Elliptic Systems of Phase Transition Type — Junctions & Vortices

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Athens, June 2025

This thesis is dedicated to my two sisters, for their constant support.

Acknowledgments

As this thesis comes to its end, I would like to thank, at the bottom of my heart, the supervisory committee, which has been supportive for the duration of the writing process. I am honored to have worked with people that not only I have high regard for, but they all have been a cornerstone during my studies. Panayotis Smyrnelis, the main supervisor of this thesis, guided me through the writing process and was the first one, along with Nicholas Alikakos, that introduced me to research. He also gave me the spark to study Calculus of Variations during my master's studies. Nicholas Alikakos tought me Partial Differential Equations from ground-zero, and I seriously owe him almost everything I know on this subject. He was also the one that made me realise how important and beautiful differential equations are and how deep the rabbit hole goes. Panagiotis Gianniotis offered me everything I know about Riemannian Geometry and showed me the connection between partial differential equations and geometry.

For these reasons, I thank them. But above all, I admire them for being Humans. Mathematics is not only academic achievements and stiff logic, it is also the connections and interactions of its members.

For reasons other than academic ones, I thank my sisters and my friends for their constant support, especially during difficult times. One relative story that comes to mind, in order to not talk a lot, is that of Thales of Melos, during a night walk with an old lady. It seems that what is stated below is a general phenomenon among mathematicians.

«Λέγεται δ᾽ ἀγόμενος ὑπό γραὸς ἐκ τῆς οἰκίας, ἵνα τὰ ἄστρα κατανοήσῃ, εἰς βόθρον ἐμπεσεῖν καὶ αὐτῷ ἀνοιμώξαντι φάναι τὴν γραῦν · “σὺ γάρ, ὦ Θαλῆ, τὰ ἐν ποσὶν οὐ δυνάμενος ἰδεῖν, τὰ ἐπὶ τοῦ οὐρανοῦ οἴει γνώσεσθαι;”»[1]

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Anastasios Fragkos,
Athens, 30th of June 2025

[1]It is said that [Thales], after he had left his house with an old lady to study the stars, he fell into a pit, and while he was screaming of pain, the old lady told him; “You Thales, while you cannot see what is in front of your feet, do you believe you will be able to learn what is high in the sky?”

Abstract

In this thesis we present some new results along with some well-known results on Elliptic Systems of Phase transition Type, and in particular on junctions and votices. Our results are joint work mainly with Panayotis Smyrnelis.

In Chapter 1 we deal with the physical origins of these equations and the intuition that comes from materials science and quantum mechanics.

In Chapter 2 we study the problem of motion of interfaces for networks. We also deal with motion by curvature with various constraints, derived from a bounded set, and we show $C^{1,a}-$convergence, along with some exponential estimates on the speed of the convergence. The mathematical problem here is related to the material science model presented in Chapter 1.

In Chapter 3 we study the Ginzburg-Landau energy minimisers in two cases, the smooth and the non-smooth. In the smooth case we show the existence of vortices (Hervé-Hervé) and in the non-smooth case, under some symmetry assumptions, we find exactly the form of the vortices in all dimensions. The mathematical problem here is related to the quantum mechanical model, presented in Chapter 1.

In Chapter 4 we deal with the correlation of phase transition problems with the calculus of variations and $\Gamma-$convergence. The basic theorems here are related to the $\Gamma-$convergence of the energy to the perimeter functional (Modica, Pacard-Ritoré). This chapter, together with parts of the first, was presented in an extended form in November 2024 at the Mathematics Club. Many thanks for the comments of those present, which helped to better shape the text.

Finally, there are two appendices, one on analysis and one on geometry, where some of the prerequisites and basic theorems used throughout the text are presented.

Περίληψη

Σε αυτήν την εργασία παρουσιάζουμε κάποια νέα αποτελέσματα μαζί με κάποια γνωστά αποτελέσματα στα Ελλειπτικά Συστήματα Τύπου Αλλαγής Φάσης, και συγκεκριμένα με τους κόμβους (junctions) και τις δίνες (vortices). Τα αποτελέσματά μας είναι σε συνεργασία κυρίως με τον Παναγιώτη Σμυρνέλη.

Στο Κεφάλαιο 1 ασχολούμαστε με τη φυσική προέλευση των εξισώσεων αυτών και τη διαίσθηση που προέρχεται από την επιστήμη των υλικών και τη κβαντομηχανική.

Στο Κεφάλαιο 2 μελετούμε το πρόβλημα της κίνησης των διεπιφανειών (interfaces) για τα δίκτυα. Επίσης, ασχολούμαστε με την κίνηση μέσω καμπυλότητας με διάφορους περιορισμούς, που προέρχονται από ένα φραγμένο σύνολο, και δείχνουμε $C^{1,a}-$σύγκλιση μαζί με κάποιες εκθετικές εκτιμήσεις όσον αφορά την ταχύτητα της σύγκλισης. Το μαθηματικό πρόβλημα εδώ σχετίζεται με το μοντέλο της επιστήμης των υλικών, που παρουσιάστηκε στο Κεφάλαιο 1.

Στο Κεφάλαιο 3 μελετούμε τους ελαχιστοποιητές της ενέργειας Ginzburg-Landau σε δύο περιπτώσεις, στη ομαλή και την μη-ομαλή. Στη ομαλή περίπτωση δείχνουμε την ύπαρξη των δινών (Hervé-Hervé) και στην μη-ομαλή, υπό κάποιες υποθέσεις συμμετρίας, βρίσκουμε ακριβώς τη μορφή των δινών σε όλες τις διαστάσεις. Το μαθηματικό πρόβλημα εδώ σχετίζεται με το μοντέλο της κβαντομηχανικής, που παρουσιάστηκε στο Κεφάλαιο 1.

Στο Κεφάλαιο 4 ασχολούμαστε με τη συσχέτηση των προβλημάτων αλλαγής φάσης με το λογισμό μεταβολών και τη $\Gamma-$σύγκλιση. Τα βασικά θεωρήματα εδώ σχετίζονται με τη $\Gamma-$σύγκλιση της ενέργειας στο συναρτησιακό της περιμέτρου (Modica, Pacard-Ritoré). Αυτό το κεφάλαιο, μαζί με τμήματα του πρώτου, παρουσιάστηκε σε μία εκτεταμένη μορφή τον Νοέμβριο του 2024 στη Λέσχη Μαθηματικών. Ευχαριστώ πολύ για τα σχόλια των παρευρισκόμενων, που βοήθησαν στην καλύτερη διαμόρφωση του κειμένου.

Τέλος, υπάρχουν δύο παραρτήματα, ένα στην ανάλυση κι ένα στη γεωμετρία, όπου παρουσιάζονται κάποια από τα προαπαιτούμενα και τα βασικά θεωρήματα που χρησιμοποιούνται σε όλο το κείμενο.

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CHAPTER 1

The Allen-Cahn, Ginzburg-Landau and related equations

1.1 Physical interpretation

1.1.1 Material science

All of the equations we are going to study are more or less related to the Allen-Cahn equation. The common denominator of all of these phenomena is some kind of **phase transition** taking place, with the more intuitive and geometric of those to probably be the phase separation in multi-component alloys, which is what we are going to study first.

When a mixture of elements A_k goes through phase separation, it is observed that pure regions appear, that is regions with almost pure phase A_k. For one to be transfered from a phase to another, he must pass through an interface of mixed phases . In the two component case, we observe a change of density from phase A_1 to phase A_2 which resembles $\tanh$. In general, one must examine how exactly the density should be defined, but in any case density is the object of our study. Phase transitions appear also in superconductivity, to explain how a superconductor transitions from a normal (non-superconducting) state to a pure (superconducting) state. Here another kind of density plays a role, that is Cooper pair density $|u|^2$, and in fact the corresponding wavefunction of this density, u, is the object of our study. We will mention superconductivity, along other phenomena, in the next section.

Getting back to alloying, suppose the number of phases is exactly 2. To quantify the amount of each phase, we use signed density, that is the difference of concentrations $u = c(A_1) - c(A_2)$. For convinience, we may assume -1 corresponds to pure phase A_1 while $+1$ to pure phase A_2. Since the two phases are seperating, it is intuitive to examine potentials with two minima at $\{\pm 1\}$, so that u is forced towards -1 or $+1$.

Figure 1.1

One case of such potential, in fact the simplest which is balanced, is $W(u) = \frac{1}{4}(1 - u^2)^2$. This is the one dimensional Ginzburg-Landau potential.

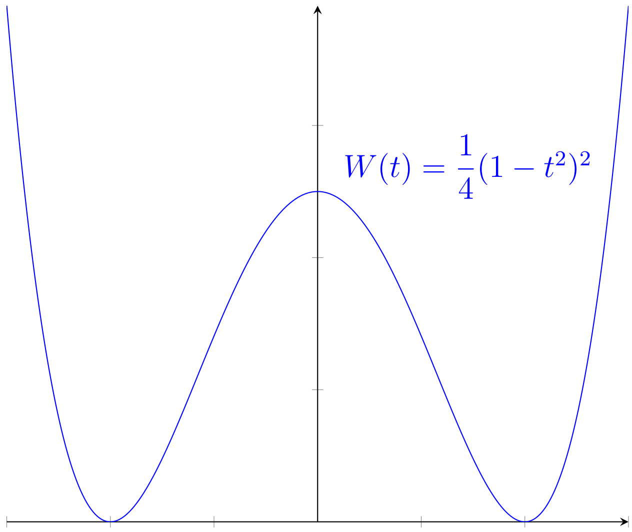


Figure 1.2: One dimensional Ginzburg-Landau potential.

We will suppose for simplicity that W has 0 as its minimum value. Then, the excess free energy density is given by:

$$e_\kappa(u) = 2\kappa|\nabla u|^2 + W(u), \ \kappa > 0 \text{ constant}$$

and the total free energy in $\Omega \subseteq \mathbb{R}$ is:

$$\mathscr{E}_\kappa(u;\Omega) = \int_\Omega 2\kappa|\nabla u|^2 + W(u)\, dx$$

Here κ is usually called gradient energy coefficient, but it is of no importance for us. Since $|\nabla u|$ penalises abrupt changes, it is a term associated with the interface energy, that is with tension.

Now, if we want to examine how the phases balance in space after infinite time, we calculate $\delta\mathscr{E}_\kappa(\cdot;\Omega)/\delta v$ for some $v \in C_c^\infty(\Omega)$, and since we expect balance we set it equal to 0. We obtain:

$$\begin{aligned}
\frac{\delta\mathscr{E}_\kappa(\cdot;\Omega)}{\delta v} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega W(u+\varepsilon v) - W(u) + 2\kappa\big[\langle\nabla(u+\varepsilon v), \nabla(u+\varepsilon v)\rangle - \langle\nabla u, \nabla u\rangle\big]\, dx \\
&= \int_\Omega \langle W_u(u), v\rangle\, dx + \int_\Omega 4\kappa\langle\nabla u, \nabla v\rangle\, dx \\
&= \int_\Omega \langle W_u(u), v\rangle\, dx - \int_\Omega 4\kappa\langle\Delta u, v\rangle\, dx \\
&= \int_\Omega \langle W_u(u) - 4\kappa\Delta u, v\rangle\, dx = 0, \quad \text{for all } v \in C_c^\infty(\Omega)
\end{aligned}$$

where W_u denotes the gradient of W. It follows that:

$$4\kappa\Delta u - W_u(u) = 0$$

and if we set $4\kappa = \varepsilon^2$ (as it is usual notation):

$$\varepsilon^2\Delta u - W_u(u) = 0$$

we get the time independent Allen-Cahn equation, or just Allen-Cahn. The associated energy is:

$$\mathscr{E}_\varepsilon(u;\Omega) = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\, dx$$

Some authors prefer to use:

$$\mathscr{E}_\varepsilon(u;\Omega) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\ dx$$

which does not change things. In fact, in Section 1.3 we will obtain Allen-Cahn equation using the second form of the energy.

This equation generalises in more dimensions. For the many dimensions, we notice that a mere signed density is not enough to describe the phenomenon, and this is a consequence of geometry.

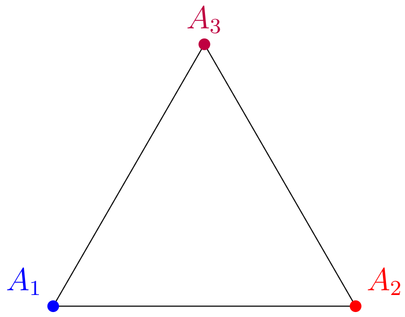


Figure 1.3

The problem lies in the inability to describe the co-exitence of more than 2 phases. In the case where the number of phases is 3, it is proper to use a vector density, where pure phases correspond to vertices of a triangle. So, if u is a vector function, we can show as above that:

$$\varepsilon^2\Delta u - W_u(u) = 0,\ \ u : \mathbb{R}^n \to \mathbb{R}^m$$

Definition 1.1 (Allen-Cahn equation). *The **Allen-Cahn equation** with $\varepsilon > 0$ is:*

$$\varepsilon^2\Delta u - W_u(u) \tag{1.1}$$

for some potential W. With W_u we denote its gradient. The associated energy is:

$$\mathscr{E}_\varepsilon(u;\Omega) = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\ dx \tag{1.2}$$

or:

$$\mathscr{E}_\varepsilon(u;\Omega) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\ dx \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^m$.

The scalar case $u : \mathbb{R}^n \to \mathbb{R}$ can be generalised to Riemannian manifolds, using the respective gradient and divergence.

$$\begin{aligned}
&\langle \nabla u, X\rangle = X(u)\\
&\nabla\bullet X = \sum_k \langle \nabla_{E_k} X, E_k\rangle\\
&\Delta u = \nabla\bullet\nabla u
\end{aligned}$$

1.1.2 Phase transitions and superconductivity

It is by now a well-known phenomenon, that under certain conditions (of pressure and temperature) some materials lose their resistance. A rather interesting thing, observed experimentally and proved theoretically, is the existence of two types of superconductors, based on the transition between normal and superconducting states. First examples of superconductors belong to the first category, of Type-I superconductors, where phase transition phenomena are not observed, as we have a spontaneous jump between normal and superconducting states when magnetic fields are applied. However, there is another kind of superconductors, that of Type-II, where a transition layer is present, in which normal and superconducting states coexist. This has lead to a description of superconductivity (unlike those first models proposed) based on the framework of phase transitions.

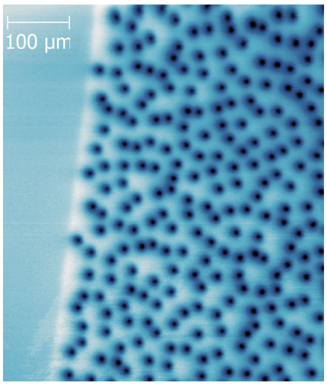


Figure 1.4: Vortices in real life.

In this intermediate mixed state, it is observed (in approximate triangular lattice form, as predicted by Abrikosov) vortex filaments, which are regions of non-superconducting state, enclosed by superconducting material. Now, this is not an a priori scheme, but nowadays it is well known -by the work of Bardeen, Cooper and Schrieffer- that the correct parameter for this phase transition is some function u (a wavefunction), which represents the density $|u|^2$ of Cooper pairs. Cooper pairs are pairs of electrons, held together by another strange phenomenon, which is a result of the interaction between electrons and the atom lattice (usually called phonon effect). Here $|u|^2 = 0$ corresponds to a non-superconducting state and $|u|^2 = 1$ to a superconducting state.

It turns out that by choosing a potential with a connected set of minima, such as $W(u) = \frac{1}{4}(1 - |u|^2)^2$, a good decription of these phase transition phenomena can be made. Such potentials force the wavefunction to “move around” the connected set of minima. The usual form of the equation is:

$$\Delta u - \frac{1}{\xi^2} W_u(u) = 0$$

which is very similar to its Allen-Cahn counterpart. This is called the Ginzburg-Landau equation. The constant $\xi = \xi(T)$ is called characteristic length and is proven to be the radius of each vortex. We emphasise that -excluding the $1-$dimensional case- the difference between Allen-Cahn and Ginzburg-Landau equations lies in the connectedness of the set of minima.

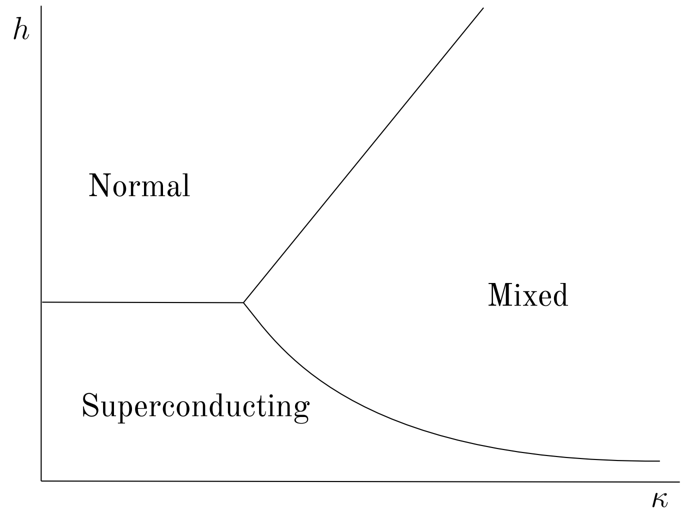


Figure 1.5

Definition 1.2 (Ginzburg-Landau equation). *The **Ginzburg-Landau equation** with $\xi > 0$ is:*

$$\Delta u - \frac{1}{\xi^2} W_u(u) = 0 \qquad (1.4)$$

The energy is defined as in the Allen-Cahn counterpart.

In Chapter 3 we review the existence of vortices for the 2−dimensional case. Later, we also present some results in the case where the potential is non-smooth.

In Figure 1.5 it is shown the dependence of the type of phase transition on the external magnetic field and on the coupling constant $\kappa = \lambda/\xi$ (where $\lambda = \lambda(T)$ is the penetration depth of the applied magnetic field).

1.2 Landau theory for phase transitions

The derivation of the Allen-Cahn equation is compatible with a general theory of phase transitions that Landau introduced. We will next describe this theory to have a vague idea of the physics, but we will not be interested in the physics of these phenomena; we will focus on the mathematical side of them only.

As we have already mentioned, density is important in all kind of phase transition problems, but is not always the only or best object of study. For example, we have mentioned that a wavefunction u related to the Cooper pair density $|u|^2$ is what encodes all information about phase transitions in superconductors. Therefore, from now on we will refer to u as an order parameter and it is generally better to compare it to a wavefunction rather than to a mere density. The "prostulates" of the theory of Landau are as follows:

The order parameter: For a given system, an order parameter has to be constructed, which is zero in the disordered phase and non-zero in the ordered phase.

The free energy functional: The free energy $\mathscr{E} = \mathscr{E}_0(T) + \mathscr{E}_1(T, u)$ depends on parameter (temperature) T and in the order parameter u. Also, the energy density f_0 of $\mathscr{E}_0$ is an analytic function.

Dependence on the order parameter: The dependence in the order parameter is encoded in $\mathscr{E}_1$, whose density f_1 is analytic too. The coefficients are determined by the symmetries associated with u.

Temperature dependence: Sometimes it is assumed that all the non-trivial temperature dependencies reside in the lower order term in the expansion of the density f_1 of $\mathscr{E}_1$.

In fact, to understand more deeply the sketch above, one has to have in mind that physicists wanted to derive a macroscopic theory inspired from statistical mechanics. It is easier to describe the phenomenon using a global thing, as a wavefunction, in contrast to summing over all possible microstates. The statistical mechanics formula for the energy:

$$e^{-\mathscr{E}/kT} = \sum_{\mu} e^{-H(\mu)/kT}$$

should be generalised by interchanging the sum over all microstates μ with a functional integral over all possible wavefunctions:

$$\begin{aligned} e^{-\mathscr{E}/kT} &\approx \int g(u) e^{-H(u)/kT} \mathscr{D}u \\ &= \int e^{-[H(u)-kT\log g(u)]/kT} \mathscr{D}u \end{aligned}$$

Here k is the Boltzmann constant, H is related to (microscopic) energy and g is a weight that has to do with the number of microstates that correspond to u. Now, near a critical point, a saddle-point approximation argument shows that:

$$\mathscr{E}/_{\text{Vol}} \approx f \approx H(u) - kT\log g(u), \ f \text{ being the density of } \mathscr{E}$$

and since in statistical mechanics entropy is related to energy as $\mathscr{E} = H - TS$, where H is the total energy, $\mathscr{E}$ is the free energy and S the entropy, the quantity $k\log g(u)$ is an analogue of entropy. We denote it by S too. By expanding H and S we obtain:

$$f \approx \sum_{\lambda=0}^{n} H_\lambda u^\lambda - T\sum_{\lambda=0}^{n} S_\lambda u^\lambda, \text{ with } H_\lambda, S_\lambda \text{ constants}$$

This indicates that the energy density can be decomposed to a part dependent on T and another on T, u. The analytic conditions are used naturally, to have a workable form of the energy.

For $n = 4$ and assuming appropriate symmetries, it is possible to get the Ginzburg-Landau potential, for some choice of H_λ, S_λ. Of course it seems that many more potentials can arise, and indeed they do in nature (for example, unbalanced potentials resembling the balanced Ginzburg-Landau potential).

1.3 Allen-Cahn and Cahn-Hilliard equations via gradient flows

Allen-Cahn equation can also be derived from Cahn-Hilliard, by some a gradient flows.

As it is usual convention in physics, we want to change u so that energy tends to a minimum. That is, it is logical for u to follow some kind of gradient $-\nabla\mathscr{E}_\varepsilon(u;\Omega)$, in $\Omega \subseteq \mathbb{R}^n$, for some notion of gradient. Note that the gradient always points at points of maximum. This is in a few words the idea of **gradient flows**.

$$\partial_t u = -\nabla_{\mathscr{G}}\mathscr{E}_\varepsilon(u;\Omega)$$

Here the appropriate choice of gradient is the variational gradient or Gâteaux differential. We give the following definition, which can also be generalised to the case of topological vector spaces.

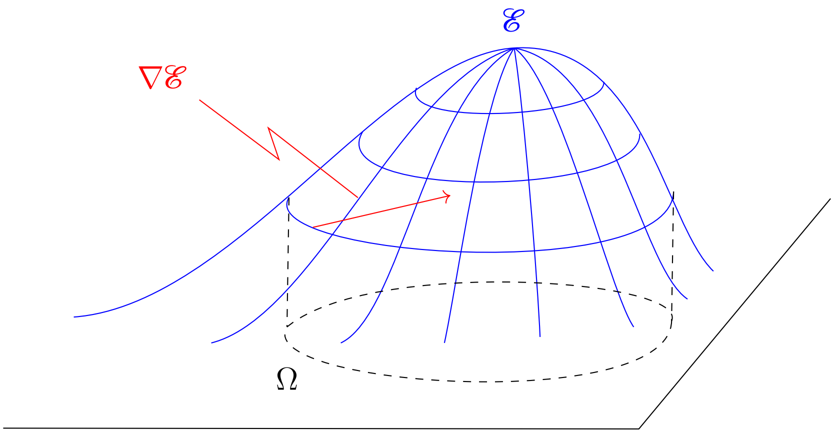


Figure 1.6

Definition 1.3 (Gâteaux differential). *Let $J : \mathscr{X} \to \mathbb{R}$ be a functional in a space of functions or distributions and $u \in \mathscr{X}$. The **Gâteaux differential** at u is the function or distribution $\nabla_{\mathscr{G}}J(u)$ for which for every v:*

$$\langle \nabla_{\mathscr{G}}J(u), v\rangle = \frac{\delta J}{\delta v}\Big|_u$$

Here $\langle\cdot,\cdot\rangle$ in $\mathscr{X}$ is either an inner product or the evaluation, if we work in a space of distributions.

In our case we will work with the homogenous Sobolev space $\dot{H}^1(\Omega)$:

$$\dot{H}^1(\Omega) = \{u \in \mathcal{S}'(\Omega) \mid \nabla u \in L^2(\Omega)\}$$

as well as with its dual $\dot{H}^{-1}(\Omega) = \dot{H}^1_0(\Omega)'$. With $\mathcal{S}'$ we denote the tempered distributions, that is distributions in Schwartz space.

As you have probably imagined, the choice of the appropriate inner product for the definition of the Gâteaux differential is of great importance for the Cahn-Hilliard equation. In what follows we explain how the choice is made and how the Cahn-Hilliard equation is derived.

Step I: In the dual of homogenous Sobolev space $\dot{H}^{-1}(\Omega)$, we can define an inner product using $L^2(\Omega)$. In fact, the defintion is given for a dense subset of $\dot{H}^{-1}(\Omega)$, and to be exact we define:

$$\langle v_1, v_2\rangle_{\dot{H}^{-1}(\Omega)} = \langle \nabla\varphi_{v_1}, \nabla\varphi_{v_2}\rangle_{L^2(\Omega)}$$

where $\varphi_{v_1}, \varphi_{v_2} \in \dot{H}^{-1}(\Omega)$ are associates of v_1, v_2, whose choice will be explained later. This is what we need to define the gradient of the energy functional.

Step II: For every v in $H_0^1(\Omega)$, there exists φ_v such that the Neumann boundary value problem has a solution:

$$\begin{cases} -\Delta\varphi_v = v, & \text{in } \Omega \\ \dfrac{\partial \varphi_v}{\partial \widehat{n}} = 0, & \text{in } \partial\Omega \\ \int_\Omega \varphi_v \, dx = 0 \end{cases}$$

This is a consequence of the Lax-Milgram theorem (cf. Theorem A.11).

For our case, we define $B : \dot{H}_0^1(\Omega) \times \dot{H}_0^1(\Omega) \to \mathbb{R}$:

$$\begin{aligned} B(\varphi_{v_1}, \varphi_{v_2}) = \langle \nabla\varphi_{v_1}, \nabla\varphi_{v_2} \rangle_{L^2(\Omega)} &= \int_\Omega \langle \nabla\varphi_{v_1}, \nabla\varphi_{v_2} \rangle \, dx \\ &= -\int_\Omega \Delta\varphi_{v_1} \cdot \varphi_{v_2} \, dx \end{aligned}$$

and we prove the bounds required by the Lax-Milgram theorem.

- For the first bound, from Hölder inequality we have:

$$|B(\varphi_{v_1}, \varphi_{v_2})| \leqslant ||\nabla\varphi_{v_1}||_{L^2(\Omega)} \cdot ||\nabla\varphi_{v_2}||_{L^2(\Omega)} \leqslant C||\varphi_{v_1}||_{\dot{H}^1(\Omega)} \cdot ||\varphi_{v_2}||_{\dot{H}^1(\Omega)}$$

- For the coercivity, we have:

$$B(\varphi_{v_1}, \varphi_{v_1}) = ||\nabla\varphi_{v_1}||_{L^2(\Omega)}$$

  and from Poincaré inequality, if $(\varphi_{v_1})_\Omega = ⨍_\Omega \varphi_{v_1} \, dx$:

$$||\varphi_{v_1}||^2_{L^2(\Omega)} = ||\varphi_{v_1} - (\varphi_{v_1})_\Omega||^2_{L^2(\Omega)} \leqslant c||\nabla\varphi_{v_1}||^2_{L^2(\Omega)} = c \cdot B(\varphi_{v_1}, \varphi_{v_1})$$

  therefore:

$$\frac{1}{c+1}||\varphi_{v_1}||^2_{\dot{H}^1(\Omega)} \leqslant B(\varphi_{v_1}, \varphi_{v_1})$$

So if one defines:

$$F(\varphi_{v_2}) = \int_\Omega v_1 \varphi_{v_2} \, dx, \;\; v_1 \in \dot{H}_0^1(\Omega)$$

then there exists unique φ_{v_1} such that:

$$B(\varphi_{v_1}, \varphi_{v_2}) = F(\varphi_{v_2}), \;\; \text{for all } \varphi_{v_2} \in \dot{H}_0^1(\Omega)$$

As a consequence, for every v_1 there exists φ_{v_1} such that $-\Delta\varphi_{v_1} = v_1$ weakly. Therefore, the inner product $\langle \cdot, \cdot \rangle_{\dot{H}^{-1}(\Omega)}$ can be defined in a dense subspace.

Step III: For every $v \in \dot{H}_0^1(\Omega)$ we calculate:

$$\begin{aligned}
\langle \nabla_{\mathscr{G}} \mathscr{E}_\varepsilon(u;\Omega), v\rangle &= \frac{d}{dt}\Big|_{t=0} \mathscr{E}_\varepsilon(u+tv;\Omega) \\
&= \lim_{t\to 0} \frac{1}{t}\big(\mathscr{E}_\varepsilon(u+tv;\Omega) - \mathscr{E}_\varepsilon(u;\Omega)\big) \\
&= \lim_{t\to 0} \frac{1}{t} \int_\Omega \frac{\varepsilon}{2} 2t\langle \nabla u, \nabla v\rangle + \int_\Omega \frac{1}{\varepsilon} W_u(u) v \, dx \\
&= \int_\Omega \varepsilon \langle \nabla u, \nabla v\rangle \, dx + \frac{1}{\varepsilon} W_u(u) v \, dx \\
&= \int_\Omega \left[-\varepsilon \Delta u + \frac{1}{\varepsilon} W_u(u)\right] v \, dx \\
&= \int_\Omega \left[\varepsilon \Delta u - \frac{1}{\varepsilon} W_u(u)\right] \Delta \varphi_v \, dx \\
&= -\int_\Omega \left\langle \nabla \left[\varepsilon \Delta u - \frac{1}{\varepsilon} W_u(u)\right], \nabla \varphi_v \right\rangle \\
&= \left\langle \nabla \left[\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u\right], \nabla \varphi_v \right\rangle_{L^2(\Omega)} \\
&= \left\langle -\nabla \cdot \nabla \left[\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u\right], -\nabla \cdot \nabla \varphi_v \right\rangle_{\dot{H}^{-1}(\Omega)} \\
&= \left\langle -\Delta \left[\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u\right], v \right\rangle_{\dot{H}^{-1}(\Omega)}
\end{aligned}$$

All of the above allow us to weakly identify:

$$\nabla_{\mathscr{G}} \mathscr{E} = -\Delta \left[\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u\right]$$

Therefore, the gradient flow becomes:

$$\partial_t u = \Delta \left[\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u\right]$$

This equation is the Cahn-Hilliard equation. If we seek time invariant solutions, we get:

$$\Delta \left[\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u\right] = 0$$

and as a special case, we get Allen-Cahn:

$$\frac{1}{\varepsilon} W_u(u) - \varepsilon \Delta u = 0 \Leftrightarrow \varepsilon^2 \Delta u - W_u(u) = 0$$

Definition 1.4 (Cahn-Hilliard equation). *The gradient flow of the energy functional:*

$$\mathscr{E}_\varepsilon(u;\Omega) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx$$

... where the Gâteaux differential is calculated in the $\langle\cdot,\cdot\rangle_{\dot{H}^{-1}}$product, gives rise to:

$$\partial_t u = \Delta\left[\frac{1}{\varepsilon}W_u(u) - \varepsilon\Delta u\right] \tag{1.5}$$

which is the ***Cahn-Hilliard equation.***

1.4 1-D solutions of the Allen-Cahn equation

In the simple one-dimensional case -with the usual Ginzburg-Landau potential- Allen-Cahn equation has a simple solution. In fact, this is an indicative profile for many other solutions, which we will decribe later.

Notice that, to find a solution, it suffices to study $\Delta u - W_u(u) = 0$. Indeed:

$$\varepsilon^2\Delta u(\varepsilon x) = W_u(u(\varepsilon x))$$

is equivalent to:

$$\Delta v = W_u(v) = v^3 - v$$

where $v(x) = u(\varepsilon x)$. Moreover:

$$\left((v')^2 - 2W(v)\right)' = 2v'\cdot v'' - 2v'\cdot W_u(v) = 0$$

and:

$$(v')^2 = 2W(v) + c,\ c \in \mathbb{R}$$

To find a solution, set $c = 0$ and suppose $v' > 0$. We have:

$$v' = \frac{1}{\sqrt{2}}(1 - v^2)$$

therefore the (heteroclinic) solution is:

$$v(x) = \hbar(x - x_0),\ \hbar(x) = \tanh(x/\sqrt{2})$$

Function $\hbar$ has some good physical intuition, both due to its asymptotic monotonicity and due to finite energy.

$$\mathscr{E}(v;\mathbb{R}) = \int_{\mathbb{R}} \frac{1}{2}(v')^2 + W(v)\ dx < \infty$$

One can show that the only solutions on one-dimension with finite energy are those of the form of $\hbar$, along with the trivial ones ± 1.

Remark 1.5 (Solutions of finite energy - $1-$dimension)**.** *The only solutions of the Allen-Cahn equation, with $\varepsilon = 1$, of finite energy are $v(x) = \pm\hbar(x - x_0)$ and $v \equiv \pm 1$.*

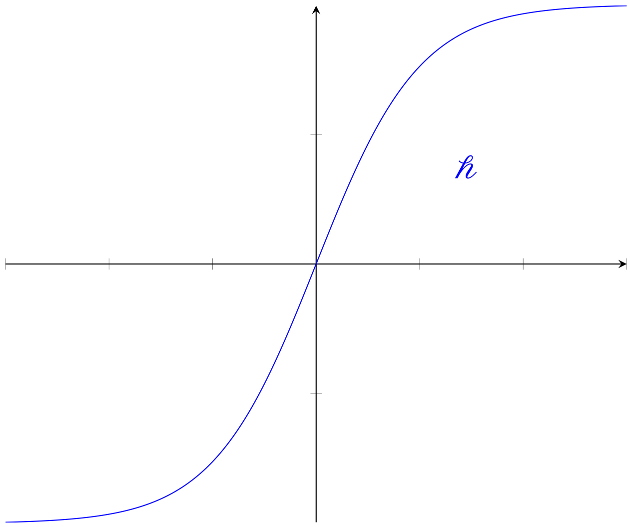


Figure 1.7

Proof. As we have already seen, it holds for some $c \in \mathbb{R}$, $(v')^2 = 2W(v) + c$. Since $(v')^2/2$, $W(v) > 0$, because of finite energy, there must exist a sequence $(x_k)_k$ such that:

$$x_k \to \infty,\ v'(x_k) \to 0,\ W(v(x_k)) \to 0$$

Therefore $(v'(x_k))^2 - 2W(v(x_k)) \to 0$, and then $c = 0$.

Now we consider some cases. If $|v(0)| < 1$, then:

$$v' = \frac{1}{\sqrt{2}}(1 - v^2) \text{ or } v' = -\frac{1}{\sqrt{2}}(1 - v^2)$$

which shows $v = \pm\hbar(x - x_0)$.

If $|v(0)| = 1$, we consider -without loss of generality- $v(0) = 1$. Suppose that there exists some x_1 such that $v(x_1) < 1$. Until x_1 our solution is of the form $\pm\hbar(x - x_0)$. Therefore, a discontinuity must appear at some point on the interval:

$$(\min\{x_1, 0\}, \max\{x_1, 0\}]$$

which is contradictory. This shows $v \geqslant 1$. Since the monotonicity of v changes only in those points where $v(x) = 1$, then v must preserves its monotonicity. If v is decreasing, $v \equiv 1$, while if it is increasing, it is of infinite energy, unless $v \equiv 1$.

Last but not least, we consider the case where $|v(0)| > 1$. Again, without loss of generality, suppose $v(0) > 1$. If $v > 1$, we can show as before that v has infinite energy. Moreover, if a point x_1 exists such that $v(x_1) = 1$, then from this point forth, $v \equiv 1$, while for $x < x_1$ function v has infinite energy. $\square$

This one-dimensional solution has a two-dimensional analogue, which we usualy call the one-dimensional solution in two dimensions. This solution is as follows: We consider $a \in \mathbb{S}^1$, $b \in \mathbb{R}$, and we define:

$$v(x) = \hbar(\langle a, x\rangle - b)$$

This simple analysis serves as a base for many problems concerning phase transition. First of all, notice that both in 1 and 2$-$dimensional case the function increases along one direction for the problem of two phases. Is this a general phenomenon for the two phase case? This is a question posed by de Giorgi, called de Giorgi's conjecture, and

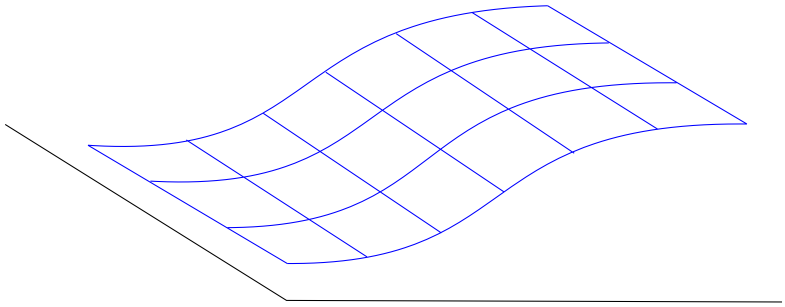


Figure 1.8

has been disproven for dimensions $n \geqslant 9$ (and proven in whole or a weak version for the smaller dimensions). We will not discuss de Giorgi's conjecture at all in this thesis. Another question is if it is possible to construct an analogue for the three phase case. In the two phase case, the transition takes place along one direction and the interface is approcimately a line (which is a minimal object). In the three component case, we expect solutions having interfaces being minimal cones (these solutions are called triple junction solutions). We will say a few words about junction solutions in Chapter 2. Last but not least, by considering the $\varepsilon-$problem and its respective solutions for two phases in $2-$dimensions (the $1-$dimesional solution on the plane), we see that u_ε becomes all the more steeper, as $\varepsilon \to 0$, and tends towars a characteristic-type fuction of a set with its boundary being a line. Is it true that the interfaces become approximately minimal surfaces as $\varepsilon \to 0$? We will mention this problem in Chapter 4.

CHAPTER 2

Junctions

2.1 Connections and junctions

As mentioned in Chapter 1, the 1−dimensional (heteroclinic) solution of the Allen-Cahn equation is indicative for many other phase transition phenomena. For example, if we change the line interface of the 1−dimensional solution with a minimal cone, we can probably formulate a similar problem for the three phase case. More detailed analysis can be found in [2].

In what follows, we consider a potential $W : \mathbb{R}^m \to \mathbb{R}$ which is $W \geqslant 0$ and vanishes only on a finite set of minima, which has cardinality bigger or equal to 2. Consider $a_\pm$ two of those (distinct) minima. A heteroclinic connection between $a_\pm$ is defined as below:

Definition 2.1 (Heteroclinic connections). *Consider the vector Allen-Cahn equation:*

$$u'' - W_u(u) = 0,\ u : \mathbb{R} \to \mathbb{R}^m$$

If a solution of the above equation exists such that:

$$\lim_{x \to \pm\infty} u(x) = a_\pm$$

*then u is called a **heteroclinic connection** between a_- and a_+.*

To define triple junction solutions, we will use heteroclinic connections parallel to the walls of a minimal cone interface. We remind the reader of minimal partitions: Consider an open, bounded set $\Omega \subseteq \mathbb{R}^n$. For every partition $P = \bigcup_{i=1}^N P_i$, we define the energy:

$$\mathscr{E}(P) = \sum_{i<j\leqslant N} \sigma_{i,j} \mathscr{H}^{n-1}(\partial P_i \cap \partial P_j)$$

for some coefficients $\sigma_{i,j} = \sigma_{j,i} > 0$, $\sigma_{i,i} = 0$. In general, if $\Omega = \mathbb{R}^n$, we can define the energy in a similar fashion, by restricting ourselves in the compact subsets of $\mathbb{R}^n$:

$$\mathscr{E}(P; V) = \sum_{i<j\leqslant N} \sigma_{i,j} \mathscr{H}^{n-1}(\partial P_i \cap \partial P_j \cap V)$$

An $N-$partition P is called **minimising** if for any compact set V and any other $N-$partition $\widetilde{P}$ with:

$$\bigcup_{i=1}^{N}(P_i \triangle \widetilde{P}_i) \subset V$$

we have $\mathscr{E}(P;V) \leqslant \mathscr{E}(\widetilde{P};V)$. It is a very well known fact that the minimising $3-$partition in $\mathbb{R}^2$ satisfying the triangle inequality $\sigma_{i,k} < \sigma_{i,j} + \sigma_{j,k},\ i \neq j \neq k \neq i$, is the unique minimal cone in $\mathbb{R}^2$. Therefore, the junction solutions will be defined for the simple triod. A very concise paper on the general theory of minimal cones can be found in [1].

Definition 2.2 (Triple junction solutions). *Consider the unique minimising $3-$partition P in $\mathbb{R}^2$, as before. Suppose $\{a_i\}_{i\leqslant 3}$ is the set of distinct minima of W. A **triple junction solution** u is a solution of the Allen-Cahn equation which satisfies the following estimates.*

i. For every $x \in P_i$, $|u(x) - a_i| \leqslant Ce^{-c\cdot \mathrm{dist}(x,\partial P_i)}$, for $c, C > 0$.

ii. Given any line parallel to $\partial P_i \cap \partial P_j$, $\mathrm{dist}(x, \partial P_i \cap \partial P_j) = \mu$, it holds:

$$\lim_{\substack{|x|\to\infty \\ \mathrm{dist}(x,\partial P_i \cap \partial P_j)=\mu}} u(x) = \mathscr{h}_{i,j}(\mu)$$

Here dist *denotes the signed distance and $\mathscr{h}_{i,j}$ the heteroclinic connection between a_i and a_j.*

We emphasise that a triple junction solution u is a stationary object. It is not something that describes motion, rather it is a balance configuration. In order to examine the motion of interfaces, one needs to follow some different way, which is either a parabolic alternative $\partial_t u = \varepsilon^2 \Delta u - W_u(u)$ to the Allen-Cahn equation or an approximation by the motion by curvature flow, as suggested by Mullins.

However, our limiting model is always the triple junction (at least locally) and some logical conditions, such as Young's law, originates from this stationary case. In what follows we describe how Young's law arises in the study of the junction solutions.

We will use the stress-energy formulation of the equation.

Remark 2.3 (The stress-energy tensor). *We define the stress-energy tensor:*

$$T_{i,j}(u, \nabla u) = \langle \partial_i u, \partial_j u \rangle - \delta_{i,j}\left(\frac{1}{2}|\nabla u|^2 + W(u)\right)$$

(by $|\cdot|$ we denote the Frobenius norm $||\cdot||_F$ of the Jacobian matrix $Ju = \nabla u$). Then the Allen-Cahn equation can be written in a divergence free form:

$$\nabla \cdot T = (\nabla T)^{\top}(\Delta u - W_u(u)) = 0$$

We note that if $T = (T_1, \cdots, T_n)$, then we define $\nabla \cdot T = (\nabla \cdot T_1, \cdots, \nabla \cdot T_n)$. (The proof is a simple calculation of each $\nabla \cdot T_j$).

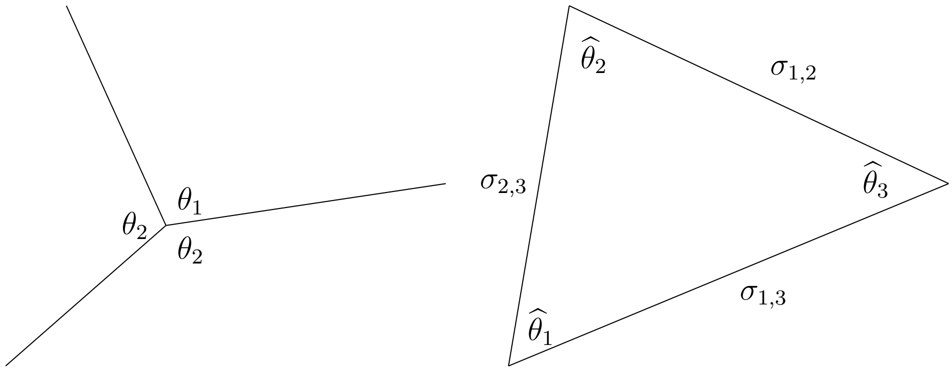


Figure 2.1: The junction configuration and the triangle formed by surface tension coefficients.

Theorem 2.4 (Young's law). *Let u be a triple junction solution, as before. We denote $\widehat{\theta}_i = \pi - \theta_i$ and and with $\sigma_{i,j}$ the interfacial energy (of the connection):*

$$\sigma_{i,j} = \int_{-\infty}^{\infty} \frac{1}{2}|\hbar'_{i,j}(\mu)|^2 + W\big(\hbar_{i,j}(\mu)\big)\, d\mu$$

Here $\hbar_{i,j}$ is the heteroclinic connection between two distinct minima a_i, a_j. Then, Young's law holds:

$$\frac{\sin\widehat{\theta}_1}{\sigma_{2,3}} = \frac{\sin\widehat{\theta}_2}{\sigma_{1,3}} = \frac{\sin\widehat{\theta}_3}{\sigma_{1,2}}$$

Proof. If $\widehat{t}_{i,j}$ is the tangent unit vector of $\partial P_i \cap \partial P_j$, we need to prove:

$$\sigma_{1,2}\widehat{t}_{1,2} + \sigma_{2,3}\widehat{t}_{2,3} + \sigma_{1,3}\widehat{t}_{1,3} = 0$$

Step I: Let our junction centre to be the origin 0. We suppose we have some disc $B_r(0)$ and we use the divergence theorem in each coordinate to obtain:

$$0 = \int_{B_r(0)} \nabla \bullet T\, dx = \int_{\partial B_r(0)} \langle T, \widehat{n}\rangle\, dS$$

In order to establish Young's relation, we will use the condition at infinity.

Step II: We cosider around $\partial P_i \cap \partial P_j$ a sector $S_{i,j}$ of angle $2\varphi(r)$, as in the following picture. We note from here on that -for simplicity purposes- we can assume that $\partial P_i \cap \partial P_j$ is parallel to the $y-$axis.

Function φ is not random and it satisfies $r\sin\varphi(r) \to \infty$, $\varphi(r) \to 0$, as $r \to \infty$. The idea is to compute the integral in the boundary $\partial B_r(0)$ in parts, which are $S_{i,j}$ and the complement $S^c = (\bigcup_{i<j} S_{i,j})^c$ in $\partial B_r(0)$. One can guess that the first three are of importance. Indeed, by elliptic estimates:

$$|\nabla u(x)| \leqslant Ce^{-c\cdot \operatorname{dist}(x, \partial P_i)}$$

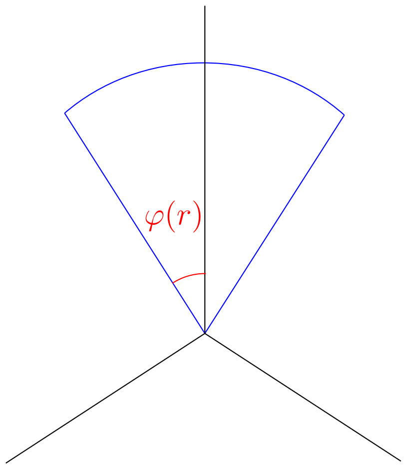


Figure 2.2

and we have:

$$\left|\int_{S^c} \langle T, \widehat{n}\rangle \, dS\right| \leqslant \int_{S^c} |T| \cdot |\widehat{n}| \, dS \leqslant \int_{S^c} e^{-2cr\sin\varphi(r)} \, dS \xrightarrow{r\to\infty} 0$$

Step III: As for the sector integrals: We write:

$$\int_{S_{i,j}} \langle T, \widehat{n}\rangle \, dS = \int_{-r\sin\varphi(r)}^{r\sin\varphi(r)} \langle T, \widehat{n}\rangle \frac{dy_1}{\cos\theta}$$

for $y_1 = r\sin\theta$ and $y_2 = r\cos\theta$. Now, notice that our convergence will follow from the dominated convergence theorem, as $|T_{i,j}| \leqslant Ce^{-|y_1|}$. Moreover, for fixed y_1 we have:

$$\begin{aligned}
&\lim_{r\to\infty} u(y_1, y_2) = \lim_{y_2\to\infty} u(y_1, y_2) = \hslash_{i,j}(y_1) \\
&\lim_{r\to\infty} \partial_{y_1} u(y_1, y_2) = \lim_{y_2\to\infty} \partial_{y_1} u(y_1, y_2) = \hslash'_{i,j}(y_1) \\
&\lim_{r\to\infty} \partial_{y_2} u(y_1, y_2) = \lim_{y_2\to\infty} \partial_{y_2} u(y_1, y_2) = 0
\end{aligned}$$

and by $\varphi(r) \to 0$:

$$\lim_{r\to\infty} \widehat{n} = (0,1) = \widehat{t}_{i,j}$$

These observations combined give us:

$$\lim_{r\to\infty} \int_{S_{i,j}} \langle T, \widehat{n}\rangle \, dS = -\int_{-\infty}^{\infty} \frac{1}{2}|\hslash'_{i,j}(x)|^2 + W\big(\hslash_{i,j}(x)\big) \, dx \, \widehat{t}_{i,j} = -\sigma_{i,j}\widehat{t}_{i,j}$$

which concludes the proof.

□

2.2 Curvature flows of networks

2.2.1 Motion by curvature

The problem of planar networks moving by curvature was proposed by Mullins in 1999, and has since been proved to be a very useful tool to describe growth of grain boundaries in polycrystalline material (cf. [4], [6], [7]). In what follows we will describe what motion by curvature is and how it can be extended to networks. Then, we see how short-time existence is proved, we see the existence of expanding networks and we examine how they converge to some limiting networks provided (space) restrictions exist.

The scheme is that, starting with a curve $X(0, s)$, we can move it by curvarture, following the vector $\kappa\widehat{n}$. This leads to a family of functions, $X(t, s)$, parametrised by t, that satisfy the equation:

$$\frac{\partial X}{\partial t} = \kappa(X)\widehat{n}(X) \tag{2.1}$$

Variable s is arc length and it depends on t. If we use the Frenet-Serret formulas $\partial_s\widehat{t} = \kappa\widehat{n}$ (where $\widehat{t} = \partial_s X$), we can obtain the other well-known form of this equation:

$$\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2} \tag{2.2}$$

which is not the heat equation, because of that dependence of s on t. However, some regularity results still hold. Equations (2.1) and (2.2) are called **curvature flow equations**. In normal velocity form, we can take the inner product $\langle \cdot, \widehat{n}\rangle$ on (2.1) to obtain:

$$V = \left\langle \frac{\partial X}{\partial t}, \widehat{n} \right\rangle = \langle \kappa\widehat{n}, \widehat{n}\rangle = \kappa$$

or, using signed curvature:

$$V_\sigma = \left\langle \frac{\partial X}{\partial t}, \widehat{n}_\sigma \right\rangle = \langle \kappa\widehat{n}, \widehat{n}_\sigma\rangle = \kappa_\sigma$$

(where $\widehat{n}_\sigma = e_3 \times \widehat{t}$ and κ_σ is the signed curvature). It is often times useful to consider uniform intervals, say $I = [0, 1]$, to define these equations, since arclength is changing with time. Then, by computing:

$$\kappa\widehat{n} = \frac{1}{|\partial_x X|}\frac{\partial}{\partial x}\frac{\partial_x X}{|\partial_x X|} = \frac{\partial_x^2 X|\partial_x X|^2 - \partial_x X\langle \partial_x^2 X, \partial_x X\rangle}{|\partial_x X|^4}$$

we obtain:

$$\frac{\partial X}{\partial t} = \frac{\partial_x^2 X|\partial_x X|^2 - \partial_x X\langle \partial_x^2 X, \partial_x X\rangle}{|\partial_x X|^4} \tag{2.3}$$

Definition 2.5 (Curvature flow equations)**.** *If $X(t, s)$ is a family of curves parametrised by t and if s denotes arc length, then the **curvature flow equations** are:*

$$\frac{\partial X}{\partial t} = \kappa(X)\widehat{n}(X) = \frac{\partial^2 X}{\partial s^2}$$

*or, in **normal velocity form:***

$$V = \kappa, \ V_\sigma = \kappa_\sigma$$

A simple example of curvature flow is the contracting circle: Suppose we have some circle $X(0, s) = \partial B_1(0)$. Motion by curvature gives a family of functions $X(t, s)$, which are all circles, that converge to a point in finite time. Indeed, it is easy to check that they are all circles, with decreasing radii $r \leqslant 1$; then notice that:

$$\frac{dA}{dt} = -\int_X \left\langle \frac{\partial X}{\partial t}, \widehat{n}_\sigma \right\rangle \, d\ell = -\int_X \kappa_\sigma \, d\ell = -2\pi$$

by Hopf's Umlaufsatz, which shows that $A(t) = A(0) - 2\pi t$ and in turn $t \leqslant A(0)/2\pi$.

The following very useful lemma indicates that curvature flows are, in some sense, gradient flows of the length functional. Similar formulas are used often in the study of networks.

Lemma 2.6 (Variation of the length). *For any family of curves $X(t,\cdot)$, parametrised by t, which moves by curvature flow, there holds:*

$$\frac{d}{dt}\mathscr{L}^1(X(t,\cdot)) = -\int_{X(t,\cdot)} \kappa^2 \, d\ell \leqslant 0$$

Proof. Indeed, we will calculate:

$$\frac{d}{dt}\mathscr{L}^1(X(t,\cdot)) = \frac{\delta \mathscr{L}^1(X(t,\cdot))}{\delta(\kappa\widehat{n})}$$

We have:

$$\begin{aligned}
\frac{\delta \mathscr{L}^1(X(t,\cdot))}{\delta(\kappa\widehat{n})} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{X(t,\cdot)} |\partial_s X(t,s) + \varepsilon \partial_s(\kappa\widehat{n})| - |\partial_s X(t,s)| \, ds \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{X(t,\cdot)} \sqrt{1 + 2\varepsilon\langle \partial_s X, \kappa \partial_s \widehat{n}\rangle + O(\varepsilon^2)} - 1 \, ds
\end{aligned}$$

and by the Frenet-Serret formulas $\partial_x \widehat{n} = -\kappa |\partial_x X| \widehat{t}$:

$$\begin{aligned}
[...] &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{X(t,\cdot)} \sqrt{1 - 2\varepsilon\kappa^2 + O(\varepsilon^2)} - 1 \, ds \\
&= \int_{X(t,\cdot)} \frac{-2\kappa^2}{2} \, ds \\
&= -\int_{X(t,\cdot)} \kappa^2 \, ds
\end{aligned}$$

□

Since curvature flows can be viewed as the gradient flow of the length functional, they can rightfully be called **curve shortening flows**.

2.2.2 Motion of networks

Curvature flows are not enough to describe the full picture in boundary motion in polycrystalline material. The main issue is the existence of non-smooth points (angles) in interfaces, which disrupt curvature flow. Intuitively, imagine two curves such that in their union point the curvature vectors do not match. Then, one curve tends to move away from the second and the second from the first, leading to discontinuity.

We resolve this issue by modifying the curvature equations, by adding conditions. First, we define some basic concepts, for networks for equal surface tension coefficients $\sigma^p_{i,j} = \sigma$ (without loss of generality, $\sigma = 1$).

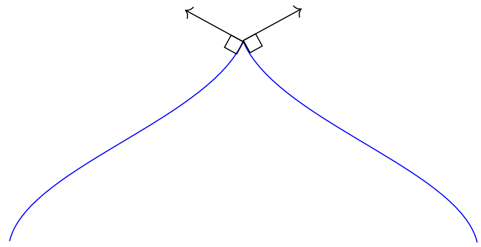


Figure 2.3

Definition 2.7 (Networks). *Suppose $\Omega \subseteq \mathbb{R}^2$ is smooth, convex and open. A* ***network*** *$\mathcal{N}$ in $\overline{\Omega}$ is a connected set constituted by a finite family of regular C^1-curves contained in $\overline{\Omega}$ such that:*

i. *All curves are simple.*

ii. *Two different curves intersect only at their end points.*

iii. *A curve intersects $\partial\Omega$ only at its end points.*

iv. *If an endpoint of a curve coinsides with some $P \in \partial\Omega$, then no other end point of any other curve can be P.*

A ***multipoint*** *O^k of* ***multiplicity*** *n is any point $O^k \in \Omega$ at which $n-$curves meet. We denote the* ***end points*** *on $\partial\Omega$ by P^k.*

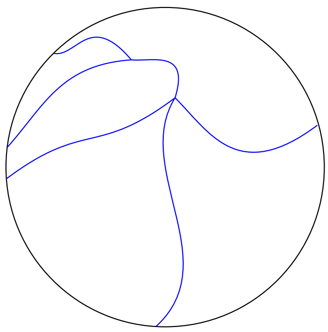


Figure 2.4

In what follows we are interested only in the so-called regular networks. Moreover, we will use the following notation:

i. If $\mathcal{N}$ is a network, with X^k we denote its curves.

ii. With $X^{p,k}$, $k \in \{1, 2, 3\}$, we denote the curves that pass from multipoint O^p.

iii. In general, indices k will denote the different curves, so for example $\widehat{t^k}$ denotes the tangent vector of X^k.

Definition 2.8 (Regular networks). *A network $\mathcal{N}$ is called **regular** if all its multipoints are triple and the sum of the unit tangent vectors of the concurring curves at each multipoint is* 0.

$$\sum_{k\leqslant 3} \widehat{t}^k = 0, \text{ at every multipoint } O^p$$

The geometric problem is to find some curvature flow such that all triple points of the network do not break. In fact, some appropriate conditions must be chosen a priori in order to prove the existence of such a flow at a first level. Some more general results, concerning short time existence with general initial data can be found in [35].

Definition 2.9 (Admissible initial data). *A network $\mathcal{N}_0$ is **admissible initial data** for the curvature flow if:*

i. It is regular.

ii. At every multipoint the sum of the curvature vectors is 0.

iii. The curvature on each end point on $\partial\Omega$ is 0 *(and therefore no motion exists on the boundary).*

iv. Each curve can be parametrised as an regular $C_x^{2,a}$−curve, $a \in (0,1)$.

By this notion of initial data, we also define the solution of the curvature flow.

Definition 2.10 (Solution of the curvature flow - Networks). *Let $\mathcal{N}_0$ be an admissible inital network. A family $\mathcal{N}_t$, parametrised by $t \in [0,T)$, is a **solution** of the curvature flow if:*

i. Every X^k can be parametrised in a $C^{1+a/2,2+a}([0,T) \times [0,1])$ and regular way.

ii. The following problem is solved:

ii.a. We have motion by curvature:

$$V^k = \langle \partial_t X^k(t,\cdot), \widehat{n}^k \rangle = \kappa^k(t,\cdot)$$

ii.b. The concurrency condition holds: $X^{p,k}(t,\cdot) = X^{p,\lambda}(t,\cdot)$ at every triple point O^p.

...

ii.c. The angle condition holds:

$$\sum_{k\leqslant 3}\widehat{t^k}=0,\ \textit{at every triple point } O^p$$

ii.d. Dirichlet boundary condition: $X^k(t,1)=P^k$.

The intuition behind the angle condition lies in the following lemma:

Lemma 2.11. *Let there be a triod* $\mathscr{N}=\mathbb{T}$ *in* Ω. *We suppose* $X^k(0)$ *coincide and we consider some variations* Ψ^k. *Then, if* $\mathscr{M}$ *is the* $\varepsilon-$*variation of* $\mathscr{N}$*:*

$$\mathscr{L}^1(\mathscr{M})=\sum_{k\leqslant 3}\mathscr{L}^1(X^k+\varepsilon\Psi^k)$$

and by some standard computation and integration by parts:

$$\frac{\delta\mathscr{L}^1(\mathscr{N})}{\delta\Psi}=-\sum_{k\leqslant 3}\int_{X^k}\kappa^k\langle\Psi^k,\widehat{n^k}\rangle\,d\ell-\sum_{k\leqslant 3}\langle\Psi^k(0),\widehat{t^k}(0)\rangle$$

If we want the boundary term to vanish, we must impose the condition:

$$\sum_{k\leqslant 3}\langle\Psi^k(0),\widehat{t^k}(0)\rangle,\ \textit{for every } \Psi\Rightarrow\sum_{k\leqslant 3}\widehat{t^k}(0)=0$$

As an important remark of the previous lemma, the importance of the angle condition lies in the formulation of the curvature equations as the gradient flow of the length functional.

For the more general case, where phases with different surface tension coefficients are present, we use the following notation:

i. If $\mathscr{N}$ is a network, with $X^{i,j}$ we denote its curves between phases i and j.

ii. With $X^{p,i,j}$, $i,j\leqslant 3$, we denote the curves between phases i,j that pass from multipoint O^p.

iii. In general, indices i,j will denote the different curves between phases i and j, so for example $\widehat{t^{i,j}}$ denotes the tangent vector of $X^{i,j}$.

The generalisation of curvature flows to the anisotropic case is made by considering the gradent flow of the energy functional (weighted perimeter).

$$\mathscr{E}(\mathscr{N})=\sum_{i<j\leqslant 3}\sigma_{i,j}\mathscr{L}^1(X^{i,j})$$

The basic conditions in anisotropic networks, such as concurrency $X^{p,i,j}(t,\cdot)=X^{p,i,j}(t,\cdot)$ at O^p, the Dirichlet boundary conditions, et cetera, remain the same. However, the equations and the angle conditions change, to take into account the existence of interfacial energy. As far as the equations go, the correct law is that of:

$$V^{i,j}=\left\langle\frac{\partial X^{i,j}}{\partial t},\widehat{n^{i,j}}\right\rangle=\sigma_{i,j}^2\kappa^{i,j}$$

Lemma 2.12. *Let there be a triod $\mathscr{N} = \mathbb{T}$ in Ω. We suppose $X^{i,j}(0)$ coincide and we consider some variations $\Psi^{i,j}$. Then, if $\mathscr{M}$ is the $\varepsilon-$variation of $\mathscr{N}$:*

$$\mathscr{E}(\mathscr{M}) = \sum_{i<j\leqslant 3} \sigma_{i,j}\mathscr{L}^1(X^{i,j} + \varepsilon\Psi^{i,j})$$

and by some standard computation and integration by parts:

$$\frac{\delta\mathscr{E}(\mathscr{N})}{\delta\Psi} = -\sum_{i<j\leqslant 3}\int_{X^{i,j}} \sigma_{i,j}\kappa^{i,j}\langle\Psi^{i,j}, \widehat{n}^{i,j}\rangle\, d\ell - \sum_{i<j\leqslant 3}\sigma_{i,j}\langle\Psi^{i,j}(0), \widehat{t}^{i,j}(0)\rangle$$

If we want the boundary term to vanish, we must impose the condition:

$$\sum_{i<j\leqslant 3}\sigma_{i,j}\langle\Psi^{i,j}(0), \widehat{t}^{i,j}(0)\rangle, \textit{ for every } \Psi \Rightarrow \sum_{i<j\leqslant 3}\sigma_{i,j}\widehat{t}^{i,j}(0) = 0$$

This suggests that the correct angle condition on the multipoints / junctions is the expected Young's law.

These anisotropic network flows are actually part of an even more general kind of anisotropic flows, as presented in [36] and [34]. An **anisotropy** is a function $\varphi : \mathbb{R}^2 \to [0, \infty)$ which is convex and:

$$\varphi(\lambda\xi) = |\lambda|\varphi(\xi),\ \varphi(\xi) \geqslant c|\xi|,\ \lambda \in \mathbb{R},\ \xi \in \mathbb{R}^2,\ c > 0$$

The dual of φ is defined as:

$$\varphi^o(\xi) = \max_{\varphi(\eta)=1}\langle\xi, \eta\rangle$$

Then, one can define the anisotropic energy (or $\varphi-$length):

$$\mathscr{E}_\varphi(\mathscr{N}) = \sum_{1<j\leqslant N}\int_{X^{i,j}} \varphi^o_{i,j}(\widehat{n}^{i,j})\, d\ell$$

where $\varphi_{i,j}$ are anisotropies associated to each phase boundary. By assuming $\varphi^o_{i,k} \leqslant \varphi^o_{i,j} + \varphi^o_{j,k}$, the angle condition is again a generalised form of Young's law:

$$\sum_{i<j\leqslant N} R\nabla\varphi^o_{i,j}(\widehat{n}^{i,j}) = 0$$

where R is a $\pi/2-$rotation. The motion by curvature equation is:

$$V^{i,j} = \langle\partial_t X^{i,j}, \widehat{n}^{i,j}\rangle = \varphi^o_{i,j}(\widehat{n}^{i,j})\langle\text{Hess}\ \varphi^o_{i,j}(\widehat{n}^{i,j})\widehat{t}^{i,j}, \widehat{t}^{i,j}\rangle\kappa^{i,j}$$

Remark 2.13. *In our case, we might take $\varphi_{i,j}(\xi) = \sigma^{-1}_{i,j}|\xi|$, and then $\varphi^o_{i,j}(\xi) = \sigma_{i,j}|\xi|$, $\nabla\varphi^o_{i,j}(\xi) = \sigma_{i,j}\widehat{\xi}$:*

$$\mathscr{E}_\varphi(\mathscr{N}) = \sum_{1<j\leqslant 3}\int_{X^{i,j}}\sigma_{i,j}\, d\ell = \sum_{i<j\leqslant 3}\sigma_{i,j}\mathscr{L}^1(X^{i,j})$$

Moreover:

$$0 = \sum_{i<j\leqslant 3} R\nabla\varphi^o_{i,j}(\widehat{n}^{i,j}) = \sum_{i<j\leqslant 3}\sigma_{i,j}R\widehat{n}^{i,j} = \sum_{i<j\leqslant 3}\sigma_{i,j}\widehat{t}^{i,j}$$

and:

$$V^{i,j} = \varphi^o_{i,j}(\widehat{n}^{i,j})\langle\text{Hess}\ \varphi^o_{i,j}(\widehat{n}^{i,j})\widehat{t}^{i,j}, \widehat{t}^{i,j}\rangle\kappa^{i,j} = \sigma^2_{i,j}\kappa^{i,j}$$

2.2.3 Short-time existence for the curvature flow of networks

Note that the $C^{1+a/2,2+a}$ condition is what was first studied, however the more natural choice of $C^{1,2}$ is enough to prove short-time existence.

The proof of short-time existence for equal surface tension coefficients $\sigma^p_{i,j} = \sigma$ is a standard procedure, which is based on some results of Solonnikov [27] (see also [38]). For general surface tension coefficients -that is, for the anisotropic problem- some work has been done too, for example in the directions of Kröner, Novaga and Pozzi [36], and Bellettini and Kholmatov [34]. The short-time existence results in those two previous references are based on the same linearisation and fixed-point arguments as in the equal surface tension coefficients case.

In what follows we describe how the short-time existence of the motion by curvature on networks is established for the case of equal surface tension coefficients. First notice that, under some suitable reframed curvature flow, we can write:

$$\frac{\partial X}{\partial t} = \kappa(X)\widehat{n}(X) + \lambda(X)\widehat{t}(X)$$

and to have some non-degenerate equation, we can choose (in view of (2.3)):

$$\lambda(X) = \frac{1}{|\partial_x X|^2}\langle \partial_x^2 X, \widehat{t}\rangle \tag{2.4}$$

to obtain:

$$\frac{\partial X}{\partial t} = \frac{\partial_x^2 X}{|\partial_x X|^2} \tag{2.5}$$

Definition 2.14 (Admissible initial parametrisation)**.** *We say that a parametrisation $\varphi = (\varphi^1, \cdots, \varphi^n)$ of an admissible initial network $\mathcal{N}_0$ (composed of $n-$curves) is* ***admissible*** *if:*

i. $\bigcup_{k=1}^n \varphi^k([0,1]) = \mathcal{N}_0$

ii. Each $\varphi^k \in C_x^{2,a}$ is regular.

iii. Concurrency: $\varphi^{p,k} = \varphi^{p,\ell}$, at each junction O^p.

iv. Angle condition:

$$\sum_{k\leqslant 3} \frac{\partial_x \varphi^{p,k}}{|\partial_x \varphi^{p,k}|} = 0, \text{ at each junction } O^p$$

v. Curvature condition:

$$\frac{\partial_x^2 \varphi^{p,1}}{|\partial_x \varphi^{p,1}|^2} = \frac{\partial_x^2 \varphi^{p,2}}{|\partial_x \varphi^{p,2}|^2} = \frac{\partial_x^2 \varphi^{p,3}}{|\partial_x \varphi^{p,3}|^2}, \text{ at each junction } O^p$$

vi. Dirichlet boundary condition: At the endpoints $\varphi^k(1) = P^k$.

vii. Curvature at the boundary: We have $\partial_x^2 \varphi^k(1) = 0$.

Theorem 2.15 (Bronsard-Reitich). *For any admissible initial parametrisation there exists some radius $R > 0$ and some time $T > 0$ such that the following system has a unique solution in $C^{1+a/2,2+a}([0,T) \times [0,1]) \cap \overline{B}_R(0)$. For $k \leqslant n$:*

i. We have motion by curvature:

$$\frac{\partial X^k}{\partial t} = \frac{\partial_x^2 X^k}{|\partial_x X^k|^2}$$

ii. Concurrency at each junction O^p, $X^{p,k} = X^{k,\ell}$.

iii. Angle condition:

$$\sum_{k \leqslant 3} \frac{\partial_x \varphi^{p,k}}{|\partial_x \varphi^{p,k}|} = 0, \text{ at each junction } O^p$$

iv. Dirichlet boundary condition $X^k(t,1) = P^k$ at the endpoints.

v. Initial data $X^k(0,x) = \varphi^k(x)$.

Proof. The sketch of the proof is demonstrated in several steps. We note that, by the parabolicity of the system and the "local" formulation around each junction, we can restrict ourselves in the case where the initial parametrisation is a triod $\mathbb{T}$.

Step I: First of all, we fix some admissible data $Y = (Y_1, Y_2, Y_3)$ and we linearise the system around Y. That is, if we set $Fu = \partial_t u - \partial_x^2 u / |\partial_x u|^2$, we write:

$$F(u + \varepsilon v) = Fu + \varepsilon \frac{\delta F}{\delta v}\Big|_u + O(\varepsilon^2)$$

and by some straight-forward calculations:

$$\frac{\delta F}{\delta v}\Big|_u = \partial_t v - \frac{\partial_x^2 v}{|\partial_x u|^2} + 2\frac{\partial_x^2 u}{|\partial_x u|^4}\langle \partial_x u, \partial_x v\rangle$$

If $u + \varepsilon v \rightsquigarrow X^k$, $u \rightsquigarrow Y^k$:

$$F(X^k) \approx \partial_t X^k - \frac{\partial_x^2 X^k}{|\partial_x Y^k|^2} - 2\frac{\partial_x^2 Y^k}{|\partial_x Y^k|^2} + 2\frac{\partial_x^2 Y^k}{|\partial_x Y^k|^4}\langle \partial_x Y^k, \partial_x X^k\rangle$$

For well-posedness, the relevant information resides in the higher order term, and this is why we will not consider the term with $\partial_x X$.

The other conditions, that being concurrency and Dirichlet boundary conditions, are linear, except the angle condition. Similarly, a linear version of it is:

$$-\sum_{k \leqslant 3} \frac{\partial_x X^k}{|\partial_x Y^k|} - \frac{\partial_x X^k}{|\partial_x Y^k|^3}\langle \partial_x X^k, \partial_x Y^k\rangle = \sum_{k \leqslant 3}\left(\frac{1}{|\partial_x X^k|} - \frac{1}{|\partial_x Y|}\right)\partial_x X^k + \\ + \frac{\partial_x Y^k}{|\partial_x Y^k|^3}\langle \partial_x X^k, \partial_x Y^k\rangle$$

The associated linear system is:

$$\begin{cases} \partial_t X^k - \frac{\partial_x^2 X^k}{|\partial_x Y^k|^2} = f^k(t,x) \\ X^1(t,0) - X^2(t,0) = X^1(t,0) - X^3(t,0) = 0 \\ -\sum_{k\leqslant 3} \frac{\partial_x X^k(t,0)}{|\partial_x X^k|} - \frac{\partial_x X^k(t,0)}{|\partial_x Y^k(t,0)|^3}\langle \partial_x X^k(t,0), \partial_x Y^k(t,0)\rangle = b(t,0) \\ X^k(t,1) = P^k \\ X^k(0,x) = \varphi^k(x) \end{cases}$$

In order to show that there exists a solution of this system, in view of Solonnikov's standard theory (cf. [27]), one needs to show that some complementary conditions hold. It is though easier to use another fact, that these conditions can be replaced by some **Lopatinskii-Shapiro** condition, which is defined below:

We suppose $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0$. The Lopatinskii-Shapiro condition for the linearised system is satisfied at the triple juction if for every solution of:

$$\begin{cases} \lambda X^k - \frac{\partial_x^2 X^k}{|\partial_x Y^k(0)|} = 0 \\ X^1(0) - X^2(0) = X^2(0) - X^3(0) = 0 \\ \sum_{k\leqslant 3} \frac{\partial_x X^k(0)}{|\partial_x X^k|} + \frac{\partial_x X^k(0)}{|\partial_x Y^k(0)|^3}\langle \partial_x X^k(0), \partial_x Y^k(0)\rangle = 0 \end{cases}$$

such that $|X^k| \to 0$, as $x \to \infty$, is trivial. Notice we have "frozen" variable t and instead replaced ∂_t by the "spectral" parameter λ. The Lopatinskii-Shapiro condition is verified by projecting the first (motion) equation to $|\partial_x^2(0)|\overline{\langle X^k, \widehat{n}^k(0)\rangle}\widehat{n}^k(0)$ (similarly to the same expression with tangents) and using the concurrency and angle conditions (the remaining equations, that is).

Step II: For $T > 0$, we define the map $L_T : \mathcal{S}_T \to \mathcal{G}_T$ between spaces:

$$\mathcal{S}_T = \left\{ X \in C^{1+a/2,2+a}([0,T]\times[0,1];\mathbb{R}^6) \mid X^1(t,0) = X^2(t,0) = X^3(t,0) \right\}$$
$$\mathcal{G}_T = \Big\{ (f,(b,c),\psi) \in C^{a/2,a}([0,T]\times[0,1];\mathbb{R}^6) \times C_t^{1/2+a/2}([0,T];\mathbb{R}^4) \times$$
$$\times\, C_x^{2,a}([0,1];\mathbb{R}^6 \mid \text{ linear compatability conditions hold} \Big\}$$

by:

$$L_T(X) = \left(\begin{array}{c} \partial_t X^k - \frac{\partial_x^2 X^k}{|\partial_x Y^k|^2} \mid k \leqslant 3 \\ -\sum_{k\leqslant 3} \frac{\partial_x X^k(t,0)}{|\partial_x X^k|} - \frac{\partial_x X^k(t,0)}{|\partial_x Y^k(t,0)|^3}\langle \partial_x X^k(t,0), \partial_x Y^k(t,0)\rangle \\ X^1(t,0) \\ X(0,x) \end{array} \right)$$

The problem now is to define some proper operator so that X is a fixed point. Then existence and uniqueness will follow from Banach-Caccioppoli theorem. In this direction, we define $N_T = (N_1, N_2, X(0,x))$, which contains all the information of the non-linearities of the problem:

$$N_1 : \mathcal{S}_T^{\varphi,P} \to C^{a/2,a}([0,T]\times[0,1];\mathbb{R}^6),\ X \mapsto f(X)$$
$$N_2 : \mathcal{S}_T^{\varphi,P} \to C_t^{1/2+a/2}([0,T];\mathbb{R}^4),\ X \mapsto (b,c)(X)$$

where $\mathcal{S}_T^{\varphi,P} = \{X \in \mathcal{S}_T \mid X(0,x) = \varphi(x),\ X^k(t,1) = P^k\}$. Then, X is a solution if and only if $X \in \mathcal{S}_T^{\varphi,P}$ and:

$$L_T(X) = N_T(X) \Leftrightarrow X = L_T^{-1} N_T(X) = K_T(X)$$

It can be shown that K_T is a contraction, which concludes the proof. $\square$

This establishes the proof of a parametric solution of the curvature flow. To deal with the geometric problem, we shall consider the case of the triod. Notice that the notion of the geometric solution is as in Definition 2.10, and the starting network is an admissible initial network rather than an admissible initial parametrisation.

Lemma 2.16. *If in a junction structure curves X^1, X^2, X^3 meet, then:*

$$\kappa^1\widehat{n}^1 + \lambda^1\widehat{t}^1 = \kappa^2\widehat{n}^2 + \lambda^2\widehat{t}^2 = \kappa^3\widehat{n}^3 + \lambda^3\widehat{t}^3$$

is satisfied if and only if $\sum_{k\leqslant 3}\kappa^k = \sum_{k\leqslant 3}\lambda^k = 0$.

Proof. If $\kappa^i\widehat{n}^i + \lambda^i\widehat{t^i} = \kappa^j\widehat{n}^j + \lambda^j\widehat{t^j}$, then we can multiply these equations by $\widehat{t}^k$, $\widehat{n}^k$, and by using the angle condition at the junction:

$$\begin{aligned}\lambda^i &= -\lambda^{i+1}/2 - \sqrt{3}\kappa^{i+1}/2\\ \lambda^i &= -\lambda^{i-1}/2 - \sqrt{3}\kappa^{i-1}/2\\ \kappa^i &= -\kappa^{i+1}/2 - \sqrt{3}\lambda^{i+1}/2\\ \kappa^i &= -\kappa^{i-1}/2 - \sqrt{3}\lambda^{i-1}/2\end{aligned}$$

(of course the convention is that the indices are considered modulo 3). The solution of this system is:

$$\lambda^i = \frac{\kappa^{i-1}-\kappa^{i+1}}{\sqrt{3}} \quad\text{and}\quad \kappa^i = \frac{\lambda^{i+1}-\lambda^{i-1}}{\sqrt{3}}$$

from which the lemma follows. □

Now, the relation between admissible initial networks and admissible parametrisations is encapsuled in the following lemma.

Proposition 2.17. *Let $\mathbb{T}_0$ be an admissible initial triod, parametrised by $Y = (Y^1, Y^2, Y^3)$. There exist smooth functions $\theta^i : [0,1] \to [0,1]$ such that reparametrisation:*

$$\varphi = (Y^1\circ\theta^1, Y^2\circ\theta^2, Y^3\circ\theta^3)$$

is an admissible initial parametrisation.

Proof. We need to check every condition in Definition 2.14. The ones in i., ii., iii., vi. are automatically satisfied.

As for iv., this is true for any θ^i, since it involves the unit tangent vectors, which are invariant under reparametrisation.

As for v.: We define (compare with (2.4)):

$$\lambda^i = \frac{1}{|\partial_x\varphi^i|^3}\langle\partial_x^2\varphi^i, \partial_x\varphi^i\rangle$$

and we have that the curvature condition becomes:

$$\kappa^1_\varphi\widehat{n}^1_\varphi + \lambda^1\widehat{t}^1_\varphi = \kappa^2_\varphi\widehat{n}^2_\varphi + \lambda^2\widehat{t}^2_\varphi = \kappa^3_\varphi\widehat{n}^3_\varphi + \lambda^3\widehat{t}^3_\varphi$$

All of the appearing geometric quantities are invariant under reparametrisation, so the dependence on φ can be omitted in all terms except λ^i. We can then utilise Lemma 2.16 to see that we must have:

$$\sum_{k=1}^{3} \kappa^i = \sum_{k=1}^{3} \lambda^i = 0$$

As for vii.: The condition $\partial_x^2 \varphi^k(1) = 0$ at the endpoints can be equivalenty written as $\kappa^i \widehat{n^i} + \lambda^i \widehat{t^i} = 0$. This is satisfied if $\kappa^i = \lambda^i = 0$. We have now reduced the problem to finding reparametrisations θ^i such that:

$$\sum_{k=1}^{3} \kappa^i = \sum_{k=1}^{3} \lambda^i = 0 \;\; \text{(at the junction)} \qquad \kappa^i = \lambda^i = 0 \;\; \text{(at the endpoints)}$$

Since in the endpoints 0, 1, we have $\theta^i(0) = 0$, $\theta^i(1) = 1$, we get:

$$\begin{aligned}
\lambda^i &= \frac{1}{|\partial_x \varphi^i|^3} \langle \partial_x^2 \varphi^i, \partial_x \varphi^i \rangle \\
&= -\partial_x \frac{1}{|\partial_x X^k \circ \theta^i| \cdot |\partial_x \theta^i|} \\
&= \frac{1}{|\partial_x X^i|^3} \langle \partial_x^2 X^i, \partial_x X^i \rangle + \frac{\partial_x^2 \theta^i}{|\partial_x Y^i| \cdot |\partial_x \theta^i|^2} \\
&= \lambda_X^i + \frac{\partial_x^2 \theta^i}{|\partial_x Y^i| \cdot |\partial_x \theta^i|^2}
\end{aligned}$$

In the end, one can choose smooth functions (say, polynomials) such that $\partial_x \theta^i(0) = \partial_x \theta^i(1)$, $\partial_x^2 \theta^i(1) = -\lambda_X^i |\partial_x X^i| \cdot |\partial_x \theta^i|^2$ and:

$$\partial_x^2 \theta^i(0) = \left(\frac{\kappa^{i-1} - \kappa^{i+1}}{\sqrt{3}} - \lambda_X^i \right) |\partial_x X^i| \cdot |\partial_x \theta^i|^2$$

□

All of this work enables us to prove existence and uniqueness of the solution for the geometric problem.

Theorem 2.18 (Geometric existence and uniqueness). *For the admissible initial triod, there exists a geometrically unique solution of the curvature flow in some time interval* $[0, T]$.

Proof. First of all, Proposition 2.17 guarantees that there exists an admissible initial parametrisation of our triod $\mathbb{T}_0$. Moreover, by Theorem 2.15 we know that there exists a unique solution $X = (X^1, X^2, X^3)$ in some time interval $[0, T_1]$. This gives us a network $\mathbb{T}_t$, $t \leqslant T_1$.

Lets suppose that there exists another solution $\widetilde{X} = (\widetilde{X}^1, \widetilde{X}^2, \widetilde{X}^3)$ in some time interval $[0, T_2]$. This gives us another network $\widetilde{\mathbb{T}}_t$, $t \leqslant T_2$. We aim to show that $\mathbb{T}_t$ and $\widetilde{\mathbb{T}}_t$ coincide up to some positive time, which can be restated as that X coincides with $\widetilde{X}$ up to reparametrisation.

We let $\varphi^k : [0,T_3] \times [0,1] \to [0,1]$ be $C^{1+a/2,2+a}-$reparametrisations. By considering $\overline{X}(t,x) = \widetilde{X}^i(t,\varphi^i(t,x))$, we have:

$$\begin{aligned}\partial_t \overline{X}^k(t,x) = &\left\langle \frac{\partial_x^2 \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|}, \widehat{n}^k(t,\varphi^k) \right\rangle \widehat{n}^k(t,\varphi^k) + \widetilde{\lambda}^k(t,\varphi^k) \frac{\partial_x \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|} \\ &+ \partial_x \widetilde{X}^k(t,\varphi^k) \partial_t \varphi^k(t,x)\end{aligned}$$

We now ask for φ^k to be solutions, in some time interval $[0,T_4]$ of the following quasilinear partial differential equation:

$$\begin{aligned}\partial_t \varphi^k(t,x) = &\frac{1}{|\partial_x \widetilde{X}^k(t,\varphi^k)|} \left\langle \frac{\partial_x^2 \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2}, \frac{\partial_x \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|} \right\rangle \\ &- \frac{\widetilde{\lambda}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|} + \frac{\partial_x^2 \varphi^k(t,x)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2 \cdot |\partial_x \varphi^k(t,x)|^2}\end{aligned}$$

such that $\varphi^k(t,0) = 0$, $\varphi^k(t,1) = 0$, $\varphi^k(0,x) = x$ and $\partial_x \varphi^k(t,x) = 0$. The existence of these kind of solutions follows from some standard theory of quasilinear parabolic equations, in [24]. In the end, we obtain:

$$\begin{aligned}\partial_t \overline{X}^k(t,x) = &\left\langle \frac{\partial_x^2 \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2}, \widehat{n}^k(t,\varphi^k) \right\rangle \widehat{n}^k(t,\varphi^k) \\ &+ \left\langle \frac{\partial_x^2 \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2}, \frac{\partial_x \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|} \right\rangle \frac{\partial_x \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|} \\ &+ \frac{\partial_x^2 \varphi^k(t,x) \partial_x \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2 \cdot |\partial_x \varphi^k(t,x)|^2} \\ = &\frac{\partial_x^2 \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2} + \frac{\partial_x^2 \varphi^k(t,x) \partial_x \widetilde{X}^k(t,\varphi^k)}{|\partial_x \widetilde{X}^k(t,\varphi^k)|^2 \cdot |\partial_x \varphi^k(t,x)|^2} \\ = &\frac{\partial_x^2 \overline{X}^k(t,x)}{|\partial_x \overline{X}^k(t,x)|^2}\end{aligned}$$

By the uniqueness of Theorem 2.15, we conclude that $\widetilde{X}^k = \overline{X}^k$ in some time interval $[0,T]$, $T = \min\{T_1,T_2,T_4\}$.

□

2.3 Angle conditions for expanding, contracting and stable networks

In what follows we present some remarks on how the interface forms certain angles in a certain anisotropic case. There are two cases (three in fact, but the intermediate case is trival) that are going to define the interface's shape. To have some geometric intuition, the general problem is of Steiner-type with some motion by curvature attached. Suppose one has a juction structure and -by force- injects some fourth material in between the other three. We expect that three simple configurations can appear, that of elliptic (convex), hyperbolic (concave) and euclidean triangles.

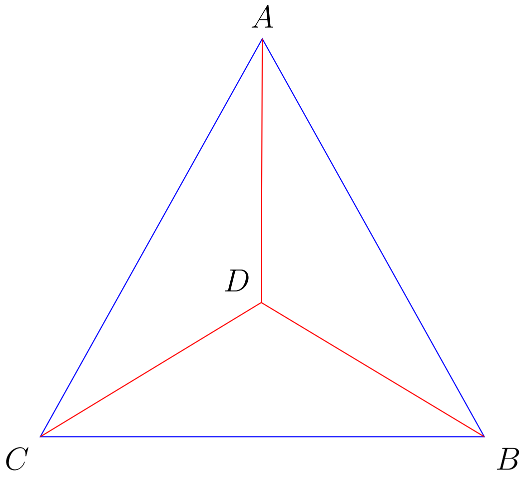


Figure 2.5

Remark 2.19. *Inside some disc $B_R(0)$, let σ_0, σ be the surface tension coefficients of the triangle T and junction J. Then J is prefered if $\sigma < \sqrt{3}\sigma_0$ and T is prefered if $\sigma > \sqrt{3}\sigma_0$.*

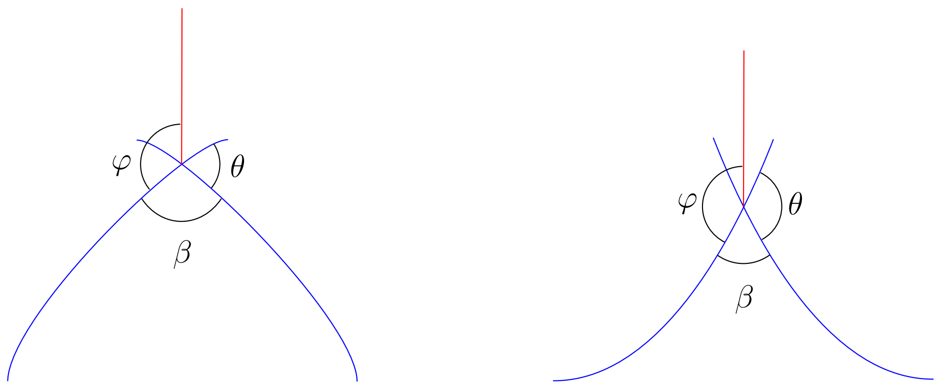
Proof. This is just a geometry fact, as is indicated below. It is simple to see that $DA = DB = DC = R$ and $AB = BC = CA = \sqrt{3}R$, so the energy of the juction is $3\sigma R$ and for the triangle $3\sqrt{3}\sigma_0 R$. □

An important lemma follows.

Lemma 2.20. *Suppose we have some junction structure inside $B_R(0)$, with materials 1, 2, 3 and surface tension coefficients $\sigma_{i,j} = \sigma$, $i \neq j \leqslant 3$. We inject material 4 right on the junction's triple point, such that the surface coefficients become $\sigma_{4,i} = \sigma_{i,4} = \sigma_0$. Then:*

i. If $\sigma < \sqrt{3}\sigma_0$, then in the interface's cusps appear angles β such that $\beta > \pi/3$ (see Figure 2.6 (a)).

ii. If $\sigma > \sqrt{3}\sigma_0$, then in the interface's cusps appear angles β such that $\beta < \pi/3$ (see Figure 2.6 (b)).



(a) Case $\sigma < \sqrt{3}\sigma_0$. (b) Case $\sigma > \sqrt{3}\sigma_0$.

Figure 2.6

Proof. We will see i., as ii. is similar. By Young's law, we have:

$$\frac{\sin\varphi}{\sigma_0} = \frac{\sin\varphi}{\sigma_0} = \frac{\sin\beta}{\sigma} \Rightarrow \frac{\sin\beta}{\sin\varphi} = \frac{\sigma}{\sigma_0} < \sqrt{3}$$

and by $2\pi = 2\varphi + \beta$, $\sin\beta = -\sin(2\varphi) = -2\sin\varphi\cos\varphi$. But then:

$$-2\cos\varphi < \sqrt{3} \Rightarrow \cos(\pi - \varphi) < \frac{\sqrt{3}}{2} \Rightarrow \varphi < \frac{5\pi}{6} \Rightarrow \beta > \frac{\pi}{3}$$

□

Considering closed curves, it is a general procedure to calculate the enclosed area to compute short-time existence. For this reason, we will need the following theorem of Neumann and Mullins, which is similar to Hopf's Umlaufsatz for piecewise smooth curves.

Theorem 2.21 (Neumann-Mullins law). *Let γ be a simple, piecewise smooth curve on the plane. Then:*

$$\int_\gamma \kappa_\sigma \, d\ell = 2\pi - \sum \theta_i$$

where θ_i are the external angles, one on each sharp point of γ.

Proof. By cutting a small region around each sharp point, we can replace it with an arc of angle ω and radius r, as in Figure 2.7. This forms a new curve γ^* without sharp points,

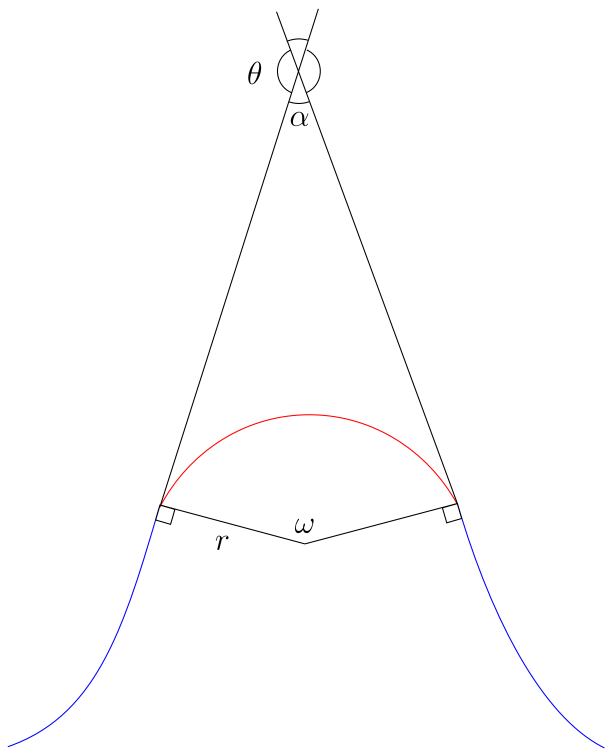


Figure 2.7

and Hopf's Umlaufsatz can be applied. Notice that these cuts can be made arbitrarily close to each sharp point, and as a consequence we can assume the black four-sided shape is approximately a kite. We can also assume, by this principle, that:

$$\int_{\gamma \setminus \text{arcs}} \kappa_\sigma \, d\ell \approx \int_\gamma \kappa_\sigma \, d\ell$$

for some arbitrarily small error. We have:

$$\int_{\gamma^*} \kappa_\sigma \, d\ell = 2\pi$$

Now, a simple observation shows $\alpha = \pi - \omega$ and $\theta = \omega$, so by integrating on the arc we obtain:

$$\int_{\text{arc}} \kappa_\sigma \, d\ell = \frac{1}{r}\omega r = \omega = \theta$$

Therefore, if ω_i are the arc angles and θ_i their respective external angles, we get:

$$2\pi = \int_{\gamma^*} \kappa_\sigma \, d\ell = \int_{\gamma \backslash \text{arcs}} \kappa_\sigma \, d\ell + \sum \theta_i$$

□

Remark 2.22 (Expanding, contracting and stationary networks). *We can calculate the change of area enclosed by the triangle formation under curvature flow. Suppose $A(t)$ is the area enclosed by X and write:*

$$\frac{dA}{dt} = -\int_X \left\langle \frac{\partial X}{\partial t}, \widehat{n}_\sigma \right\rangle \, d\ell = -\int_X V_\sigma \, d\ell = -\sigma_0^2 \int_X \kappa_\sigma \, d\ell$$

Now, by Theorem 2.21:

$$\frac{dA}{dt} = \left(-2\pi + \sum \theta_i\right) \sigma_0^2 = (-2\pi + 3\theta)\, \sigma_0^2$$

where $\theta = \pi - \beta$ (see Figure 2.6) and $dA/dt = \pi - 3\beta$. If $\sigma < \sqrt{3}\sigma_0$, then $\beta > \pi/3$ and A decreases at a constant rate. In fact, it disappears right at:

$$t = \frac{1}{\sigma_0^2} \cdot \frac{A(0)}{3\beta - \pi}$$

Similarly, if $\sigma > \sqrt{3}\sigma_0$ then $\beta < \pi/3$ and A increases at a constant rate. The intermediate case, where A remains stationary, is trivial.

2.4 Motion under constraints

Suppose -for context- that the whole flow takes place in a bounded region (say, a disc). After some expanding network hits the boundary, we expect the extremal points to stop moving, thus we must have curvature flow with fixed boundary conditions. Such a flow isn't guranteed to exist, when curvature does not vanish on the boundary.

One way to prevent this collapse is to demand for the curvature to vanish on the boundary. That is, we consider:

$$V = \left\langle \frac{\partial X}{\partial t}, \widehat{n} \right\rangle = \sigma_0^2 \kappa(X) \tag{2.6}$$

with κ vanishing on the boundary. For the sake of simplicity and without loss of generality, we will omit σ_0 from our calculations.

Remark 2.23 (On short-time existence and regularity). *Short time existence for quasilinear parabolic partial differential equations is not an easy task. We refer to [24] and our previous analysis. From now, on we will assume existence and sufficient regularity in all of our calculations.*

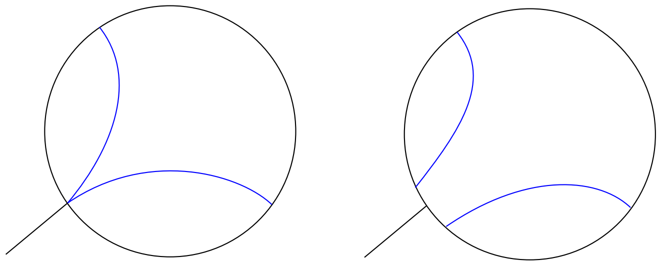


Figure 2.8: Junction collapse

It is quite useful and generally easier to work with curves that can be parametrised as a graph $\big(x, f(t,x)\big)$, say above interval $[0,1]$, such that $f(0) = f(1) = 0$.

Remark 2.24. *If $X(t,x) = (x, f(t,x))$ can be parametrised as a graph, it holds that:*

$$\frac{\partial f}{\partial t} = \frac{\partial_x^2 f}{1 + (\partial_x f)^2} \tag{2.7}$$

Proof. We write:

$$\frac{\partial X}{\partial t} = \kappa(X)\widehat{n}(X) + \lambda(X)\widehat{t}(X)$$

for some function λ to be chosen. In $x-$coordinates we have:

$$(\partial_t x, \partial_t f) = \frac{\partial_x^2 f}{\sqrt{1+(\partial_x f)^2}^4}(-\partial_x f, 1) + \frac{\lambda(X)}{\sqrt{1+(\partial_x f)^2}}(1, \partial_x f)$$

and by demanding $\partial_t x = 0$ it follows that:

$$\lambda(X) = \frac{\partial_x f \partial_x^2 f}{\sqrt{1+(\partial_x f)^2}^3}$$

Coming back to the original equation, we obtain:

$$\frac{\partial f}{\partial t} = \frac{\partial_x^2 f}{1 + (\partial_x f)^2}$$

□

First of all, there is this very general theorem, which shows -possibly for a subsequence- some good ($C^{1,a}-$)convergence towards geodesics. For its proof we need the following lemma, which is a maximum principle.

Lemma 2.25 (Maximum principle). *Let $X(t,x) = \big(x, f(t,x)\big)$ (defined on $[t_0,\infty) \times I$) be the graph of a function f. Define:*

$$\varphi_\Sigma(t) = \max_\Sigma \langle X, e_2\rangle = \max_\Sigma f(t,x) \textit{ and } \psi_\Sigma(t) = \min_\Sigma \langle X, e_2\rangle = \min_\Sigma f(t,x)$$

with Σ being a subinterval $\Sigma \subset I$. Then φ_Σ is non-increasing and ψ_Σ non-decreasing. Similarly, if:

$$\widetilde{\varphi}_\Sigma(t) = \max_\Sigma \partial_x \langle X, e_2\rangle = \max_\Sigma \partial_x f(t,x) \textit{ and } \widetilde{\psi}_\Sigma(t) = \min_\Sigma \partial_x \langle X, e_2\rangle = \min_\Sigma \partial_x f(t,x)$$

then $\widetilde{\varphi}_\Sigma$ is non-increasing and $\widetilde{\psi}_\Sigma$ non-decreasing. For general curves X, we also have that $|X|$ is non-increasing in time.

Proof. The proof is done in steps, first starting from the graph case and then treating the general case.

Step I: We will work only with φ_Σ, since ψ_Σ is similar. Consider $\Sigma^* \subseteq \Sigma$ the set of critical points of f such that $\partial_x^2 f \leqslant 0$. Then:

$$\varphi_\Sigma(t) = \max_{\Sigma^*}\langle X, e_2\rangle$$

and by differentiating:

$$\frac{d\varphi_\Sigma}{dt} = \max_{\Sigma^*}\left\langle \frac{\partial X}{\partial t}, e_2\right\rangle = \max_{\Sigma^*}\langle \kappa(X)\widehat{n}(X), e_2\rangle \leqslant \max_{\Sigma^*}\kappa = \max_{\Sigma^*}\frac{\partial_x^2 f}{\sqrt{1+(\partial_x f)^2}^3} \leqslant 0$$

Step II: As for the $\widetilde{\varphi}_\Sigma$ case ($\widetilde{\psi}_\Sigma$ is similar), by differentiating and by restricting ourselves to Σ^*, the set of critical points of $\partial_x f$, we obtain:

$$\frac{d\widetilde{\varphi}_\Sigma}{dt} = \max_{\Sigma^*}\left[\frac{\partial_x^3 f}{1+(\partial_x f)^2} - O(\partial_x^2 f)\right] \leqslant 0$$

(Since $\partial_x^3 f \leqslant 0$ and $\partial_x^2 f = 0$ on Σ^*).

Step III: For the general case, consider the set Σ^* of the most distant points of X from the origin. Considering the tangent ζ on each point on Σ^*, we can parametrise locally X as a graph above a line parallel to ζ. Then, using the graph case, each point of Σ^* does not move farther from the origin.

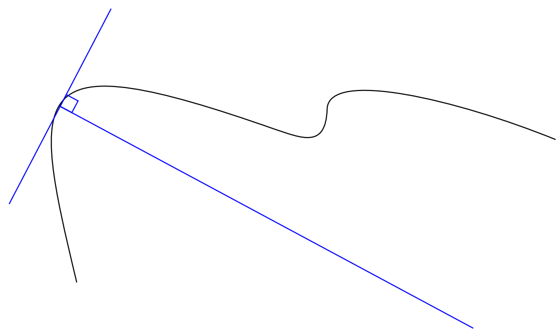


Figure 2.9

□

Theorem 2.26 (Motion towards geodesics)**.** *Suppose we have a family of curves $X(t,\cdot)$, parametrised by t, which moves by curvature flow, as above. There exists a sequence of times $(t_i)_{i=1}^\infty$, $t_i \to \infty$, such that:*

$$X(\tau + t_i, \cdot) \xrightarrow{C^{1,a}} \gamma$$

almost everywhere in an arbitrary interval $\tau \in [\tau_0, \tau_1]$, for every $a \leqslant 1/2$, where γ is a geodesic. The bounds and the convergence results persist under a possible graph reparametrisation $\big(x, f(t,x)\big)$.

Proof. We will work in several steps.

Step I: The change in length, as in Lemma 2.12, is:

$$\frac{d}{dt}\mathscr{L}^1(X(t,\cdot)) = -\int_{X(t,\cdot)} \kappa^2 \, d\ell \leqslant 0$$

Therefore, $\mathscr{L}^1(X(t,\cdot))$ is decreasing and it has a limit. It follows that if for any τ_0, τ_1, if $t \to \infty$:

$$\begin{aligned}\mathscr{L}^1(X(t+\tau_1,\cdot)) - \mathscr{L}^1(X(t+\tau_0,\cdot)) &= -\int_{t+\tau_0}^{t+\tau_1}\int_{X(\tau,\cdot)} \kappa^2 \, d\ell d\tau \\ &= -\int_{\tau_0}^{\tau_1}\int_{X(\tau+t,\cdot)} \kappa^2 \, d\ell d\tau \to 0\end{aligned}$$

Choose a sequence of $(t_i)_{i=1}^{\infty}$, $t_i \to \infty$, such that:

$$\int_{\tau_0}^{\tau_1}\int_{X(\tau+t_i,\cdot)} \kappa^2 \, d\ell d\tau \leqslant \frac{1}{2^i}$$

so this following sum converges:

$$\sum_{i=1}^{\infty}\int_{\tau_0}^{\tau_1}\int_{X(\tau+t_i,\cdot)} \kappa^2 \, d\ell d\tau < \infty$$

Then, by monotone convergence:

$$\int_{\tau_0}^{\tau_1}\sum_{i=1}^{\infty}\int_{X(\tau+t_i,\cdot)} \kappa^2 \, d\ell d\tau < \infty$$

Hence, for almost every $\tau \in [\tau_0, \tau_1]$:

$$\sum_{i=1}^{\infty}\int_{X(\tau+t_i,\cdot)} \kappa^2 \, d\ell < \infty$$

which means that:

$$\int_{X(\tau+t_i,\cdot)} \kappa^2 \, d\ell \xrightarrow{i\to\infty} 0$$

for almost every $\tau \in [\tau_0, \tau_1]$. We then expect in the limit that the curve has $\kappa = 0$, which means that it is a geodesic.

Step II: Considering all $\mathscr{L}^1(X(\tau+t_i,\cdot))$, $i \in \mathbb{N}$, this constitutes a uniformly bounded family for almost every $\tau \in [\tau_0, \tau_1]$. By using Lemma 2.25 and parametrising by arc length:

$$\int_s |X(\tau+t_i,s)|^2 \, ds \leqslant \max(|X|)\mathscr{L}^1(X(\tau+t_i,\cdot))$$

and:

$$\int_s \left|\frac{\partial}{\partial s}X(\tau+t_i,s)\right|^2 \, ds = \mathscr{L}^1(X(\tau+t_i,\cdot))$$

Moreover:

$$\int_s \left|\frac{\partial^2}{\partial s^2}X(\tau+t_i,s)\right|^2 \, ds = \int_s \kappa^2 \, ds = \int_{X(\tau+t_i,\cdot)} \kappa^2 \, d\ell$$

All of the above show uniform bounds on $(2,2)-$Sobolev norms. In order to have the same domain, we consider the change of variables $s = \ell^i_\tau \widetilde{s}$, where $\ell^i_\tau = \mathscr{L}^1(X(\tau + t_i, \cdot))$. This leads to interval $[0,1]$ and similar uniform bounds persist.

Step III: In Step III we showed uniform $(2,2)-$Sobolev bounds for the family $X(\tau + t_i, \ell^i_\tau \cdot)$, $i \in \mathbb{N}$, for almost every $\widetilde{s} \in [0,1]$. Utilising this bound for each coordinate, we obtain that possibly for a subsequence:

$$X(\tau + t_i, \ell^i_\tau \cdot) \xrightharpoonup[W^{2,2}]{w} \gamma$$

for some function γ. Since $1 < 2 \cdot 2 = 4$, by Theorem A.5:

$$X(\tau + t_i, \ell^i_\tau \cdot) \xrightarrow{C^{1,a}} \gamma$$

for every $a \leqslant 1/2$. In particular, along with $\partial_s^2 X \xrightharpoonup{w} 0$, $\partial_s^2 X \xrightharpoonup{w} \partial_s^2 \gamma$, we have that $\partial_s^2 \gamma = 0$ weakly; then $\partial_s \gamma$ is constant, which shows that γ is a geodesic.

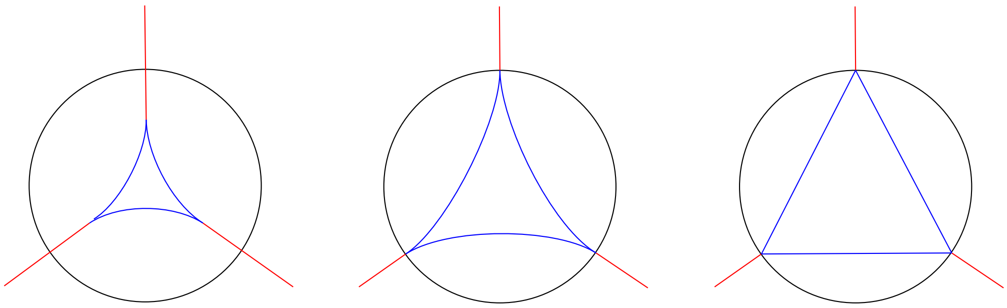


Figure 2.10: Convergence towards the triangle.

Step IV: We can show Hölder convergence in $x-$variable too, under possible graph reparametrisation. Notice we have already shown that $x(\ell^i_\tau \widetilde{s})$ converges in the $(1,a)-$Hölder sense, and we need to do the same for its inverse $\ell^i_\tau \widetilde{s}(x)$, or simply for $s(x)$. We show that $\partial_x s$ is $a-$Hölder, and to show that s is too, follow a similar but easier procedure. Notice two things, first that $\partial_x s = 1/\partial_s x$, and second that near the limit we have $|\partial_s x| \geqslant c > 0$ (by convergence to line segment γ). Moreover, $|\partial_x s| \leqslant 1/c$. Now we write:

$$\begin{aligned}
|\partial_x s(x_1) - \partial_x s(x_2)| &= \left| \frac{1}{\partial_s x(s_1)} - \frac{1}{\partial_s x(s_2)} \right| \\
&= \frac{|\partial_s x(s_1) - \partial_s x(s_2)|}{|\partial_s x(s_1) \partial_s x(s_2)|} \\
&\leqslant \frac{1}{c^2} |\partial_s x(s_1) - \partial_s x(s_2)| \\
&\leqslant \frac{M}{c^2} |s(x_1) - s(x_2)|^a
\end{aligned}$$

By the mean value theorem, we conclude:

$$\begin{aligned}
[\ldots] &= \frac{M}{c^2} |\partial_x s(\xi)| \cdot |x_1 - x_2|^a \\
&\leqslant \frac{M}{c^3} |x_1 - x_2|^a
\end{aligned}$$

These show that $x(s)$, $s(x)$ are $C^{1,a}-$diffeomorphisms (and in fact their norms can be uniformly bounded). $\square$

Remark 2.27. *Consider the graph case. By the maximum principle, we know that $||f||_{L^\infty}$ as well as $||\partial_x f||_{L^\infty}$ do not increase. In fact, the proof of Lemma 2.25 shows that, starting from any arbitrary time $t_0 \geqslant 0$, those maxima do not increase. Since for a sequence of times $(t_k)_{k=1}^{\infty}$, $t_k \to \infty$, we have $||f(\tau + t_k, \cdot)||_{C^1} \to 0$, then for any $\tau + t_k < t$:*

$$||f(t,\cdot)||_{C^1} \leqslant ||f(\tau + t_k, \cdot)||_{C^1}$$

and then $||f(t,\cdot)||_{C^1} \to 0$, as $t \to \infty$.

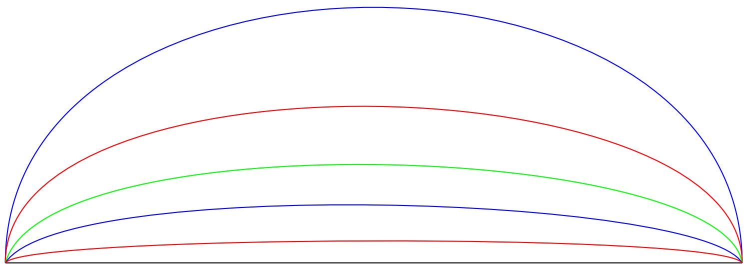


Figure 2.11

Proposition 2.28 (Motion towards the triangle - The graph case)**.** *Suppose we have a starting network $\mathscr{N}_0$, which is smooth and regular, with three junctions. We postulate that each arc is a graph on the corresponing sides of a triangle $\mathscr{T}$, formed by the junctions as vertices. Keeping each junction point fixed, under curvature flow we have smooth and exponential convergence:*

$$X(t,\cdot) \xrightarrow[t\to\infty]{C^{2,\delta}} \mathscr{T}, \;\; \delta \in (0,1)$$

towards triangle $\mathscr{T}$, in each closed subinterval. That is:

$$||X(t,\cdot) - \mathscr{T}||_{C^{2,\delta}_x} \leqslant C\big(\max\big|\partial_x X\big|, \delta, d, \varepsilon\big)||X(t,\cdot) - \mathscr{T}||_{L^\infty_x} \leqslant Ce^{-c(t-\varepsilon)}$$

where $d = \operatorname{dist}(I, [0,1])$, $\varepsilon, \delta \in (0,1)$. Moreover:

$$||X(t,\cdot) - \mathscr{T}||_{C^{1+\delta/2, 2+\delta}_{t,x}(\widetilde{Q})} \leqslant C\big(\max\big|\partial_x X\big|, \delta, d, \varepsilon\big)||X(t,\cdot) - \mathscr{T}||_{L^\infty_{t,x}(Q)}$$

for parabolic cylinders $\widetilde{Q} \Subset Q$ with $d = \operatorname{dist}(\widetilde{Q}, Q)$.

Proof. This proof will be done is several steps.

Step I: We first we will establish some $W^{1,2}-$bounds. We define:

$$\mathscr{E}(t) = \int_0^1 (\partial_x f)^2 \, dx$$

and by differentiating:

$$\frac{d\mathscr{E}}{dt} = \int_0^1 2\partial_x f \partial_{x,t} f \, dx$$

Now we use integration by parts and the boundary conditions of f to get:

$$\frac{d\mathscr{E}}{dt} = -2\int_0^1 \frac{(\partial_x^2 f)^2}{1+(\partial_x f)^2}\, dx \leqslant -2\int_0^1 (\partial_x^2 f)^2\, dx \tag{2.8}$$

Step II: The general idea now is to use Poincaré's inequality to bound $\mathscr{E}$ by its derivative, and then obtain some estimates by Grönwall's inequality. We observe that $f(t,0) = f(t,1) = 0$, $⨍_0^1 \partial_x f\, dx = 0$, and by (2.8):

$$\mathscr{E}(t) = \int_0^1 |\partial_x f|^2\, dx \leqslant C\int_0^1 |\partial_x^2 f|^2\, dx \leqslant -\frac{C}{2}\frac{d\mathscr{E}}{dt}$$

By Grönwall's inequality:

$$\mathscr{E}(t) \leqslant \mathscr{E}(0)e^{-2t/C}$$

Poincaré's inequality can by used again, since $f(t,0) = f(t,1) = 0$, to get:

$$\int_0^1 |f|^2\, dx \leqslant C\int_0^1 |\partial_x f|^2\, dx \leqslant C\mathscr{E}(0)e^{-2t/C}$$

We conclude that:

$$||f||_{W_x^{1,2}} \leqslant M_0 e^{-t/C}$$

and by Sobolev embeddings:

$$||f||_{L_x^\infty} \leqslant ||f||_{C_x^{0,a}} \leqslant M_1||f||_{W_x^{1,2}} \leqslant M_2 e^{-t/C}$$

Step III: Having $L^\infty-$bounds and assuming sufficient regularity, in view of some results of the type of Ladyženskaja-Solonnikov-Ural'ceva [24] (cf. Theorem A.9), Schauder estimates follow in each parabolic subcylinder. We use a simple trick in order to achieve these estimates up to the boundary.

We consider the following problem, which has f as a solution:

$$\begin{cases} \dfrac{\partial g}{\partial t} = \alpha(t,x)\dfrac{\partial^2 g}{\partial x^2} \\ g(t,0) = g(t,1) = 0 \end{cases}, \quad \text{where } \alpha(t,x) = \frac{1}{1+(\partial_x f)^2}$$

We can extend this problem to a bigger interval, $[-1,2]$, by an odd reflection. More specifically, we define:

$$g(t,x) = -g(t,-x) \text{ for } x \in [-1,0] \quad \text{and} \quad g(t,x) = -g(t,2-x) \text{ for } x \in [1,2]$$

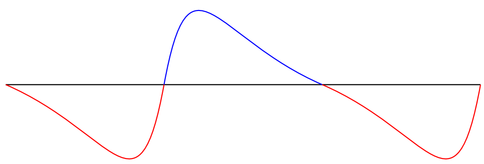


Figure 2.12: Function g extended by odd reflection.

Now consider some appropriate parabolic cylinders $\widetilde{Q} \Subset Q$ such that the $x-$projection of the smallest is $[0,1]$, as in Figure 2.13. Then, by some Schauder estimates:

$$||g||_{C^{1+\delta/2,2+\delta}_{t,x}(\widetilde{Q})} \leqslant C\left(\max_x |\partial_x f(t=0,\cdot)|, \delta, d\right) ||g||_{L^\infty_{t,x}(Q)}, \;\; \delta \in (0,1), \;\; d = \mathrm{dist}(\widetilde{Q}, Q)$$

and since $g|_{\widetilde{Q}} = f|_{\widetilde{Q}}$, $||f||_{L^\infty_{t,x}(\widetilde{Q})} = ||g||_{L^\infty_{t,x}(Q)}$:

$$||f||_{C^{1+\delta/2,2+\delta}_{t,x}(\widetilde{Q})} \leqslant C\left(\max_x |\partial_x f(t=0,\cdot)|, \delta, d\right) ||f||_{L^\infty_{t,x}(\widetilde{Q})}$$

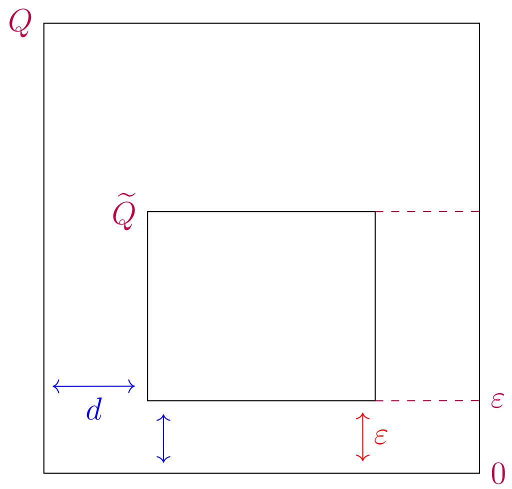


Figure 2.13

Step IV: By shifting our original problem to have starting time $t_0 - \varepsilon$, we repeat the above arguments and we get:

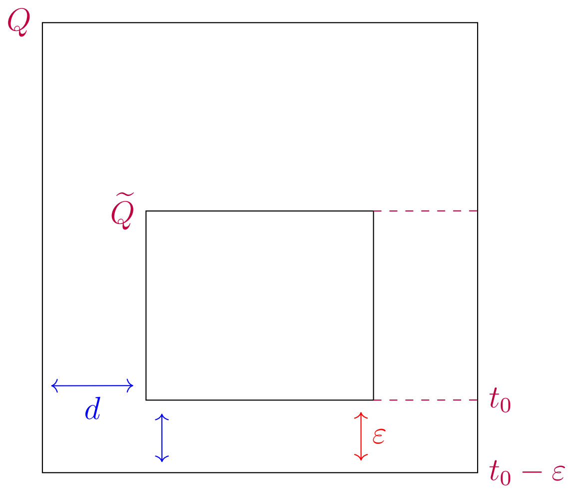


Figure 2.14

$$||f||_{C^{1+\delta/2,2+\delta}_{t,x}(\widetilde{Q})} \leqslant C\left(\max_x |\partial_x f(t=0,\cdot)|, \delta, d\right) ||f||_{L^\infty_{t,x}(Q)}$$

for parabolic cylinders $\widetilde{Q} \Subset Q$, starting from $t = t_0$ and $t = t_0 - \varepsilon$ respectively, as in Figure 2.14. We obtain:

$$||f||_{C^{2,\delta}_x(\widetilde{Q}\cap\{t=t_0\})} \leqslant ||f||_{C^{1+\delta/2,2+\delta}_{t,x}(\widetilde{Q})} \leqslant C\left(\max_x |\partial_x f(t=0,\cdot)|, \delta, d, \varepsilon\right) e^{-2(t_0-\varepsilon)/C}$$

Notice that $d = \operatorname{dist}(\widetilde{Q}, Q)$ already contains the dependence on ε; however, if we rewrite $d = \operatorname{dist}(\widetilde{Q} \cap \{t = t_0\}, Q \cap \{t = t_0\})$, this dependence on ε must appear explicitly.

$\square$

CHAPTER 3

Vortices

3.1 The smooth potential case

3.1.1 Periodic minimisers (1-D case)

We note the usual notation / convention of analysis, of $\mathbb{S}^1$: By writing $\mathbb{S}^1$ we mean either that the function really has domain the circle, or that it is $2\pi-$periodic in $\mathbb{R}$. It is not difficult to see the duality between functions $u : \mathbb{S}^1 \to \mathbb{R}^2$ and $u : \mathbb{R} \to \mathbb{R}^2$ that are $2\pi-$periodic.

Without a doubt, the simplest smooth case is the one of one dimension, $v : \mathbb{S}^1 \to \mathbb{R}^2 = \mathbb{C}$, with zero weight $\int_0^{2\pi} v\, dt = 0$. The reason we study $2\pi-$periodic functions is to overcome the usual trivialities of the calculus of variations. In minimal suface theory, it is absurd to ask for a global minimiser, since then a degenerate case -a singular point- would be the appropriate "minimiser". In our case, a non-degenerate global minimiser does not exist either. Indeed, we cannot minimise the energy over all $\mathbb{R}$, except if $R = 0$ or $|u| = 1$. If $R = 0$, the problem is trivial. We then have to impose a condition on v.

Considering only $2\pi-$periodic functions, we define:

$$\mathscr{E}_R(v) = \mathscr{E}_R(v; \mathbb{S}^1) = \int_0^{2\pi} \frac{1}{2}|v'|^2 + \frac{R^2}{4}(1 - |v|^2)^2\, dt$$

Notice on the right side the Ginzburg-Landau potential $W(v) = \frac{1}{4}(1 - |v|^2)^2$. Since this is a smooth case, it is possible to produce an equation that describes the critical points of $\mathscr{E}_R$, using the usual process with the first variation (as in Section 1.1). We consider $\varphi \in C_c^\infty(\mathbb{S}^1; \mathbb{R}^2)$ and we compute:

$$\begin{aligned}\frac{\delta\mathscr{E}}{\delta\varphi} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{2\pi} \frac{1}{2}\left(|v' + \varepsilon\varphi'|^2 - |v'|\right) + \frac{R^2}{4}\left(\left(1 - (u + \varepsilon\varphi)^2\right)^2 - (1 - u^2)^2\right)\, dt \\ &= \int_0^{2\pi} \left\langle v'' - R^2(|v|^2 - 1)u, \varphi\right\rangle\, dt\end{aligned}$$

and from the criticality condition $\delta\mathscr{E}/\delta\varphi = 0$ follows the 1-dimensional Ginzburg-Landau equation:

$$v'' - R^2(|v|^2 - 1)v = 0 \tag{3.1}$$

Several periodic solutions exists, for example those on the following remark.

Remark 3.1. *For all $n \geqslant 1$, if $R \geqslant n$, then:*

$$u_{R,n} = \sqrt{1 - \frac{n^2}{R^2}}\, e^{\pm int}$$

is a $2\pi/n-$periodic solution of (3.1).

Proof. To find these solutions, we can seach for solutions of the type Ae^{int} and then determine A. Alternatively, we can do directly the computations, since we have already "guessed" the form of the solution.

We have:

$$u''_{R,n} = -n^2\sqrt{1 - \frac{n^2}{R^2}}\, e^{\pm int}$$

and:

$$R^2(|u_{R,n}|^2 - 1)u_{R,n} = R^2\left(-\frac{n^2}{R^2}\right)\sqrt{1 - \frac{n^2}{R^2}}\, e^{\pm int} = -n^2\sqrt{1 - \frac{n^2}{R^2}}\, e^{\pm int}$$

so this remark follows □

These solutions are actually minimisers, as the next remark shows.

Remark 3.2. *For all $n \geqslant 1$, if $R > n$, then $u_{R,n}$ is a minimiser of $\mathscr{E}_R$ in the class of $H^1_{\text{loc}}(\mathbb{S}^1/n;\mathbb{R}^2)$ functions with zero weight, that is $\int_0^{2\pi/n} v\ dt = 0$. The notation $\mathbb{S}^1/n$ means $2\pi/n-$periodic, since with constant speed 1 we trasverse n times the contracted circle.*

Proof. Suppose we have another function v. We consider their difference $f = v - u_{R,n} \in H^1_{\text{loc}}(\mathbb{S}^1/n;\mathbb{R}^2)$ and we see that $\int_0^{2\pi/n} f\ dt = 0$. We have:

$$\mathscr{E}_R(v) - \mathscr{E}_R(u_{R,n}) = \int_0^{2\pi/n} \frac{1}{2}\big(|v'|^2 - |u'_{R,n}|^2\big) + \frac{R^2}{4}\big((1-|v|^2)^2 - (1-|u_{R,n}|^2)^2\big)\ dt$$

and then we must endure some computations. Using $v = f + u_{R,n}$ and an integration by parts:

$$\begin{aligned}
[\ldots] &= \int_0^{2\pi/n} \frac{1}{2}|f'|^2 + n^2 f u_{R,n} + \frac{R^2}{4}(|f|^2 + 2fu_{R,n})^2 - \frac{n^2}{2}(|f|^2 + 2fu_{R_n})\ dt \\
&\geqslant \int_0^{2\pi/n} \frac{1}{2}\big(|f'|^2 - n^2|f|^2\big)\ dt
\end{aligned}$$

Now, if f is represented by its Fourier series $f = \sum_k a_k e^{\pm inkt}$, $f' = \pm\sum_k inka_k e^{\pm inkt}$, then we deduce $|f'| \geqslant n|f|$ and as a consequence:

$$\mathscr{E}_R(v) - \mathscr{E}_R(u_{R,n}) \geqslant 0$$

□

3.1.2 Vortices on the plane (2-D case)

We will now focus our attention on functions $u : \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$. The $2-$dimensional problem comes from minimising the energy:

$$\mathscr{E}_{B_R(0)}(v) = \int_{B_R(0)} \frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1-|v|^2)^2\, dx,\ |\nabla v|^2 = |\nabla \Re v|^2 + |\nabla \Im v|^2$$

under some assumptions. Notice, once again, on the right side the Ginzburg-Landau potential $W(v) = \frac{1}{4}(1-|v|^2)^2$. The reason we restrict ourselves on the disc $B_R(0) \subseteq \mathbb{R}^2$ is as in the $1-$dimensional case. As in (3.1) or the introduction, considering the variation $\delta\mathscr{E}/\delta\varphi$, we can obtain the associated partial differential equation:

$$\Delta u = (|u|^2 - 1)u \tag{3.2}$$

What are the assumptions one utilises in these kind of problems? Surely the easiest is the radial condition $u(re^{iqt}) = \rho(r)e^{iqt}$, since it leads to an ordinary differential equation, and those solutions are basically treated in [14].

Expressing the Laplace operator in polar coordinates:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial t^2}$$

we obtain:

$$\rho''(r)e^{iqt} + \frac{1}{r}\rho'(r)e^{iqt} - \frac{q^2}{r^2}\rho(r)e^{iqt} = (\rho^2(r) - 1)\rho(r)$$

which becomes:

$$r^2\rho''(r) + r\rho'(r) - q^2\rho(r) + r^2\rho(r)(1-\rho^2(r)) = 0 \tag{3.3}$$

What can be proven is the following proposition:

Theorem 3.3. *Suppose ρ is a solution of (3.3). The following hold:*

i. All real solutions ρ in $[0, R]$ are located near 0 and they can be expressed as a series of the form:

$$\rho_a(r) = r^q \left[a + \sum_{k=1}^{\infty} P_k(a) r^{2k} \right]$$

where $a \in \mathbb{R}$ is a real parameter and P_k are odd polynomials satisfying the recursive relation:

$$4k(k+q)P_k = \sum_{\ell+m+n=k-q-1} P_\ell P_m P_n - P_{k-1},\ P_0(a) = a$$

These ρ_a can be extended analytically, so we will denote by ρ_a their respective analytic continuation.

...

ii. *Suppose* $a > 0$. *There exists a constant* $A > 0$ *such that:*

ii.a. *If* $a > A$, *then* ρ_a *is strictly monotone increasing from* 0 *to* ∞, *as* r *goes from* 0 *to some finite radius.*

ii.b. *If* $a = A$, *then* ρ_A *is strictly monotone increasing from* 0 *to* 1, *as* r *goes from* 0 *to* ∞.

ii.c. *If* $a < A$, *then* ρ_a *is oscillates between* ± 1 *indefinitely, on both sides of* 0, *in the interval* $[0, \infty)$.

In fact, Hervé and Hervé in [14] prove some more properties for radial solutions, but they are out of the scope of this presentation. Point ii.b. is important, since it guarantees the existence of **vortices** on the plane, that is radial solutions of (3.2) whose radial part is monotone increasing asymptotically from 0 to 1.

Definition 3.4 (Vortices). *A function* $u(re^{i\theta}) = \rho(r)e^{iq\theta}$ *is called a* ***vortex*** *if it is a radial solution of (3.2) that increases asymptoticaly from* 0 *to* 1, *as* r *goes from* 0 *to* ∞.

We will give some ideas about the proofs in Theorem 3.3 in what follows. Notice that throughout the proofs we often avoid treating symmetric cases. For example, observe that if ρ is a solution of (3.3), then $-\rho$ is too.

Proof. There are a couple of different forms of (3.3) which can be proven to be equivalent. Those are, for example:

$$r\frac{d}{dr}(r\rho) = (q^2 - r^2)\rho + r^2\rho^3 \tag{3.4}$$

$$\frac{d}{dr}\left[\frac{1}{r^{2q-1}}\frac{d}{dr}(r^q\rho)\right] = r^{1-q}\rho(\rho^2 - 1) \tag{3.5}$$

$$\varphi''(t) = \big(q^2 + e^{2t}(\varphi^2(t) - 1)\big)\varphi(t), \quad \text{where } \varphi(t) = \rho(e^t) \tag{3.6}$$

As a remark, (3.6) shows that φ is concave and positive or convex and negative, unless its graph lies in the open set:

$$\Omega = \{(t, \tau) \mid \tau^2 < 1 - q^2 e^{-2t}\}$$

This property is often called logarithmic convexity and set Ω is illustrated above.

Equation (3.5) gives us the system of differential equations:

$$r\rho' + q\rho = r^q g \text{ and } g' = r^{1-q}\rho(\rho^2 - 1) \tag{3.7}$$

which is, in integral form:

$$\rho = \frac{1}{r^q}\int r^{2q-1} g \, dr \text{ and } g = \int r^{1-q}\rho(\rho^2 - 1) \, dr \tag{3.8}$$

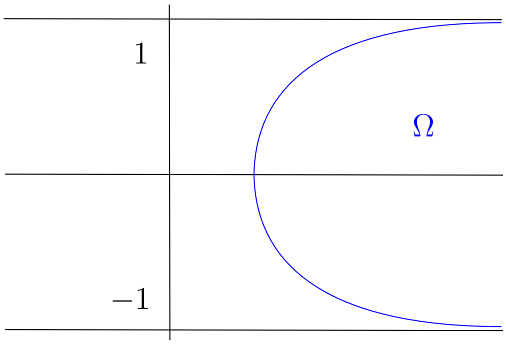


Figure 3.1

Now, if ρ, ϖ are two solutions of (3.3), we have:

$$\frac{d}{dr}\big(r(\rho'\varpi - \rho\varpi')\big) = r\rho\varpi(\rho^2 - \varpi^2) \tag{3.9}$$

which shows that the difference of two positive solutions ρ, ϖ cannot vanish more than once on $[0, \infty)$.

This proof is partitioned in several lemmas, which we will see one-by-one. First, if we set:

$$\rho(r) = \sum_{n=1}^{\infty} a_n r^n$$

then from (3.3) follows that $a_n = 0$ for all $n < q$, as well as for those n which are not of the same parity as q. Value $a = a_q$ is indetermined and also a_{q+2k} is given by $a_{q+2k} = P_k(a)$. Those P_k are polynomials given by the recursive formula:

$$4k(k+q)P_k = \sum_{\ell+m+n=k-q-1} P_\ell P_m P_n - P_{k-1}, \text{ where } P_0(a) = a$$

Our convention is that a sum of negative index is empty. P_k are actually odd and if we choose b and $\lambda > 0$ so that:

$$b^2 = 8\lambda^{q+1} - \frac{2\lambda^2}{q+1} \tag{3.10}$$

we get $|P_k(a)| \leqslant \lambda^k|a|$ for all $|a| \leqslant b$ and for all k. In turn, the series announced in Theorem 3.3 converge, with radius of convergence at least $1/\sqrt{\lambda}$.

The following lemma concerns the extendability of solutions to series of the form of ρ_a.

Lemma 3.5. *Every solution ρ of (3.3), defined on some open interval $(0, R)$, is either tending to $\pm\infty$ as $r \to 0$, or it can be extended to a solution of the form ρ_a.*

Proof. There are three cases we need to consider. First, suppose at some $r_0 = \log t_0$ we have $\rho(e^{t_0}) \geqslant 0$ and $\rho'(e^{t_0}) \leqslant 0$. Then, by logarithmic convexity, ρ tends to infinity as $t \to -\infty$. Similarly, if $\rho'(r_0) > 0$ and $\rho > 1$ for all $r \in (0, r_0]$, then there exists a limit $\rho \to \ell$ as $r \to 0$. By two integrations of (3.4) we obtain:

$$\rho \sim \frac{q^2\ell}{2}\log^2 r$$

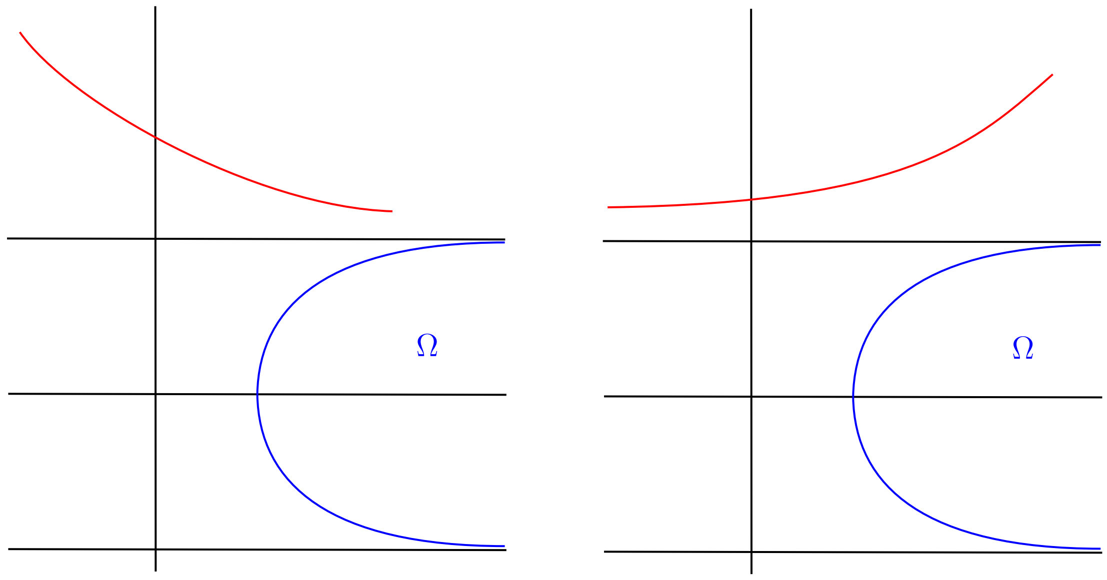


Figure 3.2

It remains to see what happens if $\rho : (0, R) \to (-1, 1)$. If g is as in (3.8) (notice that $r^q \rho \to 0$ as $r \to 0$), then using the formulas of (3.8) successively and in alteration we obtain:

$$g = O(r^{1-q}),\ \rho = O(r),\ g = O(r^{2-q}),\ \rho = O(r^2),\ \cdots, g = O(1),\ \rho = O(r^q)$$

Now, in (3.9) we can choose $\varpi = \rho_1$. Quantity $\rho'\rho_1 - \rho\rho_1'$ tends to 0 and the right hand side of (3.9) is $O(r^{4q+1})$, so that quantity in the start is $O(r^{4q+1})$ and, as $r \to 0$:

$$\frac{d}{dr}\frac{\rho}{\rho_1} \to 0$$

Then ρ/ρ_1 has a limit, say a, which is also the limit of ρ/r^q. If we choose $b > |a|$ in (3.10) and if $r_0 < 1/\sqrt{\lambda}$, then $\rho_{a'} \to \rho_a$ uniformly as $a' \to a$, in $[0, r_0]$. Since ρ lies between $\rho_{a-\varepsilon}$ and $\rho_{a+\varepsilon}$ for each $\varepsilon > 0$, we conclude $\rho = \rho_a$. $\square$

The second lemma examines how ρ increases, if there exists a point r_0 such that $\rho(r_0) \geqslant 1$.

Lemma 3.6. *If a solution of (3.3) has the property that there exists some $r_0 \in (0, \infty)$ such that $\rho(r_0) \geqslant 1$ and $\rho'(r_0) > 0$, then one $R \in (r_0, \infty)$ can be found such that ρ is strictly increasing from $\rho(r_0)$ to ∞ as r increases from r_0 to R.*

Proof. This proof is quite technical. The summary is as follows:

Function φ in (3.6) is strictly increasing after $t_0 = \log r_0$, and in fact it is increasing quite fast (as $\varphi, \varphi', \varphi''$ are all positive). For technical reasons, choose t_1 large enough such that $\varphi(t_1) \geqslant \sqrt{2}$ and:

$$R \geqslant e^{t_1}\big(1 + \varphi(t_1)/\varphi'(t_1)\big),\ (R - e^{t_1})^2 \geqslant R + 1$$

In the end, choose $R_0 \in [\sqrt{2(R+1)}, \sqrt{2}(R - e^{t_1})]$. We consider φ as a solution of:

$$\varphi''(t) = \alpha(t)\varphi^3(t), \quad \text{where } \alpha(t) = q^2\varphi^{-2}(t) + e^{2t}\big(1 - \varphi^{-2}(t)\big)$$

and we will compare it to some $\psi(t) = R_0/(R - e^t)$, which is the solution of:

$$\psi''(t) = \beta(t)\psi^3(t),\ \text{where } \beta(t) = \frac{1}{R_0^2}(Re^t + e^{2t})$$

(here $\beta(t) \leqslant e^{2t}/2 < \alpha(t)$). From the initial conditions:

$$\psi(t_1) \leqslant \varphi(t_1) \text{ and } \frac{\psi'(t_1)}{\psi(t_1)} \leqslant \frac{\varphi'(t_1)}{\varphi(t_1)}$$

we obtain $\psi < \varphi$ for $t > t_1$ and $e^t < R$. Indeed, if the analytic $\varphi - \psi$ remained positive on (t_1, t_2), yet vanished at t_2, then:

$$\frac{d}{dt}(\psi\varphi' - \varphi\psi') = \varphi\psi(\alpha\varphi^2 - \beta\psi^2)$$

would force φ/ψ to be strictly increasing on (t_1, t_2), starting from something greater or equal to 1 and arriving at 1. □

The third lemma concerns the oscillating solutions ρ_a.

Lemma 3.7. *Let ρ be a solution of (3.3) on $[r_0, r_1]$, which is negative on (r_0, r_1) but vanishes on either end. Then, $\rho > -1$ and there exists some other $r_2 \in (r_1, r_1^2/r_0)$ such that $\rho(r_2) = 0$. Moreover, $\rho > 0$ on (r_1, r_2) and in fact:*

$$\sup_{(r_1,r_2)} |f| < \sup_{(r_0,r_1)} |f|$$

Proof. If we had $\rho(r_2)$ for some $r_2 \in (r_0, r_1)$, then by Lemma 3.6 we would have that $\rho(r_1)$ cannot vanish (notice here that if $\rho(r_2) = -1$ and $\rho'(r_2) < 0$, by a reflection we get $-\rho(r_2) = 1$ and $-\rho'(r_2) > 0$). Therefore, $\rho > -1$.

Now set $t_0 = \log r_0$, $t_1 = \log r_1$ and consider φ as in (3.6). We reflect the graph of φ around $(t_1, 0)$ and we get some function ψ such that $0 < \psi < 1$ on $(t_1, 2t_1 - t_0)$, vanishing on either end, and satisfying the ordinary differential equation:

$$\psi'' = q^2\psi + e^{4t_1 - 2t}(\psi^3 - \psi)$$

If we differentiate the latter as well as (3.6), we observe that ψ and φ have the same derivatives at t_1, up to the third order. However:

$$\varphi^{[4]}(t_1) < \psi^{[4]}(t_1)$$

and then for $t > t_1$ sufficiently close to t_1, $0 < \varphi < \psi$. This holds, at first sight, only near t_1, so consider $t_2 \in (t_1, 2t_1 - t_0)$ the first time which the strict inequality fails. We would get:

$$\frac{d}{dt}(\psi\varphi' - \varphi\psi') = \varphi\psi\left[e^{4t_1 - 2t}(1 - \psi^2) - e^{2t}(1 - \varphi^2)\right]$$

with the term inside the brackets being negative, and φ/ψ to be strictly decreasing. But then $\varphi/\psi = 1$ on either end, which is contradictory. □

The fourth lemma refines Lemma 3.6 for $a > 0$. In its proof we will find strong indications about the form of the solutions, as presented in the main Theorem 3.3.

Lemma 3.8. *Let $a > 0$. If there exists some $r_0 \in (0, \infty)$ such that $\rho_a(r_0) = 1$, then there exists also some $R \in (r_0, \infty)$ such that ρ_a is strictly increasing from 0 to ∞, as r goes from 0 to R.*

Proof. Function $\varphi_a(t) = \rho_a(e^t)$ in (3.6) is strictly convex and increasing, as long as its graph never crosses the boundary of Ω. So, if φ_a does not cross the boundary of Ω, Lemma 3.6 is in effect and we can prove the assertion.

If φ_a does cross the boundary of Ω, then it becomes concave and can either remain strictly increasing or pass through a maximum. In the first case, it re-exits and, like before, the lemma is proved. In the second case, it decreases and vanishes, so it has a sequence of zeros, predicted by Lemma 3.7. Therefore, φ_a stays entirely inside Ω and it must be strictly between -1 and 1, thus never obtaining value 1. $\square$

These four lemmas are the basic ingredients for the proof (plus an existence result, based on Bessel's functions, which we will see later). What we have seen so far indicates three kind of possible solutions, those strictly monotone increasing from 0 to some finite radius, the radial parts of vortices and the oscillating solutions.

Case $a > A$**:** If $0 < \rho_a < 1$ and g_a is as in (3.7), then g_a decreases strictly from $2qa$ and the integrals of (3.8) become $r^q\rho_a$ and $g_a - 2qa$, and:

$$g_a < 2qa,\ \rho_a < ar^q,\ g_a > 2qa - ar^2/2,\ \rho_a > a\left[r^q - \frac{r^{q+2}}{4(q+1)}\right]$$

The maximum of the term inside the bracket is:

$$M = \frac{2^{q+1}}{q+2}\left(\frac{q(q+1)}{q+2}\right)^{q/2}$$

Now, if aM is greater or equal to 1, Lemma 3.8 applies. If ρ_a reaches 1 at some point, Lemma 3.8 shows that this point is unique, so $r = \rho_a^{-1}(1)$ is well defined. The same holds for $a' > a$ and in fact:

$$\rho_{a'}^{-1}(1) < \rho_a^{-1}(1),\ \text{since } a' > a \Rightarrow \rho_{a'} > \rho_a$$

Moreover, for any a' close to a, function $\rho_{a'}$ also obtains the value 1 exactly once and $\rho_{a'}^{-1}(1) \to \rho_a^{-1}(1)$ as $a' \to a$. Indeed, we have:

$$\rho_{a'} \to \rho_a,\ \rho'_{a'} \to \rho'_a$$

uniformly on $[0, r_0]$, an interval chosen at the end of Lemma 3.5. By using the continuous dependence of solutions on their data, if ρ_a extends to $[0, r_1]$, then for a' nearby, $\rho_{a'}$ also extends and $\rho_{a'} \to \rho_a$ uniformly on $[0, r_1]$. Taking:

$$r_1 = \rho_a^{-1}(1+\varepsilon),\ r_2 = \rho_a^{-1}(1-\varepsilon),\ 0 < \varepsilon < 1$$

we obtain $|\rho_{a'} - \rho_a| < \varepsilon$ on $[0, r_1]$, which forces $\rho_{a'}^{-1}(1)$ to tend to $\rho_a^{-1}(1)$, as the first is in between r_1, r_2.

Those values for $a > 0$ which ρ_a intersects 1 form an interval (A, ∞), where $A \geqslant 0$. However, we will need to prove $A > 0$, because the other remaining cases need to lie in $[0, A]$. This is the content of the next lemma, with which we basically show that near 0 oscillating solutions exist. More details on oscillating solutions later.

Lemma 3.9. *The following hold:*

i. *Given $r_1 \in (0, \infty)$, there exists $b \in (0, \infty)$ such that $\{\rho_a/a\}_{|a| \leqslant b}$ is a (of course well defined) bounded family, which is also equicontinuous on $[0, r_1]$.*

ii. *When $a \to 0$, those ρ_a/a tend uniformly towards the Bessel function J_q, on any compact subset of $[0,\infty)$. Bessel's function has an infinite number of zeros which are positive.*

$$J_q = q! r^q \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{4^k k!(q+k)!}$$

Proof. For i.: Suppose we have some b and $\lambda < 1$, as in (3.10). Then for $|a| < b$:

$$\left|\frac{\rho_a}{a}\right|, \left|\frac{\rho_a'}{a}\right| \leqslant \beta = \sum_{k=0}^{\infty}(q+2k)\lambda^k$$

(also, $|\varphi_a(0)|, |\varphi_a'(0)| \leqslant |a|\beta$). If $r_1 \leqslant 1$, this concludes the proof. If $r_1 > 1$ and ρ_a is defined on $[0, r_1]$, then write $t = \log r$ and consider φ as a solution of some:

$$\varphi''(t) = c(t)\varphi(t)$$

Choose $\gamma \geqslant 1$ so that $\gamma \geqslant |c|$, and then a standard argument (Grönwall) shows that the initial bounds imply:

$$|\varphi_a(t)|,\ |\varphi_a'(t)| \leqslant |a|\beta e^{\gamma t},\ \text{where } 0 \leqslant t \leqslant t_1$$

If we take $\gamma = q^2 + e^{2t_1}$ and if $|a| \leqslant b$, $|a|\beta e^{\gamma t_1} < 1$, then φ_a remains bounded by $|a|\beta e^{\gamma t_1} < 1$.

For ii.: All $P_k(a)/a$ tend towards $P_k'(0)$ as $a \to 0$. By differentiating the recursive formula of P_k, we obtain:

$$4k(k+q)P_k'(0) = -P_{k-1}'(0)$$

But those $P_k'(0)$ are exactly the $(q+2k)-$coefficients in the Taylor-Maclaurin series of the Bessel function J_q. If b and λ are as in i., then $|P_k(a)/a| \leqslant \lambda^k$ for all k and $|a| \leqslant b$, so it follows that:

$$\frac{\rho_a}{a} \to J_q \text{ and } \frac{\rho_a'}{a} \to J_q'$$

as $a \to 0$, uniformly in $[0,1]$. For a compact interval $[1, r_1]$, a uniform bound on ρ_a/a and ρ_a'/a, together with (3.3), give a uniform bound for ρ_a''/a. By the Arzelà-Ascoli theorem, there exists a sequence $(a_n)_{n=1}^{\infty}$, $a_n \to 0$, such that ρ_{a_n}/a_n, ρ_{a_n}'/a_n, ρ_{a_n}''/a_n converge uniformly. The limit, say ϖ, of ρ_{a_n}/a_n satisfies:

$$r^2\varpi''(r) + r\varpi'(r) + (r^2 - q^2)\varpi(r) = 0$$

so it is the Bessel function J_q. □

Case $a = A$: The image so far is that of solutions that intersect 1 at some point, which comes closer to 0 as $a \in (A, \infty)$ increases. If we consider those points $\rho_a^{-1}(1)$, we expect them to approach ∞ as $a \to A$, since the limit case ρ_A has no intersection with 1 (the limit $a \to \infty$ is 0, and this is not difficult to see). Set:

$$S = \sup_{a > A} \rho_a^{-1}(1)$$

and suppose $S < \infty$. Function ρ_A cannot take as a value 1 and cannot vanish either, since then ρ_a, $a > A$, would tend uniformly to ρ_A, as $a \to A$, on a set $[0, r_1]$ (r_1 is as before Lemma 3.9). As in the proof of Lemma 3.8, we can see that ρ_A lies between -1, 1, and tends towards some $\ell \in (0,1]$. Now, if $\ell \neq 1$, by (3.6) we get the asymptotic behaviour $\varphi_A'' \sim \ell(\ell^2 - 1)e^{2t}$, which is contradictory. We conclude that ρ_A is strictly increasing, from 0 to 1 and is defined in the whole $[0, \infty)$. If now a is sufficiently close to A, we have already seen the procedure with which ρ_a can be extended, say to $[0, S+1]$. But then ρ_a is less than 1 in this interval (decrease a even more, if needed), which is absurd, by the definition of S. We conclude that $S = \infty$ and ρ_A is strictly increasing, from 0 to 1, as r goes from 0 to ∞.

Case $0 < a < A$: In this case ρ_a cannot be ascending and having the same limits as ρ_A, since then we would have $\rho_a < \rho_A$ and (3.9) would prove ρ_a/ρ_A to be strictly decreasing, from a/A to 1. Therefore, ρ_a vanishes, and we must show that it does so more than once, since then Lemma 3.7 provides an infinite ammount of zeros. Let $r_1 > 0$ be the unique zero of ρ_a, so that $\rho_a < 0$ in (r_1, ∞). In the proof of Lemma 3.7 replace φ with φ_a, t_0 with $-\infty$ and ψ with $\varphi_a(2t_1 - t)$. Then, for $t > t_1$ we obtain:

$$0 < -\varphi_a(t) < \varphi_a(2t_1 - t)$$

with the latter tending to 0 as $t \to \infty$. Therefore, for t large enough, φ_a lies inside Ω, and $\varphi_a < 0$ would force φ_a to be convex. This is a contradiction.

This proof is now complete. □

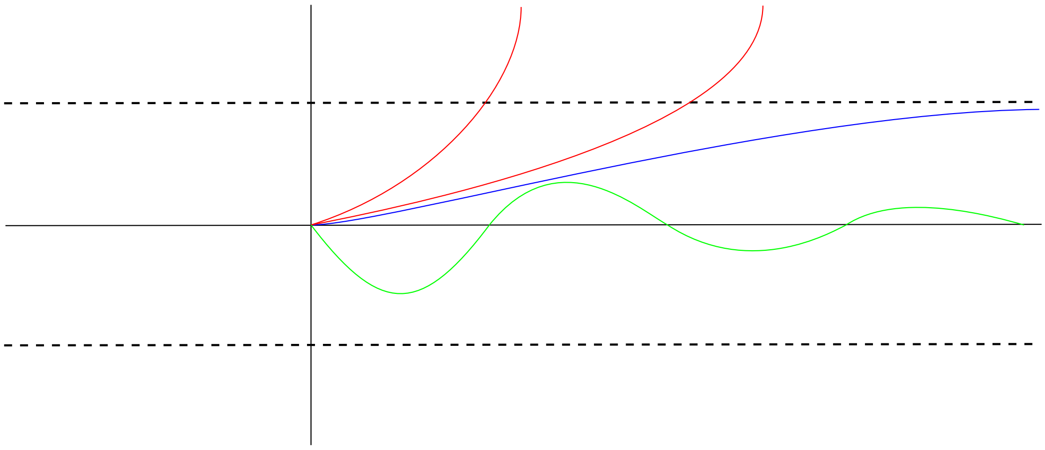


Figure 3.3: Possible profiles of the solutions of (3.3).

3.2 The non-smooth potential case

In what follows we will restrict ourselves to the case of the non-smooth potential $W : \mathbb{R}^2 \to \mathbb{R}$:

$$W(u) = \begin{cases} 0, & \text{if } |u| = 1 \\ 1, & \text{if } |u| \neq 1 \end{cases}$$

This choice has the advantage of being an explicit potential, while also exaggerating the maximum and the minimum of the usual Ginzburg-Landau potential.

What we want to examine is the minimisers of the energy functional:

$$\mathscr{E}_R(v) = \mathscr{E}_R(v; \mathbb{S}^1) = \int_0^{2\pi} \frac{1}{2}|v'|^2 + R^2 \cdot W(v)\, dt,\ \ u : \mathbb{S}^1 \to \mathbb{R}^2$$

and in particular we are interested in proving the existence of them under symmetry hypothesis, as well as finding a closed form.

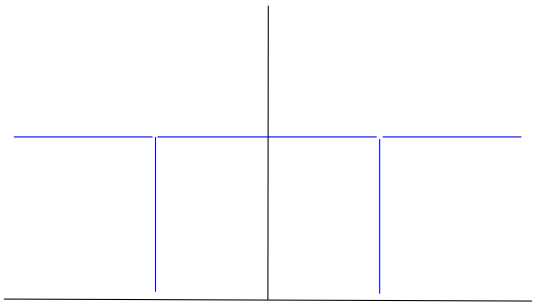


Figure 3.4

While having some advantages, potential W is discontinuous. So, it is justified to work in such a way as to avoid the discontinuity. By considering only circular trajectories, we have the following:

Remark 3.10. *Suppose $R \geqslant 0$ is given. The circular trajectories $u(t) = a_R e^{it}$ minimise the energy $\mathscr{E}_R$ in the class:*

$$\mathscr{A}_{\text{circ}} = \{ae^{it} \mid a \geqslant 0\}$$

if:

$$a_R = \begin{cases} a_R = 0, & R \leqslant 1/\sqrt{2} \\ a_R = 1, & R > 1/\sqrt{2} \end{cases}$$

Proof. If $v(t) = ae^{it}$, we have:

$$2\pi \cdot \mathscr{E}_R(v) = \frac{1}{2}a^2 + R^2 \mathbb{1}_{\{a \neq 1\}} = \begin{cases} 2\pi \cdot \frac{1}{2}, & a = 1 \\ 2\pi \cdot \left(\frac{1}{2}a^2 + R^2\right), & a \neq 1 \end{cases}$$

By $1/2 < a^2/2 + R^2$ if and only if $R > 1/\sqrt{2}$, we get that $\mathscr{E}_R(v)$ is minimised in the $R > 1/\sqrt{2}$ case if we set $a_R = 1$. If $R \leqslant 1/\sqrt{2}$, it is immediate that the minimisation occurs when $a_R = 0$. □

This remark is the starting point of finding minimisers for the more general case $u : \mathbb{S}^1 \to \mathbb{R}^2$, under some symmetry hypotheses.

3.2.1 Minimisers under symmetry hypotheses (1-D case)

The symmetry we impose to the problem is the following: Symmetric points on $\mathbb{S}^1$, under any axis x or y, map to symmetric values under u. This is known as **equivariance** for the group of $\{x, y\}-$reflections. In general, we state the following definition:

Definition 3.11 (Equivariance)**.** *Let G be a group acting on sets X and Y. We say a function $u : X \to Y$ is **equivariant** with respect to G if it respects the action, that is:*

$$u(g \cdot x) = g \cdot u(x)$$

In particular, if G is the group of reflections on x and y axes, then u maps symmetric points to symmetric points.

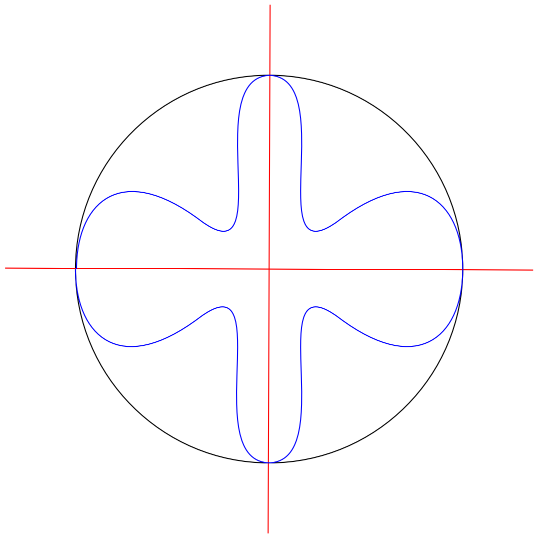


Figure 3.5: An example of a $G-$invariant u for the group of $\{x, y\}-$reflections.

This restriction forces each candidate u to repeat itself in each $\pi/2-$time interval. For this reason, sometimes we compute the energy of u not in the whole 2π interval but rather only in $(0, \pi/2)$. Then, we multiply by 4.

For the following, since equivariance with respect to $\{x, y\}-$reflections is the only form of equivariance we are interested in, we will not specify the group each time. Equivariance gives us the following remark about the trajectory u:

Remark 3.12. *If $u : \mathbb{S}^1 \to \mathbb{R}^2$ is equivariant and $L^1(\mathbb{S}^1; \mathbb{R}^2)$, then u_1 is even, u_2 is odd and:*

$$\int_0^{2\pi} u(t)\, dt = 0$$

As a consequence, $u_1(\pm\pi/2) = 0$, $u_2(0) = 0$, $u_2(\pi) = 0$.

Proof. From equivariance follows:

$$u_1(t) = u_1(-t),\ u_1(t) = -u_1(\pi - t)$$

and:

$$u_2(t) = -u_2(-t),\ u_2(t) = u_2(\pi - t)$$

(in particular $u_1(\pm\pi/2) = 0$, $u_2(0) = 0$, $u_2(\pi) = 0$), so $u(t) = -u(-t)$. Integrating gives the desired result. □

A key ingredient for the proof of the existence of minimisers, under this equivariance hypothesis, is the projection to the circle. This way we will restrict the behaviour of the minimisers, by showing that they must lie inside the closed disc $\overline{B}_1(0)$.

Lemma 3.13 (Projection on the disc). *Let $u : \mathbb{S}^1 \to \mathbb{R}^2$ be an $H^1_{\text{loc}}(\mathbb{S}^1; \mathbb{R}^2)$ function. Then the projection on the closed ball $\overline{B}_1(0)$:*

$$Pu = \begin{cases} u/|u|, & \textit{if } |u| \geqslant 1 \\ u, & \textit{if } |u| < 1 \end{cases}$$

is $1 -$ Lipschitz:

$$|Pu - Pv| \leqslant |u - v|$$

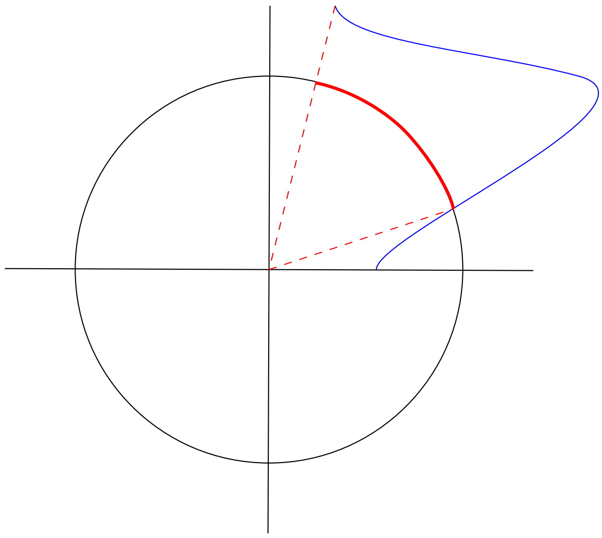


Figure 3.6

Moreover, by composition $Pu \in H^1_{\mathrm{loc}}(\mathbb{S}^1;\mathbb{R}^2)$ *and:*

$$|(Pu)'| \leqslant |u'|$$

Therefore, any function u *escaping the disc* $\overline{B}_1(0)$ *for positive measure time cannot be a minimiser for* $\mathscr{E}_R$*.*

Proof. Let $u, v \in H^1_{\mathrm{loc}}(\mathbb{S}^1;\mathbb{R}^2)$. We set:

$$M = \begin{cases} \max\left\{\frac{1}{|u|}, \frac{1}{|v|}\right\}, & \text{if } |u|, |v| \geqslant 1 \\ 1, & \text{otherwise} \end{cases}$$

and we notice that $M \leqslant 1$ and:

$$|Pu - Pv| \leqslant M|u-v| \leqslant |u-v|$$

which shows that P is $1-$Lipschitz.

Because P is Lipschitz, the composition Pu is $H^1_{\mathrm{loc}}(\mathbb{S}^1;\mathbb{R}^2)$ (cf. Theorem A.2). Now we calculate:

$$\begin{aligned}(Pu)'(t) &\overset{\star}{=} \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} (Pu)'(s)\, ds \\ &= \lim_{h\to 0} \frac{Pu(t+h) - Pu(t)}{h}\end{aligned}$$

where $(\star)$ is justified by the Lebesgue differentiation theorem. Therefore:

$$\begin{aligned}|(Pu)'(t)| &= \lim_{h\to 0}\left|\frac{Pu(t+h) - Pu(t)}{h}\right| \\ &\overset{\star\star}{\leqslant} \lim_{h\to 0}\left|\frac{u(t+h) - u(t)}{h}\right|\end{aligned}$$

where in $(\star\star)$ the Lipschitz condition of P is used. We arrive at the desired estimate for the derivative of the projection.

$$|(Pu)'| \leqslant |u'|$$

□

Theorem 3.14 (Existence of $H^1_{\mathrm{loc}}(\mathbb{S}^1;\mathbb{R}^2)$ equivariant minimisers for the non-smooth potential case)**.** *There exists a minimiser of:*

$$\mathscr{E}_R(v) = \int_0^{2\pi} \frac{1}{2}|v'|^2 + R^2 \cdot W(v)\; dt$$

in the class:

$$\mathscr{A} = \left\{ v \in H^1_{\mathrm{loc}}(\mathbb{S}^1;\mathbb{R}^2) \;\middle|\; v \text{ is equivariant} \right\}$$

Proof. The proof is an application of the direct method.

Step I: Notice that:

$$0 \leqslant \inf_{\mathscr{A}} \mathscr{E}_R(v) < \infty$$

because $v(t) = e^{it}$ has finite energy:

$$\mathscr{E}_R(e^{it}) = \int_0^{2\pi} \frac{1}{2}\; dt = \pi < \infty$$

Step II: From Step I, since we have established $\inf_{\mathscr{A}} \mathscr{E}_R(v) < \infty$, we consider a minimising sequence, that is, a sequence $\{v_k\}_{k=1}^{\infty} \subseteq \mathscr{A}$ such that:

$$\mathscr{E}_R(v_k) \xrightarrow{k\to\infty} \inf_{\mathscr{A}} \mathscr{E}_R(v)$$

We can suppose that $|v_k| \leqslant 1$, since from Lemma 3.13 escaping the disc $\overline{B}_1(0)$ only increases the energy. In other words, Pv_k is another minimising sequence for $\mathscr{E}_R$, because:

$$\inf_{\mathscr{A}} \mathscr{E}_R(v) \leqslant \mathscr{E}_R(Pv_k) \leqslant \mathscr{E}_R(v_k) \xrightarrow{k\to\infty} \inf_{\mathscr{A}} \mathscr{E}_R(v)$$

Thus, this way we have:

$$\sup_{k\in\mathbb{N}} \int_0^{2\pi} |v_k|^2\; dt \leqslant 2\pi$$

and also, for k_0 sufficienty large, since $\{v_k\}_{k=1}^{\infty}$ is a minimising sequence:

$$\sup_{k\geqslant k_0} \int_0^{2\pi} |v_\kappa'|^2\; dt \leqslant \mathscr{E}_R(e^{it}) = \pi$$

Therefore, we can suppose there exists a constant $M > 0$ such that:

$$\sup_{k\in\mathbb{N}} \int_0^{2\pi} |v_\kappa'|^2\; dt \leqslant M$$

and, in turn, $||v_k||_{H^1(\mathbb{S}^1;\mathbb{R}^2)}$ is uniformly bounded for all k.

Step III: Since $||v_k||_{H^1(\mathbb{S}^1;\mathbb{R}^2)}$ is uniformly bounded for all k, there exists a subsequence, still denoted by $\{v_k\}_{k=1}^{\infty}$, such that:

$$v_k \xrightharpoonup[H^1(\mathbb{S}^1;\mathbb{R}^2)]{w} u$$

for some $u \in H^1(\mathbb{S}^1;\mathbb{R}^2)$. By the lower semi-continuity of $\int_0^{2\pi} |\Diamond|^2\, dt$:

$$\int_0^{2\pi} |u'|^2\, dt \leqslant \liminf_{k\to\infty} \int_0^{2\pi} |v_k'|^2\, dt$$

because weak H^1-convergence implies $v_k' \overset{w}{\rightharpoonup} u'$ in $L^2(\mathbb{S}^1;\mathbb{R}^2)$. Semi-continuity is a fundamental idea in the calculus of variations and we study it a little bit in Definition 4.8, in Chapter 4, which is clearly more geometric. Moreover, H^1-convergence implies $v_k \overset{w}{\rightharpoonup} u$ in $L^2(\mathbb{S}^1;\mathbb{R}^2)$, and for a subsequence still denoted by $\{v_k\}_{k=1}^{\infty}$, we have strong convergence $v_k \to u$ in $L^2(\mathbb{S}^1;\mathbb{R}^2)$. The above are a consequence of the compact embedding:

$$H^1(\mathbb{S}^1;\mathbb{R}^2) = W^{1,2}(\mathbb{S}^1;\mathbb{R}^2) \hookrightarrow\hookrightarrow L^2(\mathbb{S}^1;\mathbb{R}^2)$$

(cf. Theorem A.4).

Potential W is also lower semi-continuous, so:

$$\int_0^{2\pi} W(u)\, dt \overset{\star}{\leqslant} \int_0^{2\pi} \liminf_{k\to\infty} W(v_k)\, dt \overset{\star\star}{\leqslant} \liminf_{k\to\infty} \int_0^{2\pi} W(v_k)\, dt$$

In $(\star)$ we use the lower semicontinuity of W and in $(\star\star)$ Fatou's lemma. All of Step III can now be summarised by the fact:

$$\mathscr{E}_R(u) = \int_0^{2\pi} \frac{1}{2}|u'|^2 + R^2 \cdot W(u)\, dt \leqslant \liminf_{k\to\infty} \int_0^{2\pi} \frac{1}{2}|v_k'|^2 + R^2 \cdot W(v_k)\, dt = \liminf_{k\to\infty} \mathscr{E}_R(v_k)$$

Therefore, u is a minimiser of $\mathscr{E}_R$.

Step IV: What remains is to show that u is indeed in the class $\mathscr{A}$. Class $\mathscr{A}$ is strongly closed and convex, since:

$$\forall v_1, v_2 \in \mathscr{A},\ \ \lambda \in \mathbb{R},\ \text{ we have } v_1 + \lambda v_2 \in \mathscr{A}$$

($\mathscr{A}$ is a subspace of $H^1(\mathbb{S}^1;\mathbb{R}^2)$), so Theorem A.10 applies. Therefore $\mathscr{A}$ is weakly closed and if $v_k \overset{w}{\rightharpoonup} u$, then $u \in \mathscr{A}$. □

Theorem 3.14 guarantees the existence of an equivariant minimiser, but it does not give any idea about the closed form of it. In the following we examine how a minimiser of this kind can be.

Lemma 3.15 (Harmonic condition inside quadrants). *Any $H^1(\mathbb{S}^1;\mathbb{R}^2)$ and equivariant minimiser of $\mathscr{E}_R$ is contained inside $\overline{B}_1(0)$ (by the projection of Lemma 3.13) and has a continuous representative. Inside $B_1(0)$, it is a (possibly countably infinite) union of line segments in each quadrant, or it is constant.*

Proof. Suppose that in the interval (t_0, t_1) minimiser u is inside $B_1(0)$ and in fact inside some quadrant. We deliberately avoid the axes (that is, we avoid going past a quadrant), since there is a possibility that there appear some regularity issues. Later, in Proposition 3.16, we will treat those cases too.

We consider a test function $\varphi \in C_c^1\big((t_0,t_1); B_1(0)\big)$ and $\varepsilon > 0$ small enough so that $|u + \varepsilon\varphi| < 1$. Because of minimality:

$$\mathscr{E}_R(u) \leqslant \mathscr{E}_R(u+\varepsilon\varphi) \Rightarrow \int_0^{2\pi} \frac{1}{2}|u'|^2 + R^2 \cdot W(u)\, dt \leqslant \int_0^{2\pi} \frac{1}{2}|u' + \varepsilon\varphi'|^2 + R^2 \cdot W(u+\varepsilon\varphi)\, dt$$

thus:

$$\int_0^{2\pi} \frac{1}{2}|u'|^2 \, dt \leqslant \int_0^{2\pi} \frac{1}{2}|u' + \varepsilon\varphi'|^2 \, dt$$

and by restricting ourselves on the integral (t_0, t_1) (outside of it $\varphi = 0$):

$$\int_{t_0}^{t_1} \frac{1}{2}|u'|^2 \, dt \leqslant \int_{t_0}^{t_1} \frac{1}{2}|u' + \varepsilon\varphi'|^2 \, dt \Rightarrow \int_{t_0}^{t_1} \varepsilon u'\varphi' + \frac{\varepsilon^2}{2}|\varphi'|^2 \, dt \geqslant 0$$

We divide by ε and we let $\varepsilon \to 0$.

$$\int_{t_0}^{t_1} u'\varphi' + \frac{\varepsilon}{2}|\varphi'|^2 \, dt \geqslant 0 \Rightarrow \int_{t_0}^{t_1} u'\varphi' \, dt \geqslant 0, \ \forall \varphi \in C_c^1\big((t_0, t_1); \mathbb{R}^2\big)$$

By integration by parts:

$$-\int_{t_0}^{t_1} u''\varphi \, dt \geqslant 0, \ \forall \varphi \in C_c^1\big((t_0, t_1); \mathbb{R}^2\big)$$

so we set $\varphi \rightsquigarrow -\varphi$ and it follows that:

$$\int_{t_0}^{t_1} u''\varphi \, dt = 0, \ \forall \varphi \in C_c^1\big((t_0, t_1); \mathbb{R}^2\big) \Rightarrow u'' = 0 \text{ weakly in } (t_0, t_1)$$

This shows that in (t_0, t_1) function u is a line segment or constant. □

Proposition 3.16 (Harmonic condition). *Let u be an $H^1(\mathbb{S}^1; \mathbb{R}^2)$ and equivariant minimiser of $\mathscr{E}_R$. Then u is harmonic under general pertubations.*

Proof. Using the principle of symmetric criticality: We can give a proof that utilises a general theorem, the principle of symmetric criticality. This on its own is quite interesting and we dedicate section B.2 for it.

Let $G < \mathrm{O}(2) < \mathrm{Isom}(2)$ be the group of $\{x, y\}-$reflections on the plane. If u is equivariant, then for every $g \in G$ we get:

$$\nabla u(gx) = \nabla u(x) \text{ and } W\big(u(gx)\big) = W\big(u(x)\big)$$

so the energy density (Lagrangian) $\frac{1}{2}|\nabla u|^2 + W(u)$ is $G-$invariant. The principle of symmetric criticality loosely states that whenever a function is $G-$invariant, where G is a subgroup of the group of isometries Isom, then the critical points under $G-$invariant pertubations that are $G-$invariant, are critical points under general pertubations. So, in our case, our equivariant minimiser u must be a minimiser under general pertubations, and not only pertubations that preserve equivariance.

Suppose $\varphi \in C_c^\infty\big((0, 2\pi); B_1(0)\big)$ and $\varepsilon > 0$ small enough such that $|u + \varepsilon\varphi| < 1$. We compute the first variation and we get, as in Lemma 3.15:

$$-\int_0^{2\pi} u'' \cdot \varphi \, dx = 0, \ \forall \varphi \in C_c^\infty\big((0, 2\pi); B_1(0)\big) \Rightarrow u'' = 0$$

There is another elementary way to prove this proposition, which avoids the criticality principle. Suppose $\varphi \in C_c^\infty\big((0,2\pi); B_1(0)\big)$ is a general pertubation. We can write $\varphi = \varphi_o + \varphi_e$, where:

$$\varphi_o(x) = \frac{\varphi(x) - \varphi(-x)}{2} \text{ is odd}$$

and:

$$\varphi_e(x) = \frac{\varphi(x) + \varphi(-x)}{2} \text{ is even}$$

Therefore, we have:

$$\int_0^{2\pi} u_1''(\varphi_o)_1 \, dx = -\int_0^{2\pi} u_1'(\varphi_o)_1' \, dx = 0$$

because u_1' is odd and $(\varphi_o)'$ is even (the product is odd). Similarly:

$$\int_0^{2\pi} u_2''(\varphi_o)_2 \, dx = -\int_0^{2\pi} u_2'(\varphi_o)_2' \, dx = 0$$

because u_1' even and $(\varphi_o)'$ is odd (the product is odd).

Now, if u is a minimiser in the equivariance class, then for every equivariant pertubation ψ (that is ψ_1 is odd and ψ_2 even) we have:

$$0 = \int_0^{2\pi} u_1''\psi_1 + u_2''\psi_2 \, dx = \int_0^{2\pi} -u_1'\psi_1' + u_2''\psi_2 \, dx$$

So if we set $\psi_1 = (\varphi_o)_1$, $\psi_2 = (\varphi_e)_1$, we have:

$$0 = \int_0^{2\pi} u_1'(\varphi_o)_1' \, dx = \int_0^{2\pi} u_1''(\varphi_e)_1 \, dx$$

and similarly, if $\psi_1 = (\varphi_o)_2$, $\psi_2 = (\varphi_e)_2$:

$$0 = \int_0^{2\pi} u_2'(\varphi_o)_2' \, dx = \int_0^{2\pi} u_2''(\varphi_e)_2 \, dx$$

Gathering:

$$\int_0^{2\pi} u_1''(\varphi_o)_1 \, dx = \int_0^{2\pi} u_1''(\varphi_e)_1 \, dx = \int_0^{2\pi} u_2''(\varphi_o)_2 \, dx = \int_0^{2\pi} u_2''(\varphi_e)_2 \, dx = 0$$

we obtain:

$$\int_0^{2\pi} u \cdot \varphi \, dx = \int_0^{2\pi} u \cdot (\varphi_o + \varphi_e) \, dx = 0$$

□

In the following proposition we follow the variation of domain techique, as in [26], Lemma 4.1, to establish a sort-of equipartition result for the non-smooth potential. This is in fact an energy conservation result for the minimiser and a Pohozaev identity in disguise.

Proposition 3.17. *Let u be an equivariant minimiser, as in Theorem 3.14. Then the following equipartition type equality holds:*

$$\frac{1}{2}|u'|^2 = R^2 \cdot W(u) + c, \;\; c \in \mathbb{R} \textit{ constant}$$

Proof. **Step I:** Let u be an equivariant minimiser, as in Theorem 3.14. We let $s \in (0, \pi/2)$ and fix $r \geqslant 0$ appropriately small, such that $0 \leqslant kr \leqslant s$, for κ in a neighbourhood of 1. We define the comparison map:

$$\widetilde{u}(t) = \begin{cases} u(t/\kappa), & 0 \leqslant t \leqslant \kappa r \\ u\left(r + (s-r)\frac{t-\kappa r}{s-\kappa r}\right), & \kappa r \leqslant t \leqslant s \end{cases}$$

which is basically a consequence of the variation of the domain of u. Map $\widetilde{u}$ extends to the whole $[0, 2\pi]$ interval, because of symmetry and with a linear continuation on $(s, \pi/2)$, that respects the equivariance of $\widetilde{u}$. Notice that minimality gives us:

$$\mathscr{E}_R(u) \leqslant \mathscr{E}_R(\widetilde{u})$$

and also, if $\kappa = 1$, then $\widetilde{u} = u$.

Now, a direct (but long) computation to $\mathscr{E}_R(\widetilde{u})$ shows the following:

$$\begin{aligned}
\mathscr{E}_R(\widetilde{u}) &= \int_0^{2\pi} \frac{1}{2}|\widetilde{u}'|^2 + R^2 \cdot W(u)\ dt \\
&= 4\left[\int_0^{\pi/2} \frac{1}{2}|\widetilde{u}'|^2 + R^2 \cdot W(u)\ dt\right] \\
&= 4\Bigg[\int_0^{\kappa r} \frac{1}{2}\left|\frac{d}{dt}u\left(\frac{t}{\kappa}\right)\right|^2 + R^2 \cdot W(u)\ dt + \int_{\kappa r}^{s} \frac{1}{2}\left|\frac{d}{dt}u\left(r + (s-r)\frac{t-\kappa r}{s-\kappa r}\right)'\right|^2 \\
&\quad + R^2 \cdot W(u)\ dt + \int_s^{\pi/2} \frac{1}{2}|\widetilde{u}'|^2 + R^2 \cdot W(u)\ dt\Bigg]
\end{aligned}$$

We use a change of variables $y = t/k$ in the first integral and $y = r + (s-r)(t-\kappa r)/(s-\kappa r)$ in the second integral, and we also set the last integral equal to $A(s)$. We have:

$$[...] = 4\left[\int_0^r \frac{1}{2\kappa}|u'|^2 + \kappa R^2 W(u)\ dy + \int_r^s \frac{s-r}{2(s-\kappa r)}|u'|^2 + \frac{s-\kappa r}{s-r}R^2 W(u)\ dy + A(s)\right]$$

Step II: We also define the function of κ:

$$\begin{aligned}
f(\kappa) = &\int_0^r \frac{1}{2\kappa}|u'|^2 + \kappa R^2 \cdot W(u)\ dy + \int_r^s \frac{s-r}{2(s-\kappa r)}|u'|^2 + \frac{s-\kappa r}{s-r}R^2 \cdot W(u)\ dy + A(s) \\
&- \int_0^s \frac{1}{2}|u'|^2 + R^2 \cdot W(u)\ dy - A(s) \\
&= \int_0^r \left(\frac{1}{2\kappa} - 1\right)|u'|^2 + (\kappa - 1)R^2 \cdot W(u)\ dy + \int_r^s \left(\frac{s-r}{2(s-\kappa r)} - \frac{1}{2}\right)|u'|^2 \\
&+ \left(\frac{s-\kappa r}{s-r} - 1\right)R^2 \cdot W(u)\ dy
\end{aligned}$$

which is positive $f(\kappa) \geqslant 0$, because u has minimal energy (therefore $\mathscr{E}_R(\widetilde{u}) - \mathscr{E}_R(u) \geqslant 0$). By differentiating:

$$f'(\kappa) = \int_0^r -\frac{1}{2\kappa^2}|u'|^2 + R^2 \cdot W(u)\ dy + \int_r^s \frac{r}{2(s-\kappa r)^2}|u'|^2 - \frac{r}{s-r}R^2 \cdot W(u)\ dy$$

and since $f(1) = 0$, the difference of the energies, f, attains its global minimum at 1. Therefore, $f'(1) = 0$. Remember that value $\kappa = 1$ corresponds to $\widetilde{u} = u$, which is a minimiser of the energy.

The above show that:

$$f'(1) = \int_0^r -\frac{1}{2}|u'|^2 + R^2 \cdot W(u)\, dy + \int_r^s \frac{r}{2(s-r)}|u'|^2 - \frac{r}{s-r}R^2 \cdot W(u)\, dy = 0 \Rightarrow$$

$$\Rightarrow \int_0^r \frac{1}{2}|u'|^2 - R^2 \cdot W(u)\, dy = r\left[\frac{1}{s-r}\int_r^s \frac{1}{2}|u'|^2 - R^2 \cdot W(u)\, dy\right]$$

Letting $s \to r$, by the Lebesgue differentiation theorem we obtain:

$$\int_0^r \frac{1}{2}|u'|^2 - R^2 \cdot W(u)\, dy = r\left(\frac{1}{2}|u'|^2 - R^2 \cdot W(u)\right)$$

and if we set $g(y) = \frac{1}{2}|u'|^2 - R^2 \cdot W(u)$, the above becomes:

$$\int_0^r g(y)\, dy = rg(r) = r\frac{d}{dr}\int_0^r g(y)\, dy \tag{3.11}$$

which is an ordinary differential equation, with respect to r, which can be solved. We note that since smoothness is not given, this differential equation is solved in the distribution sence. Equation (3.11) becomes:

$$\frac{d}{dr}\left[\frac{1}{r}\int_0^r g(y)\, dy\right] = 0 \Rightarrow \int_0^r g(y)\, dy = cr$$

and therefore:

$$g(y) = \frac{d}{dr}\int_0^r g(y)\, dy = c, \text{ constant}$$

This shows the equipartition type equality:

$$\frac{1}{2}|u'|^2 = R^2 \cdot W(u) + c$$

□

Remark 3.18. *Any $H^1(\mathbb{S}^1;\mathbb{R}^2)$ and equivariant minimiser of $\mathscr{E}_R$, restricted to $[0,\pi/2]$, cannot leave its quadrant, say for example $\overline{B}_1(0) \cap (\mathbb{R}_+ \times \mathbb{R}_+)$.*

Proof. Indeed, suppose that u has some points inside $\overline{B}_1(0) \cap (\mathbb{R}_+ \times \mathbb{R}_+)$ but it manages to escape for some non-zero time to the other quadrants. Then, with at most two reflections, one with respect to the x and one other with respect to the y axes, we map the outliers inside the first quadrant $\overline{B}_1(0) \cap (\mathbb{R}_+ \times \mathbb{R}_+)$. This new trajectory has the same energy as the first one. If line segments existed, in this new trajectory appear regularity issues, which contradict the harmonic condition established by Proposition 3.16. If u was constituted only by arcs, then its total length is more than one circle, which means that its speed $\widetilde{c}$ must be bigger the total speed c of the circle (we utilised Proposition 3.17). By Proposition 3.17, we find that the circular trajectory has less energy.

□

Remark 3.19. *From Proposition 3.17 follows that any arc of the minimiser has speed $\sqrt{2c}$, while any line has speed $\sqrt{2c+2R^2}$. Also, from Proposition 3.16 u intersects the circle at least once (for the non-trivial zero case); this allows us to suppose, for convenience purposes, that this one point is $(1,0)$. That is, the trajectory "starts" from $(1,0)$.*

Lemma 3.20. *Given Proposition 3.17, if u is an non-zero $H^1(\mathbb{S}^1;\mathbb{R}^2)$ and equivariant minimiser of $\mathscr{E}_R$ with constant c, then $0 \leqslant c \leqslant 1/2$.*

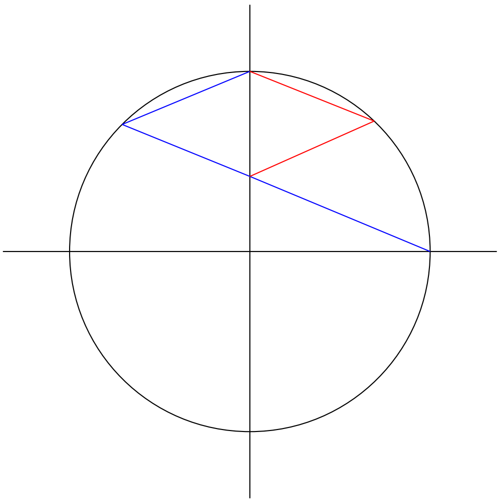


Figure 3.7: Loss of smoothness after reflection.

Proof. First observe that it is not possible to have $c > 1/2$. That is because the energy becomes:

$$\mathscr{E}_R(v) = 4\left[\frac{\pi c}{2} + 2R^2\left(\frac{\pi}{2} - t\right)\right] > \pi$$

with the right hand side being the energy of the circular trajectory.

As for $c \geqslant 0$: If u is a mixed trajectory or a circle, that is it contains an arc, then it is obvious that $c \geqslant 0$, as Remark 3.19 shows. We must examine what happens if u is a union of line segments only. Observe that in such unions corresponds a value c that depends on length, and as length increases, so does c. Indeed, if $\ell_1 \leqslant \ell_2$ are the lengths of two unions as described above, then:

$$2\pi\sqrt{2c_1 + 2R^2} \leqslant 2\pi\sqrt{2c_2 + 2R^2} \Rightarrow c_1 \leqslant c_2$$

(where c_1, c_2 are the corresponding constants). Hence, energy also increases with length, since:

$$\mathscr{E} = 2\pi\left(c + 2R^2\right)$$

If we show that the trajectory of minimal length is not a minimiser, our lemma follows.

This minimal-length trajectory is the line segment $a = [1, -1] \cup [-1, 1]$ (that is, $[1, -1]$ counted twice). By calculating its length we get:

$$2\pi\sqrt{2c + 2R^2} = 4 \Rightarrow c + R^2 = \frac{2}{\pi^2} \Rightarrow c + 2R^2 = \frac{2}{\pi^2} + R^2$$

The energy is:

$$\mathscr{E}(a) = 2\pi\left(\frac{2}{\pi^2} + R^2\right)$$

and the energy of the zero function is:

$$\mathscr{E}(0) = 2\pi R^2$$

therefore $\mathscr{E}(a) > \mathscr{E}(0)$. This shows a cannot be a minimiser. □

Lemma 3.21. *Given Proposition 3.17 and Lemma 3.20, let $\mathscr{A}_c$, $0 \leqslant c \leqslant 1/2$, be the class of functions:*

$$\mathscr{A}_c = \left\{ v \in H^1(\mathbb{S}^1; \mathbb{R}^2) \;\middle|\; v \text{ equivariant and } \frac{1}{2}|v'|^2 = R^2 \cdot W(v) + c \right\}$$

Each minimiser of:

$$\mathscr{E}_R(v) = \int_0^{2\pi} \frac{1}{2}|v'|^2 + R^2 \cdot W(v)\, dx$$

in $\mathscr{A}_c$ is a union of a single arc and a line which is perpandicular to one of the axes, in each quadrant.

Proof. Using the conservation of the energy in Proposition 3.17, one can rewrite the energy as:

$$\mathscr{E}_R(v) = 4\left[ct + (c+2R^2)\left(\frac{\pi}{2} - t\right)\right] = 4\left[\frac{\pi c}{2} + 2R^2\left(\frac{\pi}{2} - t\right)\right]$$

where t is the time v spends on the arcs. To minimise $\mathscr{E}_R$, given that c is constant in each class $\mathscr{A}_c$, it is clear that we must maximise t. In turn, we need to minimise the time v spends outside the circle, that is the time v is harmonic. But then notice that this time is directly proportional to length, since:

$$4\sqrt{2c+2R^2}\left(\frac{\pi}{2} - t\right)$$

is the total length of all of the line segments that may be present. It follows that the line segment is one and that it is perpandicular to one of the axes, since this configuration minimises length.

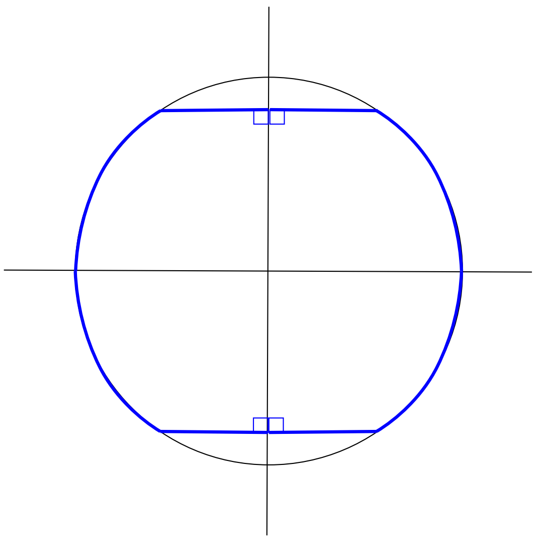


Figure 3.8: A candidate of a minimiser, a minimiser in $\mathscr{A}_c$.

□

Proposition 3.22 (Minimisers in 1−dimension, under symmetry hypotheses). *Let u be an $H^1(\mathbb{S}^1;\mathbb{R}^2)$ and equivariant minimiser of $\mathscr{E}_R$.*

i. If $R \geqslant 1/\sqrt{2}$, a circular trajectory is favoured over any other mixed trajectory, where both arcs and line segments are present. In combination with Remark 3.10, in this case the minimiser is the circular trajectory.

ii. If $R < 1/\sqrt{2}$, the zero functon is favoured over any other mixed trajectory, where both arcs and line segments are present. In combination with Remark 3.10, in this case the minimiser is the zero function.

Proof. Our previous Lemma 3.21 allows us to consider only unions of one arc and one line segment, which intersects one of the axes at a right angle. Suppose that a trajectory starts at $(1,0)$ and follows an arc of angle θ, before leaving the circle for a line segment. We want to minimise the energy $\mathscr{E}_R$, but this is impossible without imposing a restriction. The reason is because $\mathscr{E}_R$ minimises at $(t,c) = (\pi/2, 0)$ without any extra assumptions, which is problematic as $t = \pi/2$ means that only an arc exists and $c = 0$ that no arc exists. In a way, there are two problems, one that concerns times and one concerning distance, which we have not yet relate. Therefore, writing $t = \theta/\sqrt{2c}$, we set:

$$\sqrt{2c + 2R^2}\left(\frac{\pi}{2} - t\right) = \cos\theta \tag{3.12}$$

This equation, along with $t = \theta/\sqrt{2c}$, expresses the relationship between t and θ, by measuring the line segment of the trajectory two different ways. Equation (3.12) can also be written in the more usable form:

$$\sqrt{2c + 2R^2}\left(\frac{\pi}{2} - t\right) = \cos(t\sqrt{2c}) \tag{3.13}$$

This is more usable not because it is easier than (3.12), but because functions F, $\widehat{F}$ -which will be introduced later- are easier to compute. It will be useful to denote:

$$G(t,c) = \cos(t\sqrt{2c}) - \sqrt{2c + 2R^2}\left(\frac{\pi}{2} - t\right)$$

In i. we will need to compare two energies, the one of the mixed circular trajectory with that of the circular orbit. Respectively, in ii. we will compare the energy of the circular trajectory to that of the zero function. This is the reason we also consider the energy differences:

$$F(t,c) = \frac{1}{4}\left(\mathscr{E}_{\text{mixed}} - \mathscr{E}_{\text{circ}}\right) = \frac{\pi c}{2} + 2R^2\left(\frac{\pi}{2} - t\right) - \frac{\pi}{4}$$

and:

$$\widehat{F}(t,c) = \frac{1}{4}\left(\mathscr{E}_{\text{mixed}} - \mathscr{E}_{\text{zero}}\right) = \frac{\pi c}{2} + \frac{\pi R^2}{2} - 2R^2 t$$

Equations $G = 0$, $F = 0$, $\widehat{F} = 0$ define three curves (the latter two are linear) and it is not difficult to see that above F or $\widehat{F}$ we have $F > 0$ or $\widehat{F} > 0$ respectively. The one defined by $G = 0$, inside $(0, \pi/2) \times (0, 1/2)$, is exactly the collection of pairs (t,c) which are able to correctly define a mixed circular trajectory of arc $\theta = t/\sqrt{2c}$. If we prove that $G = 0$ is above $F = 0$ or $\widehat{F} = 0$ (when $R \geqslant 1/\sqrt{2}$ or $R < 1/\sqrt{2}$), then we will have shown that $G = 0$ lies exactly in the region where $F > 0$ or $\widehat{F} > 0$. This in turn means that every trajectory of arc $\theta = t/\sqrt{2c}$, $(t, c) \in (0, \pi/2) \times (0, 1/2)$, has more energy than either the circular trajectory or the zero function. Many questions arise, for example whether $G = 0$ really defines a function, rather than a union of functions (as $x^2 + y^2 = 1$ does).

Lets show that $G = 0$ defines only one function $c = c(t)$: First of all, by the chain rule we have:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial c}\frac{dc}{dt} \Rightarrow \frac{dc}{dt} = -\frac{\partial G}{\partial t} \Big/ \frac{\partial G}{\partial c}$$

where by d/dt we denote the total derivative, whereas with $\partial/\partial t$ the derivative that does not account for the dependence of c on t. Then, because:

$$\frac{\partial G}{\partial t} = -\sqrt{2c}\sin(t\sqrt{2c}) + \sqrt{2c + 2R^2}$$

and:

$$\frac{\partial G}{\partial c} = -\frac{1}{\sqrt{2c}}\sin(t\sqrt{2c}) - \frac{1}{\sqrt{2c + 2R^2}}\left(\frac{\pi}{2} - t\right)$$

we have:

$$\frac{dc}{dt} = \frac{1}{\sqrt{2c}}\sin(t\sqrt{2c}) + \frac{1}{\sqrt{2c + 2R^2}}\left(\frac{\pi}{2} - t\right) \Big/ -\sqrt{2c}\sin(t\sqrt{2c}) + \sqrt{2c + 2R^2}$$

This is a differential equation that defines $G = 0$. In fact it is easily seen that $dc/dt > 0$, which means that c increases. Notice that $(\pi/2, 1/2)$ belongs to $G = 0$, as well as some other point $(a, 0)$, with a depending on R (this is justified by the fact that if R is large, then $G = 0$ cannot be defined for all t, but only for those such that $\pi/2 - t$ is sufficiently small). So there exists a branch of $G = 0$ that passes from $(\pi/2, 1/2)$ and another that passes from $(a, 0)$. The second one cannot arrive again at $(\pi/2, 1/2)$, for reasons of ordinary differential equations. It must intersect either $c = 0$ or $c = 1/2$ or $t = \pi/2$. Intersection on $c = 0$ cannot happen again, since c increases with t. Therefore, intersection happens on $t = \pi/2$ (below $c = 1/2$) or on $c = 1/2$ (for $t < \pi/2$). The first leads to contradiction: If $t = \pi/2$, then our minimiser has an arc only, which means it is a circle and $c = 1/2$. The second is another contradiction, since if $c = 1/2$, we have:

$$G(t, 1/2) = \cos t - \sqrt{1 + 2R^2}\left(\frac{\pi}{2} - t\right) \leqslant \cos t + t - \frac{\pi}{2} < 0 \text{ for } t < \pi/2$$

Notice we have used Theorem A.1.

You may have already sensed that this is a rather calculation-heavy proof, and indeed it is. We will divide the proof in several steps. The general idea is to show that $G = 0$ does not intersect $F = 0$ or $\widehat{F} = 0$, so by continuity $G = 0$ will either be above or below. It is then easy to see that it is above both $F = 0$ and $\widehat{F} = 0$ (when $R \geqslant 1/\sqrt{2}$ in the first case and $R < 1/\sqrt{2}$ in the second).

For i.: We suppose $R \geqslant 1/\sqrt{2}$.

Step i.I: Equation $F = 0$ becomes:

$$\frac{\pi c}{2} + 2R^2\left(\frac{\pi}{2} - t\right) - \frac{\pi}{4} \Rightarrow c = \frac{4R^2}{\pi}t + \frac{1}{2} - 2R^2$$

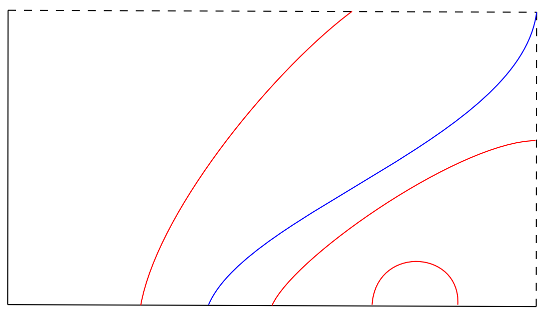


Figure 3.9: In blue $G = 0$ is sketched, while in red supposedly possible branches that do not exist.

and it is linear in t. We use this c to the general form of G and we will show it is not possible to have $G = 0$. This means that $F = 0$ and $G = 0$ cannot happen simutaneously, that is, they do not intersect.

$$\begin{aligned} G(t,c) &= \cos(t\sqrt{2c}) - \sqrt{2c+2R^2}\left(\frac{\pi}{2} - t\right) \\ &= \cos\left(t\sqrt{\frac{8R^2}{\pi}t + 1 - 4R^2}\right) - \sqrt{\frac{8R^2}{\pi}t + 1 - 2R^2}\left(\frac{\pi}{2} - t\right) \end{aligned}$$

This cosine would cause quite a lot of discomfort in our calculations, so we will use the inequalities:

$$\cos x \geqslant \begin{cases} \dfrac{-2(2-\sqrt{2})}{\pi}x + 1, & \text{when } x \in (0, \pi/4] \\ \quad \text{and} & \\ \dfrac{-2\sqrt{2}}{\pi}\left(x - \dfrac{\pi}{2}\right), & \text{when } x \in [\pi/4, \pi/2) \end{cases} \tag{3.14}$$

to replace it. These two inequalities arise naturally if one considers the two line segments that touch $\cos x$ at $x = 0$, $x = \pi/4$ and $x = \pi/4$, $x = \pi/2$. The very simple approximation $\cos x \geqslant -\frac{2}{\pi}x + 1$ does not work in our case. Therefore, we obtain:

$$G(t,c) \geqslant \begin{cases} \dfrac{-2(2-\sqrt{2})}{\pi}t\sqrt{\dfrac{8R^2}{\pi}t + 1 - 4R^2} + 1 - \sqrt{\dfrac{8R^2}{\pi}t + 1 - 2R^2}\left(\dfrac{\pi}{2} - t\right) \\ \quad \text{and} \\ \dfrac{-2\sqrt{2}}{\pi}\left(t\sqrt{\dfrac{8R^2}{\pi}t + 1 - 4R^2} - \dfrac{\pi}{2}\right) - \sqrt{\dfrac{8R^2}{\pi}t + 1 - 2R^2}\left(\dfrac{\pi}{2} - t\right) \end{cases} \tag{3.15}$$

In (3.15) it will be useful to denote with g_1, g_2 the right-hand sides.

Step i.II: We will differentiate both g_1, g_2 with respect to R, with the aim to show that $\partial g_1/\partial R$, $\partial g_2/\partial R > 0$. This would mean that g_1, g_2 increase uniformly and strictly. Setting $R = 1/\sqrt{2}$ and considering the respectives g_1, g_2, we will have $G \geqslant g_1$ and $G \geqslant g_2$. But then we hope to be able to prove $g_1, g_2 > 0$, and as a consequence, $G > 0$ in $(0, \pi/2) \times (0, 1/2)$.

We have:

$$\frac{\partial g_1}{\partial R} = \frac{R(\pi - 2t)}{\sqrt{\pi}^3}\left(\frac{\pi(\pi - 4t)}{\sqrt{8R^2t - 2\pi R^2 + \pi}} + \frac{8(2-\sqrt{2})t}{\sqrt{8R^2t - 4\pi R^2 + \pi}}\right) \tag{3.16}$$

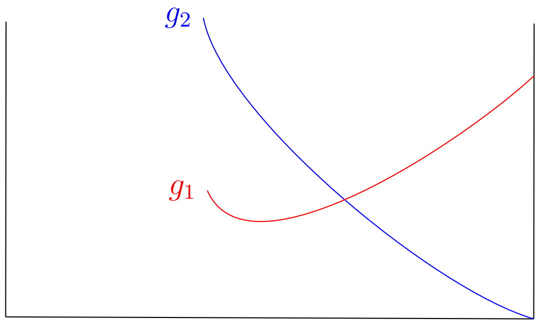


Figure 3.10: Functions g_1 and g_2 when $R = 1/\sqrt{2}$.

and:

$$\frac{\partial g_2}{\partial R} = \frac{R(\pi - 2t)}{\sqrt{\pi}^3}\left(\frac{\pi(\pi - 4t)}{\sqrt{8R^2t - 2R^2t + \pi}} + \frac{8\sqrt{2}t}{\sqrt{8R^2t - 4R^2\pi + \pi}}\right) \tag{3.17}$$

We denote f_1 and f_2 the functions inside the parentheses in (3.16) and (3.17). These are the terms that define the signs of $\partial g_1/\partial R$ and $\partial g_2/\partial R$.

We differentiate again f_1 and f_2 with respect to R. We get:

$$\frac{\partial f_1}{\partial R} = \frac{2\pi R(\pi - 4t)^2}{\sqrt{8R^2t - 2\pi R^2 + \pi}^3} + \frac{32R(2 - \sqrt{2})(\pi - 2t)t}{\sqrt{8R^2t - 4\pi R^2 + \pi}^3}$$

and:

$$\frac{\partial f_2}{\partial R} = \frac{2\pi R(\pi - 4t)^2}{\sqrt{8R^2t - 2\pi R^2 + \pi}^3} + \frac{32R\sqrt{2}(\pi - 2t)t}{\sqrt{8R^2t - 4\pi R^2 + \pi}^3}$$

Surprisingly, it is easy to see that both $\partial f_1/\partial R$, $\partial f_2/\partial R$ are strictly positive. So, we consider the case $R = 1/\sqrt{2}$, and if we show that the respectives f_1, f_2 are positive, then (3.16) and (3.17) are positive.

We have, for $R = 1/\sqrt{2}$:

$$f_1 = \frac{\pi(\pi - 4t)}{2\sqrt{t}} + \frac{8(2 - \sqrt{2})t}{\sqrt{4t - \pi}}$$

and:

$$f_2 = \frac{\pi(\pi - 4t)}{2\sqrt{t}} + \frac{8\sqrt{2}t}{\sqrt{4t - \pi}}$$

In the next step we will show that both f_1, f_2 are strictly monotone decreasing, with $f_1(\pi/2), f_2(\pi/2) \geqslant 0$, which shows that both are positive.

Coming back to (3.16), (3.17), we have shown that both are positive. We once again set $R = 1/\sqrt{2}$ and we are willing to show that the respectives g_1, g_2 are positive, which concludes i., as we indicated on the start of Step i.II. We compute:

$$g_1 = \frac{-2(2 - \sqrt{2})}{\pi}t\sqrt{\frac{4t}{\pi} - 1} + 1 - 2\sqrt{\frac{t}{\pi}}\left(\frac{\pi}{2} - t\right)$$

and:

$$g_2 = \frac{-2\sqrt{2}}{\pi}\left(t\sqrt{\frac{4t}{\pi} - 1} - \frac{\pi}{2}\right) - 2\sqrt{\frac{t}{\pi}}\left(\frac{\pi}{2} - t\right)$$

In the following step we will also prove that g_1, g_2 are positive.

Step i.III: Each one of the f_1, f_2, g_1, g_2 in this step is computed in the $R = 1/\sqrt{2}$ case.

For f_1 and f_2, we have:

$$\frac{df_1}{dt} = -\frac{\pi^2}{4\sqrt{t}^3} - \frac{8(2-\sqrt{2})(\pi - 2t)}{\sqrt{4t-\pi}^3} - \frac{\pi}{\sqrt{t}} < 0$$

and:

$$\frac{df_2}{dt} = -\frac{\pi^2}{4\sqrt{t}^3} - \frac{8\sqrt{2}(\pi - 2t)}{\sqrt{4t-\pi}^3} - \frac{\pi}{\sqrt{t}} < 0$$

so f_1, f_2 are strictly monotone decreasing.

For g_1, g_2, we differentiate with respect to t and we find:

$$\frac{dg_1}{dt} = -\frac{(\pi - 6t)[\pi\sqrt{4t-\pi} - 4(2-\sqrt{2})\sqrt{t}]}{2\sqrt{\pi}^3\sqrt{(4t-\pi)t}}$$

and:

$$\frac{dg_2}{dt} = -\frac{(\pi - 6t)[\pi\sqrt{4t-\pi} - 4\sqrt{2}\sqrt{t}]}{2\sqrt{\pi}^3\sqrt{(4t-\pi)t}}$$

The second one has its zeros outside $(\pi/4, \pi/2)$, so it retains its sign. It is not difficult to see that this sign is negative. This means that g_2 is decreasing, and by:

$$g_2(\pi/2) = 0$$

we conclude that it is positive. The first one is a little more tricky: It is negative if $t \leqslant t_0$ and positive if $t > t_0$, where:

$$t_0 = \frac{\pi^3}{4[\pi^2 - 4(\sqrt{2}-2)^2]}$$

In t_0 there exists a global minimum. Now we calculate $g_1(t_0)$ and we find:

$$g_1(t_0) = 1 - \frac{4(2-\sqrt{2})^2\pi^2 - \pi^4}{4[\pi^2 - 4(\sqrt{2}-2)^2]^{3/2}} - \frac{\pi^2}{2\sqrt{\pi^2 - 4(\sqrt{2}-2)^2}} \approx 0.15354\cdots > 0$$

which shows that $g_1 > 0$ too.

For ii.: This case is similar.

Step ii.I: By solving $\widehat{F} = 0$ we obtain:

$$\frac{\pi c}{2} + \frac{\pi R^2}{2} - 2R^2 t = 0 \Rightarrow c = \frac{4R^2}{\pi}t - R^2$$

and then, by (3.14):

$$G(t,c) \geqslant \begin{cases} \dfrac{-2(2-\sqrt{2})}{\pi} tR\sqrt{\dfrac{8t}{\pi} - 2} + 1 - R\sqrt{\dfrac{8t}{\pi}}\left(\dfrac{\pi}{2} - t\right) \\ \quad\text{and} \\ \dfrac{-2\sqrt{2}}{\pi}\left(tR\sqrt{\dfrac{8t}{\pi} - 2} - \dfrac{\pi}{2}\right) - R\sqrt{\dfrac{8t}{\pi}}\left(\dfrac{\pi}{2} - t\right) \end{cases} \tag{3.18}$$

It will, once again, be useful to denote the first and second branch as g_1 and g_2.

Step ii.II: We differentiate g_1 and g_2 with respect to R and we get:

$$\frac{\partial g_1}{\partial R} = \frac{-2(2-\sqrt{2})}{\pi} t\sqrt{\frac{8t}{\pi} - 2} - \sqrt{\frac{8t}{\pi}}\left(\frac{\pi}{2} - t\right) < 0$$

and:

$$\frac{\partial g_2}{\partial R} = \frac{-2\sqrt{2}}{\pi} t\sqrt{\frac{8t}{\pi} - 2} - \sqrt{\frac{8t}{\pi}}\left(\frac{\pi}{2} - t\right) < 0$$

(when $t < \pi/2$). This means that g_1, g_2 decrease uniformly as R approaches $1/\sqrt{2}$. If we show that the smallest of those, that is g_1, g_2 that correspond to $R = 1/\sqrt{2}$, are positive, then $G > 0$, which concludes the proof once more.

If $R = 1/\sqrt{2}$, we have:

$$g_1 = \frac{-2(2-\sqrt{2})}{\pi} t\sqrt{\frac{4t}{\pi} - 1} + 1 - 2\sqrt{\frac{t}{\pi}}\left(\frac{\pi}{2} - t\right)$$

and:

$$g_2 = \frac{-2\sqrt{2}}{\pi}\left(t\sqrt{\frac{4t}{\pi} - 1} - \frac{\pi}{2}\right) - 2\sqrt{\frac{t}{\pi}}\left(\frac{\pi}{2} - t\right)$$

We have already dealt with these functions in i., so we already know that $g_1, g_2 \geqslant 0$ and this concludes the proof in whole.

$\square$

3.2.2 Minimisers under symmetry hypotheses (2-D case)

In Section 3.2 we examined the $H^1(\mathbb{S}^1;\mathbb{R}^2)$ equivariant minimisers of:

$$\mathscr{E}_R(v) = \int_0^{2\pi} |v'|^2 + R^2 \cdot W(v)\, dt, \text{ where } W(v) = \mathbb{1}_{\{|v|\neq 1\}}$$

By identifying $\mathbb{R}^2 = \mathbb{C}$, a logical generalisation of the previous case would be to study $H^1\big(B_R(0);\mathbb{C}\big)$ minimisers of:

$$\mathscr{E}_{B_R(0)}(v) = \int_{B_R(0)} |\nabla v|^2 + W(v)\, dt, \quad \text{where } W(v) = \mathbb{1}_{\{|v|\neq 1\}}$$

Of course, if $u = \Re u + i\Im u$, we define $|\nabla u|^2 = |\nabla \Re u|^2 + |\nabla \Im u|^2$. The equivariance condition can be replaced by rotational symmetry, that is we examine radial minimisers.

Just like in the one-dimensional case, the projection on the unit ball is of critical importance for the existence of radial minimisers.

Lemma 3.23 (Projection on the disc, 2-D case)**.** *Let $u : \mathbb{R}^2 \to \mathbb{C}$ be an $H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{C})$ function. Then the projection on the closed ball $\overline{B}_1(0)$:*

$$Pu = \begin{cases} u/|u|, & \text{if } |u| \geqslant 1 \\ u & \text{if } |u| < 1 \end{cases}$$

is $1-$Lipschitz:

$$|Pu - Pv| \leqslant |u - v|$$

Moreover, by composition $Pu \in H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{C})$ and:

$$|\nabla Pu| \leqslant |\nabla u|$$

Proof. Proceeding as in proof of Lemma 3.13, it can be shown that P is $1-$Lipschitz and $Pu \in H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{C})$ (cf. Theorem A.2). Moreover, calculating the directional derivatives, $|\nabla Pu| \leqslant |\nabla u|$ can also be shown. We will be brief, since this is not too disimilar from the proof of Lemma 3.13.

$$\begin{aligned} \frac{d}{dx_i}\Big|_x (Pu)_j &\overset{\star}{=} \lim_{h\to 0}\frac{1}{h}\int_x^{x+he_i} \frac{d}{dx_i}\Big|_y (Pu)_j \, dy \\ &= \lim_{h\to 0}\frac{(Pu)_j(x+he_i)-(Pu)_j(x)}{h} \end{aligned}$$

where $(\star)$ is justified by the Lebesgue differentiation theorem. Therefore:

$$\begin{aligned} \left|\frac{d}{dx_i}\Big|_x (Pu)_j\right| &= \lim_{h\to 0}\left|\frac{Pu(x+he_i)-Pu(x)}{h}\right| \\ &\overset{\star\star}{\leqslant} \lim_{h\to 0}\left|\frac{u(x+he_i)-u(x)}{h}\right| \end{aligned}$$

where in $(\star\star)$ the Lipschitz condition of P is used. □

We now state the analog of Theorem 3.14 for the two dimensional case. We note that for the duration of this section, a very useful tool will be the polar form of the gradient ∇v, that is:

$$\nabla v = \nabla_{(r,\theta)} v = \frac{\partial v}{\partial r}\widehat{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\widehat{\theta}$$

where $\widehat{r}$, $\widehat{\theta}$ are the unit radial and angular basis elements.

Theorem 3.24 (Existence of $H^1_{\text{rad}}\big(B_R(0);\mathbb{C}\big)$ minimisers for the non-smooth potential - $2-$dimensional)**.** *There exists a minimiser of:*

$$\mathscr{E}_{B_R(0)}(v) = \int_{B_R(0)} \frac{1}{2}|\nabla v|^2 + W(u)\, dx$$

in the class:

$$\mathscr{A} = \left\{ v \in H^1_{\text{rad}}\big(B_R(0);\mathbb{C}\big) \;\middle|\; \operatorname{Tr} v(re^{i\theta}) = e^{i\theta} \text{ when } r = R \right\}$$

Proof. This is, again, a consequence of the direct method.

Step I: First we note that:

$$0 \leqslant \inf_{\mathscr{A}} \mathscr{E}_{B_R(0)}(v) < \infty$$

since $v(re^{i\theta}) = re^{i\theta}/R$, $r \leqslant R$, has finite energy:

$$\begin{aligned} \mathscr{E}_{B_R(0)}(v) &= \int_{B_R(0)} \frac{1}{2}|\nabla v|^2 + W(v)\, dx \\ &= \int_0^R \int_{\theta=0}^{2\pi} \left(\frac{1}{2}|\nabla_{(r,\theta)} v|^2 + 1\right)\; r\cdot d\theta\, dr \\ &= \int_0^R \int_{\theta=0}^{2\pi} \left(\frac{1}{R^2} + 1\right)\; r\cdot d\theta\, dr \\ &< \infty \text{ (depending on } r, R) \end{aligned}$$

Step II: From Step I we have established $\inf_{\mathscr{A}} \mathscr{E}_{B_R(0)}(v) < \infty$, so we can consider a minimising sequence $\{v_k\}_{k=1}^{\infty} \subseteq \mathscr{A}$:

$$\mathscr{E}_{B_R(0)}(v_k) \xrightarrow{k\to\infty} \inf_{\mathscr{A}} \mathscr{E}_{B_R(0)}(v)$$

We can suppose that $|v_k| \leqslant 1$, from Lemma 3.23, therefore:

$$\sup_{k\in\mathbb{N}} \int_{B_R(0)} |v_k|^2 \, dx \leqslant \pi R^2$$

and also, for k_0 sufficiently large, since $\{v_k\}_{k=1}^{\infty}$ is a minimising sequence:

$$\sup_{k\geqslant k_0} \int_{B_R(0)} |\nabla v_k|^2 \, dx \leqslant \mathscr{E}_{B_R(0)} \left(\frac{r}{R} e^{i\theta}\right) < \infty$$

So we can suppose there exists a constant $M > 0$ such that:

$$\sup_{k\in\mathbb{N}} \int_{B_R(0)} |\nabla v_k|^2 \, dx \leqslant M$$

and, in turn, $||v_k||_{H^1(B_R(0);\mathbb{C})}$ is uniformly bounded for all k.

Step III: Since $||v_k||_{H^1(B_R(0);\mathbb{C})}$ is uniformly bounded for all k, there exists a subsequence, still denoted by $\{v_k\}_{k=1}^{\infty}$, such that:

$$v_k \xrightarrow[H^1(B_R(0);\mathbb{R}^2)]{w} u$$

for some $u \in H^1\big(B_R(0);\mathbb{R}^2\big)$. By the lower semi-continuity of $\int_{B_R(0)} |\Diamond|^2 \, dx$:

$$\int_{B_R(0)} |\nabla u|^2 \, dx \leqslant \liminf_{k\to\infty} \int_{B_R(0)} |\nabla v_k|^2 \, dx$$

because weak H^1-convergence implies $\nabla v_k \overset{w}{\rightharpoonup} \nabla u$ in $L^2\big(B_R(0);\mathbb{C}\big)$. Moreover, H^1-convergence implies $v_k \overset{w}{\rightharpoonup} u$ in $L^2\big(B_R(0);\mathbb{C}\big)$, and for a subsequence still denoted by $\{v_k\}_{k=1}^{\infty}$, we have strong convergence $v_k \to u$ in $L^2\big(B_R(0);\mathbb{R}^2\big)$. The above are a consequence of the compact embedding:

$$H^1\big(B_R(0);\mathbb{C}\big) = W^{1,2}\big(B_R(0);\mathbb{C}\big) \hookrightarrow\hookrightarrow L^2\big(B_R(0);\mathbb{C}\big)$$

(cf. Theorem A.4).

Potential W is also lower semi-continuous, so:

$$\int_{B_R(0)} W(u) \, dt \overset{\star}{\leqslant} \int_{B_R(0)} \liminf_{k\to\infty} W(v_k) \, dt \overset{\star\star}{\leqslant} \liminf_{k\to\infty} \int_{B_R(0)} W(v_k) \, dt$$

In ($\star$) we use the lower semi-continuity of W and in ($\star\star$) Fatou's lemma. All of Step III can now be summarised by the fact:

$$\begin{aligned}
\mathscr{E}_{B_R(0)}(u) &= \int_{B_R(0)} \frac{1}{2}|\nabla u|^2 + R^2 \cdot W(u) \, dt \\
&\leqslant \liminf_{k\to\infty} \int_{B_R(0)} \frac{1}{2}|\nabla v_k|^2 + R^2 \cdot W(v_k) \, dt \\
&= \liminf_{k\to\infty} \mathscr{E}_{B_R(0)}(v_k)
\end{aligned}$$

Therefore, u is a minimiser of $\mathscr{E}_{B_R(0)}$. Note that u has the same boundary conditions, by the continuity of the trace operator.

$$||\text{Tr } u_n - \text{Tr } u||_{L^2(\partial B_R(0);\mathbb{C})} \leqslant C||u_n - u||_{H^2(B_R(0);\mathbb{C})}$$

Step IV: What remains is to show that u is in class $\mathscr{A}$. First, $\mathscr{A}$ is strongly closed. This is because $H^1_{\text{rad}}\big(B_R(0);\mathbb{C}\big)$ is a subspace of $H^1\big(B_R(0);\mathbb{C}\big)$, set:

$$\mathscr{B} = \left\{v \in L^2\big(B_R(0);\mathbb{C}\big) \;\middle|\; \text{Tr } v(Re^{i\theta}) = e^{i\theta}\right\}$$

is strongly closed and also:

$$\mathscr{A} = H^1_{\text{rad}}\big(B_R(0);\mathbb{C}\big) \cap \mathscr{B}$$

All of those sets are convex too. So Theorem A.10 applies and therefore $\mathscr{A}$ is weakly closed. This means that if $v_k \xrightarrow{w} u$, then $u \in \mathscr{A}$.

□

Having established the existence of such minimisers, next we will examine the form of them. Our intuition from the continuous case indicates that near the origin a cone is formed, while far away the minimiser is of constant modulus, equal to one. The advantage of the non-smooth case is the simplicity of the potential, and in that case we can be more specific about the form of the minimisers.

Proposition 3.25. *Let a minimiser $u \in H^1_{\text{rad}}\big(B_R(0);\mathbb{C}\big)$ of $\mathscr{E}_{B_R(0)}$ such that* $\text{Tr } u(Re^{i\theta}) = e^{i\theta}$. *Function u is harmonic in parts in $B_R(0)\backslash\{0\}$ and has a continuous representative in the same set. In fact, for the radial part we have:*

$$\rho(r) = c_1 r + \frac{c_2}{r} \textit{ whenever } \rho < 1$$

Proof. We will work in steps.

Step I: Lets denote with ρ the radial part of u. Because of the Sobolev embedding, ρ can be considered continuous away from the origin. Therefore, for each $0 < \varepsilon < R$:

$$H^1(\varepsilon, R) = W^{1,2}(\varepsilon, R) \hookrightarrow\hookrightarrow C([\varepsilon, R])$$

In a similar fashion, the angular part of u is continuous too. Moreover, by Lemma 3.23, ρ is bounded by 1, and in fact ρ obtains this value, because of the boundary conditions. Of course, we do not know if ρ becomes 1 right at the boundary or a little before.

Step II: Whenever $0 < \rho < 1$, u is harmonic. Our analysis uses the polar representation of the derivative, so we must restrict ourselves in an anulus of the form $A = B_{r_0}(0)\backslash B_{\delta_0}(0)$, that is $\delta_0 < \rho < r_0$. We consider $\varphi \in C^\infty_c\big(A; B_1(0)\big)_{\text{rad}}$ and $\varepsilon > 0$ small enough so that $|u + \varepsilon\varphi| < 1$. Because of minimality, we have $\mathscr{E}_{B_R(0)}(u) \leqslant \mathscr{E}_{B_R(0)}(u + \varepsilon\varphi)$, and from this follows:

$$\int_A \frac{1}{2}|\nabla u|^2 \, dx \leqslant \int_A \frac{1}{2}|\nabla(u + \varepsilon\varphi)|^2 \, dx$$

One immediate way to obtain the harmonic condition is to use the principle of symmetric criticality. Another, more elementary way, follows: Using the polar form of the derivative, we get:

$$\begin{aligned}\frac{1}{2}\int_{\delta_0}^{r_0}\left(|\rho'|^2+\frac{1}{r^2}\rho^2\right) r\,dr &\leqslant \frac{1}{2}\int_{\delta_0}^{r_0}\left(|\rho'+\varepsilon\rho_\varphi'|^2+\frac{1}{r^2}|\rho+\varepsilon\rho_\varphi|^2\right) r\,dr\\ &=\frac{1}{2}\int_{\delta_0}^{r_0}\Big[|\rho'|^2+2\varepsilon\rho'\cdot\rho_\varphi'+\varepsilon^2|\rho_\varphi'|^2\\ &\quad+\frac{1}{r^2}\left(|\rho|^2+2\varepsilon\rho\cdot\rho_\varphi+\varepsilon^2|\rho_\varphi|^2\right)\Big] r\,dr\end{aligned}$$

where ρ_φ is the radial part of φ (we omit $\int_0^{2\pi}\,d\theta=2\pi$, since it appears on both sides of the inequality). This shows:

$$0\leqslant\int_{\delta_0}^{r_0}\left[2\varepsilon\rho'\cdot\rho_\varphi'+\varepsilon^2|\rho_\varphi'|^2+\frac{1}{r^2}\left(2\varepsilon\rho\cdot\rho_\varphi+\varepsilon^2|\rho_\varphi|^2\right)\right] r\,dr=0$$

Dividing by ε and letting $\varepsilon\to 0$, we obtain:

$$0\leqslant\int_{\delta_0}^{r_0} r\rho'\cdot\rho_\varphi'+\frac{1}{r}\rho\cdot\rho_\varphi\,dr$$

Integration by parts gives:

$$0\leqslant\int_{\delta_0}^{r_0}\left(-r\rho''-\rho'+\frac{1}{r}\rho\right)\cdot\rho_\varphi\,dr$$

and interchanging $\varphi\rightsquigarrow-\varphi$:

$$\int_{\delta_0}^{r_0}\left(-r\rho''-\rho'+\frac{1}{r}\rho\right)\cdot\rho_\varphi\,dr=0,\ \text{for all }\varphi\in C_c^\infty\big(A;B_1(0)\big)_{\text{rad}}$$

Therefore:

$$-\rho''-\frac{1}{r}\rho'+\frac{1}{r^2}\rho=0,\ \text{for }r\in(\delta_0,r_0)$$

The general solution of the equation above is:

$$c_1r+\frac{c_2}{r}$$

so if A is maximal, that is $\rho(\delta_0)=\rho(r_0)=1$, then c_1,c_2 can be found.

$$c_1=\frac{1}{\delta_0+r_0}\ \text{and}\ c_2=\frac{\delta_0r_0}{\delta_0+r_0}$$

Both $re^{i\theta}$ and $e^{i\theta}/r$ are harmonic on the plane, so u is harmonic too. □

Just for the sake of convenience, the section $\rho=c_1r+c_2/r$, $c_1,c_2\neq 0$, of a minimiser will be called a **chasm**. If $c_2=0$, that is ρ is linear, we will call this part of u a **cone**. By these definitions, a minimiser can be a combination of a cone, chasms or constant modulus parts only.

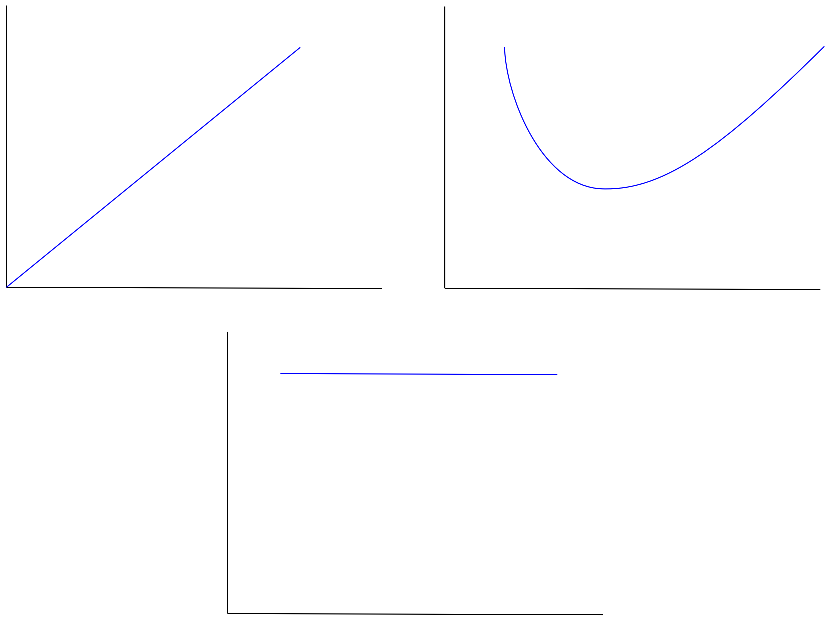


Figure 3.11: The radial profile of a cone, a chasm and a constant modulus function.

Proposition 3.26 (Minimisers in 2−dimensions, under symmetry hypotheses)**.** *Let a minimiser $u \in H^1_{\mathrm{rad}}\big(B_R(0); \mathbb{C}\big)$ of $\mathscr{E}_{B_R(0)}$ such that* $\operatorname{Tr} u(Re^{i\theta}) = e^{i\theta}$*. The following hold:*

i. If $R \geqslant 1/\sqrt{2}$, then minimiser u is a cone from the origin 0 to $\partial B_{\frac{1}{\sqrt{2}}}(0)$, followed by a unitary modulus function until $\partial B_R(0)$.

ii. If $R < 1/\sqrt{2}$, then minimiser u is a cone from the origin 0 to $\partial B_R(0)$.

Proof. The proof is rather calculation-heavy. We will work in several steps.

It is useful to do some computations before we continue with our proof. First off, suppose we have a cone $u = re^{i\theta}/\Lambda$. By using Green's formula in each coordinate, the energy in an annulus $A = B_{r_2}(0) \backslash B_{r_1}(0)$ is:

$$\begin{aligned}\int_A \frac{1}{2}|\nabla u|^2 + W(u)\, dx &= \int_{\partial A} \frac{1}{2}\left\langle u, \frac{\partial u}{\partial r}\right\rangle\, dS + \int_A W(u)\, dx \\ &= \int_{\partial B_{r_2}(0)} \frac{1}{2}\frac{r_2}{\Lambda^2}\, dS - \int_{\partial B_{r_2}(0)} \frac{1}{2}\frac{r_1}{\Lambda^2}\, dS + \pi(r_2^2 - r_1^2) \\ &= \frac{\pi}{\Lambda^2}(r_2^2 - r_1^2) + \pi(r_2^2 - r_1^2) \qquad (3.19)\end{aligned}$$

Then, we suppose u is a chasm in an annulus A as before, with:

$$\rho = c_1 r + \frac{c_2}{r}, \quad \text{where } c_1 = \frac{1}{r_1 + r_2},\ c_2 = \frac{r_1 r_2}{r_1 + r_2}$$

The energy in A is:

$$\begin{aligned}\int_A \frac{1}{2}|\nabla u|^2 + W(u)\ dx &= \int_{\partial A} \frac{1}{2}\left\langle u, \frac{\partial u}{\partial r}\right\rangle\ dS + \int_A W(u)\ dx \\ &= \int_{\partial B_{r_2}(0)} \frac{1}{2}\left(c_1 r_1 + \frac{c_2}{r_1}\right)\left(c_1 - \frac{c_2}{r_1^2}\right)\ dS \\ &- \int_{\partial B_{r_1}(0)} \frac{1}{2}\left(c_1 r_2 + \frac{c_2}{r_2}\right)\left(c_1 - \frac{c_2}{r_2^2}\right)\ dS + \pi(r_2^2 - r_1^2) \\ &= \pi r_2 \frac{r_2 - r_1}{r_2(r_1 + r_2)} - \pi r_1 \frac{r_1 - r_2}{r_1(r_1 + r_2)} + \pi(r_2^2 - r_1^2) \\ &= 2\pi \frac{r_2 - r_1}{r_1 + r_2} + \pi(r_2^2 - r_1^2) \end{aligned} \tag{3.20}$$

Finally, if u is of unitary modulus in annulus A, where A is as before, then the energy in A is:

$$\begin{aligned}\int_A \frac{1}{2}|\nabla u|^2 + W(u)\ dx &= \pi \int_{r_1}^{r_2} \frac{1}{r}\ dS \\ &= \pi \log \frac{r_2}{r_1}\end{aligned} \tag{3.21}$$

Our sketch is as follows: Firstly, we prove that sufficiently close to the origin u is a cone. Then, we show that a cone followed by a chasm can by replaced by a cone, with strictly less energy. Continuing, if a cone is followed by a constant modulus part, we prove that again it is preferable to use a single cone.

Step I: Sufficiently close to the origin, u is a cone. We need to show two things: First, no chasms are present inside some ball $B_{R_0}(0)$ and second in this ball u is not of constant modulus.

Notice that using polar coordinates we get:

$$\int_{B_R(0)} |\rho'|^2 + \frac{1}{r}\rho^2\ dx < \infty \Rightarrow \int_{B_R(0)} \frac{1}{r}\rho^2 < \infty \Rightarrow \liminf_{r\to 0} \rho = 0$$

because of the finite energy. Therefore, constant parts do not exists arbitrarily close to the origin, and if the non-existence of chasms is to be proved, it follows that near 0 minimiser u is a cone.

We suppose a sequence of chasms exists such that $\liminf_{r\to 0} \rho = 0$. Denoting by r_1, r_2 the radii of the chasm, we will calculate the minimum value of the radial component of each of them. We have:

$$0 = \frac{d\rho}{dr} = \frac{1}{r_1 + r_2} - \frac{r_1 r_2}{r^2}$$

and this is true if and only if $r = \sqrt{r_1 r_2}$. It follows that this value of r is where the local mimimum appears. Now we obtain:

$$\min_{B_{r_2}(0)\setminus B_{r_1}(0)} \rho = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2}$$

and if we denote $\lambda = r_2/r_1$:

$$\min_{B_{r_2}(0)\setminus B_{r_1}(0)} \rho = \frac{2\sqrt{\lambda}}{\lambda + 1}$$

The limit infimum $\liminf_{r\to 0}\rho = 0$ allows us to assume, without loss of generality, that $\lambda \to \infty$.

Coming back to the energies, by (3.19) and (3.20):

$$\mathscr{E} = \mathscr{E}_{\text{chasm}} - \mathscr{E}_{\text{cone}} = 2\pi\frac{\lambda-1}{\lambda+1} - \pi \xrightarrow{\lambda\to\infty} \pi$$

so chasms cannot appear arbitrarily close to 0, in some ball $B_{R_0}(0)$. Of course we can assume $R_0 \leqslant R$, because of the boundary conditions.

Step II: If a cone is followed by a chasm, the whole configuration can be replaced by a cone. Indeed, we compare the cone of radius R_0 and chasm of edges R_0, r_1, with a cone of radius r_1. By (3.19), (3.20):

$$\mathscr{E} = \mathscr{E}_{\text{cone+chasm}} - \mathscr{E}_{\text{cone}} = 2\pi\frac{\lambda-1}{\lambda+1} \geqslant 0$$

therefore, a single cone is prefered. This is basically a proof that a minimiser accends to 1 and meets only the parts of constant modulus.

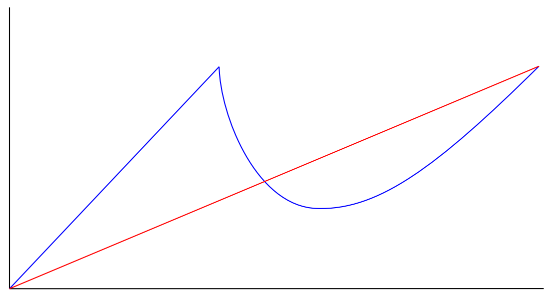


Figure 3.12

Step III: We will show that if $R \geqslant 1/\sqrt{2}$, then $R_0 = 1/\sqrt{2}$, whereas if $R < 1/\sqrt{2}$, then $R_0 = R$. Suppose then $R \geqslant 1/\sqrt{2}$: The general configuration is a cone of radius R_0, followed by a unitary constant function until r_1.

By considering all the possible values of R_0 (by varying R_0), we want to find the optimum, that is the value R_0 for which the corresponding function u minimises the energy. From (3.19), (3.21):

$$\mathscr{E} = \mathscr{E}_{\text{cone}} + \mathscr{E}_{\text{const}} = \pi + \pi R_0^2 + \pi \log\frac{r_1}{R_0} = \pi + \pi R_0^2 + \pi\log r_1 - \pi \log R_0$$

and this shows that to minimise $\mathscr{E}$ we have to minimise:

$$f(R_0) = R_0^2 - \log R_0$$

It is easy to see that if R_0 can obtain the value $1/\sqrt{2}$, that is $r_1 \geqslant 1/\sqrt{2}$, then the minimum is at $R_0 = 1/\sqrt{2}$. If $r_1 < 1/\sqrt{2}$, then similarly f minimises at the largest possible value for R_0, that is $R_0 = r_1$. But then we obtain a sequence of a cone and a chasm, which can be replaced by a cone of bigger radius, as shown before in Step II. In any case, either immediately or by "crawling" towards $\partial B_{\frac{1}{\sqrt{2}}}(0)$, we find that u is a cone from the origin to $\partial B_{\frac{1}{\sqrt{2}}}(0)$.

If we suppose $R < 1/\sqrt{2}$, the argument is as before: f minimises at the largest possible value for R_0, that is $R_0 = R$.

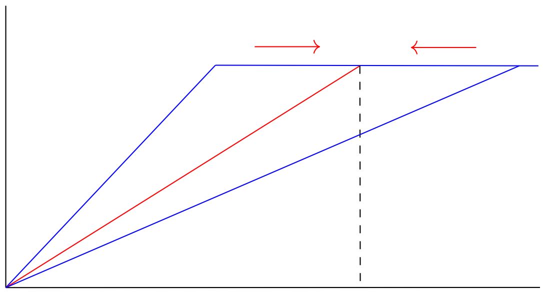


Figure 3.13

To end this proof, we will show that no chasms are present outside $B_1(0)$ and no chasms can lie on either side of $\partial B_1(0)$.

Step IV: We will show that if $r_1 > 1/\sqrt{2}$, it is not optimal to use a chasm in the complement $\mathbb{C}\backslash\overline{B}_{r_1}(0)$. Comparing the energies $\mathscr{E}_{\text{chasm}}$, $\mathscr{E}_{\text{const}}$ we get:

$$\mathscr{E} = \mathscr{E}_{\text{chasm}} - \mathscr{E}_{\text{const}} = 2\pi\frac{\lambda-1}{\lambda+1} + \frac{\pi}{2}(\lambda^2-1)r_1^2 - \pi\log\lambda \geqslant \frac{\pi}{2}(\lambda^2-1) - \pi\log\lambda$$

with the latter being monotone increasing and 0 when $\lambda = 1$. So $\mathscr{E} > 0$.

Step V: The previous step shows that no chasms exists in whole outside $B_{\frac{1}{\sqrt{2}}}(0)$. Of course, there is a possibility the chasm lies on either side of $\partial B_{\frac{1}{\sqrt{2}}}(0)$, which we must exclude. We will show that, if a chasm exists on either side of $\partial B_{\frac{1}{\sqrt{2}}}(0)$, say in $B_{r_2}(0)\backslash B_{r_1}(0)$, the function which is a cone until $\partial B_{\frac{1}{\sqrt{2}}}(0)$ and constant afterwards, has less energy.

Indeed, notice that just like in Step II this chasm configuration can be replaced by a cone until $\partial B_{r_2}(0)$. Then, as in Step III, the cone until $\partial B_{\frac{1}{\sqrt{2}}}(0)$ followed by the constant modulus has less energy. $\square$

3.2.3 Minimisers under symmetry hypotheses (n-D case)

The problem is similar in arbitrary $n-$dimensions. In this section we will examine the $H^1\big(B_R(0);\mathbb{R}^n\big)$ minimisers of:

$$\mathscr{E}_{B_R(0)}(v) = \int_{B_R(0)} \frac{1}{2}|\nabla v|^2 + W(u)\,dx,\ B_R(0) \subseteq \mathbb{R}^n$$

that are radial. The radial condition means that $v(rs) = \rho(r)s$, for every $s \in \mathbb{S}^{n-1}$ and some function $\rho : (0,\infty) \to (0,\infty)$ of one variable (which is the radial part of v). There is, of course, an ambiguity here, as to what ∇v and $|\nabla v|$ mean when v is a function $v : B_R(0) \to \mathbb{R}^n$, $B_R(0) \subseteq \mathbb{R}^n$. Some authors explicitly avoid this case, staying on $\mathbb{R}^2 = \mathbb{C}$, so to avoid computational complications. In our case, ∇v is the differential (Jacobian) and $|\nabla v|$ the Frobenius norm $||\nabla v||_F$, that is:

$$|\nabla v|^2 = ||\nabla v||_F^2 = \sum_{i,j}\left|\frac{\partial v_i}{\partial x_j}\right|^2$$

Lemma 3.23 practically remains unchainged, since its proof is identical. We restate it below:

Lemma 3.27. *Let $u : \mathbb{R}^n \to \mathbb{C}$ be an $H^1_{\text{loc}}(\mathbb{R}^n;\mathbb{C})$ function. Then the projection on the closed ball $\overline{B}_1(0)$:*

$$Pu = \begin{cases} u/|u|, & \text{if } |u| \geqslant 1 \\ u & \text{if } |u| < 1 \end{cases}$$

is 1−Lipschitz:

$$|Pu - Pv| \leqslant |u - v|$$

Moreover, by composition $Pu \in H^1_{\text{loc}}(\mathbb{R}^n;\mathbb{C})$ and:

$$|\nabla Pu| \leqslant |\nabla u|$$

Notice that in Section 3.2.2 a very useful tool was the polar form of the gradient ∇v. There is a sort of an analogue in many dimensions, which we explain thoroughly in Appendix B. If $v(rs) = \rho(r)s$ for every $s \in \mathbb{S}^{n-1}$:

$$\nabla v = \rho'(r) \cdot s \otimes s + \frac{\rho(r)}{r}(\text{Id} - s \otimes s)$$

Here ρ is the radial part, $r = |x|$ the radius, $s = x/|x|$ the angular and $\otimes$ represents the tensor product. We view it as the matrix $s \otimes s = (s_i s_j)_{i,j}$. The norm of this gradient can be computed to be:

$$|\nabla v|^2 = \big(\rho'(r)\big)^2 + \frac{n-1}{r^2}\rho^2(r)$$

The existence of such minimisers in the $n-$dimensional case follows as in Theorem 3.24. We will only state a small difference, since the whole remaining proof is identical.

Theorem 3.28 (Existence of $H^1_{\text{rad}}\big(B_R(0);\mathbb{R}^n\big)$ minimisers for the non-smooth potential - $n-$dimensional)**.** *There exists a minimiser of:*

$$\mathscr{E}_{B_R(0)}(v) = \int_{B_R(0)} \frac{1}{2}|\nabla v|^2 + W(u)\, dx,\ B_R(0) \subseteq \mathbb{R}^n$$

in the class:

$$\mathscr{A} = \left\{v \in H^1_{\text{rad}}\big(B_R(0);\mathbb{R}^n\big) \;\middle|\; \text{Tr } v(rs) = s \text{ for all } s \in \mathbb{S}^{n-1} \text{ when } r = R\right\}$$

Proof. Only a small difference exists in Step I, which is the use of $v(rs) = rs/R$ instead of $v(re^{i\theta}) = re^{i\theta}/R$. Using the general formula for the gradient:

$$\begin{aligned}
\mathscr{E}_{B_R(0)}(v) &= \int_{B_R(0)} \frac{1}{2}|\nabla v|^2 + W(v)\, dx \\
&= \int_0^R \int_{\theta=0}^{2\pi} \left(\frac{1}{2}|\nabla v|^2 + 1\right)\ r \cdot d\theta\, dr \\
&= \int_0^R \int_{\theta=0}^{2\pi} \left(\frac{n}{2R^2} + 1\right)\ r \cdot d\theta\, dr \\
&< \infty \text{ (depending on } n, r, R)
\end{aligned}$$

Therefore:

$$0 \leqslant \inf_{\mathscr{A}} \mathscr{E}_{B_R(0)}(v) < \infty$$

The remaining Steps II and III are independent of dimension and can be repeated. In Step IV we replace $\mathscr{B}$ with:

$$\mathscr{B} = \left\{ v \in L^2\big(B_R(0); \mathbb{R}^n\big) \;\middle|\; \operatorname{Tr} v(Rs) = s \right\}$$

□

In the 2−dimensional case what we essentially achieved was to reduce the problem in an 1−dimensional one, concerning the radius. If the case is the same for n−dimensions, our problem will be on a right track. In what follows we once again show that minimisers are harmonic in parts in $B_R(0)\backslash\{0\}$ and we extract a formula for the radial part, as in Proposition 3.25.

Proposition 3.29. *Let a minimiser* $u \in H^1_{\text{rad}}\big(B_R(0); \mathbb{R}^n\big)$ *of* $\mathscr{E}_{B_R(0)}$ *such that* $\operatorname{Tr} u(Rs) = s$ *for each* $s \in \mathbb{S}^{n-1}$*. Function* u *is harmonic in parts in* $B_R(0)\backslash\{0\}$ *and has a continuous representative in the same set. In fact, for the radial part we have:*

$$\rho(r) = c_1 r + c_2 r^{1-n} \text{ whenever } \rho < 1$$

Proof. As in the 2−dimensional case, for each $0 < \varepsilon < R$:

$$H^1(\varepsilon, R) = W^{1,2}(\varepsilon, R) \hookrightarrow\hookrightarrow C([\varepsilon, R])$$

In a similar fashion, the angular part of u is continuous too.

Whenever $0 < \rho < 1$, u is harmonic. Our analysis uses the spherical representation of the derivative, so we must restrict ourselves in an anulus of the form $A = B_{r_0}(0)\backslash B_{\delta_0}(0)$, that is $\delta_0 < \rho < r_0$. We consider $\varphi \in C_c^\infty\big(A; B_1(0)\big)_{\text{rad}}$ and $\varepsilon > 0$ small enough so that $|u + \varepsilon\varphi| < 1$. Because of minimality, we have $\mathscr{E}_{B_R(0)}(u) \leqslant \mathscr{E}_{B_R(0)}(u + \varepsilon\varphi)$, and from this follows:

$$\int_A \frac{1}{2}|\nabla u|^2 \, dx \leqslant \int_A \frac{1}{2}|\nabla(u + \varepsilon\varphi)|^2 \, dx$$

Once again, an immediate way to obtain the harmonic condition is to use the principle of symmetric criticality. To obtain the exact form of ρ, we use the spherical form of the gradient and we get once again:

$$\begin{aligned}
\frac{1}{2}\int_{\delta_0}^{r_0} \left(|\rho'|^2 + \frac{n-1}{r^2}\rho^2\right) r^{n-1} \, dr &\leqslant \frac{1}{2}\int_{\delta_0}^{r_0} \left(|\rho' + \varepsilon\rho_\varphi'|^2 + \frac{n-1}{r^2}|\rho + \varepsilon\rho_\varphi|^2\right) r^{n-1} \, dr \\
&= \frac{1}{2}\int_{\delta_0}^{r_0} \Big[|\rho'|^2 + 2\varepsilon\rho' \cdot \rho_\varphi' + \varepsilon^2|\rho_\varphi'|^2 \\
&\quad + \frac{n-1}{r^2}\left(|\rho|^2 + 2\varepsilon\rho \cdot \rho_\varphi + \varepsilon^2|\rho_\varphi|^2\right)\Big] r^{n-1} \, dr
\end{aligned}$$

where ρ_φ is the radial part of φ (we omit $\int_{\mathbb{S}^{n-1}} dS = \operatorname{Vol}_{n-1}\mathbb{S}^{n-1}$, since it appears on both sides of the inequality). This shows that:

$$0 \leqslant \int_{\delta_0}^{r_0} \left[2\varepsilon\rho' \cdot \rho_\varphi' + \varepsilon^2|\rho_\varphi'|^2 + \frac{n-1}{r^2}\left(2\varepsilon\rho \cdot \rho_\varphi + \varepsilon^2|\rho_\varphi|^2\right)\right] r^{n-1} \, dr = 0$$

Dividing by ε and letting $\varepsilon \to 0$, we obtain:

$$0 \leqslant \int_{\delta_0}^{r_0} r^{n-1}\rho' \cdot \rho'_\varphi + (n-1)r^{n-3}\rho \cdot \rho_\varphi \, dr$$

Integration by parts gives:

$$0 \leqslant \int_{\delta_0}^{r_0} \left(-r^{n-1}\rho'' - (n-1)r^{n-2}\rho' + (n-1)r^{n-3}\rho\right) \cdot \rho_\varphi \, dr$$

and interchanging $\varphi \rightsquigarrow -\varphi$:

$$\int_{\delta_0}^{r_0} \left(-r^{n-1}\rho'' - (n-1)r^{n-2}\rho' + (n-1)r^{n-3}\rho\right) \cdot \rho_\varphi \, dr = 0, \text{ for all } \varphi \in C_c^\infty\left(A; B_1(0)\right)_{\text{rad}}$$

Therefore:

$$-\rho'' - \frac{n-1}{r}\rho' + \frac{n-1}{r^2}\rho = 0, \text{ for } r \in (\delta_0, r_0)$$

The general solution of the equation above is:

$$c_1 r + c_2 r^{1-n}$$

Observe that if $n = 2$, we obtain the formula of Proposition 3.25:

$$c_1 r + c_2 r^{1-2} = c_1 r + \frac{c_2}{r}$$

If A is maximal, that is $\rho(\delta_0) = \rho(r_0) = 1$, then c_1, c_2 can be found, by solving this system. We get:

$$c_1 = \frac{r_0^{n-1} - \delta_0^{n-1}}{r_0^n - \delta_0^n} \text{ and } c_2 = \frac{(r_0 - \delta_0)(r_0\delta_0)^{n-1}}{r_0^n - \delta_0^n}$$

□

Once again, if the radial part of our minimiser is of the form $c_1 r + c_2 r^{1-n}$, $c_1, c_2 \neq 0$, then we will call it a **chasm** . To connect with the 2−dimensional case, if $c_2 = 0$, we will the call minimiser a **cone** . Our minimiser can be a combination of a cone, chasms or constant modulus parts.

Proposition 3.30 (Minimisers in $n-$dimensions, under symmetry hypotheses). *Let $n \geqslant 3$ and a minimiser $u \in H^1_{\text{rad}}\left(B_R(0); \mathbb{R}^n\right)$ of $\mathscr{E}_{B_R(0)}$ such that* $\operatorname{Tr} u(Rs) = s$ *for every $s \in \mathbb{S}^{n-1}$. The following hold:*

i. If $R \geqslant 1/\sqrt{2}$, then minimiser u is a cone from the origin 0 to $\partial B_{\frac{1}{\sqrt{2}}}(0)$, followed by a unitary modulus function until $\partial B_R(0)$.

ii. If $R < 1/\sqrt{2}$, then minimiser u is a cone from the origin 0 to $\partial B_R(0)$.

Proof. We will closely follow the proof of Proposition 3.26, making nessesary modifications whenever needed. There are significant differences, mostly concerning the energy of contant modulus parts. Here the associated energy is not logarithmic but polynomial.

First some computations as before. In what follows, we denote:

$$S_{n-1} = \mathrm{Vol}_{n-1}(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \text{ and } V_n = \mathrm{Vol}_n\big(B_1(0)\big) = \frac{\pi^{n/2}}{\Gamma(1+n/2)} = \frac{1}{n}S_{n-1}$$

Suppose $A = B_{r_2}(0)\backslash B_{r_1}(0)$ is an annulus, with $\lambda = r_2/r_1$. Suppose u is a cone, that is $\rho = cr$. Using Green's formula in each coordinate, the energy of the cone becomes:

$$\begin{aligned}\int_A \frac{1}{2}|\nabla u|^2 + W(u)\, dx &= \int_{\partial A} \frac{1}{2}\left\langle u, \frac{\partial u}{\partial r}\right\rangle\, dS + \int_A W(u)\, dx \\ &= \int_{\partial A} \frac{1}{2}c^2 r\, dS + V_n(r_2^n - r_1^n) \\ &= \frac{S_{n-1}c^2}{2}(r_2^n - r_1^n) + V_n(r_2^n - r_1^n) \end{aligned} \tag{3.22}$$

We suppose u is a chasm in an annulus A, with:

$$\rho = c_1 r + c_2 r^{1-n}, \quad \text{where } c_1 = \frac{r_2^{n-1} - r_1^{n-1}}{r_2^n - r_1^n},\ c_2 = \frac{(r_2 - r_1)(r_2 r_1)^{n-1}}{r_2^n - r_1^n}$$

The energy in A is:

$$\begin{aligned}\int_A \frac{1}{2}|\nabla u|^2 + W(u)\, dx &= \int_{\partial A} \frac{1}{2}\left\langle u, \frac{\partial u}{\partial r}\right\rangle\, dS + \int_A W(u)\, dx \\ &= \int_{\partial A} \frac{1}{2}\left(c_1 r + \frac{c_2}{r^{n-1}}\right)\left(c_1 + \frac{(1-n)c_2}{r^n}\right)\, dS + V_n(r_2^n - r_1^n) \\ &= \frac{1}{2}\int_{\partial A} c_1^2 r + \frac{c_1 c_2(2-n)}{r^{n-1}} + \frac{(1-n)c_2^2}{r^{2n-1}}\, dS + V_n(r_2^n - r_1^n)\end{aligned}$$

$$\begin{aligned}[...] &= \frac{S_{n-1}}{2}(r_2^n - r_1^n)\left(c_1^2 + \frac{(n-1)c_2^2}{(r_1 r_2)^n}\right) + V_n(r_2^n - r_1^n) \\ &= \frac{S_{n-1}}{2(r_2^n - r_1^n)}\big[(r_2^{n-1} - r_1^{n-1})^2 + (n-1)(r_1 r_2)^{n-2}(r_2 - r_1)^2\big] \\ &\quad + V_n(r_2^n - r_1^n) \\ &= \frac{S_{n-1}}{2(\lambda^n - 1)}\big[(\lambda^{n-1} - 1)^2 + (n-1)\lambda^{n-2}(\lambda - 1)^2\big]r_1^{n-2} + V_n(\lambda^n - 1)r_1^n \end{aligned} \tag{3.23}$$

Finally, if u is of unitary modulus in annulus A, where A is as before, then the energy in A becomes:

$$\begin{aligned}\int_A \frac{1}{2}|\nabla u|^2 + W(u)\, dx &= \frac{1}{2}\int_A \frac{n-1}{r^2}\, dx \\ &= \frac{1}{2}\int_{r_1}^{r_2}\int_{\mathbb{S}^{n-1}} (n-1)r^{n-3}\, dS dr \\ &= \frac{S_{n-1}(n-1)}{2(n-2)}(\lambda^{n-2} - 1)r_1^{n-2}\end{aligned} \tag{3.24}$$

Step I: Sufficiently close to the origin, u is a cone. We need to show two things: First, no chasms are present inside some ball $B_{R_0}(0)$ and second in this ball u is not of constant modulus.

Once again we have

$$\int_{B_R(0)} |\rho'|^2 + \frac{1}{r}\rho^2 \, dx < \infty \Rightarrow \int_{B_R(0)} \frac{1}{r}\rho^2 < \infty \Rightarrow \liminf_{r\to 0} \rho = 0$$

Therefore, constant parts do not exists arbitrarily close to the origin, and if the non-existence of chasms is to be proved, it follows that near 0 minimiser u is a cone.

We suppose a sequence of chasms exists such that $\liminf_{r\to 0}\rho = 0$. Denoting by r_1, r_2 the radii of the chasm, we will calculate the minimum value of the radial component of each of them. We have:

$$0 = \frac{d\rho}{dr} = c_1 + c_2(1-n)r^{-n} = 0$$

and this is true if and only if:

$$r = \left(-\frac{c_2(1-n)}{c_1}\right)^{1/n} = \left(\frac{(n-1)(r_2-r_1)(r_1 r_2)^{n-1}}{(r_1^{n-1} - r_2^{n-1})}\right)^{1/n}$$

Now by setting $\lambda = r_2/r_1$ we obtain:

$$r = r_1\left(\frac{(n-1)(\lambda-1)\lambda^{n-1}}{\lambda^{n-1}-1}\right)^{1/n}$$

It follows that this value of r is where the local mimimum appears. We get:

$$\begin{aligned}\min_{B_{r_2}(0)\setminus B_{r_1}(0)} \rho &= \frac{\lambda^{n-1}-1}{\lambda^n - 1}\left[\frac{(n-1)(\lambda-1)\lambda^{n-1}}{\lambda^{n-1}-1}\right]^{\frac{1}{n}} \\ &+ \frac{(\lambda-1)\lambda^{n-1}}{\lambda^n-1}\left[\frac{(n-1)(\lambda-1)\lambda^{n-1}}{\lambda^{n-1}-1}\right]^{\frac{1-n}{n}} \\ &= O(\lambda^{-1+1/n})\end{aligned}$$

The limit infimum $\liminf_{r\to 0}\rho = 0$ allows us to assume, without loss of generality, that $\lambda \to \infty$.

Coming back to the energies, by (3.22) and (3.23) one can check:

$$\begin{aligned}\mathscr{E} = \mathscr{E}_{\text{chasm}} - \mathscr{E}_{\text{cone}} &= \frac{S_{n-1}}{2(\lambda^n-1)}\left[(\lambda^{n-1}-1)^2 + (n-1)\lambda^{n-2}(\lambda-1)^2\right]r_1^{n-2} \\ &+ V_n(\lambda^n - 1)r_1^n - \frac{S_{n-1}}{2}\lambda^{n-2}r_1^{n-2} - V_n\lambda^n r_1^n \\ &= O(1)r_1^{n-2} - O(1)r_1^n > 0 \text{ (for small } r_1)\end{aligned}$$

so chasms cannot appear arbirtarily close to 0, in some ball $B_{R_0}(0)$. Of course we can assume $R_0 \leqslant R$, because of the boundary conditions.

Step II: If a cone is followed by a chasm, the whole configuration can be replaced by a cone. Indeed, we compare the cone of radius R_0 and chasm of edges R_0, r_1, with a cone of radius r_1. By (3.22), (3.23):

$$\begin{aligned}\mathscr{E} &= \mathscr{E}_{\text{cone+chasm}} - \mathscr{E}_{\text{cone}} \\ &= \frac{S_{n-1}}{2(\lambda^n-1)}\left[(\lambda^{n-1}-1)^2 + (n-1)\lambda^{n-2}(\lambda-1)^2 - (\lambda^{n-2}-1)(\lambda^n-1)\right]r_1^{n-2}\end{aligned}$$

Now denote $f_n(\lambda)$ the term inside the bracket. We have:

$$f_n(\lambda) = (\lambda-1)^2\big[(\lambda^{n-2}+\cdots+1)^2 + (n-1)\lambda^{n-2} - (\lambda^{n-3}+\cdots+1)(\lambda^{n-1}+\cdots+1)\big]$$

with the first sum product having all the terms of the last plus some. Therefore, $\mathscr{E}>0$ and a single cone is prefered.

Step III: We can show that if $R \geqslant 1$, then $R_0 = 1$, whereas if $R<1$, then $R_0 = R$. By considering the energy $\mathscr{E} = \mathscr{E}_{\text{cone}} + \mathscr{E}_{\text{const}}$, we want to find where it minimises, just like in Proposition 3.26. We have:

$$\mathscr{E} = \frac{S_{n-1}}{2n}\left(2R_0^2 - \frac{n}{n-2}\right)R_0^{n-2} + \frac{S_{n-1}(n-1)}{2(n-2)}r_1^{n-2}$$

The first term, which is of interest, is monotone decreasing until $R_0 = 1/\sqrt{2}$ and monotone increasing afterwards. It follows that, if $R \geqslant 1/\sqrt{2}$, the minimisation occurs at $R_0 = 1/\sqrt{2}$, whereas if $R<1/\sqrt{2}$, at $R_0 = R$.

Step IV: We will show that if $r_1 > 1/\sqrt{2}$, it is not optimal to use a chasm in the complement $\mathbb{C}\backslash\overline{B}_{r_1}(0)$. Comparing the energies $\mathscr{E}_{\text{chasm}}$, $\mathscr{E}_{\text{const}}$ we get:

$$\begin{aligned}
\mathscr{E} &= \mathscr{E}_{\text{chasm}} - \mathscr{E}_{\text{const}}\\
&= \frac{S_{n-1}}{2(\lambda^n-1)}\big[(\lambda^{n-1}-1)^2 + (n-1)\lambda^{n-2}(\lambda-1)^2\big]r_1^{n-2} + V_n(\lambda^n-1)r_1^n\\
&- \frac{S_{n-1}(n-1)}{2(n-2)}(\lambda^{n-2}-1)r_1^{n-2}\\
&= \frac{S_{n-1}}{2(\lambda^n-1)}\left[(\lambda^{n-1}-1)^2 + (n-1)\lambda^{n-2}(\lambda-1)^2 - \frac{n-1}{n-2}(\lambda^{n-2}-1)(\lambda^n-1)\right]r_1^{n-2}\\
&+ V_n(\lambda^n-1)r_1^n\\
&\geqslant \frac{S_{n-1}}{2(\lambda^n-1)}\left[\frac{1}{2n}(\lambda^n-1)^2 + (\lambda^{n-1}-1)^2 + (n-1)\lambda^{n-2}(\lambda-1)^2\right.\\
&\left.- \frac{n-1}{n-2}(\lambda^{n-2}-1)(\lambda^n-1)\right]r_1^{n-2}
\end{aligned}$$

Denote $g_n(\lambda)$ the term inside the brackets. By factoring-out $(\lambda-1)^2$, it is not hard to see that:

$$g_n = (\lambda-1)^2\left[\frac{1}{2n}h_n^2 + h_{n-1}^2 + (n-1)\lambda^{n-2} - \frac{n-1}{n-2}h_{n-2}h_n\right]$$

where $h_m = (\lambda^m-1)/(\lambda-1)$. Then, by $h_{n-1}^2 - h_{n-2}h_n = \lambda^{n-2}$:

$$[...] = (\lambda-1)^2\left[\frac{1}{2n}h_n^2 - \frac{1}{n-2}h_{n-1}^2 + \frac{(n-1)^2}{n-2}\lambda^{n-2}\right]$$

and by $h_{n-1} = h_n - \lambda^{n-1}$:

$$[...] = (\lambda-1)^2\left[-\frac{n+2}{2n(n-2)}h_n^2 + \frac{2\lambda^{n-2}}{n-2}h_n - \lambda^{n-2}\left(\lambda^{n-2} + \frac{(n-1)^2}{n-2}\right)\right]$$

Denote the term in the bracket as P_n. We claim that P_n is positive for $\lambda \geqslant 1$. Indeed, first notice that $P_n(1) = n/2 > 0$ (since $h_n(\lambda) = \sum_{k=0}^{n-1}\lambda^k$) and also that P_n cannot attain the value 0. This is because, if that was the case, there would be a value $h_n(\lambda_0) = y$

such that the polynomial P_n in y variable is zero. However, the discriminant is negative for all values of λ.

$$\Delta = \frac{\lambda^{n-2}}{n(n-2)^2}\Big[-((n+1)^2+3)\lambda^{n-2} - (n-1)(n+2)\Big]$$

Step V: There is a possibility the chasm lies on either side of $\partial B_{\frac{1}{\sqrt{2}}}(0)$, which we must exclude. We will show that, if a chasm exists on either side of $\partial B_{\frac{1}{\sqrt{2}}}(0)$, say in $B_{r_2}(0)\backslash B_{r_1}(0)$, the function which is a cone until $\partial B_{\frac{1}{\sqrt{2}}}(0)$ and constant afterwards, has less energy.

Indeed, notice that just like in Step II this chasm configuration can be replaced by a cone until $\partial B_{r_2}(0)$. Then, as in Step III, the cone until $\partial B_{\frac{1}{\sqrt{2}}}(0)$ followed by the constant modulus has less energy. $\square$

CHAPTER 4

The two phase interface

4.1 Minimal surfaces

Minimal surfaces are of big importance in phase transition problems. As we have already mentioned, the usual definitions of energy:

$$\mathscr{E}_\varepsilon(u;\Omega) = \int \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\ dx \text{ or } \mathscr{E}_\varepsilon(u;\Omega) = \int \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\ dx$$

contain some dependence on interface energy. The term $|\nabla u|$ penalises abrupt changes, so it is what encodes the interface energy (mostly). Potential W also plays a role away from its minima.

In what follows we will define minimal surfaces and see some examples.

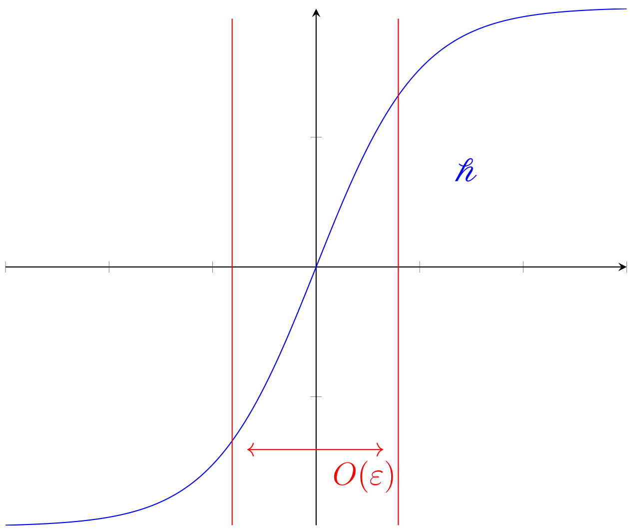


Figure 4.1

4.1.1 First variation of the area

Focusing on the linguistics of "minimal" surfaces, one might want to define them as surfaces (or generally manifolds) that locally minimise the area (or volume in general). It is however better to use the critical points of the area (or volume) functional, as they are not just geometric objects, but important elements for the calculus of variations.

The theorem which posseses the name "first variation of area" is characteristic for the definition of minimal surfaces. For its statement, we need tangential divergence. Suppose $(\mathscr{M}, \overline{g})$ is a Riemannian manifold of dimension $n+1$ and (Σ, g) and embedded submanifold of dimension n. If X is a vector field along Σ (not nessesarily tangent), we define the **tangential divergence**:

$$\overline{\nabla}^{\top} \bullet X = \sum_{E_k \| \Sigma} \langle \overline{\nabla}_{E_k} X, E_k \rangle$$

where $\overline{\nabla} = \nabla^{\overline{g}}$ and $E_k \parallel \Sigma$ means vector fields that produce the tangent spaces on Σ.

Theorem 4.1 (First variation of area). *Let $(\Sigma, g) \subseteq (\mathscr{M}, \overline{g})$ be an embedded submanifold without border. We consider a flow Θ on $\mathscr{M}$ with infinitesimal generator $X = \Theta^{\alpha} \in C_c^{\infty}(\mathscr{M}; T\mathscr{M})$, and $\Theta_t(\Sigma)$ the moving surface Σ following Θ. The first variation of $n-$volume (area) of Σ becomes:*

$$\frac{\delta \mathscr{H}^n}{\delta \Theta} = \frac{d}{dt}\Big|_{t=0} \mathscr{H}^n\big(\Theta_t(\Sigma)\big) = \int_{\Sigma} \overline{\nabla}^{\top} \bullet X \, dS = -n \int_{\Sigma} \langle X, H \rangle \, dS$$

where H is the mean curvature of Σ.

We write $\mathscr{H}^n$ (this is the Hausdorff measure) for the $n-$dimensional volume, instead of Vol_n, as it is a usual.

Proof. We can restrict ourselves in the case of a single map, since the arguments are similar in the more general case.

Step I: We will show:

$$\frac{d}{dt}\Big|_{t=0} \mathscr{H}^n\big(\Theta_t(\Sigma)\big) = \int_{\Sigma} \overline{\nabla}^{\top} \bullet X \, dS$$

Here we will calculate the metric coefficients with respect to the tangent vector fields to Σ. We can assume, without loss of generality, that $g_{i,j} = \delta_{i,j}$ and $\nabla_{E_k} E_k = 0$ on Σ. The volume element of $\Theta_t(\Sigma)$, as a function of the volume element of Σ, becomes:

$$dS^t = \frac{\sqrt{\det(g^t_{i,j})_{i,j}}}{\sqrt{\det(g_{i,j})_{i,j}}} dS$$

where g^t is the metric on $\Theta_t(\Sigma)$. Therefore, if we are to calculate the first variation, we must compute:

$$\frac{d}{dt}\Big|_{t=0} \frac{\sqrt{\det(g^t_{i,j})_{i,j}}}{\sqrt{\det(g_{i,j})_{i,j}}}$$

which, in our case, (since $g_{i,j} = \delta_{i,j}$, $g^0 = g$) becomes:

$$\frac{d}{dt}\Big|_{t=0} \sqrt{\det(g^t_{i,j})_{i,j}} = \frac{1}{2\sqrt{\det(g^0_{i,j})_{i,j}}} \frac{d}{dt}\Big|_{t=0} \det(g^t_{i,j})_{i,j} = \frac{1}{2} \frac{d}{dt}\Big|_{t=0} \det(g^t_{i,j})_{i,j}$$

Expand the determinant:

$$\det(g^t_{i,j})_{i,j} = \sum_{k=1}^{n} g^t_{1,k} \det(g^t_{i,j})_{i\neq 1, j\neq k}$$

and with a derivation:

$$\frac{d}{dt}\Big|_{t=0} \det(g^t_{i,j})_{i,j} = \cdots = \frac{d}{dt}\Big|_{t=0} g^t_{1,1} + \frac{d}{dt}\Big|_{t=0} \det(g^t_{i,j})_{i,j} = \cdots = \sum_{k=1}^{n} \frac{d}{dt}\Big|_{t=0} g^t_{k,k}$$

Now, because:

$$\frac{d}{dt}\Big|_{t=0} g^t_{k,k} = X\langle E_k, E_k\rangle = 2\langle \overline{\nabla}_X E_k, E_k\rangle = 2\langle \overline{\nabla}_{E_k} X, E_k\rangle$$

it follows:

$$\frac{d}{dt}\Big|_{t=0} \det(g^t_{i,j})_{i,j} = \sum_{E_k \| \Sigma} \langle \overline{\nabla}_{E_k} X, E_k\rangle = \overline{\nabla}^{\top} \bullet X$$

which means:

$$\frac{d}{dt}\Big|_{t=0} \mathscr{H}^n\big(\Theta_t(\Sigma)\big) = \int_{\Sigma} \overline{\nabla}^{\top} \bullet X \; dS$$

Step II: We will show the equality:

$$\int_{\Sigma} \overline{\nabla}^{\top} \bullet X \; dS = -n \int_{\Sigma} \langle X, H\rangle \; dS$$

By writing $X = X^{\top} + X^{\perp}$, where $X^{\top}$ is parallel to Σ and $X^{\perp}$ perpendicular, we have:

$$\int_{\Sigma} \overline{\nabla}^{\top} \bullet X^{\top} \; dS = 0$$

and then:

$$\int_{\Sigma} \overline{\nabla}^{\top} \bullet X \; dS = \int_{\Sigma} \overline{\nabla}^{\top} \bullet X^{\perp} \; dS$$

Consider $\{\widehat{n}_k\}_{k=1}^{N}$ a family of unitary orthogonal (to Σ) vectors, that produce the orthogonal complements of tangent spaces. We write the following:

$$\begin{aligned}
\overline{\nabla}^{\top} \bullet X^{\perp} &= \overline{\nabla}^{\top} \bullet \left(\sum_k \langle X, \widehat{n}_k\rangle \widehat{n}_k \right) \\
&= \sum_{\lambda} \left\langle \overline{\nabla}_{E_\lambda} \sum_k \langle X, \widehat{n}_k\rangle \widehat{n}_k, E_\lambda \right\rangle \\
&= \sum_{k,\lambda} \langle E_\lambda \langle X, \widehat{n}_k\rangle \widehat{n}_k + \langle X, \widehat{n}_k\rangle \overline{\nabla}_{E_\lambda} \widehat{n}_k, E_\lambda\rangle \\
&= \sum_{k,\lambda} \langle X, \widehat{n}_k\rangle \langle \overline{\nabla}_{E_\lambda} \widehat{n}_k, E_\lambda\rangle
\end{aligned}$$

But, since:

$$\langle \overline{\nabla}_{E_\lambda} \widehat{n}_k, E_\lambda\rangle = E_\lambda \langle \widehat{n}_k, E_\lambda\rangle - \langle \widehat{n}_k, \overline{\nabla}_{E_\lambda} E_\lambda\rangle = -\langle \widehat{n}_k, \overline{\nabla}_{E_\lambda} E_\lambda\rangle \widehat{n}_k$$

we obtain:

$$\begin{aligned}
\overline{\nabla}^{\top} \bullet X^{\perp} &= -\sum_{k,\lambda} \langle X, \widehat{n}_k \rangle \langle \overline{\nabla}_{E_\lambda} E_\lambda, \widehat{n}_k \rangle \\
&= -\sum_{k} \langle X, \widehat{n}_k \rangle \left\langle \sum_{\lambda} \overline{\nabla}_{E_\lambda} E_\lambda, \widehat{n}_k \right\rangle \\
&= -\sum_{k} \left\langle \langle X, \widehat{n}_k \rangle \widehat{n}_k, \left\langle \sum_{\lambda} \overline{\nabla}_{E_\lambda} E_\lambda, \widehat{n}_k \right\rangle \widehat{n}_k \right\rangle \\
&= -n \langle X, H \rangle
\end{aligned}$$

Hence:

$$\int_{\Sigma} \overline{\nabla}^{\top} \bullet X \, dS = -n \int_{\Sigma} \langle X, H \rangle \, dS$$

□

A consequence of this theorem is the following remark, that connects local minimisation of area with mean curvature.

Remark 4.2. *Suppose again we have $(\Sigma, g) \subseteq (\mathscr{M}, \overline{g})$, without border. We consider the following conditions:*

i. For every open set $U \Subset \mathscr{M}$, $U \cap \Sigma \Subset \Sigma$ and for every variation $\mathscr{N}$ with $U \cap \mathscr{N} \Subset \mathscr{N}$, $\Sigma \backslash U = \mathscr{N} \backslash U$, we have:

$$\mathscr{H}^n(\Sigma) \leqslant \mathscr{H}^n(\mathscr{N})$$

(a small "compact" variation increases the volume).

ii. It holds:

$$\int_{\Sigma} \overline{\nabla}^{\top} \bullet X \, dS = 0$$

iii. On Σ we have $H \equiv 0$.

iv. The volume functional has a critical point on Σ.

Then, i. $\Rightarrow$ ii. $\Leftrightarrow$ iii. $\Leftrightarrow$ iv.

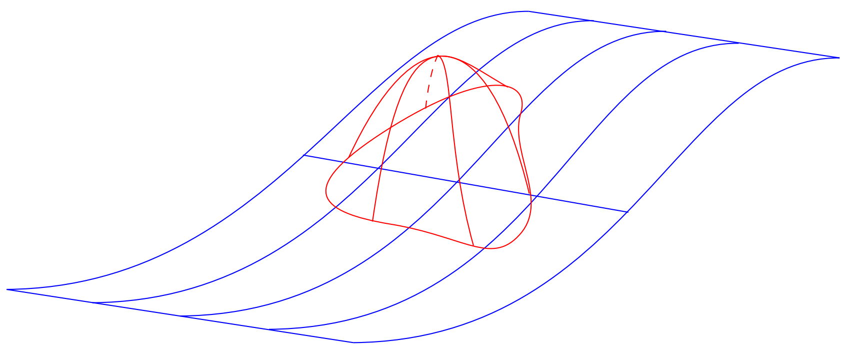


Figure 4.2

Definition 4.3 (Minimal surfaces). *Let (Σ, g) be an embedded submanifold of $(\mathscr{M}, \overline{g})$. We will say that Σ a minimal surface if it is a critical point of the volume functional. According to Remark 4.2, minimal surfaces are exactly those for which $H \equiv 0$ on $\Sigma \backslash \mathrm{bd}\Sigma$.*

4.1.2 Holomorphic representation and some examples

From here on we study surfaces Σ in $\mathbb{R}^n$, $n \geqslant 2$, and especially in $\mathbb{R}^3$, that have a parametric form. That is, we are interested in surfaces $\Sigma = \{f(x,y)\}$, $(x,y) \in \Omega \subseteq \mathbb{R}^2$.

We want to see some corellation between parametric surfaces and the real / imaginary parts of some holomorphic functions. In this direction, the following lemma is of importance. We note that a very natural metric can be obtained by Σ, which is:

$$g_{i,j}(x) = g_x(e_i, e_j) = \langle \partial_i f(x), \partial_j f(x) \rangle$$

Lemma 4.4. *Let $\Sigma = \{f(x,y)\}$, $f : \Omega \to \mathbb{R}^2$, be a parametric conformal surface in $\mathbb{R}^2$. If g is the metric defined above, then:*

i. We have:

$$\Delta_g f = 2H$$

ii. Especially, Σ is a parametric minimal surface if and only if $\Delta_g f \equiv 0$.

iii. We have:

$$\Delta f = 2\sqrt{\det(g_{i,j})_{i,j}} H$$

iv. Especially, Σ is a parametric minimal surface if and only if $\Delta f \equiv 0$.

Proof. We only need to prove i. and iii. Since the statement is local, we may assume f is an embedding. We consider the position vector $p_i(x) = x_i$ and then ∇p_i is constant in $\mathbb{R}^n$ and is equal to the $i-$th basis element. Consider the general fact of the intristic (tangential) Laplace operator $\Delta_g(p \circ f) = (\Delta^\top p) \circ f$ and the decomposition $\nabla p_i = \nabla^\top p_i + \nabla^\perp p_i$. In the proof of Theorem 4.1 we have seen that $\nabla^\top \bullet X^\perp = -n\langle X, H\rangle$, so along with our previous observations:

$$\begin{aligned}
\Delta_g f_i &= (\Delta^\top p_i) \circ f \\
&= (\nabla^\top \bullet \nabla^\top p_i) \circ f \\
&= (\nabla^\top \bullet \nabla p_i - \nabla^\top \bullet \nabla^\perp p_i) \circ f \\
&= 2\langle \nabla p_i, H \rangle \\
&= 2H_i
\end{aligned}$$

This concludes i. As for iii., by conformality we have $g_{i,j} = \lambda \delta_{i,j}$ for some $\lambda : \Omega \to [0, \infty)$ (in fact, since f is an immersion, $\lambda > 0$, and $g^{i,j} = \lambda^{-1}\delta_{i,j}$). By computing the determinant of the metric, we obtain:

$$\det(g_{i,j})_{i,j} = \lambda^2$$

By the coordinate form of the Laplace operator:

$$\Delta_g u = \frac{1}{\sqrt{\det(g_{i,j})_{i,j}}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{\det(g_{i,j})_{i,j}} g^{i,j} \partial_j u \right)$$

the result follows. □

Theorem 4.5 (Holomorphic representation). *Let $\Sigma = \{f(x,y)\}$ be a conformal parametric surface in $\mathbb{R}^n$ with $f : \Omega \to \mathbb{R}^n$, and $\Omega \subseteq \mathbb{R}^2$ simply connected. Then Σ is a parametric minimal surface if and only if $f = \Re h$ for some non-constant holomorphic function $h : \Omega \to \mathbb{C}^n$ such that:*

$$\sum_{k=1}^{n} \left(\frac{\partial \Re h_k}{\partial x} + i \frac{\partial \Im h_k}{\partial x} \right)^2 \equiv 0 \text{ in } \Omega$$

Moreover, $\Sigma^* = \{\Im h(x,y)\}$ is a parametric minimal surface, called the ***conjugate*** *of Σ.*

Proof. The proof will be done in several steps.

Step I: We first establish:

$$\Delta f \equiv 0 \text{ in } \Omega \Leftrightarrow f = \Re h, \ h : \Omega \to \mathbb{C}^n \text{ holomorphic} \tag{4.1}$$

($\Rightarrow$) Given f, we need to find some g such that the Cauchy-Riemann equations hold. That is:

$$\frac{\partial g_k}{\partial x} = -\frac{\partial f_k}{\partial y}$$
$$\frac{\partial g_k}{\partial y} = \frac{\partial f_k}{\partial x}$$

This means that there exists some function g_k with gradient:

$$\nabla G_k = \left(-\frac{\partial f_k}{\partial y}, \frac{\partial f_k}{\partial x} \right)$$

that is, the potetial of the following function exists $F_k = (-\partial f_k/\partial y, \partial f_k/\partial x)$. Therefore, for every simply connected set B:

$$0 = \int_{\partial B} \langle \nabla g_k, d\ell \rangle = \int_{\partial B} -\frac{\partial f_k}{\partial y}\, dx + \frac{\partial f_k}{\partial x}\, dy = \int_B \frac{\partial}{\partial x}\frac{\partial f_k}{\partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial f_k}{\partial y} \right) dA = \int_B \Delta f_k \, dA$$

(by Green's theorem). The above computations show that a necessary and sufficient condition for g to exist is $\Delta f \equiv 0$.

($\Leftarrow$) The other direction follows from Cauchy-Riemann equations.

Step II: Next up, we will show that if $h = f + ig$, with h being holomorphic, then f is conformal. We have:

$$\begin{aligned}
\sum_{k=1}^{n}\left(\frac{\partial \Re h_k}{\partial x} + i\frac{\partial \Im h_k}{\partial x}\right)^2 \equiv 0 \Leftrightarrow \sum_{k=1}^{n}\left(\frac{\partial f_k}{\partial x} + i\frac{\partial g_k}{\partial x}\right)^2 \equiv 0 \\
\Leftrightarrow \left|\frac{\partial f}{\partial x}\right|^2 - \left|\frac{\partial g}{\partial x}\right|^2 + 2i\cdot\left\langle\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\right\rangle \equiv 0 \\
\Leftrightarrow \left|\frac{\partial f}{\partial x}\right| = \left|\frac{\partial g}{\partial x}\right| \text{ and } \left\langle\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\right\rangle = 0 \\
\Leftrightarrow \left|\frac{\partial f}{\partial y}\right| = \left|\frac{\partial f}{\partial y}\right| \text{ and } \left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle = 0
\end{aligned}$$

The last condition is exactly what establishes the conformality of f.

Last but not least, $\Sigma^* = \{\Im h(x,y)\}$ is a conformal parametric surface, since $\Im h = \Re(-ih)$. □

Utilising Theorem 4.5, we can give a couple of examples of conformal parametric minimal surfaces.

The first example is that of the Catenoid. If:

$$h(z) = \alpha \begin{pmatrix} \cosh z \\ i\cdot\sinh z \\ z \end{pmatrix}, \ \alpha \in \mathbb{R}$$

then:

$$\sum_{k=1}^{3}\left(\frac{\partial \Re h_k}{\partial x} + i\frac{\partial \Im h_k}{\partial x}\right)^2 \equiv 0$$

and $f = \Re h$ is conformal parametric minimal surface. In fact, if we use the relations:

$$\begin{aligned}
\cosh(x+iy) &= \cosh x \cos y + i \sinh x \sin y \\
\sinh(x+iy) &= \sinh x \cos y + i \cosh x \sin y
\end{aligned}$$

we obtain:

$$f(x,y) = \alpha \begin{pmatrix} \cosh x \cos y \\ -\cosh x \sin y \\ x \end{pmatrix}$$

The second example is the conjugate of the first, that is the Helicoid. Function $g = \Im h$ is a conformal minimal surface, and as before we can show:

$$g(x,y) = \alpha \begin{pmatrix} \sinh x \sin y \\ \sinh x \cos y \\ y \end{pmatrix}$$

In the end, we mention an independent example (which is interesting due to the existence of self-intersection). This is Henneberg's suface. If

$$h(z) = \begin{pmatrix} -1 + \cosh(2z) \\ -i\big(\cosh z + 1\cosh(3z)/3\big) \\ -\sinh z + \sinh(3z)/3 \end{pmatrix}$$

we set $f = \Re h$. In real coordinates:

$$f(x,y) = \begin{pmatrix} -1+\cosh(2x)\cos(2y) \\ \sinh x \sin y + \sinh(3x)\sin(3y)/3 \\ -\sinh x \cos y + \sinh(3x)\cos(3y)/3 \end{pmatrix}$$

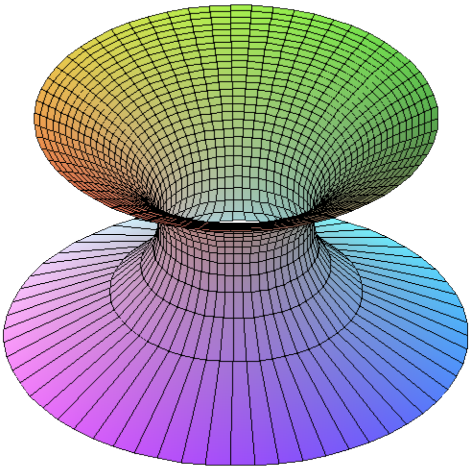


Figure 4.3: Catenoid

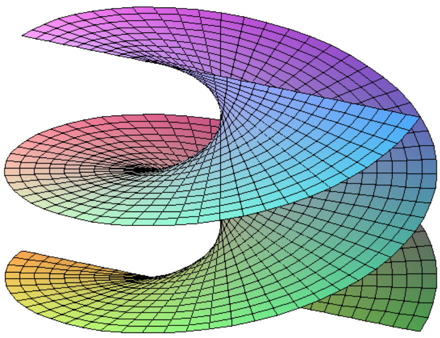


Figure 4.4: Helicoid

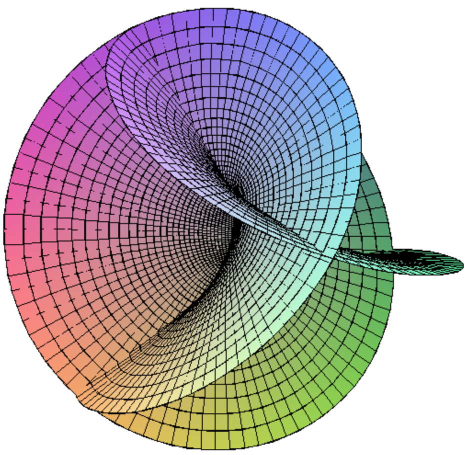


Figure 4.5: Henneberg surface

4.2 Γ-convergence

Lets suppose we have some family of functionals J_k, which converge to some other, say J. A usual problem in the calculus of variations is the following: Given a family of functions $\{u_k\}_k$, with limit u, such that u_k minimise J_k, can we conclude that u minimises J; Of course, the answer is negative.

An indicative example follows: In $H_0^1([0,1])$, of H^1-functions, vanishing on the boundary, we consider $J : H_0^1([0,1]) \to [0,\infty]$:

$$J(u) = \int_0^1 \left(u'(t)^2 - 1\right)^2 \, dt$$

and we define J_k:

$$J_k(u) = \begin{cases} J(u), \text{ if } u' \text{ constant in each } \big(i/(2k), (i+1)/(2k)\big), \; i \leqslant 2k-1 \\ \infty, \text{ otherwise} \end{cases}$$

Then, the following jagged functions u_k minimise J_k. Moreover, $u_k \to 0$ (uniformly) and

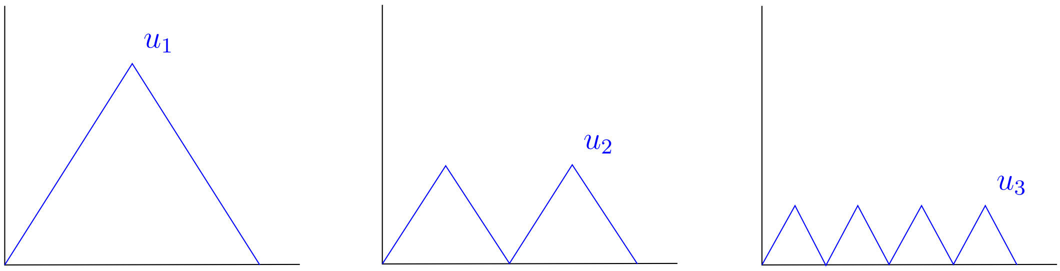


Figure 4.6

$J_k(v) \to J(v)$ almost everywhere, for every jagged or constant function. However, to our bad luck, $u \equiv 0$ does not minimise $J(u)$. The problem lies in the form of convergence, and specifically that we do not have $\Gamma-$convergence (a good reference is [29]).

Definition 4.6 ($\Gamma-$convergence)**.** *Let $\mathscr{X}$ be a Hausdorff and first countable topological space, and $J_k : \mathscr{X} \to \overline{\mathbb{R}}$ be a family of functionals in $\mathscr{X}$. We say that J_k* ***$\Gamma-$converge*** *to some $J : \mathscr{X} \to \overline{\mathbb{R}}$ if:*

i. ***There exists an asymptotic common lower bound:*** *For every $f \in \mathscr{X}$ and for every sequence $\{f_k\}_k$ with $f_k \to f$ in $\mathscr{X}$, we have:*

$$J(f) \leqslant \liminf_{k\to\infty} J_k(f_k)$$

ii. ***There exists approximating sequences:*** *For every $f \in \mathscr{X}$ there exists a sequence $\{f_k\}_k$ with $f_k \to f$ in $\mathscr{X}$ and:*

$$J(x) = \lim_{k\to\infty} J_k(f_k)$$

or, equivalently:

$$J(f) \geqslant \limsup_{k\to\infty} J_k(f_k)$$

... *If J_k $\Gamma-$converge to J, we write:*

$$J = \Gamma\text{-} \lim_{k\to\infty} J_k \text{ or } J_k \xrightarrow{\Gamma} J$$

Theorem 4.7 (Fundamental theorem of $\Gamma-$convergence). *Let $\mathscr{X}$ be a Hausdorff and first countable topological space, and $J_k : \mathscr{X} \to \overline{\mathbb{R}}$ a family of functionals in $\mathscr{X}$. If $J_k \xrightarrow{\Gamma} J$, $J : \mathscr{X} \to \overline{\mathbb{R}}$ and $f_k \in \mathscr{X}$ minimise J_k, then the limit $f_k \to f$ minimises J.*

As we saw in the previous example, there exist families of functionals which converge, but their minimisers do not converge to a minimiser of the limit. There is a phenomenon behind this discrepancy, which is lower semi-continuity.

Definition 4.8 (Lower semi-continuity). *Let $\mathscr{X}$ be a Hausdorff and first countable topological space, and $J : \mathscr{X} \to \overline{\mathbb{R}}$ a functional. We say that J is lower semi-continuous if:*

$$f_k \to f \Rightarrow J(f) \leqslant \liminf_{k\to\infty} J(f_k)$$

Equivalently, $J^{-1}\big((-\infty, t]\big)$ are closed in $\mathscr{X}$.

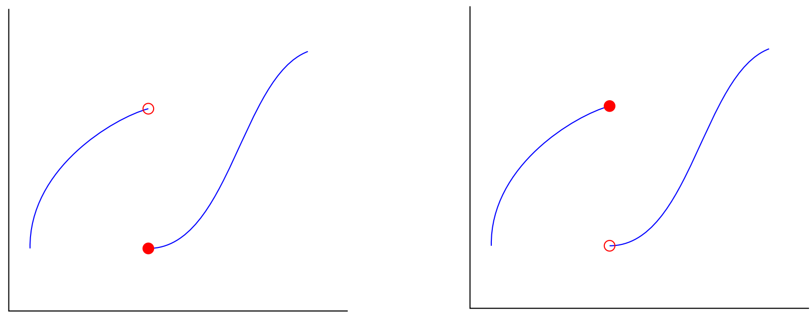


Figure 4.7: An example of a lower semi-continuous function (left) and another which is not (right).

Below we see that the limit must necessarily be lower continuous if we aim to some $\Gamma-$convergence. This is logical, if we imagine functionals J_k that approximate J, with their minimisers tending towards the non-lower semi-continuous part of J.

Definition 4.9 (Epigraph). *Let $\mathscr{X}$ be a topological space and $J : \mathscr{X} \to \overline{\mathbb{R}}$ a functional. We define the* ***epigraph:***

$$\operatorname{epi}(J) = \left\{(f, j) \in \mathscr{X} \times \overline{\mathbb{R}} \mid j \geqslant J(f)\right\}$$

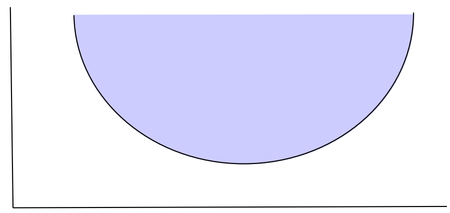


Figure 4.8

Remark 4.10. *A functional $J : \mathscr{X} \to \overline{\mathbb{R}}$ is lower semi-continuous if and only if* $\operatorname{epi}(J)$ *is closed in $\mathscr{X} \times \overline{\mathbb{R}}$.*

Definition 4.11 (Kuratowski limits). *Let $(\mathscr{X}, d)$ be a metric space and $\{A_k\}_{k=1}^{\infty}$ a sequence of sets. We define the* ***Kuratowski limit supremum:***

$$\text{K-}\limsup_{k\to\infty} A_k = \left\{x \in \mathscr{X} \;\middle|\; \liminf_{k\to\infty} d(x, A_n) = 0\right\}$$

and, respectively, the ***limit infimum:***

$$\text{K-}\liminf_{k\to\infty} A_k = \left\{x \in \mathscr{X} \;\middle|\; \limsup_{k\to\infty} d(x, A_n) = 0\right\}$$

If these two sets coincide, we say that the ***Kuratowski limit*** *exists and we write:*

$$\text{K-}\lim_{k\to\infty} A_k = \text{K-}\limsup_{k\to\infty} A_k = \text{K-}\liminf_{k\to\infty} A_k$$

We will see later how Γ−convergence relates to these kind of limits. For the time being, we mention some examples that indicates some geometric image and suggests that Kuratowksi limits are logical.

For the first example, we set:

$$A_k = \{(x, y) \in \mathbb{R}^2 \mid |x|^n + |y|^n < 1\}$$

Then the Kuratowski limit exists and:

$$\text{K-}\lim_{k\to\infty} A_k = B_{\ell^\infty} = \{(x, y) \in \mathbb{R}^2 \mid |x| \leqslant 1,\ |y| \leqslant 1\}$$

For the second example, we set:

$$A_k = \frac{1}{k}\mathbb{Z}^2$$

Then the Kuratowski limit exists and:

$$\text{K-}\lim_{k\to\infty} A_k = \mathbb{R}^2$$

The third and last example is different. Consider $B = B\big((0, 1/4), 1/2\big)$ (The ball of radius 1, translated by 1/4 on the right). By rotating B by $2\pi k/n$ around the origin, we obtain the sets:

$$A_k = e^{-2\pi ki/n} B$$

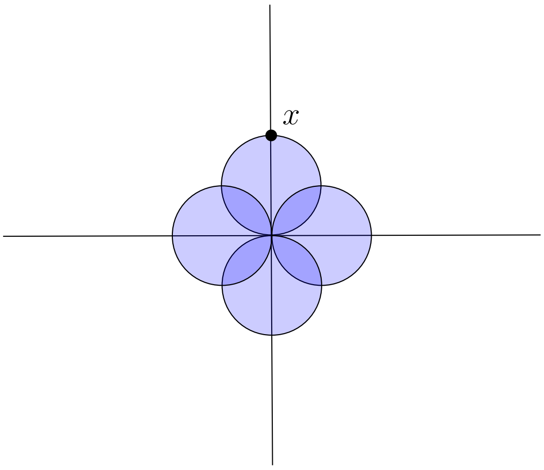
Then:

$$\text{K-}\limsup_{k\to\infty} A_k = \bigcup_{k=1}^{n} A_k$$

and:

$$\text{K-}\liminf_{k\to\infty} A_k = \{(0, 0)\}$$

An image for $n = 4$ can be found below. In what follows we examine, for the sake of simplicity, the motion of $x = (0, 1)$.



The various distances $d(x, A_k)$ become, as k varies:

$$1, \alpha, 0, \alpha, 1, \alpha, 0, \alpha, 1, \alpha, 0, \alpha, 1, \cdots$$

(where $\alpha < 1$). Therefore x belongs to $\text{K-}\limsup A_k$ but not to $\text{K-}\liminf A_k$.

Proposition 4.12. *Let $(\mathscr{X}, d)$ be a metric space and $J_k : \mathscr{X} \to \overline{\mathbb{R}}$, $J : \mathscr{X} \to \overline{\mathbb{R}}$. Then $J_k \xrightarrow{\Gamma} J$ if and only if:*

$$\text{K-}\lim \operatorname{epi}(J_k) = \operatorname{epi}(J)$$

Proposition 4.13. *Let $(\mathscr{X}, d)$ be a metric space and $J_k : \mathscr{X} \to \overline{\mathbb{R}}$, $J : \mathscr{X} \to \overline{\mathbb{R}}$. If $J_k \xrightarrow{\Gamma} J$, then J is lower semi-continuous.*

Proof. If $J_k \xrightarrow{\Gamma} J$, then:

$$\text{K-}\lim \text{epi}(J_k) = \text{epi}(J)$$

But then Kuratowski limits are always closes, which means that epi(J) is closed. In turn, J is lower semi-continuous. □

4.3 Γ-convergence of the energy to the perimeter

The relation of the study of phase transition problems with the theory of minimal surfaces was examined by Modica [9], based on some work of Modica and Mortola [31], ten years ago. In what follows we mention the basic concepts and the main theorem. Later, we also mention a similar result, established by Pacard and Ritoré [11], and later improved by Pacard [10].

First, we define the notion of bounded variation in our context, in order to define the perimeter functional next.

In real-valued scalar functions, the variation measures traverse in $y-$axes, that is:

$$\int \left|\frac{df}{dx}\right| \, dx$$

The general case can be treated similarly, by:

$$\int_\Omega |\nabla f| \, d\sigma$$

We can be more precise, by giving the following definition.

Definition 4.14 (Functions of bounded variation). *Let $(\mathscr{M}, g)$ be a Riemannian manifold and $\Omega \subseteq \mathscr{M}$ an open subset. Let $u \in L^1(\Omega)$ whose gradient can be represented by a $T\mathscr{M}-$Radon measure, in the following sense: There exists a $T\mathscr{M}-$Radon measure, denoted by Du, such that for every $X \in C_c^\infty(\Omega; T\mathscr{M})$:*

$$\int_\Omega \langle X, Du\rangle = -\int_\Omega u\nabla\bullet X \, d\sigma$$

As far as $\langle X, Du\rangle$ is concerned, one can say that there exist some Radon measures $d\mu_k$ such that:

$$Du = \sum_k d\mu_k E_k$$

and then:

$$\int_\Omega \langle X, Du\rangle = \sum_k \int_\Omega X^k d\mu_k$$

If the total variaton:

$$\int_\Omega |Du| = \sup\left\{\int_\Omega u\nabla\bullet X \, d\sigma \;\middle|\; X \in C_c^\infty(\Omega; T\mathscr{M}),\ ||X||_{L^\infty} \leqslant 1\right\}$$

... *is finite, we say that u is of bounded variation (the integral notation is not random). The set of all functions of bounded variation is denoted by* $\mathrm{BV}(\Omega)$*. The norm of this space is:*

$$||u||_{\mathrm{BV}(\Omega)} = ||u||_{L^1(\Omega)} + \int_\Omega |Du|$$

Note that $|Du|$:

$$\int_V |Du| = \sup \left\{ \int_V u \nabla \bullet X \ d\sigma \ \middle| \ X \in C_c^\infty(V; T\mathscr{M}), \ ||X||_{L^\infty} \leqslant 1 \right\}, \ V \subseteq \Omega$$

is a Radon measure, which justifies the integral notation.

The general idea behind this definition is as follows: Suppose we have some disc whose perimeter must be measured. By taking a cross-section, the resulting set is an iterval and the "perimeter" is some interval:

$$\int_{\mathbb{R}} \delta_1 + \delta_2 \ dx$$

(since the boudary consist of two points). Taking all cross-sections into account, one obtains a continuum of Dirac "functions" along the perimeter of the disc.

$$\int_{\mathbb{R}} \delta_{\partial B_1(0)} \ dA$$

This Dirac continuum, as in the one-dimensional case, can be achieved by differentiating the characteristic function of the disc, in the sense of distributions.

Definition 4.15 (Perimeter and Cacciopoli sets)**.** *Let* E *be a borel subset of a Riemannian manifold* $(\mathscr{M}, g)$*. We define the perimeter of* E*, when* $\mathbb{1}_E$ *has gradient viewed as a* $T\mathscr{M}-$*Radon measure, as:*

$$\mathrm{Per}(E;\Omega) = \int_\Omega |D\mathbb{1}_E|$$

We say that E *is a Cacciopoli set if* $\mathrm{Per}(E;\Omega) < \infty$.

It is important to clarify that the Hausdorff measure $\mathscr{H}^{n-1}$ of the boundary ∂E is -in general- not the same as the perimeter Per of E.

We restrict ourselves in the case $\Omega = \mathbb{R}^n$. The **reduced boundary** $\partial^* E$ of E is the set of all points x for which:

$$\lim_{r \searrow 0} \frac{D\mathbb{1}_E\big(B_r(x)\big)}{|D\mathbb{1}_E|\big(B_r(x)\big)} \in \mathbb{R}^n$$

has modulus 1. In fact we have $\partial^* E \subseteq \partial E$. The following remark shows that the perimeter of E is more related to the reduced boundary $\partial^* E$, rather than ∂E.

Remark 4.16 (De Giorgi's theorem). *Let $E \subseteq \mathbb{R}^n$ be a Cacciopoli set. It holds that:*

$$\mathrm{Per}(E; \mathbb{R}^n) = \mathscr{H}^{n-1}(\partial^* E)$$

Other that this theorem above, the equality of perimeter and Hausdorff measure can be achieved under weak assumptions on the regularity of the boundary (cf. [28]). One known condition is that of the Lipschitz boundary.

We are now able to state the theorem of Modica, concerning $\Gamma-$convergence (cf. [9] and [5]).

Theorem 4.17 (Modica). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipshitz Cacciopoli set. Consider the family of functionals $\mathscr{E}_\varepsilon$ of the energies with interfacial constant ε (as in the first chapter):*

$$\mathscr{E}_\varepsilon(u; \Omega) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u)\, dx$$

Potential W must be a two-well potential with two minima at ± 1, where $W(\pm 1) = 0$ and $W > 0$ everywhere else. Then $\mathscr{E}_\varepsilon$ $\Gamma-$converge, with respect to the $L^1(\Omega)-$topology, to the weighted perimeter functional $\sigma \mathrm{Per}(\{\Diamond = 1\}; \Omega)$ (where $\sigma = \int_{-1}^{1} \sqrt{2W(x)}\, dx$, constant).

$$\mathscr{E}_\varepsilon(\Diamond; \Omega) \xrightarrow{\Gamma} \mathrm{Per}(\{\Diamond = 1\}; \Omega)$$

This is a theorem that is proven using almost exclusively methods of the calculus of variations and analysis. There is another similar theorem to this one, which is proven for Riemannian manifolds and uses techiques from differential geometry and differential equations. First proven by Parard and Ritoré [11], its proof was significantly improved some years later by Pacard [10] (using ideas that emmerged in the meantime).

Theorem 4.18 (Pacard-Ritoré). *Suppose $(\mathscr{M}, g)$ is a compact Riemannian manifold of dimension $n+1$, without boundary. Let $\Sigma \subseteq \mathscr{M}$ be a non-degenerate, oriented minimal surface of dimension n, which seperates the manifold $\mathscr{M} \backslash \Sigma = \mathscr{M}^+ \cup \mathscr{M}^-$. The normal vector that indicates the orientation of Σ points towards $\mathscr{M}^+$. Then, there exists some $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a solution u_ε of the $\varepsilon-$problem (of the Allen-Cahn equation) such that:*

$$u_\varepsilon \to \mathbb{1}_{\mathscr{M}^+} - \mathbb{1}_{\mathscr{M}^-}$$

uniformly in compact subsets of $\mathscr{M}^+$, $\mathscr{M}^-$, as $\varepsilon \to 0$. Moreover, for the energy we have:

$$\lim_{\substack{\varepsilon \to 0^+ \\ \varepsilon < \varepsilon_0}} \mathscr{E}_\varepsilon(u_\varepsilon) = \frac{1}{\sqrt{2}} \mathscr{H}^n(\Sigma)$$

Appendices

CHAPTER A

Appendix: Analysis

A.1 Differential equations

Theorem A.1 (Extension lemma, [22], Thm. 2.1, pp. 17). *If D is any open set in $\mathbb{R}^{n+1}$, $f : D \to \mathbb{R}^n$ is continuous and bounded on D, then any solution of:*

$$x'(t) = f(t, x)$$

defined on some interval (a, b) is such that $x(a+)$, $x(b-)$ exist. Also, if from now on f is just continuous, there exists a continuation of $x(t)$ to a maximal interval of existence. If (a, b) is maximal, then $(t, x(t))$ tends to the boundary of D as $t \to a$ and $t \to b$.

Theorem A.2 (Composition with Lipschitz, [23], Cor. 4.14, pp. 47). *Suppose X, Y are Banach spaces and $\Omega \subseteq \mathbb{R}^n$ is open. Let $1 < p \leqslant \infty$, $u \in W^{1,p}(\Omega; X)$ and $F : X \to Y$ be Lipschitz. If Y has the Radon-Nikodym property, then $F \circ u \in W^{1,p}(\Omega; Y)$. In particular, $||u|| \in W^{1,p}(\Omega; \mathbb{R})$*

Remark A.3. *All Euclidean spaces X have the Radon-Nikodym property, which can be stated in these two equivalent ways:*

i. For any $\sigma-$finite, complete measure space $(\Omega, \mathscr{A}, \mu)$ the following holds: For any vector measure $\nu : \mathscr{A} \to X$ with bounded variation that is absolutely continuous with respect to μ, there exists a function $f \in L^1(\Omega, X, \mu)$ such that:

$$\nu(A) = \int_A f \; d\mu, \quad \textit{for all } A \in \mathscr{A}$$

ii. Every Lipschitz function $f : I \to X$ (I being an interval) is differentiable almost everywhere.

Theorem A.4 (Sobolev embeddings, [20], Thm. 8.8, pp. 212). *Let I be an interval. There exists a constant $C = C(\mathscr{L}^1(I))$ such that:*

$$||u||_{L^\infty(I)} \leqslant C||u||_{W^{1,p}(I)},\ u \in W^{1,p}(I),\ 1 \leqslant p \leqslant \infty$$

If I is bounded:

i. The injection $W^{1,p}(I) \hookrightarrow C(\overline{I})$ is compact for all $1 < p \leqslant \infty$.

ii. The injection $W^{1,1}(I) \hookrightarrow L^q(I)$ is compact for all $1 \leqslant q < \infty$.

Theorem A.5 (General Sobolev inequalities, [21], Chpt. 5, Thm. 6, pp. 284). *Let U be a bounded open subset of $\mathbb{R}^n$, with C^1-boundary. Assume $u \in W^{k,p}(U)$.*

i. If $n > kp$, then $u \in L^q(U)$, where:

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

Moreover:

$$||u||_{L^q(U)} \leqslant C(k,p,n)||u||_{W^{k,p}(U)}$$

ii. If $n < kp$, then $u \in C^{k-\lfloor n/p \rfloor -1,\gamma}(\overline{U})$, where:

$$\gamma = \begin{cases} \left\lfloor \dfrac{n}{p} \right\rfloor + 1 - \dfrac{n}{p}, & \textit{if } \dfrac{n}{p} \notin \mathbb{N} \\ < 1, & \textit{if } \dfrac{n}{p} \in \mathbb{N} \end{cases}$$

Moreover:

$$||u||_{C^{k-\lfloor n/p \rfloor -1,\gamma}(\overline{U})} \leqslant C(k,p,n,\gamma)||u||_{W^{k,p}(U)}$$

Remark A.6 (A generalisation of Theorem A.5, [19], Theorem 4.12, pp. 85). *A generalisation of Theorem A.5 exists, for sets satisfying some strong local Lipschitz condition.*

Theorem A.7 (Poincaré-Wirtinger inequality and Poincaré inequality). *Assume $1 \leqslant p < \infty$, let $\Omega \subseteq \mathbb{R}^n$ be bounded, connected and open, with Lipschitz boundary. Then there exists a constant $C = C(\Omega, p)$ such that if $u \in W^{1,p}(\Omega)$ and $u_\Omega = ⨍_\Omega u(y)\,dy$:*

...

$$||u - u_\Omega||_{L^p(\Omega)} \leqslant C||\nabla u||_{L^p(\Omega)}$$

Moreover, if $u \in W_0^{1,p}(\Omega)$*, then:*

$$||u||_{L^p(\Omega)} \leqslant C||\nabla u||_{L^p(\Omega)}$$

Definition A.8 (Parabolic Hölder spaces, [25], Sect. 2.1)**.** *Consider the parabolic metric* $|p| = |t|^{1/2} + |x|$, $p = (t, x)$*. For a function* $u : Q \to \mathbb{R}$*, defined on some parabolic cylinder, we define the* $(\delta/2, \delta)-$*seminorm:*

$$[u]_{\delta/2,\delta;Q} = \sup_{p \neq q \in Q} \frac{|u(p) - u(q)|}{|p - q|^\delta}$$

The $(\delta/2, \delta)-$*Hölder norm is defined as:*

$$||u||_{C^{\delta/2,\delta}(Q)} = ||u||_{L^\infty(Q)} + [u]_{\delta/2,\delta;Q}$$

Moreover, the $(1 + \delta/2, 2 + \delta)-$*seminorm is defined as:*

$$[u]_{1+\delta/2,2+\delta;Q} = [\partial_t u]_{\delta/2,\delta;Q} + \sum_{i,j} [\partial_{i,j} u]_{\delta/2,\delta;Q}$$

and, similarly, the $(1 + \delta/2, 2 + \delta)-$*Hölder norm becomes:*

$$||u||_{C^{1+\delta/2,2+\delta}(Q)} = ||u||_{L^\infty(Q)} + ||\partial_t u||_{L^\infty(Q)} + \sum_i ||\partial_i u||_{L^\infty(Q)} +$$

$$+ \sum_{i,j} ||\partial_{i,j} u||_{L^\infty(Q)} + [u]_{1+\delta/2,2+\delta;Q}$$

Another equivalent definition is that of:

$$[[u]]_{\delta/2,\delta;Q} = \sup_{(t,x) \neq (s,x) \in Q} \frac{|u(t,x) - u(t,x)|}{|t - s|^{\delta/2}} + \sup_{(t,x) \neq (t,y) \in Q} \frac{|u(t,x) - u(t,y)|}{|x - y|^\delta}$$

Theorem A.9 (Ladyženskaja-Solonnikov-Ural'ceva [24] and [25] - Combined results)**.** *Suppose* $u \in C_{t,x}^{1+\delta/2,2+\delta}(Q)$ *is a solution to a uniformly parabolic partial differential equation:*

...

$$\begin{cases} \partial_t u + Lu = 0 \\ u(0,x) \in C_x^{2+\delta}([a,b]) \end{cases}$$

where $Q = [a,b] \times [0,T)$ *is a parabolic cylinder. For every other parabolic cylinder* $\widetilde{Q} \Subset Q$ *we have estimates of the form:*

$$||u||_{C_{t,x}^{1+\delta/2,2+\delta}(\widetilde{Q})} \leqslant C(\Lambda, \delta, d)||u||_{L^\infty(Q)}, \ \delta \in (0,1)$$

Here Λ *is the uniform parabolicity constant and* $d = \operatorname{dist}(Q, \widetilde{Q})$*. It is also true that, as far as d is concerned,* $C = O(d^{-2-\delta})$.

A.2 Functional analysis

Theorem A.10 ([20], Thm. 3.7, pp. 60)**.** *Let* X *be a Banach space and* $A \subseteq X$*. It is not nessecerily true that every strongly closed set is weakly closed, but what is true is that if* A *is convex:*

$$A \text{ is strongly closed } \Leftrightarrow A \text{ is weakly closed}$$

Theorem A.11 (Lax-Milgram)**.** *Let* H *be Hilbert space and* $B : H \times H \to \mathbb{R}$ *a bilinear form. Suppose:*

i. There exists constant $C > 0$ *such that:*

$$|B(u,v)| \leqslant C||u|| \cdot ||v||$$

ii. There exists constant $c > 0$ *such that:*

$$c||u||^2 \leqslant B(u,u)$$

Let $F : H \to \mathbb{R}$ *be a bounded functional. Then there exists a unique* $u \in H$ *so that:*

$$B(u,v) = \langle F, v \rangle, \ \forall v \in H$$

CHAPTER B

Appendix: Geometry

B.1 Calculations

We remind the reader that ∇v means either the gradient of v, if v is scalar, or the differential (Jacobian), if v is vector valued $v : \mathbb{R}^n \to \mathbb{R}^n$. In the latter case, the norm $|\nabla v|$ that appears when calculating energies is the Frobenius norm, which in general in is defined as:

$$||A||_F = \left(\sum_{i,j} A_{i,j}^2\right)^{1/2} \quad \text{for } A \in \mathbb{R}^{n,n}$$

This norm is induced by the Frobenius inner product:

$$\langle A, B\rangle_F = \sum_{i,j} A_{i,j}^\top B_{i,j}, \; A, B \in \mathbb{R}^{n,n}$$

Proposition B.1 (Spherical form of the gradient). *Suppose $v \in H^1_{\text{rad}}\big(B_R(0);\mathbb{R}^n\big)$, that is, it is differentiable in the $(1,2)-$Sobolev sense and also $v(rs) = \rho(r)s$ for all $s \in \mathbb{S}^{n-1}$. Then we have:*

$$\nabla v = \rho'(r) \cdot s \otimes s + \frac{\rho(r)}{r}(\text{Id} - s \otimes s)$$

and:

$$|\nabla v|^2 = \big(\rho'(r)\big)^2 + \frac{n-1}{r^2}\rho^2(r)$$

Here ρ is the radial part, depending on $r = |x|$, with $s = x/|x|$ we denote the angular part and $\otimes$ is the tensor product. It can be viewed as the matrix $s \otimes s = (s_i s_j)_{i,j}$.

Proof. It is immediate that:

$$\nabla v = \nabla \rho \otimes s + \rho \otimes \nabla s$$

But since:

$$\nabla \rho = \rho' \nabla r = \rho' s \text{ and } \nabla s = \nabla \frac{x}{r} = \frac{1}{r}(\text{Id} - s \otimes s)$$

we obtain:

$$\nabla v = \rho'(r) \cdot s \otimes s + \frac{\rho(r)}{r}(\mathrm{Id} - s \otimes s)$$

As for the modulus, notice the following: First, $||s \otimes s||_F = 1$, since:

$$\sum_{i,j} s_i^2 s_j^2 = \sum_i s_i^2 \sum_j s_j^2 = |s|^4 = 1$$

Second, $s \otimes s$ and $\mathrm{Id} - s \otimes s$ are orthogonal in the Frobenius inner product:

$$\langle s \otimes s, \mathrm{Id} - s \otimes s \rangle_F = \sum_{i,j} s_i s_j (\delta_{i,j} - s_i s_j) = \sum_i s_i^2 - \sum_{i,j} s_i^2 s_j^2 = 0$$

Third, the norm of $\mathrm{Id} - s \otimes s$ is:

$$||\mathrm{Id} - s \otimes s||_F^2 = \sum_{i,j} (\delta_{i,j} - s_i s_j)^2 = \sum_{i,j} \delta_{i,j}^2 - 2\delta_{i,j} s_i s_j + s_i^2 s_j^2 = n - 2 + 1 = n - 1$$

Gathering all of these, we obtain the desired result:

$$|\nabla v|^2 = \left(\rho'(r)\right)^2 + \frac{n-1}{r^2}\rho^2(r)$$

□

B.2 The principle of symmetric criticality

There was a general belief in physics, that to find the critical points of a functional $F : \mathscr{M} \to \mathbb{R}$, one needed just to study F not in the whole space, but just in all symmetric points. That is, to find all points such that $(F_*)_p = 0$, just find the critical points of $F|_S$, where $S = \{p \in \mathscr{M} \mid gp = p,\ \forall g \in G\}$, for some group of symmetries G acting on $\mathscr{M}$. In this generality, this naive principle does not hold. It does hold, however, for certain cases of symmetry groups G and spaces $\mathscr{M}$.

From now on, $\mathscr{M}$ will be a Riemann-Hilbert manifold, that is a manifold that is locally diffeomorphic to a finite or even infinite dimensional Hilbert space, and there exists an inner product $\langle \cdot, \cdot \rangle_p$ on each tangent space $T_p\mathscr{M}$, varying smoothly with p in the tangent bundle $T\mathscr{M}$. Also, G will be a group acting by diffeomorphisms on $\mathscr{M}$ and a group of isometries $G \leq \mathrm{Isom}(\mathscr{M})$. The last definition means that the pushforward $(g_*)_p : T_p\mathscr{M} \to T_{gp}\mathscr{M}$ preserves inner products.

The main theorem is the following (cf. [32]):

Theorem B.2 (The principle of symmetric criticality). *Let G and $\mathscr{M}$ be as before, and $F : \mathscr{M} \to \mathbb{R}$. Then the set of symmetric points:*

$$S = \{p \in \mathscr{M} \mid gp = p,\ \forall g \in G\}$$

is a totally geodesic smooth submanifold of $\mathscr{M}$, and if $p \in S$ is a critical point of $F|_S$, then it is a critical point of F too.

Proof. It is a general fact that $\exp(g_*)_p = g \circ \exp$, so the representation $g^{\exp} = \exp^{-1} \circ g \circ \exp$ is linear and orthogonal. In particular, S intersects the domain of geodesic normal coordinates at p in a linear subspace, which is:

$$\{v \in T_p\mathscr{M} \mid (g_*)_p(v) = v,\ \forall g \in G\}$$

It is therefore clear that S is a smooth submanifold of $\mathscr{M}$. This also shows that any geodesic with tangent v at p is left pointwise fixed by all $g \in G$.

Now, by assumption, $(\nabla F)_p$ is orthogonal to T_pS and if we show $(\nabla F)_p \in T_pS$, we will conclude the proof. Why $(\nabla F)_p$ is orthogonal to S, is a consequence of the definitions. Remember that in general:

$$(F_*)_p(v) = \langle v, (\nabla F)_p\rangle_p, \text{ for all } v \in T_p\mathscr{M}$$

(in our case, the manifold is S). Then observe that $(g_*)_p(\nabla F)_p = (\nabla F)_{gp} = (\nabla F)_p$ for all $g \in G$, and therefore the geodesic emanating from p in the direction $(\nabla F)_p$ is pointwise fixed under all $g \in G$. The formula before is a consequence of the chain rule for $G-$invariant F, $(F_*)_p = (F \circ g)_{*p} = (F_*)_{gp} \circ (g_*)_p$, and the fact that $(g_*)_p$ maps $T_p\mathscr{M}$ onto $T_{gp}\mathscr{M}$ isometrically: Indeed:

$$(F_*)_p(v) = \langle v, (\nabla F)_p\rangle_p \text{ and } (F_*)_{gp} \circ (g_*)_p(v) = \langle (g_*)(v), (\nabla F)_{gp}\rangle_p$$

hence $(g_*)_p(\nabla F)_p = (\nabla F)_{gp}$.

In conclusion, the geodesic lies inside S and $(\nabla F)_p \in T_pS$. $\square$

Some counterexamples for the completely general principle can be found in [32].

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