Chapter 1

1.2

Theorem 1.2.1: Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. Some vector $\vec{v}_i, 1 \leq i \leq k$ can be written as a linear combination of $\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k$ if and only if $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$.

Theorem 1.2.2: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly dependent iff $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$, for some $1 \leq i \leq k$.

Basis: A linearly independent spanning set. If $S = \text{Span}\,\mathcal{B}$, then $\mathcal{B}$ is a basis for S. Basis for $\{\vec{0}\}$ is the empty set. Basis is linearly independent so that each vector which is spanned by $\mathcal{B}$ has a unique representation.

Standard basis: The set $\{\vec{e}_1, \dots, \vec{e}_n\}$, where $\vec{e}_i$ is the vector whose i-th component is 1 and all other components are 0.

Line in $\mathbb{R}^n$: Let $\vec{v}, \vec{b} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. A line in $\mathbb{R}^n$ through $\vec{b}$ has vector equation $\vec{x} = t\vec{v} + \vec{b}, t \in \mathbb{R}$.

Plane in $\mathbb{R}^n$: Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^n$ with $\{\vec{v}, \vec{w}\}$ being linearly independent. A plane in $\mathbb{R}^n$ through $\vec{b}$ has vector equation $\vec{x} = r\vec{v} + s\vec{w} + \vec{b}, s, r \in \mathbb{R}$.

Hyperplane in $\mathbb{R}^n$: Let $\vec{v}_1, \dots, \vec{v}_{n-1}, \vec{b} \in \mathbb{R}^n$ with $\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ being linearly independent. The set with vector equation $\vec{x} = c_1\vec{v}_1 + \dots + c_{n-1}\vec{v}_{n-1} + \vec{b}, c_1, \dots, c_{n-1} \in \mathbb{R}$ is a hyperplane in $\mathbb{R}^n$ through $\vec{b}$.

K-Flat in $\mathbb{R}^n$: Let $\vec{v}_1, \dots, \vec{v}_k, \vec{b} \in \mathbb{R}^n$ with $\{\vec{v}_1, \dots, \vec{v}_k\}$ being linearly independent. The set with vector equation $\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k + \vec{b}, c_1, \dots, c_k \in \mathbb{R}$ is a k-flat in $\mathbb{R}^n$ through $\vec{b}$. A hyperplane in $\mathbb{R}^n$ is an $(n-1)$-flat.

1.3

Theorem 1.3.1: Subspace Test: If a non-empty subset of $\mathbb{R}^n$ is closed under addition and multiplication, then it is a subspace of $\mathbb{R}^n$. Note that all subspaces contain $\vec{0}$.

Theorem 1.3.2: If $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of $\mathbb{R}^n$. This implies that if a subset of $\mathbb{R}^n$ has a basis, then it is a subspace.

Proving that $\mathcal{B}$ is a basis for a subspace S:

1. Prove that $\text{Span}\,\mathcal{B} \subseteq S$ by showing that each vector in $\mathcal{B}$ is in S then since S is a subspace and closed under linear combination, we have that $\text{Span}\,\mathcal{B} \subseteq S$.

2. Prove that $S \subseteq \text{Span}\,\mathcal{B}$ by creating a general form of a vector in S then matching the components to a linear combination of $\mathcal{B}$ then solving the system. Show that $c_1, c_2, \dots$ of $\text{Span}\,\mathcal{B}$ can be expressed with $x_1, x_2, \dots$ of the general S vector.

1.4

Dot product: $\vec{x} \cdot \vec{y} = x_1y_1 + \dots + x_ny_n$. Takes in two vectors and returns a scalar. Is the length of $\vec{x}$ multiplied by the length of $\vec{y}$ projected onto $\vec{x}$ in R^2.

The length or norm of a vector is $||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$. A vector of length 1 is a unit vector.

Theorem 1.4.1: If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and θ is the angle between them, then $\vec{x} \cdot \vec{y} = ||\vec{x}|| ||\vec{y}|| \cos\theta$.

Theorem 1.4.2: If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ then

(1) $\vec{x} \cdot \vec{x} \geq 0, \vec{x} \cdot \vec{x} = 0$ iff $\vec{x} = \vec{0}$.

(2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

(3) $\vec{x} \cdot (s\vec{y} + t\vec{z}) = s(\vec{x} \cdot \vec{y}) + t(\vec{x} \cdot \vec{z})$

Theorem 1.4.3: If $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then

(1) $||\vec{x}|| \geq 0, ||\vec{x}|| = 0$ iff $\vec{x} = \vec{0}$.

(2) $||c\vec{x}|| = |c|||\vec{x}||$

(3) $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||||\vec{y}||$

(4) $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$

Cross product: The cross product of two vectors in $\mathbb{R}^3$, $\vec{v} \times \vec{w} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$, orthogonal to $\vec{v}, \vec{w}$

Theorem 1.4.5: Let $\vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

(1) If $\vec{n} = \vec{v} \times \vec{w}$, then for any $\vec{y} \in \text{Span}\{\vec{v}, \vec{w}\}$, $\vec{y} \cdot \vec{n} = 0$

(2) $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

(3) $\vec{v} \times \vec{v} = \vec{0}$

(4) $\vec{v} \times \vec{w} = \vec{0}$ iff either $\vec{v} = \vec{0}$, or $\vec{w}$ is a scalar multiple of $\vec{v}$

(5) $\vec{v} \times (\vec{w} + \vec{x}) = \vec{v} \times \vec{w} + \vec{v} + \vec{x}$

(6) $(c\vec{v}) \times (\vec{w}) = c(\vec{v} \times \vec{w})$

(7) $||\vec{v} \times \vec{w}|| = ||\vec{v}||||\vec{w}||(|\sin\theta|)$, where θ is the angle between $\vec{v}, \vec{w}$

Theorem 1.4.6: Scalar Equation of a Plane: Let $\vec{v}, \vec{w}, \vec{b} \in \mathbb{R}^3$, with $\{\vec{v}, \vec{w}\}$ being linearly independent and let P be a plane in $\mathbb{R}^3$ with vector equation $\vec{x} = s\vec{v} + t\vec{w} + \vec{b}, s, t \in \mathbb{R}$. If $\vec{n} = \vec{v} \times \vec{w}$, then an equation for the plane is $(\vec{x} - \vec{b}) \cdot \vec{n} = 0$. Notice $(\vec{x} - \vec{b})$ is a vector which lies on the plane. As well, we can expand this equation to $\vec{x} \cdot \vec{n} = \vec{b} \cdot \vec{n}$.

Scalar equation of P: Let P be a plane in $\mathbb{R}^3$ which passes through the point $B(b_1, b_2, b_3)$. If $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \in \mathbb{R}^3$ is a vector such that $n_1x_1 + n_2x_2 + n_3x_3 = b_1n_1 + b_2n_2 + b_3n_3$ is an equation for P, then $\vec{n}$ is a normal vector for P.

Scalar equation of a hyperplane: If $\{\vec{v}_1, \dots, \vec{v}_{m-1}\}$ is a linearly independent set of vectors in $\mathbb{R}^m, m \geq 2$, and $\vec{b} \in \mathbb{R}^m$, then $\vec{x} = c_1\vec{v}_2 + \cdots + c_{m-1}\vec{v}_{m-1} + \vec{b}$ is a vector equation of a hyperplane in $\mathbb{R}^m$. If there is a non-zero vector $\vec{n}$ orthogonal to all $\vec{v}_1, \dots, \vec{v}_{m-1}$, then the scalar equation of the hyperplane is $n_1x_1 + \cdots + n_mx_m = n_1b_1 + \cdots + n_mb_m$.

1.5

Projection: Let $\vec{v}, \vec{u} \in \mathbb{R}^n, \vec{v} \neq \vec{0}$. Projection of $\vec{u}$ onto $\vec{v}$ is $\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v}$.

Perpendicular of $\vec{u}$ onto $\vec{v}$ is $\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u})$

Projection and perpendicular onto a plane: Let P be a plane in $\mathbb{R}^3$ that passes through the origin and has normal vector $\vec{n}$. The projection of $\vec{x} \in \mathbb{R}^3$ onto P is: $\text{proj}_P(\vec{x}) = \text{perp}_{\vec{n}}(\vec{x})$ and the perpendicular is the projection onto the normal.

Chapter 2

2.1

Geometrically, a system of m linear equations in n variables represents m hyperplanes in $\mathbb{R}^n$, a solution to the system is a vector in $\mathbb{R}^n$ which lies on all m hyperplanes.

Theorem 2.1.1: If a system has two distinct solutions $\vec{s}, \vec{t}$, then $\vec{x} = \vec{s} + c(\vec{s} - \vec{t})$ is a distinct solution for each $c \in \mathbb{R}$.

2.2

Two systems of equations which have the same solution set are equivalent.

The coefficient matrix can be viewed in two ways: The i-th row represents the coefficients of the i-th equation in the system, the j-th column is the coefficients of x_j in all equations.

Row equivalent: Two matrices are row equivalent if there is a sequence of EROs which transform one into the other.

If two augmented matrices are row equivalent, then the systems of linear equations associated with each system are equivalent.

Theorem 2.2.2: All matrices have unique reduced row echelon forms.

Theorem 2.2.3: The solution set of a homogeneous system of m linear equations in n variables is a subspace of $\mathbb{R}^n$. This solution set is called the solution space of the system.

Theorem 2.2.5: System-Rank Theorem: Let A be the coefficient matrix of a system of m linear equations in n unknowns $[A|\vec{b}]$

(1) rank $A <$ rank $[A|\vec{b}]$ iff the system is inconsistent

(2) if $[A|\vec{b}]$ is consistent, then the system contains (n-rank$A$) free variables. Used for proving linear independence.

(3) rank $A =$ m iff $[A|\vec{b}]$ is consistent for every $\vec{b} \in \mathbb{R}^m$. Used to prove spanning.

Theorem 2.2.6: Solution Theorem: Let $[A|\vec{b}]$ be a consistent system of m linear equations in n variables with RREF $[R|\vec{c}]$. If rank$A = k < n$, then a vector equation of the solution set of $[A|\vec{b}]$ has the form $\vec{x} = \vec{d} + t_1\vec{v}_1 + \cdots + t_{n-k}\vec{v}_{n-k}, t_1, \ldots, t_{n-k} \in \mathbb{R}$, where $\vec{d} \in \mathbb{R}^n$ and the set of vectors is linearly independent. In particular, the solution set is an $n - k$ flat.

Theorem 2.2.7: A set of n vectors in $\mathbb{R}^n$ is linearly independent iff it spans $\mathbb{R}^n$. t

Chapter 3

3.1

Transpose: $(A^T)_{ij} = (A)_{ji}$. Use it to represent rows as the transpose of column vectors.

Theorem 3.1.2: If $A, B \in M_{m\times n}(\mathbb{R}), c \in \mathbb{R}$, then:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$

Matrix-Vector Multiplication: Coefficients are rows of A in 1, columns in 2

1. Let $A \in M_{m\times n}(\mathbb{R})$ whose rows are denoted $\vec{a}_i^T$ for $1 \le i \le m$. For any $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = \begin{bmatrix} \vec{a}_i \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix}$
2. Let $A = [\vec{a}_1 \cdots \vec{a}_n] \in M_{m\times n}(\mathbb{R})$. For any $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ \vec{x}_n \end{bmatrix} \in \mathbb{R}^n, A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$.

Note: If A is an $m \times n$ matrix, then $A\vec{x}$ is defined iff $\vec{x} \in \mathbb{R}^n$ and $A\vec{x} \in \mathbb{R}^m$. As well, from the second definition, we see that $A\vec{x}$ is a linear combination of the columns of A.

Theorem 3.1.3: If $\vec{e}_i$ is the ith standard basis vector and $A = [\vec{a}_1 \cdots \vec{a}_n]$, then $A\vec{e}_i = \vec{a}_i$.

Theorem 3.1.4: If $A \in M_{m\times n}(\mathbb{R}), \vec{x}, \vec{y} \in \mathbb{R}^n, c \in \mathbb{R}$, then

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
2. $c(A\vec{x}) = (cA)\vec{x} = A(c\vec{x})$
3. $(A\vec{x})^T = \vec{x}^T A^T$
4. $\vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$ (like matrix-vector multiplication)

Observe that if $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $\vec{x} \cdot \vec{y} = y_1x_1 + \cdots + y_nx_n = [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{x}^T\vec{y} = \vec{y}^T\vec{x}$

Matrix-Matrix Multiplication: For an $m \times n$ matrix A and $n \times p$ matrix $B = [\vec{b}_1 \cdots \vec{b}_p]$, AB is an $m \times p$ matrix and $AB = [A\vec{b}_1 \cdots A\vec{b}_p]$

Theorem 3.1.5:

1. $A(B + C) = AB + AC$
2. $t(AB) = (tA)B = A(tB)$
3. $A(BC) = (AB)C$
4. $(AB)^T = B^T A^T$

Theorem 3.1.6: $A, B \in M_{m\times n}(\mathbb{R})$ are equal if $A\vec{x} = B\vec{x}\ \forall \vec{x} \in \mathbb{R}^n$

Identity matrix: $AI = A = IA$, A must be square.

3.2 Linear Mappings

Linear Mappings: $L : R^n \to R^m$ is a linear mapping if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$, we have $L(b\vec{x} + c\vec{y}) = bL(\vec{x}) + cL(\vec{y})$. Notice that matrix mappings are linear mappings. A linear operator maps R^n to R^n. Two linear mappings are equal if they map to the same value for all vectors in the domain. Each input has a unique output.

To prove that something is a linear mapping, decompose $L(b\vec{x} + c\vec{y})$ into $bL(\vec{x}) + cL(\vec{y})$.

Note: If L is a linear mapping, then $L(\vec{0}) = \vec{0}$.

Theorem 3.2.2: Every linear mapping $L : R^n \to R^m$ can be represented as a matrix mapping, where $L(\vec{x}) = [L]\vec{x}$ and $[L] = [L(\vec{e}_1) \cdots L(\vec{e}_n)]$. Then $[L]$ is the standard matrix of L. L is $m \times n$.

Two linear mappings:

1. Rotation matrix: Define $R_\theta : R^2 \to R^2$ to be the function which rotates a vector about the origin (counter-clockwise) by the angle θ. Then $[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
2. Reflection: Reflects a vector over the hyperplane P and $\operatorname{refl}_P(\vec{x}) = \vec{x} - 2\operatorname{proj}_{\vec{n}}(\vec{x})$

3.3 Special Subspaces

$\text{Range}(L) = \{L(\vec{x}) \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$. To determine whether a vector is in the range, set up a system of linear equations and determine whether the system is inconsistent (not in range) or if there is a solution (in range).

To find a basis for the range, create a general vector in the range.

Theorem 3.3.2: If $L : R^n \to R^m$ is a linear mapping, then $\text{Range}(L)$ is a subspace of $\mathbb{R}^m$.

Kernel: Let $L : R^n \to R^m$ be a linear mapping. The kernel (nullspace) is the set of all vectors in the domain which are mapped to the zero vector in the codomain. $\text{Ker}(L) = \{\vec{x} \in \mathbb{R}^n | L(\vec{x}) = \vec{0}\}$. As well, since $L(\vec{x}) = [L]\vec{x}$, we define the nullspace of an $m \times n$ matrix A to be all vectors in $\mathbb{R}^n$ for which $A\vec{x} = \vec{0}$.

Note: If the nullspace of a matrix is only $\vec{0}$, then the rank of the matrix is equal to the number of columns in the matrix.

To find a basis for the kernel, find the solution set of a matrix represented by the coefficients of the range augmented with the zero vector. To find a basis for the nullspace of a matrix, row reduce the matrix.

Theorem 3.3.3: If $L : R^n \to R^m$ is a linear mapping, then $\text{Ker}(L)$ is a subspace of $\mathbb{R}^n$.

Theorem 3.3.6: If $L : R^n \to R^m$ is a linear mapping with $[L] = [\vec{a}_1 \cdots \vec{a}_n]$, then $\text{Range}(L) = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$.

Column space: The column space of a matrix is the subspace defined by the span of its columns: $\text{Col}(A) = \text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \{A\vec{x} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$.

Row space: Let A be an $m \times n$ matrix. The row space of A is the subspace of $\mathbb{R}^n$ defined by $\text{Row}(A) = \{A^T\vec{x} \in \mathbb{R}^n | \vec{x} \in \mathbb{R}^m\}$, or the span of the rows of A.

Left nullspace: Let A be an $m \times n$ matrix. The left nullspace of A is the subspace of $\mathbb{R}^m$ defined by $\text{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m | A^T\vec{x} = \vec{0}\}$.

3.4 Operations on Linear Mappings

Let $L : R^n \to R^m$, $L : R^n \to R^m$ be linear mappings. Then $(L + M)(\vec{x}) + L(\vec{x}) + M(\vec{x})$ and $(cL)(\vec{x}) = cL(\vec{x})$.

The set of all linear mappings from R^n to R^m is a vector space.

Composition: Let $L : R^n \to R^m$, $L : R^m \to R^p$ be linear mappings. Then $M \circ L : R^n \to R^p$ and $(M \circ L)(\vec{x}) = M(L(\vec{x}))$

Identity mapping is $\mathrm{Id} : R^n \to R^n$ such that $\mathrm{Id}(\vec{x}) = \vec{x}$.

Chapter 4

4.1 Vector Spaces

A set $\mathbb{V}$ with addition and scalar multiplication is called a vector space over $\mathbb{R}$ and for every $\vec{v}, \vec{x}, \vec{y} \in \mathbb{V}$ and $c, d \in \mathbb{R}$, we have:

1. $\vec{x} + \vec{y} \in \mathbb{V}$
2. $(\vec{x} + \vec{y}) + \vec{v} = \vec{x} + (\vec{y} + \vec{v})$
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
4. $\exists \vec{0} \in \mathbb{V}$ such that $\vec{x} + \vec{0} = \vec{x}$
5. $\exists (-\vec{x}) \in \mathbb{V}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
6. $c\vec{x} \in \mathbb{V}$
7. $c(d\vec{x}) = (cd)\vec{x}$
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
9. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
10. $1\vec{x} = \vec{x}$

Polynomials: $P_n(\mathbb{R})$ is a vector space of all polynomials of degree at most n with real coefficients, $P_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n | a_0, a_1, \ldots, a_n \in \mathbb{R}\}$

Theorem 4.1.1: If $\mathbb{V}$ is a vector space, then $0\vec{x} = \vec{0} \forall \vec{x} \in \mathbb{V}$ and $(-\vec{x}) = (-1)\vec{x} \forall \vec{x} \in \mathbb{V}$. This means the zero vector and additive inverse of any vector is unique. So to find the zero vector and additive inverse, just multiply by 0 or -1.

Let $\mathbb{V}$ be a vector space. If S is a subset of $\mathbb{V}$ and S is a vector space under the same operations as $\mathbb{V}$, then S is a subspace.

Subspace test (same as before): Non-empty subset closed under the operations of addition and scalar multiplications defined in the vector space.

Skew symmetric matrix: $A^T = -A$.

Theorem 4.1.3: If $\mathcal{B}$ is a set of vectors in a vector space, then Span $\mathcal{B}$ is a subspace of the vector space.

4.2 Bases and Dimension

Finding a basis: determine general form of a vector in the vector space, write it as a linear combination of vectors. If this set is linearly independent, then stop as it is a basis. Otherwise, remove a vector and repeat. Or, we find n vectors which are linearly independent and in the vector space.

Theorem 4.2.1: a basis is a maximally linearly independent set and a minimal spanning set. The dimension of a vector space is the number of vectors in a basis. So a set of n vectors in a n-dimensional vector space is linearly independent iff it spans the vector space.

Theorem 4.2.4: We can always extend a linearly independent set to a basis for a finite dimensional vector space. For example, adding one vector to the basis of a 3 dimensional vector space so that the vectors span $\mathbb{R}^4$.

Corollary 4.2.5: Subspaces of a finite dimensional vector space have smaller or equal dimensions.

Coordinates

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for a vector space $\mathbb{V}$. If $\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$, then $b_1, \ldots, b_n$ are the $\mathcal{B}$-coordinates of $\vec{v}$ and $[\vec{v}]_\mathcal{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$. This allows us to work with vectors in $\mathbb{R}^n$ instead of vectors in a vector space. This function which takes in vectors and outputs a vector in $\mathbb{R}^n$ is linear.

Theorem 4.3.2: If $\mathbb{V}$ is a vector space with basis $\mathcal{B}$, then for all vectors in $\mathbb{V}$ and scalars, $[s\vec{v}+t\vec{w}]_\mathcal{B} = s[\vec{v}]_\mathcal{B}+t[\vec{w}]_\mathcal{B}$.

Change of coordinates matrix. Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{C}$ be bases for $\mathbb{V}$. Then ${}_\mathcal{C}P_\mathcal{B} = \begin{bmatrix} [\vec{v}_1]_C & \cdots & [\vec{v}_n]_C \end{bmatrix}$ and for any $\vec{x} \in \mathbb{V}$, $[\vec{x}]_\mathcal{C} = {}_\mathcal{C}P_\mathcal{B}[\vec{x}]_\mathcal{B}$. Note that $\vec{x} = [\vec{x}]_\mathcal{S}$, standard basis.

Theorem 4.3.3: If $\mathcal{B}, \mathcal{C}$ are basis for $\mathbb{V}$, then ${}_\mathcal{C}P_{\mathcal{B}\mathcal{B}}P_\mathcal{C} = I = {}_\mathcal{B}P_{\mathcal{C}\mathcal{C}}P_\mathcal{B}$

Chapter 5

5.1 Matrix Inverses

If $AB = I$, then A is the left inverse of B and B is the right inverse of A. In order for $AB = I$ to work, we need the rank of A to equal the number of rows, since the system $A\vec{b_i} = \vec{e_i}$ where A is the coefficient matrix needs to be consistent.

Theorem 5.1.1: If A is an $m \times n$ matrix with $m > n$, then A cannot have a right inverse. As well, if $m < n$, then A cannot have a left inverse. Only square matrices can have a left and right inverse.

Theorem 5.1.3: If $A, B, C \in M_{n\times n}(\mathbb{R})$ such that $AB = I = CA$, then $B = C$. As well, if $AB = I = BA$, then B is the inverse of $A, B = A^{-1}$ and A is invertible. All matrices have unique inverses.

Theorem 5.1.4: If A, B are $n \times n$ matrices such that $AB = I$, then A, B are invertible and both have rank n. Thus, we only need to show that $AB = I$ to conclude that they are inverses instead of showing $AB = I$ and $BA = I$.

Theorem 5.1.5: If A, B are invertible matrices and $c \in \mathbb{R}, c \neq 0$, then:

1. $(cA)^{-1} = \frac{1}{c}A^{-1}$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(AB)^{-1} = B^{-1}A^{-1}$: product of two invertible matrices is invertible

Theorem 5.1.6: If A is an $n \times n$ matrix with rank n, then A is invertible. To find this inverse, row reduce $[A|I] \sim [I|A^{-1}]$. If rank is not n, then A is not invertible.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ab-cd}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and we need $ab - cd \neq 0$.

Invertible Matrix Theorem: For any $n \times n$ matrix A, these are equivalent:

1. A is invertible
2. RREF of A is I
3. rank $A = n$
4. $A\vec{x} = \vec{b}$ is consistent with a unique solution for all $b \in \mathbb{R}^n$. Then $\vec{x} = A^{-1}\vec{b}$
5. nullspace of A is $\{\vec{0}\}$
6. columns of A form a basis for $\mathbb{R}^n$
7. rows of A form a basis for $\mathbb{R}^n$
8. A^T is invertible
9. $\lambda = 0$ is not an eigenvalue

Note: To prove something is invertible, prove that its rank is n. That is, show that $A\vec{x} = \vec{0}$ has a unique solution, which is $\vec{x} = \vec{0}$.

5.2 Elementary Matrices

Elementary matrix: Obtained from the $n \times n$ identity matrix by performing one ERO.

Theorem 5.2.1: Every elementary matrix is invertible and E^{-1} is also an elementary matrix by performing the opposite ERO. Determine the ERO needed to bring E back to I then perform this ERO on E to find E^{-1}.

Theorem 5.2.2: If A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix, then EA is the matrix obtained by performing the ERO associated with E on A.

Corollary 5.2.5: Since ERO do not change the rank of a matrix, then if A is an $m \times n$ matrix and E is an $m \times m$ elementary matrix, rank of EA is equal to rank of A.

Theorem 5.2.6: If $A \in M_{m\times n}(\mathbb{R})$ with RREF R, then $\exists E_1, \ldots, E_k$ of $m \times m$ elementary matrices such that $E_k \cdots E_2 E_1 A = R$ and $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} R$, where E_1 corresponds to the first ERO to get A into RREF, and E_k is the last.

Corollary 5.2.7: If A is an invertible matrix then A, A^{-1} can be written as a product of elementary matrices. If $E_1, \ldots, E_k$ correspond to the EROs to reduce A to I, where $E_k \cdots E_1 A = I$ then $A = E_1^{-1} \cdots E_k^{-1}$ and $A^{-1} = E_k \cdots E_1$. This makes sense if we think about how we find A^{-1}. We reduce $[A|I]$ to $[I|A^{-1}]$. Therefore, if $E_k \cdots E_1 A = I$, then we are applying the same operations to I on the right side so $E_k \cdots E_1(I) = A^{-1}$

Theorem 5.2.8: If E is an $m \times m$ elementary matrix, then E^T is an elementary matrix. We prove this by considering each of the three ERO in separate cases.

5.3 Determinants

Determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

Cofactor: Let A ba an $n \times n$ matrix, $n \geq 2$. Then $C_{ij} = (-1)^{i+j} \det A(i,j)$, where $A(i,j)$ is the matrix A with the ith row and jth column removed.

Cofactor expansion: $\det A = \sum_{k=1}^{n} a_{ik} C_{ik}$ across the ith row, or $\det A = \sum_{k=1}^{n} a_{kj} C_{kj}$ across the jth column.

Upper and lower triangular: An $m \times n$ matrix is upper triangular if $a_{ij} = 0$ whenever $i > j$ or is lower triangular if $a_{ij} = 0$ whenever $i < j$.

Theorem 5.3.2: If a square matrix is upper or lower triangular, then its determinant is just the product of the diagonal entries. Thus, $\det I = 1^n = 1$

If B is obtained from A by:

1. multiplying one row or column by $c \in \mathbb{R}$, then $\det B = c \det A$
2. swapping two rows or columns, then $\det B = -\det A$, if A has two identical rows, $\det A = 0$
3. adding a multiple of one row to another or multiple of a column, then $\det B = \det A$

Corollary 5.3.7: If $A, E \in M_{n\times n}(\mathbb{R})$ and E is an elementary matrix, then $\det EA = \det E \det A$. As well, $\det E \neq 0$ since $\det I = 1$ and none of the elementary row operations will turn this 1 into a 0.

Theorem 5.3.8: A is invertible iff $\det A \neq 0$. Proof involves breaking A down into its elementary matrices and RREF, if A is invertible, then RREF is I which has $\det I = 1$. As well, $\det E_i \neq 0$.

Theorem 5.3.9: If A, B are $n \times n$ matrices, then $\det(AB) = \det A \det B$.

Corollary 5.3.10: If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$

Theorem 5.3.11: If A is a square matrix, then $\det A = \det A^T$

5.4 Determinants and Systems of Equations

False Expansion Theorem: If we do a cofactor expansion along one row or column but take cofactors of another row or column, the determinant will always be 0.

Theorem 5.4.2: If A is invertible, then $(A^{-1})_{ij} = \frac{1}{\det A} C_{ji}$

Cofactor matrix: Let A be an $n \times n$ matrix. The cofactor matrix, cof A, is $(\text{cof } A)_{ij} = C_{ij}$

Adjugate: $(\text{adj } A)_{ij} = C_{ji}$ and $\text{adj } A = (\text{cof } A)^T$. Thus, $A^{-1} = \frac{1}{\det A} \text{adj } A$

Cramer's Rule: If A is invertible, then $\vec{x}$ of $A\vec{x} = \vec{b}$ has a unique solution and is given by $x_i = \frac{\det A_i}{\det A}, 1 \leq i \leq n$ where A_i is the matrix obtained from A by replacing the ith column of A by $\vec{b}$. This is useful for matrices where there are many variables, we can just replace variables with numbers.

5.5 Area and Volume

If $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$, then they induce an n dimensional parallelotope and the n volume of this parallelotope is $V = |\det[\vec{v}_1 \cdots \vec{v}_n]|$

Chapter 6

$\mathcal{B}$-Matrix: Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. The $\mathcal{B}$-matrix of L is: $[L]_{\mathcal{B}} = \begin{bmatrix}[L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}}\end{bmatrix}$ and satisfies $[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$.

Theorem 6.1.1: If A, B are $n \times n$ matrices such that $P^{-1}AP = B$ for some invertible matrix P, then

1. $\operatorname{rank} A = \operatorname{rank} B$
2. $\det A = \det B$
3. $\operatorname{tr} A = \operatorname{tr} B$, where the trace is $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$

If $P^{-1}AP = B$ for some invertible matrix P, then A is similar to B. To prove that two matrices are not similar, we can show that P does not exist, or that their rank, det, or trace are not equal.

For an $n \times n$ matrix, if $A\vec{v} = \lambda\vec{v}, \vec{v} \neq 0$, then λ is an eigenvalue of A, $\vec{v}$ is an eigenvector of A corresponding to λ and $(\lambda, \vec{v})$ is an eigenpair.

To find eigenvectors, we need to solve $(A - \lambda I)\vec{v} = \vec{0}$. Since $\vec{v} \neq \vec{0}$, we need the system to have infinite solutions which means we need λ such that $\det(A - \lambda I) = 0$.

For an $n \times n$ matrix A, the characteristic polynomial is the nth degree polynomial $C(\lambda) = \det(A - \lambda I)$. This means we will have n eigenvalues, assuming the polynomial is fully factorable.

Eigenspace: The nullspace of $A - \lambda I$ is the eigenspace of λ. This is the set of all eigenvectors for λ, excluding $\vec{0}$ which we want to include so that the eigenspace is a subspace.

Multiplicity: a_{λ_i} is the number of times λ_i is a root of $C(\lambda)$ and g_{λ_i} is the dimension of the eigenspace of λ_i. As well, for all eigenvalues, $1 \leq g_\lambda \leq a_\lambda$.

If $[L]$ is similar to a diagonal matrix D, then the columns in P must be eigenvectors of $[L]$ and the diagonal entries in D must be eigenvalues of $[L]$.

A square matrix is diagonalizable if it is similar to a diagonal matrix, $P^{-1}AP = D$. Here, P diagonalizes A. In this course, if A has a non-real eigenvalue, then it is not diagonalizable.

Theorem 6.3.1: A is diagonalizable iff $\exists\{\vec{v}_1, \dots, \vec{v}_n\}$ a basis for $\mathbb{R}^n$ of eigenvectors of A. Then, $P = [\vec{v}_1 \cdots \vec{v}_n]$ and $D = [\lambda_1 \cdots \lambda_n]$.

Theorem 6.3.2: If A has eigenpairs $(\lambda_1, \vec{v}_1), \dots, (\lambda_k, \vec{v}_k)$ where $\lambda_i \neq \lambda_j$ for $i \neq j$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

Theorem 6.3.3: The basis vectors for all the eigenspaces of distinct eigenvalues for a matrix together form a linearly independent set.

Theorem 6.3.4: A is diagonalizable iff the geometric multiplicity equals the algebraic multiplicity for all eigenvalues. This is because the This means if a matrix has n distinct eigenvalues, then we know immediately it is diagonalizable.

Every matrix has n eigenvalues, including repeats. The determinant of A is the product of all its eigenvalues (including repeats) and the trace is the sum (including repeats).

Powers of matrices: If A is diagonalizable, then $A^k = PD^kP^{-1}$

Extra Theorems

Theorem 1: An orthogonal set in $\mathbb{R}^n$ where none of the vectors are the zero vector is linearly independent.

Proof: Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthogonal set where none are the zero vector. Suppose for contradiction this set is linearly dependent. Then

$$\begin{aligned}
\vec{v}_i &= c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \ldots + c_k\vec{v}_k \\
\vec{v}_1 \cdot \vec{v}_i &= \vec{v}_1 \cdot (c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \ldots + c_k\vec{v}_k) \text{ suppose WLOG } c_1 \neq 0 \\
0 &= c_1\vec{v}_1 \cdot \vec{v}_1 + \cdots 0 \text{ since this is an orthogonal set}
\end{aligned}$$

But this means $\vec{v}_1 = \vec{0}$ which is a contradiction. Therefore, the set must be linearly independent.

Theorem 2: If the dimension of a subspace is equal to the dimension of the vector space which it is a subspace of, then the subspace is equal to the vector space.

Theorem 3: If we can conclude that $\vec{x} = \vec{y}$ if $A\vec{x} = A\vec{y}$ then A has rank equal to the number of columns.

Theorem 4: For matrices A, B, C, if $AC = BC$, then A does not necessarily equal B.

Theorem 5: If $AB = C$ and C is invertible, then A, B are also invertible.

$$\begin{aligned}
AB &= C \\
\det AB &= \det C \\
\det A \det B &= \det C
\end{aligned}$$

Since C is invertible, then $\det C \neq 0$ therefore $\det A \neq 0, \det B \neq 0$.

As well, if A, B are invertible, then AB is invertible.

Theorem 6: Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis. Then ${}_{\mathcal{S}}P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$

Theorem 7: A square matrix whose rows are linearly independent cannot have a row of zeros. However, if this matrix is not square, we can have a row of zeros.

A matrix whose cols are linearly independent cannot have free variables.

Theorem 8: If A, B are similar, then they have the same characteristic polynomial and hence, the same eigenvalues.

Proof: Since A, B are similar, then $P^{-1}AP = B$.

$$\begin{aligned}
\det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\
&= \det(P^{-1}AP - \lambda P^{-1}P) \\
&= \det(P^{-1}(AP - \lambda P)) \\
&= \det(P^{-1}(A - \lambda I)P) \\
&= \det P^{-1} \det(A - \lambda I) \det P \\
&= \det(A - \lambda I)
\end{aligned}$$