On Sparse Discretization for Graphical Games

Luis E. Ortiz   LEORTIZ@CS.STONYBROOK.EDU

Department of Computer Science, Stony Brook University,
Stony Brook, NY 11794-4400 USA

Abstract

This short paper concerns discretization schemes for representing and computing approximate Nash equilibria, with emphasis on graphical games, but briefly touching on normal-form and poly-matrix games. The main technical contribution is a representation theorem that informally states that to account for every exact Nash equilibrium using a nearby approximate Nash equilibrium on a grid over mixed strategies, a uniform discretization size linear on the inverse of the approximation quality and natural game-representation parameters suffices. For graphical games, under natural conditions, the discretization is logarithmic in the game-representation size, a substantial improvement over the linear dependency previously required. The paper has five other objectives: (1) given the venue, to highlight the important, but often ignored, role that work on constraint networks in AI has in simplifying the derivation and analysis of algorithms for computing approximate Nash equilibria; (2) to summarize the state-of-the-art on computing approximate Nash equilibria, with emphasis on relevance to graphical games; (3) to help clarify the distinction between sparse-discretization and sparse-support techniques; (4) to illustrate and advocate for the deliberate mathematical simplicity of the formal proof of the representation theorem; and (5) to list and discuss important open problems, emphasizing graphical-game generalizations, which the AI community is most suitable to solve.

1. Introduction

There has been quite a bit of work over last 10 years, mostly, but not exclusively, in the theoretical computer science community, on the problem of computing approximate Nash equilibria in games (see, e.g., the work of Kearns, Littman, & Singh, 2001; Vickrey & Koller, 2002; Ortiz & Kearns, 2003; Lipton, Markakis, & Mehta, 2003; Singh, Soni, & Wellman, 2004; Soni, Singh, & Wellman, 2007; Daskalakis, Mehta, & Papadimitriou, 2007; Tsaknakis & Spirakis, 2007; Feder, Nazerzadeh, & Saberi, 2007; Daskalakis & Papadimitriou, 2008; Daskalakis, Mehta, & Papadimitriou, 2009; Kontogiannis, Panagopoulou, & Spirakis, 2009; Hémon, de Rougemont, & Santha, 2008; Awasthi, Balcan, Blum, Sheffet, & Vempala, 2010; Bosse, Byrka, & Markakis, 2010; Ponsen, de Jong, & Lanctot, 2011 and the references therein; see also the book by Nisan, Roughgarden, Éva Tardos, & Vazirani, 2007 for additional references). [1]

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1. To keep the focus and length of this short paper, the presentation and motivation of the significance and broader scientific and practical impact the study of Nash equilibria is beyond the scope of this paper. Similarly, the main focus here is on *graphical games (GGs)* and *absolute approximations* of Nash equilibria, the most common type of approximation found in the literature. However, the curious reader can go to Appendix A to find a brief discussion of the significance of Nash equilibria (Appendix A.1), other classes of graphical models for game theory (Appendix A.2), and other types of approximations of Nash equilibria (Appendix A.3).

This short paper revisits the simple *uniform-discretization scheme* that Kearns et al. (2001) originally introduced in the context of n-player 2-action *graphical games (GGs)*. (Formal definitions of concepts from game theory, graph theory, AI, and uniform-discretization appear in Section 3. The presentation in the Introduction will remain informal.)

Kearns et al. (2001) showed that if the size of the individual grid that the uniform-discretization scheme induces over the probability of playing an action was $O(2^k/\epsilon)$, where k is the size of the largest set of neighbors of any player and ϵ is an approximation quality parameter, then for each exact Nash equilibrium of the GG, its closest joint mixed-strategy in the resulting regular *joint* grid is an approximate ϵ-Nash equilibrium of the GG. [2] Kearns et al. (2001) used that discretization to design a special type of dynamic-programming algorithm tailored to computing approximate Nash equilibria in GGs with tree-structured graphs they called **TreeNash** that runs in time linear in the number of players and $O(2^{k^2})$, assuming a fixed ϵ. The size of the input representation of the n-player 2-action GG is $O(n\,2^k)$. In collaboration with Prof. Kearns, we later extended **TreeNash** as a heuristic for GGs with loopy graphs, leading to an algorithm we called **NashProp**, which stands for "Nash Propagation" (Ortiz & Kearns, 2003).

An unpublished note drafted back in December 2002 (Ortiz, 2002), later posted online[3] as part of a course on computational game theory taught by Prof. Michael Kearns during the Spring 2003 at the University of Pennsylvania, provided a significantly sharper bound of $O(k/\epsilon)$ on the size of the discretization required to achieve the same approximation result. The revised bound was *logarithmic* in the representation size of the game, as opposed to the previous *linear* bound that Kearns et al. (2001) derived. [4] The revised, significantly tighter upper-bound led to an improved running time of $O(k^k)$ for **TreeNash** in terms of just k, which meant that, when using the sparser discretization derived in the old note (Ortiz, 2002), **TreeNash** becomes a quasi-polynomial time approximation scheme (quasi-PTAS) to compute an ϵ-Nash equilibrium. Daskalakis and Papadimitriou (2008) independently rediscovered this result, about five years later, using a considerably more complex approach to the proof/derivation than the simple, algebraic approach presented in the old note (Ortiz, 2002) and here.

Similarly, some of the results regarding algorithmic implications presented here that followed from the old note (Ortiz, 2002), particularly for normal-form games, have also been independently rediscovered in the literature using different approaches throughout the years (see, e.g., some of the results of Lipton et al., 2003, and Daskalakis & Papadimitriou, 2008). Those particular results, discussed in more detail in the technical sections of this paper, followed immediately from the improved discretization bound given in the old note (Ortiz, 2002), combined with previously known results from the CSP and graphical-models literature in AI.

The present short paper extends the old note (Ortiz, 2002) and shows how the improved discretization-size bounds for GGs, and some specializations, fall off immediately as corollaries of a theorem that holds for GG generalizations.

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2. In an ϵ-Nash equilibrium, players tolerate losing expected gains, up to an ϵ amount, from not unilaterally deviating.

3. `http://www.cis.upenn.edu/~mkearns/teaching/cgt/revised_approx_bnd.pdf`

4. Note that Kearns et al. (2001) only considered the case of *binary* actions, i.e., $m = 2$, hence m does not play a role in the results within the context of that paper.

In particular, the current paper presents a result on the sufficient size for uniform discretization to capture, in a formal sense, every possible Nash equilibrium of the game via another close approximate Nash equilibria in the induced grid over the space of mixed strategies of players in the game. As example corollaries of the main result, for graphical games with largest neighborhood size k, the sufficient size is $O(km/\epsilon)$, which implies, $O(nm/\epsilon)$ for standard normal-form games, while for n-player m-action poly-matrix games, the sufficient size is $O(m/\epsilon)$.

Similarly, the representation result yields immediate computational results based on connection to algorithms for *constraint satisfaction problems (CSPs)*. Section 6 presents and discusses several results on polynomial and quasi-polynomial time algorithms for several interesting subclasses of graphical and normal-form games that fall off from that connection.

The paper ends with a discussion of the algorithmic implications that the main technical result may have on other work in computational game theory. It also lists several open problems.

2. Motivation

This section provides additional motivation for the necessity and significance of the study of the problem of computing approximate Nash equilibria, particularly in GGs, and for the development and dissemination of this short paper, with emphasis on its particular relevance to the AI community.

2.1 The Computational Complexity of Nash Equilibria: State of Affairs

Chen and Deng (2005b) settled the complexity of computing an exact Nash equilibrium in bimatrix games, a 50-years-old open problem, proving the problem to be PPAD-complete (see also later work by Chen & Deng, 2006 and Chen, Deng, & Teng, 2009). [5] Daskalakis, Goldberg, and Papadimitriou (2006) later proved the same result (see also the presentation of Daskalakis, Goldberg, & Papadimitriou, 2009a, 2009b). This means that a polynomial-time algorithm for this problem is unlikely. This seminal result was the culmination of a series of results that same year for 3-player (Daskalakis & Papadimitriou, 2005; Chen & Deng, 2005a), and 4-player (Daskalakis, Goldberg, & Papadimitriou, 2005) normal-form games.

Within the game-theory community, Bubelis (1979) devised a polynomial-time reduction of any n-player normal-form game to some 3-player normal-form game; in the theoretical computer science community, Goldberg and Papadimitriou (2005) rediscovered a similar result, reducing n-player normal-form games to 4-player normal-form games. This last result rendered the computational complexity of computing a Nash equilibrium in arbitrary n-player normal-form games as PPAD-complete (Goldberg & Papadimitriou, 2005).

Several results on the PPAD-completeness of computing Nash equilibria in arbitrary GGs served as lemmas to the proofs for normal-form games (Daskalakis et al., 2005; Daskalakis & Papadimitriou, 2005). In fact, GGs played a very important role in finally settling the computational complexity of bimatrix games.

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5. Papadimitriou (1994) introduced the complexity class PPAD to characterize fixed-point-type problems such as those of Nash equilibria in which a solution always exists. A discussion on the complexity class PPAD is beyond the scope of this short paper.

There is other important, seminal work regarding the computation of Nash equilibria prior to the hardness results discussed above, including some regarding other equilibrium notions such as pure-strategy Nash equilibria and correlated equilibria (Aumann, 1974, 1987) in graphical and normal-form games (see, e.g., the work of Gilboa and Zemel (1989), McKelvey and McLennan (1996), Kakade, Kearns, Langford, and Ortiz (2003), Gottlob, Greco, and Scarcello (2005), Conitzer and Sandholm (2008), Papadimitriou and Roughgarden (2008), Roughgarden (2009), Jiang and Leyton-Brown (2011), and the references therein). But the focus of this short paper is on representing and computing *approximate Nash equilibria in GGs*.

2.2 A Caveat: Exact vs. Approximate Nash Equilibria in Multiplayer Games

Technically, the result for more than 3 players is for the problem of computing an *approximate* Nash equilibrium with an approximation quality *exponentially small* in the representation size of the game. This is because, as is well-known since Nash's journal version (Nash, 1951) of his original paper (Nash, 1950), Nash presents a poker-inspired game of 3 players, built in collaboration with Lloyd Shapley, a 2012 Nobel Laureate in Economics, in which all payoff hypermatrix entries/values are *rational* numbers, but whose *unique* Nash equilibrium contains mixed strategies with *irrational* numbers.

Chen, Deng, and Teng (2006) also proved that there does not exist a fully-polynomial-time approximation scheme (FPTAS) for computing an approximate Nash equilibrium, unless PPAD=P (see also the presentation of Chen et al., 2009). [6] For this result, they proved that the approximation quality can not go below an *inverse-polynomial* function of the representation size of the game. They stated in their paper, without proof, that it is easy to extend the result for 2-player normal-form games to n-player normal-form games and n-player graphical games. *Hence, computing a Nash equilibrium (or more formally, an approximate Nash equilibrium with approximation quality inversely-exponential in the representation size of the game), in an arbitrary n-player GG is PPAD-complete.* Yet, it is still open whether there exists a polynomial-time approximation scheme (PTAS) for n-player games in normal-form and arbitrary GGs.

The result of Daskalakis et al. (2009a) also implied that the problem of computing an approximate Nash equilibrium in n-player normal-form games, for $n \geq 2$, with approximation quality inversely exponential on the game's representation size is PPAD-complete.

All of the above helps us motivate the study of approximate Nash equilibria in arbitrary GGs from a technical perspective:

(a) there are multiplayer games with payoff matrices represented entirely using rational numbers but with unique Nash equilibria formed of mixed strategies with *irrational* numbers;

(b) computing "exact" Nash equilibria is likely to be computationally intractable in general; and

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6. Formal definitions of the different approximation schemes, which textbooks such as those of Vazirani (2001) and Williamson and Shmoys (2011) cover, are beyond the scope of this short paper. See Appendix D for a general description of some of the approximation schemes.

(c) while there is no FPTAS to compute an approximate Nash equilibrium with approximation quality below a value inversely polynomial of the representation size of the game, the existence of a PTAS is open.

Here, we provide an array of results, from polynomial time, to FPTAS and quasi-PTAS, for a variety of GGs under reasonable conditions on the parameters and network characteristics of the GG.

2.3 "Why revisit an old research note now?"

There are several motivations for reviving the old note (Ortiz, 2002).

- The note has gained significance and relevance with time. The sparse discretization and main representation results are proving to be particularly useful in establishing polynomial running times for a specific class of dynamic-programming and propagation algorithms.

- Many algorithmic results immediately follow from the connection to the previous work on constraint networks in AI. Thus, this paper highlights the usefulness and significance of those results, coming from the AI community, regarding the use of constraint networks to solve network-structured CSPs.

- The derivation of the proof of the result on sparse discretization is *deliberately simple,* relative to considerably more mathematically complex proofs of the same results (see, e.g., the derivations by Daskalakis & Papadimitriou, 2008): *no fancy mathematical sophistication needed when simple algebraic manipulations suffice.* Similarly, this short paper highlights the important, but often ignored, role that the work on constraint networks in AI has in simplifying the derivation and analysis of algorithms to compute approximate Nash equilibria in GGs and some of their specializations, including normal-form games.

- This paper helps clarify the distinct pros and cons of the two approaches most often used to compute approximate Nash Equilibria: sparse support vs. sparse discretization.

- This paper seeks to present the state-of-the-art on computing approximate Nash equilibria in GGs, and lists several important open problems, particular for GG generalizations, with important practical consequences, and for which the AI community is particularly well-suited to eventually solve.

The following discussion expands on the first two bullet points, with some emphasis on prior and current work in my research group or with collaborators, but mentioning briefly potential implications for the work of other AI researchers.

2.3.1 Games as CSPs

The computation of Nash equilibria in games is inherently a CSP, an area of great relevance to AI research. In the game-induced CSP, we have one variable for each player's mixed strategy, one domain for each player corresponding to the simplex over the player's

actions, and a set of constraints for each player's Nash equilibrium conditions. For example, for binary-action games, recall that a mixed-strategy is just a probability, real-valued number between 0 and 1; thus, the number of mixed-strategies available to each player is *uncountable*. Hence, the corresponding CSP would have real-valued variables, each with an uncountably infinite domain size: an *infinite* or *continuous CSP*. The same holds for multi-action games.

As stated previously, Kearns et al. (2001) introduced the idea in AI of using a uniform discretization of the mixed strategies of each player which leads to a regular grid. This turns the problem of computing approximate Nash equilibria into a *discrete CSP* (Vickrey & Koller, 2002), i.e., a CSP with finite, discrete domains. This paper considers the same type of discretization.

One motivation to revisit the old note (Ortiz, 2002) is the implications it has on extensions of **NashProp** (Ortiz & Kearns, 2003) based on *survey propagation (SP)* (Mezard, 2003; Parisi, 2003d; Parisi, 2003a; Braunstein & Zecchina, 2004; Braunstein, Mézard, & Zecchina, 2005; Maneva, Mossel, & Wainwright, 2007), a message-passing technique with roots in physics and introduced to computer science to solve boolean satisfiability problems, such as random 3-SAT formulas (Mézard, Parisi, & Zecchina, 2002; Parisi, 2003c; Mézard, 2004), and more generally, a variety of CSPs, such as graph 3-coloring (Parisi, 2003b). Ortiz (2008) introduced an axiomatic way to view the standard SP algorithm for boolean satisfiability, and an approach called *constraint propagation relaxation (CPR)* that facilitates the derivation of SP-like techniques tailored to specific CSPs. In the case of GGs, the computation of each message-passing, between players, takes *exponential* time in the size of the discretization of the space of mixed strategies. (A thorough description of the SP-like version of **NashProp** is beyond the scope of this paper.) But the improved bound on the discretization size for each player is *logarithmic* in the representation size of the GG, *assuming a constant number of actions*, leading to a running time that is polynomial in $1/\epsilon$, where ϵ is the approximation-quality parameter, and either quasi-polynomial in the representation size of the game in the case of arbitrary maximum neighborhood size k, or simply polynomial in the case k is bounded by a constant, independent of the number of players n. The representation size of each message is simply *quadratic* in k, m and $1/\epsilon$, even if k and m are free parameters.

Another motivation for considering GG generalizations is that they include the class of *graphical poly-matrix games*, which in the case of 2-action games relates to *linear (or generalized linear) influence games* (Irfan & Ortiz, 2011, 2013; Irfan, 2013). In part because of their very compact representations (i.e., of size $O(n\,m^2)$), *poly-matrix games* (Janovskaja, 1968) are an important class of games within game theory. We revisit their graphical version when we consider generalizations of GGs in Section 7.

In summary, particular interest in *the application of CPR to derive better algorithms for computing and counting Nash equilibria in arbitrary, loopy GGs,* as well as more specific GG classes such as *linear or generalized-linear influence games* (Irfan & Ortiz, 2011, 2013; Irfan, 2013), motivates this paper in large part.

Similarly, the new bounds may also provide improvements to previous discretization-based schemes for computing ϵ-Nash equilibria in other similar models (e.g., those of Singh et al., 2004; Soni et al., 2007).

3. Preliminaries

This section introduces the basic technical notation and concepts necessary to understand the upcoming technical sections of the research note.

3.1 Basic Notation

Denote by $x \equiv (x_1, x_2, \ldots, x_n)$ an n-dimensional vector and by $x_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ the same vector without component i. Similarly, for every set $S \subset [n] \equiv \{1, \ldots, n\}$, denote by $x_S \equiv (x_i : i \in S)$ the (sub-)vector formed from x using only components in S, such that, if $S^c \equiv [n] - S$ denotes the complement of S, $x \equiv (x_S, x_{S^c}) \equiv (x_i, x_{-i})$ for every i. If $A_1, \ldots, A_n$ are sets, denote by $A \equiv \times_{i \in [n]} A_i$, $A_{-i} \equiv \times_{j \in [n]-\{i\}} A_j$ and $A_S \equiv \times_{j \in S} A_j$.

If $G = (V, E)$ is an undirected graph, then for each $i \in V$ denote by $\mathcal{N}_i \equiv \{j \mid (j, i) \in E\}$ the *neighbors of node/vertex i* in G, and $N_i \equiv \mathcal{N}_i \bigcup \{i\}$ the *neighborhood of node/vertex i* in G. Note that we have $i \notin \mathcal{N}_i$ but $i \in N_i$ for all $i \in V$.

3.2 Graphical Games in Local Normal-Form Payoff Representations

This section formally defines *graphical games (GGs)*, which are graphical models for compact representations of classical game representations in game theory (Kearns et al., 2001). GGs extend and generalize *normal-form games.* In particular, a normal-form game is a GG with a complete/fully-connected graph.

Definition 1. *A* graphical game (GG) *consists of an* undirected graph $G = (V, E)$, [7] *where each node $i \in V$ in the graph corresponds to a player i in the game, and for each player i, we have a set of* actions *or* pure strategies A_i *and a local payoff hypermatrix/function $M_i' : A_{N_i} \to \mathbb{R}$, where N_i is the* neighborhood *of player i in the graph defined with respect to the edges E of the graph. The (global) payoff hypermatrix/function M_i of player i is such that, for each joint-action $x \in A \equiv A_V$, we have $M_i(x) \equiv M_i'(x_{N_i})$. That is, the payoff that each individual player i receives when all players, including i, take joint-action/pure-strategy x is a function of the joint-actions x_{N_i} of player i's neighborhood N_i only, thus conditionally independent of x_{V-N_i} given x_{N_i}. It is convention to let $V = \{1, \ldots, n\} \equiv [n]$, so that $n \equiv |V|$. The representation size of each local payoff hypermatrix M_i' is $\Theta(|A_{N_i}|) = O(m^k)$, where $m \equiv \max_{i \in V} |A_i|$ and $k \equiv \max_{i \in V} |N_i|$. The representation size of the GG is $\Theta(\sum_{i \in V} |A_{N_i}|) = O(nm^k)$. If for all i we have $N_i = V$, then the GG is a standard* normal-form game, *also called strategic- or matrix-form game, which has a representation size $\Theta(n|A|) = O(nm^n)$.*

A GG achieves considerable representation savings whenever $k \ll n$.

3.3 Solution Concepts

A *joint mixed strategy* $p \equiv (p_1, \ldots, p_n)$ in a game is formed from each individual *mixed strategy* $p_i \equiv (p_i(x_i) : x_i \in A_i)$ for player i, which is a probability distribution over the players actions A_i (i.e., $p_i(x_i) \geq 0$ for all $x_i \in A_i$ and $\sum_{x_i \in A_i} p_i(x_i) = 1$). Denote by $\mathcal{P}_i \equiv \{p_i \mid p_i(x_i) \geq 0,$ for all $x_i \in A_i$ and $\sum_{x_i \in A_i} p_i(x_i) = 1\}$ the set of all possible mixed

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7. It is easy to extend the same result to GGs with *directed* graphs.

strategies of player i (i.e., all possible probability distributions over A_i). Similar to the vector notation introduced above, for all i and any clique/set $S \subset V$, denote by p_{-i} and p_S the mixed strategies corresponding to all the players *except* i and all the players in clique S, respectively, so that $p \equiv (p_i, p_{-i}) \equiv (p_S, p_{V-S})$. A joint mixed strategy p induces a joint (product) probability distribution over the joint action space A, such that, for all $x \in A$, $p(x) \equiv \prod_{i \in V} p_i(x_i)$ is the probability, with respect to joint mixed strategy p, that joint action x is played.

The *expected payoff* of player i with respect to joint mixed strategy p is denoted by $M_i(p) \equiv \sum_{x \in A} p(x) M_i(x)$.

Definition 2. *For any $\epsilon \geq 0$, a joint mixed-strategy p^* is called an ϵ-Nash equilibrium if for every player i, and for all $x_i \in A_i$, $M_i(p_i^*, p_{-i}^*) \geq M_i(x_i, p_i^*) - \epsilon$. That is, no player can increase its expected payoff more than ϵ by unilaterally deviating from its mixed strategy part p_i^* in the equilibrium, assuming the others play according to their respective parts p_{-i}^*. A* Nash equilibrium, *or more formally, a mixed-strategy Nash Equilibrium, is then a 0-Nash equilibrium.*

Note that, for all $p_{-i} \in \mathcal{P}_{-i}$, $\max_{p_i \in \mathcal{P}_i} M_i(p_i, p_{-i}) = \max_{x_i \in A_i} M_i(x_i, p_{-i}) \geq M_i(x_i', p_{-i})$, for all $x_i' \in A_i$. Also, note that the equilibrium conditions are invariant to affine transformations. In the case of GGs with local payoff matrices represented in tabular/matrix/normal-form, it is convention to assume, without loss of generality, that the payoff values are such that, for each player $i \in V$, we have $\min_x M_i(x) = \min_{x_{N_i}} M_i'(x_{N_i}) = 0$ and $\max_x M_i(x) = \max_{x_{N_i}} M_i'(x_{N_i}) = 1$. Note that in the case of GGs using such "tabular" representations, we do not lose generality by assuming the maximum and minimum local payoff values of each player are 0 and 1, respectively, because we can compute them both *efficiently*. This will *not* be the case for GG generalizations, in the worst case. Section 7 revisits this point.

4. Discretization Schemes

The discretization scheme considered here is similar to that of (Kearns et al., 2001), except that we allow for the possibility of different discretization sizes for the mixed strategies of players.

Definition 3. *In an* (individually-uniform) *discretization scheme, the uncountable set $I = [0, 1]$ of possible value assignments to the probability $p_i(x_i)$ of each action x_i of each player i is approximated by a finite grid defined by the set $\widetilde{I}_i = \{0, \tau_i, 2\tau_i, \ldots, (s_i - 1)\tau_i, 1\}$ of values separated by the same distance $\tau_i = 1/s_i$ for some integer s_i. Thus the* discretization size *is $|\widetilde{I}_i| = s_i + 1$. Then, we would only consider mixed strategies q_i such that $q_i(x_i) \in \widetilde{I}_i$ for all x_i, and $\sum_{x_i \in A_i} q_i(x_i) = 1$. The* induced discretized space *of joint mixed strategies is $\widetilde{I} \equiv \times_{i \in V} \widetilde{I}_i^{|A_i|}$, subject to the individual normalization constraints.*

5. Sparse Discretization

The obvious question is, how small can we make s_i and still guarantee that there exists an ϵ-Nash equilibrium in the induced discretized space of joint mixed strategies? The following corollary provides a stronger answer for GGs: it provides values for the s_i's that

guarantee that for *every* Nash equilibrium, its closest point in the induced grid is an ϵ-Nash equilibrium. An interesting aspect of the result is that s_i depends only on information local to player i's neighborhood: the number of actions $|A_i|$ available to player i and the largest number of neighbors $|\mathcal{N}_j|$ *of the neighbors* $j \in \mathcal{N}_i$ of player i.

Note that the corresponding discretization bound provided in (Kearns et al., 2001) in the context of GGs is *exponential* in the largest neighborhood size k. In contrast, the bound here is *linear* in k, a substantial reduction.

The corollary is a GG instantiation of the Nash-equilibria Representation Theorem based on sparse discretization, Theorem 3, which holds for a broader class of GG generalizations. The statement and discussion of the theorem is in Section 7. The statement of the corollary uses notation introduced above.

Corollary 1. (Sparse Discretization for Graphical Games) *For any m-action graphical game and any $\epsilon > 0$, a (individually-uniform) discretization with*

$$s_i = \left\lceil \frac{2\,|A_i|\,\max_{j \in \mathcal{N}_i} |\mathcal{N}_j|}{\epsilon} \right\rceil = O\left(\frac{m\,k}{\epsilon} \right)$$

for each player i is sufficient to guarantee that for every *true (i.e., not approximate) Nash equilibrium of the game, its closest (in ℓ_∞ distance) joint mixed strategy in the induced discretized space is also an ϵ-Nash equilibrium of the game.*

6. Algorithmic Implications

One objective of this research note is to clarify the distinction between sparse discretization and sparse support in the context of approximate Nash equilibria. The next subsection deals with that objective in the context of normal-form games. Then, the second subsection presents the algorithmic implications of the sparse-discretization approach, borrowing heavily from standard results in AI, and highlighting their simplifying usefulness and powers.

Before starting the presentation, it is important to note a crucial distinction between the parameters of interest in GGs versus those of normal-form games, a specialization of GGs. In the study of GGs, the main objects of interest, in terms of input-game representation size, is the number of players n and the graph structure (e.g., the maximum number of neighbors k, the graph tree-width, or the hypertree-width of the game-induced constraint network), and their effect in both representation and computations of (approximate) Nash equilibria; the maximum number of actions m of each player plays a lesser role, and is often assumed constant. Yet, the statements of the results here still sometimes include the explicit dependence on m, so that such dependence is clear should m also be an important component/aspect of interest in evaluating the quality of the approximations and the resulting algorithms.

6.1 Sparse Discretization vs. Sparse/Small Support

Before starting this subsection, it is important to clarify that the work of Althöfer (1994) does not imply, and cannot be used to derive, the sparse representation result presented as the main technical contribution here. Instead, the approximation bounds presented

in Lipton et al. (2003) follow immediately from those of Althöfer (1994); in fact, both use the same approach based on applying Hoëffding bounds (Hoeffding, 1963) and the probabilistic method (Alon & Spencer, 2004). More recently, Hémon et al. (2008) derived a slightly improved small-support upper-bound result in multi-player games by using a more general large deviation bound than Hoëffding's, due to McDiarmid (1989).

It is equally important to understand the distinctions between the two types of approximations. Each type has their own pros and cons.

As a warmup to the results and discussion regarding GGs and their generalization, let us first consider GG simplifications, starting with the simple case of 2-player games in normal form, and then moving to n-player games in normal-form.

6.1.1 BIMATRIX GAMES

In the small-support approach to approximation, we perform a brute-force support-enumeration search over all possible $\binom{m}{r}$ subsets of strategies, where r is the minimum support necessary to guarantee the desired absolute approximation quality $\epsilon > 0$; thus, it is important to keep in mind that r depends on ϵ. Hence, the running time of approximation algorithms of that type is $\binom{m}{r} = O(m^r)$, thus polynomial in m but *exponential* in a function of the approximation parameter. Given $\epsilon > 0$, the support size that results from the approach of Althöfer (1994) and Lipton et al. (2003) is $r = O\left(\frac{\ln(m\,n)}{\epsilon^2}\right)$. It is a well-known fact that given the supports for each player in a Nash equilibrium, one can set up a simple linear program (LP) to find the Nash equilibrium. Each call to the LP runs in polynomial time in r. This leads to a naive brute-force algorithm, also called "support enumeration," for computing ϵ-Nash equilibrium in bimatrix games: for each possible support set of size r, run an LP to either find an ϵ-Nash equilibrium or decide there is none with that particular set. There are $\binom{m}{r} = O(m^r)$ possible support for each player. Hence, for 2-player games, the algorithm runs in time $m^{O(r)} = m^{O\left(\frac{\ln m}{\epsilon^2}\right)}$, thus *exponential* in $1/\epsilon^2$, and also exponential in $(\ln m)^2$, which means *quasi-polynomial* in m.

But, there is actually an alternative to the LP. Because of the probabilistic way in which the result for sparse supports arises (i.e., using a combination of Hoeffding's or other large deviation bound and invoking the probabilistic method), we only need to search over mixed strategies in the support whose individual probability value has the form z_k/r, for each action $k = 1, \ldots, m$, and $\sum_k z_k/r = 1$, $0 \leq z_k \leq r$. Lipton et al. (2003) call such "discretization scheme" r-uniform. Another way of viewing this is as a uniform-discretization scheme of size $r + 1$, which does not seem to guarantee to find all Nash equilibria, but it does guarantee to find at least one. Thus, in that case, the worst-case running time to find at least one ϵ-Nash equilibrium is $(m\,r)^{O(r)}$. Recalling that $r = O\left(\frac{\ln m}{\epsilon^2}\right)$, we have that the worst-case running time is still *exponential* in $1/\epsilon^2$ and exponential in $(\ln m)^2$, thus *quasi-polynomial* in m.

Regardless, because the representation size of the game is $N = O(m^2)$, both algorithms run in time $N^{O(\ln N)}$, thus *both are quasi-PTASs for computing approximate Nash equilibria in bimatrix games* (Lipton et al., 2003).

In contrast, suppose that s is the size of the uniform-grid sparse discretization presented here. A simple analysis reveals $s = O\left(\frac{m}{\epsilon}\right)$. Hence, using the sparse discretization, a naive brute-force search runs in time $O\left(\left(\frac{m}{\epsilon}\right)^m\right)$, thus *polynomial* in $1/\epsilon$, but *exponential* in $m \ln m$.

6.1.2 MULTIPLAYER GAMES

For multiplayer games, one cannot use an LP to compute an ϵ-Nash equilibrium even if one were to know the support of the ϵ-Nash equilibrium strategies of all players. This is because now the ϵ-Nash equilibrium conditions are highly non-linear, involving the product of n variables, where n is the number of players, each variable corresponding to the probability of a particular action in the mixed strategy of a player. (Recall that a Nash equilibrium is a product distribution.) Thus, it might first appear unclear how to efficiently perform this computation efficiently in multiplayer games. It would seem, on the surface, that for sparse support, an algorithm based on support enumeration would run in worst-case time $O\left(m^{r\,n}\text{rtime}(n,m,r)\right)$, where $\text{rtime}(n,m,r)$ is the worst-case running time of the algorithm that attempts to compute an ϵ-Nash equilibrium with the given players' supports, and is a direct function of r, m, and n, and of course, indirectly, of ϵ too.

But, as mentioned in the case of bimatrix games, because of the probabilistic way in which the result for sparse supports arises, we only need to search over the set of r-uniform mixed strategies in the support. Thus, using this approach of support enumeration the worst-case running time becomes $\text{rtime}(n,m,r) = O\left(\binom{m}{r} r^r\right)$. Thus, the total running time is $O\left((mr)^{r\,n}\right)$. Lipton et al. (2003) proved that using $r = O\left(\frac{n^2 \ln (n^2 m)}{\epsilon^2}\right)$ is sufficient. [8] Substituting that value of r, the expression of the worst-case running time becomes $m^{O\left(\left(n^3 \ln (n^2 m)\right)\left(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}\right)\right)}$. Now, to obtain a quasi-PTAS, we would have to impose a condition on n, such as $n = O(\ln m)$. This is unlike the bimatrix-game case in which no restriction was necessary.

In contrast, for sparse discretization in multiplayer games, the value of $s = O\left(\frac{m\,n}{\epsilon}\right)$. Hence, a brute-force search algorithm applied to a normal-form game, for example, would run in worst-case time $\left(\frac{m\,n}{\epsilon}\right)^{O(m\,n)}$, thus *polynomial* in $1/\epsilon$, but *exponential* in $m\,n \ln m\,n$. (The results for GGs are in a later section.) To obtain a quasi-PTAS, we would have to impose a condition on $m = poly(n)$, not n.

6.1.3 TRADEOFF

The tradeoff is now clear. If the interest is m, and ϵ is fixed, then the approach of Althöfer (1994) and Lipton et al. (2003) is better. On the other hand, if the interest is ϵ, and m is fixed, which is often the case for GGs as discussed above, then the sparse discretization wins. It is important to note that the sparse discretization guarantees the computation of *all* ϵ-Nash equilibria, while the sparse-support approach can only guarantee *one* ϵ-Nash equilibrium. It is also important to note that in the case of multiplayer games, the sparse-support approach yields a quasi-PTAS only after bounding n, which is the quantity of most interest in GGs. The sparse-discretization approach bounds m, which is the number of actions of each player to obtain a quasi-PTAS, but m is not as significant in a GG setting as n is.

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8. The reader should be aware that the notation of Lipton et al. (2003) is the exact opposite of the one here: here n denotes the number of players, while there it denotes the maximum number of actions of any player; also, here m denotes the maximum number of actions of any player, while there it denotes the number of players.

6.2 Reductions to Consistency in CSPs

This section involves concepts from AI and graph theory; in the interest of space, the reader is referred to appropriate standard references (see, e.g., the textbooks of Russell & Norvig, 2003, for AI and Dechter, 2003 for graph theory as used in AI; or Bollobas, 1979 for graph theory).

The representation result of the last section has several immediate computational consequences for the problem of computing approximate Nash equilibria in GGs with local payoff matrices represented in tabular form, and in turn, also for multi-player games represented in standard normal (tabular) form.

To simplify the presentation, let us assume that payoff values are in $[0, 1]$, all the players in the GG have the same number of actions m and the largest neighborhood size in the graph of the game is k, so that the representation size of the GG is $\Theta(nm^k)$. For this case, the uniform discretization presented in Theorem 3 has size $s = s_i = O(mk/\epsilon)$ for all i.

6.2.1 SPARSE-DISCRETIZATION GG-INDUCED CSP

Once we introduce a discretization over the space of mixed strategies, then it is natural to formulate the problem of computing ϵ-Nash equilibria on the induced discretized space as a CSP, or more specifically in the case of GGs, as a special type of constraint network (Dechter, 2003). (In the interest of keeping this paper short, please see Russell & Norvig, 2003, or other introductory textbook on AI, for general information on CSPs. The presentation here contains only the CSP concepts necessary exclusively within the context of the paper's topic.) Several researchers have taken this or related approaches either explicitly or implicitly (Kearns et al., 2001; Vickrey & Koller, 2002; Ortiz & Kearns, 2003; Soni et al., 2007). The CSP for the game has one variable, domain and constraint for each player. Each variable corresponds to mixed strategy p_i for each player i. Each variable's domain corresponds to the discretized set $\widetilde{I}_i^m$ of mixed strategies for each player i, properly corrected to account for normalization. The *approximate best-response equilibrium conditions* are the following: for each player i, each constraint function (table) $c_i : \widetilde{I}_i^{mk} \to \{0, 1\}$ is defined such that, for all $p_{N(i)} \in \widetilde{I}_i^{mk}$, $c_i(p_{N(i)}) = 1$ if and only if $M_i(p_i, p_{-i}) \geq \max_{x_i' \in A_i} M_i(x_i', p_{-i}) - \epsilon$. Each constraint has arity at most k and encodes the approximate best-response equilibrium conditions, each represented in tabular form using $s^{mk} = O((mk/\epsilon)^{mk})$ bits. The transformation takes time $O(poly(n, (mk/\epsilon)^{mk}))$ and the size of the resulting CSP is $T = \Theta(n(mk/\epsilon)^{mk})$.

As previously discussed, for GGs, it is natural to consider the number of players n as being the "free" parameter of the representation. Hence, if $mk \log(mk) = O(\log(n))$ and $\epsilon = n^{\Omega(-1/(mk))} = (mk)^{-\Omega(1)} = (\log n)^{-\Omega(1)}$, then both the time to perform the CSP transformation and its resulting representation size are polynomial in the representation size of the game (i.e., polynomial in the number of players). Similarly, if m and k are bounded (by a constant, independent of n), then the representation size of the game is linear in n and the running time of the transformation is polynomial in n and $1/\epsilon$. Note that this is a natural restriction on the game parameters because otherwise the representation size would be exponential in the number of players, thus defeating the main purpose of the GG representation in the first place: *succinctness*.

At this point, we can apply any of a large number of existing off-the-shelf techniques for solving the induced game-CSP, or apply techniques such as **NashProp** (Ortiz & Kearns, 2003) that take advantage of the particular properties of the game best-response constraints.

Instead, in the next section, we will see how standard results for solving constraint networks (Dechter, 2003) lead immediately to simple derivations of algorithmic results for GGs, some of which had been independently re-discovered by employing more sophisticated mathematical tools (Daskalakis & Papadimitriou, 2008). Facilitated in large part by the knowledge acquired for solving constraint networks and other graphical models in the AI community over the last 50 years, the derivations here are quite straightforward and do not require complex mathematics. By building on existing AI knowledge, fancy mathematical derivations for the same result become unnecessary despite their elegance.

6.2.2 An approach based on the GG-induced CSP hypergraph

An approach to solving CSPs in AI, now over 10 years olds, works on the *hypergraph induced by the game CSP* (Gottlob, Leone, & Scarcello, 2001). If T is the representation size of the game-CSP, w is the hypertree width of the hypergraph, and the corresponding hypertree decomposition for the CSP has been computed, then solving the CSP takes time $O(T^{w+1} \log(T))$ (Gottlob, Leone, & Scarcello, 2000, 2002; Gottlob et al., 2001) (see also page 158 of "Bibliographical and Historical Notes" Section in Chapter 5 of Russell & Norvig, 2003). In the case of GGs, we can compute the hypertree decomposition that the algorithm would use in time $O(n^{2w+2})$.

The following theorem summarizes the discussion. As a preamble to a discussion on primal graphs and treewidths later in the section, note that it is known that a CSP might have hypergraphs with bounded hypertree width, but whose *primal* graph has *unbounded* treewidth. However, the treewidth *always bounds* the hypertree width. Thus, the restriction that the hypertree width be bounded by a constant may not be as limiting to the application of the results as it first appears.

Theorem 1. *There exists an algorithm that, given as input a number $\epsilon > 0$ and a GG with n players, maximum neighborhood size k and maximum number of actions m, and whose corresponding CSP has a hypergraph with hypertree width w, computes an ϵ-Nash equilibrium of the GG in time $[n \, (mk/\epsilon)^{mk}]^{O(w)}$.*

The following corollary characterizes the computational complexity of the approximation schemes resulting from instances of the last theorem.

Corollary 2. *There exists an algorithm that, given as input a GG with corresponding hypergraph of hypertree width w bounded by a constant independent of the number of players n, with a logarithmic function of n restricting the maximum number of actions m and the maximum neighborhood size k as $mk \log(mk) = O(\log(n))$, and given $\epsilon = n^{\Omega(-1/(mk))} = (mk)^{-\Omega(1)} = (\log n)^{-\Omega(1)}$, outputs an ϵ-Nash equilibria of the game in time polynomial in the representation size of the input. If, in particular, both m and k are bounded by constants independent of n, then the algorithm runs in polynomial time in n and $1/\epsilon$, for any $\epsilon > 0$; hence, the algorithm is a* fully polynomial time approximation scheme (FPTAS). *If, instead, the expression constraining m and k as a logarithmic function of n holds, and $w = polylog(n)$, then the algorithm is a* quasi-polynomial time approximation scheme (quasi-PTAS).

6.2.3 A SIDE NOTE ON NORMAL-FORM GAMES

For normal-form games, $k = n$ and $w = 1$. This leads to the following corollary.

Corollary 3. *There exists a quasi-PTAS for computing an ϵ-Nash equilibrium of n-player m-action normal-form games with $m = O(poly(n))$ that runs in time $N^{O(polylog(N)\log(1/\epsilon))} = \left(\frac{1}{\epsilon}\right)^{O(polylog(N))}$, where $N = n^{\Theta(n)}$ is the representation size of the game. If, in particular, m is bounded by a constant independent of n, then the running time is $N^{O(\log \frac{\log(N)}{\epsilon})}$, where $N = 2^{\Theta(n)}$ is the corresponding representation size of the game.*

As briefly mentioned in Section 6.1.2, we can also obtain the same result by using the sparse-support approach of Lipton et al. (2003), even if $m = 2^{O(n)}$, but the dependence is *exponential* in $1/\epsilon^2$ (i.e., $N^{O(\text{polylog}(N)(1/\epsilon)^2)}$; or $N^{O(\frac{\log(N)}{\epsilon^2})}$, if m is bounded by a constant). The result of the last corollary for the case of m fixed or bounded by a constant is stated by Daskalakis and Papadimitriou (2008). The same result for m fixed or bounded by a constant also follows from Theorem 4 of Kearns (2007). While no formal proof appears for Theorem 4 of Kearns (2007), the theorem follows immediately from the proof in the original note of Ortiz (2002).

But, it is important to emphasize, as touched upon in Section 6.1.2, that in the case of normal-form games, we could have obtained the result directly by using an exhaustive search over the induced grid over mixed strategies, which is essentially what the algorithm referred to in the corollary reduces to in this case. Hence, we could output not just one ϵ-Nash equilibrium, but *all* ϵ-Nash equilibria in the induced grid in the worst-case running time given in the corollary. The algorithms based on the sparse-support approach can only guarantee to output *one* ϵ-Nash equilibrium among all mixed strategies of a given maximum support. Algorithms based on sparse support that would output (a compact representation of) *all* ϵ-Nash equilibria do not seem to exist, even for the given maximum support size; let alone (a compact representation of) all ϵ-Nash equilibria in the game, as the exhaustive search that uses the sparse discretization result presented here does for normal-form games. Section 6.1 here discusses the distinctions between the sparse-support and the sparse-discretization approaches.

6.2.4 AN APPROACH BASED ON THE GG-INDUCED CSP PRIMAL GRAPH

Another approach is to build a clique (or join) tree from the *primal graph* of the game-CSP, which in the case of the GGs is the graph created by forming cliques of every neighborhood. Then, one applies a dynamic programming (or message-passing) algorithm on the clique tree. Once the clique tree is built, the running time is linear in the number of nodes of the join tree and exponential in the size of the largest clique associated to a node in the clique tree. If the primal graph has treewidth w', the largest clique associated to the optimal clique tree is $w' + 1$. It has been common knowledge in the graphical-models community for quite a while now, almost two decades, at least (see, e.g., the textbooks of Dechter, 2003; Russell & Norvig, 2003, and the references therein, for a recent accounting of previous work in this area) that if the primal graph of a CSP has treewidth w' that is logarithmic in the number of nodes n, then one can solve the CSP in polynomial time if the CSP is represented in tabular form. The following theorem and corollary follow from careful

application of previously known results for solving constraint networks and other related graphical models. Note that the treewidth w'' of the original graph of the game is always no smaller than the hypertree width w of its hypergraph (Gottlob et al., 2005). In addition, the GG's primal graph treewidth $w' \leq (w'' + 1)k$ (Daskalakis & Papadimitriou, 2006). So the interesting bounds in the hypertree case is as given in the corollary. Also, this means that the results can be easily extended to GGs with (original) graphs that have $O(\log(n))$ treewidth as long as k is bounded. Finally, if a graph with n nodes has treewidth w', then the graph has at most $(w' + 1)n$ edges (see, e.g., the work of Becker & Geiger, 2001). Because the number of edges of a GG primal graph is $O(k^2 n)$, $w' = O(\log(n))$ implies $k = O(\sqrt{\log(n)})$.

Theorem 2. *There exists an algorithm that, given as input a number $\epsilon > 0$ and an n-player m-action GG with maximum neighborhood size k and primal-graph treewidth w', computes an ϵ-Nash equilibrium of the game in time $2^{O(w')} n \log(n) + n[(mk/\epsilon)^{mk}]^{O(w')}$.*

Corollary 4. *There exists a PTAS for computing an approximate Nash equilibria in n-player GGs with bounded maximum number of actions, bounded neighborhood size and primal-graph treewidth $w' = O(\log(n))$.*

The new discretization bounds also provide significant improvements on the representation results and running times for **NashProp** and its variants (Kearns et al., 2001; Ortiz & Kearns, 2003).

7. Graphical Multi-hypermatrix Games: Generalizing Graphical Games

This section introduces *graphical multi-hypermatrix games (GMhGs)*, a class of games that extends and generalizes GGs while capturing many classical game-theoretic model representations, as discussed below. This class of games is not some theoretical concoction: they are not only convenient in their generality, covering a large number of existing models, but also practical. Indeed, Yu and Berthod (1995) used the same type of games to establish an equivalence between *local* maximum-*a-posteriori* (MAP) inference in Markov random fields and the Nash equilibria of the induced game. [9]

This section also states and discuss the core theorem on sparse discretization in this broader class generalizing GGs. The simplicity, broadness and strength of impact makes this extension theorem the major technical contribution of this research note.

Definition 4. *A* graphical multi-hypermatrix game (GMhG) *is defined by a set V of n players, and for each player $i \in V$, a set of* actions, *or pure strategies, A_i; a set $\mathcal{C}_i \subset 2^V$ of* cliques, *or hyperedges, such that if $C \in \mathcal{C}_i$ then $i \in C$; and a set $\{M'_{i,C} : A_C \to \mathbb{R} \mid$*

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9. Unbeknown at the time, the MRF-induced game is a *potential game* (Monderer & Shapley, 1996); therefore *sequential* (or *synchronous*) best-response dynamics converges. Because, in addition, the goal in MAP inference is to obtain a *global* optimum configuration, in an attempt to avoid local minima, Yu and Berthod (1995) proposed a Metropolis-Hastings-style algorithm, which is also similar to simulated annealing algorithms used for solving satisfiability problems, and other local methods such as WalkSAT. We can view their proposed method as a kind of *learning-in-games* scheme (Fudenberg & Levine, 1999) based on best-response with random exploration, or "trembling hand" best-response, in which, at every round, each player individually play some best-response, with some probability; otherwise the player replays the player's previous/last response.

$C \in \mathcal{C}_i\}$ *of* local-clique payoff matrices. *For each player $i \in V$, the sets $N(i) \equiv \cup_{C \in \mathcal{C}_i} C$ and $\mathcal{N}_i \equiv \{j \in V \mid i \in N(j), j \neq i\}$ are the clique of players affecting i's payoff including i (i.e., i's neighborhood) and those affected by i not including i, respectively. The* local *and* global payoff matrices $M_i' : A_{N(i)} \to \mathbb{R}$ and $M_i : A \to \mathbb{R}$ of i are (implicitly) defined as $M_i'(x_{N(i)}) \equiv \sum_{C \in \mathcal{C}_i} M_{i,C}'(x_C)$ and $M_i(x) \equiv M_i'(x_{N(i)})$, respectively.*

Connections to other game classes. If for each player, each clique set is a singleton, we obtain a *graphical game*, and the single clique in the set defines the neighborhood of the player (i.e., in that case, $\mathcal{C}_i = \{N(i)\}$ for all i). Furthermore, if, in addition, each clique is the complete set of players, then the game is a standard *normal-form game*, also called strategic- or matrix-form game (i.e., in that case, $N(i) = V$ for all i). A GMhG becomes a classical, standard *polymatrix game* (Janovskaja, 1968) if for each player i, $\mathcal{C}_i = \{\{i,j\} \mid j \in V, j \neq i\}$, which is the set of cliques of pairs of nodes involving the player and every other player. [10] In contrast to *hypergraphical games* (Papadimitriou, 2005), a GMhG is more expressive, in part because a GMhG does not require that the same "sub-game" (i.e., local-clique payoff hypermatrix) be shared among all players in the clique of the "sub-game." For example, a local-clique payoff hypermatrix may appear in the summation defining the local payoff hypermatrix of exactly one player. [11] A GMhG has the polynomial intersection property and thus a polynomial correlated equilibrium scheme (Papadimitriou, 2005).

Representation size. The representation size of a GMhG is $\Theta(\sum_{i \in V} \sum_{C \in \mathcal{C}_i} \prod_{j \in C} |A_j|) = O(n\, l\, m^c)$, where $l \equiv \max_{i \in V} |\mathcal{C}_i|$ and $c \equiv \max_{i \in V} \max_{C \in \mathcal{C}_i} |C|$. Hence, the size is dominated by the representation of the local-clique payoff matrices, which are each of size exponential in their respective clique size. However, this representation size could be considerably smaller than for a graphical game, which is exponential in the neighborhood size. For example, if for each i, we have $|\mathcal{C}_i| \leq k$, and for each $C \in \mathcal{C}_i$, we have $|C| = 2$, then the GMhG becomes a *graphical poly-matrix game*, with representation size $O(n\, k\, m^2)$, *linear* in the maximum number of neighbors k, compared to $O(nm^k)$ for a standard graphical game with "tabular" representations, which is *exponential* in k.

Payoff Scale. Normalizing the payoff of a GG in standard local strategic/normal-form takes linear time in the representation size of the game, because we can find the minimum and maximum local payoff values for each local payoff hypermatrix (which, for each player, is exponential in the size of the player's neighborhood) simply by going over each payoff value in the hypermatrix in sequence. However, such an approach is intractable in GMhGs in general. Denote the maximum and minimum payoff values for each player $i \in V$, $u_i \equiv \max_{x_{N_i}} \sum_{C \in \mathcal{C}_i} M_{i,C}'(x_C)$ and $l_i \equiv \min_{x_{N_i}} \sum_{C \in \mathcal{C}_i} M_{i,C}'(x_C)$, respectively. Computing *both* u_i and l_i is NP-hard. To see this, first note that u_i and l_i are the result of a max and min operation over an additive function of the set of the player's hyper-edges and its possible joint-actions. It is easy to reduce the problem of finding a solution to an arbitrary contraint network to that of computing *both* u_i and l_i for each player i. Hence, in general, we do not have much of a choice but to assume that the payoffs of all players are in the same scale, so that using a global approximation-quality value ϵ is meaningful; and to compute

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10. Note that this is the standard definition of polymatrix games. It requires *symmetry* in the hyperedges: for all pair of players $i, j \in V, i \neq j, \{i,j\} \in \mathcal{C}_i \cap \mathcal{C}_j$.

11. Appendix B expands on the relation between GMhGs and hypergraphical games.

the individual maximum and minimum values of each hypermatrix payoff of each player as a way to set up the sparse uniform-discretization.

Some additional notation is necessary before stating the theorem. Denote by $u_{i,C} \equiv \max_{x_C \in A_C} M'_{i,C}(x_C)$ and $l_{i,C} \equiv \min_{x_C \in A_C} M'_{i,C}(x_C)$ the largest and smallest payoff values achieved by the local-grid payoff hypermatrix $M'_{i,C}$, respectively; and by $R_{i,C} \equiv u_{i,C} - l_{i,C}$ its *largest range* of values.

Theorem 3. (Sparse Nash-Equilibria Representation) *For any graphical multi-hypermatrix game and any ϵ such that*

$$0 < \epsilon \leq 2 \min_{i \in V} \frac{\sum_{C \in \mathcal{C}_i} R_{i,C} \left(|C| - 1\right)}{\max_{C' \in \mathcal{C}_i} |C'| - 1},$$

a (uniform) discretization with

$$s_i = \left\lceil \frac{2 |A_i| \max_{j \in \mathcal{N}_i} \sum_{C \in \mathcal{C}_j} R_{j,C} \left(|C| - 1\right)}{\epsilon} \right\rceil$$

for each player i is sufficient to guarantee that for every *Nash equilibrium of the game, its closest (in ℓ_∞ distance) joint mixed strategy in the induced discretized space is also an ϵ-Nash equilibrium of the game.*

The proof of the theorem appears in Appendix C, *deliberately.* This is not only because it is simple, based on algebraic manipulations only, but also because the proof is *irrelevant* to the evaluation of the *significance* of the actual result stated in the theorem. Stating the proof here would simply be distracting, getting in the way of the really important objective: *determining the importance of the theorem on its own.*

The theorem establishes minimal sizes for each player's individually uniform-discretization to capture a compact representation of *all* Nash equilibria with respect to the resulting grid over joint mixed strategies in GMhGs. *Hence, as a result of the theorem, we can represent every exact Nash equilibria of any GMhG via a sparse multi-dimensional grid over the joint mixed-strategies induced by a sparse uniform-discretization of the individual probabilities of each action of each mixed strategy of each player individually.* This sparse discretization can lead not only to compact representations of the set of exact NE, but also to tractable algorithms or effective methods for the computation of approximate Nash equilibria in some classes of games. We have already seen *examples of important representational and computational implications that result from the theorem's application* in the previous section, which focused *on GGs* and some of their specializations, such as *normal-form games.*

Also, the generalized bound has already proved useful to derive and analyze dynamic-programing algorithms for computing approximate Nash equilibria in non-trivial special cases of *interdependent defense (IDD) games* with specific, practical graph structures. Chan, Ceyko, and Ortiz (2012) recently introduced IDD games to model, study, and analyze security and defense mechanism for deterrence in network-structured interdependent security systems under the risk of a deliberate attack from an external agent. One objective is the potential to study the effect of minimal interventions in overall system security for deterrence purposes.

Another example in which the generalized bounds have proven useful is in the design of specific instantiations of CPR that lead to versions of survey propagation for arbitrary GGs, which generalize **NashProp** (Ortiz & Kearns, 2003), and for linear influence games (Irfan & Ortiz, 2011, 2013; Irfan, 2013). In the particular case of **Survey NashProp**, the theorem helped us jump a major hurdle. Each message-passing process in most general CPR instances require time exponential in the size of the largest variable domains in the CSP; this is in contrast to **NashProp** whose running time is linear in the domain size. In the case of **Survey NashProp**, that would be *exponential* in the number of mixed-strategies it considers due to the discretization of the mixed-strategy space; for **NashProp**, on the other hand, that would be quasi-linear. But using the generalized version of the sparse discretization result given in the last theorem, we can now perform each message-passing in time exponential in k^2, which means *quasi-linear* in the size of the input with respect to k, while keeping the size of the messages *quadratic* in the representation size of the game, assuming m is bounded by a constant, independent of n. In short, this means that we can now perform each message-passing step in time *quasi-linear in the size of the game*!

Unlike for standard GGs, the broader algorithmic and computational implications of the sparse discretization in GMhGs remains wide open for the most part. The next section lists and briefly discusses several of such open problems.

8. Concluding Remarks and Open Problems to the AI Community

While sparse-discretization result for *graphical games (GGs)* and its algorithmic implications are actually old, going back to a simple unpublished note wrote in 2002 (Ortiz, 2002), posted online a few months later as part of a course on computational game theory at the University of Pennsylvania, [12] it has gained considerable relevance over the last few years because of the need to deal with even more compact representations than GGs provide, and algorithms/heuristics that require running times exponential in the size of the discretization, thus linear in the size of the model, under many reasonable conditions. Of particular relevance along this line is prior and current exploration of extensions, based on constraint propagation relaxation (Ortiz, 2008), of survey propagation to the problem of computing approximate Nash equilibria in GGs and recent work on linear influence games, and their generalizations (Irfan & Ortiz, 2011, 2013; Irfan, 2013). As stated at the end of Section 2, the new bounds may also provide improvements to previous discretization-based schemes for computing ϵ-Nash equilibria in other similar models, such as those in the work of Singh et al. (2004) and Soni et al. (2007).

Several open questions remain, perhaps most important of which are those related to computation in GG generalizations such as GMhGs. The following is a partial list of open problems for which the AI community, and in particular, the community on constraint networks (Dechter, 2003) for CSPs, are very well equipped to solve.

- Can we combine the sparse-support and sparse-discretization approach to obtain a better algorithm? For example, can we reduce dependency on the number of actions or the accuracy parameter, without unduly increasing the dependency on either? A naive

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12. `http://www.cis.upenn.edu/\homedirmkearns/teaching/cgt/`

approach does not seem to work, but maybe a more sophisticated approach would. Are the sparse-discretization and sparse-support approaches simply incompatible?

- Can we do away with the *uniform* discretization scheme and still guarantee that every Nash equilibria is near a grid point? Would it help if the objective is to find a single approximate solution only? What if we want our approximate Nash equilibria to also be "stable" (i.e., near an actual Nash equilibrium)?

- Computing *exact* Nash equilibria, or more formally, approximate Nash equilibria with exponentially small approximation parameter, in n-player 2-action poly-matrix games, which is a GG generalization, is PPAD-complete. This result is an implicit corollary of the work of Daskalakis et al. (2009b), although they do not provide any explicit, formal proof. Yet, the design of efficient algorithms or effective heuristics for computing *approximate* Nash equilibria in such GG generalizations is wide open! A naive application of the sparse-discretization approach does not seem to help. Can sparse discretization or a sparse-support approach help? If so, how?

- There is no substantive work on using the sparse-support approach for GGs, at least not directly. *Action-graph games* (Jiang, Leyton-Brown, & Bhat, 2011) contain GGs. Thompson, Leung, and Leyton-Brown (2011) designed methods based on the sparse-support approach to compute approximate Nash equilibria in the context of action-graph games. How exactly do these methods apply to GGs directly, or indirectly, if at all? Can we adapt those methods to obtain direct versions, still based on sparse support, for computing approximate Nash equilibria in GGs? Is the sparse-support approach simply incompatible with GGs, when it comes to the design of efficient or effective algorithms for computing approximate Nash equilibria that are also natural within the GG context?

- One can view a sparse discretization as generating an equivalent game in which the discretized mixed-strategy space of the original game becomes the pure strategies of the equivalent game. It seems possible to extend the computational results of Gottlob et al. (2005) by considering the computation of pure-strategy Nash equilibria of the equivalent game, which corresponds to approximate mixed-strategy Nash equilibria of the original game. How to do this exactly requires further research and thus remains open.

Acknowledgement

Appendix A. What is Beyond the Scope of This Short Paper?

This section briefly discusses some motivation for the study of Nash equilibria and other technical aspects that, while interesting, addressing them within this short paper will make it lose its focus.

A.1 Significance of Non-cooperative Game Theory and Nash Equilibria

At this point, it would not be an exaggeration to say that game theory and the concept of Nah equilibrium have touched almost all areas of science and engineering. The list of the multitude of areas and applications would be too long to present here. It is beyond the scope of this note to discuss the theoretical and practical relevance that non-cooperative game theory and the concept of a Nash equilibrium have to real-world problem. But, here are a few quotes from the organization that awards what most people call the "Nobel Prize in Economics" every year. They give a taste for the broader impact of non-cooperative game theory and the Nash equilibria as a solution concept.

> *Summary:* "The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994 was awarded jointly to John C. Harsanyi, John F. Nash Jr. and Reinhard Selten *"for their pioneering analysis of equilibria in the theory of non-cooperative games"."* [13]

> *Press Release:* "Game theory is a mathematical method for analyzing strategic interaction." [14]

> *Press Release:* "Many interesting economic issues, such as the analysis of oligopoly, originate in non-cooperative games. In general, firms cannot enter into binding contracts regarding restrictive trade practices because such agreements are contrary to trade legislation. Correspondingly, the interaction among a government, special interest groups and the general public concerning, for instance, the design of tax policy is regarded as a non-cooperative game. Nash equilibrium has become a standard tool in almost all areas of economic theory. The most obvious is perhaps the study of competition between firms in the theory of industrial organization. But the concept has also been used in macroeconomic theory for economic policy, environmental and resource economics, foreign trade theory, the economics of information, etc. in order to improve our understanding of complex strategic interactions." [15]

A.2 Other Game-theoretic Graphical Models

This note does not consider other kinds of graphical models for game theory such as *multiagent influence diagrams (MAIDs)* (Koller & Milch, 2003), which provide graphical models for extensive-form games, *action-graph games* (Jiang et al., 2011), [16] which exploit "context-sensitive" conditional expected-payoff independence, and *expected utility networks (EUNs)* (La Mura, 2000), which focuses on models for the "strategic" decision-making aspects of a single, individual player.

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13. `http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1994/`

14. `http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1994/press.html`

15. `http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1994/press.html`

16. Thompson et al. (2011) considers enumeration methods based on sparse support to compute approximate Nash equilibria in the context of action-graph games.

A.3 Relative and Constant Approximations

This note only considers *absolute approximations*, the most commonly studied form of approximation of Nash equilibria. Relative approximations have also been the subject of study, but mostly for non-graphical models (Hémon et al., 2008).

A very recent interest is to provide polynomial-time algorithms for computing Nash equilibria of *specific, constant* approximation quality (see, e.g., work by Tsaknakis & Spirakis, 2007; Daskalakis et al., 2009; Hémon et al., 2008; Bosse et al., 2010 and the references therein). Most results of this kind are in 2-player games. Several authors have shown how to turn such polynomial algorithms for a constant approximation quality from 2-player to results for n-player, with a *larger, still constant* approximation quality. Most of that work uses a sparse-support approach. This type of approximate Nash equilibria problem does not seem to have been studied for arbitrary GGs, just specializations such as normal-form games. This paper does not consider such problems to compute approximate Nash equilibria with constant approximation quality here, except for revisiting it in the list of open problems at the end.

Appendix B. Expanding on the Relation between GMhGs and Hypergraphical Games

In the standard definition of hypergraphical games we have a hypergraph $(V, \mathcal{E})$, where each vertex $i \in V$ corresponds to a player i in the game and $\mathcal{E} \subset 2^V$ is a set of hyperedges (i.e., sets of subsets of V). Each player $i \in V$ has a finite set of actions A_i. There is "game" for each hyperedge $C \in \mathcal{E}$. By "game" here we mean that each player $i \in C$ has the same set of actions A_i in all (local) "games" defined by $\mathcal{E}$, and a local payoff hypermatrix $M''_{i,C}(x_C)$ that is a function of the joint-actions $x_C \in A_C$ of all the players in C. Let $\mathcal{C}''_i \equiv \{C \in \mathcal{E} \mid i \in C\}$ be the (local) set of hyperedges that the hypergraphical game induces for each player $i \in V$. The final/global payoff functions of each player i in the hypergraphical game is $M_i(x) \equiv \sum_{C \in \mathcal{C}''_i} M''_{i,C}(x_C)$.

Note that by definition, for all pairs of players $i, j \in V$, we have that, for any hyperedge $C \in \mathcal{E}$ such that $i, j \in C$, the following symmetry property must hold in a hypergraphical game: $C \in \mathcal{C}''_i$ if and only if $C \in \mathcal{C}''_j$. *GMhGs do not require this symmetry condition.*

Of course, from a *mathematical* perspective, one can always take a GMhG with a set of player's hyperedges $\mathcal{C}_i$ and local hypermatrices $M'_{i,C}$ for each player $i \in V$ and hyperedge $C \in \mathcal{C}_i$, and turn it into a hypermatrix game; just as one can take a GG and turn it into a normal-form game. In the case of the GMhG transformation, we can set the set of hyperedges $\mathcal{E}$ of the GMhG-induced hypergraphical game to $\mathcal{E} \equiv \cup_{i \in V} \cup_{C \in \mathcal{C}_i} C$, and, letting the GMhG-induced hyperedges $\mathcal{C}'_i \equiv \{C \in \mathcal{E} \mid i \in C\}$ for all $i \in V$. Then, for every $i \in V$, we must either create a local *zero-valued* hypermatrix $M''_{i,C}(x_C) \equiv 0$, for all $x_C \in A_C$ if $C \notin \mathcal{C}_i$; otherwise, if $C \in \mathcal{C}_i$, set the local hypermatrix of the hypergraphical game to $M''_{i,C} \equiv M'_{i,C}$.

Indeed, one can take any game with a finite number of players and actions and always turn it back into a normal-form game. But, from a *computational* perspective, such transformations could take an exponential amount of time and space, defeating the main purpose for introducing compact, tractable, and flexible representations of significant expressive power in the first place! Although, from a *theoretical* standpoint, the given transformation from

GMhGs to hypergraphical games is computationally efficient, from a *practical* standpoint, the loss in expressive power is clear, and significant.

Appendix C. Proof of Theorem 3

To simplify notation, given any joint mixed-strategy (i.e., a product distribution) p, for all $B \subset V$, and $x_B \in A_B$, we denote by $p(x_B) \equiv \prod_{i \in B} p_i(x_i) = \sum_{x_{-B}} p(x_B, x_{-B})$ the joint mixed-strategy over players in B only (i.e., marginal product-distributions of p over the joint-actions of players in B). Let p and q be two joint mixed strategies and, for each player i and each action x_i, denote by $\Delta_i(x_i) \equiv p_i(x_i) - q_i(x_i)$. In a slight abuse of notation, let $\Delta(x_S) \equiv \prod_{k \in S} \Delta_k(x_k)$.

The following very simple lemma is a cornerstone of the proof.

Lemma 1. (Product-Distribution Differences) *For any clique $B \subset V$ of players, for any clique joint action x_B,*

$$p(x_B) - q(x_B) = \sum_{S \in 2^B - \emptyset} \Delta(x_S) \, q(x_{B-S}) \, .$$

Proof. The lemma follows by applying a binomial expansion:

$$p(x_B) = \prod_{j \in B} (q_j(x_j) + \Delta_j(x_j))$$
$$= \sum_{S \in 2^B} \Delta(x_S) \, q(x_{B-S})$$
$$= q(x_B) + \sum_{S \in 2^B - \emptyset} \Delta(x_S) \, q(x_{B-S}) \, .$$

$\square$

To further simplify the presentation of the proof it is convenient to introduce a slight abuse of notation: for all, $i \in C, C \in \mathcal{C}_i, B, S \subset C, B \cap S = \emptyset, x_S \in A_S, p_{C-B-S} \in \mathcal{P}_{C-B-S}$, let $M'_{i,C}(x_S, \Delta_B, p_{C-B-S}) \equiv \sum_{x_B \in A_B} \Delta(x_B) M'_{i,C}(x_S, x_B, p_{C-B-S})$.

The following useful claim follows immediately from the last lemma of joint product distribution differences (Lemma 1).

Claim 1. *Under the conditions of Lemma 1, for all $i \in V$, $C \in \mathcal{C}_i$, $B \subset C$, $x_B \in A_B$ and $p_{C-B}, q_{C-B} \in \mathcal{P}_{C-B}$, we have*

$$M'_{i,C}(x_B, p_{C-B}) - M'_{i,C}(x_B, q_{C-B}) = \sum_{S \in 2^{C-B} - \emptyset} M'_{i,C}(x_B, \Delta_S, q_{C-B-S}) \, .$$

Proof. Applying the last lemma on the differences between product distributions (Lemma 1), we have

$$M'_{i,C}(x_B, p_{C-B}) - M'_{i,C}(x_B, q_{C-B}) = \sum_{x_{C-B}} \left[\sum_{S \in 2^{C-B} - \emptyset} \Delta(x_S) q(x_{C-B-S}) \right] M'_{i,C}(x_C)$$

$$= \sum_{S \in 2^{C-B} - \emptyset} \sum_{x_S} \Delta(x_S) \sum_{x_{C-B-S}} q(x_{C-B-S}) M'_{i,C}(x_C)$$

$$= \sum_{S \in 2^{C-B} - \emptyset} M'_{i,C}(x_B, \Delta_S, q_{C-B-S}) .$$

$\square$

Using some algebra we can show another useful claim.

Claim 2. *Under the conditions of Lemma 1, for all $i \in V$, and $C \in \mathcal{C}_i$, we have*

$$\sum_{S \in 2^C - \emptyset} M'_{i,C}(\Delta_S, q_{C-S}) = \sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(p_i, \Delta_B, q_{C-B-\{i\}}) .$$

Proof. First note that we can decompose the left-hand side of the equation in the claim as

$$\sum_{S \in 2^C - \emptyset} M'_{i,C}(\Delta_S, q_{C-S}) = \sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(q_i, \Delta_B, q_{C-B-\{i\}}) +$$

$$\sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(\Delta_i, \Delta_B, q_{C-B-\{i\}}) .$$

Now note that, using the definition of Δ_i, we have

$$M'_{i,C}(\Delta_i, \Delta_B, q_{C-B-\{i\}}) = M'_{i,C}(p_i, \Delta_B, q_{C-B-\{i\}}) - M'_{i,C}(q_i, \Delta_B, q_{C-B-\{i\}}) .$$

The claim follows after making the appropriate substitution for that expressions and simplifying:

$$\sum_{S \in 2^C - \emptyset} M'_{i,C}(\Delta_S, q_{C-S}) = \sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(q_i, \Delta_B, q_{C-B-\{i\}}) +$$

$$\sum_{B \in 2^{C-\{i\}} - \emptyset} (M'_{i,C}(p_i, \Delta_B, q_{C-B-\{i\}}) - M'_{i,C}(q_i, \Delta_B, q_{C-B-\{i\}}))$$

$$= \sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(p_i, \Delta_B, q_{C-B-\{i\}}) .$$

$\square$

Suppose p is a Nash equilibrium of the game, which must exist by Nash's Theorem (Nash, 1951). Applying the last two claims above, we obtain

$$
\begin{aligned}
M_i(p) &= M_i(q) + \sum_{C \in \mathcal{C}_i} \sum_{S \in 2^C - \emptyset} M'_{i,C}(\Delta_S, q_{C-S}) \\
&= M_i(q) + \sum_{C \in \mathcal{C}_i} \sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(p_i, \Delta_B, q_{C-B-\{i\}}) \\
&\geq \max_{x'_i} M_i(x'_i, p_{-i}) \\
&= \max_{x'_i} \left[M_i(x'_i, q_{-i}) + \sum_{C \in \mathcal{C}_i} \sum_{B \in 2^{C-\{i\}} - \emptyset} M'_{i,C}(x'_i, \Delta_B, q_{C-B-\{i\}}) \right] .
\end{aligned}
$$

Rearranging and simplifying, we obtain the following expression:

$$
\begin{aligned}
M_i(q) \geq \max_{x'_i} M_i(x'_i, q_{-i}) + \\
\sum_{C \in \mathcal{C}_i} \sum_{B \in 2^{C-\{i\}} - \emptyset} \left(M'_{i,C}(x'_i, \Delta_B, q_{C-B-\{i\}}) - M'_{i,C}(p_i, \Delta_B, q_{C-B-\{i\}}) \right) .
\end{aligned}
$$

Let q be the closest (in ℓ_∞ distance) joint mixed strategy in $\widetilde{I}$, defined using sizes s_i as given in the statement of the theorem, to exact Nash equilibrium p. Hence, we have

$$
|\Delta_i(x_i)| \leq \frac{\epsilon}{2 |A_i| \max_{j \in \mathcal{N}_i} \sum_{C \in \mathcal{C}_j} R_{j,C} (|C| - 1)} .
$$

Consider the conditional expected hypermatrix payoff difference in parenthesis within the innermost summation inside the maximization of the equilibrium condition above. Noting that

$$
|M'_{i,C}(x'_i, x_B, q_{C_{-i,B}}) - M'_{i,C}(p_i, x_B, q_{C_{-i,B}})| \leq R_{i,C},
$$

we obtain the following lower bound on that innermost summation:

$$
\begin{aligned}
\left(M'_{i,C}(x'_i, \Delta_B, q_{C_{-i,B}}) - M'_{i,C}(p_i, \Delta_B, q_{C_{-i,B}}) \right) &\geq \sum_{x_B} \left[\prod_{k \in B} |\Delta_k(x_k)| \right] (-R_{i,C}) \\
&= -R_{i,C} \sum_{x_B} \prod_{k \in B} |\Delta_k(x_k)| . \quad (1)
\end{aligned}
$$

We can upper bound the last factor in the right-hand side of the last expression as

$$
\begin{aligned}
\sum_{x_B} \prod_{k \in B} |\Delta_k(x_k)| &= \prod_{k \in B} \sum_{x_k} |\Delta_k(x_k)| \\
&\leq \prod_{k \in B} \sum_{x_k} \frac{\epsilon}{2 |A_k| \max_{j \in \mathcal{N}_k} \sum_{C' \in \mathcal{C}_j} R_{j,C'} (|C'| - 1)} \\
&= \prod_{k \in B} \frac{\epsilon}{2 \max_{j \in \mathcal{N}_k} \sum_{C' \in \mathcal{C}_j} R_{j,C'} (|C'| - 1)} \\
&\leq \prod_{k \in B} \frac{\epsilon}{2 \sum_{C' \in \mathcal{C}_i} R_{i,C'} (|C'| - 1)} \\
&= \left(\frac{\epsilon}{2 \sum_{C' \in \mathcal{C}_i} R_{i,C'} (|C'| - 1)} \right)^{|B|} .
\end{aligned}
$$

Thus, using the resulting lower bound on the expression given in (1), we obtain

$$
M_i(q) \geq \max_{x_i'} M_i(x_i', q_{-i}) - \sum_{C \in \mathcal{C}_i} \sum_{B \in 2^{C - \{i\}} - \emptyset} R_{i,C} \left(\frac{\epsilon}{2 \sum_{C' \in \mathcal{C}_i} R_{i,C'} (|C'| - 1)} \right)^{|B|} .
$$

The second term in the right-hand side of the last expression equals

$$
\begin{aligned}
&\sum_{C \in \mathcal{C}_i} R_{i,C} \sum_{B \in 2^{C - \{i\}} - \emptyset} \left(\frac{\epsilon}{2 \sum_{C' \in \mathcal{C}_i} R_{i,C'} (|C'| - 1)} \right)^{|B|} \\
&= \sum_{C \in \mathcal{C}_i} R_{i,C} \left[\left(1 + \frac{\epsilon}{2 \sum_{C' \in \mathcal{C}_i} R_{i,C'} (|C'| - 1)} \right)^{|C| - 1} - 1 \right] \\
&\leq \sum_{C \in \mathcal{C}_i} R_{i,C} \left[\frac{\epsilon (|C| - 1)}{\sum_{C' \in \mathcal{C}_i} R_{i,C'} (|C'| - 1)} \right] = \epsilon .
\end{aligned}
$$

The last inequality follows from using the upper bound condition on ϵ given in the statement of the theorem and applying the well-known inequality, $1 + z \leq \exp(z) \leq 1 + z + z^2$ for $|z| < 1$ (Cormen, Leiserson, & Rivest, 1990). This completes the proof of the theorem.

Appendix D. Approximation Schemes

This appendix presents a brief, general description of the different approximation schemes. To begin, it is important to note that it is most common to define approximations in terms of *relative* factors. Here, given the standard/traditional mathematical definition of approximate Nash equilibria in a game setting, the description assumes that the approximation quality is in *absolute* terms. The description assumes that the problem is that of computing absolute approximations of Nash equilibria in games, as defined in the main body of the text.

Definition 5. *Let N be the representation size of the input game $\mathcal{G}$ and $\epsilon > 0$ the (absolute) approximation quality. An algorithm* A *is a* polynomial-time approximation scheme (PTAS)

if, for every $\epsilon > 0$, on input $\mathcal{G}$, A *runs in time polynomial in N. If, in addition,* A *runs in time polynomial in $\frac{1}{\epsilon}$, then* A *is a* fully-polynomial-time approximation scheme (FPTAS). *If, instead,* A *runs in time $N^{O(polylog(N))}$, where $polylog(N)$ denotes a polynomial function of $\log N$, then* A *is a* quasi-polynomial-time approximation scheme (quasi-PTAS).

References

Alon, N., & Spencer, J. H. (2004). *The Probabilistic Method* (Second edition). John Wiley & Sons.

Althöfer, I. (1994). On sparse approximations to randomized strategies and convex combinations. *Linear Algebra and its Applications, 199, Supplement 1*(0), 339 – 355. Special Issue Honoring Ingram Olkin.

Aumann, R. (1974). Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics, 1*.

Aumann, R. (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica, 55*.

Awasthi, P., Balcan, M.-F., Blum, A., Sheffet, O., & Vempala, S. (2010). On Nash-equilibria of approximation-stable games. In Kontogiannis, S., Koutsoupias, E., & Spirakis, P. (Eds.), *Algorithmic Game Theory*, Vol. 6386 of *Lecture Notes in Computer Science*, pp. 78–89. Springer Berlin Heidelberg.

Becker, A., & Geiger, D. (2001). A sufficiently fast algorithm for finding close to optimal clique trees. *Artificial Intelligence, 125*(1-2), 3 – 17.

Bollobas, B. (1979). *Graph Theory*. Springer.

Bosse, H., Byrka, J., & Markakis, E. (2010). New algorithms for approximate Nash equilibria in bimatrix games. *Theoretical Computer Science, 411*(1), 164 – 173.

Braunstein, A., Mézard, M., & Zecchina, R. (2005). Survey propagation: an algorithm for satisfiability. *Random Structures and Algorithms, 27*, 201.

Braunstein, A., & Zecchina, R. (2004). Survey propagation as local equilibrium equations. *Journal of Statistical Mechanics: Theory and Experiment, 6*, 7.

Bubelis, V. (1979). On equilibria in finite games. *International Journal of Game Theory, 8*(2), 65–79.

Chan, H., Ceyko, M., & Ortiz, L. (2012). Interdependent defense games: Modeling interdependent security under deliberate attacks. In *Proceedings of the Twenty-Eighth Conference Annual Conference on Uncertainty in Artificial Intelligence (UAI-12)*, pp. 152–162, Corvallis, Oregon. AUAI Press.

Chen, X., & Deng, X. (2005a). 3-NASH is PPAD-complete. Tech. rep. 134, Electronic Colloquium on Computational Complexity (ECCC). `http://eccc.hpi-web.de/eccc-reports/2005/TR05-134/index.html`.

Chen, X., & Deng, X. (2005b). Settling the complexity of 2-player Nash-equilibrium. Tech. rep. 140, Electronic Colloquium on Computational Complexity (ECCC). `http://eccc.hpi-web.de/eccc-reports/2005/TR05-140/index.html`.

Chen, X., & Deng, X. (2006). Settling the complexity of two-player Nash equilibrium. In *Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*.

Chen, X., Deng, X., & Teng, S.-H. (2006). Computing Nash equilibria: Approximation and smoothed complexity. In *FOCS*, pp. 603–612. IEEE Computer Society.

Chen, X., Deng, X., & Teng, S.-H. (2009). Settling the complexity of computing two-player Nash equilibria. *J. ACM*, *56*(3), 14:1–14:57.

Conitzer, V., & Sandholm, T. (2008). New complexity results about Nash equilibria. *Games and Economic Behavior*, *63*(2), 621 – 641. Second World Congress of the Game Theory Society.

Cormen, T., Leiserson, C., & Rivest, R. (1990). *Introduction to Algorithms*. The MIT Press, Cambridge, MA.

Daskalakis, C., Goldberg, P. W., & Papadimitriou, C. H. (2006). The complexity of computing a Nash equilibrium. In *STOC*.

Daskalakis, C., Goldberg, P. W., & Papadimitriou, C. H. (2009a). The complexity of computing a Nash equilibrium. *Commun. ACM*, *52*(2), 89–97.

Daskalakis, C., Goldberg, P. W., & Papadimitriou, C. H. (2009b). The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, *39*(1), 195–259.

Daskalakis, C., Mehta, A., & Papadimitriou, C. (2007). Progress in approximate Nash equilibria. In *EC '07: Proceedings of the 8th ACM conference on Electronic commerce*, pp. 355–358, New York, NY, USA. ACM.

Daskalakis, C., Mehta, A., & Papadimitriou, C. (2009). A note on approximate Nash equilibria. *Theoretical Computer Science*, *410*(17), 1581 – 1588. Internet and Network Economics.

Daskalakis, C., & Papadimitriou, C. H. (2006). Computing pure Nash equilibria in graphical games via Markov random fields. In *EC '06: Proceedings of the 7th ACM conference on Electronic commerce*, pp. 91–99, New York, NY, USA. ACM.

Daskalakis, C., & Papadimitriou, C. H. (2008). Discretized multinomial distributions and Nash equilibria in anonymous games. In *FOCS*.

Daskalakis, K., Goldberg, P. W., & Papadimitriou, C. H. (2005). The complexity of computing a Nash equilibrium. *Electronic Colloquium on Computational Complexity (ECCC)*.

Daskalakis, K., & Papadimitriou, C. H. (2005). Three-player games are hard. *Electronic Colloquium on Computational Complexity (ECCC)*.

Dechter, R. (2003). *Constraint Processing*. Morgan Kaufmann.

Feder, T., Nazerzadeh, H., & Saberi, A. (2007). Approximating Nash equilibria using small-support strategies. In *Proceedings of the 8th ACM Conference on Electronic Commerce*, EC '07, pp. 352–354, New York, NY, USA. ACM.

Fudenberg, D., & Levine, D. (1999). *The Theory of Learning in Games*. MIT Press.

Gilboa, I., & Zemel, E. (1989). Nash and correlated equilibria: some complexity considerations. *Games and Economic Behavior*, *1*, 80–93.

Goldberg, P. W., & Papadimitriou, C. H. (2005). Reducibility among equilibrium problems. *Electronic Colloquium on Computational Complexity (ECCC)*.

Gottlob, G., Greco, G., & Scarcello, F. (2005). Pure Nash equilibria: Hard and easy games. *JAIR*, *24*, 357–406.

Gottlob, G., Leone, N., & Scarcello, F. (2000). A comparison of structural CSP decomposition methods. *Artificial Intelligence*, *124*(2), 243 – 282.

Gottlob, G., Leone, N., & Scarcello, F. (2001). Hypertree decompositions: A survey. In *MFCS '01: Proceedings of the 26th International Symposium on Mathematical Foundations of Computer Science*, pp. 37–57, London, UK. Springer-Verlag.

Gottlob, G., Leone, N., & Scarcello, F. (2002). Hypertree decompositions and tractable queries. *Journal of Computer and System Sciences*, *64*, 579–627.

Hémon, S., de Rougemont, M., & Santha, M. (2008). Approximate Nash equilibria for multi-player games. In Monien, B., & Schroeder, U.-P. (Eds.), *Algorithmic Game Theory*, Vol. 4997 of *Lecture Notes in Computer Science*, pp. 267–278. Springer Berlin Heidelberg.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, *58*(301), 13–30.

Irfan, M., & Ortiz, L. (2011). A game-theoretic approach to influence in networks. In *AAAI Conference on Artificial Intelligence*.

Irfan, M. T. (2013). *Causal Strategic Inference in Social and Economic Networks*. Ph.D. thesis, Stony Brook University, Department of Computer Science. `http://www.bowdoin.edu/~mirfan/papers/Mohammad_Tanvir_Irfan_Dissertation.pdf`.

Irfan, M. T., & Ortiz, L. E. (2013). On influence, stable behavior, and the most influential individuals in networks: A game-theoretic approach. *CoRR*, *abs/1303.2147*.

Janovskaja, E. B. (1968). Equilibrium situations in multi-matrix games. *Litovsk. Mat. Sb.*, *8*, 381–384.

Jiang, A. X., & Leyton-Brown, K. (2011). Polynomial computation of exact correlated equilibrium in compact games. *SIGecom Exchanges*, *10*(1), 6–8.

Jiang, A. X., Leyton-Brown, K., & Bhat, N. A. (2011). Action-graph games. *Games and Economic Behavior*, *71*, 141–173. DOI:10.1016/j.geb.2010.10.012.

Kakade, S., Kearns, M., Langford, J., & Ortiz, L. (2003). Correlated equilibria in graphical games. In *EC '03: Proceedings of the 4th ACM conference on Electronic commerce*, pp. 42–47, New York, NY, USA. ACM.

Kearns, M. (2007). Graphical games. In Nisan, N., Roughgarden, T., Éva Tardos, & i, V. V. V. (Eds.), *Algorithmic Game Theory*, chap. 7, pp. 159–180. Cambridge University Press.

Kearns, M. J., Littman, M. L., & Singh, S. P. (2001). Graphical models for game theory. In *UAI '01: Proceedings of the 17th Conference in Uncertainty in Artificial Intelligence*, pp. 253–260, San Francisco, CA, USA. Morgan Kaufmann Publishers Inc.

Koller, D., & Milch, B. (2003). Multi-agent influence diagrams for representing and solving games. *Games and Economic Behavior*, *45*(1), 181–221.

Kontogiannis, S. C., Panagopoulou, P. N., & Spirakis, P. G. (2009). Polynomial algorithms for approximating Nash equilibria of bimatrix games. *Theor. Comput. Sci.*, *410*(17), 1599–1606.

La Mura, P. (2000). Game networks. In *Proceedings of the 16th Conference on Uncertainty in Artificial Intelligence (UAI)*, pp. 335–342.

Lipton, R. J., Markakis, E., & Mehta, A. (2003). Playing large games using simple strategies. In *Proceedings of the 4th ACM Conference on Electronic Commerce*, EC '03, pp. 36–41, New York, NY, USA. ACM.

Maneva, E., Mossel, E., & Wainwright, M. J. (2007). A new look at survey propagation and its generalizations. *J. ACM*, *54*(4).

McDiarmid, C. (1989). On the method of bounded differences. *Surveys in Combinatorics*, *141*, 148–188.

McKelvey, R. D., & McLennan, A. (1996). Computation of equilibria in finite games. In *Handbook of Computational Economics*, Vol. I, pp. 87–142.

Mézard, M., Parisi, G., & Zecchina, R. (2002). Analytic and Algorithmic Solution of Random Satisfiability Problems. *Science*, *297*(5582), 812–815.

Mezard, M. (2003). PHYSICS/COMPUTER SCIENCE: Passing Messages Between Disciplines. *Science*, *301*(5640), 1685–1686.

Mézard, M. (2004). Message passing in random satisfiability problems..

Monderer, D., & Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, *14*(1), 124 – 143.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, *54*, 286–295.

Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences of the United States of America*, *35*(1), 48–49.

Nisan, N., Roughgarden, T., Éva Tardos, & Vazirani, V. V. (Eds.). (2007). *Algorithmic Game Theory*. Cambridge University Press.

Ortiz, L. E. (2002). Approximation bound refinement of KLS. Unpublished Note, `http://www.cis.upenn.edu/~mkearns/teaching/cgt/revised_approx_bnd.pdf`.

Ortiz, L. E. (2008). CPR for CSPs: A probabilistic relaxation of constraint propagation. In Platt, J., Koller, D., Singer, Y., & Roweis, S. (Eds.), *Advances in Neural Information Processing Systems 20*, pp. 1113–1120. Curran Associates, Inc.

Ortiz, L. E., & Kearns, M. (2003). Nash propagation for loopy graphical games. In Becker, S. B., Thrun, S. T., & Obermayer, K. (Eds.), *Advances in Neural Information Processing Systems 15*, pp. 817–824.

Papadimitriou, C. H. (1994). On the complexity of the parity argument and other inefficient proofs of existence. *J. Comput. Syst. Sci.*, *48*(3), 498–532.

Papadimitriou, C. H. (2005). Computing correlated equilibria in multi-player games. In *STOC '05: Proceedings of the thirty-seventh annual ACM symposium on Theory of computing*, pp. 49–56.

Papadimitriou, C. H., & Roughgarden, T. (2008). Computing correlated equilibria in multi-player games. *J. ACM, 55*(3), 14:1–14:29.

Parisi, G. (2003a). A backtracking survey propagation algorithm for K-satisfiability. *eprint arXiv:cond-mat/0308510*.

Parisi, G. (2003b). Constraint optimization and statistical mechanics. *CoRR, cs.CC/0312011*.

Parisi, G. (2003c). On the probabilistic approach to the random satisfiability problem. *CoRR, cs.CC/0308010*.

Parisi, G. (2003d). Some remarks on the survey decimation algorithm for k-satisfiability. *CoRR, cs.CC/0301015*.

Ponsen, M., de Jong, S., & Lanctot, M. (2011). Computing approximate Nash equilibria and robust best-responses using sampling. *J. Artif. Int. Res., 42*(1), 575–605.

Roughgarden, T. (2009). Computing equilibria: A computational complexity perspective. *Economic Theory*.

Russell, S. J., & Norvig, P. (2003). *Artificial Intelligence: A Modern Approach*. Pearson Education.

Singh, S. P., Soni, V., & Wellman, M. P. (2004). Computing approximate Bayes-Nash equilibria in tree-games of incomplete information. In Breese, J. S., Feigenbaum, J., & Seltzer, M. I. (Eds.), *ACM Conference on Electronic Commerce*, pp. 81–90. ACM.

Soni, V., Singh, S. P., & Wellman, M. P. (2007). Constraint satisfaction algorithms for graphical games. In Durfee, E. H., Yokoo, M., Huhns, M. N., & Shehory, O. (Eds.), *AAMAS*, p. 67. IFAAMAS.

Thompson, D., Leung, S., & Leyton-Brown, K. (2011). Computing Nash equilibria of action-graph games via support enumeration. In Chen, N., Elkind, E., & Koutsoupias, E. (Eds.), *Internet and Network Economics*, Vol. 7090 of *Lecture Notes in Computer Science*, pp. 338–350. Springer Berlin Heidelberg.

Tsaknakis, H., & Spirakis, P. (2007). An optimization approach for approximate Nash equilibria. In Deng, X., & Graham, F. (Eds.), *Internet and Network Economics*, Vol. 4858 of *Lecture Notes in Computer Science*, pp. 42–56. Springer Berlin Heidelberg.

Vazirani, V. V. (2001). *Approximation Algorithms*. Springer.

Vickrey, D., & Koller, D. (2002). Multi-agent algorithms for solving graphical games. In *Proceedings of the Eighteenth National Conference on Artificial Intelligence (AAAI-02)*, pp. 345–351.

Williamson, D. P., & Shmoys, D. B. (2011). *The Design of Approximation Algorithms*. Cambridge.

Yu, S., & Berthod, M. (1995). A game strategy approach for image labeling. *Computer Vision and Image Understanding, 61*(1), 32 – 37.