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Title

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Abstract

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CHAPTER 1

Introduction

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Outline

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CHAPTER 2

Compressive Sensing

In an underdetermined system of linear equations there are fewer equations than unknowns. In mathematical terms this can we stated as the matrix equation $A\mathbf{x} = \mathbf{y}$ where $A \in \mathbb{C}^{M \times N}$, $\mathbf{x} \in \mathbb{C}^N$, $\mathbf{y} \in \mathbb{C}^M$ and $M < N$. This equation is unsolvable in the general case, however, under certain conditions, it is possible to find exact or estimated solutions. The underlying assumptions that make it possible is sparsity and compressibility. The research area associated with these assumptions is called compressive sensing. The goal of this chapter is to show that by using compressive sensing techniques it possible to construct a stable and robust mapping $\mathbf{B} : \mathbb{C}^M \to \mathbb{C}^N$ such that an estimate or solution would exist for the *inverse problem* $\mathbf{x} = A^{-1}\mathbf{y}$.

2.1 Sparsity and Compressibility

This section introduces the reader to the main assumptions and their notions in CS. Necessary terminology will be introduced appropriately.

Definition 2.1.1. The support of a vector $\mathbf{x} \in \mathbb{C}^N$ is the index set of its non-zero entries, that is:

$$supp(\mathbf{x}) := \{i \in [N] : x_i \neq 0\}.$$

In Compressive Sensing it is customary to abuse the l_0 notation to denote the cardinality of the support set , i.e. the number of non-zero entries of a vector. The $\|\cdot\|_0$ fails to be a norm due to not satisfying the scaling property of norms. We can now define *s-sparse* vectors.

Definition 2.1.2. The vector $\mathbf{x} \in \mathbb{C}^N$ is called *s-sparse* if it has no more than s non-zero entries, that is if: $\|\mathbf{x}\|_0 \leq s$

The notion of sparsity is an ideal one, meaning that in the real world it may very well be that our vector is only close to being sparse. In order for CS to tackle more problems, we may also be interested in the following notion of *compressibility*.

Definition 2.1.3. For $p > 0$, the measure of a vectors compressibility is given by the l_p - error of best s-term approximation to $\mathbf{x} \in \mathbb{C}^N$ defined by

$$\sigma(\mathbf{x})_p := inf\{\|\mathbf{x} - \mathbf{z}\|_p, \mathbf{z} \in \mathbb{C}^N iss - sparse\}$$

Informally, a vector is compressible if $l_p - error$ decays quickly in s.

2.2 Algorithms

In line with our goal to construct the mapping $\mathbf{B} : \mathbb{C}^M \to \mathbb{C}^N$ it will be useful to restate the compressive sensing problem as an optimization problem and to show that by solving the compressive sensing problem, that is to find the s-sparse vector consistent with Ax=y, we also solve the map construction problem.

$$\min \|\mathbf{z}\|_0 \text{ subject to } A\mathbf{z} = A\mathbf{x} \tag{2.1}$$

(P_0) is a combinatorial optimization problem, but the problem is generally NP-hard. Since this makes (P_0) intractable, we will solve the convex relaxation of (P_0) instead.

$$\min\|\mathbf{z}\|_1 \text{ subject to } A\mathbf{z} = A\mathbf{x} \tag{2.2}$$

Solving the inverse problem by solving (P_1) is known as Basis Pursuit.

Basis Pursuit

In order to show why BP can solve (P_0) we need to introduce the Null Space Property of a matrix A:

Definition 2.2.1. A matrix $A \in \mathbb{C}^{M \times N}$ is said to satisfy the *Null Space Property (NSP)* relative to a set $S \subset \{1, 2, ..., N\}$ if

$$\min \|\mathbf{v}_S\|_1 < \|\mathbf{v}_{\overline{S}}\|_1 \text{ for all } \mathbf{v} \in \ker A \setminus \{\mathbf{0}\}$$

A is said to satisfy the *Null Space Property of order s* if it satisfies the null space property relative to any set S $\subset \{1, 2, ..., N\}$ with $|\mathrm{S}| \leq s$

The following theorem shows that the NSP of a matrix is a sufficient condition in order to solve (P_0).

Theorem 2.2.2. *Given a matrix $A \in \mathbb{C}^{M \times N}$, every vector $\mathbf{x} \in \mathbb{C}^N$ supported on a set S is the unique solution (P_1) with $A\mathbf{x} = \mathbf{y}$ if and only if A satisfies the NSP relative to S.*

Furthermore, if the set S varies, then every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ is the unique solutions to (P_1) with $A\mathbf{x} = \mathbf{y}$ if and only if A satisfies the NSP of order s.

Proof. Let S be a fixed index set, and assume that every vector $\mathbf{x} \in \mathbb{C}^N$ supported on this set, is the unique minimizer of (P_1). From the assumption it follows that for $\mathbf{v} \in \ker A \setminus \{\mathbf{0}\}$, the vector $\mathbf{v}_S$ is the unique minimizer of (P_1). Since $A(\mathbf{v}_S + \mathbf{v}_{\overline{S}}) = \mathbf{0}$ and $-\mathbf{v}_S \neq \mathbf{v}_{\overline{S}}$, from the minimality assumption we must have that $\|-\mathbf{v}_S\|_1 < \|\mathbf{v}_{\overline{S}}\|_1$. This established the NSP relative to S.

Conversely, assume that NSP relative to S holds. Let $\mathbf{x} \in \mathbb{C}^N$ be supported on S and a vector $\mathbf{z} \in \mathbb{C}^N$, $\mathbf{z} \neq \mathbf{x}$, such that $A\mathbf{z} = A\mathbf{x}$. Following the rules for norms and taking complements for the support of a set , we obtain

$$\|\mathbf{x}\|_1 \leq \|\mathbf{x}-\mathbf{z}_S\|_1+\|\mathbf{z}_S\|_1 = \|\mathbf{v}_S\|_1+\|\mathbf{z}_S\|_1 < \|\mathbf{v}_{\overline{S}}\|_1+\|\mathbf{z}_S\|_1 = \|-\mathbf{z}_{\overline{S}}\|_1+\|\mathbf{z}_S\|_1 = \|\mathbf{z}\|_1.$$

Which shows that $\mathbf{x}$ obtains the unique minimum.

To prove the second part of the theorem,let S vary and assume that every s-sparse vector $\mathbf{x}$ is found by solving (P_1) subject to $A\mathbf{x} = \mathbf{y}$. Let $\mathbf{z}$ be the solution to P_0 subject to $A\mathbf{x} = \mathbf{y}$ then $\|\mathbf{z}\|_0 \leq \|\mathbf{x}\|_0$ so that also $\mathbf{z}$ is s-sparse. But since every s-sparse vector is the unique minimizer of (P_1), we have that $\mathbf{x} = \mathbf{z}$ and the result follows. ■

Appendices