 + 5 = 12$ " contains three singular terms, namely " $7$ ", " $5$ " and " $12$ ". According to (M), the referents of those expressions can only be abstract mathematical objects. But the above conclusion C entails that none of our expressions refer to abstract mathematical objects; hence " $7$ ", " $5$ " and " $12$ " cannot refer to anything – they are *empty* singular terms. But if " $7$ ", " $5$ " and " $12$ " are empty singular terms, then " $7 + 5 = 12$ " is not true. So even if there are abstract mathematical objects, (S) and (A) are not both true, and neither is mathematical platonism as a whole.

The semantic argument is often seen as a reason for adopting mathematical fictionalism.<sup>47</sup> My goal in this chapter is to investigate the plausibility of this claim. I will ultimately argue that a mathematical fictionalist can endorse the conclusion of the semantic argument only on pain of adopting other assumptions that are just as questionable, if not more so, than the conclusion of the semantic argument against mathematical platonism. While I will

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<sup>47</sup> For instance, at Field (1989) p.23-4 and p.68.

not ultimately show that the fictionalist *cannot* endorse the conclusion of the semantic argument, I do hope to deepen our understanding of what exactly the costs of holding such a position are.

Here's the plan. In section 2 I describe the relevant components of Field's mathematical fictionalism (hereon simply "fictionalism"<sup>48</sup>), including his appeal to the semantic argument and the important result that the fictionalist is committed to the claim that mathematical theories are possibly true in some sense. In section 3 I take a pass at arguing that fictionalism is in tension with the semantic argument: if mathematical theories are possibly true, then reference to the objects required to make them true is possible. However this first pass relies on a claim that the fictionalist does not have to accept: that mathematical theories are possibly true in the *same* sense that the semantic argument says it is not possible for our expressions to refer to abstract mathematical objects. In particular, the mathematical fictionalist can hold that it is *logically* possible for mathematical statements like " $7 + 5 = 12$ " and theories containing such statements to be true, but *conceptually* impossible (in a sense I will explain) for any of our expressions to refer to abstract mathematical objects. I begin section 5 by noting that this requires that the fictionalist can distinguish logical possibility from conceptual possibility. This in turn commits them to a distinction between logical expressions and all other expressions. I argue that the best way for a fictionalist to distinguish logical expressions from all other expressions is in terms of what it takes to grasp the inferential role of an expression. However adopting this kind of view of logical expressions places the fictionalist in danger of committing themselves to intuitionistic logic, and which in turn my commit them to global anti-realism. In Section 7 I conclude by

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<sup>48</sup> There are varieties of mathematical fictionalism quite distinct from Field's, such as that described in Balaguer (1998). I will not consider those distinct versions of fictionalism.



briefly reflecting on what I have shown.

Let's begin with the details of fictionalism.

## 2: *Fictionalism and Mathematical Contingency*

Here is Hartry Field endorsing the semantic argument against mathematical platonism (and its epistemological sibling, though that better-known argument is beyond the scope of this chapter):

Probably the **main ground** for suspicion about mathematical entities is the difficulty that these entities raise for the theory of knowledge and for the theory of reference or theory of belief content. [. . .] There are no causal connections between the entities in the platonic realm and ourselves; how then can [. . .] a particular word like “two”, or a particular belief state of our brains, *stand for* or *be about* a particular one of the absolute infinity of objects in that realm? (Field 1989, p.68; italic emphasis original, boldface emphasis mine.)

Field's rhetorical question “how can a particular word like “two” stand for a particular one of a causally isolated infinity of objects?” appears to invite a pessimistic response: no particular word like “two” could “stand for or be about” a particular one of an infinity of causally isolated objects. But if none of our expressions can stand for any abstract mathematical objects (objects ‘in the platonic realm’), then none of our statements can be true in virtue of what such objects are like. Thus Field is using the semantic argument to urge us away from mathematical platonism – and ultimately towards his own anti-platonist fictionalism.

We now move to the salient details of that fictionalism and why it entails that mathematical statements like “ $7 + 5 = 12$ ” are possibly true, even if they are actually not true.

Field's positive view is a *fictionalist* view about mathematics. It is a constitutive feature of fictionalist views of mathematics that they endorse the conjunction of the semantic component of platonism (S) with the denial of its alethic component (A). (S) has it that the numerals "7", "5" and "12" are singular terms, and hence that " $7 + 5 = 12$ " entails " $\exists x \exists y \exists z (x + y = z)$ ". According to fictionalism, *if* " $7 + 5 = 12$ " were true, this would be because some abstract mathematical objects stand in a certain relation. Thus fictionalism contrasts with anti-platonist views that deny (S) and thus *reinterpret* statements like " $7 + 5 = 12$ " so their truth does not require the existence of any mathematical objects.

Field's fictionalism is an explicit response to the *indispensability argument* for mathematical platonism that we saw in section 2.3 of chapter one. The indispensability argument says that because many of our most successful scientific theories entail the existence of abstract mathematical objects (when read at face-value), and we are committed to the truth of these theories, we are committed to the existence of abstract mathematical objects. In other words, we can't just pick-and-choose what parts of a successful scientific theory to believe in; at least, not without further philosophical ado.<sup>49</sup> Views which deny (S) have access to a straightforward response to the indispensability argument: successful scientific theories *are* (approximately) true, but this doesn't entail the existence of abstract mathematical objects. But Field affirms (S); he grants that the indispensability of pure mathematical statements entails the indispensability of abstract mathematical objects. Thus he needs a different response.

Field's response to the indispensability argument is to show that the scientific utility of mathematical theories does not require that such theories be true. The first ingredient is the

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<sup>49</sup> Field (1980) explicitly acknowledges this motivation for his programme: "If one *just* advocates fictionalism about a portion of mathematics, without showing how that part of mathematics is dispensable in applications, then one is engaging in intellectual doublethink..." (p.2, emphasis original.)

notion of a *conservative extension*. Let  $T$  be a theory in a language  $L$  and  $T^*$  a theory in a language  $L^*$ .<sup>50</sup> ( $L$  may or may not be a sub-language of  $L^*$ .) Then  $T^* \cup T$  (the union of  $T$  and  $T^*$ ) conservatively extends  $T$  if and only if any well-formed formula  $p$  of  $L$  that is a logical consequence of  $T \cup T^*$  is also a logical consequence of  $T$  alone. Symbolically:

$$\forall p \in L [(T^* \cup T \Rightarrow p) \text{ iff } (T \Rightarrow p)].^{51}$$

In such cases, we say that  $T^* \cup T$  “conservatively extends”  $T$ . Field uses the notion of a conservative extension to respond to the indispensability argument as follows.

Let  $L_N$  be a *nominalistic* language – a language with an intended interpretation<sup>52</sup> according to which the singular terms of  $L_N$  refer only to concrete objects, and the quantifiers of  $L_N$  range over only concrete objects. Let  $N$  be a theory in  $L_N$ . Seeing as  $N$  is a theory in the nominalistic language  $L_N$ , no statement in  $N$  entails the existence of any abstract object. Let  $P$  be a theory in a non-nominalistic language  $L$  whose intended interpretation takes  $L$  to contain expressions purportedly referring to and quantifying over abstract mathematical objects. Then the mixed mathematical-physical theory  $N \cup P$  conservatively extends the purely nominalistic theory  $N$  if and only if any statement in the nominalistic language  $L_N$  following logically from  $N \cup P$  also follows logically from  $N$  alone.<sup>53</sup> In other words:

$$\forall p \in L_N [(N \cup P \Rightarrow p) \text{ iff } (N \Rightarrow p)].$$

Thus there are no purely nominalistic logical consequences of the mixed mathematical-physical

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<sup>50</sup> A theory is a set of well-formed formulae, for instance, a set of axioms. A language is (for now) a set of well-formed formulae closed under the formation rules (e.g. if  $A$  and  $B$  are well-formed formulae, then  $\lceil A \rightarrow B \rceil$  is a well-formed formula).

<sup>51</sup> I'm using “ $\Rightarrow$ ” for logical consequence instead of either the single- or double-turnstile because I want to stay neutral between identifying logical consequence with either syntactic (i.e. proof-theoretic) or semantic (model-theoretic) consequence. It will soon become apparent why.

<sup>52</sup> Thus  $L_N$  is not only a language in the mere formal sense described in footnote 50.

<sup>53</sup> Cf. Field (1980), p.11-12.

theory  $N \cup P$  that are not also consequences of the purely nominalistic theory  $N$ . This is important because it means that we can go about deriving consequences about the physical world from  $N \cup P$  whilst safe in the knowledge that we could in principle derive all those very same consequences from  $N$  alone, if we had to.<sup>54</sup> Hence the mathematical theory  $P$  is not *indispensable* to our uses of  $N \cup P$ , it is merely very *useful*.  $P$  is acting as a collection of ‘inferential shortcuts’ allowing us to derive statements that are consequences of  $N$  more *efficiently* than if we were confined to using  $N$  alone.<sup>55</sup> (Note that in order for  $P$  to have this kind of utility it will need to contain ‘bridge laws’ linking the abstract vocabulary of  $P$  with the nominalistic vocabulary of  $N$ .) And so using  $N \cup P$  to reason about the purely nominalistic subject-matter of  $N$  does not force a commitment to the truth of  $P$ . Thus if we can show that the platonistic theory  $N \cup P$  conservatively extends its nominalistic fragment  $N$ , then we are free use  $N \cup P$  whilst denying that there are any abstract mathematical objects. In *Science Without Numbers* Field applies the above strategy to Newtonian Gravitational Theory (hereon “NGT”). NGT is a mixed mathematical-physical theory; it contains many pure mathematical statements like “ $7 + 5 = 12$ ”, as well as bridge laws containing expressions for both abstract and concrete objects, such as “for any points  $x, y$ , and  $z$ ,  $y$  is between  $x$  and  $z$  if and only if  $d(x, y) + d(y, z) = d(x, z)$ ” (here  $d$  is a function from pairs of spacetime points to non-negative real numbers; hence “ $d(x, y)$ ” is an expression for a number).<sup>56</sup> Field identifies a purely nominalistic sub-theory of NGT, which we’ll call “NGT<sub>N</sub>”, such that there is no statement purely about the physical world following from NGT that does not also follow from NGT<sub>N</sub>. So using mathematical statements

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<sup>54</sup> Field (1980), p.14. Shapiro (1983) heavily criticizes this way of describing the significance of  $N \cup P$ ’s conservatively extending  $N$  by observing that the details of the relationship between conservativeness and *deductive efficiency* depends on whether  $P$  is a first- or second-order theory. However these issues are independent of my argument.

<sup>55</sup> The details of the relationship between conservativeness and *deductive efficiency* depends on whether  $P$  is a first- or second-order theory; cf. Shapiro (1983). However these issues are independent of my argument.

<sup>56</sup> The example is from Field (1980), p.26.

appearing in NGT when deriving purely physical consequences from NGT does not commit us to the existence of the abstract mathematical objects postulated by NGT.

Note as we're going past that Field's strategy requires that we be able to *identify* the nominalistic fragment  $NGT_N$  conservatively extended by NGT. This presents a difficulty because identifying  $NGT_N$  requires *nominalizing* NGT; that is, reinterpreting NGT in a purely nominalistic language.<sup>57</sup> And while Field has managed to nominalize NGT (given the assumption that spatiotemporal points are concrete entities),<sup>58</sup> it is not widely accepted that we can nominalize *any* given successful scientific theory, in particular theories like Quantum Mechanics that are mathematically more complex than NGT.<sup>59</sup> However if the argument I'm about to give does go through, it goes through even if every successful scientific theory can be nominalized. Thus I grant that every scientifically successful theory can be nominalized in the way required to apply Field's conservative extension strategy.

Having gotten the bare mechanics of fictionalism on the table, I will now show why the claim that NGT conservatively extends  $NGT_N$ , plus a general background rejection of abstract mathematical objects, entails that the mathematical statements in NGT must be possibly true.

Now, the notion of a conservative extension was explained in terms of the notion of logical consequence. There are two standard ways of analyzing the notion of logical consequence. One characterizes what it is for one statement to be a logical consequence of another in terms of whether the latter can be *true* when the former is not. The other characterizes logical consequence in terms of whether there is a *proof* of one statement from another. Both

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<sup>57</sup> Ibid., p.2.

<sup>58</sup> Ibid., chapter 8.

<sup>59</sup> Malament (1982) presses this point; Balaguer (1996) responds.

analyses are up to their necks in commitments to abstract mathematical objects. Here's why.

Let  $\phi$  be a formula and  $\Gamma$  a set of formulae in a language  $L$ . According to standard (platonistic) accounts, we can *prove*  $\phi$  from  $\Gamma$  if and only if there is a finite sequence  $s$  of formula-types of  $L$  ending with  $\phi$ , every element of which is either an axiom, a member of  $\Gamma$ , or is the result of an application of a rule of inference to previous members of  $s$ . In this case  $\phi$  is a *syntactic* consequence of  $\Gamma$ . The second standard notion of logical consequence – *semantic* consequence – is that the truth of  $\Gamma$  suffices for the truth of  $\phi$ : every model  $M = \langle D, I \rangle$  (where  $D$  is a set of objects and  $I$  is a function assigning semantic values to the constants, predicates, and function symbols of  $L$ ) that makes every member of  $\Gamma$  true also makes  $\phi$  true. A soundness theorem for a logic guarantees that if  $\phi$  is a syntactic consequence of  $\Gamma$ , then  $\phi$  is a semantic consequence of  $\Gamma$ ; a completeness theorem guarantees that if  $\phi$  is a semantic consequence of  $\Gamma$ , then  $\phi$  is a syntactic consequence of  $\Gamma$ .

Viewing logical consequence in either of these standard ways requires dealing with abstract objects: ordered sequences of formula-types for syntactic consequence, sets and functions for semantic consequence.<sup>60</sup> But if  $\phi$ 's being a logical consequence of  $\Gamma$  is a matter of whether certain abstract objects have certain properties or stand in certain relations, then whether NGT conservatively extends  $\text{NGT}_N$  is itself a matter of facts about abstract objects. Appealing to conservative extensions to avoid committing to abstract objects when formulating and applying mixed mathematical-physical theories requires committing to them in metalogic. So even if NGT *is* a conservative extension of  $\text{NGT}_N$ , this doesn't mean the fictionalist has managed to avoid commitment to abstract mathematical objects altogether. For without a non-platonistic understanding of logical consequence, the fictionalist's belief that NGT

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<sup>60</sup> Cf. Field (1989), p.30/1, p.98.

conservatively extends  $\text{NGT}_N$  commits them to abstract mathematical objects.

Thus the fictionalist needs a notion of logical consequence that does not deal in abstracta. Field himself adopts a *modal* notion of logical consequence:  $\phi$  is a consequence of  $\Gamma$  if and only if it is not possible for every member of  $\Gamma$  to be true and  $\phi$  not true.<sup>61</sup> In other words:

$$\Gamma \Rightarrow \phi \text{ iff } \sim \Diamond(\Gamma \ \& \ \sim \phi)$$

Moreover Field claims that this modal understanding of logical consequence is nominalistically acceptable because we can refrain from cashing it out in terms of possible worlds or models.<sup>62</sup>

I will now show why interpreting logical consequence in modal terms entails that for a theory  $\Gamma$  to conservatively extend a possibly true  $T$ ,  $\Gamma$  must itself be possible. Let  $\Gamma$  be a theory,  $T$  a consistent sub-theory of  $\Gamma$ , and  $\phi$  a statement in the language of  $T$  such that  $T \not\Rightarrow \phi$ .

0: $\sim \Diamond \Gamma$	[assumption]
1: $(\Gamma \Rightarrow \phi) \text{ iff } \sim \Diamond(\Gamma \ \& \ \sim \phi)$	[modal defn. logical consequence]
2: if $\sim \Diamond \Gamma$ , then $\sim \Diamond(\Gamma \ \& \ \sim \phi)$	[impossibility preserved by conjunction]
3: if $\sim \Diamond \Gamma$ , then $\forall \psi(\sim \Diamond(\Gamma \ \& \ \sim \psi))$	[2, choice of $\phi$ is arbitrary]
4: if $\sim \Diamond \Gamma$ , then $\forall \psi(\Gamma \Rightarrow \psi)$	[1, 3, substitution]
5: $\forall \psi(\Gamma \Rightarrow \psi)$	[0, 4, detachment]

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<sup>61</sup> “[T]he relevant notion of consequence can be explained modally (it is not possible for the premises to be true and the conclusion false)” (Field 1989, p.127.) Hale (1987), p.108 also states Field’s view in this way. In ‘Metalogic and Modality’ (1990) Field argues that there are reasons aside from fictionalism for preferring a modal analysis of logical consequence. Note also that Field accepts that a fictionalist cannot “literally believe the talk about a set of sentences”, but claims that such talk “is easier to eliminate” than the talk about abstract resulting from standard analyses of logical consequence. (1989, p.29/30.)

<sup>62</sup> “[T]he relevant notion of [...] modality can be understood without explanation in terms of platonistic entities (possible worlds, models, etc.).” (Field 1989, p.127; cf. p.98)

Thus if  $\sim\Diamond\Gamma$ , then  $\Gamma$  entails every formula (in the language of  $T$ ). Hence  $\Gamma$  entails a formula not entailed by  $T$  (we're assuming that  $T \not\Rightarrow \phi$ ). So if  $\sim\Diamond\Gamma$  then  $\Gamma$  is not a conservative extension of  $T$ . Thus if  $\Gamma$  is conservative over  $T$ , then  $\Diamond\Gamma$ .<sup>63</sup>

So if the fictionalist wants a mixed mathematical-physical theory like NGT to be conservative over a purely nominalistic (and presumably consistent) theory  $\text{NGT}_N$ , then they need to hold  $\Diamond\text{NGT}$ . We know that NGT entails that  $7 + 5 = 12$ . Hence  $\Diamond\text{NGT}$  entails that  $\Diamond(7 + 5 = 12)$ . So for the Fieldian fictionalist it is possible for abstract mathematical objects to exist, in whatever sense of “possible” is required for NGT to conservatively extend  $\text{NGT}_N$ . I will now take a first pass at showing that if the Fieldian fictionalist believes that  $\Diamond(7 + 5 = 12)$ , then they should believe that we can refer to abstract objects, *contra* the semantic argument against mathematical platonism.

### 3: Should Fictionalists Believe We Can Refer to Abstract Objects?

We have that if NGT is conservative over its nominalistic fragment  $\text{NGT}_N$ , then  $\Diamond\text{NGT}$ . If  $\Diamond\text{NGT}$ , then  $\Diamond(7 + 5 = 12)$ . Now, according to the claim (S) stated at the outset, the truth of “ $7 + 5 = 12$ ” requires that “7”, “5” and “12” refer to objects of some kind or another; and Field agrees with the platonist that if those expressions refer, they refer to abstract objects.<sup>64</sup> (We also saw some arguments for this claim in section 3.3 of chapter one, and will see a further argument in chapter three.) Hence the *possible* truth of a face-value reading of “ $7 + 5 = 12$ ” requires that

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<sup>63</sup> Field is aware of this consequence; e.g. (1993), p.43

<sup>64</sup> “[A] mathematical theory, taken at face value, is a theory that is primarily about some postulated realm of abstract entities: numbers, functions, or sets or whatever ...” (Field 1989, p.2.) The context makes it obvious that by a theory’s being ‘about’ mathematical objects he means that that theory’s singular terms refer to and its quantifiers range over abstract mathematical objects.



it be *possible* that “7”, “5” and “12” refer to abstract mathematical objects. But now, if it is possible for “5” to refer to an abstract mathematical object, then reference to abstract mathematical objects is possible. This contradicts the conclusion of the semantic argument against mathematical platonism. Thus Field's fictionalism – in particular, the conjunction of the claim that NGT conservatively extends  $\text{NGT}_N$  with the modal notion of logical consequence described in the previous section – contradicts the conclusion of the semantic argument against mathematical platonism.

The reader might be thinking that this looks all too easy; and they would be right. The fictionalist does have the resources to neutralize this argument as I've just given it. In the next section I will show how.

#### *4: Logical Possibility and Conceptual Impossibility*

The fictionalist who endorses the semantic argument against mathematical platonism appears to be in the uncomfortable position of trying to maintain two apparently contradictory claims: NGT is possible, but reference to abstract objects is not possible. However they can eliminate this discomfort by taking these occurrences of the word “possible” as indicating distinct kinds of possibility. In particular, all that's required for NGT to conservatively extend  $\text{NGT}_N$  is that NGT be *logically* possible in a certain austere sense, whereas the semantic argument shows that reference to abstract mathematical objects is impossible in a more restricted sense. (I've been using ‘metaphysical’ possibility in this connection, but we'll soon see that Field himself is not happy with this.) Thus the situation is similar to that with the possibility of travelling at superluminal speeds: travelling faster than the speed of light, while physically impossible (as

far as we know), is nevertheless logically possible.<sup>65</sup>

The first move is to claim that the “ $\Diamond$ ” in “ $\Gamma \Rightarrow \phi$  iff  $\sim\Diamond(\Gamma \ \& \ \sim\phi)$ ” stands for Field’s *austere* notion of logical possibility. For Field, a statement  $P$  that contains no modal operators is logically possible if and only if  $P$  is logically consistent. In fact, Field takes logical possibility and logical consistency to be interdefinable;<sup>66</sup> hence I will freely slide between them in what follows.  $P$  is logically consistent if and only if is an instance of a consistent logical form.<sup>67</sup> (I will for the sake of the argument grant that Field can get away with talking about ‘logical forms’ without being committed to abstract objects.<sup>68</sup>) Thus  $\phi$  is a logical consequence of  $\Gamma$  if and only if “ $\Gamma \ \& \ \sim\phi$ ” is an instance of an inconsistent logical form. Recall that it is a necessary condition for  $\Gamma$  to be a conservative extension of any of its sub-theories that not every formula (of the relevant sub-theory) is a logical consequence of  $\Gamma$ . We saw that this in turn requires  $\Diamond\Gamma$ , i.e. that  $\Gamma$  is logically possible. Hence given Field’s identification of (austere) logical possibility with consistency, we now have that in order for NGT to be conservative over NGT<sub>N</sub>, NGT must be consistent.

Now, Field takes logical consistency to be a *primitive* notion, whose meaning can’t be clarified by giving a definition or reductive explanation in more basic terms.<sup>69</sup> (Note that he also takes logical consistency and logical possibility to be the same thing; in the rest of this

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<sup>65</sup> My thanks to both Justin Dallman and an anonymous referee for the University of Western Ontario 2014 Philosophy of Logic, Mathematics and Physics conference for separately raising points that ultimately brought this move to my attention.

<sup>66</sup> Field (1990), p.8.

<sup>67</sup> This follows from the “logical form principle” described in Field (1990), p.9 and (1993), p.298. Seeing as I’m not going to discuss statements with embedded modal operators I won’t worry about how Field deals with those. Note also that Field is operating with the Davidsonian notion of logical form – where the logical form of a statement determines its inferential properties – that I described in section 1 of chapter one.

<sup>68</sup> Recall that Field thinks that talk of sets of sentences is “easier to eliminate” than talk of mathematical entities like models. (1989, p.29/30.)

<sup>69</sup> Field (1990), p.5; (1993), p.298. In other places he takes logical implication as primitive, and defines logical consistency in terms of implication; but I won’t worry about that complication in this chapter.

chapter I will follow suit.) However one way we can tell whether a given statement (or theory) is consistent is if that statement (or the conjunction of every statement in the theory) is an instance of a logical form that also has an instance that we know is true. (The reader may wonder if taking logical consistency as primitive forces one to take the notion of logical form as primitive too. This will become important in the next section, but for now I'll set it aside.) For example, we know the logical form of “there is a male dog” is consistent because we know “there is a male dog” is true. Thus seeing as “there is a male dog” is an instance of a consistent logical form, it is logically consistent – i.e. logically possible – as is every statement sharing that logical form. Let us grant that NGT is an instance of a consistent logical form and hence that NGT meets the necessary condition on conservatively extending  $NGT_N$  that we identified in section 2.<sup>70</sup>

It remains to show that the (austere) logical possibility of NGT does not conflict with the conclusion of the semantic argument against mathematical platonism. To do this we first need to get clear on how the fictionalist should interpret that argument's conclusion (C): that it is not possible for any singular term to refer to any abstract mathematical object. That is a modal claim. But what modality should the fictionalist read into (C)? (I have been assuming that we read *metaphysical* possibility into (C), but shall now drop that assumption.)

Now, the claim that we cannot refer to abstract objects is not supposed to mean that, as a matter of everyday contingent fact, we just *don't* refer to abstracta. Nor is it that reference to abstracta is *medically* impossible, as if some physiological feature of ours prevents us from referring to abstracta in the same way that physiological features of the human eye prevent us

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<sup>70</sup> One way to show this would be to cite a true statement with the same logical form as (the conjunction of every statement in) NGT. Given that Field takes consistency as primitive, it seems the only other way to show this would be by something like intuition. MacBride (1999) raises doubts about this route (p.446-7).

from seeing infra-red light. Nor is it that reference to abstracta is *technologically* impossible; it's not like we could invent 'abstracta goggles' that would allow us to refer to numbers in the same way that night-vision goggles allow us to see infra-red light. Nor does it look like (C) is the claim that the *laws of nature* prevent us from referring to abstracta. Presumably a proponent of the semantic argument against mathematical platonism would hold that causation constrains reference even in 'nomologically impossible' worlds where a difference in the laws of nature results in a difference in the behaviour and perhaps even the reach of causal relations. (I leave aside what would happen in worlds where there is no causation.<sup>71</sup>)

So (C) is not the claim that reference to abstracta just doesn't happen (as if by chance), nor that it is merely medically, technologically, or nomologically impossible. The modality operating in (C) is stronger than any of these.

On the other hand, the fictionalist cannot take (C) to be saying that reference to abstract mathematical objects is logically impossible in Field's austere sense. For Field a statement *P* is logically possible if and only if it is an instance of a consistent logical form. Consider now the statement "'7" refers to an abstract object", or more perspicuously, " $\exists x[\text{abstract}(x) \ \& \ \text{refers}('7', x)]$ ". The logical form of that statement is:

$$\ulcorner \exists x[\Phi x \ \& \ \Psi(\alpha, x)] \urcorner.$$

Here  $\Psi$  ranges over binary predicates,  $\Phi$  ranges over unary predicates, and  $\alpha$  ranges over singular terms. But we know that the logical form  $\ulcorner \exists x[\Phi x \ \& \ \Psi(\alpha, x)] \urcorner$  is consistent because it has true instances, for instance "Toronto is south of a national capital". So if "Toronto is south of a national

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<sup>71</sup> I also leave aside whether the fictionalist should interpret (C) as a claim about *epistemic* possibility, i.e. that reference to abstract objects is inconsistent with what we know. Such a reading strikes me as wildly off the mark.

capital” is true, and has the same logical form as ““7” refers to an abstract object”, then the latter is logically possible. Therefore reference to abstract objects is, at least, logically possible (in Field’s austere sense).

Having seen which modalities the fictionalist should *not* read into the claim that we cannot refer to abstract objects, we are left with the question of which modality the Fieldian fictionalist *should* read into that claim. That modality needs to be weaker than medical, technological or nomological possibility, but stronger than (austere) logical possibility.

Now, one natural choice of modality weaker than nomological possibility but stronger than logical possibility is *metaphysical* possibility. But Field himself is quite wary of the notion of metaphysical possibility:

To say that the modal notion one has in mind is *metaphysical* possibility is to give a pseudo-clarification, the practical effect of which I think is to serve as a license to equivocate. (1989, p.39; emphasis original.)

Hence it behooves the fictionalist (or at the very least, Field himself) to find a modal notion intermediate between nomological and logical possibility that is also distinct from metaphysical possibility. Here is my take on how a fictionalist can do this using a resource that Field himself has appealed to in discussions of modal logic: the Carnapian notion of a *meaning postulate*.

Consider the statement “there is a married bachelor”. This statement has the same logical form as “there is a male dog”; thus “there is a married bachelor” is logically possible. But “there is a married bachelor” is *the* paradigm case of something that ‘cannot’ be true. It seems impossible that there could be a married bachelor, and not just merely medically, technologically or nomologically impossible. One rough way to describe the situation is that the ‘meanings’ of the

English language predicates “married” and “bachelor” ensure that nothing can satisfy both (simultaneously). Field captures this idea by appealing to Carnap’s notion of a *meaning postulate*.<sup>72</sup> A meaning postulate is a statement of a ‘logical’<sup>73</sup> relation between two or more predicates. So we might think that one of the meaning postulates for the English language predicate “married” is that anyone who satisfies “married” cannot satisfy “bachelor”. Let us follow Carnap<sup>74</sup> in writing this meaning postulate as a universally quantified conditional:  $\forall x[\text{bachelor}(x) \rightarrow \sim \text{married}(x)]$ .

Field says that a statement  $P$  is *conceptually possible* if and only if the ‘combination’ of  $P$  with the meaning postulates for the predicates and singular terms appearing in  $P$  is logically possible.<sup>75</sup> Let us take this to mean that it is sufficient for  $P$  to be conceptually impossible that the conjunction of  $P$  with the meaning postulates governing the predicates occurring in  $P$ , or anything that follows from those meaning postulates, is an instance of an inconsistent logical form. Then while “there is a married bachelor” is logically possible, it is conceptually impossible. For the conjunction of “there is a married bachelor” with the meaning postulates for “married” and “bachelor” entails:

$$(*): \exists x[\text{married}(x) \ \& \ \text{bachelor}(x)] \ \& \ \forall x[\text{married}(x) \rightarrow \sim \text{bachelor}(x)].$$

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<sup>72</sup> Field (1989), p.87/8, esp. f/n 9. Carnap discusses meaning postulates in his essay ‘Meaning Postulates’, reprinted as supplement B to the second edition of his *Meaning and Necessity* (1956).

<sup>73</sup> Note that there is something *slightly* strange about calling such relations ‘logical’ relations, especially with Field’s austere notion of logical possibility in the background. Hence I place ‘logical’ in scare-quotes because Field does (1989, p.87), and whenever I use ‘logical’ with scare-quotes I mean whatever Field means by it.

<sup>74</sup> Carnap (1956), p.224. Carnap writes this claim as ‘ $(x)(Bx \supset \sim Mx)$ ’ but the differences between this and our formulation are obviously merely symbolic.

<sup>75</sup> “[A] set of sentences is *conceptually possible* iff it in combination with all the definitions and other meaning relations among its vocabulary items is logically consistent.” (Field 1993, p.292; emphasis original.) See also Field (1989), p.87, though he does not use the phrase “conceptually possible” there. There is an interesting question about what a ‘meaning postulate’ for a singular term would look like, but I won’t address that question here.

This conjunction has the following logical form:

$$\lceil \exists x[\Phi x \ \& \ \Psi x] \ \& \ \forall x[\Phi x \rightarrow \sim \Psi x] \rceil.$$

Here  $\Phi$  and  $\Psi$  range over unary predicates. Everyone will agree that this logical form is inconsistent. Hence (\*) is logically impossible. Thus “there is a married bachelor” is conceptually impossible. (Or at least, it is conceptually impossible in English. There could of course be other languages where the symbols “married” and “bachelor” are governed by other meaning postulates.) And so if  $\forall x[\text{married}(x) \rightarrow \sim \text{bachelor}(x)]$  really *is* a meaning postulate governing “married” and “bachelor”, then (\*) is not conceptually possible (in English).

So there are logical possibilities that are (in English<sup>76</sup>) not conceptual possibilities (like “there is a married bachelor”). And so long as nomological impossibilities are not by that very token also conceptual impossibilities – i.e. if there are statements that are nomologically impossible, but yet conceptually possible – then conceptual possibility is a modal notion both intermediate in strength between logical possibility and nomological possibility and distinct from metaphysical possibility. This notion of conceptual possibility looks like exactly what the fictionalist requires. I will now show how to read the conclusion of the semantic argument against mathematical platonism as a claim about conceptual possibility.

The idea is that the conclusion of the semantic argument (C) – the claim that it is impossible for any of our expressions to refer to any abstract mathematical object – amounts to a claim about the meaning postulates for the predicates “refers” and “abstract”. (We take “refers” to be a 2-place relational predicate.) That claim is that the meaning postulates governing “refers” and “abstract” are such that following claim (R) is a consequence of those meaning postulates

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<sup>76</sup> I will hereon leave the qualification “in English” implicit when discussing conceptual possibility.

(the range of the second quantifier includes English language expressions like “Toronto”, “7”, “the even prime” etc.):

$$(R): \forall x \forall y [\text{abstract}(x) \rightarrow \sim \text{refers}(y, x)].$$

In other words, nothing that is the referent of anything is abstract. (If we wanted the more modest claim that no *human* expression refers to anything abstract, we would restrict the range of the second quantifier accordingly.)

We just saw that for the fictionalist reference to abstract objects is logically possible because ““7” refers to an abstract object” is an instance of a logical form with a true instance. But if we conjoin ““7” refers to an abstract object” with the proposed consequence of the meaning postulates for “refers” and “abstract” (R), we have:

$$(**): \exists x [\text{refers}(\text{“7”}, x) \ \& \ \text{abstract}(x)] \ \& \ \forall x \forall y [\text{abstract}(x) \rightarrow \sim \text{refers}(y, x)].$$

The logical form of (\*\*) is:

$$\lceil \exists x [\Psi(\alpha, x) \ \& \ \Phi x] \ \& \ \forall x \forall y [\Phi x \rightarrow \sim \Psi(y, x)] \rceil.$$

Here  $\Phi$  ranges over unary predicates,  $\Psi$  over binary predicates, and  $\alpha$  over singular terms. Now, the above logical form is inconsistent. Therefore (\*\*) is logically impossible, and hence ““7” refers to an abstract object” is conceptually impossible. So it looks like the fictionalist can take the semantic argument as showing that reference to abstracta is conceptually impossible by taking it to show that (R) is a meaning postulate governing “abstract” and “refers”.

Moreover it looks like the conceptual impossibility of ““7” refers to an abstract object” and the like poses no threat to the logical possibility of NGT. For the logical possibility of NGT requires only that (the conjunction of every member of) NGT is an instance of a consistent logical



form, and nothing about the conceptual impossibility of “‘7’ refers to an abstract object” appears to threaten that.

### *5: Fictionalism and Primitivism about Logical Possibility*

If what I’ve said so far is right, the fictionalist who endorses the conclusion of the semantic argument ought to take statements like “‘7’ refers to an abstract object” as logically possible (i.e. logically consistent), but conceptually impossible. This requires that the fictionalist be able to distinguish logical and conceptual possibility. For if there is *no* distinction between logical possibility and conceptual possibility, then all logical possibilities are conceptual possibilities. And if reference to abstract objects is logically possible then it is conceptually possible too, *contra* the conclusion of the semantic argument against mathematical platonism (according to the reading of that claim which I’ve recommended to the fictionalist).

I am going to press on the fictionalist’s ability to distinguish logical from conceptual possibility by arguing that that distinction requires a distinction between logical expressions like “&” and “v” and non-logical expressions like “bachelor”. But because the fictionalist takes logical possibility to be *primitive*, the best account of the distinction between logical and non-logical expressions available to them is an *inferentialist* distinction. An ‘inferentialist’ distinction between logical expressions and all others says, to put it roughly, that the logical expressions are those expressions whose meaning can be given by means of introduction- and elimination-rules satisfying certain constraints. But inferentialism about logical expressions raises well-known philosophical problems. The one problem I will mention is that there are reasons to think that inferentialism leads to the rejection of the law of the excluded middle, and this in turn leads to

anti-realism. Thus if the fictionalist adopts inferentialism about logical expressions, then they need to take stands on philosophical questions that at first glance appear quite remote from the concerns that motivated mathematical fictionalism in the first place.

Consider the two statements “grass is green and grass is not green” and “snow is white or snow is not white”. Assuming for the sake of the example that the mass nouns “grass” and “snow” are singular terms, we can write these statements slightly more formally as:

(g)  $\text{Green}(\text{grass}) \ \& \ \sim\text{green}(\text{grass})$ .

(s)  $\text{White}(\text{snow}) \vee \sim\text{white}(\text{snow})$ .

These two statements are supposed to differ with respect to logical possibility: (s) is logically possible (or even logically true) whereas (g) is logically impossible. If this is right then (g) and (s) must have distinct logical forms. Now, the process of moving from a statement  $P$  to the logical form of  $P$  involves ‘abstracting away’ the specific meanings of the expressions occurring in  $P$ . This way of characterizing logical form, in terms of the results of abstracting away from specific meanings of expressions, might make it look like logical form is a purely *syntactic* matter. One *might* be tempted to think that the logical form of  $P$  is the result of abstracting away the meanings of all the expressions occurring in  $P$ . But if (g) and (s) are to have distinct logical forms then logical form cannot be a matter of pure syntax. Here’s why.

At first glance the statement (g) looks to be an instance of the logical form  $\lceil\Phi\alpha \ \& \ \sim\Phi\alpha\rceil$ , where  $\Phi$  ranges over unary predicates and  $\alpha$  over singular terms. So (g) looks like an instance of an inconsistent logical form. But if we consider the syntactical form of (g) after abstracting away from the specific meanings of *all* the expressions occurring in (g), we don’t get  $\lceil\Phi\alpha \ \& \ \sim\Phi\alpha\rceil$ . This is because in addition to the meanings of “grass” and “green”, we are *also* abstracting away

the specific meanings of “&” and “~”. And if we abstract away from the specific meanings of “&” and “~” (as well as “grass” and “green”), rather than  $\ulcorner \Phi\alpha \ \& \ \sim\Phi\alpha \urcorner$ , we get:

$$\ulcorner \Psi(\Phi\alpha, \S(\Phi\alpha)) \urcorner.$$

Here  $\Psi$  ranges over binary operators,  $\S$  ranges over unary operators,  $\Phi$  ranges over unary predicates and  $\alpha$  ranges over singular terms. Call this the *bare syntactic form* of (g). Note that if we abstract away from the specific meanings of all the expressions occurring in (s), including those for “v” and “~”, then we do not get  $\ulcorner \Phi\alpha \vee \sim\Phi\alpha \urcorner$ , but the bare syntactic form  $\ulcorner \Psi(\Phi\alpha, \S(\Phi\alpha)) \urcorner$ , which is the same as that of (g).

Thus (g) and (s) have the same bare syntactic form. So if having the same syntactic form as a true statement suffices for being logically possible, then (g) is logically possible. This is the wrong result.

Hence the fictionalist cannot let sharing a bare syntactic form with a true statement be sufficient for being logically possible. They need a way of telling when two statements differ with respect to logical form despite sharing the same bare syntactic form. Now, if statements are to differ with respect to logical form whilst sharing the same bare syntactic form, then there needs to be a privileged class of expressions whose meanings are not abstracted away when moving from a statement to its logical form; that is, a class of expressions identified as *logical* expressions. Pre-theoretic intuitions suggest that when we move from (g) or (s) to their logical forms, but stop short of their common bare syntactic form, we abstract away from the specific meanings of expressions like “grass”, “green”, “now” and “white”, but we do *not* abstract away the specific meanings of “&”, “v”, and “~”. Thus pre-theoretic intuition has it that the latter expressions are logical expressions. Moreover if logical possibility is to not be a vacuous notion – if at least some well-

formed statements are to be logically impossible – then at least *some* expressions have to be identified as logical expressions.

Now, it may seem obvious that “&” is a logical expression but “bachelor”, for instance, is not. But further consideration reveals the existence of more borderline cases: the quantifier “ $\forall$ ” is often considered a logical expression, but at least one well-known logic textbook disagrees: in *A Mathematical Introduction to Logic* Herbert Enderton takes “ $\forall$ ” to be a parameter whose assignment – the domain of quantification – varies from model to model. Thus for Enderton the meaning of “ $\forall$ ” can vary between models whereas the meaning of “&” does not.<sup>77</sup> Indeed, the *general* question of exactly which expressions are logical expressions is a notoriously deep and fraught one. But if what I’ve said so far in this section is correct, then this is a question on which the mathematical fictionalist who endorses the conclusion of the semantic argument against mathematical platonism needs to take some kind of stand.

Thus we are led to the question of what kind of stand the fictionalist can take on the question of exactly which expressions are logical expressions. Recall from section 4 that the fictionalist who is eager to avoid commitment to abstract proofs or models when explicating logical consequence takes the notion of logical possibility as *primitive*. One important consequence of this primitivism about logical possibility is that logical possibility cannot be reductively explained or defined in more basic terms (like models or possible worlds).

Nevertheless we also have the following relationship between logical possibility and the logicity of an expression, though the fictionalist won’t take this relationship to constitute a reduction of logical possibility to which expressions are logical expressions. The relationship is:

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<sup>77</sup> Enderton (2001), p.70.

a statement  $P$  is logically possible if and only if  $P$  is an instance of a consistent logical form, and whether or not  $P$  is an instance of a consistent logical form will help determine which expressions are logical expressions. But then, if there is no reductive explanation or definition of logical possibility, then there is no reductive explanation or definition of what it is for an expression to be a logical expression that goes beyond the following: the logical expressions are just those expressions that need to be counted as logical in order for all and only the logical possibilities to come out logically possible, and which statements are logically possible is a primitive matter. So if the fictionalist is a primitivist about logical possibility then they are, for all intents and purposes, a primitivist about which expressions are the logical expressions. (Perhaps the fictionalist could renounce their primitivism about logical possibility, analyse that notion in terms of which expressions are logical expressions, and either take an expression's being logical as a primitive matter or give a reductive account. But any such reductive account had better not involve any mathematics or possible worlds. Unfortunately it would take us too far afield to pursue this question further; though the interested reader should consult Cohnitz & Rossberg (2009) for a reductive yet nominalistically acceptable account of which expressions are logical.) But now it looks like the fictionalist has been moved to fairly unpersuasive table-thumping on the matter: *these* expressions are logical because that is what's required for logical possibility to come out *this* way, and it is a *primitive* matter that logical possibility comes out this way.

Fortunately for the fictionalist, taking logical possibility as primitive need not reduce them to table-thumping on the matter of which expressions are logical. Field himself has claimed that while there is no good reductive explanation or definition of logical possibility in other terms, the meaning of that primitive notion can be conveyed "by specifying the procedural rules involved in

inferring with it.”<sup>78</sup> One example of such a procedural rule would be: ‘if  $P$  is true, then  $P$  is logically possible’ (there will of course be others). Importantly, Field claims that supplying some procedural inference rules for the expression “it is logically possible that” need not be taken as analysing the meaning of that phrase in other terms, but rather as *elucidating* it.<sup>79</sup> And elucidating logical possibility by specifying relevant rules of inference or usage is not to deny that that notion is primitive.

Now, logical possibility is not the only notion that Field thinks can be conveyed by specifying procedural rules for inference:

[N]egation and conjunction and universal quantification are primitive notions. To say that these notions are primitive is to say that their meaning is not to be conveyed by definition. The meaning is to be conveyed, I think, by specifying the procedural rules involved in inferring with it. (Field 1989, p.32.)

Field is claiming that one can give the meanings of “&”, “~” and “ $\forall$ ” by means of inference rules without thereby impugning the claim that those notions are primitive. Now, I leave aside the question of whether Field’s taking those expressions as primitive could allow him to dispense with taking logical possibility as primitive, and instead defining it in terms of “&”, “~” and “ $\forall$ ” via the notion of logical form. But note that the claims of Field’s we’ve seen suggest that he takes *both* logical possibility *and* “&”, “~” and “ $\forall$ ” as primitive.

The important thing for us though is that there is a view of the distinction between logical expressions and all others that is very close to Field’s claim that the meanings of “&”, “~” and

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<sup>78</sup> Field (1989), p.32. Compare Field (1993), p.298: “How then are the meanings of “it is logically consistent that” and “it is logically true that” conveyed, if not in terms of conceptual possibility and conceptual necessity, or in terms of proofs or models? That’s easy: in terms of the laws governing them, i.e. in terms of rules of use.”

<sup>79</sup> Field (1993), p.298.

“ $\forall$ ” can be given by specifying inference rules. Consider, for example, the introduction- and elimination rules for “ $\&$ ”:

$$\begin{array}{ccc}
 A & B & \\
 & & A \& B \quad A \& B \\
 \&-introduction: \quad \underline{\hspace{2cm}} & \&-elimination: \quad \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\
 & & A \quad B \\
 A \& B & & 
 \end{array}$$

One supposedly suggestive fact about these statements of the introduction- and elimination-rules for “ $\&$ ” is that they each contain only schematic variables for statements, the horizontal line denoting the drawing of an inference, and a single instance of “ $\&$ ”. The idea is that this shows the meaning of “ $\&$ ” to be topic-neutral, in the sense that these introduction- and elimination-rules can be understood by anyone capable of rational inference (and who understands that the horizontal line denotes the drawing of an inference) no matter what specific antecedent knowledge they have.<sup>80</sup>

Call an inference rule understanding of which requires grasp of no specific notions (other than the drawing of an inference) a *purely* inferential rule. (This is a very broad characterization that can be made precise in many distinct ways. However rather than becoming embroiled in such details I will press on with the main point.) Then *inferentialism* about the logical expressions is the view that what distinguishes logical expressions from others is that the meanings of logical expressions can be conveyed by means of purely inferential rules.<sup>81</sup> Thus “ $\&$ ” is a logical

<sup>80</sup> MacFarlane (2015), section 6: “Thus the meaning of “ $\&$ ” can be grasped by anyone who understands the significance of the horizontal line in an inference rule. [...] Anyone who is capable of articulate thought or reasoning at all should be able to understand these inference rules, and should therefore be in a position to grasp the meaning of “ $\&$ ”. Or so the thought goes.”

<sup>81</sup> Compare Cohnitz & Rossberg (2009), p.153: “Inferentialism insists that the meaning of the logical constants [i.e. logical expressions] is determined by their introduction- and elimination-rules, and that these rules (so far as they are the correct ones) are self-justifying. No further appeal to model-theoretic semantics, truth-tables or the like is needed in order to argue for the validity of the rules.” I will not comment on whether there is any

expression because its introduction- and elimination-rules are purely inferential. And presumably “bachelor” is not a logical expression because its meaning cannot be conveyed via purely inferential rules, though I won’t pause to investigate whether this is actually so.

Thus inferentialism about logical expressions may be the best bet for the fictionalist who wants to both hold logical possibility as primitive, but also be able to give a distinction between logical and non-logical expressions that is more satisfying than ‘the logical expressions are just those expressions which must be counted as logical in order for all and only the logical possibilities to come out as logically possible (and which statements are logically possible is a primitive matter)’. However inferentialism about logical expressions raises certain questions, to which we now turn.

#### *6: Inferentialism, Intuitionism, and Anti-Realism*

In this section I sketch, in broad strokes, an argument from inferentialism about logical expressions to the rejection of realism. (The argument is originally due to Michael Dummett, though my presentation will be slightly different from his.<sup>82</sup>) Moreover, resisting this line of reasoning requires the fictionalist to take stands on philosophical issues that *prima facie* appear to be far removed from the desire to resist the inference from scientific utility to truth that originally motivated mathematical fictionalism. The argument says that inferentialism about logical expressions leads to giving up the law of the excluded middle (the claim that for every statement  $P$ ,  $\lceil P \vee \sim P \rceil$  is a logical truth), and that giving up the law of excluded middle is

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connection between this notion of introduction- and elimination-rules being self-justifying and the notion that those rules can be understood by anyone capable of rational inference.

<sup>82</sup> Dummett makes the argument in ‘The Philosophical Basis of Intuitionistic Logic’ and ‘The Justification of Deduction’, both reprinted in his *Truth and Other Enigmas*.



tantamount to giving up realism. While I will leave many specific details unarticulated, my goal is to show that the fictionalist needs to say something about how they will resist this line of reasoning if they are to retain both the semantic argument against platonism and realism about concrete objects.

I begin sketching the line from inferentialism about logical expressions to the move away from realism by noting that the inferentialist characterization of logical expressions as those expressions whose introduction- and elimination-rules are purely inferential cannot be correct as it stands. This is forcefully showed by an example of Arthur Prior's.<sup>83</sup> Let "*tonk*" be a sentential connective governed by the following introduction- and elimination-rules:

A	A <i>tonk</i> B
" <i>tonk</i> "-introduction: _____	" <i>tonk</i> "-elimination: _____
A <i>tonk</i> B	B

"*Tonk*"-introduction and "*tonk*"-elimination are purely inferential rules by the criteria described in the previous section: they contain only schematic sentence letters, the horizontal, and one occurrence each of the expression being introduced. But "*tonk*" cannot be a logical expression because it can be used to infer anything from anything. From "grass is green" we can, by "*tonk*"-introduction, move to "grass is green *tonk* snow is white". And now from "grass is green *tonk* snow is white" we can, by "*tonk*"-elimination, move to "snow is white". So using "*tonk*" we can 'infer' "grass is green" from "snow is white". So we should not count "*tonk*" as a logical expression, on pain of trivialising logical entailment.

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<sup>83</sup> 'The Runabout Inference-Ticket' *Analysis* (1960) 21 (2): 38-39. Note that Dummett does not present his argument in terms of Prior's "*tonk*".

Thus the inferentialist needs a further restriction on which introduction- and elimination-rules yield logical expressions beyond their being purely inferential. One source of such a restriction is the natural picture of logical inference as the activity of manipulating the information we possess to yield only what is already *implicit* in that information.<sup>84</sup> (This characterization of logical inference is only as good as our notion of information, which I will not take the time to make more precise here. But the intuitive notion should suffice for the argument-sketch I'm about to give.) One way to describe what is wrong with “*tonk*” is that ‘inferring’ using “*tonk*”-introduction and “*tonk*”-elimination can lead one to acquire *genuinely new* information; the information that snow is white is not implicitly contained in the information that grass is green. That this is not so for “&” can be seen by noting that successive applications of “&”-introduction and “&”-elimination will only ever lead one ‘in a circle’, from the premises one starts with, back to those premises. From A and B we infer ‘A & B’, and then from ‘A & B’, we infer A and B. Thus the introduction- and elimination-rules *conserve information*, in that successive applications of them will not lead us to genuinely new information. But with “*tonk*” from A we infer ‘A *tonk* B’, and then from ‘A *tonk* B’, we infer B, which is *not* what we started with, and may actually be something quite different. Thus the introduction- and elimination-rules for “*tonk*” do not conserve information.

With this picture of logical inference as the manipulation of antecedently possessed information in play, the inferentialist can place a constraint on an expression’s being logical that rules out “*tonk*”: an expression is logical if and only if its introduction- and elimination-rules are

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<sup>84</sup> This picture of deductive inference can be read into Frege’s enigmatic remark that logical consequences are contained in premises “as plants are contained in their seeds, not as beams are contained in a house.” (*Foundations of Arithmetic*, section 88.) See Dummett (1991), pp. 37-38 for a development of this idea.

both purely inferential and conserve information. “&” meets these criteria; “*tonk*” does not.<sup>85</sup>

Now to problems with the broadly sketched inferentialist proposal on the table. I will briefly describe two. The first is to do with the introduction-rule for “ $\sim$ ”. The second is to do with the (classical<sup>86</sup>) elimination-rule for “ $\sim$ ”. The introduction- and (classical) elimination rules for “ $\sim$ ” are:

$$\begin{array}{ccc}
 [A], X & & \\
 \vdots & & \\
 \perp & & \sim\sim A \\
 \text{“}\sim\text{”-introduction: } \underline{\hspace{2cm}} & \text{“}\sim\text{”-elimination: } \underline{\hspace{2cm}} & \\
 \sim A & & A
 \end{array}$$

The “ $\sim$ ”-introduction rule says that if from A and some auxiliary premises X we can deduce a contradiction, then we can infer  $\lceil \sim A \rceil$ . (The square brackets “[A]” indicate that we discharge A as an assumption when moving to  $\lceil \sim A \rceil$ .) The “ $\sim$ ”-elimination rule says that from  $\lceil \sim\sim A \rceil$  we can infer A.

The first problem is that the rule for “ $\sim$ ”-introduction is arguably not purely inferential, because it contains “ $\perp$ ”, an expression for contradiction. But then if knowledge of the notion of a contradiction is counted as *specific* knowledge in the sense required by the notion of a purely

<sup>85</sup> We would need to precisify the ‘no new information’ requirement in order to for it to do substantial philosophical work. But given that my aim here is to show that the fictionalist faces certain unforeseen questions, rather than with whether they can definitively answer those questions, I will not try to precisify the ‘no new information’ requirement. But the interested reader may like to know that one way this requirement has been developed is as the notion of *harmony* between introduction- and elimination-rules; see (e.g.) Dickie (2010) for a description of the harmony requirement.

<sup>86</sup> The distinction between the classical and *intuitionistic* “ $\sim$ ”-elimination rules will be made in due course. The classical rule is indeed ‘classical’ in the sense that it is the presumed standard.

inferential rule then “ $\sim$ ”-introduction is not a purely inferential rule, and thus “ $\sim$ ” is not a logical expression. Now, there is a standard way of writing “ $\sim$ ”-introduction that does not contain “ $\perp$ ”, namely:<sup>87</sup>

$$\begin{array}{c}
 [A] \quad [A] \\
 \vdots \quad \vdots \\
 X \quad \sim X \\
 \text{“}\sim\text{”-introduction: } \underline{\hspace{2cm}} \\
 \sim A
 \end{array}$$

This version of “ $\sim$ ”-introduction violates the requirement that introduction-rules be purely inferential because it contains instances of the expression being characterized in both the premise and the conclusion. Either way, the standard version of “ $\sim$ ”-introduction appear to violate the requirement that logical expressions have purely inferential introduction-rules.

Now, perhaps this is not so serious. For even purely inferential introduction- and elimination-rules contain the horizontal line denoting the drawing of an inference, and this was allowed because the ability to draw inferences was assumed to be possessed by anyone capable of rational thought. Thus if “ $\perp$ ” also denotes a notion possessed by anyone capable of rational thought, then its presence in a statement of an introduction- or elimination-rule will not thereby prevent that rule from being purely inferential. Or, perhaps, we should not require that a purely inferential rule contain only a single instance of the expression whose meaning is being conveyed. Either way, a fictionalist who endorses inferentialism about logical expressions will need to say something along these lines if they are to retain “ $\sim$ ”-introduction. Having noted that the

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<sup>87</sup> This is the formulation in Cohnitz & Rossberg (2009), p.153.

inferentialist fictionalist has an obligation here, let us set aside worries about “~”-introduction and move to “~”-elimination.

The second problem is that the classical elimination-rule for “~” arguably conflicts with the picture of logical inference as the manipulation of antecedently possessed information. Classical “~”-elimination says ‘from  $\lceil \sim\sim A \rceil$ , infer  $A$ ’. But a statement of form  $\lceil \sim\sim A \rceil$  can arguably sometimes contain strictly *less* information than the categorical statement  $A$ . And if that is right then the inference from  $\lceil \sim\sim A \rceil$  to  $A$  can result in acquiring genuinely new information. Here is a sketch of why  $\lceil \sim\sim A \rceil$  can contain strictly less information than  $A$ .<sup>88</sup>

Suppose we have derived  $\lceil \sim\sim A \rceil$  via the introduction-rule for “~”. That rule is ‘if  $A$  along with assumptions  $X$  entails a contradiction, infer  $\lceil \sim A \rceil$  (and discharge  $A$ )’. Then the information we have is ‘the claim that  $\lceil \sim A \rceil$  (with  $X$ ) entails a contradiction itself entails a contradiction’, or more simply: ‘ $A$  (with  $X$ ) does not entail a contradiction’. But the information that  $A$  (with  $X$ ) does not entail a contradiction is – or at least, *can* be – strictly less information than that  $A$  is *true*. (Indeed this is particularly true for our fictionalist; our fictionalist believes that “ $7 + 5 = 12$ ” does not entail a contradiction, but nevertheless is not true.) But so long as  $\lceil \sim\sim A \rceil$  *can* contain less information than  $A$ , “~”-elimination is an inference-rule that does not conserve information.<sup>89</sup>

Thus we have two inferentialist reasons for rejecting “~” as a logical expression: its introduction rule is arguably not purely inferential, and its elimination-rule arguably does not conserve information. The line of reasoning from inferentialism about logical expressions towards a move away from *realism* begins with the observation that while classical “~”-elimination may not conserve information, *intuitionistic* “~”-elimination *does* conserve information. Intuitionistic

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<sup>88</sup> See Dickie (2010), p.164-165 for a full statement of the argument.

<sup>89</sup> Cf. Ibid., p.165.

“ $\sim$ ”-elimination is ‘from  $A$  and  $\lceil \sim A \rceil$ , infer  $\perp$ ’.<sup>90</sup> Replacing classical “ $\sim$ ”-elimination with its intuitionistic counterpart has the result that the law of the excluded middle fails:  $\lceil P \vee \sim P \rceil$  is no longer a logical truth for every  $P$ .<sup>91</sup>

Now, it has been argued (most notably by Michael Dummett) that giving up the law of the excluded middle for a given area of discourse is tantamount to moving away from realism for the subject matter of that discourse.<sup>92</sup> The idea is that realism about a given subject-matter requires regarding all statements about that subject-matter as either determinately true or determinately false (setting aside vagueness worries), irrespective of any ability of ours to ever come to know whether any such statement is true or false. To give up the law of the excluded middle is to allow that there may be statements that are neither determinately true nor determinately false. Note that the fictionalist is a primitivist about logical possibility *in general*, including the logical possibility of claims about concrete objects. Thus the rejection of the law of excluded middle to which that primitivism leads (or may lead) also potentially applies to claims about concrete objects: there may be claims about concrete objects which are neither determinately true nor determinately false. And then, if it is right to characterize realism as the thesis that the law of the excluded middle holds, the fictionalist is in danger of being committed to anti-realism about concrete objects.

There are at least three broad options for the fictionalist to resist the argument from primitivism about logical possibility to anti-realism about concrete objects that I have just sketched. The first is to resist the move from primitivism about logical possibility to inferentialism about logical expressions. Note though that this may reduce the fictionalist to table-thumping

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<sup>90</sup> Ibid., f/n 5.

<sup>91</sup> *Vague* predicates can seem to not be subject to the law of the excluded middle: perhaps “ $a$  is bald or  $a$  is not bald” is not true for borderline-bald people. But I explicitly set vagueness worries aside here.

<sup>92</sup> Dummett (1978), pp.227-230; cf. Dickie (2010), pp.166-7.

about the distinction between logical and non-logical expressions. The second is to resist the move from inferentialism to the rejection of the law of the excluded middle. This may require moving away from the natural picture of logical inference as the manipulation of antecedently possessed information, or the adoption of alternative introduction- and/or elimination-rules for “ $\sim$ ”.<sup>93</sup> The third is to resist the move from the rejection of the law of the excluded middle to anti-realism. This will require a characterization of realism that makes realism compatible with the failure of the law of the excluded middle. Now, it would take us too far afield to fully investigate these options to see which the fictionalist can adopt. But again my claim is not that the fictionalist is incapable of resisting this line of thought from primitivism about logical possibility to anti-realism, but merely that they *must* do so; and that this is not obvious given the characterization of mathematical fictionalism we saw in section 2. On the other hand, if the fictionalist gives up the semantic argument against mathematical platonism (read as a claim about conceptual possibility), then they do not need the distinction between logical and conceptual possibility that I described in section 4. Indeed, if the fictionalist gives up the semantic argument against mathematical platonism then they are not obliged to say *anything* about the difference (if any) between logical possibility and conceptual possibility. And then they do not need to endorse inferentialism about logical expressions.

## 7: Conclusion

I have argued that rather than motivating a move away from mathematical platonism and towards fictionalism, the semantic argument raises problems for the fictionalist. Or at least,

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<sup>93</sup> See Dickie (2010), pp.170-174 for a description of one version of this latter move.

endorsing the conclusion of the semantic argument requires the fictionalist to engage in debates that appear at first glance to be quite far removed from the initial concern of fictionalism, which was to explain how pure mathematical statements could be scientifically useful without being true. Therefore I conclude that if the fictionalist wants to retain their fictionalism *without* engaging in such debates, they should give up the semantic argument against mathematical platonism. In that case, the fictionalist should join the platonist in accepting that reference to abstract mathematical objects *is* possible, and in seeking after a viable account of how reference to abstract mathematical objects could work. The goal of the rest of this dissertation is to construct just such an account.



### Chapter Three

#### *Semantic Indeterminacy and Abstract Objects*

##### *Abstract*

In this chapter I consider an argument for the CCR – the claim that for an expression  $a$  to refer to an object  $o$ , it must be metaphysically possible for  $o$  to stand in at least one causal relation. The argument I will consider is a *semantic indeterminacy* argument: the lack of causal contact between ourselves and abstract mathematical objects – natural numbers in particular – means that there are infinitely many distinct yet equally correct interpretations of the language of arithmetic; that is, interpretations that make all the statements comprising the axioms of Peano Arithmetic and their logical consequences come out true. But then expressions from the language of arithmetic like “7” and “prime” are semantically indeterminate: “7” does not have a unique referent, and “prime” does not stand for a unique property. Hence statements containing those expressions, for instance “7 is prime”, cannot be true. I will explore the prospects of using David Lewis’s notion of a *reference magnet* to resist the charge that the language of arithmetic is semantically indeterminate. While my conclusions are programmatic rather than definite, I will draw an interesting connection between the notion of a reference magnet and the indispensability argument for mathematical platonism.

## Contents

- 1: Introduction
- 2: What Numbers Could Not Be
- 3: From Benacerraf to Semantics
- 4: Semantic Indeterminacy and the CCR
- 5: Indeterminacy and Arbitrariness
- 6: The Permutation Argument
- 7: Reference Magnets?
- 8: Conclusion

### *1: Introduction*

The causal constraint on reference (CCR) says that for an expression  $a$  to refer to an object  $o$ , it needs to be metaphysically possible for  $o$  to stand in at least one causal relation. This chapter is about one argument for the CCR. The argument goes roughly like this. Because no abstract object can stand in any causal relation, there is no fact of the matter about which abstract object is the referent of an expression like “7”. But if there is no fact of the matter about which abstract mathematical object “7” refers to, then “7” has no unique referent – it is *semantically indeterminate*. But if “7” has no unique referent, then it has no referent at all; it is an empty singular term. Thus we have the CCR. And now the semantic argument runs in the familiar way: if “7” is an empty singular term, then statements like “ $7 + 5 = 12$ ” cannot be true (when read at

face value). Thus mathematical platonism cannot be true.

This chapter investigates the prospects for resisting this semantic indeterminacy argument for the CCR. Here is the plan. The next two sections give a detailed description of one way to mount this semantic indeterminacy argument. The first section describes a well-known argument that numbers cannot be objects due to Paul Benacerraf.<sup>94</sup> The second section shows how to transform Benacerraf's argument from an argument that numbers cannot be objects into an argument that there is no unique correct interpretation of the language of arithmetic (hereon "LA"). That is, there is no uniquely correct assignment of objects, properties and operations to the singular terms, predicates and functional expressions of LA (respectively). Hence LA is semantically indeterminate. So even if numbers *are* objects (*pace* Benacerraf's original conclusion), we cannot refer to them. Section 4 observes that Benacerraf's argument is a specific instance of more general point, famously associated with Quine: that there are insufficient grounds for selecting a unique correct interpretation of *any* language whatsoever, no matter whether that language concerns (or purports to concern) abstract objects like numbers or concrete objects like birds, boats and bosons. Section 5 describes a response to this general semantic indeterminacy challenge: *causal* connections between our uses of expressions and concrete objects can serve as a basis on which to decide upon a (relatively) unique correct interpretation of a theory concerning concreta. This clarifies that it is our lack of causal contact with numbers that results in the semantic indeterminacy of LA. Hence we have an argument for the CCR: causal contact is required for avoiding semantic indeterminacy, and if LA is semantically indeterminate, then (S) – the claim that pure mathematical statements should be read at face value – and (A) – the claim that some

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<sup>94</sup> 'What Numbers Could Not Be' *The Philosophical Review* (1965), pp.47-73. Reprinted in Benacerraf & Putnam (eds.) (1983) *Philosophy of Mathematics: Selected Readings*. Page references are to the reprint.

pure mathematical statements that are not negative existentials are non-vacuously true – cannot both be true (for LA).

In section 6 I begin exploring how to resist this semantic indeterminacy argument. First I observe that the semantic adaptation of Benacerraf's argument described in section 3 itself provides the resources for ruling out many interpretations of LA, in particular, any set-theoretic interpretation. However there remains a more specific version of the semantic indeterminacy argument called the *permutation argument* which shows that there (still) are infinitely many equally correct arithmetical interpretations of LA. Thus the mathematical platonist needs a non-causal constraint on the correctness of an interpretation. In section 7 I explore the prospects of applying David Lewis's notion of a *reference magnet* to resist the permutation argument. While I will not mount a full defence of reference magnet theory, I *will* make an intriguing and (as far as I am aware) hitherto unnoticed connection between reference magnet theory and the indispensability argument for mathematical platonism: insofar as the notion of a reference magnet can be used to resist the permutation argument, reference magnet theory receives support from the indispensability argument. Thus while the conclusions I ultimately reach in this chapter will be programmatic rather than definitive, I hope to point the way towards fruitful areas of future research. Section 9 concludes this chapter by briefly recapping the discussion.

We begin by finding our way into the semantic indeterminacy argument for the CCR through an argument about the ontology of LA: Paul Benacerraf's well-known argument that numbers cannot be objects.

## 2: *What Numbers Could Not Be*

I will present Benacerraf's argument in terms of the logician's notion of an *interpretation*.<sup>95</sup> An *interpretation*  $I$  of the language  $L$  of a theory  $T$  in the logician's sense is an ordered pair  $\langle D, i \rangle$  where  $D$  is a set of objects for the quantifiers of  $L$  to range over and  $i$  a function assigning to the singular terms,  $n$ -place predicates and functional expressions of  $L$  members of  $D$ ,  $n$ -ary relations (i.e. properties when  $n = 1$ ), and  $n$ -ary operations, respectively.<sup>96</sup>

Consider the claim that the natural numbers are sets. If the natural numbers are sets then there is an interpretation  $I$  of LA where the domain  $D$  is a set of sets, such that this  $I$  is the *correct* interpretation of LA.<sup>97</sup> Consider now two well-known interpretations of LA in set theory: the Zermelo and von Neumann interpretations. Both take zero to be the empty set  $\emptyset$ ; i.e., both assign  $\emptyset$  to the singular term "0". The Zermelo interpretation takes each number to be the singleton of its predecessor; for example  $3 = \{2\} = \{\{1\}\} = \{\{\{0\}\}\} = \{\{\{\emptyset\}\}\}$ . (The domain is thus the set of all sets obtained by taking successive singletons beginning with the empty set.) The predicates and functional expressions of LA receive the interpretation required to make the axioms of Peano Arithmetic and all the logical consequences<sup>98</sup> thereof (hereon simply "PA") come out true. So, for example, the extension of the binary operation assigned to "+" includes the ordered triple required to make " $7 + 5 = 12$ " come out true (for the brave:  $\langle \{\{\{\{\{\{\{\{\emptyset\}\}\}\}\}\}\}, \{\{\{\{\{\emptyset\}\}\}\}\}, \{\{\{\{\{\{\{\{\{\{\{\{\{\{\emptyset\}\}\}\}\}\}\}\}\}\}\rangle$ ).

The von Neumann interpretation, on the other hand, takes each number to be the *union* of its predecessors. Hence  $3 = \{0, 1, 2\} = \{0, \{0\}, \{0, 1\}\} = \{0, \{0\}, \{0, \{0\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ . (The domain is thus the set of successive powersets beginning with the empty set.) Just as

<sup>95</sup> My presentation differs slightly from Benacerraf's in certain details. But the overall point will be the same.

<sup>96</sup> Cf. Enderton (2001), p.80/1. Formal semanticists often eschew talk of relations, properties and operations in favour of sets of ordered  $n$ -tuples. But I will speak with the vulgar.

<sup>97</sup> Benacerraf (1965/1983), p.280.

<sup>98</sup> In this chapter I will stay officially neutral on whether logical consequence is syntactic consequence, semantic consequence, or something else; no such decision should have any effect on the discussion.

with the Zermelo interpretation, the von Neumann interpretation assigns to each  $n$ -place predicate and functional expression the  $n$ -place relation or  $n$ -ary operation required to make the axioms of PA (and their logical consequences) come out true. (I hope the reader will forgive me for sparing them the details of the construction of an example.)

As mentioned above, if numbers are sets, then one interpretation of LA in set theory is correct. Benacerraf states that *if* there is a correct interpretation of LA in set theory, then there must be grounds on which we can *decide* which such interpretation is the correct one:

[I]f there exists such a “correct” account [i.e. interpretation], do there also exist arguments which will show it to be the correct one? Or does there exist a particular set of sets  $b$ , which is *really* the numbers, but such that there exists no argument one can give to establish that it, and not, say, [the domain of the Zermelo interpretation], is really the numbers? [...] No, if such a question has an answer, there are arguments supporting it, and if there are no such arguments, then there *is* no “correct” account [.]. (Benacerraf (1965/1983), p.281; emphasis original.)

What are the grounds on which we could decide in favour of any one set-theoretic interpretation of LA over any other, for instance, of the Zermelo over the von Neumann interpretation? One obvious such ground would be that the only correct interpretation of LA makes PA come out *true*. Consider for example the interpretation of LA whose domain is empty. Under this interpretation PA come out false. For it is a logical consequence of the axioms of Peano Arithmetic that the domain of the quantifiers of LA is not empty. So insofar as PA is supposed to be a true theory, the *correct* interpretation of LA is one that makes PA come out true. Thus we have at least one constraint on the correctness of an interpretation of LA, which I’ll call the *truth constraint*:

*Truth constraint*: an interpretation  $I$  of LA is correct only if it makes PA come out true.

The truth constraint allows us to disqualify the “empty” interpretation, as well as any

interpretation with a finite domain (because PA entails that there are infinitely many numbers). The truth constraint also disqualifies any interpretation under which the addition operation is not defined for every number, any interpretation where some number lacks a successor, and so on. Thus the truth constraint gives us grounds on which to disqualify a great many of the set-theoretic interpretations of LA as incorrect.

However the truth constraint does not give us sufficient grounds on which to disqualify either of the Zermelo or von Neumann interpretations in favour of the other. Both of those interpretations satisfy the truth constraint: they both make PA come out true. It is at this point that Benacerraf's argument that numbers cannot be sets comes into view. In the absence of any other finer-grained constraint on the correctness of an interpretation of LA, we are still faced with an 'embarrassment of riches': there are multiple distinct interpretations of LA all meeting the truth constraint (the Zermelo and von Neumann interpretations, plus (infinitely) many others<sup>99</sup>). But we've seen Benacerraf's claim that if numbers are sets, then not only must there must be a fact of the matter about which sets the numbers are, i.e. which set-theoretic interpretation of PA is correct; there must also be a cogent *argument* for the correctness of that interpretation over all others.<sup>100</sup> But there is no cogent argument from the truth constraint alone for the correctness of any set-theoretic interpretation of LA (meeting the truth constraint) over any other. Now, if every set-theoretic interpretation of LA is *equally* correct, then none is *uniquely* correct. If no set-theoretic interpretation of LA is uniquely correct, then there is no fact of the matter about which sets are the numbers. Therefore, numbers are not sets.

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<sup>99</sup> Benacerraf (1965/1983), p.279.

<sup>100</sup> "[I]f the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that it is an unknowable truth is hardly tenable. But there seems to be little to choose among the accounts. [...] If all the above is cogent, then there is little to conclude except that any feature of an account that identifies 3 with a set is a superfluous one – and that therefore 3, and its fellow numbers, could not be sets at all." (Ibid., p.284/5)

We get from the interim conclusion that numbers are not sets to Benacerraf's final conclusion that numbers are not objects of any kind as follows. Just as there are infinitely many set-theoretic interpretations of LA meeting the truth constraint, there are also infinitely many *non*-set-theoretic interpretations of LA meeting the truth constraint.<sup>101</sup> For example, there are interpretations of PA that take zero to be some given spacetime point and the successor function to be some suitable operation from points to points. Multiple such 'spacetime' interpretations of LA can be shown to satisfy the truth constraint.<sup>102</sup> Indeed, *any* collection of objects can be made the domain of some interpretation of LA that satisfies the truth constraint, provided that collection is big enough (i.e. at least countably infinite) and the right *n*-tuples and operations are assigned to the predicates and functional expressions of LA. And again, Benacerraf maintains that if any such interpretation is (uniquely) correct, there must be a cogent argument for its (unique) correctness. The truth constraint provides no such argument. Thus as long as the truth constraint is the only constraint on the correctness of an interpretation of LA, there is no uniquely correct interpretation of LA. Thus we have Benacerraf's conclusion that numbers are not objects of any kind.<sup>103</sup>

So far this is an argument about what kinds of things numbers are (or are not). We are interested in a slightly different question: given the mathematical platonist's thesis (M) that the referents of pure mathematical singular terms can only be abstract objects, how can we refer to them? The next section shows how to turn Benacerraf's argument about what numbers are (or are not) into an argument for the CCR, and hence ultimately into an argument that our pure

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<sup>101</sup> A non-set-theoretic interpretation of a language *L* is officially an interpretation assigning to at least one singular term of LA an object that is not a set.

<sup>102</sup> Cf. Shapiro (1983), p.526: "since the class of space-time points is isomorphic to  $\mathbb{R}^4$ , it is possible to model the natural numbers in space-time..."

<sup>103</sup> "I therefore argue, extending the argument that led to the conclusion that numbers could not be sets, that numbers could not be objects at all; for there is no more reason to identify any individual number with any one particular object than with any other (not already known to be a number)." (Ibid., p.291)



mathematical singular terms do not refer.

### *3: From Benacerraf to Semantics*

It is trivial that if Benacerraf's original argument proves its conclusion then our pure mathematical singular terms cannot refer to abstract objects – for our singular terms cannot refer to what does not exist (or so I am assuming in this dissertation<sup>104</sup>). But recall that the terms of debate are slightly different when it comes to the semantic argument against mathematical platonism. In the context of that debate the platonist is entitled to assume that is that abstract mathematical objects like the natural numbers *do* exist. Hence for us Benacerraf's original argument is slightly beside the point.

However it is not too hard to see how the proponent of the semantic argument can appropriate some of the resources found in Benacerraf's argument for their own purposes. We begin by giving Benacerraf's argument in a shortened premise-conclusion form as follows:

P1: If no interpretation of LA is uniquely correct, then numbers are not objects.

P2: No interpretation of LA is uniquely correct.

Therefore,

C: Numbers are not objects.

The first step toward turning this into an argument that none of our singular terms refer to abstract objects, that is consistent with the platonist's allowed assumption of (M) is to modify P1 (and hence also C) as follows:

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<sup>104</sup> Some people think that reference to nonexistent objects is possible, for instance in the case of proper names for fictional characters. This is the kind of claim I am setting aside for the purposes of this dissertation.

P1\*. If no interpretation of LA is uniquely correct, then no expression refers to any number.

P2. No interpretation of LA is uniquely correct.

Therefore,

C\*. No expression refers to any number.

Let us suppose, for now, that the opponent of reference to abstract objects can retain P2 unmodified. We thus turn to the question of what reasons the opponent of reference to abstract objects can give us for accepting P1\*. What is the connection between the existence of a uniquely correct interpretation of LA and our expressions being able to refer to natural numbers?

The opponent of reference to abstracta can mount an argument for P1\* by modifying Benacerraf's claim that if there is a fact of the matter about which objects are the numbers then we must be able to mount a cogent argument for this fact as follows. The modified version begins with the observation that it is a presupposition of classical or 'referential' semantics that if a singular term does not have a *unique* referent, then it has *no* referent.<sup>105</sup> Next we are told that in order for arithmetical singular terms like "7" and "the even prime"<sup>106</sup> to have *unique* referents, there must be a uniquely correct interpretation of LA: the uniquely correct interpretation of LA is the one assigning "7" and the like its (actual) referent.

The argument now runs as expected. We know that if the truth constraint is the only

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<sup>105</sup> This presupposition can be questioned. See (e.g.) Hartry Field 'Theory Change and Indeterminacy of Reference' (reprinted as chapter 6 of his *Truth and the Absence of Fact* (Clarendon Press: Oxford (2001)) for a dissenting view. Burgess & Rosen (1997), p.56/7 also outline the rudiments of a semantics not employing this presupposition.

<sup>106</sup> Recall from the introduction (sect. 2.1) that I am counting definite descriptions as singular terms. The resulting distinction between descriptively singular terms (like "the 44<sup>th</sup> president of the United States") and *singularly* singular terms (like "Barack Obama") is the focus of chapter 4.

constraint on the correctness of an interpretation of LA, then there is no unique correct interpretation of LA; LA is semantically indeterminate. If arithmetical singular terms do not uniquely refer, then they have no referents. And if arithmetical expressions have no referents, then well-known arithmetical ‘truths’ like “ $7 + 5 = 12$ ” cannot be true (on a face value reading). This undermines mathematical platonism.

The reader may be wondering what the connection is between this semantic indeterminacy argument against platonism and the CCR; for nothing has been said about causal contact (or a lack thereof). The next two sections reveal that connection. First I observe that the semantic indeterminacy argument as described applies not just to LA but to all languages, including those we use to talk about ordinary concrete objects like trees. Second I give a standard counter-response: the fact that we are in *causal contact* with concrete objects suffices for rendering semantically determinate the language(s) we use to talk about concrete objects. Thus it is our lack of causal contact with the natural numbers (as per the platonist thesis (M)) that results in LA’s being semantically indeterminate.

#### *4: Semantic Indeterminacy and the CCR*

Our semantic adaptation of Benacerraf’s argument trades on the idea that the *only* constraint on the acceptability of an interpretation of LA is the truth constraint. The truth constraint is not strong enough to deliver a unique correct interpretation of LA. Hence LA has no correct interpretation. But LA is not the only language that can be given multiple distinct interpretations satisfying the truth constraint.<sup>107</sup> Our languages concerning concrete objects can be given many distinct

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<sup>107</sup> Burgess & Rosen (1997), pp.53-4

interpretations satisfying the truth constraint.<sup>108</sup> (I here assume the truth constraint can be formulated generally so as to apply to all languages, not just LA.)

Suppose we have the following two candidate interpretations of the fragment of our language we use to talk about some kind of particulars – trees, for example. The first candidate interpretation,  $I_1$ , is the supposedly ‘natural’ one: it assigns to the predicate “tree” the property of being a tree, to “California” the state of California, and to “Hyperion” the tree Hyperion (the tallest living tree in the world). The second candidate  $I_2$  is a ‘gerrymandered’ interpretation. It assigns to “tree” the property of being a tortoise, to “California” the Republic of Seychelles, and to “Hyperion” the tortoise Jonathan (the world’s oldest living tortoise). Consider now the statement:

(H): Hyperion is a tree in California.

According to  $I_1$  a sincere literal assertion of (H) is true if and only if Hyperion is a tree in California. According to  $I_2$  a sincere literal assertion of (H) is true if and only if Jonathan is a tortoise in the Seychelles. As things turn out, both are true: Hyperion *is* a tree in California and Jonathan *is* a tortoise in the Seychelles. Moreover this agreement between  $I_1$  and  $I_2$  can be made totally general: for any statement  $P$  in the language we use to talk about trees,  $P$  is true according to  $I_1$  (plus the way the world is) if and only if  $P$  is true according to  $I_2$  (plus the way the world is).<sup>109</sup> Thus  $I_1$  and  $I_2$  both satisfy the truth constraint.

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<sup>108</sup> Classic references include Quine *Word and Object* (1960) and Davidson ‘The Inscrutability of Reference’ (1979). Quine took his thesis of the general semantic indeterminacy of all natural language to show that ontology is relative to language. Davidson took Quine’s argument to show that the thesis of ontological relativity cannot be truly expressed.

<sup>109</sup> Note that if the language we use to talk about trees contains *modal* statements, then  $I_1$  and  $I_2$  may not be equivalent in this way. This is because there are possible worlds where Hyperion exists but Jonathan does not, and vice-versa. But we can still generate alternatives to  $I_1$  displaying this ‘modal equivalence’. For instance, an interpretation assigning to “Hyperion” an object that exists if and only if Hyperion does, such as the set {Hyperion}, and so on for the other non-logical expressions of the tree-language. (Button 2013, p.24-5.)

Unless there is a way of restricting the indeterminacy charge so that it applies to LA but *not* to the language(s) we use to talk about other kinds of particulars, what we have is not a reason to doubt that we can refer to abstract mathematical objects like numbers in particular. Rather, what we have is reason to doubt that we can refer to any particulars whatsoever. If the truth constraint is the only constraint on the correct interpretation of any one of our languages, indeterminacy worries do not afflict platonism in particular, but realism about our ability to think and speak about particulars in general.<sup>110</sup>

This is the point at which the notion of causal contact enters the picture. An argument that a language *L* is indeterminate given along the lines just described requires the premise that the truth constraint is the only constraint on the correctness of an interpretation of *L*. But the realist about concrete objects can claim that the language(s) we use to talk about concrete objects are not subject only to a truth constraint, but also to some version of the following *causal constraint on correct interpretation*, or *CCI* for short:

CCI: if the entities (objects, properties and operations) *I* assigns to expressions in *L* do not stand in the right causal relations to our uses of those expressions, then *I* is not a correct interpretation of *L*.<sup>111</sup>

Two caveats: first, I assume there are no problems with regarding properties of objects standing in causal relations as standing in causal relations to our uses of predicates; likewise for operations

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<sup>110</sup> Wright (1983), section xv (especially p.127) argues that if Benacerraf's original (1965) argument succeeds in showing that numbers can't be objects then it must lead to general scepticism about all objects.

<sup>111</sup> E.g. Field (1989), p.24: "advocates of causal theories of reference would claim that causal considerations do much to constrain the reference of "rabbit", and it is precisely the fact that such causal considerations seem inapplicable in the case of numbers and sets that makes those so much more problematic." Cf. also Burgess & Rosen (1997), p.55: "'Adam' refers to Adam and 'Eve' to Eve, and not vice versa, because the relevant causal chains connect our use of 'Adam' with Adam, not Eve, and our usage of 'Eve' with Eve, not Adam." (Note that Burgess & Rosen are merely describing this proposal, not endorsing it.)

on objects standing in causal relations and our uses of functional expressions. Second, I say that interpretations of languages are subject to *some version* of the CCI because philosophers have proposed different accounts of what the ‘right causal relations’ are for grounding reference.<sup>112</sup> Obviously our concern is not with the *nature* of the role played by causation in fixing reference, but whether causal contact is a precondition for reference in the first place. Hence from now on I’ll speak of ‘the’ CCI, on the understanding that I mean whichever version of the CCI is generated by the reader’s favourite causal theory of reference.

Recall our competing interpretations  $I_1$  and  $I_2$  of the language we use to talk about trees. The CCI allows us to rule out  $I_2$  in favour of  $I_1$  because the entities  $I_1$  assigns to the expressions of our arboreal language stand in the right causal relations to our uses of those expressions, whereas the entities  $I_2$  assigns to those expressions do not. It is Hyperion, not Jonathan, that stands in the right causal relations (whatever they may be) to our uses of “Hyperion”; it is California, not the Seychelles, that stands in the right causal connections with our uses of “California”; and it is trees, not tortoises, that stand in the right causal relations to our uses of “tree”. Hence we can use the CCI to reject  $I_2$  in favour of  $I_1$  as the correct interpretation of the language we use to talk about trees.

Note also that this line of thought generates an argument from the CCI to the CCR. We’ve seen that the truth constraint alone is not enough to rule out, or even substantially restrict, indeterminacy. So if we want to say that there are domains of discourse in which we succeed in referring, and if appeal to the CCI is the only way to explain how this might be, that is a reason to think that the CCR should be accepted.

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<sup>112</sup> E.g. Evans (1973), Devitt (1981), Bach (1987), and Recanati (1993), to name but a few.

Let us return to the case of LA as construed by the mathematical platonist. The mathematical platonist takes the natural numbers to be *abstract* objects, metaphysically incapable of being spatiotemporally located or standing in causal relations. So the natural numbers cannot be causally connected in any way to anything. The platonist thus faces a dilemma. Either LA is subject to the causal CCI or it is not. If LA is subject to the CCI, then no candidate platonistic interpretation of LA is correct, because none satisfies the CCI: all the candidate interpretations assign abstract objects to the singular terms of LA. On the other hand, if LA is not subject to the CCI, then – for all we’ve said so far – the only constraint LA is subject to is the truth constraint. But as we’ve said, the truth constraint is not strong enough to render LA semantically determinate. So if *L* has no correct interpretation, then the singular terms of *L* do not have unique referents; which according to classical semantical presuppositions entails that those expressions have no referents at all.

We now begin asking after the options available to the mathematical platonist for resisting this line of reasoning.

### *5: Indeterminacy and Arbitrariness*

We’ve seen that the truth constraint fails to ensure that LA has a uniquely correct interpretation. We’ve also seen that the CCI cannot limit indeterminacy for LA in the same way as it might do for the language(s) we use to talk about concrete objects. Therefore if the platonist is to maintain that we are capable of referring to numbers, then they need a *non-causal* constraint on correct interpretation; one that is finer-grained than the truth constraint.

It turns out that the platonist can make use of a premise appearing in both Benacerraf’s

original argument and our semantic adaptation of it to go *some* of the way towards eliminating some interpretations of LA as incorrect. One common way of putting Benacerraf's original point is that the choice between the competing set-theoretic interpretations of PA is *arbitrary*. For example:

[I]f one wants to identify natural numbers with sets, it seems rather arbitrary which sets one picks. (Field (1989), p.20.)

We can firm this notion of 'arbitrariness' up a little bit as follows: an interpretation  $I$  of a language  $L$  is *equally arbitrary* to another interpretation  $I^*$  if and only if there is no good reason to choose  $I$  over  $I^*$  and vice-versa (thus *being equally arbitrary to* is a symmetric relation). I will leave this notion of a 'good reason' somewhat vague, but suffice it to say that matters of personal preference (e.g. "I was taught *this* reduction of arithmetic to set theory") do not qualify as good reasons. Using this notion of one interpretation being equally arbitrary to another we can lay out our adaptation of Benacerraf's argument as follows:

P1: If two interpretations  $I$  and  $I^*$  of a language  $L$  are equally arbitrary to each other, then neither is correct.

P2: Every set-theoretic interpretation of LA is equally arbitrary to every other.

Therefore

C1: No set-theoretic interpretation of LA is correct.

The analogue of Benacerraf's extension of his argument to all kinds of objects is as follows:

P3: Every interpretation of LA is equally arbitrary to every other.

Therefore



C2: No interpretation of LA is correct.

It follows from C2 that LA is indeterminate and thus we cannot refer to numbers.

Now, P1 of the above argument is tantamount to the following constraint on the correctness of an interpretation of a language  $L$ , which I'll call the *arbitrariness constraint*:

*Arbitrariness Constraint*: for every interpretation  $I$  of  $L$ , if there is another interpretation  $I^*$  such that  $I$  is equally arbitrary to  $I^*$ , then  $I$  is not a correct interpretation of  $L$ .

Here is how the platonist can use the arbitrariness constraint to rule out every *set-theoretic* interpretation of LA. While we'll ultimately see that the platonist needs to at least supplement the arbitrariness constraint with some further constraint on the correctness of an interpretation of LA, we will reach an important conclusion about what *kind* of objects a mathematical platonist anxious to avoid issues of semantic indeterminacy should take the natural numbers to be.

Some philosophers have noted that Benacerraf's original argument is, strictly speaking, consistent with the view that numbers are *sui generis* objects: objects that are not sets or objects any other kind, and whose only (genuine<sup>113</sup>) properties are the relational properties they have in virtue of the relations they stand in to other numbers: for instance being prime, greater than 17, being divisible by a Fibonacci number, and so on.<sup>114</sup> (Note that these are all *arithmetical* properties; this will be important very soon.) Recall that within the context of debates over the possibility of *reference* to abstract objects, the mathematical platonist is entitled to their assumption (M) that numbers do exist (see section 4 of the Introduction to this dissertation).

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<sup>113</sup> I insert the parenthetical "genuine" to allow that numbers can have non-arithmetical 'mere-Cambridge' properties like being such that Obama is president on the 22<sup>nd</sup> of July 2015. (See footnote 16 of chapter one.)

<sup>114</sup> For instance Balaguer (1998) p.78, Field (1998/2001) p.326-7, Linsky & Zalta (1995) p.23, and Maddy (1990) p.86 have all made this observation.

Seeing as the *sui generis* position is consistent with Benacerraf's ontological argument, the platonist can (and I think should) adopt the view that the natural numbers are *sui generis* objects that have only arithmetical properties. (However as Field notes,<sup>115</sup> the *sui generis* move may be more plausible for numbers than for so-called "algebraic" mathematical entities like groups. Hence it may not be advisable to make the *sui generis* move across the board. Maybe the arithmetical platonist should not be a platonist about *every* branch of mathematics. Nevertheless I will elide this detail in what follows.)

Note as we're going past that one important consequence of taking numbers to be *sui generis* in this sense is that we have a new argument for the component thesis (M) of mathematical platonism, as applied to natural numbers: it is metaphysically impossible for any natural number to stand in any causal relations. It seems that which kinds of causal relations an object is capable of standing in has something to do with which genuine properties it has. Objects with chemical composition H<sub>2</sub>O can cause certain things, objects with chemical composition HCl can cause certain other things. We could go one step further and say that objects have the causal powers that they have, and hence stand in the causal relations that they do, in virtue of having the properties that they do. Objects made of H<sub>2</sub>O can cause certain things *because* they are made of H<sub>2</sub>O; or perhaps more suggestively, being made of H<sub>2</sub>O *bestows* certain causal powers on the objects that have that property. (Indeed we could go a step further and say that properties just *are* clusters of (conditional) causal powers. This is Sydney Shoemaker's 'structuralist' view of properties.<sup>116</sup> But the point I'm about to make does not require this more extreme view.) Now, it is at least *prima facie* plausible that arithmetical properties, like being prime, less than 17, or a multiple of a

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<sup>115</sup> 'Mathematical Objectivity and Mathematical Objects' in *Truth and the Absence of Fact* (2001), p.326-7.

<sup>116</sup> Shoemaker (2003) 'Causality and Properties'.

Fibonacci number do not bestow any causal powers. (Though it is not completely clear that the intuition behind this is any different from the intuition that numbers are non-causal and only numbers can have arithmetical properties. But I'll leave this aside.) If that's right, and numbers are *sui generis* objects that can only have arithmetical properties – that is, anything that has even one non-arithmetical genuine property is by that token not a natural number – then it follows that numbers are necessarily incapable of standing in causal relations. For standing in a causal relation requires having a non-arithmetical genuine property, and we know from Benacerraf's argument that this is one thing numbers cannot do.

Note also as an interesting historical aside that if the platonist does make the move of regarding numbers as *sui generis* rather than being sets or classes or any other non-arithmetical mathematical object, then they are making a break with the logicist tradition associated with Frege, Russell, and some of the logical positivists (e.g. Hempel<sup>117</sup>). That tradition has tended to favour the identification of the natural numbers with sets or classes of certain kinds. Frege and Russell both took numbers to be equivalence classes of equinumerous sets.<sup>118</sup> The contemporary *neo-logicism* of Bob Hale and Crispin Wright has tended to follow suit.<sup>119</sup> The course I am recommending goes against this, and hence the platonism I am exploring cannot be exactly the same as the logicism of Frege, Russell, and their heirs.<sup>120</sup> (While I believe that neo-logicism about arithmetic is ultimately consistent with the move of taking the natural numbers to be *sui generis* objects, I don't have space to argue for that claim here.)

Let us return to matters of semantics. Recall that the indeterminacy argument from the

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<sup>117</sup> 'On the Nature of Mathematical Truth' (reprinted in Benacerraf & Putnam 1983, pp.377-93), section 8.

<sup>118</sup> This is a bit anachronistic, to be sure; Frege made no distinction between classes and sets. But such historical niceties should not detract from the point at hand.

<sup>119</sup> Wright 1983, section xv and Hale 1987, chapter 8 section II, both defend the claim that numbers are classes.

<sup>120</sup> Thanks to Imogen Dickie for pointing this historical observation out to me.

CCI to the CCR that we are considering proceeds, like Benacerraf's original argument, in two stages. The first stage establishes that every set-theoretic interpretation of LA is equally arbitrary to every other, and hence no set-theoretic interpretation can be correct. The second stage extends that conclusion to all interpretations of LA via the claim that every interpretation of PA whatsoever is equally arbitrary to every other (whatsoever). However once the platonist has adopted a *sui generis* conception of the natural numbers, they might wonder whether they can stall the semantic indeterminacy argument between the first and second stages, much as that same move stalls Benacerraf's original argument between its first and second stages. Perhaps they could argue that some *sui generis* interpretation of LA can be shown to be uniquely correct.

However it turns out that there is a version of the semantic indeterminacy argument that even the *sui generis* platonist about numbers is susceptible to. It is the *permutation argument*, which I will now describe.

### 6: The Permutation Argument

The *permutation argument* is so called because it generates new interpretations of LA by permuting, i.e. rearranging, the domain of a given interpretation and making compensatory adjustments to the assignments to the predicates and functional expressions of LA required for the new interpretation to satisfy the truth constraint. Consider the following two interpretations of LA. The first,  $I_1$ , assigns to each numeral the number that is twice that which the numeral is 'supposed' to stand for. The referent of "0" is 0, that of "1" is 2, that of "2" is 4, etc.  $I_1$  also assigns the right properties and operations to the predicates and functional expressions of LA to make (the logical consequences of the axioms of) PA come out true. Hence  $I_1$  assigns to "prime" the

property of being the double of a prime, to “successor” the operation of taking a successor and then doubling it, and so on. According to  $I_1$  the statement “the successor of 4 is prime” is true if and only if the double of the successor of 10 (i.e. 22) is the double of a prime (which it is: 22 is twice 11, which is prime.)

The second interpretation  $I_2$  assigns to each numeral the number that is five places ahead of the ‘intended’ referent. So “0” refers to 5, “1” to 6, “2” to 7, etc.  $I_2$  also assigns to the predicates and functional expressions of LA the requisite properties and operations to make PA come out true: it assigns to “prime” the property of being the result of adding 5 to a prime, to “successor” the operation of taking the successor of the number 5 places ahead, and so forth. According to  $I_2$  the statement “the successor of 4 is prime” is true if and only if the successor of the number five spots ahead of 4 (i.e. 10) is five more than a prime. (This is also true: 10 is five more than 5, which is prime.)

Here is the permutation argument in complete generality. Let  $p_1$  and  $p_2$  be arbitrary (but distinct) functions from numbers to numbers, from arithmetical properties to arithmetical properties, and from arithmetical operations to arithmetical operations. Let  $I_{p_1}$  assign to “1” the number  $p_1(1)$ , to “2”  $p_1(2)$ , and so on; to each arithmetical predicate “ $F$ ” the arithmetical property  $p_1(F)$ ; and each arithmetical operation  $f$  the arithmetical operation  $p_1(f)$ . Likewise for the interpretation  $I_{p_2}$ . Assume that  $I_{p_1}$  and  $I_{p_2}$  both make PA true but that for every  $n$ ,  $p_1(n) \neq p_2(n)$ . According to  $I_{p_1}$  the truth-conditions of “the successor of 4 is prime” are that the  $p_1$ (successor) of  $p_1(4)$  is  $p_1$ (prime). Similarly for  $I_{p_2}$ : “the successor of 4 is prime” is true if and only if the  $p_2$ (successor) of  $p_2(4)$  is  $p_2$ (prime).  $I_{p_1}$  and  $I_{p_2}$  are distinct – by hypothesis we have that  $p_1(4) \neq p_2(4)$  – and they both satisfy the truth constraint. Neither satisfies the CCI. Moreover, they are both *sui generis* interpretations – the domains of both are comprised only of *sui generis* numbers.

But it looks like there is no ‘good’ reason to choose one over the other. Hence  $I_{p1}$  is equally arbitrary to  $I_{p2}$ . So it looks like  $I_{p1}$  and  $I_{p2}$  both pass exactly the same constraints on the correctness of an interpretation of LA that have been introduced so far: they both satisfy the truth constraint but fail the causal and arbitrariness constraints. Hence neither can be uniquely correct. And this argument is completely general –  $I_{p1}$  and  $I_{p2}$  could be *any* two interpretations of LA. In particular,  $I_{p1}$  and  $I_{p2}$  could be any two *sui generis* interpretations of LA. Therefore, for all I’ve said so far, no *sui generis* interpretation of LA is uniquely correct. Call this instance of the permutation argument specific to *sui generis* interpretations of LA the *local* permutation argument against arithmetical platonism. The local permutation argument is bad news for the *sui generis* platonist.

The platonist needs to resist the local permutation argument. In order to do so, they need a non-causal constraint on the correctness of an interpretation of LA. In the rest of this chapter I will canvass one candidate (though all too briefly): a *reference magnet* constraint, adapted from the work of David Lewis.

### 7: Reference Magnets?

Lewis’s notion of a “reference magnet” is his well-known but controversial response to a semantic indeterminacy argument mounted by Hilary Putnam.<sup>121</sup> Putnam’s conclusion is the same as that of the generalized indeterminacy argument briefly floated in section 4 (though his method for generating new interpretations is different): there are no grounds on which to decide which of an indefinitely large range of interpretations of our mathematical, scientific and everyday languages

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<sup>121</sup> Putnam (1980) ‘Models and Reality’ (reprinted in Benacerraf & Putnam 1983, pp.421-44); Lewis (1984) ‘Putnam’s Paradox’, *Australasian Journal of Philosophy* 62:3, pp.221-36. Lewis did not himself use the term “reference magnet” in the latter article, but it has become a standard appellation for the notion I’m about to describe.

is correct, and hence there is no correct interpretation of those languages. Lewis's response to Putnam is that we can rule out most interpretations of our scientific and everyday languages as incorrect on the grounds that they fail to assign *natural* properties to the predicates of those languages. A *natural* property is a property shared by individuals if and only if those individuals are *objectively similar* to each other.<sup>122</sup> For example, two green things bear more objective similarity to each other than a previously examined green thing and a yet-to-be-examined blue thing. Hence the property of being green is a *more natural* property than the property of being grue.<sup>123</sup> (Note that Lewis takes naturalness to come in degrees; being green is more natural than being grue, which is in turn more natural than some even more gerrymandered property.) Thus the property of being green is a more eligible candidate assignment to the predicate "green" than that of being grue. And so we have a reference magnet constraint on correct interpretation:

*Reference Magnet Constraint:* given two interpretations *I* and *I\** of a language *L* that both satisfy the truth constraint and are equally arbitrary to each other, the one which assigns the more natural properties to the predicates of *L* is more correct.

Hence given the choice between an interpretation of the language we use to talk about green things that assigns the property of being green to "green" and another assigning the property of being grue to "green", such that both interpretations satisfy the truth constraint and are equally arbitrary, we can choose the first interpretation as more correct on the grounds that it is more natural.

Now, this is not the time or the place to mount a comprehensive defense of reference magnet theory against all comers. Suffice it to say that Lewis's view is not the orthodox one.

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<sup>122</sup> "Among all the countless things and classes there are, most are miscellaneous, gerrymandered, ill-demarcated. Only an elite minority are carved at the joints, so that their boundaries are established by *objective sameness and difference* in nature." (Lewis (1984), p.227; my emphasis.)

<sup>123</sup> Something is *grue* if and only if it is green and observed before the 1<sup>st</sup> of January 2020 (say) or it is blue and not observed before the 1<sup>st</sup> of January 2020.

Hence I will restrict my goal to defending a conditional thesis: *if* reference magnet theory holds, *then* reference to abstract mathematical objects – that natural numbers in particular – is possible. This at least gives the platonist an option in responding to the permutation argument.

Lewis proposes the notion of a reference magnet as an explicit alternative to a purely causal constraint on correct interpretation.<sup>124</sup> Hence Lewis intends the naturalness constraint to replace the CCI. We need not be so radical; we can retain both the CCI and naturalness constraints if we wish, though perhaps it will turn out that an interpretation of a language for concrete objects satisfies the CCI if and only if it satisfies the naturalness constraint (though I won't argue for that here). Moreover in another article<sup>125</sup> Lewis talks about the possibility that the addition operation is more natural than the “quaddition” operation (an operation differing from addition only when it comes to numbers above a certain threshold).<sup>126</sup> If that's right then the former is a more eligible candidate assignment to “+” than the latter. So it seems clear that Lewis thinks that a given entity's being referentially magnetic to a given expression in virtue of that entity's being more natural than its competitors does not require that that entity stand in any causal relations.

Here is a more detailed sketch of how a platonist can put the reference magnet constraint to work. We start by backing off slightly from the natural numbers and considering abstract mathematical objects in general. There is a good case for the claim that abstract mathematical objects from different theories are less objectively similar to each other than they are to objects from their home theories. For instance the numbers 7 and 5 are more objectively similar to each

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<sup>124</sup> Lewis (1984), p.227: “This constraint looks not to the speech and thought of those who refer, and *not to their causal connections to the world*, but rather to the referents themselves.” (Emphasis mine.) Note that Lewis uses the term “reference” for the relation between a predicate and the property it stands for as well as that between a singular term and its referent.

<sup>125</sup> ‘New Work for a Theory of Universals’ *Australasian Journal of Philosophy* 61:4 (1983), pp.343-77.

<sup>126</sup> *Ibid.*, p.375-6. The “quaddition” example is from Kripke (1982) *Wittgenstein on Rules and Private Language* Harvard University Press: Cambridge, MA.



other than either is to a graph. This is because those numbers share (genuine) arithmetical properties with each other that neither shares with any graph. Both are less than 17, for instance, while no graph is less than 17. Indeed, asking whether a given graph is less than 17 seems to be a category mistake. Hence there is at least one genuine property shared by 7 and 5 that neither shares with any graph. Now, it seems obvious that if two objects share more genuine properties with each other than either shares with another object  $z$ , then  $x$  and  $y$  are more objectively similar to each other than either is to  $z$ . If that is right, and 5 and 7 share more genuine properties with each other than either shares with any graph, then 5 and 7 are more objectively similar to each other than either is to any graph. But then if 5 and 7 are objectively similar to some degree, then they share some amount of natural properties. Therefore some properties of 5 and 7 are natural properties. (This argument would be much shorter if we took the categories of *genuine* property and *natural* property to be the same category. But I won't take a stand on the relationship between the two categories here.) And so, if some properties of 5 and 7 are natural properties, then perhaps the platonist can invoke the reference magnet constraint to begin ruling out interpretations of LA on the basis that they assign unnatural semantic values to the expressions of LA.

However this does not by itself go far enough for it to be completely effective against the local permutation argument. Recall that the local permutation argument introduced in section 6 is an argument constructed to show that even if we grant that the subject-matter of a domain of discourse is a particular domain of objects, we still face indeterminacy. For the case of LA the permutation argument, if it succeeds, shows that even if numbers are *sui generis*, there are multiple competing interpretations of LA and therefore no correct interpretation.

Hence what the platonist needs is a reason to think that some arithmetical properties are more natural *than other arithmetical properties*. Here is a sketch of a route toward one such

reason. The platonist could make the following appeal to intuition: the property of being prime is obviously less gerrymandered – hence more natural – than the property of being double-some-prime, just as the property of being green is less gerrymandered than that of being grue. Thus the property of being prime is more referentially magnetic than the property of being double-some-prime, and so, the interpretation of LA assigning to “prime” the property of being double-some-prime is not a correct interpretation. If this is on the right track – if the property of being prime really *is* more natural than the property of being double-some-prime – then the platonist can use the reference magnet constraint to resist the permutation argument.

Hence if the above appeal to intuition is on the right track – if some arithmetical properties and operations are more natural than others, and reference magnetism does not require causal contact – then the platonist can use the reference magnet constraint to argue that most of the interpretations generated by the permutation argument assign very unnatural arithmetical properties to the predicates of LA. Hence the scope of the semantic indeterminacy that LA is subject to is now drastically limited.

However this does not yet suffice to show that LA is not subject to *any* problematic indeterminacy. There may yet be multiple interpretations that are equally correct, *even* by the standards set by the reference magnet constraint. But note that this bare possibility does not by itself constitute a damaging case against mathematical platonism. Recall the CCI: the correct interpretation of a language is the one assigning the entities standing in the right causal relations to the right expressions of that language. While the CCI serves to rule out many deviant interpretations of the languages we use to talk about concrete objects, it does not by itself suffice to narrow down *unique* interpretations of those languages. For insofar as it is indeterminate whether it is individual time slices of Hyperion (the tallest living tree), or some sequence of

causally continuous time-slices of Hyperion (i.e. Hyperion's 'time-slice worm'), or Hyperion itself that stands in causal relations, the CCI will fail to narrow down a unique referent for "Hyperion" and "the tallest living tree". Indeed, so long as it is allowing that it is *more than one* of these candidates (or any others) that stands in causal relations to our uses of "Hyperion", the CCI will not narrow down the range of competing interpretations of our arboreal language to uniqueness.

Now, I am not raising this as a problem for the CCI. Rather I am pointing out that even the causal theorist should accept some degree of indeterminacy with respect to the possibly correct interpretations of the language(s) we use to talk about concrete objects. So then if there is residual indeterminacy present for LA even after the application of the reference magnet constraint, the presence of this residual indeterminacy need not present a special problem for the platonist. At least, not if the platonist can show that the indeterminacy afflicting LA is no *worse* than that afflicting the language(s) we use to talk about concrete objects. Now, whether this can be shown is something that remains to be seen; for now, the best the platonist can offer is a promissory note. Nevertheless it is worthwhile to note that the platonist's goal need not be the *complete* determinacy of LA, but only determinacy to whatever extent that the causal theorist believes suffices for us to talk and think about concrete objects.

Let us return to the platonist's contention that the property of being prime is more natural than the property of being double-some-prime. The obvious problem with that claim is that it looks like the platonist does not have much in the way of an *argument* for it. In the absence of such an argument, the idea that some arithmetical properties are more natural than others, and that this suffices for the kind of reference magnetism required to resist the permutation argument, is a

“metaphysical article of faith”.<sup>127</sup> The important thing to note here is that this is not a problem specific to reference magnetism about *sui generis* numbers and their properties. It is a problem for reference magnetism about all objects and properties whatsoever, including ordinary concrete objects and their properties. The claim that the property of being green is more natural than the property of being grue, in the way required for the former to be a more eligible candidate assignment to “green” than the latter, is just as much a ‘metaphysical article of faith’ as the platonist’s claim that the property of being prime is more natural than that of being double-some-prime and that this suffices for the former’s being a better assignment to “prime” than the latter. And if that is right, then abstract mathematical objects do not present any *special* problems when it comes to the determinacy of the language(s) we use to talk about particulars; any problems presented by mathematical abstracta are problems presented by concrete objects too. This while reference magnet theory does require some table-thumping, it does not require any *more* table-thumping from the mathematical platonist than from the general realist.

This shows that the platonist’s appeal to reference magnet theory to resist the permutation argument stands or falls with reference magnet theory in general. Nevertheless before concluding this chapter I will make an observation regarding the relationship between reference magnet theory and mathematical platonism. I have briefly sketched a defence of the conditional that *if* reference magnet theory is right, *then* the mathematical platonist can fend off the local permutation argument; and this may suffice, in conjunction with the move to taking numbers to be *sui generis* objects, for fending off indeterminacy arguments in general. Recall the indispensability argument from chapter one: if pure mathematical theories form indispensable parts of our best scientific theories (the truth of which we are committed to), then we are

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<sup>127</sup> The expression is from Button (2013) *The Limits of Realism* (Oxford University Press: Oxford), p.22.

committed to the truth of those mathematical theories. This is an argument for thesis (A) of mathematical platonism, restricted to the theories finding indispensable application in science: (some of) our pure mathematical theories are true. In conjunction with thesis (S) that we should read pure mathematical theories at face value, the indispensability argument provides support for platonism.<sup>128</sup> On the other hand, we've seen that the semantic argument *against* platonism can be supported with semantic indeterminacy arguments. And, finally, I have floated the suggestion that reference magnet theory can be used by the platonist to resist semantic indeterminacy arguments against platonism (at least, I have floated this claim for arithmetic). What this means is that if reference magnet theory is wrong, and there are no other ways for the platonist to resist indeterminacy arguments of the kind we have been discussing (though this is *not* something I have argued for – there may be alternative strategies that I have not addressed), then platonism is incorrect. And insofar as platonism is incorrect, face value readings of pure mathematical theories *cannot* be indispensable to our successful scientific practices. But by the same token, insofar as face value readings of pure mathematical theories *are* indispensable to our successful scientific practices, platonism must be correct – and if the only way platonism *could* be correct is if reference magnet theory were true, then the indispensability argument can be used to argue for reference magnet theory. From afar this is a surprising connection, and one that I am not aware that anyone else has articulated. So while I have not done anything more than sketch one possible route towards a defence of mathematical platonism from semantic indeterminacy, and have not examined all possible responses to the charge of semantic indeterminacy that may be available to the platonist, I nevertheless hope to have advanced the conversation somewhat.

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<sup>128</sup> Some philosophers, e.g. Hartry Field, believe the indispensability argument is the *only* good argument for platonism. (Field 1980, p.4.)

## 8: Conclusion

In this chapter I described one argument for the CCR: if we cannot get into causal contact with abstract mathematical objects, then our utterances of “ $7 + 5 = 12$ ” or “the successor of 4 is prime” are semantically indeterminate. One particularly resilient version of this argument is the permutation argument. I briefly sketched a resistance strategy for the platonist, centred on David Lewis’s notion of a reference magnet. This revealed an interesting connection between reference magnet theory and the indispensability argument: if reference magnet theory offers the platonist a response to worries about semantic indeterminacy, the local permutation argument in particular, and the indispensability argument supports platonism, then that latter argument supports reference magnet theory. Nevertheless I did not defend reference magnet theory in general. Hence for all I’ve said in this chapter the platonist’s position remains at least as tenuous as overall reference magnet theory.

In the next chapter I will set indeterminacy worries aside, and move to my final topic of discussion: the question of whether we can refer *singularly*, or only ever *descriptively*, to abstract mathematical objects. For we are supposing, with the proponent of the semantic argument against mathematical platonism, that there can be no causal contact with abstract mathematical objects. So the answer to this question – whether we can refer singularly to abstract mathematical objects – will turn out to be of wider significance for the theories of singular reference and singular thought: if we can refer singularly to abstract mathematical objects, there can be singular reference and singular thought in the absence of causal connection with the object referred to and thought about.

## *Chapter Four*

### **Singular Reference and Abstract Objects**

#### *Abstract*

In this chapter I take up the question of whether our expressions can refer *singularly* to abstract mathematical objects. After distinguishing singular from descriptive reference I show why the claim that we can only refer descriptively to abstract mathematical objects is damaging to mathematical platonism. I then give an argument for a causal constraint on singular reference: singular reference requires the possibility of singular thought, and singular thought requires acquaintance; but acquaintance relations are causal relations, hence acquaintance with abstract mathematical objects is not possible. Therefore singular thought about such objects is not possible. Thus the platonist needs a non-acquaintance-theoretic view of singular thought. However the best-known non-acquaintance-theoretic view of singular thought – semantic instrumentalism – is implausible because it entails voluntarism about singular thought: an agent can have singular thoughts about any given object just by deciding to do so. I then describe a view that aims to be a middle ground between acquaintance theory and instrumentalism: Robin Jeshion’s *cognitivism about singular thought*. I argue that cognitivism allows for singular thought about some natural and real numbers.

Then I consider three objections. The first is that ordinary speakers do not meet the cognitivist conditions for being able to have singular thoughts about numbers. I reply that mathematical experts meet these conditions, and this is good enough for the platonist. The second objection is that cognitivism entails that some perceptual demonstrative thought is descriptive. I

respond by recommending the move a *hybrid* cognitivism combining acquaintance theory for perceptual demonstrative thought with cognitivism for singular thought about objects that are not currently being perceived. The third objection is that the apparent instances of ‘acquaintanceless’ singular thought that motivate cognitivism are better explained by *future* acquaintance, which still rules out singular thought about abstract mathematical objects. I respond by arguing that it is implausible that future acquaintance is always sufficient for singular thought, and the best way to avoid this result collapses future acquaintance theory into cognitivism.



## Contents

- 1: Introduction
- 2: Singular Reference and Descriptive Reference
- 3: Singular Reference and the Semantic Argument
- 4: Cognitivism about Singular Thought
- 5: Names as Bearers of Significance: “ $\pi$ ” and “ $e$ ”
- 6: Do We Have Mental Files For Numbers?
- 7: Perceptual Demonstrative Thought About Insignificant Objects
- 8: Significance and Anticipated Acquaintance
- 9: Conclusion

### *1: Introduction*

Many philosophers distinguish *descriptive* reference from *singular* reference. The latter is often taken to require more direct or substantial contact with the referent than the former. So the distinction between singular and descriptive reference may be another place at which the anti-platonist can mount a version of the semantic argument against platonism. They could argue that even if it is possible to definitely describe abstract mathematical objects, the fact that such objects are non-causal and non-spatiotemporal means that our expressions cannot refer *singularly* to them. But if our expressions cannot refer singularly to abstract mathematical objects then the surface syntax of the statements “ $7 + 5 = 12$ ” and “ $e^{i\pi} = -1$ ” is misleading to at least some extent. For if

we can only definitely describe abstract mathematical objects then the actual logical form<sup>129</sup> of “ $7 + 5 = 12$ ” is not

$$(\dagger) \ulcorner \S(\alpha, \beta) = \gamma \urcorner.$$

Here  $\S$  ranges over binary functional expressions and  $\alpha$ ,  $\beta$  and  $\gamma$  range over singular terms. Rather it is something more like:

$$(\ddagger) \ulcorner \exists x \exists y \exists z [\Phi x \ \& \ Xy \ \& \ \Psi z \ \& \ x \neq y \neq z \ \& \ \forall w ((\Phi w \rightarrow w = x) \ \& \ (Xw \rightarrow w = y) \ \& \ (\Psi w \rightarrow w = z)) \ \& \ x + y = z] \urcorner.$$

Here  $\Phi$ ,  $X$  and  $\Psi$  range over open sentences with one free variable. But this opens the way for undercutting component thesis (S) of mathematical platonism: that we should read statements like “ $7 + 5 = 12$ ” at face-value. Hence it would be good for the platonist if they could defend the claim that our expressions can refer singularly to abstract mathematical objects. The purpose of this chapter is to mount such a defence.

Here’s the plan. I begin by distinguishing descriptive from singular reference. Then I flesh out the argument that if our expressions cannot refer singularly to abstract mathematical objects then this threatens component thesis (S) of mathematical platonism. I also note the following relatively uncontroversial connection between singular reference and singular *thought*: an expression  $a$  refers singularly to an object  $o$  if and only if token uses of  $a$  express singular thoughts about  $o$ .<sup>130</sup> Thus singular reference to abstract objects requires singular thought about abstract objects. Once these preliminaries are out of the way I describe two currently prominent theories

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<sup>129</sup> I am here using “logical form” in the sense of Davidson ‘The Logical Form of Action Sentences’ (reprinted in *The Essential Davidson* (2006)), where to give the logical form of a statement is to “describe it in a way that explicitly determines what sentences it entails and what sentences it is entailed by.” (p.64). See section 1 of chapter one for more discussion.

<sup>130</sup> Recanati (1993), p.54/5. I explicitly set aside indexical expressions like “I”, “here” and “yesterday”.

of singular thought: acquaintance theory and semantic instrumentalism. According to acquaintance theory, singular thought requires *acquaintance* with the referent. But acquaintance relations are usually regarded as causal relations. Hence acquaintance theories are hostile to the possibility of singular thought about abstracta. Semantic instrumentalist theories, on the other hand, seem to entail that we can have singular thoughts about any given object simply by *deciding* to do so. This looks very implausible. And so either way it looks implausible that we can have singular thoughts about, and hence is likewise implausible that any of our expressions refer singularly to, abstract mathematical objects.

I respond to this argument by describing a new theory of singular reference which makes room for singular reference to abstract mathematical objects without entailing the implausible consequences of semantic instrumentalism. This theory of singular reference is built around Robin Jeshion's *cognitivism* about singular thought (hereon simply "cognitivism"). After arguing that cognitivism facilitates singular thought to abstract mathematical objects – specifically, some natural and real numbers – I will consider three objections. First, ordinary speakers do not satisfy the cognitivist's requirements for singular thought about numbers. I reply that expert mathematicians satisfy those requirements, and given that the presence of experts in our linguistic community to which we can defer when using expressions like "7" and " $\pi$ " suffices for those expressions to refer singularly, this suffices for the platonist's purposes. Second, cognitivism entails that some perceptual demonstrative thought is descriptive. I respond by proposing a move to a hybrid theory that combines something like acquaintance theory for perceptual demonstrative thought with cognitivism about all other cases of singular thought. The third objection is that the apparent cases of acquaintanceless singular thought that Jeshion uses to motivate her cognitivism are better explained by a liberalized acquaintance theory, which nevertheless does not allow for

singular thought about abstracta. I argue that such liberal acquaintance theories either lead to implausible consequences or are indistinguishable from cognitivism.

I begin by distinguishing singular from descriptive reference.

## 2: Singular Reference and Descriptive Reference

Consider the following two statements:

(v) The tallest building in Toronto was built with the help of a Soviet helicopter.

(vi) *That* [points at the CN Tower]<sup>131</sup> was built with the help of a Soviet helicopter.

Both (v) and (vi) are ‘about’ the CN Tower in the sense that they are true (or not) in virtue of whether the CN Tower was built with the help of a Soviet helicopter. But this ‘aboutness’ seems to come about in different ways. (v) is about the CN Tower in virtue of that building’s satisfying a descriptive condition (“the tallest building in Toronto”). This is not the case for (vi). (vi) is about the CN Tower in virtue of something like the fact that the agent is attending to the CN Tower when thinking or uttering (vi). Now, the specific details of *how* that agent’s attending to the CN Tower manages to make their thought or utterance of (vi) true (or not) in virtue of the history of the CN Tower need not concern us. All that’s important for our purposes is that there *is* such a connection enabling (vi) to be about the CN Tower *without* the CN Tower’s satisfying a descriptive condition that is part of the content of (vi). This is one intuitive gloss on the distinction between descriptive and singular reference: statement (v) makes descriptive reference to the CN Tower, whilst (vi) makes singular reference to the CN Tower.

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<sup>131</sup> I am using David Kaplan’s (1977/1989) convention of using square brackets to describe physical gestures made uttering statements.

One way that we can philosophically precisify this intuitive distinction is by following Recanati<sup>132</sup> in distinguishing descriptive from singular *thought* in terms of a Russellian metaphysics of the propositional content of thought:

*[S]ingular thoughts* [are] thoughts that are directly about individual objects, and whose content is a singular proposition – a proposition involving individual objects as well as properties. (Recanati 2012, p.5; emphasis original.)<sup>133</sup>

A Russellian proposition is a structured entity comprised either of properties alone – as for instance in the proposition expressed by “ $\forall x[x \text{ is triangular} \rightarrow x \text{ is trilateral}]$ ” – or of both individuals and properties, as in the proposition expressed by “ $[a \text{ is triangular} \rightarrow a \text{ is trilateral}]$ ”. The former are *descriptive* or *general* propositions, the latter *singular* propositions. A singular thought is a thought whose content is a singular proposition. Descriptive thoughts are thoughts whose contents are descriptive propositions.<sup>134</sup> We can then distinguish singular from descriptive *reference* on behalf of an expression *a* in terms of singular thought by once again following Recanati, this time in taking it as uncontroversial that *a* refers singularly to an object *o* if and only if a mental tokening of “*a* is  $\Phi$ ” by a user of *a* expresses a singular thought about *o*.<sup>135</sup> (I explicitly set aside indexical expressions like “I”, “here” and “now”. Nothing I will have to say turns on any special features of indexicality; in order to facilitate discussion I will assume that this close tie

<sup>132</sup> In *Mental Files* (2012) Oxford: Oxford University Press.

<sup>133</sup> See also Jeshion (2010) p.108 for the same characterization. Jeshion follows Kaplan (1977/1989) and Russell (1910). There are, of course, other ways of doing this, for instance in terms of Fregean propositions and object-dependent versus object-independent senses. Nothing that matters for my purposes hangs on setting things up in a Russellian versus a Fregean framework. (Note that both frameworks assume that propositions are structured entities, as opposed to (e.g.) sets of possible worlds. I will not pause to defend this assumption.)

<sup>134</sup> I leave aside complications arising from statements containing functional names like “Nelson’s mother”. (“Nelson’s mother” contains a singular term “Nelson” but intuitively I have a descriptive thought by thinking “Nelson’s mother must have been a proud woman”.)

<sup>135</sup> “It is also fairly uncontroversial that, in literal communication, the proposition expressed coincides with the truth-conditional content of the speaker’s thought: the same state of affairs is represented by the speaker’s thought and by the utterance which (literally) expresses that thought.” (Recanati 1993, p.54/5.)

between singular reference and singular thought is universal.) Importantly, this entails that expressions purportedly referring to abstract mathematical objects like “7” and “the even prime” can refer *singularly* to such objects if and only if we are capable of having singular thoughts about abstract mathematical objects.

Having distinguished singular from descriptive reference in terms of the distinction between singular and descriptive thought, we now move to what that distinction has to do with the semantic argument against mathematical platonism.

### *3: Singular Reference and the Semantic Argument*

Once the distinction between descriptive and singular reference is made in terms of the distinction between singular and descriptive thought, we face the question of under what conditions an agent *S* can have singular thoughts about a given object *o*. One answer to this question is that *S* to have singular thoughts about *o*, *S* must be *acquainted* with *o*. There is room to quibble over what exactly acquaintance is, and what it takes to be acquainted with a given object. However Jeshion (2010) describes the ‘standard’ acquaintance constraint as functioning as follows:

One can be acquainted with an object *O* only by perception, memory, and communication chains. To have a singular thought about *O*, someone in one’s linguistic community must have perceived *O*. (Jeshion 2010, p.109.)

Call any theory of singular thought maintaining this condition an *acquaintance theory*. Recent acquaintance theories include Evans (1982), Bach (1987), Sawyer (2012), and Recanati (1993) and (2012).<sup>136</sup> And if we follow Grice (1961) in holding that for *S* to perceive *o*, it is necessary

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<sup>136</sup> This is not an exhaustive list.

that  $S$  and  $o$  stand in at least one causal relation, Jeshion's 'standard' acquaintance constraint entails that acquaintance requires causal contact.

The particular acquaintance theory that I discuss in this chapter is Recanati's (1993) and (2012) *Mental Files* based theory.<sup>137</sup> A mental file is a store of information an agent takes to be about a single object. According to Recanati mental files are devices for singular thought:  $S$ 's thinking of ' $\alpha$  is  $\Phi$ ' is a singular thought about  $o$  if and only if the mental file labelled  $\alpha$  – the " $\alpha$ -file" for short – that  $S$  deploys in thinking ' $\alpha$  is  $\Phi$ ' refers to  $o$ . Exactly which object a given mental file refers to is a function of the genetic origin of the information that file contains:  $S$ 's  $\alpha$ -file refers to  $o$  if and only if  $o$  is the dominant source of the information in that file. However Recanati has a specific notion of information in mind:

I rely on standard accounts of information, according to which there is no information without a causal link between the information-bearer and that which the information concerns. (Recanati 1993, p.119)

This entails that for  $S$  to have a mental file that refers to  $o$ , there must be a causal link between  $o$  and  $S$ . Recanati even calls these causal links 'acquaintance relations'.<sup>138</sup> Hence it looks like Recanati takes singular thought to be constrained by acquaintance. There are some nice questions about what exactly counts as acquaintance, if acquaintance is informative contact – am I acquainted with an airplane in virtue of perceiving its vapour trails? – but I won't address those here. What matters for us is whether singular thought requires causal contact of *any* kind. For if

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<sup>137</sup> Recanati (1993) *Direct Reference: From Language to Thought* and (2012) *Mental Files*. The theory presented in the latter book is "a sequel to that in *Direct Reference*" (p.viii). Hence the two theories share many of the same framework presuppositions, including the causal notion of information I'm about to describe. Thus I shall speak as if they form a unified theory.

<sup>138</sup> "The perceptual relation is what enables us to gain (perceptual) information from the object. In communication too we are related to the object we hear about, albeit in a more indirect manner (via communication chains). In general there is acquaintance with an object whenever we are so related to that object that we can gain information from it, on the basis of that relation." (Recanati 2012, p.20.)

singular thought about  $o$  requires causal contact with  $o$ , and singular reference to  $o$  requires the ability to have singular thoughts about  $o$ , then singular reference to  $o$  requires causal contact with  $o$ .

Recall the semantic argument against mathematical platonism: reference requires causal contact, but abstract mathematical objects are non-causal, hence reference to abstract mathematical objects is not possible. We now have the tools to mount a version of the semantic argument specific to singular reference. The first premise is a causal constraint on singular reference (hereon “CCSR”):

*Causal Constraint on Singular Reference*: in order for an expression  $e$  to refer singularly to an object  $o$ ,  $o$  must stand in at least one causal relation.<sup>139</sup>

If acquaintance relations are causal relations, then the CCSR follows from an acquaintance constraint on singular thought plus the singular thought requirement on singular reference. For example in Recanati’s mental files theory, an agent  $S$  can have singular thoughts about an object  $o$  if and only if  $S$  has a mental file that refers to  $o$ , and this requires causal contact between  $o$  and  $S$ . This in turn requires that  $o$  stand in at least one causal relation to something; thus we have the CCSR. The rest of the new, restricted semantic argument against mathematical platonism goes like this.

P1: CCSR: for an expression  $a$  to refer singularly to an object  $o$ ,  $o$  must stand in at least one causal relation.

P2: Abstract mathematical objects like the natural and real numbers cannot stand in any

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<sup>139</sup> Note that the CCSR does not need to be a ‘modalized’ constraint like the original CCR (the original CCR is: for an expression  $e$  to refer to an object  $o$ , it must be metaphysically possible for  $e$  to stand in causal relations).



causal relations.

Therefore,

C: Singular reference to abstract mathematical objects is impossible.

This spells trouble for the platonist because they believe claim (S) that pure mathematical statements should be read at face-value. Hence they believe that “7”, “5” and “12” are singular terms.<sup>140</sup> But this leaves it open whether they refer singularly or descriptively. If those expressions refer singularly, then the logical form of “ $7 + 5 = 12$ ” is something like:

$$(\dagger) \ulcorner \S(\alpha, \beta) = \gamma \urcorner$$

Here  $\S$  ranges over binary functional expressions and  $\alpha$ ,  $\beta$  and  $\gamma$  range over singular terms. This is what we should expect from the surface syntactical form of “ $7 + 5 = 12$ ”. But if “7”, “5” and “12” refer *descriptively* – if statements containing “7”, “5” and “12” are about those numbers in virtue of their satisfying some descriptive conditions specified by “7”, “5” and “12” – then the logical form of “ $7 + 5 = 12$ ” is not  $(\dagger)$ , but something like:

$$(\ddagger): \ulcorner \exists x \exists y \exists z [\Phi x \ \& \ Xy \ \& \ \Psi z \ \& \ x \neq y \neq z \ \& \ \forall w [(\Phi w \rightarrow w = x) \ \& \ (Xw \rightarrow w = y) \ \& \ (\Psi w \rightarrow w = z)] \ \& \ x + y = z \urcorner$$

Here  $\Phi$  is a descriptive condition uniquely satisfied by 7,  $X$  is a descriptive condition uniquely satisfied by 5, and  $\Psi$  is a descriptive condition uniquely satisfied by 12. This is a problem for the platonist because if we allow that the surface syntax of “ $7 + 5 = 12$ ” can mislead us as to its logical form to *that* sort of extent – if the syntactic form of that statement can suggest that its logical form

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<sup>140</sup> There is an interesting question about whether “12” is syntactically simple – it is after all the concatenation of “1” and “2”. However I will concentrate on showing that singular reference to *some* numbers is possible, and hence won’t address the status of “12” here. (But the issue will re-surface near the end of section 5.)

is ( $\dagger$ ), but its actually ( $\ddagger$ ) – then we are allowing that the surface syntactical form of a mathematical statement can fail to be a good guide to its logical form to at least *some* extent. But then this could be the thin end of a wedge whose thicker portions include the claim that the surface syntax of that statement misleads us to the extent that its logical form is *actually* something like:

$$(H): \forall X \forall Y [((\exists_5 x)Xx \ \& \ (\exists_7 x)Yx \ \& \ \sim(\exists x)(Xx \ \& \ Yx)) \rightarrow (\exists_{12} x)(Xx \vee Yx)].^{141}$$

(H) basically says that if you have five things, and seven other things, then you have twelve things in total. But if the logical form of “ $7 + 5 = 12$ ” is (H), then it does not entail “ $\exists x(x + 5 = 12)$ ”; and so the truth of “ $7 + 5 = 12$ ” does not require the existence of any abstract objects whatsoever. Hence (H) is a non-platonist reconstrual of “ $7 + 5 = 12$ ”. So if the platonist allows reconstrual of arithmetical statements to the extent that the logical form of “ $7 + 5 = 12$ ” is ( $\ddagger$ ), then they may be on the way to allowing reconstrual of arithmetical statements to the point where their truth neither proves nor requires the existence of numbers.

We should note here a certain subtlety regarding the above argument. The problematic conclusion that if one construes “ $7 + 5 = 12$ ” as having the logical form ( $\ddagger$ ), then we may as well *reconstrue* it as having the logical form (H), follows only if that reconstrual takes place at the level of what it takes for “ $7 + 5 = 12$ ” to be *true*. Someone who believed in something like a Fregean sense/reference distinction and held that the *senses* of “7”, “5” and “12” are descriptive – perhaps following from understanding of the Peano axioms – would not *necessarily* be forced by that consideration alone to conclude that the logical form of “ $7 + 5 = 12$ ” is that given by ( $\ddagger$ ) rather than ( $\dagger$ ). What matters is not whether for an agent *S* to *understand* uses of “7”, “5” and “12” *S* must grasp some descriptive condition, but whether “7”, “5” and “12” introduce *objects*

<sup>141</sup> The example is from Hodes (1984), p.140. “ $\exists_7$ ”, “ $\exists_5$ ” and “ $\exists_{12}$ ” are *numerical quantifiers*, defined in the standard way:  $\exists_n x(Fx) \Leftrightarrow_{def} \exists x_1, \dots, \exists x_n(Fx_1 \ \& \ \dots \ \& \ Fx_n \ \& \ x_1 \neq \dots \neq x_n \ \& \ \forall y(Gy \rightarrow y = x_1 \vee \dots \vee y = x_n))$ .

(rather than descriptive conditions) as relevant to the truth or falsity of statements in which they occur. It may be possible to maintain that not all expressions with descriptive senses are disguised definite descriptions.<sup>142</sup> (Though I won't defend that tantalizing claim here.) So officially a descriptively referring expression is an expression whose contribution to characterizing the way the world must be for a statement in which it occurs to be true is in the first instance a definite description (and which only contributes an object in a secondary sense). A singularly referring expression is an expression whose contribution to characterizing the way the world must be for a statement it occurs in to be true is in the first instance an object. (Thus 'descriptively referring expression' and 'singularly referring expression' are semantic categories, while 'singular term' is (for me) a grammatical category. A singular term can thus refer singularly or descriptively. Our question is that of which semantic category the likes of "7" and " $\pi$ " belong, given that they are singular terms.)

We've just seen that if the platonist allows that the surface syntactical form of " $7 + 5 = 12$ " (and the like) to deceive us to the extent that their logical forms are better given by ( $\ddagger$ ) rather than ( $\dagger$ ), then they may be at the top of a slippery slope, the bottom of which is a non-platonist reconstrual of " $7 + 5 = 12$ " not requiring the existence of any mathematical objects. It is for this reason that I think the platonist should try to defend the claim that the likes of "7" and "5" are singularly referring expressions that directly contribute *objects* as relevant to the truth or falsity of the statements in which they occur, rather than descriptively referring ones.

Note as we're going past that the reference-magnet view described in the previous chapter

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<sup>142</sup> Such a view can be maintained in the framework of Kaplan (1977/1989). In that framework for *S* to understand uses of "I", *S* must grasp the descriptive condition "the speaker of this utterance of 'I'", but uses of "I" nevertheless introduce *objects* (namely speakers) as relevant to the truth or falsity of statements in which they occur (e.g. "I am hungry"), *not* the descriptive condition "the speaker of this utterance of 'I'". ("[T]he descriptive meaning of a directly referential term is no part of the propositional content" Kaplan 1977/1989, p.497.)

will not help the platonist resist the above use of the CCSR. For reference-magnet theory is consistent with global descriptivism, the view that all reference is descriptive reference<sup>143</sup>. But if reference-magnet theory is consistent with global descriptivism, then all the platonist's use of reference-magnet theory can get them is that there is reference to abstract mathematical objects *of some kind* – but it won't deliver any conclusions about what the nature of that reference is, i.e. whether it is singular or descriptive. So if the platonist is to resist the above use of the CCSR, they need to say something in addition to endorsing reference-magnet theory.

### *3.1: Descriptive Names*

Recall the upshot of Jeshion's characterization of the 'standard' notion of acquaintance: we can only be acquainted with entities that have been perceived by members of our linguistic community. However there are cases where ordinary speaker intuition appears to push us in the direction of the possibility of singular thought in the absence of perception by any member of the relevant linguistic community.

Consider the following slightly fictionalized version of the story of Urbain Le Verrier's discovery of the planet Neptune.<sup>144</sup> On the 31<sup>st</sup> of August 1846 Le Verrier announced that his calculations concerning certain observed irregularities in the orbit of Uranus predicted the existence of a hitherto unobserved planet as the cause of those irregularities. This new planet was subsequently observed by telescope, in the location predicated by Le Verrier, on the 23<sup>rd</sup> of September 1846. Suppose now that Le Verrier coined the name "Neptune" prior to that observation, thinking to himself "I hereby dub the cause of the irregularities in the orbit of Uranus,

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<sup>143</sup> Lewis was himself a global descriptivist: cf. (1984), p.227.

<sup>144</sup> The example was first used in Kripke (1980) *Naming and Necessity*.

which my calculations predict will be in location *L* on the 23<sup>rd</sup> of September 1846, *Neptune*.” Or perhaps more colloquially: “there’s a planet there [points to the inscribed output of his calculations], I’ll call it *Neptune*.” (This is the fictionalized part. Neptune wasn’t so named until December 1846, and even then not by Le Verrier, who wanted to name the planet after himself.) Immediately after this naming ‘ceremony’ Le Verrier thinks to himself (the 19<sup>th</sup> century French translation of):

(DN): My discovery of Neptune will make me famous.

In thinking (DN), Le Verrier uses “Neptune” to refer to Neptune. So Le Verrier has managed to coin a singular term, indeed a proper name, for Neptune without being acquainted with Neptune. Philosophers have called expressions which function grammatically as proper names but whose reference is fixed by (and full understanding of which requires association of that name with) a definite description *descriptive names*.<sup>145</sup> Note that while an expression may begin life as a descriptive name it could become an acquaintance-based proper name (i.e. a name full understanding of which requires acquaintance with the referent) if members of the community using it become acquainted with its referent. So Le Verrier used “Neptune” as a descriptive name when thinking (DN) prior to the 23<sup>rd</sup> of September 1846. But after the planet was observed, “Neptune” became an acquaintance-based name; and insofar as Le Verrier was able to use “Neptune” as an acquaintance-based name, he was acquainted with Neptune either through perception via telescope or being communicatively linked to someone so acquainted with Neptune.

(There is a small subtlety here to do with Recanati’s mental files-based acquaintance

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<sup>145</sup> E.g. Evans (1982), p.31; Jeshion (2004), p.591, Recanati (1993), p.107 and (2012), chapter 12.

theory. Officially Recanati's theory requires causal contact with *o* for singular thought about *o*. And arguably Le Verrier *was* in causal contact with Neptune prior to observing it, via the eccentricities in Uranus's orbit. However Recanati appears to hold that Le Verrier was not acquainted with Neptune prior to observing it (though he does admit that acquaintance may be a matter of degree).<sup>146</sup> I will assume throughout the rest of this chapter that Le Verrier was not acquainted with Neptune prior to observing it.)

Given that the mathematical platonist holds that we cannot be acquainted with abstract mathematical objects, it seems that they should probably hold that expressions like "7" and "5" are descriptive names. Thus the question of whether "7", "5" etc. are singularly referring expressions is an instance of the more general question of whether descriptive names can be singularly referring expressions.

Now, prior to the 23<sup>rd</sup> of September 1846, no member of Le Verrier's linguistic community was acquainted with Neptune. So according to acquaintance theories, he could not have had a singular thought about or referred singularly to Neptune when thinking (DN). Hence "Neptune" must have been a descriptively referring expression (though it would subsequently become a singularly referring expression, once that planet was observed). So according to acquaintance theories the logical form of (DN) is not:

⌈My discovery of *a* will make me famous⌋.

(Here *a* is a singularly referring expression). Rather, because "Neptune" is a descriptively referring expression, the logical form of (DN) is something like:

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<sup>146</sup> Recanati (2012), p.151, p.154/5. On p.158 he says that Le Verrier knew Neptune "only by description".

$\ulcorner \exists x[\Phi x \ \& \ \forall y(\Phi y \rightarrow y = x) \ \& \ \text{my discovery of } x \text{ will make me famous}] \urcorner$ .

( $\Phi$  will be a descriptive condition like “causes the eccentricities in the orbit of Uranus”.) However Jeshion contends that ordinary speaker intuition pushes us in the direction of the claim that Le Verrier *did* have singular thoughts about Neptune when thinking (DN) prior to the 23<sup>rd</sup> of September 1846.<sup>147</sup> Note that Recanati agrees about the existence of this intuitive pull (though not about its significance).<sup>148</sup> And if this ordinary speaker intuition is right, then descriptive names can be used to have singular thoughts, and hence singular thought does not require acquaintance. This looks like good news for the mathematical platonist anxious to defend themselves from the CCSR-based argument described above.

Now, the *problem* with the claim that expressions like “7” and “ $\pi$ ” (and “Neptune” prior to the 23<sup>rd</sup> of September 1846) are singularly referring expressions is that it seems to require a view of singular thought that not many philosophers have found plausible: *semantic instrumentalism*.

### 3.2: *Semantic Instrumentalism*

I follow Jeshion (2009, 2010) in taking a *semantic instrumentalist* theory of singular thought to be any theory entailing that there are no substantial constraints on singular thought.<sup>149</sup> I say ‘substantial’ here because according to the instrumentalist view I’m about to introduce one must have a certain amount of linguistic *competence* in order to have acquaintanceless singular

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<sup>147</sup> Jeshion (2010), p.117/8.

<sup>148</sup> Recanati (2012), p.154. In later sections of this chapter we will see Recanati liberalizing his acquaintance constraint on singular thought, e.g.: “I am now open to the view that, in *certain* cases [...] one can think a singular thought by mentally tokening a descriptive name.” (2012, p.157.)

<sup>149</sup> Cf. Jeshion 2010 p.106.

thoughts. I will also assume an existence constraint: one cannot have singular thoughts about an object that doesn't exist. But given that the mathematical platonist is entitled to assume that abstract mathematical objects exist when defending themselves from semantic arguments against their position this shouldn't affect the discussion too much. Nevertheless these are not 'substantial' constraints in the sense that an acquaintance constraint is supposed to be substantial.

The best-known semantic instrumentalist theories are those described by Harman (1977) and Kaplan (1977/1989). (The differences between those theories are minimal, though Kaplan's is the more developed one.) According to those views, an agent *S* can get into the position to have singular thoughts about *o* solely by formulating a definite description "*D*", uniquely satisfied by *o* and then thinking to themselves "let '*dn*' refer to the satisfier of *D*". (Perhaps *S* also has to be justified in believing that *D* has a unique satisfier, but I won't pause to dwell on whether this counts as a 'substantial' constraint in the relevant sense.) Then if *S* goes on to think ' $\ulcorner dn \text{ is } \Phi \urcorner$ ', *S* has a singular thought about *o*.<sup>150</sup>

Before moving on to what's wrong with semantic instrumentalism I want to vaguely gesture in the direction of one thing that may be right about it. Kaplan's semantic instrumentalism grew out of his investigations of a non-Fregean theory of reference: one where some expressions refer directly to their referents, as opposed to via an intermediary 'sense'. He was particularly interested in demonstrative expressions like "that", use of which require supplementation with an

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<sup>150</sup> "There is nothing inaccessible to the mind about the semantics of direct reference, even when the reference is to that which we know any by description. What allows us to take various propositional attitudes toward singular propositions is not the form of our acquaintance with the object but is rather our ability to manipulate the conceptual apparatus of direct reference." (Kaplan 1977/1989, p.536) Also: "If Mary believes there is a certain unique thing satisfying certain conditions *C*<sub>1</sub>, *C*<sub>2</sub>, *C*<sub>3</sub>, she can introduce a new mental name *a* into her system by forming the beliefs that *a* is *C*<sub>1</sub>, that *a* is *C*<sub>2</sub>, and that *a* is *C*<sub>3</sub>. This name functions as a name of the unique thing satisfying these conditions if there is one; otherwise it does not name anything. Moreover, the name continues to name this thing, as long as Mary uses it, even if nothing or something different should be[come] the unique thing satisfying those of her beliefs involving the name *a*." (Harman 1977, p.174)



element of the context of use to refer – the most familiar such element being a pointing gesture.<sup>151</sup> The role of the pointing gesture is to single something out as the referent of a use of “that”. Importantly a pointing gesture is not itself part of the content of a use of “that” – the only content is the relevant object itself. The content of “that [points at the CN Tower] was built with the help is a Soviet helicopter” is not for Kaplan that the thing I am pointing at was built with the help of a Soviet helicopter. (The actual story is a lot more complicated, involving Kaplan’s well-known distinction between content and ‘character’<sup>152</sup> – the context-invariant rules for determining the referents of uses of context sensitive expressions, such as ‘the referent of a use of “that” is the object demonstrated’ – but in the interests of brevity I won’t get into those details.) Note that pointing is not the only way to single out an object for demonstration. One can nod toward the object, or perhaps not gesture toward it at all if it is already salient to the context; if it is creating a lot of noise, for instance (in which case the object may not even need to be visible to be demonstrated). And perhaps more controversially one can single out an object for demonstration by gesturing at something else, such as by holding up a book and saying “this author is really very good”. (Some call this ‘deferred’ demonstration.<sup>153</sup>) If this is correct then it seems that what matters to successful use of a demonstrative is not how one singles out an object, but that one manages to do so.

Now, all parties agree that demonstrative expressions can be used to give a name to a demonstrated object. For instance:

(C1) That ship [points] shall be called “HMS Canterbury”.

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<sup>151</sup> Kaplan (1977/1989), p.492.

<sup>152</sup> Ibid., p.505-507.

<sup>153</sup> E.g. Quine (1969), p.39-40 (though he uses the phrase ‘ostension’ instead of ‘demonstration’); Borg (2004), p.155/6.

The role of the demonstrative is to single out a unique object (a ship in this case); the rest of the statement bestows the name “HMS Canterbury” on the object singled out by the demonstrative. If the naming ceremony is successful, then the speaker and audience members have gained a new ability: they can go on to use the name “HMS Canterbury” to refer to the relevant ship without that ship being present. We might describe this situation using the suggestive metaphor of ‘cognitive restructuring’: the successful introduction of a name ‘restructures the cognition’ of the dubber and their audience.

Now, we might think that just as successfully demonstrating an object does not necessarily require pointing at the object, the physical act of pointing at the object is not essential to gaining the ability to use a name to refer to it in its absence. In the (C1) case the speaker could have nodded toward the ship instead. Or the ship could be named without being gestured at, or even being visible, if it is otherwise salient to the context; for instance, if it were undergoing a very noisy test-run of its guns. And, perhaps, the speaker could successfully name the ship by pointing not at it, but at the relevant entry in the ship delivery schedule, and uttering “that ship shall be called “HMS Canterbury””.

Hence it seems that if what matters to the successful demonstration of an object is not how one singles that object out, but that one manages to do so, then the same point seems to carry over to naming: what matters to successful naming is not how one singles out the future bearer of the name, but that one manages somehow to do so. But demonstratives are not the only way to single objects out uniquely. Descriptions can be used to do that too. Consider now an apparent bestowing of a name using a definite description, such as:

(C2) The ship that will be delivered tomorrow shall be called “HMS Canterbury”.

Here the description “the ship that will be delivered tomorrow” singles out a unique object (provided there is one and only ship that gets delivered the day after utterance). If that’s right, then the demonstrative and the description both single out a unique ship. But then if all that is required for bestowing a name on an object is that it be singled out uniquely (and an appropriate phrase uttered) then it might seem plausible that one should be able to introduce names using definite descriptions. And in this case introducing a name using a definite description accomplishes the same ‘cognitive restructuring’ as introducing a name by demonstration. This at least appears to have been Kaplan’s view at one point:

The introduction of a new proper name by means of a dubbing in terms of description [is a] mechanism [...] for providing direct reference to the denotation of an arbitrary definite description – [it] constitute[s] a form of cognitive restructuring; [it] broaden[s] our range of thought. (Kaplan 1977/1989, p.560 f/n 76.)

Thus the feature of semantic instrumentalism that we are interested in – the thesis that descriptive names are (or at least *can be*) singularly referring expressions – can perhaps be motivated by one way of considering how demonstrative phrases like “that” interact with features of the context of utterance in order to refer without intermediary Fregean senses.

Nevertheless semantic instrumentalism has been widely judged implausible.<sup>154</sup> Here’s one reason why.

One thing that most criticisms of semantic instrumentalism share is that they take semantic instrumentalism to make singular thought implausibly easy to attain.<sup>155</sup> One way to put this is that semantic instrumentalism appears to entail *voluntarism* about singular thought: to have a singular

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<sup>154</sup> “Almost all theorists think that Semantic Instrumentalism is false – indeed, wildly off.” (Jeshion 2010, p.106-7.)

<sup>155</sup> Ibid., p.107.

thought about a given object, it suffices that one *choose* to do so (provided one is a competent enough speaker to be able to introduce a descriptive name for that object, and that the object can be definitely described).<sup>156</sup> If an agent decides they want to have singular thoughts about a given object, all they need to do is introduce a descriptive name for that object using the ‘dubbing’ process analogous to (C2) described above; they thereby put themselves in a position to have singular thoughts about that object. But it is implausible that I could have singular thoughts about the satisfier of some arbitrary definite description – such as the satisfier of “the pebble whose mass is closest to the average of the masses of all the pebbles on the oldest beach in the Earth’s eastern hemisphere” – just by *choosing* to do so. Hence insofar as voluntarism about singular thought is implausible, the fact that semantic instrumentalism entails it is a *reductio* of the view.

So far we have seen two theories of singular thought. Let us now return to the question of singular thought about abstract mathematical objects.

### 3.3: Singular Thought and Abstract Objects

Here is why the stand-off between semantic instrumentalism and acquaintance theory is bad news for the mathematical platonist. Acquaintance theories do not allow singular thought about abstracta because acquaintance relations are causal relations and abstracta cannot stand in causal relations. So only semantic instrumentalist theories allow singular thought about abstracta. But as we saw above, semantic instrumentalist theories make singular thought implausibly easy to attain. Thus semantic instrumentalism must be wrong. Therefore we cannot have singular thoughts about abstract mathematical objects. And given that singular reference requires singular thought, our

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<sup>156</sup> Ibid., p.125.

expressions cannot refer singularly to abstract mathematical objects.

The rest of this chapter is about the prospects of using Robin Jeshion's *cognitivism* about singular thought to resist the above argument.<sup>157</sup> After briefly describing the main features of cognitivism I will show how to argue from one component of cognitivism, the *names as bearers of significance* principle,<sup>158</sup> to the possibility of singular thought about, and hence singular reference to, abstract mathematical objects. Then I consider three objections. The first is that ordinary speakers do not satisfy the cognitivist requirements for singular thought about abstract mathematical objects. The second is that cognitivism entails that some perceptual demonstrative thoughts – thoughts involving perceptual attention to an object and (possibly) the use of a demonstrative phrase like “this” or “that” – are descriptive thoughts. The third is that the apparent cases of acquaintanceless singular thought that motivate cognitivism (such as the fictionalized Le Verrier case from section 3.1) are better accounted for by a *liberalized* acquaintance constraint that still precludes singular thought about abstracta.<sup>159</sup>

#### 4: *Cognitivism about Singular Thought*

Robin Jeshion's cognitivism about singular thought is in many ways a successor view to Kaplan's instrumentalism view. In particular it retains the notion that the kind of ‘cognitive restructuring’ that accompanies gaining the ability to have singular thought does not require being acquainted with the object of reference. However Jeshion differentiates her view from Kaplan's on one

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<sup>157</sup> The primary expositions of Jeshion's cognitivist view are in ‘The Significance of Names’ *Mind and Language* (2009) and ‘Singular Thought: Acquaintance, Semantic Instrumentalism, and Cognitivism’ (2010) in Jeshion (ed.) *New Essays on Singular Thought* (2010a). Precursors to the view can be found in ‘Acquaintanceless *De Re* Belief’ (2002) and ‘Descriptive Descriptive Names’ (2004).

<sup>158</sup> (2009), p.374; (2010), p.136.

<sup>159</sup> Cf. Recanati (2010) section 7 and (2012) chapter 6.

important point: it seeks to avoid the voluntarism about singular thought that makes Kaplan's instrumentalism implausible. According to cognitivism, whether an agent *S*'s thoughts about an object *o* are singular or descriptive is *not* under *S*'s voluntary conscious control. Rather, if the conditions for singular thought are present, then *S*'s thoughts about *o* will be singular, whether *S* likes it or not.<sup>160</sup>

Thus Jeshion needs an account of the conditions under which *S* is able to have singular thoughts about *o*. She observes that, as a matter of fact, we do not *name* every individual we encounter.<sup>161</sup> She then asks what the individuals we do name have in common.<sup>162</sup> She argues that what is common to all the individuals we name is that they are *significant* to us. (We follow Jeshion in leaving the relevant notion of 'significance' unanalyzed, though what a good analysis requires will become important in section 8.) Hence she takes *o*'s being significant to *S* as necessary for *S* to be able to coin and maintain (i.e. remember) a singularly referring expression for *o*.<sup>163</sup>

So far this shows only that significance constrains naming. Jeshion extends this conclusion to singular thought as follows.<sup>164</sup> To name an individual is to initiate a *mental file* for that individual.<sup>165</sup> Thus significance is necessary for initiating a mental file. According to Jeshion, thinking of an individual from a mental file is necessary and sufficient for referring singularly to

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<sup>160</sup> "One cannot simply choose to have a mental name for an individual. One can have such intentions only in the right conditions." (2010, p.125) Jeshion takes 'mental names' to be *singular representations* (2009, p.372. Presumably she means *mental* singular representations).

<sup>161</sup> (2009), p.370. Note that Jeshion believes that mental names are prior to verbal names: successfully verbally naming an individual requires that one also mentally name that individual, but not vice-versa. (Cf. 2010, p.123.)

<sup>162</sup> Hence the method is example-driven: we enumerate apparent examples of singular thought and find the common factor.

<sup>163</sup> "An agent can name an individual only if she accords intrinsic or relational significance to that individual." (2009), p.374; "A mental name can be initiated only if the individual-to-be-named is in the relevant way significant to the speaker." (2010), p.126.

<sup>164</sup> (2009), pp.393-4; (2010), pp.132-6.

<sup>165</sup> Like Recanati, Jeshion takes a mental file to be "a repository of information the agent takes to be about a particular individual" (2009, p.393) Essentially the same gloss is given in (2010), p.131.

that individual.<sup>166</sup> Now, this step in the argument is very controversial. But note that it is not actually required by the cognitivist view. What's really required is the identification of singular thought with thought about significant objects. Jeshion uses the identification of singular thought with thought from mental files to derive that conclusion, but there may be other ways to derive it. The notion that singular thought essentially involves deploying a mental file may be just one of many ways into the idea that what matters for singular thought is not what relations that the agent stands in, but the structure their thoughts have. So if it turns out that we should *not* identify singular thought with thought from mental files,<sup>167</sup> then the cognitivist will need another way of linking significance with the conditions for singular thought. Unfortunately it would take us too far afield to discuss the possibilities for such alternatives here. Hence I won't question Jeshion's identification of singular thought with thought from mental files either.<sup>168</sup>

So, because significance is necessary for naming, hence for initiating a mental file, it is also necessary for singular thought. Thus the second component of Jeshion's cognitivism, the *significance condition*: *S* can have singular thoughts about *o* only if *o* is *significant* to *S*.<sup>169</sup> Note that while the significance condition is phrased as a necessary condition, Jeshion appears to regard *o*'s significance to *S* as *sufficient* for *S*'s being able to have singular thoughts about *o*. For instance:

Important individuals that our goals are directed toward, whether for tracking, discovering, or constructing, *will be* individuals for which cognition creates mental files. (2009, p.394; my emphasis.)

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<sup>166</sup> Arguments for this can be found at (2009), pp.393-4 and (2010), pp.130-6.

<sup>167</sup> See Rachel Goodman 'Against the Mental Files Conception of Singular Thought' (forthcoming in *Review of Philosophy and Psychology*) for arguments to this effect.

<sup>168</sup> Note that Jeshion intends her claim as a substantive thesis about both singular thought and thought from mental files, not a re-definition of the terms "singular thought" and "mental file". Cf. (2010), p.130.

<sup>169</sup> (2009), p.394; (2010), p.136.

Importantly for the mathematical platonist, Jeshion explicitly states that *o* can be significant to *S* even if *S* is not acquainted with (or otherwise informatively related to) *o*.<sup>170</sup>

This gives cognitivism an advantage over acquaintance theory because cognitivism can explain apparent cases of acquaintanceless singular thought like the Le Verrier case from section 3.1 above. Jeshion believes ordinary speaker intuition predicts that the thought Le Verrier has in thinking (DN) is a singular thought, even before the first observation of Neptune, which is to say before any member of Le Verrier's linguistic community was acquainted with Neptune. Our target acquaintance theorist, Recanati (1993 and 2012) agrees with Jeshion on this count. A similar case discussed by Jeshion is that of the Unabomber. The name "the Unabomber"<sup>171</sup> was introduced in 1978 for the perpetrator of a series of mailbombings. The police and the larger public did not become acquainted with the Unabomber (*qua* postal terrorist) until he was apprehended in 1996. Suppose that between 1978 and 1996 some members of the FBI think and utter:

(UD) The Unabomber is deranged.

Again Jeshion contends the FBI agents thinking (UD) referred singularly to the Unabomber even before they became acquainted with him.<sup>172</sup> And again Recanati agrees that there are ordinary speaker intuitions pointing in this direction.<sup>173</sup> Jeshion's explanation for why the FBI agents and concerned members of the public are able to have singular thoughts about the Unabomber before apprehending him is that the unique perpetrator of the relevant mailbombings was significant to

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<sup>170</sup> "Cognitivism dispenses with an acquaintance condition on singular thought, supplanting it with a significance condition." (2009), p.392. Cf. (2010), p.127/8.

<sup>171</sup> The presence of the word "the" might suggest that "the Unabomber" is a description, equivalent to something like "he who unabombs". But this is wrong: "the Unabomber" is a name in the same way that "the Holy Roman Empire" is a name, rather than a description. (Note that the Holy Roman Empire was neither holy, nor Roman, nor an empire. Hence if "the Holy Roman Empire" were a description, statements like "the Holy Roman Empire was ruled by Charlemagne" would not be true.)

<sup>172</sup> (2010), p.127.

<sup>173</sup> Recanati (2012), p.155.



those people: in particular, they *feared* him.<sup>174</sup> Likewise, Le Verrier was able to have singular thoughts about Neptune before observing it because that planet was significant to him – he thought his discovery of it would make him famous.

Thus one motivation for cognitivism is that insofar as the Neptune and Unabomber cases are plausible cases of acquaintanceless singular thought, cognitivism is better placed to explain this than acquaintance theory. This raises the prospect that cognitivism, like instrumentalism, allows for singular thought ‘on the cheap’. Jeshion explicitly denies this. She claims that whether an object is significant is not under an agent’s voluntary control, but is a matter of the behaviour of the agent’s sub-agential ‘cognition’, which is *not* under the agent’s control. (I’m placing ‘cognition’ in scare-quotes to indicate that I am using that term to mean whatever Jeshion uses it to mean. This holds throughout the rest of this chapter, whether or not scare-quotes are present, unless otherwise noted.) One cannot make an object significant to oneself by believing, deciding or willing it so; and once an object *is* significant, one cannot make that object *insignificant* by willing it so.<sup>175</sup> Hence because cognitivism entails *anti*-voluntarism about significance, it is an anti-voluntarist theory of singular thought.

I will now show that cognitivism allows for singular thought about abstract mathematical objects. If I am right then cognitivism can be pressed into service by the mathematical platonist wishing to avoid having to construe numerals and expressions like “ $\pi$ ” and “ $e$ ” as descriptively referring expressions.

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<sup>174</sup> (2010), p.116; cf. p.137 for the claim that fear of *o* means that *o* is significant.

<sup>175</sup> “An agent making a judgment “this is significant” is not sufficient for engendering the significance needed for singular thought.” (2010, p.136.)

5: *Names as Bearers of Significance: “ $\pi$ ” and “ $e$ ”*

Jeshion believes that whether  $o$  is significant to  $S$  is not a matter under  $S$ ’s conscious, voluntary control.<sup>176</sup> Rather, significance is controlled by  $S$ ’s sub-agential (hence non-voluntary) ‘cognition’.<sup>177</sup> Importantly, Jeshion believes that  $S$  can gain the ability to have singular thoughts about  $o$  purely by taking something to be a name for  $o$ .<sup>178</sup> Hence the following principle is a component of Jeshion’s cognitivism:

*Names as bearers of significance:* An agent’s construing a term as a name causes that agent to take the name’s referent as an individual accorded significance. (2009, p.374)

My reading of *names as bearers of significance* is as follows.  $S$ ’s construing an expression  $a$  as a name causes  $S$ ’s cognition to initiate a mental file labelled  $a$ , i.e. an “ $a$ -file”. Hence if the expression  $a$  refers to or abbreviates a description satisfied by a unique object – so long as  $a$  singles out an object for the statements in which it occurs to say something about –  $S$  can use  $a$  to have singular thoughts about the object (if any) that  $a$  describes or refers to. On the other hand, if contextual cues lead  $S$ ’s cognition to construe  $a$  as anything other than a name – for instance as a pronoun anaphoric on a description<sup>179</sup> –  $S$ ’s cognition will not initiate an  $a$ -file.<sup>180</sup> In such cases  $S$ ’s mental tokenings of ‘ $a$  is  $\Phi$ ’ (where  $\Phi$  ranges over predicates) are not singular thoughts.

Here is why this reading of *names as bearers of significance* allows for the significance of abstract mathematical objects. The expressions “ $\pi$ ” and “ $e$ ”, as used by mathematicians studying the real numbers, share some important features with paradigmatic proper names (like “James”). They appear as the subject-terms in well-formed statements, for example, “ $\pi$  is

<sup>176</sup> (2009), p.391-2; (2010), p.126-9.

<sup>177</sup> (2009), p.393/4; (2010), p.136-7.

<sup>178</sup> (2009), p.395.

<sup>179</sup> Like the “he” in “the richest man in the room is European, and he is old money”.

<sup>180</sup> Jeshion (2010), p.126.

transcendental”. And they seem to facilitate existential generalization, for example from “ $\pi$  is transcendental” to “ $\exists x(x \text{ is transcendental})$ ”. However these tests do not provide much evidence that “ $\pi$ ” and “ $e$ ” refer singularly rather than descriptively, because descriptively referring expressions like “the ratio of a circle’s circumference to its diameter” have the same features.

However “ $\pi$ ” and “ $e$ ” also appear to have a feature that expressions like “the ratio of a circle’s circumference to its diameter” do not have: they facilitate existential quantification into the scope of a negation. The inference from “ $\sim(\pi \text{ can be expressed as a fraction})$ ” to “ $\exists x \sim(x \text{ can be expressed as a fraction})$ ” is intuitively valid. The validity of that inference requires that the logical form of “it is not the case that  $\pi$  can be expressed as a fraction” be  $\neg\Phi\alpha$ . But if “ $\pi$ ” is a descriptively referring expression, then the logical form of “ $\sim(\pi \text{ can be expressed as a fraction})$ ” is not  $\neg\Phi\alpha$ , but:

$$(*) \neg[\exists x(\Psi x \ \& \ \forall y(\Psi y \rightarrow x = y) \ \& \ \Phi x)].$$

Here  $\Psi$  ranges over open sentences of one free variable and  $\Phi$  over predicates. But a statement of the form (\*) does not entail anything of the form  $\neg\exists x \sim\Phi x$ . The closest thing that it does entail is  $\neg\exists x[\Psi x \ \& \ \Phi x]$ . So if “ $\pi$ ” were a descriptively referring expression, we would not be able to validly infer “ $\exists x \sim(x \text{ can be expressed as a fraction})$ ” from “ $\sim(\pi \text{ can be expressed as a fraction})$ ”. We could only validly infer “ $\sim\exists x(x \text{ is the ratio of a circle’s circumference to its diameter} \ \& \ x \text{ can be expressed as a fraction})$ ”. This conflicts with our ordinary speaker intuition, which tells us that from “ $\pi$  cannot be expressed as a fraction” we *can* validly infer “something cannot be expressed as a fraction”. Indeed, “ $\sim\exists x(x \text{ is the ratio of a circle’s circumference to its diameter} \ \& \ x \text{ can be expressed as a fraction})$ ” is strictly speaking consistent with there being no thing that is the ratio of a circle’s circumference to its diameter, and hence it is also consistent with the claim that everything can be expressed as a fraction. But ordinary speaker intuition tells us that “ $\pi$  cannot be

expressed as a fraction” is definitely *inconsistent* with “everything can be expressed as a fraction”. Thus “ $\pi$ ” has a feature that “the ratio of a circle’s circumference to its diameter” does not have. (Note that a structurally identical argument can be run for “ $e$ ”.)

Perhaps even more suggestively, the expressions “ $\pi$ ” and “ $e$ ” are uncommon amongst expressions for real numbers in that they appear to be *simple syntactic units*: “ $\pi$ ” and “ $e$ ” do not appear to have parts with their own semantic values. This contrasts with both decimal representations (e.g. “3.14159...” and “5.000...”) and concatenations of expressions for operators with expressions for numbers (e.g. “ $\sqrt{\pi}$ ”). Now, this is not decisive *proof* that “ $\pi$ ” and “ $e$ ” are singular terms. For the claim that syntactically simple expressions are singular terms admits of counterexamples in both directions. “The Holy Roman Empire” is syntactically simple, despite appearances: “The Holy Roman Empire was rule by Charlemagne” does not entail “something Holy was ruled by Charlemagne”. Likewise apparently syntactically simple expressions that have been introduced as *abbreviations* for syntactically complex expressions are, despite appearances, syntactically complex: if “Julius” abbreviates “the actual inventor of the zip”, then “Julius was Swedish” entails “some inventor was Swedish”. Nevertheless despite failing to constitute definitive proof, the apparent syntactic simplicity of “ $\pi$ ” and “ $e$ ” – in contrast with the vast majority of our other expressions for real numbers – is at least suggestive *evidence* that they are singularly referring expressions.

Given that “ $e$ ” and “ $\pi$ ” appear to have some of the paradigmatic features of names, and fail to share these features with other expressions for real numbers, it is plausible that an agent’s cognition could construe uses of “ $e$ ” and “ $\pi$ ” as uses of names. If “ $\pi$ ” and “ $e$ ” can be construed as names then *names as bearers of significance* entails that the objects “ $\pi$ ” and “ $e$ ” refer to or describe can be significant to us.

There are also further reasons to think that the satisfiers of “the ratio of a circle’s circumference to its diameter” and “the limit of  $(1 + (1/n))^n$  as  $n$  approaches infinity” (for example) are significant to us in ways that most other real numbers are not. Let  $S$  be a mathematician working within a well-established mathematical theory of the real numbers.  $S$  reads, writes, hears, utters and thinks many tokens of statements containing “ $e$ ”, e.g. “ $e$  is transcendental”, “ $e$  is the limit of  $(1 + (1/n))^n$  as  $n$  approaches infinity”, “ $e^{i\varphi} = \cos\varphi + i \sin\varphi$ ”, etc.  $S$  deals with such statements a lot more often than with statements about any nearby real numbers like  $e + 1$ ,  $e - 0.05$ , etc. And, moreover, we can suppose that statements like “ $e$  is transcendental” and “ $e^{i\varphi} = \cos\varphi + i \sin\varphi$ ” figure in  $S$ ’s plans for future research activities more than statements about  $e + 1$  and  $e - 0.05$ . All this points towards  $e$ ’s being significant to  $S$  in ways that most other real numbers are not.

For Jeshion, significance is necessary and sufficient for initiating and maintaining a mental file. Thus insofar as it is plausible that  $e$  is significant to  $S$  in a way that most other real numbers are not, and given that  $S$  has reasons to systematically store the information encoded in “ $e^{i\varphi} = \cos\varphi + i \sin\varphi$ ” and the like, we should expect  $S$  to initiate and maintain a mental file for  $e$  in which to store that information. It is highly plausible that professional mathematicians working in established theories of the real numbers do store information about some real numbers (and not others) in this way; that is, in the same way as most people store information about significant persons and everyday objects, i.e. using mental files. Thus it is highly plausible that  $S$  has an “ $e$ ”-file containing information like “ $e$  is transcendental”, “ $e$  is the limit of  $(1 + (1/n))^n$  as  $n$  approaches infinity”, “ $e^{i\varphi} = \cos\varphi + i \sin\varphi$ ”, etc.

Therefore there are reasons for thinking that at least some agents – professional mathematicians working in certain areas – maintain mental files for some real numbers. For

Jeshion, where there are files, there are names. We've already seen some reasons for thinking that at least some real numbers have names. Note that the real numbers for which professional mathematicians appear to maintain mental files are the *same* numbers that we appear to have given names to. At least, it is plausible that both these claims hold for  $\pi$  and  $e$ . These claims form distinct though mutually supporting sets of reasons for supposing that we have mental files for some real numbers. And thus, by Jeshion's (controversial) argument that singular thought just is thought from mental files, our mathematician  $S$  can use their " $e$ "-file to have singular thoughts about the object satisfying (e.g.) "the limit of  $(1 + 1/n)^n$  as  $n$  approaches infinity". Structurally identical reasoning nets the same conclusion for " $\pi$ ". Thus Jeshion's *names as bearers of significance* principle allows that our cognition can be caused to accord significance to – and thereby initiate mental files for – at least some real numbers. Therefore by the identification of thought from mental files with singular thought, we can have singular thoughts about some real numbers (provided that they exist). And so, under the assumption that the ability to have singular thoughts about an object  $o$  suffices for our expressions to be able to refer singularly to  $o$ , our expressions can refer to abstract mathematical objects.

Before moving onto objections I want to note three things. First, the application of *names as bearers of significance* to " $\pi$ " and " $e$ " to obtain the consequence that  $\pi$  and  $e$  can be significant to us relied on three observations, and these observations appear to carry over to some natural numbers too. Those observations were that " $\pi$ " and " $e$ " share with ordinary proper names the appearance of being syntactically simple units; that they do not share this feature with many other expressions for real numbers; and that the numbers those expressions refer to (singularly or descriptively) play for more substantial roles in our mathematical practices than most other real numbers. Now, some expressions for natural numbers, in particular 'simple' numerals like "5"

and “7”, also appear to be syntactically simple units. Numerals for every large numbers, on the other hand, do not seem to share this feature: “3,502,751” does *not* seem to be a simple syntactic unit, but the result of concatenating a selection of such units. This suggests that some elements of the argument from a few paragraphs back for the conclusion that “ $\pi$ ” and “ $e$ ” are names can be transposed to the context of the natural numbers as follows. Because “5” and “7” are simple syntactic units, and the majority of expressions for natural numbers are not, our cognition can construe “5” and “7” as names. Thus by *names as bearers of significance* the referents of “5” and “7” can be significant to us, and hence we can initiate mental files for them; whence by the identification of singular thought with thought from mental files, we can have singular thoughts about some natural numbers.

Note, however, that this argument faces an obvious objection, arising from Frege’s observation that there is no sharp boundary between which numbers count as small and which do not.<sup>181</sup> The objection is that the notion of a numeral’s being ‘syntactically simple’ may admit of borderline cases. I said that “3” is definitely syntactically simple and “3, 042, 967” is definitely not. But it is not clear where the line between syntactically simple and syntactically complex numerals is supposed to lie. Perhaps “9” is simple, but “10” is complex, because it is the concatenation of “1” and “0”. But if we then turn to the words “nine” and “ten”, the latter appears to the syntactically simple, especially when compared to “three hundred thousand four hundred and fifty-six”.

Now, there usually is assumed to be a sharp distinction between singular and descriptive

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<sup>181</sup> Frege (1884/1950), section 5: “Yet it is awkward to make a fundamental distinction between small and large numbers, especially as it would scarcely be possible to draw any sharp boundary between them.” The sorites argument that every number is small because 0 is small and for every  $n$  is  $n$  is small, then  $n+1$  is small, has been called ‘Wang’s Paradox’. (E.g. by Dummett in ‘Wang’s Paradox’, in his *Truth and Other Enigmas* (1978).)

thoughts; the category of singular thought does not admit of borderline cases. If there is a sharp distinction between singular and descriptive thought, then there is a sharp distinction between singular and descriptive thought about natural numbers. If there is a sharp distinction between singular and descriptive thought about natural numbers, then we had better not *identify* singular thought about natural numbers with thought expressed by uses of syntactically simple expressions for numbers. For then we have to posit a *sharp* distinction between the syntactically simple expressions for numbers and all the rest, and we've just seen that there are reasons to doubt there is such a sharp distinction. Therefore it would be to the platonist's benefit if they could find reasons for thinking some natural numbers are more significant to us than others, that does not rely directly on the notion that some of our expressions for numbers are syntactically simple and some are not.

Fortunately for the platonist there are also other reasons for thinking that some natural numbers are more significant to us than others that are not do with the simplicity or complexity of numerals. The claim that some natural numbers are more significant to us than others – or at the very least, the idea that some natural numbers may play roles in our arithmetical practices (of counting collections of things and performing calculations) from others – may be supported by the phenomenon known as *subitizing*. Humans (and some non-human animals) can estimate the cardinalities of collections visual objects, when that cardinality is less than 5 or 6, with a speed and accuracy suggesting that we do not *count* those objects, but apprehend the cardinality in some more direct manner.<sup>182</sup> This difference between the natural numbers less than 5 or 6 and all others may be important enough to make the former significant to us in a way that the latter are not. Now, I will not pursue this intriguing suggestion further here – rather I will be concerned to defend

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<sup>182</sup> The *locus classicus* is Stanislas Dehaene (2011) *The Number Sense*, especially pp.55-59.



my proposed account of singular thought about some natural and real numbers from some objections – but suffice it to say that the notion of a connection between subitizing and sub-agential ‘cognitive’ significance is very suggestive and worthy of investigation by both mathematical platonists and cognitivists about singular thought alike.

Before turning to objections I will note two things. First I’ll re-iterate that this cognitivist account of singular thought about (and hence singular reference to) some abstract mathematical objects – the real numbers  $\pi$  and  $e$ , and the natural numbers less than 10 (or maybe just those less than 5 or 6) – preserves the anti-voluntarism that distinguishes Jeshion’s cognitivism from semantic instrumentalism. So while the above story does generate singular thought about abstract mathematical objects, it does not thereby generate singular thought ‘on the cheap’. Whether  $S$ ’s ‘cognition’ initiates “ $e$ ”- or “ $\pi$ ”-files (or does whatever other cognitive restructuring resulting from the accordance of significance) is not something that  $S$  has conscious control over. It is a matter of the behaviour of  $S$ ’s sub-agential ‘cognitive’ system. But nor need it be a matter of which objects  $S$  is causally related to, as acquaintance theories would have it.

The second thing I note is that my account of singular thought about some natural numbers departs from the standing (though perhaps implicit) platonist assumption that if some numerals are singularly referring expressions, then *all* numerals are singularly referring expressions.<sup>183</sup> The platonist view I have just described holds that only *some but not all* numerals are singularly referring expressions, belonging to the same semantic kind as (genuine) proper names, whilst other highly complex numerals like “3,042,860” are descriptively referring expressions, more akin to “the tallest building in Toronto”. There is, I think, a reason to take this departure from

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<sup>183</sup> This assumption is implicit in Frege (1884/1950) and (1892/1952). The assumption is more (though not always completely) explicit in the writings of his followers Bob Hale (1987), pp.180-193 and Crispin Wright (1983), pp.97-103.

traditional platonism as a good idea for the platonist. Not every concrete object is a possible object of singular thought for a given agent  $S$  at a given time  $t$ : some concrete objects are *privileged* candidates for singular thought by  $S$  at  $t$  (setting aside semantic instrumentalism). My account carries this plausible claim over to the abstract mathematical case: some abstract mathematical objects are privileged candidates for singular thought for  $S$  at  $t$ . Exactly which abstract mathematical objects these are is a function of  $S$ 's history prior to  $t$  and their goals, interests, and abilities, as well as the history and goals etc. of  $S$ 's linguistic community at  $t$ . (For instance  $\pi$  and  $e$  got their names well before any of us came on the scene.) In this, according to my account, abstract mathematical objects are *just like* concrete objects. But that my account has this consequence should be seen as an advantage, not a defect; for my account makes singular thought about abstract mathematical objects and ordinary concrete objects a more unified phenomenon than traditional platonism might lead us to expect.

I now turn to several objections to my proposed account of singular thought about and singular reference to abstract mathematical objects.

#### *6: Do We Have Mental Files for Numbers?*

I consider three objections to my account of singular thought about and singular reference to abstract mathematical objects. The first is a general objection to using any mental-files-based theory of singular thought to argue that some of our thought about natural and real numbers are singular. The second two are objections against my use of cognitivism in particular.

A mental file is a store of information an agent takes to be about a single object. But one might think that our thought and speech involving reference to the natural and real numbers does

not involve storing and retrieving information taken to be about those objects. Rather, when we refer to natural and real numbers, we are primarily concerned with using reference to those entities to make inferences: comparing cardinalities, calculating rates of change, etc. If that's right then we don't so much maintain stores of information taken to be about natural and real numbers as we perhaps remember a select few *rules* governing how to use those entities in inferences. But remembering a few select rules might not require or be sufficient for maintaining a mental file; we all tacitly know the introduction and elimination rules for conjunction, but it is not plausible that we maintain mental files for that sentential connective. And if we do not in fact maintain mental files for natural and real numbers, then a mental files analysis of singular thought (like Jeshion's or Recanati's) is all but useless to the platonist's task of showing that we can have singular thoughts about natural and real numbers.<sup>184</sup> And if that's right, then the platonist will need an analysis of singular thought in terms of something other than mental files if they are to have a hope of defending the claim that we can have singular thoughts about natural and real numbers. (Perhaps, for example, the platonist could use analyse singular thought in terms of the inferences agents are disposed to, or justified in, making when thinking singularly versus descriptively.)

I will say one brief thing in response. While it may possibly be that non-expert agents do not maintain large stores of information about individual natural or real numbers, it is also plausible that expert mathematicians *do* maintain such stores of information. It is quite plausible that Leonhard Euler knew a lot of information about the number  $e$ , and that he stored and was able to retrieve this information. This is sufficient for the platonist's purposes of showing that singular thought about abstract mathematical objects is *possible*. Moreover, given the presence of

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<sup>184</sup> Thanks to Imogen Dickie for raising this objection to me.

mathematical experts in our linguistic community, non-expert agents can be regarded as using expressions like “7” and “ $\pi$ ” deferentially to those expert mathematicians, much as non-experts use “Gödel” in deference to expert logicians. However the fact that non-expert ordinary speakers do not maintain large stores of information they take to be about the referent of “Gödel” does not impugn the status of that expression as a singularly referring expression in a language spoken by a community that does contain experts maintaining such stores of information.<sup>185</sup> The platonist can say that it is the same with the likes of “7” and “ $e$ ”. (Note, though, that it would be to the platonist’s advantage if they could also find a convincing way to develop a non-mental files-based defence of the claim that we can have singular thoughts about abstract mathematical objects. This is something I hope to pursue in future research.)

I will now move on to specific objections against my proposed use of Jeshion’s cognitivism about singular thought to argue for the possibility of singular reference to natural and real numbers.

### *7: Perceptual Demonstrative Thought About Insignificant Objects*

The first objection is to the significance constraint. According to the anticipated objector, if significance is necessary for singular thoughts, then many of our perceptual demonstrative thoughts are *descriptive* thoughts. For many of our perceptual demonstrative thoughts are about objects that are not significant to us in any plausible ordinary sense of ‘significant’. For example, suppose I am momentarily distracted by a speck on the window next to me, and wonder whether

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<sup>185</sup> I here ignore complications arising from distinctions between ‘consumers’ and ‘producers’ in name-using practices according to which consumers’ uses of singular terms are metalinguistic, as in “the person the experts call “NN””. Cf. Evans (1982), pp.376-8.

it is a small piece of matter resting on the window or a larger one floating on the breeze outside. I quickly see that the speck is resting on the window and then go about my life without ever thinking of it again. It seems incredible to say that this speck is *significant* to me. It certainly seems that the speck is not significant to me in Jeshion's sense: it has basically no effect on my plans, projects, affective states or motivations.<sup>186</sup> But if the speck is not significant to me then cognitivism entails that when I think "what is *that*" whilst visually attending to the speck I am having a *descriptive* thought about the speck. But perceptual demonstrative thought is a paradigm example of singular thought. The claim that my perceptual demonstrative thought about the speck is descriptive is implausible to anyone except a global descriptivist (someone who believes that all thought and/or reference is descriptive rather than singular<sup>187</sup>). But if perceptual demonstrative thought about insignificant objects can be singular, then significance is not necessary for singular thought, contrary to Jeshion's significance condition. Acquaintance theories of singular thought do not face this consequence because they can say that perceptual demonstrative thought about insignificant objects is singular because perceptual demonstrative thought is acquaintance-based, and acquaintance-based thought is singular. But as we've seen, acquaintance theories of singular thought appear to make no room for singular thought about abstract mathematical objects.

I will first consider a response to this problem of perceptual demonstrative thought about insignificant objects drawn from Jeshion's own writings. After describing some misgivings I will give my own response.

Jeshion claims that "Because of our visual system's object-orientation [...] the cognitive system is concrete-object oriented. Concrete objects are, consequently, privileged as candidates

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<sup>186</sup> Cf. (2010) p.136.

<sup>187</sup> As in Lewis (1984), Kroon (1987).

for being the subject of singular thought.”<sup>188</sup> We can turn this claim of Jeshion’s into an answer to the worry about perceptual demonstrative thought about insignificant objects as follows. We hypothesize that in addition to our visual system being object-oriented, our *cognitive* system is *perceptual*-object-oriented: our ‘cognition’ pays special attention to currently perceived objects.<sup>189</sup> Thus perceived objects are *automatically* significant. And so if some given object *o* is an object of perceptual demonstrative thought, then *o* is significant; whence the cognitivist can maintain a global significance constraint on singular thought.

Here is a potential problem with this proposal. To take every perceived object as automatically “significant” in virtue of being perceived is to stretch the meaning of the term “significant” far beyond its ordinary meaning. Not every thing we see or hear is ‘significant’ to us, as that word is ordinarily used. This can mean one of two things. First, “significance” has become a term of art with a special theoretical meaning particular to cognitivism about singular thought. But Jeshion has not said anything indicating what that theoretical meaning might be. Indeed, she appears to be appealing to our implicit understanding of ordinary uses of the term “significant” when citing examples of significant objects as objects of acquaintanceless singular thought. She says that the Unabomber was significant to the FBI and the wider public because they *feared* him. The second possibility is that “significance” is acting as a place-holder term, meaning something like “whatever it is that causes an agent’s cognitive system to initiate a mental file”. But now the term “significance” has no explanatory power. Saying that *S* can have singular thoughts about *o* because *o* is “significant” to *S* does nothing to advance our philosophical investigation beyond the thesis that singular thought involves deploying mental files.

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<sup>188</sup> (2010), p.137.

<sup>189</sup> This is very metaphorical, but I hope the gist is clear enough. At any rate this is not a proposal I will ultimately be endorsing.

Here is my attempt at a better response. We want an account of singular thought facilitating such thought about abstract objects (*contra* acquaintance theory) but avoiding voluntarism about singular thought. This does not require that significance be *necessary* for singular thought. It only requires that significance be *sufficient* for singular thought. Hence the platonist could adopt a hybrid view according to which either significance *or* a perceptual link (perhaps even one sustaining acquaintance) can suffice for singular thought. So long as significance is sufficient for singular thought, and abstract objects can be significant, singular thought about abstract objects is possible. Hence allowing that we can have singular thoughts about insignificant objects need not prevent us from using cognitivist resources to explain how we could have singular thoughts about abstract mathematical objects.

The move to a hybrid significance-or-perceptual link view may look *ad hoc*. What reasons can the cognitivist have to bifurcate the requirements for singular thought in this way, other than to avoid the problem of perceptual demonstrative thought about insignificant objects? However, it turns out that this bifurcation is not *ad hoc* but actually quite well-motivated.

The core move is to note the plausibility of the claim that an agent's mental file for an object *persists over time* only when that object is significant to that agent. Consider again my perceptual demonstrative thought about the speck on my window. If the speck really is not significant to me, we should expect that I will not retain any information about the speck. If I retain no information about the speck, then I also cease to retain my "that speck"-file. In this case, according to cognitivism, I will not retain the ability to have singular thoughts about the speck. So if the speck is not significant to me, my ability to have singular thoughts about the speck will not long outlast my perceptual link with the speck.

On the other hand, if I *do* retain information about the speck – if a few days hence I am

able to recall the speck and whether it was in fact resting on the window or floating outside, what I was doing when I saw it, what time of day it was, and so on – then it seems that my “that speck”-file has persisted over the intervening period. According to cognitivism, if the file persists, then so does my ability to use it to have singular thoughts. But note that in the case that I retain information about the speck over a reasonably lengthy time period (e.g. a few days or longer), the claim that the speck is significant to me is very plausible.

So while it is implausible that perception of *o* entails that *o* is significant, it *is* plausible that the ability to have singular thoughts about an object in the *absence* of a current or very recent perceptual link does require that the object is significant, in something like the ordinary sense of “significant”. This shows that far from being *ad hoc*, bifurcating the requirements for singular thought so that perceptual demonstrative thought does not require significance but persistence of a mental file and hence singular thought in the absence of a perceptual link does require significance is well-motivated.<sup>190</sup>

Moreover, the cognitivist account of acquaintanceless singular thought runs the same as before, as does my argument that some real and natural numbers can be significant. And, lastly, the hybrid view which I am proposing does not entail voluntarism about singular thought. Hence the hybrid view I have described is also a middle ground between acquaintance theory and instrumentalism. It is just as useful to the platonist as Jeshion’s ‘pure’ cognitivism, but faces fewer objections.

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<sup>190</sup> Note moreover that this kind of observation could be the source of a response to Sarah Sawyer’s (2012, p.274) objection to cognitivism that significance is a matter of degree, whereas whether a thought is singular presumably is not. The beginning of the response is that while whether a thought is *singular* is not a matter of degree, the length of time for which the agent retains the ability to have singular thoughts about a given object *is* a matter of degree.



### 8: Significance and Anticipated Acquaintance

The second objection against my use of cognitivism to accommodate singular reference to abstract mathematical objects is that the apparent instances of acquaintanceless singular thought Jeshion uses to motivate her cognitivism – e.g. the Le Verrier and Unabomber cases – are *better* explained in terms of extended or liberalized acquaintance. In particular, Recanati has argued that Jeshion's cases of acquaintanceless singular thought are best explained not as cases where the object is significant to the thinking agent, but as cases where the agent is correct in anticipating becoming acquainted with the object in the future:<sup>191</sup>

On my view, Jeshion's significance requirement works only because, when an object is relevant to our plans and so on, we expect to come into various sorts of [acquaintance] relation[s] with it, hence *the significance requirement can hardly be met without the potential acquaintance requirement also being satisfied*. (2010, p.183; my emphasis.)

Recanati explains this 'potential acquaintance requirement' as follows:

[O]ne can think a singular thought (content<sup>[192]</sup>) by opening a mental file even if, for the time being, one has only the description to reply on, provided one is *right* in anticipating that one will come into relation to the denotation of the description and be in a position to gain information from it. (Unless one is *correct* in one's anticipation, one can think of the object only descriptively.) (2012, p.169; my emphasis.)

Here is my formulation of this liberalization of his acquaintance constraint. *S* can have singular thoughts about *o* at time *t<sub>i</sub>*, despite not being acquainted with *o* at *t<sub>i</sub>*, provided the following conditions are met:

- a) *S* possesses at *t<sub>i</sub>* a descriptive name *a* introduced in terms of a definite description *D* uniquely satisfied by *o*;
- b) At *t<sub>i</sub>* *S* anticipates becoming acquainted with the satisfier of *D*; and

<sup>191</sup> (2010) p.170-173, p.183; (2012) chapter 12.

<sup>192</sup> Recanati distinguishes the vehicle of a thought from its content; mental files are thought vehicles. I will largely gloss over this distinction, in the hope that doing so will not detract from my argument.

c) There is a time  $t_j$  later than  $t_i$  such that  $S$  is acquainted with  $o$  at  $t_j$ .

Hence Le Verrier's mental tokening of (the 19<sup>th</sup> century French translation of) "my discovery of Neptune will make me famous" prior to the 23<sup>rd</sup> of September 1846 involved singular thought about Neptune (rather than descriptive thought about the cause of the eccentricities in the orbit of Uranus) because even though he wasn't acquainted with Neptune on August 31<sup>st</sup> 1846, he satisfied (a), (b) and (c) with respect to Neptune at that time.

So the notion of correctly anticipated acquaintance can be used to explain the apparent instances of acquaintanceless singular thought that Jeshion uses to motivate cognitivism. But importantly for the platonist, correctly anticipated acquaintance does *not* allow singular thought about abstract mathematical objects. For if acquaintance is a causal relation then no-one can ever be correct in anticipating becoming acquainted with any abstract object. Thus correctly anticipated acquaintance theory is the ideal option for anyone attracted by the idea that Le Verrier did manage to have a singular thought about Neptune when thinking (DN), but who also wants to deny that we can have singular thoughts about abstract mathematical objects.

I will now argue that Recanati's correctly anticipated acquaintance as articulated in conditions (a)-(c) above cannot be sufficient for singular thought. There are cases where  $S$  rightfully expects future acquaintance with  $o$  but where intuition says that  $S$  can only have descriptive thoughts about  $o$ . Consider the following definite description:

(D<sub>1</sub>) The next object that I will become perceptually acquainted with (that I am not acquainted with already).

Suppose as before that at time  $t_0$ ,  $S$  coins a descriptive name  $d_1$  with associated reference-fixing description (D<sub>1</sub>): they think to themselves "let " $d_1$ " refer to the next object that I will become

perceptually acquainted with (that I'm not acquainted with already)". Suppose also that *S* anticipates surviving long enough to become perceptually acquainted with a brand new object, and that this does occur – *S* becomes acquainted with a new object *o* at a time  $t_1$  later than  $t_0$ . That *o* thus satisfies *S*'s use of (D<sub>2</sub>) at  $t_0$ . Thus it looks like *S* satisfies (a), (b) and (c) with respect to *o* at  $t_0$ . *S* satisfies (a) because they possess at  $t_0$  a descriptive name "*d*<sub>1</sub>" introduced via a description (D<sub>1</sub>) uniquely satisfied by *o*. *S* satisfies (b) because they expect to survive long enough to become acquainted with a new object. And *S* satisfies (c) because it is true that there is a time later than  $t_0$  – namely  $t_1$  – such that *S* is acquainted with *o* at  $t_1$ . Suppose now that *S* thinks the following thought at  $t_0$ :

(V) I wonder if *d*<sub>1</sub> will be animal, vegetable or mineral.

If satisfying (a)-(c) with respect to *o* is sufficient for being able to have singular thoughts about *o* prior to becoming acquainted with *o*, then *S* thinks a singular thought about *o* by thinking (V) at  $t_0$ .

However I submit that it is very implausible that *S* should be able to have singular thoughts about *o* at  $t_0$  using the descriptive name "*d*<sub>1</sub>". To see why, consider the following description:

(D<sub>117</sub>) The one hundred and seventeenth object that I will become newly acquainted with (that I'm not acquainted with already).

Suppose that *S* expects to survive long enough to become acquainted with at least one hundred and seventeen new objects. Suppose also *S* does survive long enough – and has enough new experiences – to become acquainted with a great many new objects, the one hundred and seventeenth being the oldest streetlamp in St Catharines, Ontario. Then if *S* introduces a descriptive name "*d*<sub>117</sub>" via (D<sub>117</sub>) by thinking to themselves "let "*d*<sub>117</sub>" refer to the satisfier of

$D_{117}$ ”, then  $S$  satisfies (a)-(c) with respect to the oldest streetlamp in St Catharines. Suppose that  $S$  performs this inner mental dubbing and then goes on to think:

(V\*) I wonder if  $d_{117}$  will be animal, vegetable or mineral.

If satisfying (a)-(c) with respect to an object  $o$  is sufficient for being able to have singular thoughts about  $o$ , then when  $S$  thinks (V\*) they have a singular thought about the oldest streetlamp in St Catharines. Again, I submit that this is highly implausible that  $S$  has a singular thought about the oldest streetlamp in St Catharines when thinking (V\*). (Or at least, this should be highly implausible to anyone that is not already a committed semantic instrumentalist.)

If this is right then the liberal acquaintance theorist needs to restrict the conditions under which anticipated acquaintance can sustain singular thought. Before describing one way in which Recanati could do this I will first show that the cognitivist about singular thought has an easy explanation of why  $S$  does not have a singular thought in the (V) and (V\*) cases.

The cognitivist about singular reference can say that whether  $S$  has a singular thought when thinking (V) or (V\*) depends on whether the satisfiers of ( $D_1$ ) or ( $D_{117}$ ) are *significant* to  $S$ . So long as those entities are not significant to  $S$ ,  $S$  does not successfully initiate a mental file when introducing the descriptive names “ $d_1$ ” and “ $d_{117}$ ”, and so does not have a singular thought when using “ $d_1$ ” and “ $d_{117}$ ” to think (V) and (V\*). Now, given our minimal description of these examples, it is hard to see why the satisfiers of ( $D_1$ ) or ( $D_{117}$ ) would be significant to  $S$ . The cognitivist can maintain that this explains the intuition that (V) and (V\*) are not instances of singular thought. Therefore it looks like cognitivism is better placed than Recanati’s liberalized acquaintance theory to account for the intuitive difference between the Le Verrier and Unabomber cases as cases of singular thought versus (V) and (V\*) as instances of descriptive thought.

We now ask how a liberalized acquaintance theorist like Recanati could try to restrict the conditions under which anticipated acquaintance sustains singular thought to respect the intuition that (V) and (V\*) are not instances of singular reference, whilst retaining the claim that the Le Verrier and Unabomber cases are cases of singular thought. They could claim that in order for *S* to use a descriptive name to think singularly, it is not enough that *S* be *correct* in anticipating future acquaintance with the satisfier of the relevant reference-fixing description. Rather, *S*'s anticipated acquaintance needs to *justify* the initiation of a mental file for the satisfier of the reference-fixing description. Despite Recanati's use of the phrases "right" and "correct" in liberalizing his acquaintance constraint, he also occasionally uses the phrase "justifies", as in the following:

The (expected) existence of an information link is what *justifies* opening a file. (Recanati 2012, p.167 f/n 9; emphasis original.)

Similarly:

I, too, think we don't open a mental file unless we have a *good reason* to do so. (Ibid., p.167; my emphasis.)

The idea then is that merely knowing that one will become acquainted with the next object one will become acquainted with – or knowing that one will become acquainted with the one hundred and seventeenth object that one will become acquainted with – does not by itself give one a *good reason* to initiate a mental file. So the difference between the "*d*<sub>1</sub>" and "*d*<sub>117</sub>" cases on the one hand and the Neptune and Unabomber cases on the other is that in the latter cases, Le Verrier and the FBI's anticipation of acquaintance with Neptune and the Unabomber gave them *good reasons* to initiate mental files for those objects. But in the "*d*<sub>1</sub>" and "*d*<sub>117</sub>" cases, *S*'s anticipated

acquaintance with those objects of future acquaintance does *not* give them a good reason to initiate a “ $d_1$ ”- or “ $d_{117}$ ” file. Or so the story goes.

The cognitivist (or my hybrid cognitivist-perceptual link theorist) can respond by trying to assimilate this notion of a ‘good reason’ for opening a mental file to Jeshion’s notion of *significance*. We have been told that acquaintanceless singular thought is possible only when anticipated acquaintance gives a good reason to initiate a mental file; that is, when anticipated acquaintance gives the agent a good reason to begin systematically storing information about the object. But what happens if we ask for more details about the conditions under which anticipated acquaintance gives us good reasons to begin systematically storing information about an object? The answer cannot be that one has a good reason to initiate a file for an object when one is correct in anticipating acquaintance with that object, nor even when one *knows* that one is correct in such anticipation. For both of these obtain in the “ $d_1$ ” and “ $d_{117}$ ” cases. It is more plausible that one has a ‘good reason’ to initiate a file for an object if one anticipates wanting to recall that information at a later date. But it seems that whether *S* anticipates wanting to recall information they take to be about a single object is a matter of whether *S cares* enough about that information that they keep it clustered together and ready to be retrieved rather than forgotten. And now Recanati’s liberalized acquaintance constraint is starting to look a lot like Jeshion’s significance constraint. An agent anticipates wanting to recall information about an object – that is, an agent is justified in initiating a mental file for an object – if and only if that object is significant to them. Moreover, we have seen that expert mathematicians may well store and cluster information about abstract mathematical objects in exactly this way. If that’s right then it seems that mathematicians can have good reasons for initiating mental files for storing information like “ $\pi$  is transcendental” and “ $e^{i\varphi} = \cos\varphi + i \sin\varphi$ ”. And so, if Recanati’s liberalized acquaintance theory is to successfully

differentiate itself from cognitivism in the way required to be of use as an argument against singular thought about abstract objects, then the liberalized acquaintance theorist needs to say more to differentiate anticipated acquaintance that justifies the initiation of a file from Jeshion's notion of significance.

### *9: Conclusion*

I conclude that the platonist who is anxious to defend the possibility of singular reference to abstract mathematical objects should adopt the following theory of singular reference. The theory is a hybrid, where perceptual links can sustain perceptual demonstrative singular thought and significance sustains singular thought in non-perceptual cases. This hybrid theory has many virtues. First it allows for singular thought about abstract mathematical objects, in particular certain real numbers like  $\pi$  and  $e$  and certain natural numbers, perhaps those less than 6. Second it allows that all perceptual demonstrative thought is singular thought, which Jeshion's cognitivism arguably does not. Third, it is at least as capable as Recanati's liberalized acquaintance theory of accommodating the intuition that Le Verrier had singular thoughts about Neptune prior to observing it, and that various FBI agents and scared members of the public had singular thoughts about the Unabomber prior to his being apprehended. Moreover, Recanati's liberalized acquaintance theory appears to have implausible consequences of its own; consequences which it can avoid only by running the risk of making itself indistinguishable from cognitivism (or the hybrid theory).

Therefore I conclude that the mathematical platonist has excellent reasons to be a hybrid cognitivist about singular thought. For if we can have singular thoughts about abstract

mathematical objects, then given that  $a$  refers singularly to an object  $o$  if and only if token uses of  $a$  express singular thoughts about  $o$ , our expressions can refer singularly to abstract mathematical objects. Thus the platonist who is adopts Jeshion's cognitivism, or my hybrid cognitivism, has the resources to resist the restricted semantic argument against mathematical platonism from the causal constraint on singular reference.



### *Conclusion*

In the first introductory chapter I introduced the semantic argument against mathematical platonism. I took mathematical platonism to be the conjunction of the following semantic, alethic, metaphysical and epistemological theses:

(S): Pure mathematical statements should be read at *face-value*, that is, such that their surface syntactical forms are a good guide to their (Davidsonian) logical forms.<sup>193</sup>

(A): Some pure mathematical statements (that are not negative existentials or material conditionals) are true.

(M): The only possible referents of mathematical singular terms are abstract objects.

(A): Some pure mathematical statements (that are not negative existentials or material conditionals) are known (by us).

The formulation of the semantic argument against mathematical platonism that I ultimately came up with was as follows:

P1: In order for an expression *e* to refer to an object *o*, it must be metaphysically possible for *o* to stand in at least one causal relation. (This is the *causal constraint on reference* or CCR.)

P2: It is not metaphysically possible for any abstract mathematical object to stand in any causal relations. (This is a consequence of (M), provided some pure mathematical singular terms have referents.)

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<sup>193</sup> See section 1 of chapter one.

Therefore,

C: No expression can refer to any abstract mathematical object.

This conclusion (C) entails that not all of (S), (A) and (M) are true. Moreover because the denial of (A) entails the denial of (E), if not all of (S), (A) and (M) are true, then not all of (S), (A), (M) and (E) are true. Thus if both the CCR and thesis (M) of mathematical platonism are true, then mathematical platonism as a whole is false.

One important result I reached in the introduction was that the semantic argument is strictly stronger than its better-known sibling, the *epistemological* argument against mathematical platonism. The epistemological argument said that if there is a causal constraint on *knowledge* – if knowledge that *P* requires causal contact with *P*'s subject-matter (or some such) – then not all of (S), (M) and (E) can be true: we cannot know any pure mathematical statements (if such statements are read at face-value).

Now, the denial of (A) entails the denial of (E): if pure mathematical statements cannot be true, they cannot be known. Thus if (S), (A) and (M) are not all true, then (S), (M) and (E) are not all true. Hence the conclusion of the semantic argument entails the conclusion of the epistemological argument. But the converse does not hold: the denial of (E) is consistent with (A). Thus even if (S), (M) and (E) are not all true, (S), (A) and (M) might still all be true. In this case, some pure mathematical statements that are not negative existentials or vacuously true material conditionals would be true but unknowable. Therefore the conclusion of the epistemological argument does not entail the conclusion of semantic argument. Hence the semantic argument is strictly stronger than the epistemological argument.

In chapter two I investigated the relationship between the semantic argument against

mathematical platonism and mathematical *fictionalism*, the view that (S) holds – we should read pure mathematical statements at face-value – but (A) does not: pure mathematical statements are not *true* (except negative existentials and vacuously true material conditionals). Because fictionalism denies (A), it faces the task of explaining why pure mathematical statements are so extremely useful – and perhaps even indispensable – in scientific contexts. I outlined Hartry Field’s response, which proceeds in two steps. First, we show that our chosen mixed mathematical-physical theory *conservatively extends* its purely nominalistic fragment; hence there are no results about the physical world that we can infer from the mixed mathematical-physical theory that we could not, in principle, infer from the purely physical sub-theory alone. Second, when a mixed mathematical theory conservatively extends its purely physical sub-theory, use of the mixed theory to reach conclusions about the physical world does not bring with it commitment to the abstract mathematical objects postulated by the mixed theory.

Fictionalists often use the semantic argument to motivate the move away from mathematical platonism and toward mathematical fictionalism (for those who do not want to give up claim (S)). I argued that making this use of the semantic argument commits the fictionalist to taking certain stands in debates that, at first glance, might appear far removed from the concerns original to mathematical fictionalism. In particular, the fictionalist who endorses the conclusion of the semantic argument needs to maintain that reference to abstract mathematical objects is conceptually impossible, but logically possible. But because the fictionalist who wants to avoid commitment to abstract mathematical objects in metalogic must be a primitivist about logical possibility (or logical consistency, which Field takes to be the same thing), the best way for the fictionalist to maintain the required distinction between logical and conceptual possibility is to endorse *inferentialism* about logical expressions. But, I argued, endorsing inferentialism about

logical expressions requires that one take a stand against at least one of the following two arguments: that inferentialism about logical expressions leads to the rejection of the law of the excluded middle, and that rejecting the law of the excluded middle is to take a step away from realism. For if the fictionalist does not successfully resist at least one of these arguments, then they are in danger of being charged with anti-realism about ordinary concrete objects as well as abstract mathematical objects.

In chapter three I gave an argument for the CCR: a semantic indeterminacy argument, adapted from Benacerraf's (1965) argument that the natural numbers cannot be objects. I noted that Benacerraf's argument does not actually entail the conclusion that numbers cannot be objects *at all*, but only that if they are objects, they must be *sui generis* arithmetical objects: objects whose genuine properties can only be arithmetical properties (like being less than 17). This in turn entails that numbers cannot stand be spatiotemporally located or have causal powers. Nevertheless the move to taking numbers to be *sui generis* in this way does nothing to help the platonist avoid a specific kind of semantic indeterminacy argument – the *local permutation* argument – for the CCR. I then argued that one good bet for the platonist here is to endorse Lewis's *reference magnetism* view, according to which the relative 'naturalness' of an entity can by itself serve as grounds for the selection of that entity as the referent of an expression from a range of otherwise equally qualified alternatives. Lastly I noted that if we can establish the uniqueness claim that reference-magnetism is the only way the platonist can avoid the local permutation argument, then any reasons there are for being a platonist – such as the (supposed) indispensability of pure mathematical theories to natural science – thereby become reasons for adopting reference-magnetism.

In chapter four I moved to the question of *singular* as opposed to merely *descriptive*

reference to abstract mathematical objects. I argued that even the half-way view that we can only *definitely describe* abstract mathematical objects – which is all the reference magnet view floated at the end of chapter 3 can guarantee – may still be unfavourable to the platonist who wants to endorse claim (S). Thus it behooves the platonist to find an account of singular reference permitting such reference to abstract mathematical objects. Given the assumption that for a (non-indexical<sup>194</sup>) expression *e* to refer singularly to an object *o*, the thought ‘*e* is  $\Phi$ ’ needs to express a singular *thought* about *o*, one way to construct an account of singular reference to abstract mathematical objects is to build it around an account of singular thought.

I then described two contemporary views of singular thought: acquaintance theory and semantic instrumentalism. Acquaintance theory, according to which singular thought requires acquaintance, cannot allow for singular thought about abstract objects because acquaintance is standardly taken to require causal contact. And while semantic instrumentalism does allow for singular thought about abstract mathematical objects, it is widely viewed as implausible, because it entails *voluntarism* about singular thought: the claim that an agent can put themselves in a position to have singular thoughts about any object they can definitely describe just by *choosing* to do so.

Thus I investigated the prospects of using a new theory of singular thought that aims to carve out a middle ground between acquaintance theory and semantic instrumentalism, Robin Jeshion’s *cognitivism*, as a beginning point for an account of singular reference to abstract mathematical objects. According to Jeshion, an agent *S* can have singular thoughts about and object *o* if and only if *o* is *significant* to *S*. I then argued that abstract mathematical objects can be

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<sup>194</sup> Recall that I set aside indexical expressions like “I” which may be counterexamples to this assumption as I’ve naïvely stated it.

significant to us in the way required by cognitivism. Thus the platonist wanting to explain how we could refer singularly to abstract mathematical objects would do well to adopt cognitivism about singular thought.

The rest of the chapter was concerned with defending the platonist's use of cognitivism from three objections. In response to the first, that abstract mathematical objects are not significant to many people, I claimed that for us to be able to refer singularly to those objects requires only that they be significant to some *experts* present in our linguistic community to which we can defer when using expressions like “7” and “the even prime”. In response to the second objection, that Jeshion's cognitivism entails that some perceptual demonstrative thought is descriptive, I described a hybrid view according to which perceptual demonstrative thought is acquaintance-based, but all singular thought about objects that are not perceptually present at the time of thought – including singular thought based on memory – is significance-based. I also defended my hybrid view against the charge that it is *ad hoc*. The third objection was that the apparent instances of acquaintanceless singular thought that motivate accepting cognitivism over acquaintance theory can be better explained by a liberalized *future acquaintance* constraint that still rules out singular thought about abstract mathematical objects. In response I argued that the future acquaintance constraint itself has implausible consequences; and the best way to avoid these consequences reduces the future acquaintance constraint to Jeshion's significance constraint.

Therefore I ultimately conclude that the package of views consisting of Lewis's reference magnetism, plus my hybrid cognitivism about singular thought, allows the mathematical platonist to resist the semantic argument against platonism. Thus insofar as that package of views is plausible, mathematical platonism is plausible. And if the following uniqueness claim (which I have not argued for) turns out to be true – the package I've described is the *only* package of views

allowing the platonist to plausibly defend their view against the semantic argument – then insofar as platonism is plausible, so is the conjunction of reference magnetism and hybrid cognitivism.

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