1 Combinatorial Analysis

1.1 Introduction

1.2 Sums and Products

Union: If A, B are disjoint, then $|A \cup B| = |A| + |B|$, where $|S|$ is the number of elements in S.

Cartesian product: $A \times B = \{(a, b) : a \in A, b \in B\}$ and $|A \times B| = |A||B|$ if A, B are finite.

Cartesian power: $A^k = \{(a_1, \ldots, a_k) : a_1, \ldots, a_k \in A\}$. If A is finite, then $|A^k| = |A|^k$.

1.3 Binomial Coefficients

Enumeration problems: involve counting various kinds of combinatorial objects.

Theorem 1.3.1: For $n, k \geq 0$, there are $\frac{n!}{(n-k)!k!} = \binom{n}{k} = \binom{n}{n-k}$ subsets of size k from an n element set. There are $\frac{n!}{(n-k)!}$ ordered lists of size k from an n element set and $k!$ arrangements within each.

As well, $\binom{n}{k} = 0$ when $k > n$.

1.4 Bijections

Let S, T be sets. If $f : S \to T$ is a bijection, then $|S| = |T|$. A bijection is both:

1. Injective/one to one: For any $x_1, x_2 \in S, f(x_1) = f(x_2)$ implies $x_1 = x_2$. All elements in S map to a unique element in T. Implies that $|S| \leq |T|$.
2. Surjective/onto: For all $y \in T, \exists x \in S$ such that $f(x) = y$. Every element in T is mapped to from some element in S. Implies that $|S| \geq |T|$.

Inverse of f is $f^{-1} : T \to S$ such that for all $x \in S, f^{-1}(f(x)) = x$ and for all $y \in T, f(f^{-1}(y)) = y$.

Theorem 1.4.1: $f : S \to T$ has an inverse if and only if it is a bijection. Thus, to prove a bijection holds, give an inverse. Also, there must exist a bijection between any two sets of the same size.

Using a bijection between the set of all subsets of $\{1, \ldots, n\}$ and the set of all binary strings of length n, it can be found that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

1.5 Combinatorial Proofs

A combinatorial proof involves some kind of counting argument.

Theorem 1.5.1: For any $n \geq 0$, $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ (Binomial Theorem).

Theorem 1.5.2: For any integers $1 \leq k \leq n$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Theorem 1.5.3: For $n, k \geq 0$, $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$.

1.6 Generating Series

Let S be a set of configurations with a weight function w. The generating series for S with respect to w is $\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{k \geq 0} a_k x^k$, where a_k is the number of elements in S with weight k.

Theorem 1.6.3: Let $\Phi_S(x)$ be the generating series for a finite set S with weight function w. Then,

1. $\Phi_S(1) = |S|$
2. Sum of the weights of the elements in S is $\Phi'_S(1)$ (derivative)
3. Average weight of an element in S is $\Phi'_S(x)/\Phi_S(1)$

Given a counting problem, define an appropriate set and weight function so that the answer is the number of elements in the set of a certain weight.

1.7 Formal Power Series (FPS)

Formal power series: $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, where $(a_0, a_1, a_2, \ldots)$ is a sequence of rational numbers and $[x^n]A(x) = a_n$.

Addition of power series: add the coefficients of matching terms

Multiplication of power series: (Let $k = i, n = i + j$)

$$A(x)B(x) = \left(\sum_{i \geq 0} a_i x^i\right)\left(\sum_{j \geq 0} b_j x^j\right) = \sum_{i \geq 0}\sum_{j \geq 0} a_i b_j x^{i+j} = \sum_{n \geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$$

The coefficient of x^n in $A(x)B(x)$ is $\sum_{k=0}^{n} a_k b_{n-k}$.

Theorem 1.7.2: Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, $P(x) = p_0 + p_1 x + p_2 x^2 + \cdots$, and $Q(x) = 1 - q_1 x - q_2 x^2 - \cdots$ be formal power series. Then, $Q(x)A(x) = P(x)$ iff for $n \geq 0$,

$$a_n = p_n + q_1 a_{n-1} + q_2 a_{n-2} + \cdots + q_n a_0$$

Theorem 1.7.3: Let $P(x), Q(x)$ be formal power series. If the constant term of $Q(x)$ is non-zero, then there is a formal power series $A(x)$ satisfying $Q(x)A(x) = P(x)$ and $A(x)$ is unique.

Inverse: $B(x)$ is the inverse of $A(x)$ if $A(x)B(x) = 1$. Denoted, $B(x) = A(x)^{-1}, B(x) = 1/A(x)$.

Geometric series: $1 + x + x^2 + \cdots = \frac{1}{1-x}$

Finite geometric series: $1 + x + x^2 + \cdots + x^k = \frac{1-x^{k+1}}{1-x}$

Negative binomial series: $\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$

Theorem 1.7.8: A FPS has an inverse iff it has a non-zero constant term. If the constant term is non-zero, then the inverse is unique. Note that an inverse is equivalent to dividing 1 by the FPS.

Composition of FPS: $A(B(x)) = a_0 + a_1 B(x) + a_2 (B(x))^2 + \cdots$

Theorem 1.7.10: If $A(x), B(x)$ are FPS with the constant term of $B(x)$ equal to 0, then $A(B(x))$ is a FPS. If the constant term of $B(x)$ is not 0, then $A(B(x))$ might be a power series or might not be. This is because $A(B(x))$ might have infinite many constant terms so the constant term of $A(B(x))$ might not be finite.

Note: $[x^n]x^k A(x) = [x^{n-k}]A(x)$ if $n \geq k$, or 0 if $n < k$.

1.8 Sum and Product Lemmas

Sum Lemma: Let (A, B) be a partition of $S, S = A \cup B, A \cap B = \emptyset$. Then, $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$. If $A \cap B \neq \emptyset$, then $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A \cap B}(x)$.

Product Lemma: Let A, B be sets with weight functions α, β, respectively. If $w(\sigma) = \alpha(a) + \beta(b)$ for all $\sigma = (a, b) \in A \times B$, then $\Phi_{A \times B}(x) = \Phi_A(x)\Phi_B(x)$. This also works for k-tuples.

Theorem 1.8.5: For any $k > 0, (1 - x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$.

2 Compositions and Strings

2.1 Compositions of an Integer

For $n, k \geq 0$, a composition of n with k parts is an ordered list of k positive integers such that they sum to n. Each integer is called a part. The empty composition is a composition of 0 with 0 parts. Since compositions are ordered, they may be thought of as a k-tuple. There are $\binom{n-1}{k-1}$ compositions of n with k parts, for $n \geq k \geq 1$. There are 2^{n-1} compositions of n, for $n \geq 1$.

To find the number of compositions of n with certain properties,

1. Construct set S of all compositions with the properties
2. Find generating series of S using sum and product lemmas
3. Find coefficient of x^n of the generating series

2.2 Binary Strings

ϵ is the empty string, of length 0. In general, for generating series concerning binary strings, the weight is the length.

Let A, B be sets of binary strings, possibily infinite. Then,

1. $AB = \{ab : a \in A, b \in B\}$, concatenation
2. $A^* = \{\epsilon\} \cup A \cup AA \cup AAA \cup \cdots = \bigcup_{k \geq 0} A^k$

Blocks are maximal nonempty substrings of only 0's or only 1's. ϵ has no blocks.

2.3 Unambiguous Expressions

Let A, B be sets of strings. AB is ambiguous iff there are distinct pairs (a_1, b_1) and (a_2, b_2) in $A \times B$ such that $a_1 b_1 = a_2 b_2$. If A, B are finite sets, then AB is unambiguous iff $|AB| = |A \times B|$. In cases where AB is ambiguous, $|AB| < |A \times B|$. As well, $A \cup B$ is unambiguous iff $A \cap B = \emptyset$.

2.4 Some Decomposition Rules

Expressions which follow from unambiguous decompositions are also unambiguous. Common unambiguous expressions for the set of all binary strings are:

1. Decompose after each 0 or 1: $\{0, 1\}^*$
2. Decompose after each 0: $(\{1\}^*\{0\})^*\{1\}^*$
3. Decompose before each 0: $\{1\}^*(\{0\}\{1\}^*)^*$
4. Decompose after each block of 0's: $\{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^*$
5. Other example: strings where all blocks of 1's have length divisible by 3: $\{0, 111\}^*$

2.5 Sum and Product Rules for Strings

Let A, B be sets of strings. Then,

1. If $A \cup B$ is unambiguous, meaning $A \cap B = \emptyset$, then $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$
2. If AB is unambiguous, then $\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$
3. If A^* is unambiguous, then $\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)}$

Note that if A^* is ambiguous but AB is unambiguous, rule 2 still applies.

2.6 Decomposition Using Blocks

2.7 Recursive Decompositions of Binary Strings

Let S be the set of all binary strings. S can be defined recursively as $S = \{\epsilon\} \cup \{0, 1\}S$.

Common problem to solve with recursive definition: Find the genearting series for L, the set of strings that do not contain 1010 as a substring. To do this, define M to be the set of strings with exactly one copy of 1010 at their right end. Then, $\{\epsilon\} \cup L\{0, 1\} = L \cup M$ and $L\{1010\} = M \cup M\{10\}$.

Another common problem: Find the recursive definition for all strings with no 000: For each string, decompose after the first 1. For example, 00101010 becomes 001, 01010. Then,

$$S = \{1, 01, 001\} \cup \{\epsilon, 0, 00\}$$

The $\{\epsilon, 0, 00\}$ represents the base case, where there are no 1's in the string.

3 Recurrences, Binary Trees, and Sorting

3.1 Solutions to Recurrence Equations

Given a recurrence such as $a_n + c_1 a_{n-1} + \cdots c_k a_{n-k} = 0$, we want a closed form equation for a_n.

1. Find the characteristic polynomial $x^k + c_1 x^{k-1} + \cdots + c_k$
2. Factor the polynomial
3. For each root r with multiplicity k, add $p(n) r^n$ to the general form where $p(n)$ is an arbitrary polynomial of degree $k-1$
4. Use the initial conditions to solve for the unknowns in the general form

It is possible for the roots of the characteristic polynomial to be complex. It is not possible for there to be more unknowns than equations, since the next term can always be added to the system.

3.2 Coefficients of Rational Functions

We want to calculate $[x^n]\frac{f(x)}{g(x)}$. For example, calculate $[x^n]\frac{6x^2}{1-2x-x^2+2x^3}$.

1. From the denominator $g(x)$, pull out the recurrence. In this case, it is $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3}$. The coefficient of a_{n-i} in the recurrence is the coefficient of x^i in $g(x)$.
2. The initial conditions come from the numerator.
3. Using this recurrence and initial conditions, solve as above.

4 Introduction to Graph Theory

4.1 Definitions

A graph G is a finite nonempty set $V(G)$ of vertices and a finite set $E(G)$ of unordered pairs of distinct vertices. $E(G)$ are the edges. The edge $\{1, 2\}$ is the same as $\{2, 1\}$ and a vertex cannot have an edge to itself (loops). Graphs where edges are unordered pairs are undirected. There cannot be more than one edge between any two vertices. Graphs with loops and more than one edge per two vertices are called multigraphs.

Vertices u, v are adjacent if $e = \{u, v\} \in E(G)$. The edge e is incident with u and v. Vertices adjacent to u are called neighbours of u. The set of neighbours of u is denoted $N(u)$.

Planar graph: can be represented with no edges crossing each other.

4.2 Isomorphism

Two graphs G_1, G_2 are isomorphic if there is a bijection $f : V(G_1) \to V(G_2)$ such that $f(u), f(v)$ are adjacent in G_2 iff u, v are adjacent in G_1. f is called an isomorphism and it preserves adjacency.

If two graphs are isomorphic, then they must have the same number of edges and vertices. However, two graphs may have the same number of edges and vertices and not be isomorphic. An easy way to prove non-isomorphism is to find a structure (eg a 3-cycle) that is in one graph but not the other.

Isomorphism class of G: collection of graphs that are isomorphic to G. Most of the time, G and all graphs in its isomorphism class have some sort of property.

The identity map on $V(G)$ is an isomorphism. As well, any isomorphism from G to itself is an automorphism of G.

4.3 Degree

Degree of v: $\deg(v)$ is the number of edges incident with a vertex v. Thus, $\deg(v) = |N(v)|$.

Theorem 4.3.1: For any graph $G, \sum_{v \in V(G)} \deg(v) = 2|E(G)|$ (Handshaking lemma).

Corollary 4.3.2: The number of vertices of odd degree in a graph is even.

Corollary 4.3.3: The average degree of a vertex in G is $\frac{2|E(G)|}{|V(G)|}$.

k-regular graph: A graph in which every vertex has degree k, for some fixed k.

Complete graph: A graph in which all pairs of distinct vertices are adjacent. The complete graph with p vertices is denoted $K_p, p \geq 1$. As well, K_p is $(p-1)$-regular. There are $\binom{p}{2}$ edges in K_p.

4.4 Bipartite Graphs

Bipartite graph: One with a bipartition (A, B), such that all vertices are in A or B and all edges join a vertex in A to a vertex in B. $K_{m,n}$ is the complete bipartite graph with all vertices in A adjacent to all vertices in B and $|A| = m, |B| = n$.

K_3 is not bipartite, anything containing K_3 is not bipartite.

n-cube: For $n \geq 0$, the graph where the vertices are binary strings of length n and two strings are adjacent iff they differ in exactly one position. A n-cube has 2^n vertices and $n2^{n-1}$ edges. As well, n-cubes are bipartite where the bipartition is vertices with an even number of 1's and vertices with an odd number of 1's.

4.5 Paths and Cycles

Subgraph S of graph G is a graph where $V(S) \subseteq V(G)$ and $E(S)$ is a subset of edges of G that have both vertices in $V(S)$. If $V(S) = V(G)$, then S is a spanning subgraph of G. However, it does not imply that $E(S) = E(G)$. If S is not the same graph as G, then S is a proper subgraph.

Isomorphic graphs of subgraphs are also subgraphs.

A walk in G from v_0 to $v_n, n \geq 0$ is a sequence of vertices and edges of G: $v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$. Since each pair of vertices can be joined by at most one edge, this can also be expressed as $v_0 v_1 \cdots v_n$. This is a v_0, v_n-walk. The length is the number of edges, in this case, n. A walk is closed if $v_0 = v_n$. $v_0 v_0$ is a trivial, closed walk of length 0.

A path in G is a walk where all vertices are distinct. Since all vertices are distinct, so are all the edges. A single vertex is a trivial walk and path. All paths are walks.

In a graph G, there is a well-defined shortest walk between two vertices and longest path. The longest walk is infinite. A common proof technique is to use shortest walk or longest path and derive a contradiction.

Theorem 4.6.2: If there is a walk from x to y in G, then there is a path.

Theorem 4.6.3: Let $x, y, z \in V(G)$. If a xy path exists and a yz path exists, then a xz path exists.

A $n-$cycle in G is a subgraph with n distinct vertices, where $n \geq 3$ and n distinct edges. Also, a non-trivial closed walk with no repeated vertices or edges, except at the start and end. Or, a connected graph that is regular of degree two. A single vertex is not a cycle, since it would require an edge to itself. Two vertices cannot be a cycle either, since it would require two edges between the two vertices. The shortest cycle is of length 3. There are $2n$ closed walks of length n associated with a certain $n-$cycle since you can start at any of the n vertices then go either forwards or backwards.

An $n-$cycle is denoted $v_0, v_1, \ldots, v_{n-1}, v_0$. Note v_0 at the start and end.

Theorem 4.6.4: If every vertex in G has degree at least 2, then G contains a cycle. However, the converse is not necessarily true. All vertices that are part of the cycle have degree at least 2 but not all vertices are in the cycle.

The girth of G is the length of the shortest cycle in G, denoted $g(G)$.

A Hamilton cycle is a spanning cycle.

4.6 Connectedness

G is connected iff for each two vertices x, y, there is a path from x to y. Single vertex graph is connected.

Theorem 4.8.2: Let G be a graph and $v \in V(G)$. If for all $w \in V(G)$, there is a path from v to w, then G is connected.

A component of G is a non-empty subgraph C of G such that C is connected and no subgraph of G that properly contains C is connected. In other words, a component is maximal and connected. It is not possible to add other vertices or edges to C to get another connected non-empty subgraph.

For a subset X of vertices of G, the cut induced by X is the set of edges that have exactly one end in X.

Theorem 4.8.5: G is not connected iff there is a proper, non-empty subset X of $V(G)$ such that the cut induced by X is empty.

No edge joins a vertex inside a component with a vertex outside the component. Thus, the vertices of a component induces an empty cut. Also, the cut induced by the empty set is empty. The cut of all vertices in the graph is empty.

4.7 Eulerian Circuits

An Eulerian circuit of a graph G is a closed walk that contains every edge of G exactly once. Since this is a closed walk, vertices can be visited more than once, but not edges.

Theorem 4.9.2: Let G be connected. Then, G has an Eulerian circuit iff every vertex has even degree. This is proved with strong induction on the number of edges in G.

A disconnected graph has an Eulerian circuit iff one component has all the edges and every vertex in that component has even degree and the other components have no edges.

Additional interesting theorem: If G has k vertices of odd degree, then all vertices can be visited in k separate walks.

4.8 Bridges

For some edge $e \in E(G), G - e$ (or $G \backslash e$) is the graph whose vertex set is $V(G)$ and edge set is $E(G) \backslash \{e\}$. This removes an edge from G.

An edge e of G is a bridge iff $G - e$ has more components than G.

Theorem 4.10.2: If $e = xy$ is a bridge of a connected graph G, then $G - e$ has exactly two components and x, y are in different components.

Theorem 4.10.3: e is a bridge of G iff it is not in any cycle of G. Useful contrapositive: e is in a cycle iff it is not a bridge.

Theorem 4.10.4: If there are two distinct paths from vertex u to v in G, then G contains a cycle. Useful contrapositive: If G has no cycles, then each pair of vertices is joined by at most one path.

5 Trees

5.1 Trees

A tree is a connected graph with no cycles. Or, a tree is a minimally connected graph; removing any edge makes it disconnected. A tree has the minimum number of edges needed to connect all the vertices. A single vertex is a tree.

A forest is a graph with no cycles. May or may not be connected. A tree is also a forest. Every component of a forest is a tree.

Theorem 5.1.3: If u, v are vertices in a tree T, then there is a unique $u, v-$path in T.

Theorem 5.1.4: Every edge of a tree T is a bridge.

Theorem 5.1.5: If T is a tree, then $|E(T)|= |V(T)|-1$. This can be proved with strong induction on the number of edges or on the number of vertices.

Theorem 5.1.6: If G is a forest with k components, then $|E(G)|= |V(G)|-k$.

A leaf in a tree is a vertex of degree 1.

Theorem 5.1.8: A tree with at least two vertices has at least two leaves. This can be proved with a longest path argument.

Additionally, if a tree T contains a vertex of degree r, where $r \geq 3$, then there are at least r leaves.

5.2 Spanning Trees

A spanning tree is a spanning subgraph which is also a tree. Out of all spanning subgraphs, spanning trees have the fewest edges while remaining connected and reaching all vertices.

Theorem 5.2.1: G is connected iff it has a spanning tree. The forwards part of this proof uses induction on the number of cycles in the graph.

Theorem 5.2.2: If G is connected with p vertices and $q = p - 1$ edges, then G is a tree. Alternatively, could prove that G has no cycles.

Theorem 5.2.3: If T is a spanning tree of G and e is an edge not in T, then $T + e$ contains exactly one cycle C. Moreover, if e' is any edge on C, then $T + e - e'$ is also a spanning tree of G. Regardless of whether e is in G, $T + e$ contains exactly one cycle. However, in order for $T + e - e'$ to be a spanning tree of G, e must be in G.

Theorem 5.2.4: If T is a spanning tree of G and e is an edge in T, then $T - e$ has 2 components. If e' is in the cut induced by one of the components, then $T - e + e'$ is also a spanning tree of G.

Overall, if a graph G with n vertices satisfies any two of the following: G is connected, G has no cycles, G has n-1 edegs, then it is a tree. If any two of the three are true, then the third must be true as well (proof at the end of this document).

5.3 Characterizing Bipartite Graphs

Theorem 5.3.1: An odd cycle is not bipartite. An odd cycle is a cycle on an odd number of vertices.

Theorem 5.3.2: A graph is bipartite iff it has no odd cycles. Prove this using the fact that all connected graphs have a spanning tree and trees are bipartite.

5.4 Minimum Spanning Tree

MST problem: Given a connected graph G and a weight function on the edges $w : E(G) \to \mathbb{R}$, find a spanning tree in G whose total edge weight is minimized.

Prim's algorithm:

1. Let v be any vertex in G. Let T be the tree of just v
2. While T is not a spanning tree of G, and breaking ties in step b) arbitrarily,
    a) Look at all the edges in the cut induced by $V(T)$
    b) Let $e = uv$ be an edge with the smallest weight in the cut, where $u \in V(T), v \notin V(T)$
    c) Add v to $V(T)$ and add e to $E(T)$

Prim's algorithm is a greedy algorithm. For any graph H, let $w(H) = \sum_{e \in E(H)} w(e)$

Theorem 5.6.1: Prim's algorithm produces a MST for G. Prove this by induction on the size of T produced in each iteration of step 2 of the algorithm.

6 Planar Graphs

6.1 Planarity

A graph is planar if it has a planar embedding. A planar embedding is a drawing of the graph in the plane so that its edges intersect only at their ends and no two vertices coincidence. Or, no edge crosses another edge. A single graph may have both planar and non planar embeddings. As long as it has at least one planar embedding, it is planar. A graph is planar iff all its components are planar.

A planar embedding partitions the plane into connected regions called faces. The outer face is unbounded. The vertices and edges around the face is the boundary. Two faces are adjacent if they are incident with a common edge. A closed walk on the edges of the boundary is a boundary walk. The total number of edges, not necessarily all unique, in the boundary walk is the degree of the face.

A bridge of a planar embedding is incident with only one face, and is contained in the boundary walk of that face twice, one for each side. An edge which is not a bridge is in a cycle and is incident with exactly two faces and is contained in the boundary walk of each face once.

Handshaking Lemma for Face: $\sum_{i=1}^{s} \deg(f_i) = 2|E(G)|$

Theorem 7.1.3: If the connected graph G has a planar embedding with f faces, the average degree of each face is $\frac{2|E(G)|}{f}$

For faces in disconnected graphs whose boundaries are in several components, there is no closed walk around the boundary. Thus, the degree is the sum of the lengths of the boundaries around each component.

If e is a bridge, then the two sides of e are in the same face. Otherwise, e is in a cycle, so the two sides are in different faces.

6.2 Euler's Formula

Euler's Formula: Let G be a connected graph with n vertices, m edges. If G has a planar embedding with f faces, then $n - m + f = 2$. Thus, all planar embeddings of a certain planar graph have the same number of faces. This can be proved by induction on m, for a fixed n.

6.3 Platonic Solids

Platonic solids are polyhedra where all the faces have the same degree and all vertices have the same degree. A platonic embedding can be obtained from each platonic solid where each vertex has the same degree or at least 3 and each face has the same degree of at least 3. A connected planar graph is platonic if it has a platonic embedding.

Theorem 7.4.1: There are five platonic solids:

1. Tetrahedron: vertices have degree 3, faces have degree 3, 4 vertices, 6 edges, 4 faces
2. Cube: vertices have degree 3, faces have degree 4, 8 vertices, 12 edges, 6 faces
3. Octahedron: vertices have degree 4, faces have degree 3, 6 vertices, 12 edges, 8 faces
4. Icosahedron: vertices have degree 5, faces have degree 3, 12 vertices, 60 edges, 20 faces
5. Dodecahedron: vertices have degree 3, faces have degree 5, 20 vertices, 30 edges, 12 faces

6.4 Nonplanar Graphs

Theorem 7.5.1: If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle. The boundary is not necessarily itself a cycle. Intuitively, each face must be separated from other faces by a cycle, unless there is only one face, which is a tree which does not contain a cycle.

Theorem 7.5.2: Let G be a planar embedding with n vertices and m edges. If each face has degree at least d, then $(d-2)m \leq d(n-2)$. G does not need to be connected.

Theorem 7.5.3: In a planar graph G with $n \geq 3$ vertices and m edges, $m \leq 3n-6$. G does not need to be connected. To show a graph is non planar, show that it has more than $3n-6$ edges. The converse is not necessarily true. Just because a graph has at most $3n-6$ edges does not mean it is planar. For example, $K_{3,3}$ or K_5 with an extra vertex.

Theorem 7.5.4: K_5 is not planar.

Theorem 7.5.5: A planar graph has a vertex of degree at most 5. Prove this using Theorem 7.5.3.

Theorem 7.5.6: In a connected or disconnected bipartite planar graph G with $n \geq 3$ vertices and m edges, $m \leq 2n-4$.

6.5 Kuratowski's Theorem

Edge subdivision: Replacing every edge in G by a path of length 1 or more. If the path has length $m > 1$, then there are $m-1$ new vertices, each with degree 2 within the subdivision, and $m-1$ new edges are created. If the path has length $m = 1$, then the edge is unchanged. G is planar iff all edge subdivisions of G are planar. If G has a nonplanar subgraph, G is nonplanar.

Kuratowski's Theorem: A graph is nonplanar iff it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$. Proof outside of scope of this course.

Finding subdivisions: Do not repeat vertices or edges in the subdivision, usually $K_{3,3}$ is found mroe often than K_5, requires trial and error.

Thus, all nonplanar graphs must have at least 5 vertices.

6.6 Coloring and Planar Graphs

A $k-$coloring of G is a function from $V(G)$ to a set of at most k colours, so that adjacent vertices always have different colors. A graph with a $k-$coloring is $k-$colorable. Every graph is at the very least $|V(G)|-$colorable and if G is $k-$colorable, it is also $(k+1)-$colorable, etc.

Theorem 7.7.2: G is $2-$colorable iff it is bipartite.

Theorem 7.7.3: K_n is $n-$colorable and not $k-$colorable for any $k < n$.

Theorem 7.7.4: Every planar graph is $6-$colorable. Proof by induction on the number of vertices, and note that every planar graph has a vertex of degree atm ost 5.

Edge contraction: G/e is obtained from G by contracting the edge $e = xy$. The edge e is not in G/e, and vertices x, y have been replaced by a single vertex u. All edges which used to lead to x, y now lead to u. G/e has $n-1$ vertices and at most $m-1$ edges and if G is planar, G/e is planar. However, it is possible that G/e is planar but G is nonplanar. If G/e can have multiple edges to the same pair of vertices, then there is exactly $m-1$ edges. If not, there are at most $m-1$ edges.

Theorem 7.7.6, 7.7.7: Every planar graph is $5-$colorable and in fact, $4-$colorable.

7 Matching

7.1 Matching

A matching in G is a set M of edges of G such that no two edges in M have a common end. M matches certain pairs of adjacent vertices. The empty set or set of one edge is a matching.

A vertex v of G is saturated by M, or M saturates v, if v is incident with an edge in M. Every edge has a matching, the empty set, we want to find a largest matching, the maximum matching.

A perfect matching is a special maximum matching of size $\frac{n}{2}$, every vertex is saturated.

Not all graphs have a perfect matching. Graphs with an odd number of vertices do not have perfect matchings but not all graphs with an even number of vertices has perfect matchings either.

Useful application: Let A, B be sets, let G be a bipartite graph with bipartition (A, B). A maximum matching represents the most matches between an element in A and an element in B where each element is matched to at most one other element.

Alternating path with respect to M: A path $v_0v_1 \cdots v_n$ such that the edges in the path alternate between being in M and not being in M. For example, $v_0v_1, v_2v_3, \ldots \in M, v_1v_2, v_3v_4, \ldots \notin M$.

Augmenting path with respect to M: alternating path joining two distinct vertices, neither of which is saturated by M. Augmenting paths have odd length.

Theorem 8.1.1: If M has an augmenting path, it is not a maximum matching. To get a larger matching, put the edges in the path which are not in M into M and remove the edges which are in M from M.

7.2 Covers

A cover of G is a set C of vertices such that every edge of G has at least one end in C. It is easy to find large covers, $V(G)$ is a cover. If G is bipartite with bipartition A, B, then A and B are both covers.

Theorem 8.2.1: If M is a matching of G and C is a cover of G, then $|M| \leq |C|$.

Theorem 8.2.2: Let M, C be any matching and cover of G. If $|M| = |C|$, then M is a maximum matching and C is a minimum cover. However, some graphs do not have a maximum matching of equal size to a minimum cover. For example, a 3-cycle has maximum matching of size 1 but minimum cover of size 2.

7.3 Konig's Theorem

Konig's Theorem: In a bipartite graph, the maximum size of a matching is equal to the minimum size of a cover. Prove this using $XY-$construction. In this algorithm, with bipartition (A, B),

1. X_0 is the set of unsaturated vertices in A
2. X starts off equal to X_0 and by the end, contains all vertices in A reachable via an alternating path starting with a vertex in X_0
3. Y starts empty and by the end, contains all vertices in B reachable via an alternating path starting with a vertex in X_0

When the algorithm stops, the matching is maximum and the cover $Y \cup (A \backslash X)$ is minimum. Note here that X, Y in $Y \cup (A \backslash X)$ are the X, Y of the last iteration of the algorithm which causes termination.

7.4 Applications of Konig's Theorem

Let $N(D)$ denote the set of all vertices in G which are adjacent to at least one vertex in the subset D.

Hall's Theorem: A bipartite graph with bipartition (A, B) has a matching saturating every vertex in A iff every subset D of A satisfies $|N(D)| \geq |D|$.

7.5 Perfect Matchings in Bipartite Graphs

Theorem 8.6.1: A bipartite graph G with bipartition (A, B) has a perfect matching iff $|A| = |B|$ and every subset D of A satisfies $|N(D)| \geq |D|$.

Theorem 8.6.2: If G is a $k-$regular bipartite graph with $k \geq 1$, then G has a perfect matching. This is proved with the observation that $\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$, so $k|A| = k|B|$ and similarly, $k|D| \leq k|N(D)|$.

8 Other

Key equations, summations, and theorems:

1. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
2. For any $n \geq 0, (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$
3. For any $1 \leq k \leq n, \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
4. For any $n, k \geq 0, \binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$
5. $[x^n]A(x)B(x) = \sum_{k=0}^{n} a_k b_{n-k}$
6. If $[x^0]Q(x) \neq 0$, then there is a unique FPS $A(x)$ such that $Q(x)A(x) = P(x)$
7. $1 + x + x^2 + \cdots = \frac{1}{1-x}$
8. $1 + x + x^3 + \cdots + x^k = \frac{1-x^{k+1}}{1-x}$
9. $\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$
10. A FPS has a unique inverse iff the constant term is non-zero
11. If $[x^0]B(x) = 0$, then $A(B(x))$ is a FPS. If $[x^0]B(x) \neq 0$, then $A(B(x))$ might be a FPS
12. Sum lemma requires partition, product lemma requires sum of the weights of the parts of the tuple is eqaul to the weight of the entire tuple
13. There are $\binom{n-1}{k-1}$ compositions of n of size k
14. $A \cup B$ is unambiguous iff $A \cap B \neq \emptyset$

Theorem: If $\epsilon \in A$, then A^* is ambiguous.

Since $\epsilon = \epsilon\epsilon$, then $\epsilon \in A$ and $\epsilon \in A^2$, so they are not disjoint, so $\{\epsilon\} \cup A \cup A^2 \cup \cdots = A^*$ is ambiguous.

Theorem: There are 2^{n-1} compositions of n.

A proof with generating series is in page 33 of the textbook. However, there is an obvious bijection between the set of compositions of n and the set of compositions of $n-1$, which shows that there are twice as many compositions of n. Since there is 1 composition of 0, the proof follows inductively.

Theorem: Not all k-regular graphs are isomorphic to each other.

Theorem: If $m = n$, then $k_{m,n}$ is a $m-1 = n-1$ regular graph. Also, $k_{m,n}$ is the same as $k_{n,m}$.

Theorem: Let S, T be sets. Let w be a weight function for S. If there is a bijection $f : S \to T$, and there is a weight function w^* for T such that $\forall \sigma \in S, w(\sigma) = w^*(f(\sigma))$, then $\Phi_S(x) = \Phi_T(x)$ with respect to w, w^*.

Proof: $\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\sigma \in S} x^{w^*(f(\sigma))} = \sum_{\delta \in T} x^{w^*(\delta)} = \Phi_T(x)$

Theorem: K_n is the smallest $(n-1)-$regular graph. Smallest meaning least number of vertices.

Proof: Let G be $(n-1)-$regular. Then, each vertex has degree $n-1$. Thus, there must be at least n vertices in total. Also, K_n is $(n-1)-$regular, by definition.

Theorem: All $k-$regular graphs with n vertices have $\frac{nk}{2}$ edges. Proof by handshaking lemma.

Theorem: A graph with no edges is trivially bipartite. Take A to be all vertices, B to be empty.

Theorem: The complete bipartite graph $K_{m,n}$ has mn edges.

Theorem: A connected graph has one component. A disconnected graph has at least two.

Theorem: If $v_0, v_1, \ldots, v_n$ is a path where v_n is adjacent to v_0, then $v_0, v_1, \ldots, v_n, v_0$ is a cycle. Moreover, the length of the cycle is one more than the length of the path.

Theorem: If G is connected, then removing an edge e from G results in either one or two components.

Proof: If e is not a bridge, then $G - e$ is still connected. If it is a bridge, by Theorem 4.10.2, $G - e$ has exactly two components.

Theorem If G is connected and has an Eulerian circuit, then there is a cycle. However, the cycle is not necessarily a spanning cycle (Hamilton cycle).

Proof: Since G is connected and has an Eulerian circuit, then every vertex has even degree. Since G is connected, no vertex has degree 0. Thus, every vertex has degree at least 2. Thus, there is a cycle in G. However, as seen on page 113, this cycle is not necessarily a Hamilton cycle.

Theorem: All edges in a tree T are bridges.

Proof: Let $e \in E(T)$. If e were not a bridge, then by theorem 4.10.3, e would be in a cycle. However, trees cannot have cycles. Thus, e is a bridge.

Theorem: The cut induced by vertices in a spanning subgraph S of G is empty.

Proof: Let $e = xy$ be in the cut induced by vertices in S. Suppose without loss of generality $x \in V(S), y \notin V(S)$. But, S is a spanning subgraph. Thus, $y \in V(S)$.

Theorem: Alternate proof for Theorem 5.1.8 Suppose G has m leaves and $n - m$ non-leaves. Then, each non-leaf has degree at least 2. As well, trees have $n - 1$ leaves. By handshaking lemma,

$$\begin{aligned} n - 1 &\geq \frac{m + 2(n-m)}{2} \\ 2n - 2 &\geq m + 2n - 2m \\ -2 &\geq -m \\ 2 &\leq m \end{aligned}$$

Therefore, there are at least two leaves in every tree of at least two vertices.

Theorem: All spanning trees of a connected graph G contain all bridges of G

Proof: By theorem, since G is connected, it has a spanning tree. Let T be any spanning tree of G. Suppose for contradiction $e \notin V(T)$. By theorem, $T + e$ contains exactly one cycle. Since T is a tree, it has no cycles. Thus, e must be part of the cycle, but it is a bridge, a contradiction. Thus, all spanning trees of G contain all bridges of G.

Theorem: A tree T has at most one perfect matching.

Proof: Strong induction on the number of vertices n: When $n = 0$, then there is a trivial perfect matching. When $n = 1$, there is no perfect matching since 1 is odd. Suppose all trees with k vertices, where $n > k \geq 1$ have at most one perfect matching. Suppose T has n vertices. If n is odd, then T has no perfect matchings. If n is even, since $n > 1$, there is at least one leaf a which is adjacent to exactly one vertex b through the edge $e = ab$. Let F be T with a, b and all incident edges removed. F may or may not be connected, but there are no cycles, so it is a forest. By strong induction, each component of F has at most one perfect matching. If there is a perfect matching of T, it must include the perfect matching of each component of F plus the edge e, since there is no other way to saturate a. Therefore, T has at most one perfect matching.

Theorem: Any connected planar graph G_1 with n vertices and m edges has fewer faces than any disconnected planar graph G_2 with n vertices and m edges.

Proof: By Euler's formula, G_1 has $2 - n + m$ faces. Suppose the components of G_2 are $C_1, \ldots, C_k$ and suppose C_i has n_i vertices and m_i edges. Since each component is connected, by Euler's formula, C_i has

$f_i = 2 - n_i + m_i$ faces. So,

$$\sum_{i=1}^{k} f_i = 2k - \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} m_i = 2k - n + m$$

However, the outer face is counted k times, once in each f_i. The non-outer faces of each component are self-contained. Thus, the total number of faces in G_2 is the sum of the number of non-outer faces in each component plus one,

$$\left(\sum_{i=1}^{k} f_i\right) - (k-1) = \left(\sum_{i=1}^{k} f_i\right) - k + 1 = 2k - n + m - k + 1 = k - n + m + 1$$

Since G_2 is disconnected, $k \geq 2$. So, the total number of faces in G_2 is,

$$k - n + m + 1 \geq 2 - n + m + 1 = 3 - n + m > 2 - n + m$$

Theorem: Any disconnected planar graph with n vertices can be transformed into a connected planar graph with n vertices.

Proof:

Theorem: A graph with n vertices and fewer than $n-1$ edges is disconnected.

Proof: Suppose for contradiction it is connected. Then there is a spanning tree of $n-1$ edges. But there are fewer than $n-1$ edges in the graph, a contradiction.

Theorem: K_p has minimum cover of size $p-1$ and maximum matching of size floor of $\frac{p}{2}$.

Theorem: If $V(G)$ is partitioned into four sets A, B, C, D and $A \cup B$ is a cover, then there is no edge joining a vertex in C and a vertex in D.

Proof: Since $A \cup B$ is a cover, all edges in G have at least one end in either A or B.

Theorem: Suppose G has n vertices, is connected, and has no cycles. Then, it has $n-1$ edges.

Proof: At least $n-1$ edges are needed to connect n vertices. Suppose there were more than $n-1$ edges. Let T be a spanning tree of G. Then, T has $n-1$ edges, so there is some edge e which is not in T. However, $T+e$ is a subgraph of G and has a cycle, a contradiction. Therefore, there must be exactly $n-1$ edges.

Theorem: Suppose G has n vertices, has no cycles, and has $n-1$ edges. Then, it is connected.

Proof: Suppose for contradiction G is not connected. Then, it is a forest with k components. Then, there are $n-k$ edges, so $k=1$ and G is connected.

Theorem: Suppose G has n vertices, is connected, and has $n-1$ edges. Then, it has no cycles.

Proof: Suppose for contradiction G has a cycle C. Suppose e is some edge in C. Then, e is not a bridge, so $G-e$ is still connected. However, $G-e$ has n vertices and $n-2$ edges and it is not possible to connect n vertices with fewer than $n-1$ edges. Therefore, G has no cycles.

Theorem: All connected graphs G with n vertices and more than $n-1$ edges have at least one cycle.

Proof: Let T be a spanning tree of G. Then, T has $n-1$ edges, so there is some edge $e \in E(G)$ which is not in T and $T+e$ has a cycle. Therefore, G has a cycle.

Theorem: If M is a maximum matching of G and C is a minimum cover, $|C| \leq 2|M|$.

Interesting things:

1. Characteristic polynomial can have complex roots

2. There are sets of strings which cannot be expressed unambiguously
3. A set of strings can be empty. If $A = \{1\}, B = \emptyset$, then $AB = \{1\epsilon\} = \{1\}$
4. If A is an unambiguous expression for S, then any part of A is also unambiguous
5. Generally, use recursive decomposition when trying to find strings without a certain substring. Otherwise, block decomposition is more useful.
6. A single set of strings can never be ambiguous or unambiguous
7. When asked to give an unambiguous decomposition, it is allowed to use set union. For example, the set of strings where each block has length at least 2 is:

$$(\{00\}\{0\}^* \cup \{\epsilon\})(\{11\}\{1\}^*\{00\}\{0\}^*)^*(\{11\}\{1\}^* \cup \{\epsilon\})$$

8. To represent blocks of length not divisible by 3, do $\{1, 11\}\{111\}^*$.