# Geometric Calculus

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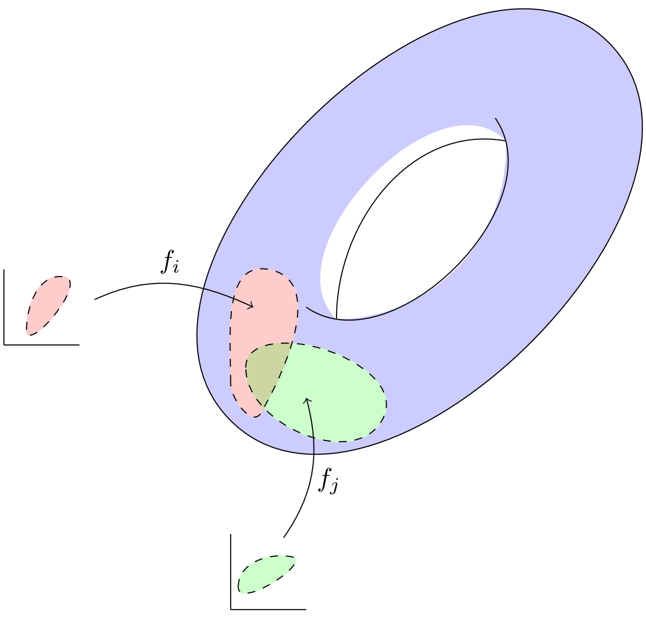


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### 1.3.1 Isometric transformations of $\mathbb{R}^n$

We already saw (in [...]) the definition of isometries in metric spaces. Especially in spaces with inner product, those isometries have a very strong geometric "image". We will show that isometries in spaces with inner products are some combination of translations, reflections and rotations.

**Definition 1.3.1** (Linear functions). *Let $X, Y$ be two $\mathbb{R}-$vector spaces. We say a function $f : X \to Y$ is linear if for every $\lambda \in \mathbb{R}$ and $x_1, x_2 \in X$:*

$$f(x_1 + \lambda x_2) = f(x_1) + \lambda f(x_2)$$

*We use the notation:*

$$L(X \to Y) := \{f : X \to Y \mid f \text{ is linear}\}$$

**Definition 1.3.2** (Orthogonal functions). *Let $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ be two spaces with inner product. We say a function $\tau : X \to Y$ is orthogonal if for every $x_1, x_2 \in X$:*

$$\langle x_1, x_2\rangle_X = \langle \tau(x_1), \tau(x_2)\rangle_Y$$

*We use the notation:*

$$O(X \to Y) := \{\tau : X \to Y \mid \tau \text{ is orthogonal}\}$$

**Corollary 1.3.1.** *From Definition 1.3.2 follows that a function $\tau : X \to Y$ is orthogonal if and only if it preserves the forms of the inner products.*

$$\tau \in O(X \to Y) \Leftrightarrow T_X = T_Y \circ \tau$$

**Lemma 1.3.1.** *Let $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ be two spaces with inner product. Every $\tau \in O(X \to Y)$ is linear, thus $O(X \to Y) \subseteq L(X \to Y)$.*

*Proof.* Let $\tau \in O(X \to Y)$. We need to show that for every $\lambda \in \mathbb{R}$ and $x_1, x_2 \in X$ we have:

$$\tau(x_1 + \lambda x_2) = \tau(x_1) + \lambda\tau(x_2) \tag{1.1}$$

Instead, we will show that:

$$T_Y\big(\tau(x_1 + \lambda x_2) - \tau(x_1) - \lambda\tau(x_2)\big) = 0 \tag{1.2}$$

which equivalent to (1.1). $T_Y$ is the form of the inner product $\langle \cdot, \cdot \rangle_Y$ of $Y$.

With many calculations to the left side of (1.2) we will find $0$. Indicatively, we write:

$$\big\langle \tau(x_1 + \lambda x_2) - \tau(x_1) - \lambda\tau(x_2),\ \tau(x_1 + \lambda x_2) - \tau(x_1) - \lambda\tau(x_2)\big\rangle_Y =$$
$$\big\langle \tau(x_1 + \lambda x_2),\ \tau(x_1 + \lambda x_2)\big\rangle_Y - 2\big\langle \tau(x_1 + \lambda x_2),\ \tau(x_1)\big\rangle_Y + \big\langle \tau(x_1),\ \tau(x_1)\big\rangle_Y + \lambda^2\big\langle \tau(x_2),\ \tau(x_2)\big\rangle_Y$$

and by the orthogonality of $\tau$ we get:

$$\langle x_1 + \lambda x_2,\ x_1 + \lambda x_2\rangle_X - 2\langle x_1 + \lambda x_2,\ x_1\rangle_X + \langle x_1, x_1\rangle_X + \lambda^2\langle x_2, x_2\rangle_X$$

which is:

$$\langle x_1 + \lambda x_2 - x_1 - \lambda x_2,\ x_1 + \lambda x_2 - x_1 - \lambda x_2\rangle_X = \langle 0_X, 0_X\rangle_X = 0$$

□

**Lemma 1.3.2.** *Let $(X, \langle \cdot, \cdot \rangle_X)$, $(Y, \langle \cdot, \cdot \rangle_Y)$ be two spaces with inner product. Linear isometries ($\sigma \in I(X \to Y) \cap L(X \to Y)$) are exactly those isometries $\sigma$ for which $\sigma(0_X) = 0_Y$.*

$$\sigma \in I(X \to Y) \cap L(X \to Y) \Leftrightarrow \sigma \in I(X \to Y) \text{ and } \sigma(0_X) = 0_Y$$

*Proof.* The direction ($\Rightarrow$) holds. For the other direction ($\Leftarrow$), we notice that, for every $x_1, x_2 \in X$:

$$\sigma \in I(X \to Y) \Rightarrow ||x_1 - x_2||_X = ||\sigma(x_1 - x_2)||_Y \overset{\star}{\Rightarrow} T_X(x_1 - x_2) = T_Y\big(\sigma(x_1) - \sigma(x_2)\big)$$

where $T_X$, $T_Y$ are the forms of the inner products in $X$ and $Y$ respectively, and $||\cdot||_X$, $||\cdot||_Y$ are their induced normes. In implication ($\star$) the linearity of $\sigma$ is used. Moreover, if we set $x_1 = 0_X$ or $x_2 = 0_X$:

$$T_X(x_1) = T_Y\big(\sigma(x_1)\big) \text{ and } T_X(x_2) = T_Y\big(\sigma(x_2)\big) \tag{1.3}$$

Now we write:

$$T_X(x_1 - x_2) = T_Y\big(\sigma(x_1) - \sigma(x_2)\big) \Leftrightarrow \langle x_1 - x_2, x_1 - x_2\rangle_X = \big\langle \sigma(x_1) - \sigma(x_2),\ \sigma(x_1) - \sigma(x_2)\big\rangle_Y \Leftrightarrow$$
$$\Leftrightarrow \langle x_1, x_1\rangle_X - 2\langle x_1, x_2\rangle_X + \langle x_2, x_2\rangle_X = \big\langle \sigma(x_1), \sigma(x_1)\big\rangle_Y - 2\big\langle \sigma(x_1), \sigma(x_2)\big\rangle_Y + \big\langle \sigma(x_2), \sigma(x_2)\big\rangle_Y$$

and from (1.3) we get:

$$\langle x_1, x_2\rangle_X = \big\langle \sigma(x_1), \sigma(x_2)\big\rangle_Y$$

which shows that $\sigma \in O(X \to Y)$. Using Lemma 1.3.1 we prove that $\sigma \in L(X \to Y)$. □

**Theorem 1.3.1** (Structure theorem). *Let $(X, \langle \cdot, \cdot \rangle_X)$, $(Y, \langle \cdot, \cdot \rangle_Y)$ be two spaces with inner product. Then:*

$$I(X \to Y) \cap L(X \to Y) = O(X \to Y)$$

*Proof.* While proving Lemma 1.3.2 we showed that every $\sigma \in L(X \to Y)$ with $\sigma(0) = 0$ is orthogonal. So, using Lemma 1.3.2, we have shown that $I(X \to Y) \cap L(X \to Y) \subseteq O(X \to Y)$. It remains for inclusion "$\supseteq$" to be shown.

For this we will now prove that every orthogonal function $\tau$ is isometry with $\tau(0_X) = 0_Y$ (and we will use, once more, Lemma 1.3.2). So let $\tau \in O(X \to Y)$. Because for every $x_1, x_2 \in X$, $\langle x_1 - x_2\rangle_X = \big\langle \tau(x_1 - x_2)\big\rangle_Y$, the norms are equal:

$$||x_1 - x_2||_X = ||\tau(x_1 - x_2)||_Y$$

(which are induced by our inner products). By the linearity of $\tau$ (which has been proven in Lemma 1.3.2) an equality of metrics follows:

$$||x_1 - x_2||_X = ||\tau(x_1) - \tau(x_2)||_Y$$

which shows that $\tau$ is isometry. By letting $x_1 = x_2$ we also see that $\tau(0_X) = 0_Y$, and this concludes the proof. □

**Theorem 1.3.2** (Isometries as compositions of orthogonal functions and translations)**.** *Let* $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ *be two spaces with inner products and* $\sigma \in I(X \to Y)$. *Then* $\sigma$ *can be written in a unique way as a composition of an orthogonal function and a translation:*

$$\sigma = \mu \circ \tau$$

*Here* $\tau \in O(X \to Y)$ *and* $\mu : Y \to Y$, $\mu(y) = y + a$ *(for some* $a \in Y$*).*

*Proof.* First, notice that every translation is an isometry -so function $\tau(x) = \sigma(x) - \sigma(0_X)$ is an isometry, and especially $\tau(0_X) = 0_Y$. From Lemma 1.3.2 we have that $\tau \in I(X \to Y) \cap L(X \to Y)$, and then, from Theorem 1.3.1, $\tau \in O(X \to Y)$. If we now consider the translation $\mu(y) = y + \sigma(0_X)$, our previous analysis shows that $\mu \circ \tau = \sigma$.

We will prove the "unique" part of the theorem too. If we suppose that there exists some $\tau^* \in O(X \to Y)$ and translation $\mu^* : Y \to Y$, $\mu^*(y) = y + a^*$ such that $\sigma = \mu \circ \tau = \mu^* \circ \tau^*$, then -because $\tau$, $\tau^*$ are linear (Lemma 1.3.1):

$$\sigma(0_X) = \mu(0_Y) = \mu^*(0_Y) \Rightarrow a = a^*$$

From this follows that $\mu = \mu^*$, and as an extension:

$$\sigma(x) = \tau(x) + a = \tau^*(x) + a \Rightarrow \tau = \tau^*$$

The proof is complete. □

**Lemma 1.3.3.** *Let* $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ *be two spaces of the same finite dimension, with inner products. Then every* $\tau \in O(X \to Y)$ *is a linear isomorphism.*

*Proof.* In Theorem 1.3.1 we saw that $O(X \to Y) = I(X \to Y) \cap L(X \to Y)$, so every $\tau \in O(X \to Y)$ is linear and isometric. Because it is isometric, we have also shown that is injective. Now, using the kernel-image dimension theorem -because of the same finite dimension of spaces $X$, $Y$- it follows that $\tau$ is surjective too. □

**Theorem 1.3.3** (Isometries of $\mathbb{R}^2$)**.** *Let* $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ *be the Euclidean plane with the usual inner product. Every* $\sigma \in I(\mathbb{R}^2 \to \mathbb{R}^2)$ *can be written as:*

$$\sigma\big((x,y)\big)^T = \begin{cases} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \quad or \\ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{cases}$$

*where* $\alpha^2 + \beta^2 = 1$ *and* $\gamma, \delta \in \mathbb{R}$.

*Proof.* In Theorem 1.3.2 we proved that every $\sigma \in I(\mathbb{R}^2 \to \mathbb{R}^2)$ can be written in a unique way as a composition of an orthogonal function and a translation.

$$\sigma = \mu \circ \tau, \quad \text{where } \tau \in O(\mathbb{R}^2 \to \mathbb{R}^2) \text{ and } \mu : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a translation}$$

So first we will examine the orthogonal group $O(\mathbb{R}^2 \to \mathbb{R}^2)$. In Lemma 1.3.3 we saw that every $\tau \in O(\mathbb{R}^2 \to \mathbb{R}^2)$ is a linear isomorphism, therefore it is defined by a matrix:

$$M_\tau = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{ with non-zero determinant } a_1a_4 - a_2a_3 \neq 0$$

and $\tau$ can be written as:

$$\tau\big((x,y)\big)^T = M_\tau \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1x + a_2y \\ a_3x + a_4y \end{pmatrix}$$

Using Theorem 1.3.1 one can see that $\tau$ preserves the norms $||\cdot||_{\mathbb{R}^2} = |\cdot|$ (that are induced from their respective inner products) and thus:

$$|(x,y)| = |\tau(x,y)| \Rightarrow x^2 + y^2 = (a_1^2 + a_3^2)x^2 + 2(a_1a_2 + a_3a_4)xy + (a_2^2 + a_4^2)y^2 \Rightarrow \begin{cases} a_1^2 + a_3^2 = 1 \\ a_1a_2 + a_3a_4 = 0 \\ a_2^2 + a_4^2 = 1 \end{cases}$$

This way some restrictions on $a_1$, $a_2$, $a_3$, $a_4$ have been made:

i. $a_1a_4 - a_2a_3 \neq 0$

ii. $a_1^2 + a_3^2 = 1$

iii. $a_1a_2 + a_3a_4 = 0$

iv. $a_2^2 + a_4^2 = 1$

These restrictions will determine the $\alpha$, $\beta$ elements. Lets first assume that $a_1 = 0$. Then from i. and iii. $a_2a_3 \neq 0$ and $a_3a_4 = 0$, which shows that $a_1 = a_4 = 0$ and $a_2, a_3 \neq 0$. Moreover, from i. and iv., $a_2, a_3 \in \{\pm 1\}$, and matrix $M_\tau$ becomes:

$$M_\tau = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

If $a_1 \neq 0$, then from iii. we can write $a_2 = -a_3a_4/a_1$. If we now let $\lambda = a_4/a_1$, we notice that $a_2 = -\lambda a_3$ and $a_4 = \lambda a_1$. From ii. and iv., $a_1^2 + a_3^2 = 1 = \lambda^2(a_1^2 + a_3^2) \Rightarrow \lambda \in \{\pm 1\}$. Depending on the value of $\lambda$:

$$M_\tau = \begin{pmatrix} a_1 & -a_3 \\ a_3 & a_1 \end{pmatrix} \text{ if } \lambda = -1, \text{ and } M_\tau = \begin{pmatrix} a_1 & a_3 \\ a_3 & -a_1 \end{pmatrix} \text{ if } \lambda = -1$$

In any case, there exist $\alpha, \beta \in \mathbb{R}$ such that:

$$M_\tau = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \text{ or } M_\tau = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

with $\alpha^2 + \beta^2 = 1$. To end this proof we consider the translations:

$$\mu\big((x,y)\big)^T = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix}, \ \gamma, \delta \in \mathbb{R}$$

and from Theorem 1.3.2 follows that every $\sigma \in I(\mathbb{R}^2 \to \mathbb{R}^2)$ has the form:

$$\sigma\big((x,y)\big)^T = \begin{cases} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \qquad or \\ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{cases}$$

where $\alpha^2 + \beta^2 = 1$, $\gamma, \delta \in \mathbb{R}$. □

**Corollary 1.3.2.** *The following notation is as in Theorem 1.3.3. Matrix:*

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

*can be written as:*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

*for some angle $\theta$. Also, matrix:*

$$\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

*can be written as:*

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*for some angle $\theta$. So every isometry $\sigma \in I(\mathbb{R}^2 \to \mathbb{R}^2)$ is some combination of rotations, reflections and translations.*

*Proof.* Is immediate from the fact that $a^2 + b^2 = 1$ and Theorem 1.3.3. □

Corollary 1.3.2 shows what we wanted to show, for the two dimensional case: "every isometry is some combination of rotations, reflections and translations". What we will show next is that in every Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ isometries are combinations of rotations, reflections and translations, if rotations are defined correctly.

In the following results $\mathbb{R}^n$ will be considered with its usual inner product. Also, $T$ will be the form of said inner product.

**Lemma 1.3.4.** *Let $\tau \in O(\mathbb{R}^2 \to \mathbb{R}^n)$ and $M_\tau$ its matrix. If $M_\tau$ has a real eigenvalue $\lambda$, then $\lambda \in \{\pm 1\}$.*

*Proof.* Indeed, if $v^T$ is an eigenvector to eigenvalue $\lambda$, then (by the orthogonality of $\tau$):

$$\tau(v) = \lambda v^T \Rightarrow T(\tau(v)) = \lambda^2 T(v) = \lambda^2 T(\tau(v)) \Rightarrow \lambda^2 = 1 \text{ or } \lambda = 0$$

The $\lambda = 0$ case cannot hold: If $\mathscr{X}_{M_\tau}$ is the characteristic polynomial of $M_\tau$, then $\mathscr{X}_{M_\tau}(0) = \det(M_\tau - 0 \cdot \mathrm{Id}_n) = \det M_\tau \neq 0$, so $0$ cannot be an eigenvalue. The inequality holds because of Lemma 1.3.3. □

**Lemma 1.3.5.** *Let $\tau \in O(\mathbb{R}^2 \to \mathbb{R}^n)$ and $M_\tau$ its matrix. If $M_\tau$ has a complex eigenvalue $\lambda = a + bi$ with eigenvector $v^T = x^T + iy^T$, then:*

$$\tau(x) = ax^T - by^T \text{ and } \tau(y) = bx^T + ay^T$$

*Proof.* Because $\lambda$ is an eigenvalue of $M_\tau$ with eigenvector $v$:

$$\tau(v) = \lambda v^T \Rightarrow M_\tau \cdot v^T = \lambda v^T \Rightarrow (M_\tau - \lambda \cdot \mathrm{Id}_n) \cdot v^T = 0 \tag{1.4}$$

Now (1.4) can be seen as an linear system with the coordinates of $v^T$ as unknowns. This system has determinant:

$$\mathscr{X}_{M_\tau}(\lambda) = \det(M_\tau - \lambda \cdot \mathrm{Id}_n) = 0$$

(because $\lambda$ is an eigenvalue), therefore there exists non-zero eigenvector $v^T = x^T + iy^T$. So from (1.4) we have:

$$\big(M_\tau - (a + ib) \cdot \mathrm{Id}_n\big) \cdot (x^T + iy^T) = 0 \Rightarrow$$
$$\big[(M - a \cdot \mathrm{Id}_n) \cdot x^T + b \cdot \mathrm{Id}_n \cdot y^T\big] + i\big[(M - a \cdot \mathrm{Id}_n) \cdot y^T - b \cdot \mathrm{Id}_n \cdot x^T\big] = 0$$

which shows that both real and imaginary parts equal zero. Therefore:

$$\tau(x) = M_\tau \cdot x^T = ax^T - by^T \text{ and } \tau(y) = M_\tau \cdot y^T = bx^T + ay^T$$

□

CHAPTER 2

# Differentiation

## 2.1 Derivatives, directed and partial derivatives

The study of derivatives in many dimensions is, in its core, a problem of finding a “good” definition. The generalisation of single variable derivative in many dimensions faces obstacles that are integrated to the definition of the single variable case, so a “special treatment” is needed. Fist, lets remember the known derivative definition.

**Definition 2.1.1** (Derivative - Single variable)**.** *Let $f : A \to \mathbb{R}$ be a real valued function defined on the open set $A \subseteq \mathbb{R}$, and $x_0 \in A$ be a point of $A$. We say that $f$ is differentiable at $x_0$ if the limit:*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

*exists and is a real number. We also define the derivative of $f$ at $x_0$ be the before-mentioned limit (we use the symbol $f'(x_0)$).*

Of course one usually generalises this definition by noticing that $A$ does not need to be open. The derivative is a local property of functions and, thus, it suffices that $x_0$ is located in an “open region” of $A$, or that it can be reached through $A$.

Now, for the many dimensions, note that the transition cannot be done so easily. Even in $\mathbb{R}^2$, the quotient:

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is not defined, as the usual algebraic structure of $\mathbb{R}^2$ does not have multiplicative inverses. It is possible to enrich the $\mathbb{R}^2$ structure so that division is possible. By writing every $(x, y) \in \mathbb{R}^2$ in its complex form $x + iy$ and using the structure of $\mathbb{C}$, it is possible to define the derivative of $f : \mathbb{C} \to \mathbb{C}$ at $x_0$ using the limit:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \quad \text{where } x_0, h \in \mathbb{C}$$

The study of such differentiable functions $f : \mathbb{C} \to \mathbb{C}$ of the complex plane (of the “holomorphic funtions”) is interesting but it does not give a “good” definition, that can be extended in more dimensions.

One constantly needs extensions of the real numbers (as complex numbers are). In just four dimensions, the extensions of the reals -the quaternions $\mathbb{H}$- have lost some of their structure -they are a division ring and an integer domain, but not a field. So this “dream” of extending the derivative definition this way fails.

Here is where analysis and geometry come. First, a geometer might have given a physical meaning to the derivative, making it possible to extend the definition in curves. So, let $\gamma : I \to \mathbb{R}^n$ be an n-dimensional curve. By writing $\gamma = (\gamma_1, \cdots, \gamma_n)$, where $\gamma_k : I \to \mathbb{R}$, $k \in [n]$ are the “coordinate” functions of $\gamma$, we can say that $\gamma$ has "speed" to a point $t_0 \in I$ if its coordinates have "speed". This notion of speed is widely used in geometry and physics. Then, if for every $k \in [n]$ the limit:

$$\lim_{h\to 0} \frac{\gamma_k(t_0+h)-\gamma_k(t_0)}{h}$$

exists (in the reals), the derivative of $\gamma$ at $t_0$ can be defined as:

$$\gamma'(t) = \left(\lim_{h\to 0} \frac{\gamma_k(t_0+h)-\gamma_k(t_0)}{h}\right)_{k\in[n]}$$

or equivalently:

$$\gamma'(t) = \lim_{h\to 0}\left[\frac{1}{h}\big(\gamma(t_0+h)-\gamma(t_0)\big)\right]$$

This definition in curves also has another geometrically intuitive property:

**Proposition 2.1.1.** *Let $\gamma : I \to \mathbb{R}^n$ be a differentiable curve at $t_0 \in I$. Then the tangent line at $t_0$ is the line:*

$$\varepsilon : \gamma(t_0) + s\cdot\gamma'(t_0), \ \ \text{as } s \in \mathbb{R}$$

*This means that the "limit position" of lines passing through $\gamma(t_0)$, $\gamma(t_0+h)$ (as $h \to 0$) is the before-mentioned line.*

*Furthermore, if $\gamma$ has continuous derivative at $t_0$, the before-mentioned line is also the line that is the limit of lines approximating $\gamma$, which are passing through any two (different) points of $\gamma$ that are "close" to $t_0$ timewise.*

*Proof.* Lets approximate the tangent line by the line passing through $\gamma(t_0)$, $\gamma(t_0+h)$ for sufficiently small $h$. In this proof we will first approximate the tangent with these kind of lines.

The approximation that passes through $\gamma(t_0)$, $\gamma(t_0+h)$ is a line of the form:

$$\gamma(t_0) + s\cdot\big(\gamma(t_0+h)-\gamma(t_0)\big), \ \ \text{where } s \in \mathbb{R}$$

because it is parallel to vector $\gamma(t_0+h)-\gamma(t_0)$. One might want to take limits to find the tangent line, thought this is not possible to do so, as $\gamma(t_0+h)-\gamma(t_0) \to 0_{\mathbb{R}^3}$. So instead we write:

$$\gamma(t_0) + s\cdot\frac{1}{h}\big(\gamma(t_0+h)-\gamma(t_0)\big), \ \ \text{where } s \in \mathbb{R}$$

because it is also parallel to vector $1/h\cdot\gamma(t_0+h)-\gamma(t_0)$. Now, by taking limits as $h \to 0$:

$$\gamma(t_0) + s\cdot\gamma'(t_0), \ \ \text{where } s \in \mathbb{R}$$

We will also see the more general case where the lines pass through $\gamma(t_0-\ell)$, $\gamma(t_0+h)$. We again approximate our tangent by the lines:

$$\gamma(t_0-\ell) + s\cdot\frac{1}{h+\ell}\big(\gamma(t_0+h)-\gamma(t_0-\ell)\big), \ \ \text{where } s \in \mathbb{R}$$

or equivalently:

$$\gamma(t_0-\ell) + s\cdot\left(\frac{\gamma_k(t+h)-\gamma_k(t-\ell)}{h+\ell}\right)_{k\in[n]}$$

Notice that here we cannot just take limits as in our previous case. Instead, we use the mean value theorem (of one variable), finding $\xi_k \in (\min\{t_0-\ell, t_0+h\}, \max\{t_0-\ell, t_0+h\})$ such that:

$$\gamma_k'(\xi_k) = \frac{\gamma_k(t+h)-\gamma_k(t-\ell)}{h+\ell}$$

Our approximation of the tangent line now becomes:

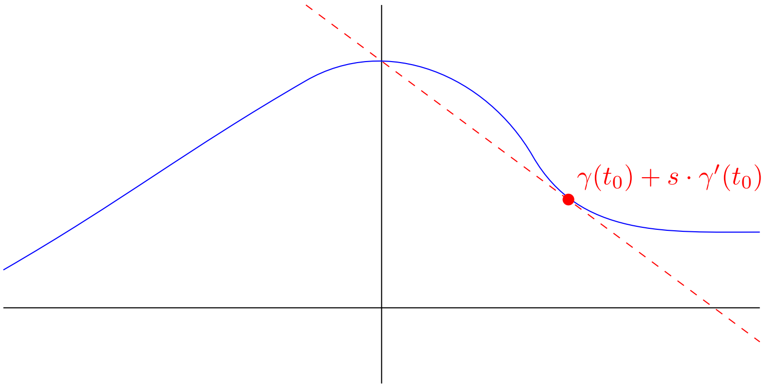
$$\gamma(t_0-\ell) + s\cdot\gamma'(\xi), \ \text{where } s \in \mathbb{R} \text{ and } \xi = (\xi_k)_{k\in[n]}$$

These $\xi$, even if it is not "labeled", depend on $h$, $\ell$ -do not forget that its coordinates $\xi_k$ belong to an interval that depends on $h$, $\ell$. Now we let $h, \ell \to 0$, $h \neq -\ell$ and we get $\xi \to t_0$ (by definition). Then, by the continuity of the derivative, the approximations have the limit:

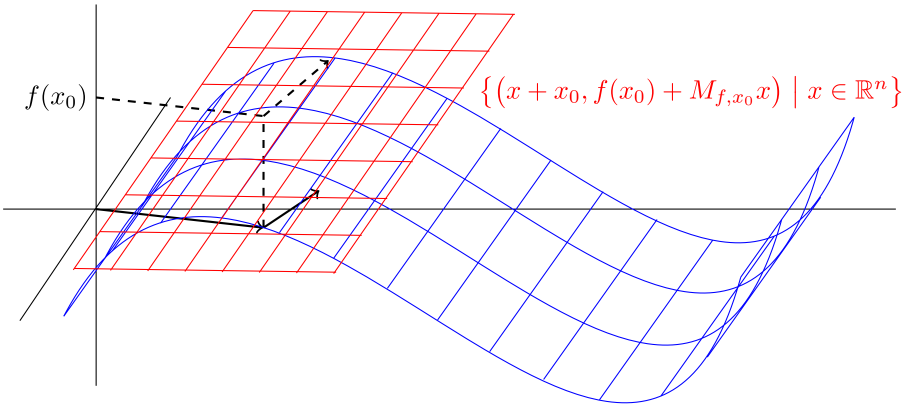
$$\gamma(t_0) + s \cdot \gamma'(t_0), \quad \text{where } s \in \mathbb{R}$$

Thus the tangent line in this case is the before-mentioned line, and the proof is complete. $\square$

This idea of "tangent lines", or generally "tangent shapes", will in fact give our definition of the derivative. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$, which can be visualised as a surface in $\mathbb{R}^3$ (a strange surface, in most cases). If we assume $f$ has some kind of differentiability, it should get approximated by some "multidimensional" lines. These "lines" -in our case- are planes (affine linear spaces).




*Tangent line on a graph of $\gamma(t) = (t, f(t))$ (where $t \in \mathbb{R}$).*




*Tangent plane to a surface $\big(x, f(x)\big)$ (where $x \in \mathbb{R}^2$).*

We give the following definition:

**Definition 2.1.2** (Derivative - Many variables). *Let $f : A \to \mathbb{R}^m$ be a function, $A \subseteq \mathbb{R}^n$ an open set and $x_0 \in A$. We say that $f$ is differentiable at $x_0$ if there exists a linear function $M_{f,x_0}$ such that:*

$$\lim_{h \to 0_{\mathbb{R}^n}} \frac{|f(x_0 + h) - f(x_0) - M_{f,x_0}h|}{|h|} = 0$$

*In this case, $f(x + x_0)$ is approximated by $f(x_0) + M_{f,x_0}x$, and the graph of $f$ is approximated by the affine linear space $\big\{\big(x + x_0, f(x_0) + M_{f,x_0}x\big) \bigm| x \in \mathbb{R}^n\big\}$, near $x_0$. The linear function $M_{f,x_0}$ is called the derivative of $f$ at $x_0$. We also say that $f$ is differentiable if for every $x_0 \in A$ is differentiable.*

Here $M_{f,x_0}$ is linear and thus we use, instead of $M_{f,x_0}(x)$, $M_{f,x_0}x$ (in a way, we use the symbol $M_{f,x_0}x$ as if $M_{f,x_0}$ was a matrix).

The derivative of $f$ at $x_0$ (as in Definition 2.1.2) is a linear function. If we gather all of these $M_{f,x_0}$, we can make another function $Df$, such that $(Df)(x_0) = M_{f,x_0}$, which is called the derivative of $f$ (notice that the "at $x_0$" is not used). Now, in the one variable case, we remembered that the derivative $f'$ can give us an operator in the differentiable functions. We saw that the function:

$$\frac{d}{dx} : \mathcal{D}(\mathbb{R} \rightharpoonup \mathbb{R}) \to (\mathbb{R} \rightharpoonup \mathbb{R}), \text{ with } \frac{d}{dx}(f) = \frac{df}{dx} = f'$$

is a linear operator. $\mathcal{D}(\mathbb{R} \rightharpoonup \mathbb{R})$ is the set of all differentiable functions $f$ with domain $A \subseteq \mathbb{R}$ and image $f(A) \subseteq \mathbb{R}$, and $(\mathbb{R} \rightharpoonup \mathbb{R})$ is the set of all functions $g$ with domain $B \subseteq \mathbb{R}$ and image $g(B) \subseteq \mathbb{R}$. We will also show that:

$$D : \mathcal{D}(\mathbb{R}^n \rightharpoonup \mathbb{R}^m) \to (\mathbb{R}^n \rightharpoonup \mathbb{R}^m), \text{ with } D(f) = Df$$

is a linear operator (which also gives meaning to the symbol $Df$). $\mathcal{D}(\mathbb{R}^n \rightharpoonup \mathbb{R}^m)$ is the set of all differentiable functions $f$ with domain $A \subseteq \mathbb{R}^n$ and image $f(A) \subseteq \mathbb{R}^m$, and $(\mathbb{R}^n \rightharpoonup \mathbb{R}^m)$ is the set of all functions $g$ with domain $B \subseteq \mathbb{R}^n$ and image $g(B) \subseteq \mathbb{R}^m$.

First, we need a definition.

**Definition 2.1.3** (Sets of differentiable functions)**.** *We define:*

- $\mathcal{D}(S \to T) := \{f : S \to T \mid f \text{ is differentiable}\}$
- $\mathcal{D}^k(S \to T) := \{f : S \to T \mid f \text{ is differentiable } k-\text{times}\}$, $k \in \overline{\mathbb{N}}$
- $\mathcal{D}(S \rightharpoonup T) := \{f : A \to B \mid f \text{ is differentiable and } A \subseteq S,\ B \subseteq T\}$
- $\mathcal{D}^k(S \rightharpoonup T) := \{f : A \to B \mid f \text{ is differentiable } k-\text{times and } A \subseteq S,\ B \subseteq T\}$, $k \in \overline{\mathbb{N}}$

*$S$ and $T$ are subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.*

It is easy to check that all of these sets are $\mathbb{R}-$vector spaces. For example $\mathcal{D}(S \to T)$ is an $\mathbb{R}-$vector space. To show this, lets show that for every $f, g \in \mathcal{D}(S \to T)$ we have $f + g \in \mathcal{D}(S \to T)$ (all the other axioms can be also be proved relatively easily).

If $Df$ and $Dg$ are the derivatives of $f$ and $g$, then:

$$0 = \lim_{h \to 0_{\mathbb{R}^n}} \frac{|f(x_0+h) - f(x_0) - (Df)(x_0)h|}{|h|} + \lim_{h \to 0_{\mathbb{R}^n}} \frac{|g(x_0+h) - g(x_0) - (Dg)(x_0)h|}{|h|} =$$

$$= \lim_{h \to 0_{\mathbb{R}^n}} \frac{|(f+g)(x_0+h) - (f+g)(x_0) - (Df+Dg)(x_0)h|}{|h|}$$

So $f+g$ is differentiable and $D(f+g) = Df + Dg$ (as we have shown $\big(D(f+g)\big)(x_0) = (Df)(x_0) + (Dg)(x_0)$ for some random $x_0 \in S$).

The previous proof also shows that:

**Proposition 2.1.2.** *The operator:*

$$D : \mathcal{D}(\mathbb{R}^n \rightharpoonup \mathbb{R}^m) \to (\mathbb{R}^n \rightharpoonup \mathbb{R}^m), \text{ with } D(f) = Df$$

*is linear.*

## 2.2 Exercises

**Exercise 2.1.** *Show that the "continuous derivative at $t_0$" hypothesis is needed in Proposition 2.1.1. The problem lies when we try to approximate the tangent line by not having point $\gamma(t_0)$ as one point of the approximation lines. If that hypothesis is not taken into account, tangent lines get approximated by lines that pass through $\gamma(t_0)$ and some other point on $\gamma$.*

*Hint: Consider the function:*

$$\gamma(t) = \begin{cases} x^2 \sin(1/x), \ \textit{if}\, x \neq 0 \\ 0, \ \textit{if}\, x = 0 \end{cases}$$

*which has discontinuous derivative:*

$$\gamma'(t) = \begin{cases} 2x \sin(1/x) - \cos(1/x), \ \textit{if}\, x \neq 0 \\ 0, \ \textit{if}\, x = 0 \end{cases}$$

CHAPTER 3

# Notation

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