The DeGiorgi-Nash-Moser Estimates

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1 Introduction

Eine der begrifflich merkwürdigsten Thatsachen in den Elementen der Theorie der analytischen Funktionen erblicke ich darin, daß es Partielle Differentialgleichungen giebt, deren Integrale sämtlich notwendig analytische Funktionen der unabhängigen Variabeln sind, die also, kurz gesagt, nur analytischer Lösungen fähig sind - David Hilbert , 1900 [1]

In this set of notes, we provide a proof of the Degiorgi-Nash-Moser C^α estimates for weak solutions of divergence form uniformly elliptic equations. We will also provide proof of a Harnack inequality for solutions to such equations. The proof has two parts, the first being to show local L^∞ boundedness of weak solutions, and the second being to prove C^α estimates through an iterative, oscillation improvement type technique. Finally, we prove the Harnack inequality through a similar technique.

2 Preliminaries

Throughout this work, B_R refers to the ball of radius R centered at 0 in the set $\mathbb{R}^d$. That is to say:

$$B_R = \{x \in \mathbb{R}^d : \|x\|_{l^2} \leq R\}$$

We examine solutions to the elliptic equation:

$$-\mathrm{div}(A(x)\nabla u) = 0 \quad \text{in } B_2\text{, in the sense of distributions} \tag{1}$$

We say $u \in \mathscr{H}^1(B_2)$ solves (1) if for all $\varphi \in \mathscr{H}_0^1(B_2)$ we have:

$$\int_{B_2} \langle A(x)\nabla u, \nabla\varphi\rangle \mathrm{d}x = 0 \tag{2}$$

where, writing $A(x) = [a_{ij}(x)]$, all $a_{ij}(x)$ are measurable functions[2], $\langle \dot{;} \rangle$ is the standard Euclidean inner product , and A is *uniformly elliptic*, i.e $\exists\ \lambda, \Lambda > 0$ such that

$$\lambda|v|^2 \leq \langle A(x)v, v\rangle \leq \Lambda|v|^2 \tag{3}$$

for all $v \in \mathbb{R}^d$, with λ, Λ independent of the choice of x. Other than this, we make no regularity assumptions on A. Since we are interested in weak solutions, henceforth, all derivatives are understood to be in the sense of distributions unless strengthened explicitly.

We say that u is a *subsolution* to (1) if for every nonnegative $\varphi \in \mathscr{H}_0^1(B_2)$ we have:

$$\int_{B_2} \langle A(x)\nabla u, \nabla\varphi\rangle \mathrm{d}x \leq 0 \tag{4}$$

[1]English translation by Mary Frances Winston Newson:-"One of the most remarkable facts in the elements of the theory of analytic functions appears to me to be this: that there exist partial differential equations whose integrals are all of necessity analytic functions of the independent variables, that is, in short, equations susceptible of none but analytic solutions"

[2]The assumption that a_{ij} are measurable is not necessary, as the fact is not used anywhere in the proof. The point (we believe) is to emphasize that we need very little regularity on the a_{ij}, (making the result very general) as measurable functions are recognizable as a very large class of functions.

We say that u is a *supersolution* if the converse inequality holds, i.e:

$$\int_{B_2} \langle A(x)\nabla u, \nabla\varphi\rangle \mathrm{d}x \geq 0 \tag{5}$$

for all $\phi \in \mathscr{H}_0^1(B_2)$. Note that a solution is a function that is both a super and sub-solution.

The family of subsolutions is amenable to several useful techniques, and it is an important fact that a convex, increasing function of a subsolution is again a subsolution. In particular, the maximum of two subsolutions is again a subsolution. We record this fact below.

Lemma 1. *Let $F : \mathbb{R} \to \mathbb{R}$ be an convex, increasing function. Then for any subsolution u of (4), $F \circ u$ is also a subsolution.*

To prove this, we need another result from Sobolev space theory.

Lemma 2. *If $F : \mathbb{R} \to \mathbb{R}$, F is C^1 with $F(0) = 0$, and $u \in \mathscr{H}^1(\Omega)$ for Ω a domain in $\mathbb{R}^d$, then $F \circ u \in \mathscr{H}^1(\Omega)$ and $\nabla(F \circ u) = F'(u)\nabla u$*

The proof of Lemma 2 is a standard approximation argument. This in hand, let us prove Lemma 1:

Proof. Let us first examine the case where F is twice continuously differentiable, so that $F' \geq 0$ and $F'' \geq 0$. Observe that

$$\int_\Omega \langle A\nabla u, F^{'}(u)\nabla\phi\rangle = \int_\Omega \langle A\nabla u, \nabla(F'\phi)\rangle - \int_\Omega \langle A\nabla u, \nabla u\rangle F^{''}(u)\phi$$

The first integral is less than or equal to 0, due to $F'\phi$ being a nonnegative test function. Now, $F^{''}(u)$ is greater than or equal to 0, and we have the uniform ellipticity condition giving us the bound:

$$-\int_\Omega \langle A\nabla u, \nabla u\rangle F^{''}(u)\phi \leq -\lambda \int_\Omega |\nabla u|^2 F^{''}(u)\phi$$

And this integrand is positive on the right hand side, so applying a minus sign to it makes it negative.
For the general case, we see that we may approximate convex function with mollifiers and pass to the limit. More precisely, let η_ϵ be the standard mollifier, and set $F_\epsilon(t) = \eta_\epsilon * F(t)$. Computing derivatives, we have that $F_\epsilon'(t) = \eta_\epsilon * F'(t) \geq 0$ and $\eta_\epsilon * F''(t) \geq 0$. Thus, by what we have shown, $F_\epsilon(t)$ is a subsolution. Now, $F_\epsilon'(t) \to F'(t)$ almost everywhere as $\epsilon \to 0^+$, and because η has compact support, we have:

$$0 \geq \int_\Omega \langle AF_\epsilon'(u)\nabla u, \nabla\phi\rangle \to \int_\Omega \langle AF'(u)\nabla u, \nabla\phi\rangle = \int_\Omega \langle A\nabla(F \circ u), \nabla\phi\rangle$$

as we need. □

Next, we record two more results that are used to prove the Harnack inequality. Both hold generally in Sobolev spaces, but proved here particularly for $\mathscr{H}^1$ functions. The first says that nontrivial characteristic functions (indicator functions over borel measureable sets) do not have L^p distributional derivatives, and therefore, are not $\mathscr{H}^1$. The second says roughly that an $\mathscr{H}^1$ function cannot be "concentrated" at two distinct real numbers, without achieving intermediate values; more precisely, if an $\mathscr{H}^1$ function achieves two distinct real values on a set of positive measure, then it achieves the interval of intermediate values on a set of positive measure as well.

Fact 2: Let $u \in \mathscr{H}^1(\Omega)$ such that u is the characteristic function χ_S of some set $S \subset \Omega$. Then, $S = \Omega$ or $S = \emptyset$.

Proof. Take χ_S for some $S \subset \Omega$. Now, if χ_S is an element of $\mathcal{H}^1$, it must satisfy the difference quotient criterion, i.e:

$$\int_{\mathbb{R}^d} (\chi_S(x) - \chi_S(x+h))^2 \mathrm{d}x \leq C|h|^2$$

Now, in the case of χ_S, notice that the square in the above integral is irrelevant, hence, the condition is now:

$$\|\chi_S - \tau_h\chi_S\|_{L^1} = \int_{\mathbb{R}^d} |\chi_S(x) - \chi_S(x+h)| \mathrm{d}x \leq C|h|^2$$

Subdividing h into m blocks of equal length and using the triangle inequality, we have:

$$\|\chi_S - \tau_h\chi_S\|_{L^1} \leq \sum_{j=0}^{n-1} \|\tau_{\frac{j}{m}h}\chi_S - \tau_{\frac{j+1}{m}h}\|_{L^1} \leq Cm\left|\frac{h}{m}\right|^2 = \frac{C|h|^2}{m}$$

by assumption. Moreover, this is true for all h. Thus, if we subdivide finely enough, the left hand side tends to 0 for all h, no matter how large. On the other hand, the right hand side tends to $2\mu(S)$ as h tends to infinity. Thus, $\chi_S = \tau_h\chi_S$ for a.e. x, that is $\chi \equiv 0$ a.e. or $\chi \equiv 1$ a.e., proving our statement. □

Fact 3: Let $u \in \mathscr{H}^1(B_1)$, $u\colon B_1 \longrightarrow [0,1]$ with $\|u\|_{H^1(B_1)} \leq C$ and suppose that $\exists\delta_0, \delta_1$ so that

$$|\{u \in B_1 \mid u = 1\}| \geq \delta_0, |\{u \in B_1 \mid u = 0\}| \geq \delta_1$$

Then, $\exists\ \epsilon$ depending only on C, δ_0, δ_1 so that

$$|\{x \in B_1 \mid 0 < u(x) < 1\}| > \epsilon$$

Proof. Suppose the statement failed. Then we can find a sequence of functions $u_n : B_1 \to [0,1]$ s.t. $\|u_n\|_{H^1(B_1)} \leq C$, $|\{u_n = 0\}| \geq \delta_0$, and $|\{u_n = 1\}| \geq \delta_1$, and $|\{0 < u_n(x) < 1\}| < \frac{1}{n}$ for each n.

By the Rellich-Kondrachov theorem, we can assume without loss of generality that our whole sequence converges in L^2 to some function u. By weak compactness, a subsequence of u_n has a weak limit in $\mathscr{H}^1$; hence $u \in H^1$. Moreover, by passing to a further subsequence, we can assume again without loss of generality that the entire sequence u_n converges to u pointwise.

Then by dominated convergence, $|\{x \in B_1 \mid 0 < u(x) < 1\}| = 0$ and $|\{x \in B_1 \mid u(x) < 0, 1 < u(x)\}| = 0$, Hence $u = 0$ or $u = 1$ almost everywhere, contradicting Lemma 1. □

We now prove the principal inequality on which our proofs depend, Cacciopoli's Inequality. It will allow us to control the integral of $|\nabla u|^2$ over a ball, by the integral of u^2 over a larger ball. This is the only result in which the elliptic structure in (1) is invoked explicitly [3].

Lemma 3. *(Cacciopoli Inequality) Let u be a nonnegative subsolution in the sense of (4) on the ball $B_{1+\delta}$. Let $\eta : B_{1+\delta}$ be a smooth, nonnegative cutoff function, with $\eta \equiv 0$ on $\partial B_{1+\delta}$. Then, there exists a constant $C = C(d, \lambda, \Lambda)$ such that:*

$$\int_{B_{1+\delta}} \eta^2|\nabla u|^2 \mathrm{d}x \leq \int_{B_{1+\delta}} u^2|\nabla\eta|^2 \mathrm{d}x$$

Proof. Let u be a subsolution and let η be a test function as above. Consider $\psi = u\eta^2$. Note that this is a nonnegative cutoff function in $\mathscr{H}_0^1(\Omega)$ and thus may be used as a test function as in the subsolution definition. Specifically, we have:

$$\int_{B_{1+\delta}} \langle A\nabla u, \nabla\psi\rangle \mathrm{d}x \leq 0 \tag{6}$$

Let us expand the above integral. We get:

$$\int_\Omega \eta\langle A(x)\nabla u(x), \nabla u(x)\rangle \mathrm{d}x + 2\int_\Omega u\langle A(x)\nabla u(x), \nabla\eta(x)\rangle \mathrm{d}x \leq 0$$

and thus:

$$\int_\Omega \eta\langle A(x)\nabla u(x), \nabla u(x)\rangle \mathrm{d}x \leq -2\int_\Omega u\langle A(x)\nabla u(x), \nabla\eta(x)\rangle \mathrm{d}x$$

The use of the product rule here to compute the derivative is allowed thanks to closure of certain Sobolev spaces under multiplication. The right hand side may be bounded using the smaller ellipticity constant λ. Specifically, we have that:

$$\lambda\int_\Omega |\nabla u|^2\eta^2 \leq \int_\Omega \eta^2\langle A(x)\nabla u(x), \nabla u(x)\rangle \mathrm{d}x \tag{7}$$

[3] It is a bit disingenuous to say this is the only place where the structure is used, as nearly all results proved after rely on this inequality

For the right hand term, we use a combination of Young's inequality and the Schwarz inequality. Specifically:

$$2u\langle A(x)\nabla u(x), \nabla\eta(x)\rangle \leq u|A\nabla u||\nabla\eta| \leq \Lambda|\nabla u||\nabla\eta|u$$

here we use the upper bound on the uniformly elliptic matrix. Hence:

$$2\int_\Omega u\langle A(x)\nabla u(x), \nabla\eta(x)\rangle\eta \mathrm{d}x \leq 2\int_\Omega \Lambda|\nabla u||\nabla\eta|u\eta \mathrm{d}x$$

Applying the Schwarz inequality to the right hand side (splitting terms $\eta\nabla u$, $|\nabla\eta|u$ we get the bound:

$$2\Lambda\int_\Omega |\nabla u||\nabla\eta|u\mathrm{d}x \leq 2\Lambda\left(\int_\Omega |\nabla u|^2\eta^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla\eta|^2u^2\mathrm{d}x\right)^{1/2}$$

Applying Young's inequality we arrive at:

$$2\Lambda\left(\int_\Omega |\nabla u|^2\eta^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla\eta|^2u^2\mathrm{d}x\right)^{1/2} \leq \Lambda\left(\int_\Omega |\nabla\eta|^2u^2\mathrm{d}x + \int_\Omega \eta^2|\nabla u|^2\mathrm{d}x\right)$$

Subtracting from both sides and normalizing gives the result. □

Corollary 4. *For any nonnegative subsolution on $B_{1+\delta}$ we have:*

$$\int_{B_1} |\nabla u|^2\mathrm{d}x \leq \frac{C}{\delta^2}\int_{B_{1+\delta}} u^2\mathrm{d}x$$

Proof. Apply Lemma 5 with a smooth cutoff function $\phi = 1$ on B_1 and $\phi = 0$ on $\partial_{B_{1+\delta}}$ whose gradient is norm bounded by $\frac{1}{\delta}$. The existence of such a function is standard, and drawing a function that is flat on two concentric balls, connected by a linear interpolation on the annulus in between (i.e., a plateau) will convince the skeptical reader of its existence. □

3 Local Boundedness of Weak Solutions

We now show that weak solutions to (1) are locally bounded. The main result we prove is the following:

Theorem 5. *Let u be a weak solution to (1), then there exists a constant $C_2 = C_2(d, \lambda, \Lambda)$ such that:*

$$\|u\|_{L^\infty(B_1)} \leq C_2(d, \lambda, \Lambda)\|u\|_{L^2(B_2)} \tag{8}$$

To show this result, we will prove a result that is equivalent to it, but more tractable.

Lemma 6. *There exists some $\delta_0 = \delta_0(d, \lambda, \Lambda)$ such that $\|u\|_{L^2}(B_2) \leq \delta_0 \implies \|u\|_{L^\infty(B_1)} \leq 1$*

To prove this lemma, we use an iterative procedure. Consider the following truncations:

$$\begin{aligned} l_k &= 1 - 2^{-k} \\ r_k &= 1 + 2^{-k} \\ u_k &= (u - l_k)_+ \\ A_k &= \|u_k\|_{L^2(B_{r_k})} \end{aligned}$$

Note that each u_k is a nonnegative subsolution with $u_k \leq u_{k+1}$ and thus $A_k \leq A_{k+1}$. Moreover, throughout this section, we let $C = C(\lambda, \Lambda, d)$, that is, C depends only on the uniform ellipticity constants and dimension.

Lemma 7. *Theorem 4 is equivalent to the statement: if $A_0 \leq \delta_0$, then $\lim_{k\to\infty} A_k = 0$*

Proof. ($\implies$) First, we show that this result implies Lemma 8.

Note that:

$$A_k^2 = \int_{B(1+1/2^k)\cap\{u>(1-2^{-k})\}} (u - l_k)^2$$

Since $u \in L^2(B_2)$, and $(u-l_k)_+^2 \xrightarrow{k\to\infty} (u-1)_+^2$ pointwise a.e., the dominated convergence theorem gives

$$\lim_{k\to\infty} \int\limits_{B(1+1/2^k)\cap\{u>1-1/2^k\}} (u-l_k)_+^2 = \int\limits_{B_1\cap\{u>1\}} (u-1)_+^2 = 0$$

Where the limit is zero by hypothesis. This implies $m(B_1 \cap \{u>1\}) = 0$ or $(u-1)_+ \equiv 0$ on $B_1 \cap \{u>1\}$, so that $u \leq 1$ a.e. in B_1. That is $\|u\|_{L^\infty(B_1)} \leq 1$. □

Lemma 8. *We have:*

$$\|u_{k+1}\|_{L^p(B_{r_{k+1}})} \leq C(d,\lambda,\Lambda)2^{k+1}\|u_{k+1}\|_{L^2(B_{r_k})}$$

where $p = 2^*$ is the Sobolev conjugate of 2 in $\mathbb{R}^d$

Proof. By lemma 5 (the Cacciopoli inequality), we have the bound:

$$\int_{B_r} |\nabla u|^2 \mathrm{d}x \leq \frac{C}{(R-r)^2}\int_{B_R} u^2 \implies \|\nabla u\|_{L^2(B_r)} \leq \frac{\sqrt{C}}{R-r}\|u\|_{L^2(B_R)}$$

here $C = C(d,\lambda,\Lambda) > 0$, and u is an a.e. positive subsolution on the set B_R (here $R > r$). Thus, consider $R = 1 + \frac{1}{2^k}$ and $r = 1 + \frac{1}{2^{k+1}}$ so that $R - r = \frac{1}{2^{k+1}}$. Hence, we have the inequality:

$$\|\nabla u_{k+1}\|_{L^2(B_{1+\frac{1}{2^{k+1}}})} \leq C2^{k+1}\|u\|_{L^2\left(B_{1+\frac{1}{2^k}}\right)}$$

Applying the Sobolev inequality[4] to $2^* = p$, we conclude that:

$$\|u\|_{L^p(B_{1+\frac{1}{2^{k+1}}})} \leq C'\|\nabla u_{k+1}\|_{L^2(B_{1+\frac{1}{2^{k+1}}})}$$

Multiplying constants together, we conclude:

$$\|u\|_{L^p(B_{1+\frac{1}{2^{k+1}}})} \leq C2^{k+1}\|u\|_{L^2\left(B_{1+\frac{1}{2^k}}\right)}$$

as we wanted. □

Lemma 9. *We have:*

$$\|u_{k+1}\|_{L^2(B_{r_{k+1}})} \leq C2^{k+1}\|u_{k+1}\|_{L^2(B_{r_k})}$$

Proof. The sobolev conjugate implies a simple Holder inequality which can be combined with the above to give the following relation between the L^2 norm of the $k+1$ subsolution on $B_{r_{k+1}}$, and the L^2 norm u_{k+1} on B_r

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{d} \implies \frac{1}{\frac{p}{2}} + \frac{1}{\frac{d}{2}} = 1$$

$$\int\limits_{B_{r_{k+1}}} u^2 \leq \left(\int\limits_{B_{r_{k+1}}} u^p\right)^{\frac{2}{p}} \cdot \left(\int\limits_{B_{r_{k+1}}} 1^{d/2}\right)^{\frac{2}{d}} \implies \|u\|_{L^2(B_{r_{k+1}})} \leq \|u\|_{L^p(B_{r+1})} \cdot |\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{\frac{1}{d}}$$

Combining with the p-norm result in Lemma 10 gives

$$\|u_{k+1}\|_{L^2} \leq C2^{k+1}\|u_{k+1}\|_{L^2(B_{r_k})}|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{\frac{1}{d}}$$

□

Lemma 10. *We have the following recurrence relation between the A_k:*

$$A_{k+1} \leq C2^{k+4k/d}A_k^{1+4/d}$$

Moreover, this recurrence allows us to conclude that $A_k \to 0$ as $k \to \infty$ if A_0 is chosen sufficiently small.

[4] It may seem suspect to apply the Sobolev inequality because there are often restrictions to its application, specifically the function need some boundary/trace regularity. However, note that we are working on the larger ball $B_{1+1/2^k}$ and our function(after cutoff multiplication) is 0 on the complement. Specifically, this occurs in the proof of the Cacciopoli inequality.

Proof. First, observe the equality $\{x \in B_2 \colon u_{k+1} > 0\} = \{x \in B_2 \colon u_k > \frac{1}{2^{k+1}}\}$. Since $B_{k+1} \subset B_k$, Tchebychev's inequality gives

$$|\{u_{k+1} > 0\} \cap B_{k+1}| \leq |\{u_k > 2^{k+1}\} \cap B_k| \leq 2^{k+1}\|u_k\|_{L^2(B_k)}$$

Then, since $u_{k+1} \leq u_k$, we have the simple comparison

$$\|u_{k+1}\|_{L^2(B_k)} \leq \|u_k\|_{L^2(B_k)}$$

Applying the two inequalities above to the result of Lemma 11 gives:

$$\|u_{k+1}\|_{L^2(B_{k+1})} \leq C2^{k+1}\|u_{k+1}\|_{L^2(B_k)}(2^{k+1}\|u_k\|_{L^2(B_k)})^{1/d} \leq C2^{(1+1/d)(k+1)}\|u_k\|_{L^2(B_k)}^{1+\frac{1}{d}}$$

that is

$$\dagger\ A_{k+1} \leq C2^{(k+1)(1+\frac{1}{d})}A_k^{1+\frac{1}{d}}$$

With $\alpha = 1 + 1/d > 1$

$$A_1 \leq C2^{(0+1)(\alpha)}A_0^{\alpha}$$

$$A_2 \leq C^1 2^{((1)+1)\alpha}(A_1)^{\alpha} \implies A_2 \leq C^{1+\alpha}2^{2\alpha+\alpha^2}A_0^{\alpha^2}$$

By induction, the following general formula is verified:

$$\implies *\ A_{k+1} \leq C^{\sum\limits_{i=0}^{k}\alpha^i}2^{\sum\limits_{i=1}^{k+1}(k+2-i))\alpha^i}A_0^{\alpha^{k+1}}$$

Now, given $\epsilon > 0$, we can always choose k sufficiently large so that $(\sum\limits_{i=1}^{k+1}(k+2-i)\alpha^i - \alpha^k) < \epsilon$; in this case

$$A_{k+1} \leq C^{\alpha^k+\epsilon}2^{\alpha^{k+1}+\epsilon}A^{\alpha^{k+1}} \leq C^{\alpha^{k+1}}2^{\alpha^{k+1}}A_0^{\alpha^{k+1}} \cdot (2^{\epsilon}C^{\epsilon}) = (C2A_0)^{\alpha^{k+1}}(C2)^{\epsilon}$$

For $A_0 < \frac{1}{2C}$, this estimate vanishes as $k \to \infty$. Therefore, choosing $\delta > 0$ sufficiently small shows $A_0 < \delta \implies$

$$\lim_{k\to\infty} A_{k+1} \leq \lim_{k\to\infty} 2C(C2A_0)^{\alpha^{k+1}} = 0$$

□

With this procedure, we complete the proof of theorem 4.

4 Hölder Continuity of Solutions, Harnack Inequality

Having confirmed local boundedness of solutions, we now move to prove Hölder continuity of solutions. The main result is:

Theorem 11.

$$\|u\|_{C^{\alpha}(B_1)} \leq C(d, \lambda, \Lambda)\|u\|_{L^2(B_2)} \tag{9}$$

The first step is to prove the Weak Harnack Inequality:

Lemma 12. *(Weak Harnack Inequality)*

Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution. Assume that $|\{x \in B_2 : u(x) \geq 1\}| \geq \delta$. Then there exists $\theta = \theta(\lambda, \Lambda, d)$ with:

$$\mathrm{ess}-\inf_{B_1} u \geq \theta \tag{10}$$

We prove this using two lemmas.

Lemma 13. *(Density) Let $u : B_2 \to \mathbb{R}$ be a positive supersolution. Then, there exists $\epsilon_0 > 0$ such that if*

$$|\{x \in B_2 : u(x) \geq 1\}| \geq (1 - \epsilon_0)|B_2|$$

then $u(x) \geq \frac{1}{2}$ a.e. in B_1

Proof. Consider $v = (1-u)^+ = \max(1-u, 0)$. This is the maximum of two subsolutions and hence a subsolution as well. Now, take ϵ_0 with:

$$|\{x \in B_2 : u(x) \geq 1\}| \geq (1-\epsilon_0)|B_2|$$

Suppose that $\|v\|_{L^\infty}(B_1) > 1/2$ (i.e so that $\|u\|_{L^\infty}(B_1) \leq 1/2$. Moreover, note that because u is positive, v cannot exceed 1. Then, applying theorem 2.1 to v, which is nonnegative by definition, we see that:

$$\|v\|_{\infty(B_1)} \leq C \left(\int_{B_2} v^2 \right)^{1/2}$$

Now,

$$\int_{B_2} v^2 = \int_{\{v \neq 0\} \cap B_2} v^2$$

v is only not equal to 0 if and only if $u < 1$, thus the integral above is:

$$\int_{\{u<1\} \cap B_2} v^2 \leq |\{u < 1\} \cap B_2|$$

thanks to the fact that $v \leq 1$ by definition. Finally, we know by assumption that:

$$|\{u < 1\} \cap B_2| \leq \epsilon_0 |B_2|$$

and we may send ϵ_0 to 0 so that the right hand side is smaller than $1/2$, which contradicts our initial assumption ☐

Lemma 14. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution. If $|\{x \in B_2 \mid u(x) \geq 1\}$, then*

$$\delta_k = |\{x \in B_{\frac{3}{2}} \mid u(x) < 2^{-k}\}| \to 0$$

with a rate of convergence independent of u,

Proof. Choose $r > \frac{3}{2}$ s.t. $|B_2 \setminus B_r| < \delta$. Then for some δ_0, $|\{x \in B_r \mid u \geq 1\}| \geq \delta_0$. Note that r and δ_0 depended on δ and not on u, so it is enough to prove the estimate with $\delta_k = \{x \in B_r \mid u < 2^{-k}\}$. Define supersolutions $v_k : B_2 \to [0, 2]$ by

$$v_k = 2^{k+1}(\min(u, 2^{-k}) - 2^{-(k+1)}) + 1$$

Then $2 - v_k$ is a subsolution, so we may apply Cacciapoli on B_r to $\|(\nabla 2 - v_k)\|_{L^2(B_r)} = \|v_k\|_{L^2(B_r)}$: for a constant C depending on r (and hence on δ), our ellipticity constants, and dimension, we have

$$\|\nabla v_k\|_{L^2(B_r)} = \|\nabla(2 - v_k)\|_{L^2(B_r)} \leq C\|2 - v_k\|_{L^2(B_2)} \leq 2C|B_2|$$

Hence we have uniform control on $\|v_k\|_{H^1(B_r)}$. Suppose that $\delta_k \to \delta_1 > 0$. Note that $\delta_1 = \{x \in B_r \mid u = 0\}$.

$$|\{x \in B_r \mid h_k = 0\}| = |\{x \in B_r \mid u = 0\}| = \delta_k \geq \delta_1$$

We also have

$$|\{x \in B_r \mid h_k = 2\}| = |\{x \in B_r \mid u > 2^{-k}\}| \geq |\{x \in B_r \mid u \geq 1\}| \geq \delta_0$$

Hence by Corollary 4, for some $\epsilon > 0$ we have, uniformly in k:

$$\epsilon < |\{x \in B_r \mid 0 < h_k < 2\}| = |\{x \in B_r \mid 0 < u < 2^{-k}\}| = \delta_k - \delta_1$$

giving us the result ☐

We are now in a position to prove the Weak Harnack Inequality (Lemma 14.)

Proof. Using Lemma 15, we choose $\epsilon_0 > 0$ small enough (independent of u) such that

$$|\{x \in B_2 \mid u(x) \geq 1\}| \geq (1-\epsilon_0)|B_2| \implies \text{ess-inf}_{B_1}(u) > \frac{1}{2}$$

And pick r large enough that

$$|B_r| > (1-\epsilon_0)|B_2|$$

From (18),

$$|\{x \in B_r \mid u \geq 2^{-k}\}| = |B_r| - \delta_k \to |B_r|$$

So for k sufficiently large, depending on δ and the ellipticity condition but not on u,

$$|\{x \in B_r \mid u \geq 2^{-k}\}| \geq (1 - \epsilon_0)|B_2|$$

Let $v = 2^k u$. Then

$$|\{x \in B_2 \mid v \geq 1\}| \geq |\{x \in B_r \mid v \geq 1\}| \geq (1 - \epsilon_0)|B_2|$$

Hence, from (16),

$$\text{ess-inf}_{B_1} v \geq \frac{1}{2} \text{ and ess-inf}_{B_1} u \geq 2^{k+1} = \theta$$

Since k was independent of u, θ depends only on ϵ_0, dimension, and our ellipticity constants. ☐

The weak Harnack inequality almost immediately furnishes the following oscillation estimate, from which we will obtain Theorem 13 by an iterative argument.

Lemma 15. *Let $u : B_2 \to [0, 1]$ be a solution. Then for any ball $B_r(x)$ s.t. $B_{2r}(x) \subset B_2$, one has*

$$osc_{B_r(x)} u \leq (1 - \theta) osc_{B_2 r(x)} u$$

Proof. By rescaling, it is enough to prove this when $r = 1$, and $\text{osc} B_{2r} u = \text{osc}_{B_2} u = 1$. Shifting u by a constant will not effect our oscillation estimate, so we can take $u : B_2 \to [0, 1]$.

Let $\delta = \frac{|B_2|}{2}$. Then either $|\{x \in B_2 \mid u(x) \geq \frac{1}{2}\}| \geq \delta$ or $|\{x \in B_2 \mid u(x) \leq \frac{1}{2}| \geq \delta$.

In the first case, from Lemma 14, for some $\theta > 0$ independent of u we have $\text{ess-}\inf_{B_1} u \geq \frac{\theta}{2}$ and $\text{ess-}\sup_{B_1} \leq 1$. Hence

$$\text{osc}_{B_1} u \leq (1 - \frac{\theta}{2}) = (1 - \frac{\theta}{2}) \text{osc}_{B_2} u$$

In the second case, $v = 1 - u$ is a solution with

$$|\{x \in B_2 \mid v \geq \frac{1}{2}\}| = |\{x \in B_2 \mid u \leq \frac{1}{2}\}| \geq \delta$$

Then from the above,

$$(1 - \frac{\theta}{2}) \text{osc} B_2 u = (1 - \frac{\theta}{2}) \geq \text{osc}_{B_1} v = \text{osc}_{B_1} u$$

observing that $\theta_0 = \frac{\theta}{2} > 0$ is independent of u, we conclude. ☐

We can now give quick proof of Theorem 13:

Proof. We can assume that $\|u\|_{L^2(B_2)} < \infty$. It is easy to see that by extending Theorem 7 slightly, we can then take $\|u\|_{L^\infty(B_{\frac{3}{2}})} < \infty$. Similarly, it is easy to see that Lemma 17 holds if we replace 2 by $\frac{3}{2}$. The upshot is that it is enough for us to bound $\|u\|_{C^\alpha(B_{\frac{1}{2}})}$ assuming $\|u\|_{L^\infty}(B_1) < \infty$.

We already know that $\|u\|_{L^\infty}(B_{\frac{1}{2}}) \leq C\|u\|_{L^2(B_2)}$, so we need only show that for some α, $[u]_{C^\alpha(B_{\frac{1}{2}})} \leq \|u\|_{L^\infty(B_1)}$. If $u(B_2) \subset [a, b]$, then let $v = \frac{u-a}{b-a}$. Then $v : B_2 \to [0, 1]$, and it is enough to show that for some C independent of u, $[v]_{C^\alpha(B_{\frac{1}{2}})} \leq C$.

Fix some x_0. If $r_n = 2^{-(n+1)}$, (just iterate.) ☐

Theorem 16. *(Harnack Inequality) Let u be a nonnegative solution to 1. Then, we have that:*

$$\sup_{B_{1/4}} u \leq C(B_2) u(0)$$

here, the constant C is written as $C(B_2)$ to indicate that it is domain dependent.

Heuristically, we will prove this by showing that if the ratio $\frac{\sup_{B1/4} u}{u(0)}$ was large, then we could construct a sequence x_n in $B_{1/2}$ with $u(x_n)$ an unbounded sequence. This of course contradicts the boundedness of u in B_1, as we showed previously. We prove it through a sequence of lemmas.

Lemma 17. *Let u be a nonnegative super solution defined on B_2. Fix $x_0 \in B_2$, and consider the nested balls $B_r(x_0) \subset B_R(x_0) \subset B_{2R}(x_0) \subset B_2$. We have that:*

$$\inf_{B_R(x_0)} u \geq c\left(\frac{r}{R}\right)^q \inf_{B_r(x_0)}$$

where c, q are universal constants.

Proof. We prove this using a scaled iteration of the weak Harnack inequality. Fix some $r < R$ and consider $a = \inf_{B_r(x_0)} u$. We know $a > 0$, and hence on B_r we have that $u/a \geq 1$. It follows that:

$$|\{u/a \geq 1\} \cap B_{2r}| \geq |B_r| = \frac{1}{2^d}|B_{2r}|$$

Thus, applying the weak Harnack inquality to u/a, we have some θ such that $u/a \geq \theta$ on B_{2r} and hence $u \geq a\theta$. Applying this same argument (dividing through etc.), we see that $u \geq a\theta^n$ on the ball $B_{2^n r}$ for all n with $2^n r < R$. Now, there exists a unique integer N such that:

$$2^{N-1}r \leq R \leq 2^N r$$

Hence, $u \geq a\theta^N$ on the ball $B_{2^N r}$ and thus on the ball B_R. Now, N is of the order $\log_2(R/r) = \log_\theta(R/r) \cdot \log_2(\epsilon)$, we conclude that:

$$inf_{B_R} u \geq a\theta^N \geq ca\theta^{\log_\theta(R/r)\cdot\log_2(\theta)} = ca\left(\frac{r}{R}\right)^q$$

where c satisfies:

$$\frac{N}{c} \leq \log_2(R/r) \leq cN$$

and $q = -\log_2 \theta$.

□

Corollary 18. *For any $x_k \in B_{1/2}$ and r_k sufficiently small, we have $u(0) \geq cr_k^q \inf_{B_{r_k}(x_k)} u$*

Proof. Applying the above techniques on a slightly smaller domain to the shifted solution, $\tilde{u}(x) = u(x - x_k)$ we have:

$$u(0) = \tilde{u}(x_k) \geq \inf_{B_1} \tilde{u} \geq cr_k^q \inf_{B_{r_k}} \tilde{u} = cr_k^q \inf_{B_{r_k}(x_k)} u$$

□

Corollary 19. *Take x_0 s.t. $u(x_0) = \sup_{B_{1/4}} u$, and take x_{k+1} s.t $u(x_{k+1}) = \sup_{B_{r_k}(x_k)} u$. Then:*

$$u(x_{k+1}) \geq \frac{u(x_k) - cr_k^{-q}u(0)}{1-\theta}$$

Proof. We have that $u(x_k) - \frac{1}{c}r_k^{-q}u(0) \leq u(x_k) - \inf_{B_{r_k}(x_k)} u \leq (1-\theta) \leq (1-\theta)u(x_{k+1})$. Multiplying through gives the result. □

Lemma 20. *If $\frac{\sup_{B_{1/4}} u}{u(0)}$ was sufficiently large, then there exists $\beta > 1$ and r_k such that:*

$$u(x_{k+1}) \geq \beta u(x_k)$$

and:

$$\sum_k r_k \leq \frac{1}{2}$$

Proof. By assumption, we have $u(x_0) \geq Mu(0)$ for M large. We will show that this works inductively. Suppose that we have the result up to k. By this we mean that there exists $\beta > 1$ such that $u(x_k) \geq \beta^{k-1}u(x_0)$ (this build up of powers happens iteratively, ie $u(x_k) \geq \beta u(x_{k-1}) \geq \beta^2 u(x_{k-2})$... etcetera. Now, by assumption:

$$u(x_0) \geq Mu(0)$$

And $\beta > 1$, thus, we may write $\beta = 1 + \beta_0$. The above inequality is thus modified to:

$$u(x_k) \geq (1+\beta_0)^{k-1}Mu(0)$$

Moreover, we may take β_0 arbitrarily small, due to it being on the right hand side of the inequality. Thus, take β_0 small enough such that $(1-\theta)(1+\beta_0) \leq 1-\beta_0$. Pick $0<\delta<1$ and $a>0$ small. To prove the inductive step, it suffices to show:

$$\frac{(1+\beta_0)^{k-1}Mu(0) - c^{-1}a^{-q}\delta^{-kq}}{1-\theta} \geq M(1+\beta_0)^k$$

thanks to Corollary 21. Simplifying the inequality, we see this occurs if and only if:

$$1 \geq \frac{1}{(1-\theta)(1+\beta_0)} - \frac{1}{ca^q\delta^{kq}(1-\theta)M(1+\beta_0)^k}$$

Thus, take $\delta \in (0,1)$ such that $\delta^q(1+\beta_0) > 1$, and a sufficiently small such that $\sum_k \delta_k < 1/2$. By our choice of β_0 we have that:

$$\frac{1}{(1-\theta)(1+\beta_0)} \geq \frac{1}{1-\beta_0} > 1$$

Taking k sufficiently large such that:

$$\frac{1}{1-\beta_0} - 1 > \frac{1}{ca^q(1-\theta)M(\delta(1+\beta_0))^k}$$

Finally, for all k small, we may pick M as large as we need for this (finite) set of cases, and finish the proof. (Recall that M was not fixed in the beginning of the proof). Having shown this result, we complete the proof (by contradiction): we have found an unbounded sequence of $u(x_k)$, contradicting the local boundedness of u. $\square$