1.2 Riemann Sums and the Definite Integral

Partition: A finite increasing sequence of numbers of the form $a = t_0 < t_1 < \cdots < t_n = b$, which divides the interval $[a, b]$ into n subintervals $[t_0, t_1], \ldots, [t_{n-1}, t_n]$. The norm of a partition is the length of the widest subinterval.

Riemann Sum: Given a bounded function f on $[a, b]$, a partition P of $[a, b]$ and a set $\{c_1, \ldots, c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then a Riemann sum for f with respect to P is a sum S $= \sum_{i=1}^{n} f(c_i)\Delta t_i$.

Regular n-Partition: A partition where each subinterval has the same length $\Delta t_i = \frac{b-a}{n}$

Right-hand Riemann sum: For regular partition, choose c_i to be the right endpoint of each subinterval. $R_n = \sum_{i=1}^{n} f(t_i)\Delta t_i = \sum_{i=1}^{n} f\big(a + i\big(\frac{b-a}{n}\big)\big)\big(\frac{b-a}{n}\big)$

Left-hand Riemann sum: For regular partition, choose c_i to be the left endpoint of each subinterval. $L_n = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i = \sum_{i=1}^{n} f\big(a + (i-1)\big(\frac{b-a}{n}\big)\big)\big(\frac{b-a}{n}\big)$

Integrable: A bounded function f is integrable on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} ||P_n|| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, then $\lim_{n\to\infty} S_n = I$. Then $I = \int_a^b f(t)\ dt$.

Integrability Theorem for Continuous Functions: Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$ and $\int_a^b f(t)\ dt = \lim_{n\to\infty} S_n$ where $S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$ is any Riemann sum associated with the regular n-partitions. In particular, choose $S_n = R_n$ or $S_n = L_n$. This theorem also holds if f is bounded and has finitely many discontinuities on $[a, b]$.

1.3 Properties of the Definite Integral

Assume that f, g are integrable on $[a, b]$. Then,

(1) If $m \leq f(t) \leq M \forall t \in [a, b]$, then $m(b - a) \leq \int_a^b f(t)\ dt \leq M(b - a)$

(2) The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t)\ dt| \leq \int_a^b |f(t)|\ dt$

Assume that f is integrable on an interval I containing a, b, c : $\int_a^b f(t)\ dt = \int_a^c f(t)\ dt + \int_c^b f(t)\ dt$

1.4 The Average Value of a Function

If f is continuous on $[a, b]$, the average value of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(t)\ dt, \exists c \in [a, b], f(c) =$av.

1.5 FTC Part 1

Assume f is continuous on an open interval I containing a point a. Let $G(x) = \int_a^x f(t)\ dt$. Then $G(x)$ is differentiable at each $x \in I$ and $G'(x) = f(x)$. Equivalently, $G'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x)$.

Extended FTC: Assume f is continuous and g, h are differentiable. Let $H(x) = \int_{g(x)}^{h(x)} f(t)\ dt$. Then $H(x)$ is differentiable and $H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$.

1.6 FTC Part 2

Common functions and their antiderivative:

1) $\frac{1}{1+x^2} \to \arctan(x) + c$
2) $\frac{1}{\sqrt{1-x^2}} \to \arcsin(x) + c$
3) $-\frac{1}{\sqrt{1-x^2}} \to \arccos(x) + c$
4) $\sec(x)\tan(x) \to \sec(x) + c$
5) $a^x, a > 0, a \neq 1 \to \frac{a^x}{\ln(a)} + c$
6) $\sec(x) \to \ln(|\sec(x) + \tan(x)|) + c$
7) $\frac{1}{x^2+b^2}\ dx \to \frac{1}{b}\arctan\left(\frac{x}{b}\right) + c$
8) $\sec^2(x) \to \tan(x) + c$
9) $-\csc^2(x) \to \cot(x) + c$
10) $-\csc(x)\cot(x) \to \csc(x) + c$

FTC 2: Assume f is continuous and F is any antiderivative of f. Then, $\int_a^b f(t)\ dt = F(b) - F(a)$.

An indefinite integral represents the family of all functions that are antiderivatives of the given function, which is why it requires the $+c$.

1.7 Change of Variables

$\int f(g(x)) \cdot g'(x)\ dx = \int f(u)\ du$, where $u = g(x)$.

Good choices for u: If you see a function and its derivative, let u be the function. Or, choose u is the base of an ugly power, the denominator, the function inside $\sin, \cos, e, \ln, etc.$

Substitution for definite integrals: If g' is continuous on $[a, b]$ and f is continuous between $g(a), g(b)$, then $\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)\ dx = \int_{u=g(a)}^{u=g(b)} f(u)\ du$

2.1 Inverse Trigonometric Substitutions

1) $\sqrt{a^2 - x^2} \to x = a\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
2) $\sqrt{a^2 + x^2} \to x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
3) $\sqrt{x^2 - a^2} \to x = a\sec\theta, 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$

May require completing the square then u-substitution to replace the square. As well, when we root a squared trig function, first do absolute value, then just take the positive and say that we are on a range such that the trig function is positive.

2.2 Integration by Parts

$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$ or $\int U dV = UV - \int V dU$.

Definite IBP: Assume f, g are such that f', g' are continuous on an interval containing a, b. Then, $\int_a^b f(x)g'(x)\ dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\ dx$

Pick U to be the first thing on this list ITATE: inverse trig functions, logarithms, algebraic functions, trig functions, exponential functions.

2.3 Partial Fractions

Requires: degree of numerator integral is less than the degree of denominator. Or else, do long division.

Distinct linear factors: $\frac{A}{ax+b}$

Repeated linear factors: $\frac{A}{ax+b} + \cdots + \frac{A_n}{(ax+b)^n}$

Distinct irreducible quadratic factors: $\frac{Ax+B}{ax^2+bx+c}$

Repeated irreducible quadratic factors: $\frac{Ax+B}{ax^2+bx+c} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$

2.4 Introduction to Improper Integrals

Type I Improper Integral: Let f be integrable on $[a, b]$ for each $a \leq b$. Then $\int_a^\infty f(x)\ dx$ converges if $\lim_{b\to\infty} \int_a^b f(x)\ dx$ exists. Otherwise, it diverges. As well, $\int_{-\infty}^\infty f(x)\ dx$ converges if both $\int_{-\infty}^c f(x)\ dx$ and $\int_c^\infty f(x)\ dx$ converge for some $c \in \mathbb{R}$.

P-Test for Type I Improper Integrals: $\int_a^\infty \frac{1}{x^p}\ dx$ converges iff $p > 1$. If $p > 1$, then $\int_a^\infty \frac{1}{x^p}\ dx = \frac{1}{p-1}$

Property: If $\int_a^\infty f(x)\ dx$ converges and $a < c < \infty$, then $\int_c^\infty f(x)\ dx$ converges and $\int_a^\infty f(x)\ dx = \int_a^c f(x)\ dx + \int_c^\infty f(x)\ dx$.

Monotone Convergence Theorem for Functions: Assume f is non-decreasing on $[a, \infty)$.

1) If $\{f(x)|x \in [a, \infty)\}$ is bounded above, then $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) = L = lub(\{f(x)|x \in [a, \infty)\})$.
2) If $\{f(x)|x \in [a, \infty)\}$ is not bounded above, then $\lim_{x\to\infty} f(x) = \infty$

Comparison Test for Type I Improper Integrals: Assume $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and both f, g are continuous on $[a, \infty)$.

1) If $\int_a^\infty f(x)\ dx$ converges, then so does $\int_a^\infty g(x)\ dx$
2) If $\int_a^\infty g(x)\ dx$ diverges, then so does $\int_a^\infty f(x)\ dx$

Compare to P-test or: $\int_0^\infty e^{-x}\ dx$ converges.

Absolute Convergence for Type I Improper Integrals: Let f be integrable on $[a, b)$ for all $b \geq a$. Then $\int_a^\infty f(x)\ dx$ converges absolutely if $\int_a^\infty |f(x)|\ dx$ converges. Useful for comparison theorem with trig functions because they can be negative sometimes. Take the absolute value to maintain the requirements for Comparison Theorem $0 \leq g(x) \leq f(x)$. The converse of this theorem is false.

Type II Improper Integral: Contains a vertical asymptote. Let f be integrable on $[t, b]$ for every $t \in (a, b]$ with either $\lim_{x\to a^+} f(x) = \pm\infty$. Then $\int_a^b f(x)\ dx$ converges if $\lim_{t\to a^+} \int_t^b f(x)\ dx$ exists. Otherwise, it diverges. A similar argument for the upper bound b.

P-Test for Type II Improper Integrals: $\int_0^1 \frac{1}{x^p}\ dx$ converges iff $p < 1$. If $p < 1$, then $\int_0^1 \frac{1}{x^p}\ dx = \frac{1}{1-p}$.

3 Applications of Integration

Let f, g be continuous on $[a, b]$. Let A be the region bounded by the graphs of f, g, the line $t = a$ and the line $t = b$. Then $A = \int_a^b |g(t) - f(t)|\, dt$

Volumes of Revolution: Disk Method I: Let f be continuous on $[a, b]$ with $f(x) \geq 0 \forall x \in [a, b]$. Let W be the region bounded by the graphs of f, the x-axis, and the lines $x = a, x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is $V = \int_a^b \pi f(x)^2\, dx$

Volumes of Revolution: Disk Method II: Let f, g be continuous on $[a, b]$ with $0 \leq f(x) \leq g(x) \forall x \in [a, b]$. Let W be the region bounded by the graphs of f and g and the lines $x = a, x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is $V = \int_a^b \pi(g(x)^2 - f(x)^2)\, dx$

Volumes of Revolution: Shell Method: Let $a \geq 0$. Let f, g be continuous on $[a, b]$ with $f(x) \leq g(x) \forall x \in [a, b]$. Let W be the region bounded by the graphs of f and g and the lines $x = a, x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the y-axis is $V = \int_a^b 2\pi x(g(x) - f(x))\, dx$

Summary: if we are rotating around a horizontal line, use washer/disc to work with x, use cylindrical shells to work with y. If we are rotating around a vertical line, use cylindrical shells to work with x, washer/disc to work with y.

4 Differential Equations

Order of an ODE is the order of the highest derivative that appears in the equation.

Linear: An ODE is linear if it contains only linear functions in y, y', y'', etc. Pretend all functions of x are constants, a linear ODE is in the form $A(x)y'' + B(x)y' + C(x)y = D(x)$

General solution of an ODE: set of all possible solutions, including arbitrary constants.

Particular solution: All arbitrary constants have been determined. To do this, we need extra info such as certain values. This info is the initial conditions and an ODE with initial conditions is called an initial value problem. Linear DEs have only one solution whereas Separable can have many.

Separable: Has form $y' = g(x)h(y)$, can separate the y on one side and x on the other. Some DE such as $y' = 3x(y-1)$ may be both separable and linear.

Solving a separable DE: Write in the form $y' = g(x)h(y)$, consider constant solutions which are all values of y for which $h(y) = 0$ (eg $y \equiv 0, y \equiv 1$). Next, integrate $\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$ and rearrange for y. If we are solving an IVP and a constant function for y is a solution, we can stop because IVP only has one solution.

Solving a linear DE: Write in the form $y' = f(x)y + g(x)$, then solutions are $y = \frac{1}{\mu(x)}\left(\int g(x)\mu(x)\, dx\right)$, where $\mu(x) = e^{-\int f(x)\, dx}$.

Existence and Uniqueness Theorem for First Order Linear Differential Equations: Assume f, g are continuous functions on an interval I. Then for $x, y \in I$, the IVP $y' = f(x)y + g(x)$ has exactly one solution on the interval I. This is not necessarily true for Separable DEs.

Mixing problem: Let $X(t)$ be the amount of salt in the tank at time t. Then $X'(t)$ is rate in minus rate out.

Newton's Law of Cooling: Let T_a be the ambient temperature, T_0 be the start temperature, and $T(t)$ be the temperature at time t. Then: $T' = k(T - T_a)$ and $T(t) = (T_0 - T_a)e^{kt} + T_a$

5 Numerical Series

The k-th partial sum S_k of $\sum_{n=1}^{\infty} a_n$ is $\sum_{n=1}^{k} a_n$. The series converges if the sequence of partial sums converges.

5.4 Arithmetic of Series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} ca_n$ converges for every $c \in \mathbb{R}$ and $\sum_{n=1}^{\infty} a_n + b_n$ converges. As well, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ converges for each j and if $\sum_{n=j}^{\infty} a_n$ converges for some j, then $\sum_{n=1}^{\infty} a_n$ converges.

5.5 Positive Series

Monotone Convergence Theorem: Let $\{a_n\}$ be a non-decreasing sequence. Then $\{a_n\}$ converges iff it is bounded above.

Positive series: All terms are greater or equal to 0. The partial sums of this series is non-decreasing so by MCT, if the sequence of partial sums is not bounded, the series would diverge.

Comparison Test for Series: Assume $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$, or eventually. Then if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. The contrapositive is also true.

If the first series of an alternating series is a positive, then all odd partial sums are an overestimate.

A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. The series converges conditionally if $\sum_{n=1}^{\infty} |a_n|$ divererges but $\sum_{n=1}^{\infty} a_n$ converges. For positive series, there is no such thing as conditional convergence.

Absolute Convergence Theorem: If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$

Rearrangement of a Series: Putting all the same elements of a series in a different order

Rearrangement Theorem: If $\sum_{n=1}^{\infty}$ is absolutely convergent, then all rearrangements also converge and have the same sum. However, if $\sum_{n=1}^{\infty}$ is conditionally convergent, we can rearrange the series to produce a sum of any number we want (including infinity).

Evaluating a series steps: Try div test, then check absolute convergence with integral/comparison/LCT, then check conditional convergence with AST.

Suppose a_n, f are such that $a_n = f(n) \forall n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$ iff $\lim_{x\to\infty} f(x)$ exists.

Orders of magnitude: constants $< (\ln(n))^p < n^p < p^n < n! < n^n$

6 Power Series

Power series centered at $x = a : \sum_{n=0}^{\infty} a_n(x-a)^n$.

Fundamental Convergence Theorem for Power Series: Let R be the radius of convergence.

1. $R = 0$: series converges at the centre, diverges for all other x
2. $0 < R < \infty$: series converges absolutely for $(a-R, a+R)$ and diverges if $|x-a| > R$, may converge absolutely or conditionally at endpoints.
3. $R = \infty$: series converges absolutely for all x.

Abel's Theorem: Continuity of Power Series: Assume $\sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for each $x \in I$. Then $f(x)$ is continuous on I.

Power Series Differentiation: Assume $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for all $x \in (a-R, a+R)$. Then f is differentiable on $(a-R, a+R)$ and on this interval, $f'(x) = \sum_{n=0}^{\infty} na_n(x-a)^{n-1} = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ is continuous on I. As well, we can re-apply this theorem on the derivative to see that $f''(x)$ exists on the interval, etc. The function has derivatives of all orders on $(a-R, a+R)$.

Differentiating and integrating a power series does not change radius of convergence, but may change endpoints. In general, multiplying a series by a polynomial in n does not change radius.

Uniqueness of Power Series Representations: Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for all $x \in (-R, R)$ where $R > 0$. Then $a_n = \frac{f^{(n)}(a)}{n!}$ and this power series representation is unique.

Integration of Power Series: Assume $\sum_{n=0}^{\infty} a_n(x-a)^n$ has $R > 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for all $x \in (a-R, a+R)$. Then $\sum_{n=0}^{\infty} \int a_n(x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$ also has radius R and its derivative is the original power series. As well, if $[c, b] \subset (a-R, a+R)$, then $\int_c^b f(x) \, dx = \int_c^b \sum_{n=0}^{\infty} a_n(x-a)^n \, dx = \sum_{n=0}^{\infty} \int_c^b a_n(x-a)^n \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}((b-a)^{n+1} - (c-a)^{n+1})$.

Taylor Series: Assume f has derivatives of all orders at $a \in \mathbb{R}$. Then the Taylor Series for f centered at $x = a$ is $\sum_{n=0}^{\infty} \frac{f^{(n}(a)}{n!}(x-a)^n$. However, not all points in the interval of convergence of this Taylor Series will have its function equal to the series. Look at page 274 for an example.

Convergence Theorem for Taylor Series: Assume $f(x)$ has derivatives of all orders on an interval I containing $x = a$. Assume all derivatives are bounded, $|f^{(k)}(x)| \leq M$ for all $x \in I$. Then the function equals its Taylor series for all $x \in I$.