Tree Coloring Notebook

July 8, 2017

```
In [167]: import random
          import numpy as np
          import scipy.spatial
          import matplotlib.pyplot as plt
          import progressbar
          %matplotlib inline


          from scipy.stats import multivariate_normal
          from scipy.stats import mvn
          from scipy import integrate
```

```
In [228]: # Some parameters for the program.
          NUM_ITER = 100000
          CLASSIFY_ROOT_RRT = False
```

This is the base class for the tree coloring problem. Both `RandomRecursiveTree` and `DaryTree` inherit from `ColoringTree`.

```
In [229]: class ColoringTree(object):
              def __init__(self, k, h, num_iterations=NUM_ITER, init_color=None):
                  assert(type(k) == int, "k should be an integer.")
                  assert(k > 0, "k should exceed 0.")
                  assert(type(h) == int, "h should be an integer.")
                  assert(h > 0, "h should exceed 0.")

                  self.k = k
                  self.h = h

                  if init_color:
                      assert init_color in range(1, self.k + 1)
                      self.root_color = init_color
                  else:
                      # Root color is chosen uniformly at random.
                      self.root_color = random.randint(1, self.k)

                  self.num_iterations = num_iterations
                  self.urn_size = 1
```

```
<ipython-input-229-00f093215cb5>:3: SyntaxWarning: assertion is always true, perhap
  assert(type(k) == int, "k should be an integer.")
<ipython-input-229-00f093215cb5>:4: SyntaxWarning: assertion is always true, perhap
  assert(k > 0, "k should exceed 0.")
<ipython-input-229-00f093215cb5>:5: SyntaxWarning: assertion is always true, perhap
  assert(type(h) == int, "h should be an integer.")
<ipython-input-229-00f093215cb5>:6: SyntaxWarning: assertion is always true, perhap
  assert(h > 0, "h should exceed 0.")
```

In [230]:
```
class LimitingFrequencyPlotter(object):
    """

    This class utilizes a dictionary data structure that resembles the ur
    but tracks the movement of relative frequencies over all iterations i
    This allows us to plot results related to limiting relative frequenci
    of the class.

    """

    def __init__(self, k, max_h):
        self.k = k
        self.max_h = max_h
        self.plotter_dict = {i: {j: 0 for j in range(1, self.k + 1)} for
        for i in range(self.max_h):
            for j in range(1, self.k + 1):
                self.plotter_dict[i][j] = []

        for j in range(1, self.k + 1):
            self.plotter_dict[self.max_h][j] = {}
            self.plotter_dict[self.max_h][j]["rf"] = []
            self.plotter_dict[self.max_h][j]["total"] = 0

    def add_data(self, urn_dictionary, new_urn_height, new_urn_color, ite
        for i in range(self.max_h):
            for j in range(1, self.k + 1):
                self.plotter_dict[i][j].append((urn_dictionary[i][j] * 1.

        if new_urn_height >= self.max_h:
            self.plotter_dict[self.max_h][new_urn_color]["total"] += 1

        for j in range(1, self.k + 1):
            self.plotter_dict[self.max_h][j]["rf"].append(
                (self.plotter_dict[self.max_h][new_urn_color]["total"] *
            )

    def plot(self, iter_cnt):
        """Add labels to the dominating frequencies for the final plot.""
```

```
            for i in range(self.max_h):
                for j in range(1, self.k + 1):
                    plt.plot(range(iter_cnt), self.plotter_dict[i][j])

            for j in range(1, self.k + 1):
                plt.plot(
                    range(iter_cnt),
                    self.plotter_dict[self.max_h][j]["rf"],
                    label="h > %d, Color %d" % (self.max_h, j)
                )

            plt.xlabel("Iterations")
            plt.ylabel("Relative Frequency of Nodes in Tree")
            plt.legend(loc='upper right')
            plt.show()
```

```
In [253]: class RandomRecursiveTree(ColoringTree):

        def __init__(self, k, h, report_conf=False):
            self.report_conf = report_conf
            super(RandomRecursiveTree, self).__init__(k, h)

            # Initialize urn data structure. Contains tuples (height, color).
            self.urn = [(0, self.root_color)]

            # The urn dictionary is indexed by (i, j), and contains the numbe
            # a particular color k at a particular height h.
            self.urn_dictionary = {i: {j: 0 for j in range(1, self.k + 1)} fo
            self.urn_dictionary[0][self.root_color] = 1

            # Used to compute confidence.
            self.parent_dictionary = {i: {j: 0 for j in range(1, self.k + 1)}

        def simulate(self, limiting_freq_plot=False, limiting_freq_max_h=None

            assert(bool(limiting_freq_plot) == bool(limiting_freq_max_h),
                "Either both or neither of limiting_freq_plot and limiting_fr

            if limiting_freq_plot:
                plotter = LimitingFrequencyPlotter(self.k, limiting_freq_max_

            for i in range(1, self.num_iterations + 1):
                # Randomly pick a ball from the urn.
                urn_draw = random.randint(0, self.urn_size - 1)
                urn_height, urn_color = self.urn[urn_draw]

                # Generate a new ball. This essentially just defines the bal
                # and draws randomly.
```

```
            new_urn_height = urn_height + 1
            possible_new_urn_colors = [j for j in range(1, self.k + 1) if
            new_urn_color = random.choice(possible_new_urn_colors)

            new_urn_ball = (new_urn_height, new_urn_color)
            self.urn.append(new_urn_ball)
            self.urn_dictionary[new_urn_height][new_urn_color] += 1

            self.parent_dictionary[new_urn_height][urn_color] += 1

            if limiting_freq_plot:
                plotter.add_data(self.urn_dictionary, new_urn_height, new

            self.urn_size += 1

        if limiting_freq_plot:
            plotter.plot(self.num_iterations)

        if tvd_plot:
            self.compile_tvd_statistics()

    def freq_at_level(self, h):
        """Helper method that retrieves relative frequencies from the urn
        colors_at_h = self.urn_dictionary[h].values()
        total_at_h = sum(colors_at_h)
        normalized_colors_at_h = (1.0 / total_at_h) * np.array(colors_at_
        return normalized_colors_at_h

    def prediction_confidence(self, h):
        """
        Utilizes a multivariate normal approximation to compute confidenc
        prediction (from classify_root_from_level/Algorithm 1.6.2). See p
        """
        mu = np.zeros(self.k)
        mu[self.root_color - 1] = 1

        b_h_inv = np.linalg.inv(np.linalg.matrix_power(self.markov_matrix
        colors_at_h = self.urn_dictionary[h].values()
        n_h = sum(colors_at_h)

        cov = np.zeros((self.k, self.k))
        for i in range(self.k):
            n_hi = self.parent_dictionary[h][i + 1]
            for j in range(self.k):
                n_hj = self.parent_dictionary[h][j + 1]
                if i == j:
                    cov[i][j] = (1. * ((self.k - 2) * (n_h - n_hi))) / ((
                else:
```

```
                cov[i][j] = (-1. * (n_h - n_hi - n_hj)) / ((n_h * (se

        sigma = np.dot(np.dot(b_h_inv, cov), b_h_inv.T)

        if self.k == 2 or h == 1:
            # Since the covariance matrix is singular, the sampling
            # will fail. But the confidence is really 1.0 in this case.
            confidence = 1.0
        else:
            num_samples = 100000
            samples = np.random.multivariate_normal(mu, sigma, size=num_s
            argmax_samples = np.argmax(samples, axis=1)
            correct_predictions_samples = np.sum(argmax_samples == self.r
            confidence = (1. * correct_predictions_samples) / num_samples

        return confidence

    def classify_root_from_level(self, h):
        """This is Algorithm 1.6.2 in the paper."""
        w_h = self.freq_at_level(h)
        v_0 = np.dot(np.linalg.inv(np.linalg.matrix_power(self.markov_mat
        prediction = np.argmax(v_0)

        if self.report_conf:
            confidence = self.prediction_confidence(h)
            return prediction + 1, confidence
        else:
            return prediction + 1

    def markov_matrix(self):
        # Construct the Markov transition probability matrix
        B = 1. / (self.k - 1) * np.ones((self.k, self.k))
        for i in range(self.k):
            B[i, i] = 0
        return B

    def theoretical_markov_approximation(self):
        theoretical_dist = np.zeros(self.k)
        theoretical_dist[self.root_color - 1] = 1
        uniform_dist = [1.0 / self.k] * self.k

        B = self.markov_matrix()

        total_variation_distances = []
        dists = []

        for h in range(self.h):
            dists.append(theoretical_dist)
```

```
            probability_matrix = np.array([theoretical_dist, uniform_dist
            total_variation_distance = scipy.spatial.distance.pdist(proba
            total_variation_distances.append(total_variation_distance[0])

            theoretical_dist = np.dot(B, theoretical_dist)

        return dists, total_variation_distances

    def compile_tvd_statistics(self):
        dists, total_variation_distances_theoretical = self.theoretical_m
        uniform_dist = [1.0 / self.k] * self.k

        total_variation_distances = []
        total_variation_distances_empirical_theoretical = []

        num_levels = 0
        for h in range(self.h):
            colors_at_h = self.urn_dictionary[h].values()
            total_at_h = sum(colors_at_h)

            if not h <= 1 and total_at_h < 100:
                break

            normalized_colors_at_h = (1.0 / total_at_h) * np.array(colors
            probability_matrix = np.array([normalized_colors_at_h, unifor

            total_variation_distance = scipy.spatial.distance.pdist(proba
            total_variation_distances.append(total_variation_distance[0])

            probability_matrix_empirical_theoretical = np.array([normaliz
            total_variation_distance_empirical_theoretical = scipy.spatia
            total_variation_distances_empirical_theoretical.append(total_

            num_levels += 1
            print "The total variation distance at height H = %d is %.4f"

        plt.plot(range(num_levels), total_variation_distances, color="red
        plt.plot(range(num_levels), total_variation_distances_theoretical
        plt.legend()
        plt.xlabel("Height (h)")
        plt.ylabel("Total Euclidean Variation Distance")
        plt.title("Total Variation Distance to Uniformity")
        plt.show()

        plt.plot(range(num_levels), total_variation_distances_empirical_t
        plt.xlabel("Height (h)")
        plt.ylabel("Total Euclidean Variation Distance")
```

```
            plt.title("Total Variation Distance between Theoretical and Empir
            plt.show()
```

```
<ipython-input-253-d2003bd2378b>:20: SyntaxWarning: assertion is always true, perha
  assert(bool(limiting_freq_plot) == bool(limiting_freq_max_h),
```

```
In [254]: class daryTree(ColoringTree):

              def __init__(self, d, k, h):
                  assert(d >= 1 and type(d) == int, "d must be an integer exceeding
                  self.d = d

                  super(daryTree, self).__init__(k, h)

                  # Initialize (external node) urn data structure. Contains tuples
                  self.external_nodes = [(1, self.root_color) for i in range(self.c
                  self.external_nodes_size = d

                  # Initialize (internal node) urn data structure. Contains tuples
                  self.urn = [(0, self.root_color)]

                  # The urn dictionary is indexed by (i, j), and contains the numbe
                  # a particular color k at a particular height h.
                  self.internal_node_dictionary = {i: {j: 0 for j in range(1, self.
                  self.internal_node_dictionary[0][self.root_color] = 1

              def simulate(self, limiting_freq_plot=False, limiting_freq_max_h=None

                  assert(bool(limiting_freq_plot) == bool(limiting_freq_max_h),
                  "Either both or neither of limiting_freq_plot and limiting_freq_h

                  if limiting_freq_plot:
                      plotter = LimitingFrequencyPlotter(self.k, limiting_freq_max_

                  for i in range(1, self.num_iterations + 1):
                      if i % 10000 == 0:
                          print("Finished iteration %d" % i)

                      # Randomly pick a ball from the urn.
                      urn_draw = random.randint(0, self.external_nodes_size - 1)
                      external_urn_draw = self.external_nodes[urn_draw]
                      urn_height, urn_color = external_urn_draw

                      # Generate a new ball. This essentially just defines the bal
                      # and draws randomly.
                      possible_new_urn_colors = [j for j in range(1, self.k + 1) if
                      new_urn_color = random.choice(possible_new_urn_colors)
```

```
            new_urn_ball = (urn_height, new_urn_color)
            self.urn.append(new_urn_ball)
            self.internal_node_dictionary[urn_height][new_urn_color] += 1

            new_external_urn_height = urn_height + 1
            new_external_urn_balls = [(new_external_urn_height, new_urn_c
            self.external_nodes.remove(external_urn_draw)
            self.external_nodes.extend(new_external_urn_balls)

            if limiting_freq_plot:
                plotter.add_data(self.internal_node_dictionary, urn_heigh

            self.urn_size += 1
            self.external_nodes_size += (self.d - 1)

        if limiting_freq_plot:
            plotter.plot(self.num_iterations)

        if tvd_plot:
            self.compile_tvd_statistics()

    def compile_tvd_statistics(self):
        uniform_dist = [1.0 / self.k] * self.k

        total_variation_distances = []
        total_variation_distances_empirical_theoretical = []

        num_levels = 0
        for h in range(self.h):
            colors_at_h = self.internal_node_dictionary[h].values()
            total_at_h = sum(colors_at_h)

            if not h <= 1 and total_at_h <= 1:
                break

            normalized_colors_at_h = (1.0 / total_at_h) * np.array(colors
            probability_matrix = np.array([normalized_colors_at_h, unifor

            total_variation_distance = scipy.spatial.distance.pdist(proba
            total_variation_distances.append(total_variation_distance[0])

            num_levels += 1
            print "The total variation distance at height H = %d is %.4f'

        plt.plot(range(num_levels), total_variation_distances, color="rec
        plt.xlabel("Height (h)")
        plt.ylabel("Total Euclidean Variation Distance")
```

```
                plt.title("Total Variation Distance to Uniformity")
                plt.show()
```

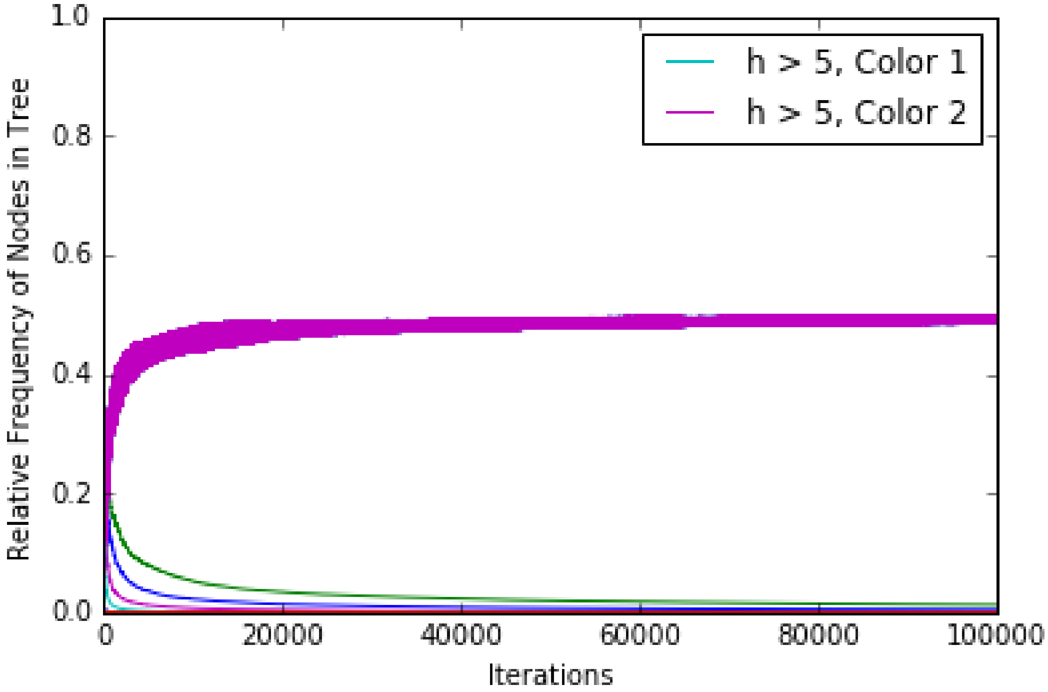
```
<ipython-input-254-836cd896357a>:4: SyntaxWarning: assertion is always true, perhap
  assert(d >= 1 and type(d) == int, "d must be an integer exceeding 0")
<ipython-input-254-836cd896357a>:23: SyntaxWarning: assertion is always true, perha
  assert(bool(limiting_freq_plot) == bool(limiting_freq_max_h),
```

Utilize the `RandomRecursiveTree` class to perform meaningful simulations.

1. Limiting frequencies plot.
2. TVD plots for Markov Chain "approximation."
3. Root classification.

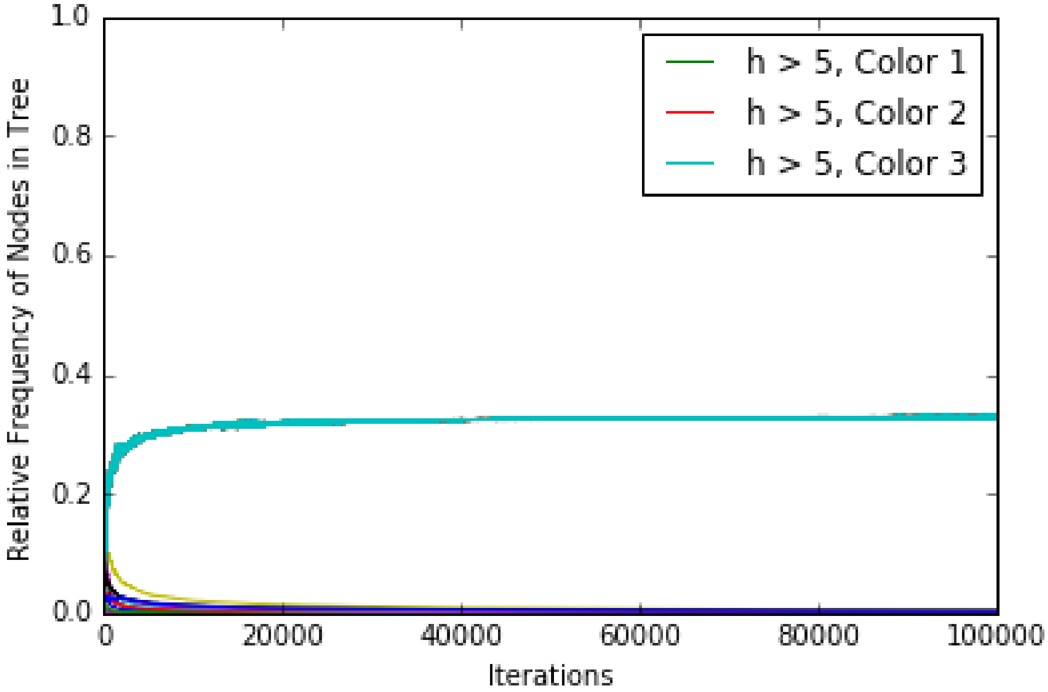
```
In [83]: two_color_rrt = RandomRecursiveTree(2, 100)
         limiting_freq_sim = two_color_rrt.simulate(limiting_freq_plot=True, limiti
```

```
Running iteration 10000
Running iteration 20000
Running iteration 30000
Running iteration 40000
Running iteration 50000
Running iteration 60000
Running iteration 70000
Running iteration 80000
Running iteration 90000
Running iteration 100000
```



```
In [73]: three_color_rrt = RandomRecursiveTree(3, 100)
         limiting_freq_sim = three_color_rrt.simulate(limiting_freq_plot=True, limi
```

```
Running iteration 10000
Running iteration 20000
Running iteration 30000
Running iteration 40000
Running iteration 50000
Running iteration 60000
Running iteration 70000
Running iteration 80000
Running iteration 90000
Running iteration 100000
```



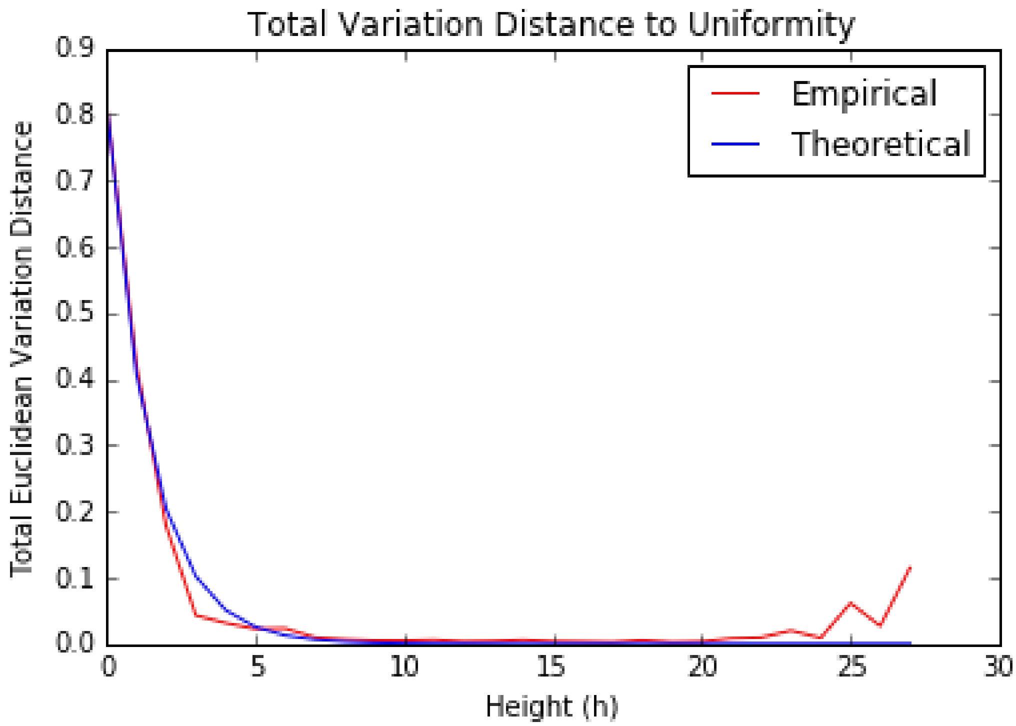

```
In [92]: three_color_rrt = RandomRecursiveTree(3, 100)
         tvd_sim = three_color_rrt.simulate(tvd_plot=True)
```

```
Running iteration 10000
Running iteration 20000
Running iteration 30000
Running iteration 40000
```

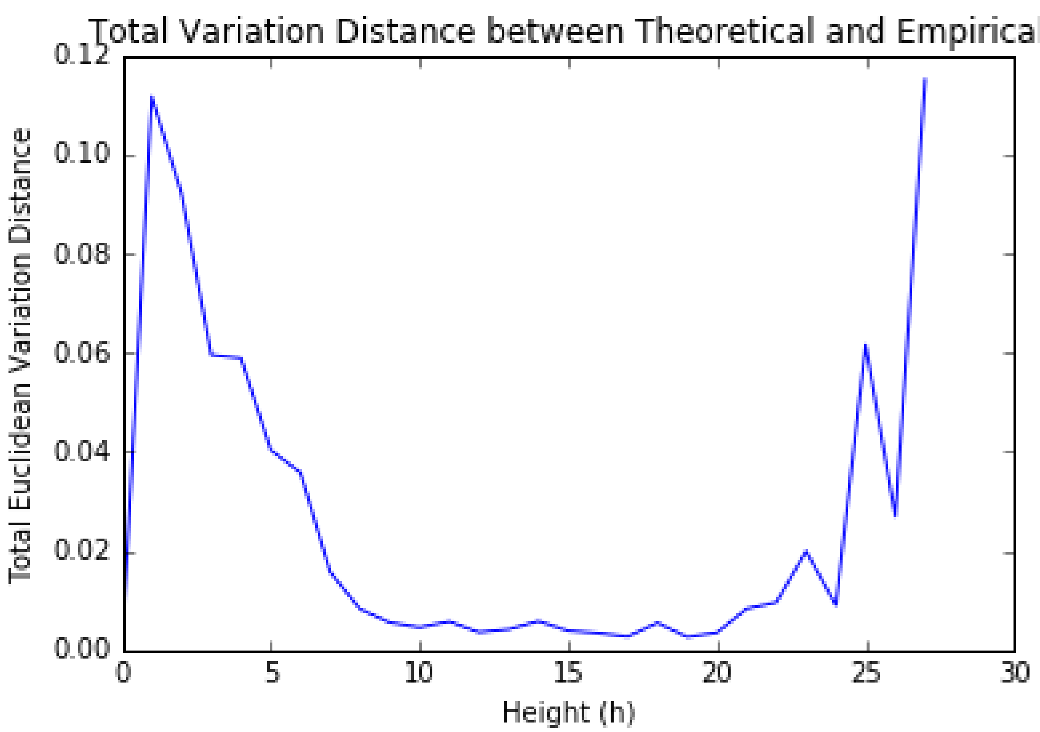
```
Running iteration 50000
Running iteration 60000
Running iteration 70000
Running iteration 80000
Running iteration 90000
Running iteration 100000
Running iteration 110000
Running iteration 120000
Running iteration 130000
Running iteration 140000
Running iteration 150000
Running iteration 160000
Running iteration 170000
Running iteration 180000
Running iteration 190000
Running iteration 200000
Running iteration 210000
Running iteration 220000
Running iteration 230000
Running iteration 240000
Running iteration 250000
Running iteration 260000
Running iteration 270000
Running iteration 280000
Running iteration 290000
Running iteration 300000
Running iteration 310000
Running iteration 320000
Running iteration 330000
Running iteration 340000
Running iteration 350000
Running iteration 360000
Running iteration 370000
Running iteration 380000
Running iteration 390000
Running iteration 400000
Running iteration 410000
Running iteration 420000
Running iteration 430000
Running iteration 440000
Running iteration 450000
Running iteration 460000
Running iteration 470000
Running iteration 480000
Running iteration 490000
Running iteration 500000
Running iteration 510000
Running iteration 520000
```

```
Running iteration 530000
Running iteration 540000
Running iteration 550000
Running iteration 560000
Running iteration 570000
Running iteration 580000
Running iteration 590000
Running iteration 600000
Running iteration 610000
Running iteration 620000
Running iteration 630000
Running iteration 640000
Running iteration 650000
Running iteration 660000
Running iteration 670000
Running iteration 680000
Running iteration 690000
Running iteration 700000
Running iteration 710000
Running iteration 720000
Running iteration 730000
Running iteration 740000
Running iteration 750000
Running iteration 760000
Running iteration 770000
Running iteration 780000
Running iteration 790000
Running iteration 800000
Running iteration 810000
Running iteration 820000
Running iteration 830000
Running iteration 840000
Running iteration 850000
Running iteration 860000
Running iteration 870000
Running iteration 880000
Running iteration 890000
Running iteration 900000
Running iteration 910000
Running iteration 920000
Running iteration 930000
Running iteration 940000
Running iteration 950000
Running iteration 960000
Running iteration 970000
Running iteration 980000
Running iteration 990000
Running iteration 1000000
```

```
The total variation distance at height H = 0 is 0.8165
The total variation distance at height H = 1 is 0.4232
The total variation distance at height H = 2 is 0.1785
The total variation distance at height H = 3 is 0.0430
The total variation distance at height H = 4 is 0.0315
The total variation distance at height H = 5 is 0.0226
The total variation distance at height H = 6 is 0.0233
The total variation distance at height H = 7 is 0.0097
The total variation distance at height H = 8 is 0.0068
The total variation distance at height H = 9 is 0.0052
The total variation distance at height H = 10 is 0.0050
The total variation distance at height H = 11 is 0.0060
The total variation distance at height H = 12 is 0.0037
The total variation distance at height H = 13 is 0.0043
The total variation distance at height H = 14 is 0.0058
The total variation distance at height H = 15 is 0.0039
The total variation distance at height H = 16 is 0.0034
The total variation distance at height H = 17 is 0.0027
The total variation distance at height H = 18 is 0.0055
The total variation distance at height H = 19 is 0.0026
The total variation distance at height H = 20 is 0.0035
The total variation distance at height H = 21 is 0.0083
The total variation distance at height H = 22 is 0.0096
The total variation distance at height H = 23 is 0.0200
The total variation distance at height H = 24 is 0.0090
The total variation distance at height H = 25 is 0.0616
The total variation distance at height H = 26 is 0.0269
The total variation distance at height H = 27 is 0.1151
```



Total Variation Distance to Uniformity
Empirical
Theoretical
Total Euclidean Variation Distance
Height (h)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
20
25
30



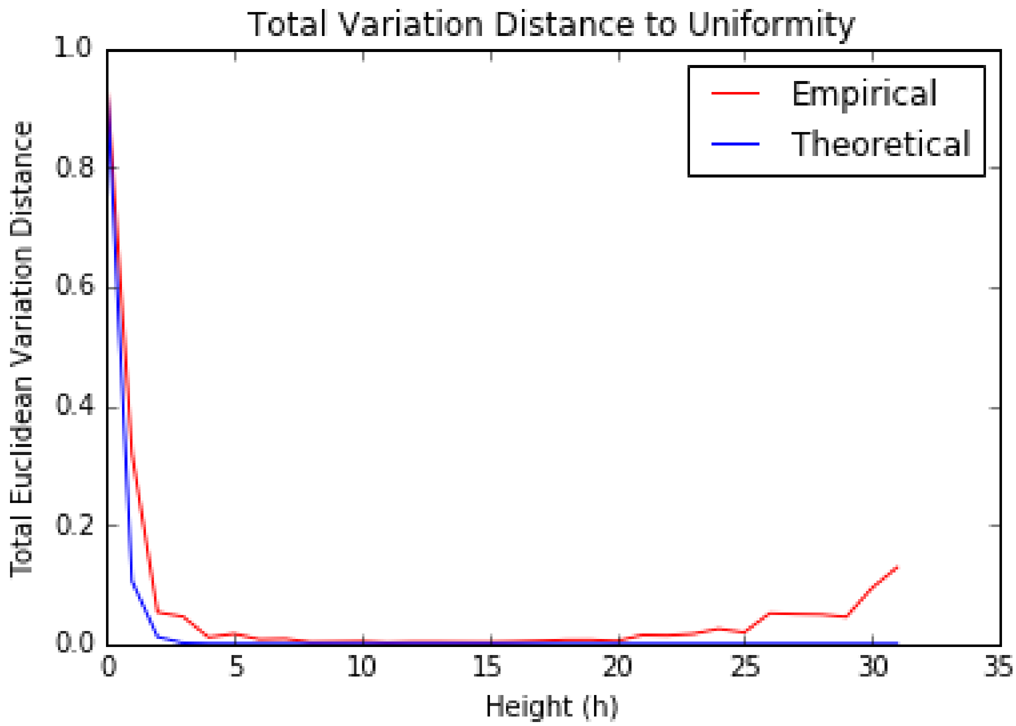
Total Variation Distance between Theoretical and Empirical
Total Euclidean Variation Distance
Height (h)
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
5
10
15
20
25
30

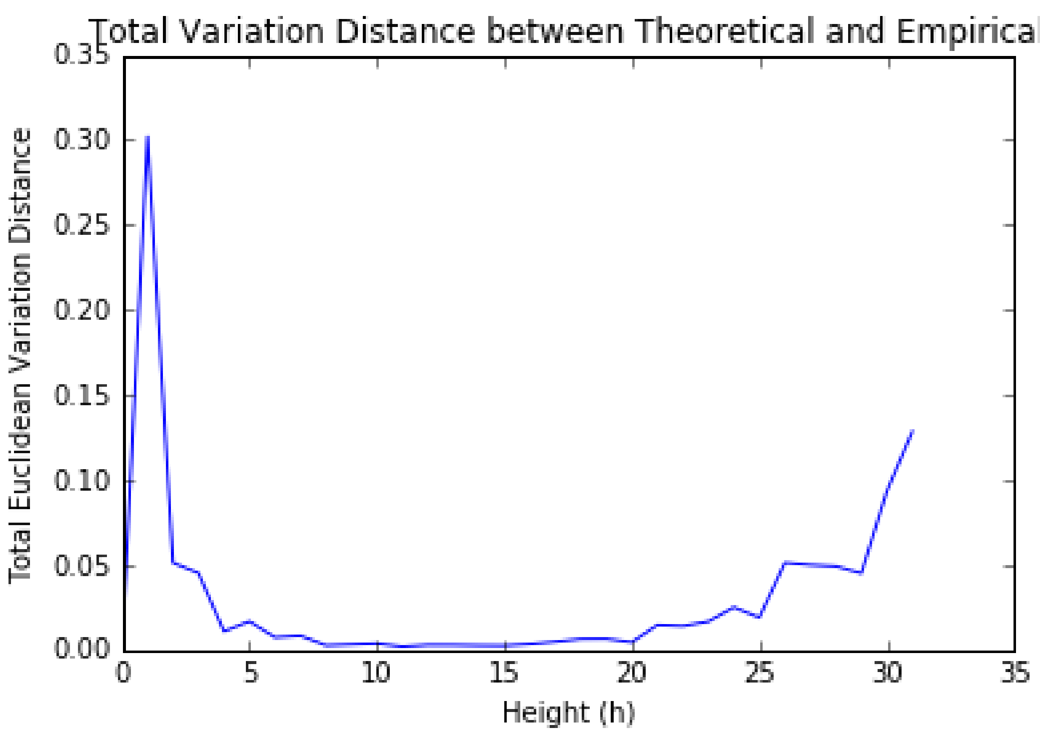
```
In [93]: ten_color_rrt = RandomRecursiveTree(10, 100)
         tvd_sim = ten_color_rrt.simulate(tvd_plot=True)
```

```
Running iteration 10000
Running iteration 20000
Running iteration 30000
Running iteration 40000
Running iteration 50000
Running iteration 60000
Running iteration 70000
Running iteration 80000
Running iteration 90000
Running iteration 100000
Running iteration 110000
Running iteration 120000
Running iteration 130000
Running iteration 140000
Running iteration 150000
Running iteration 160000
Running iteration 170000
Running iteration 180000
Running iteration 190000
Running iteration 200000
Running iteration 210000
Running iteration 220000
Running iteration 230000
Running iteration 240000
Running iteration 250000
Running iteration 260000
Running iteration 270000
Running iteration 280000
Running iteration 290000
Running iteration 300000
Running iteration 310000
Running iteration 320000
Running iteration 330000
Running iteration 340000
Running iteration 350000
Running iteration 360000
Running iteration 370000
Running iteration 380000
Running iteration 390000
Running iteration 400000
Running iteration 410000
Running iteration 420000
Running iteration 430000
Running iteration 440000
Running iteration 450000
```

```
Running iteration 460000
Running iteration 470000
Running iteration 480000
Running iteration 490000
Running iteration 500000
Running iteration 510000
Running iteration 520000
Running iteration 530000
Running iteration 540000
Running iteration 550000
Running iteration 560000
Running iteration 570000
Running iteration 580000
Running iteration 590000
Running iteration 600000
Running iteration 610000
Running iteration 620000
Running iteration 630000
Running iteration 640000
Running iteration 650000
Running iteration 660000
Running iteration 670000
Running iteration 680000
Running iteration 690000
Running iteration 700000
Running iteration 710000
Running iteration 720000
Running iteration 730000
Running iteration 740000
Running iteration 750000
Running iteration 760000
Running iteration 770000
Running iteration 780000
Running iteration 790000
Running iteration 800000
Running iteration 810000
Running iteration 820000
Running iteration 830000
Running iteration 840000
Running iteration 850000
Running iteration 860000
Running iteration 870000
Running iteration 880000
Running iteration 890000
Running iteration 900000
Running iteration 910000
Running iteration 920000
Running iteration 930000
```

```
Running iteration 940000
Running iteration 950000
Running iteration 960000
Running iteration 970000
Running iteration 980000
Running iteration 990000
Running iteration 1000000
The total variation distance at height H = 0 is 0.9487
The total variation distance at height H = 1 is 0.3198
The total variation distance at height H = 2 is 0.0531
The total variation distance at height H = 3 is 0.0457
The total variation distance at height H = 4 is 0.0109
The total variation distance at height H = 5 is 0.0169
The total variation distance at height H = 6 is 0.0075
The total variation distance at height H = 7 is 0.0084
The total variation distance at height H = 8 is 0.0027
The total variation distance at height H = 9 is 0.0033
The total variation distance at height H = 10 is 0.0038
The total variation distance at height H = 11 is 0.0020
The total variation distance at height H = 12 is 0.0030
The total variation distance at height H = 13 is 0.0030
The total variation distance at height H = 14 is 0.0027
The total variation distance at height H = 15 is 0.0026
The total variation distance at height H = 16 is 0.0035
The total variation distance at height H = 17 is 0.0048
The total variation distance at height H = 18 is 0.0065
The total variation distance at height H = 19 is 0.0066
The total variation distance at height H = 20 is 0.0046
The total variation distance at height H = 21 is 0.0147
The total variation distance at height H = 22 is 0.0141
The total variation distance at height H = 23 is 0.0167
The total variation distance at height H = 24 is 0.0252
The total variation distance at height H = 25 is 0.0191
The total variation distance at height H = 26 is 0.0514
The total variation distance at height H = 27 is 0.0501
The total variation distance at height H = 28 is 0.0493
The total variation distance at height H = 29 is 0.0452
The total variation distance at height H = 30 is 0.0934
The total variation distance at height H = 31 is 0.1286
```

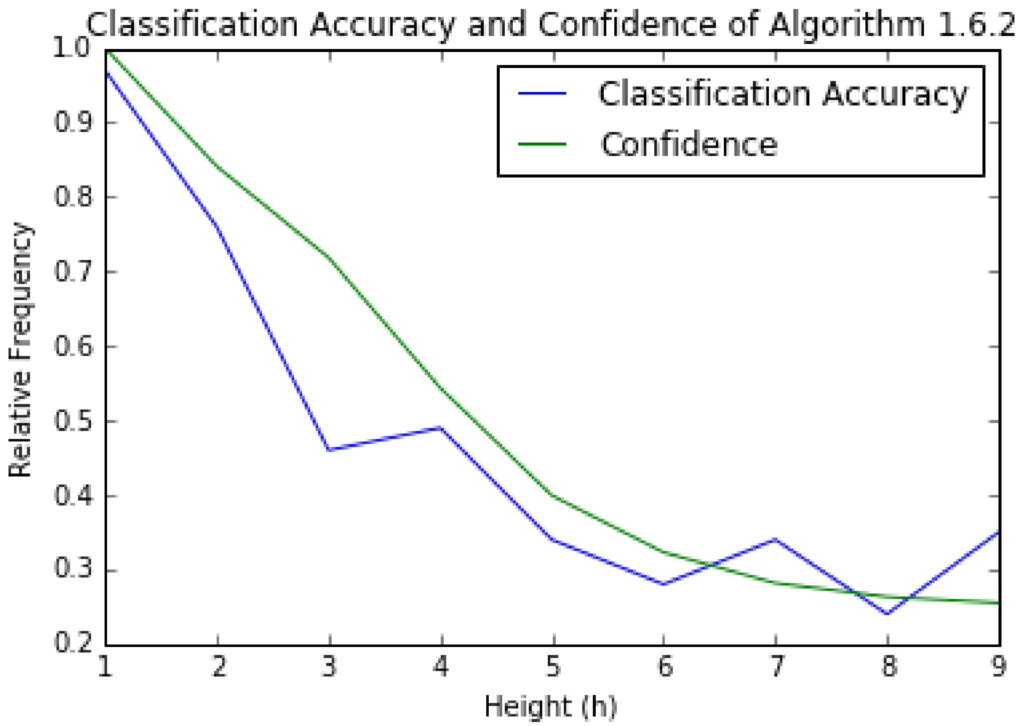

Total Variation Distance to Uniformity
Empirical
Theoretical
Total Euclidean Variation Distance
Height (h)
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
35


Total Variation Distance between Theoretical and Empirical
Total Euclidean Variation Distance
Height (h)
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0
5
10
15
20
25
30
35

```
In [110]: # This is one of the longer running scripts.
          accuracy_list = []
          num_samples = 100

          for k in range(4, 5):
              for h in range(1, 10):
                  correct = 0
                  confidence_values = []
                  bar = progressbar.ProgressBar()
                  for i in bar(range(num_samples)):
                      k_color_rrt = RandomRecursiveTree(k, 100, report_conf=True)
                      k_color_rrt.simulate()
                      classification, confidence = k_color_rrt.classify_root_from_l
                      confidence_values.append(confidence)
                      if classification == k_color_rrt.root_color:
                          correct += 1

                  confidence_average = np.mean(np.array(confidence_values))
                  accuracy_list.append((k, h, (correct * 1.)/num_samples, confidenc

100% (100 of 100) |#######################| Elapsed Time: 0:10:57 Time: 0:10:57
100% (100 of 100) |#######################| Elapsed Time: 0:11:04 Time: 0:11:04
100% (100 of 100) |#######################| Elapsed Time: 0:10:34 Time: 0:10:34
100% (100 of 100) |#######################| Elapsed Time: 0:16:22 Time: 0:16:22
100% (100 of 100) |#######################| Elapsed Time: 0:17:20 Time: 0:17:20
100% (100 of 100) |#######################| Elapsed Time: 0:14:39 Time: 0:14:39
100% (100 of 100) |#######################| Elapsed Time: 0:11:57 Time: 0:11:57
100% (100 of 100) |#######################| Elapsed Time: 0:14:04 Time: 0:14:04
100% (100 of 100) |#######################| Elapsed Time: 0:14:28 Time: 0:14:28
```

```
In [112]: print accuracy_list

[(4, 1, 0.97, 1.0), (4, 2, 0.76, 0.84149889999999994), (4, 3, 0.46, 0.71868750000000
```

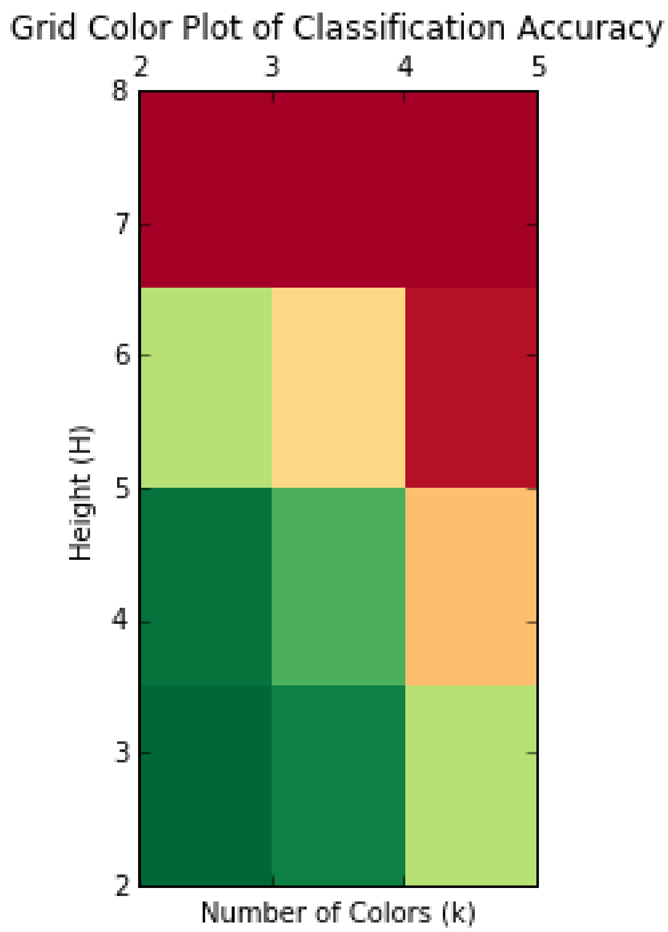
```
In [129]: k_list, h_list, acc_list, conf_list = zip(*accuracy_list)
          plt.plot(h_list, acc_list, color="blue", label="Classification Accuracy")
          plt.plot(h_list, conf_list, color="green", label="Confidence")
          plt.xlabel("Height (h)")
          plt.ylabel("Relative Frequency")
          plt.title("Classification Accuracy and Confidence of Algorithm 1.6.2")
          plt.legend()
          plt.show()
```



In [137]:
```
original_accuracy_list = [
    (2, 2, 1.0, 1.0),
    (2, 5, 1.0, 1.0),
    (2, 8, 1.0, 1.0),
    (3, 2, 0.97, 0.99368529999999977),
    (3, 5, 0.7, 0.98770829999999998),
    (3, 8, 0.48, 0.69833870000000009),
    (4, 2, 0.74, 0.84456799999999987),
    (4, 5, 0.35, 0.40276969999999999),
    (4, 8, 0.24, 0.26326249999999995),
    (5, 2, 0.48, 0.61759680000000006),
    (5, 5, 0.26, 0.232075),
    (5, 8, 0.22, 0.20147180000000003)
]

matrix_viz = np.zeros((4, 3))
for entry in original_accuracy_list:
    k, h, acc, conf = entry
    if h == 2:
        matrix_viz[k - 2][2] = acc
    elif h == 5:
        matrix_viz[k - 2][1] = acc
    else:
```

```
        matrix_viz[k - 2][0] = acc

    # How can I label the axes in matshow
    plt.matshow(matrix_viz, cmap="RdYlGn_r", extent=[2, 5, 2, 8])
    plt.xlabel("Number of Colors (k)")
    plt.ylabel("Height (H)")
    plt.title("Grid Color Plot of Classification Accuracy")
```

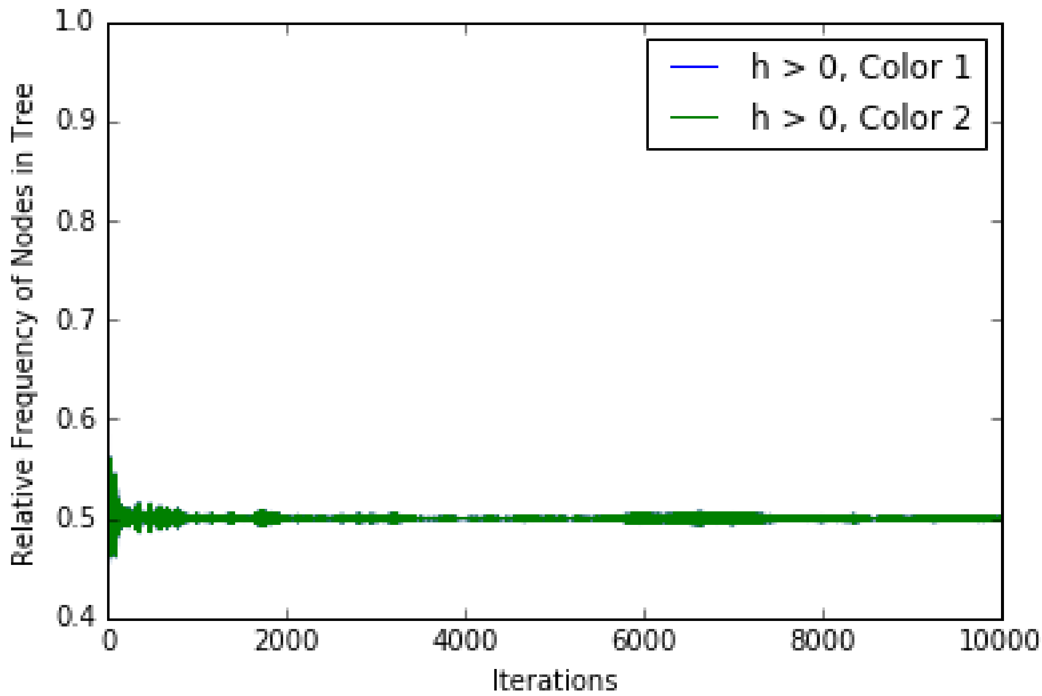
Out[137]: <matplotlib.text.Text at 0x1146bf0d0>



Utilize the `daryTree` class to perform meaningful simulations. 1. Limiting frequencies plot. 2. TVD plots - "distance to uniformity"

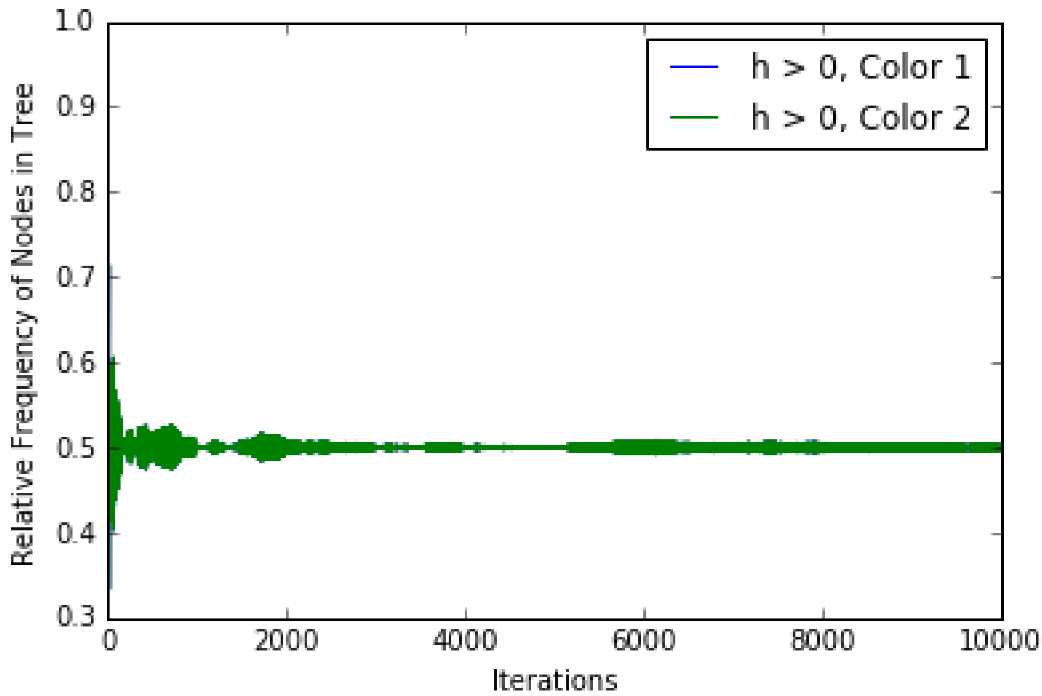
```
In [213]: two_color_two_ary = daryTree(2, 2, 100)
          limiting_freq_sim = two_color_two_ary.simulate(limiting_freq_plot=True, l
```

```
Running iteration 10000
```



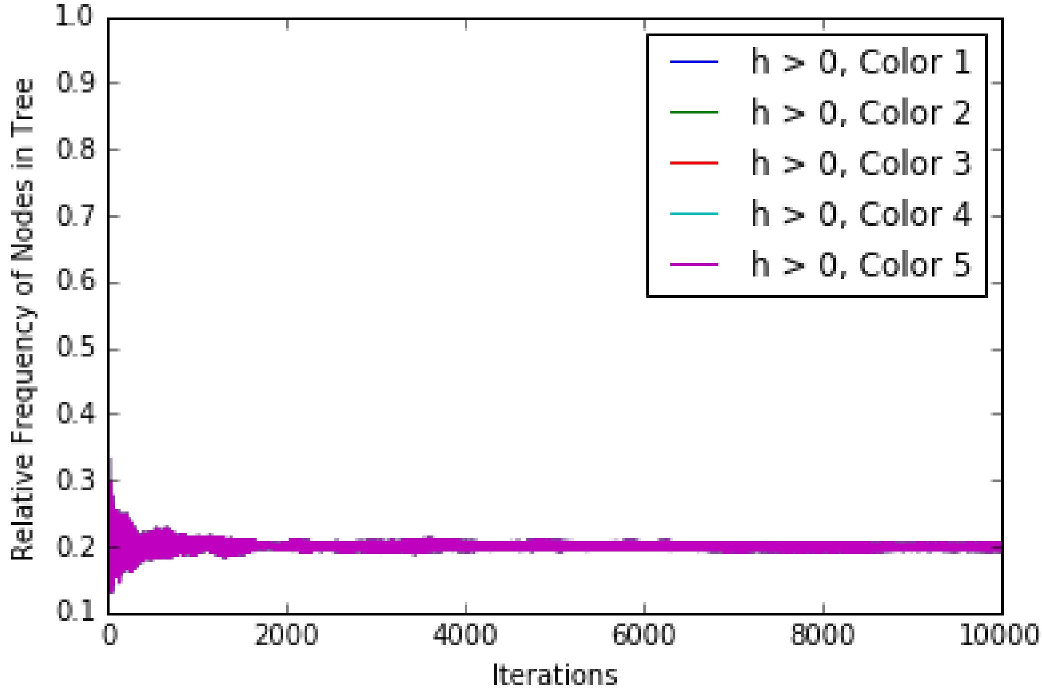

```
In [214]: two_color_ten_ary = daryTree(10, 2, 100)
          limiting_freq_sim = two_color_ten_ary.simulate(limiting_freq_plot=True, I
```

```
Running iteration 10000
```



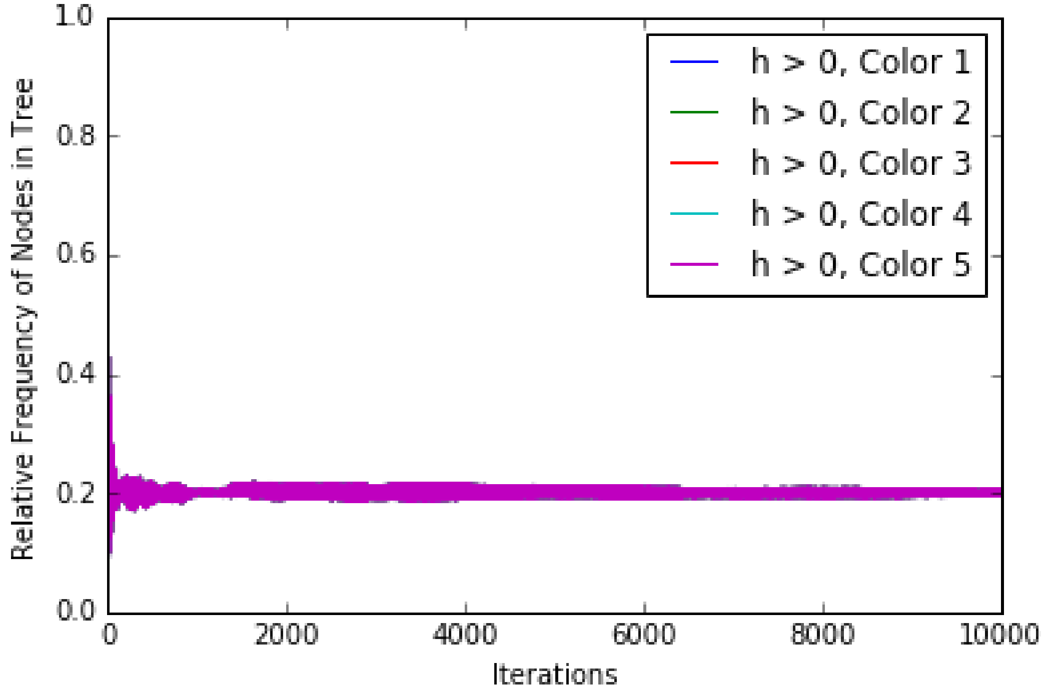

```
In [215]: five_color_two_ary = daryTree(2, 5, 100)
          limiting_freq_sim = five_color_two_ary.simulate(limiting_freq_plot=True,
```

```
Running iteration 10000
```




```
In [217]: five_color_ten_ary = daryTree(10, 5, 100)
          limiting_freq_sim = five_color_ten_ary.simulate(limiting_freq_plot=True,
```

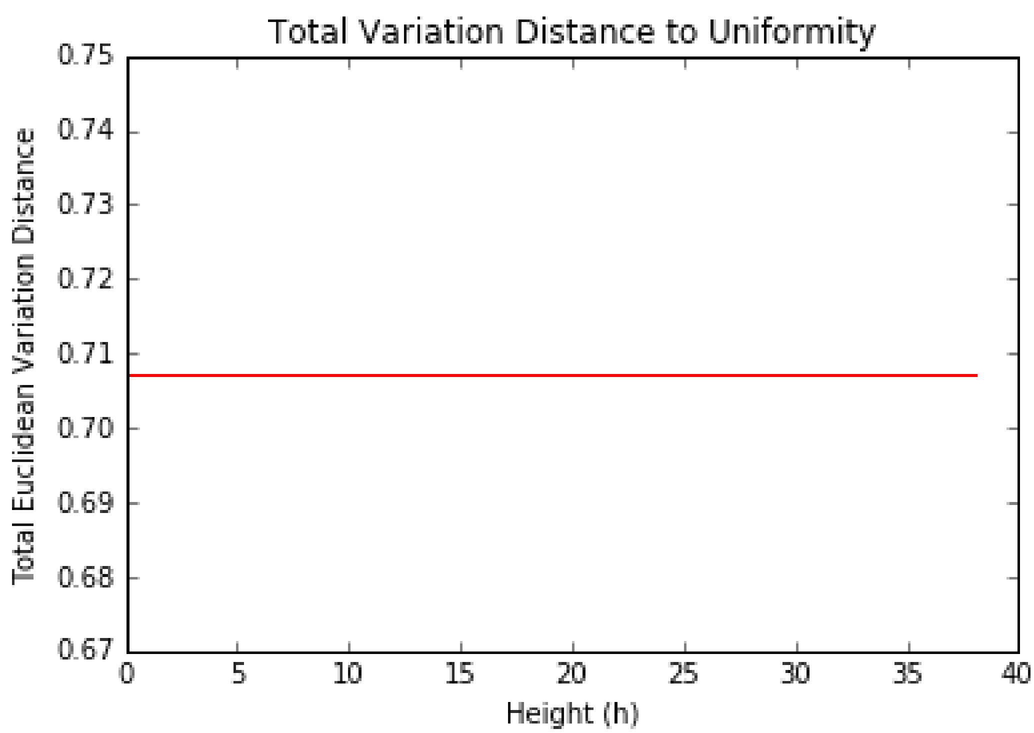
```
Running iteration 10000
```




```
In [246]: two_color_two_ary = daryTree(2, 2, 100)
          limiting_freq_sim = two_color_two_ary.simulate(tvd_plot=True)
```

```
The total variation distance at height H = 0 is 0.7071
The total variation distance at height H = 1 is 0.7071
The total variation distance at height H = 2 is 0.7071
The total variation distance at height H = 3 is 0.7071
The total variation distance at height H = 4 is 0.7071
The total variation distance at height H = 5 is 0.7071
The total variation distance at height H = 6 is 0.7071
The total variation distance at height H = 7 is 0.7071
The total variation distance at height H = 8 is 0.7071
The total variation distance at height H = 9 is 0.7071
The total variation distance at height H = 10 is 0.7071
The total variation distance at height H = 11 is 0.7071
The total variation distance at height H = 12 is 0.7071
The total variation distance at height H = 13 is 0.7071
The total variation distance at height H = 14 is 0.7071
The total variation distance at height H = 15 is 0.7071
The total variation distance at height H = 16 is 0.7071
The total variation distance at height H = 17 is 0.7071
The total variation distance at height H = 18 is 0.7071
The total variation distance at height H = 19 is 0.7071
The total variation distance at height H = 20 is 0.7071
```

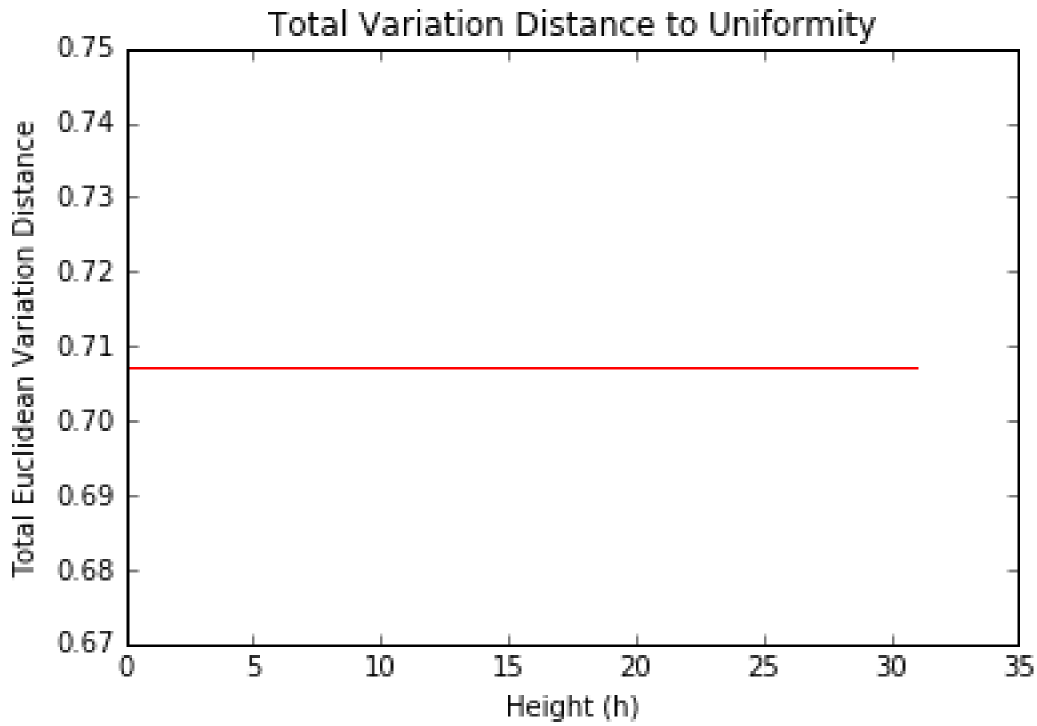
```
The total variation distance at height H = 21 is 0.7071
The total variation distance at height H = 22 is 0.7071
The total variation distance at height H = 23 is 0.7071
The total variation distance at height H = 24 is 0.7071
The total variation distance at height H = 25 is 0.7071
The total variation distance at height H = 26 is 0.7071
The total variation distance at height H = 27 is 0.7071
The total variation distance at height H = 28 is 0.7071
The total variation distance at height H = 29 is 0.7071
The total variation distance at height H = 30 is 0.7071
The total variation distance at height H = 31 is 0.7071
The total variation distance at height H = 32 is 0.7071
The total variation distance at height H = 33 is 0.7071
The total variation distance at height H = 34 is 0.7071
The total variation distance at height H = 35 is 0.7071
The total variation distance at height H = 36 is 0.7071
The total variation distance at height H = 37 is 0.7071
The total variation distance at height H = 38 is 0.7071
```




```
In [255]: three_color_ten_ary = daryTree(10, 3, 100)
          tvd_sim = two_color_ten_ary.simulate(tvd_plot=True)
```

```
Finished iteration 10000
Finished iteration 20000
```

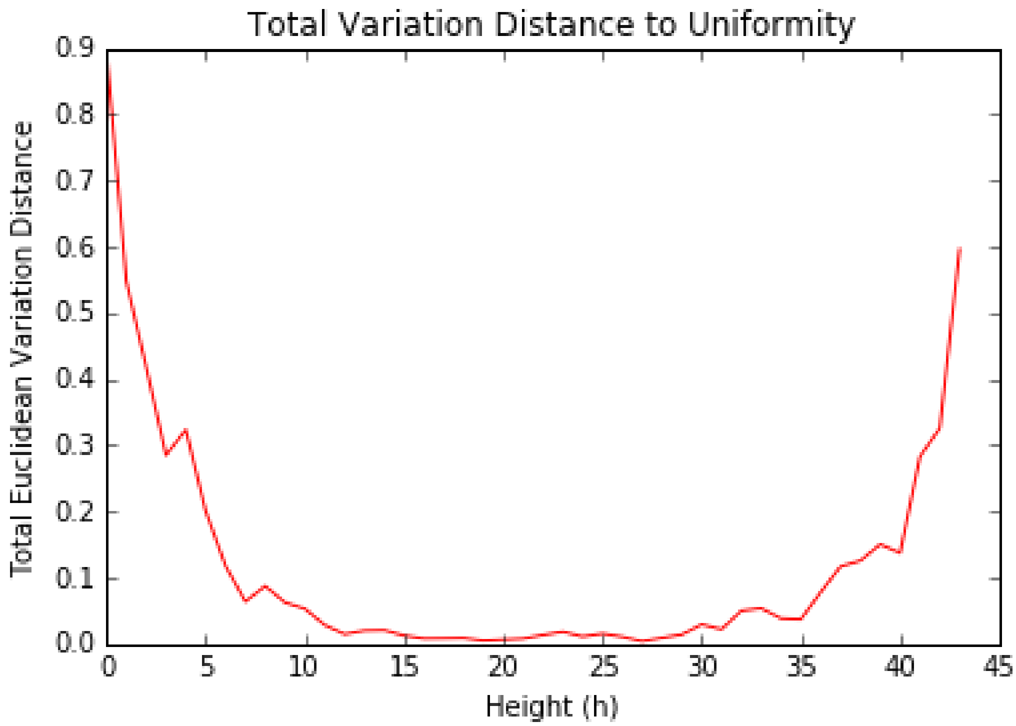
```
Finished iteration 30000
Finished iteration 40000
Finished iteration 50000
Finished iteration 60000
Finished iteration 70000
Finished iteration 80000
Finished iteration 90000
Finished iteration 100000
> <ipython-input-250-86a8b29eb782>(65)compile_tvd_statistics()
-> uniform_dist = [1.0 / self.k] * self.k
(Pdb) c
The total variation distance at height H = 0 is 0.7071
The total variation distance at height H = 1 is 0.7071
The total variation distance at height H = 2 is 0.7071
The total variation distance at height H = 3 is 0.7071
The total variation distance at height H = 4 is 0.7071
The total variation distance at height H = 5 is 0.7071
The total variation distance at height H = 6 is 0.7071
The total variation distance at height H = 7 is 0.7071
The total variation distance at height H = 8 is 0.7071
The total variation distance at height H = 9 is 0.7071
The total variation distance at height H = 10 is 0.7071
The total variation distance at height H = 11 is 0.7071
The total variation distance at height H = 12 is 0.7071
The total variation distance at height H = 13 is 0.7071
The total variation distance at height H = 14 is 0.7071
The total variation distance at height H = 15 is 0.7071
The total variation distance at height H = 16 is 0.7071
The total variation distance at height H = 17 is 0.7071
The total variation distance at height H = 18 is 0.7071
The total variation distance at height H = 19 is 0.7071
The total variation distance at height H = 20 is 0.7071
The total variation distance at height H = 21 is 0.7071
The total variation distance at height H = 22 is 0.7071
The total variation distance at height H = 23 is 0.7071
The total variation distance at height H = 24 is 0.7071
The total variation distance at height H = 25 is 0.7071
The total variation distance at height H = 26 is 0.7071
The total variation distance at height H = 27 is 0.7071
The total variation distance at height H = 28 is 0.7071
The total variation distance at height H = 29 is 0.7071
The total variation distance at height H = 30 is 0.7071
The total variation distance at height H = 31 is 0.7071
```




```
In [256]: five_color_two_ary = daryTree(2, 5, 100)
          tvd_sim = five_color_two_ary.simulate(tvd_plot=True)
```

```
Finished iteration 10000
Finished iteration 20000
Finished iteration 30000
Finished iteration 40000
Finished iteration 50000
Finished iteration 60000
Finished iteration 70000
Finished iteration 80000
Finished iteration 90000
Finished iteration 100000
The total variation distance at height H = 0 is 0.8944
The total variation distance at height H = 1 is 0.5477
The total variation distance at height H = 2 is 0.4183
The total variation distance at height H = 3 is 0.2850
The total variation distance at height H = 4 is 0.3236
The total variation distance at height H = 5 is 0.2006
The total variation distance at height H = 6 is 0.1178
The total variation distance at height H = 7 is 0.0639
The total variation distance at height H = 8 is 0.0873
The total variation distance at height H = 9 is 0.0625
The total variation distance at height H = 10 is 0.0530
```

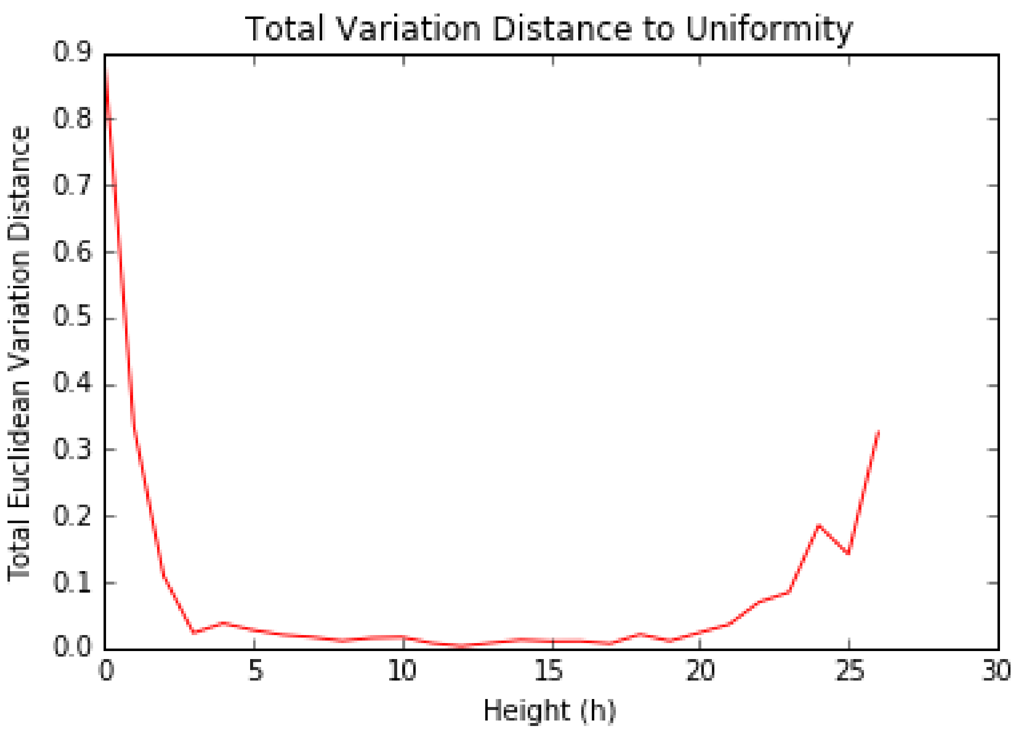
```
The total variation distance at height H = 11 is 0.0283
The total variation distance at height H = 12 is 0.0146
The total variation distance at height H = 13 is 0.0200
The total variation distance at height H = 14 is 0.0206
The total variation distance at height H = 15 is 0.0124
The total variation distance at height H = 16 is 0.0080
The total variation distance at height H = 17 is 0.0083
The total variation distance at height H = 18 is 0.0088
The total variation distance at height H = 19 is 0.0048
The total variation distance at height H = 20 is 0.0063
The total variation distance at height H = 21 is 0.0078
The total variation distance at height H = 22 is 0.0133
The total variation distance at height H = 23 is 0.0182
The total variation distance at height H = 24 is 0.0110
The total variation distance at height H = 25 is 0.0153
The total variation distance at height H = 26 is 0.0106
The total variation distance at height H = 27 is 0.0039
The total variation distance at height H = 28 is 0.0090
The total variation distance at height H = 29 is 0.0139
The total variation distance at height H = 30 is 0.0293
The total variation distance at height H = 31 is 0.0220
The total variation distance at height H = 32 is 0.0500
The total variation distance at height H = 33 is 0.0537
The total variation distance at height H = 34 is 0.0383
The total variation distance at height H = 35 is 0.0371
The total variation distance at height H = 36 is 0.0782
The total variation distance at height H = 37 is 0.1170
The total variation distance at height H = 38 is 0.1257
The total variation distance at height H = 39 is 0.1505
The total variation distance at height H = 40 is 0.1372
The total variation distance at height H = 41 is 0.2828
The total variation distance at height H = 42 is 0.3258
The total variation distance at height H = 43 is 0.5963
```




```
In [257]: five_color_ten_ary = daryTree(10, 5, 100)
          tvd_sim = five_color_ten_ary.simulate(tvd_plot=True)
```

```
Finished iteration 10000
Finished iteration 20000
Finished iteration 30000
Finished iteration 40000
Finished iteration 50000
Finished iteration 60000
Finished iteration 70000
Finished iteration 80000
Finished iteration 90000
Finished iteration 100000
The total variation distance at height H = 0 is 0.8944
The total variation distance at height H = 1 is 0.3354
The total variation distance at height H = 2 is 0.1091
The total variation distance at height H = 3 is 0.0226
The total variation distance at height H = 4 is 0.0372
The total variation distance at height H = 5 is 0.0270
The total variation distance at height H = 6 is 0.0198
The total variation distance at height H = 7 is 0.0162
The total variation distance at height H = 8 is 0.0106
The total variation distance at height H = 9 is 0.0152
The total variation distance at height H = 10 is 0.0162
```

```
The total variation distance at height H = 11 is 0.0072
The total variation distance at height H = 12 is 0.0026
The total variation distance at height H = 13 is 0.0074
The total variation distance at height H = 14 is 0.0120
The total variation distance at height H = 15 is 0.0100
The total variation distance at height H = 16 is 0.0102
The total variation distance at height H = 17 is 0.0065
The total variation distance at height H = 18 is 0.0205
The total variation distance at height H = 19 is 0.0104
The total variation distance at height H = 20 is 0.0231
The total variation distance at height H = 21 is 0.0357
The total variation distance at height H = 22 is 0.0693
The total variation distance at height H = 23 is 0.0841
The total variation distance at height H = 24 is 0.1855
The total variation distance at height H = 25 is 0.1416
The total variation distance at height H = 26 is 0.3258
```




```
In [ ]:
```