Nonmodal growth of MHD shear flows with stabilizing magnetic fields

Adrian Fraser[1], Jeff Oishi[2], Alexis Kaminski[3]

[1]UC Santa Cruz; [2]Bates College; [3]UC Berkeley

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Follow our progress and check out our source code at
`github.com/afraser3/nonmodal-MHD-KH`
My email: `adfraser@ucsc.edu`

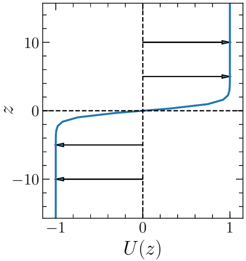
Summary

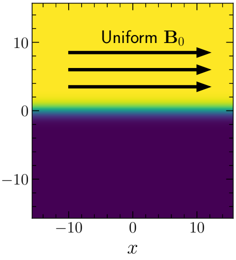
- Shear flows in MHD are often linearly unstable
  - $\rightarrow$ Perturbations grow as $e^{\sigma t}$; growth rate σ depends on parameters
  - $\rightarrow$ Instability can drive turbulence, transport
  - $\rightarrow$ Presumably no growth in parameter regimes where $\sigma \leq 0$
- However, shear flow systems are *non-normal*, implying:
  - $\rightarrow$ σ very sensitive to minor perturbations,
  - $\rightarrow$ Perturbations can transiently grow faster than $e^{\sigma t}$, and
  - $\rightarrow$ *Significant growth and mixing possible in stable regimes*
- **This work: exploring consequences of non-normality for the Kelvin-Helmholtz (KH) instability in MHD**

Linear stability analyses: useful for predicting disruptions, turbulence, etc.

Common reasons to do linear stability analyses:

- To check if a given plasma configuration is a stable equilibrium
- If unstable: how fast do perturbations grow? What do they look like?
- For what physical parameters is the equilibrium stable?
  - $\rightarrow$ Hopefully, can count on **no turbulence/mixing** for these parameters!



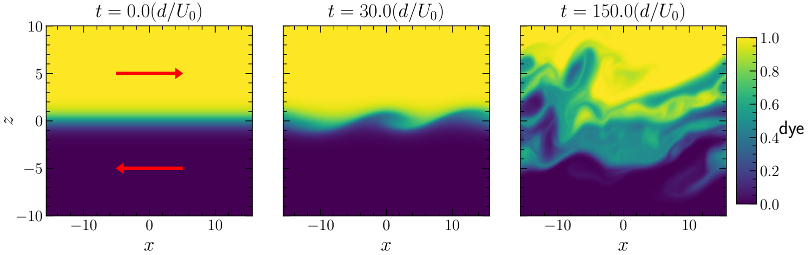


Example: KH instability in MHD
$\mathbf{V}_0 = U(z)\hat{\mathbf{x}} = U_0 \tanh(z/d)\hat{\mathbf{x}}$
Applied field: $\mathbf{B}_0 = B_0\hat{\mathbf{x}}$

Ideal MHD: magnetic tension ensures the flow is **stable** if it is sub-Alfvénic, i.e., $U_0 \lesssim v_A$

Linear stability analyses: useful for predicting disruptions, turbulence, etc.

For sufficiently strong U_0 or weak B_0, this system is linearly unstable
$\rightarrow$ turbulence ensues



What happens if U_0 is weak, and the system is stable ($\sigma \leq 0$)?
$\rightarrow$ First, review results from similar problem: KH with stratification

Analogous system: KH stabilized by density gradients

Miles-Howard theorem: **KH stabilized** by gravity + stable density stratification $\partial\rho_0/\partial z$ unless

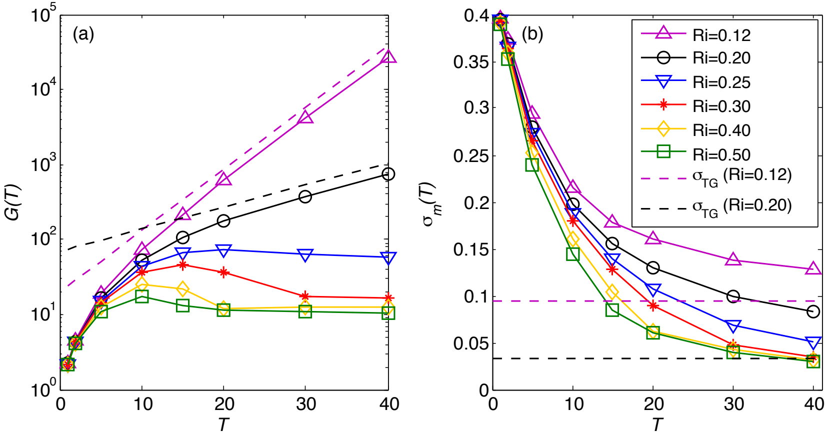
$$\mathrm{Ri} < \frac{1}{4}$$

where $\mathrm{Ri} \propto \frac{\partial\rho_0/\partial z}{(\partial U_0/\partial z)^2}$

Kaminski, Caulfield, & Taylor JFM 2014: significant transient growth even for $\mathrm{Ri} > 1/4$

Analogous system: KH stabilized by density gradients

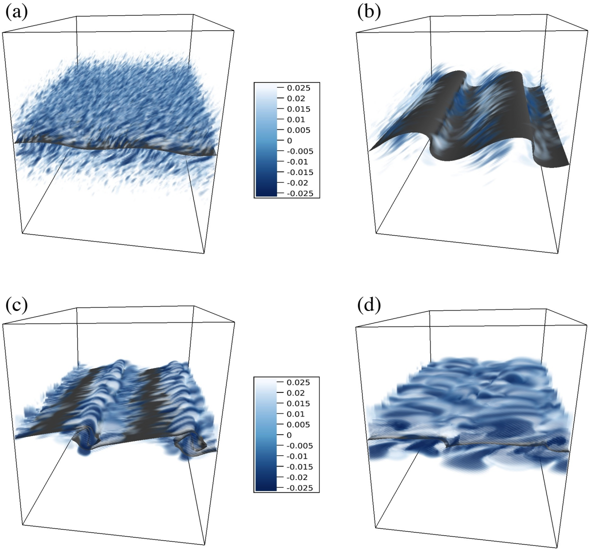
Miles-Howard: KH stabilized unless $\mathrm{Ri} < 1/4$



(Kaminski *et al.* 2014)

Left: optimal perturbation growth G by time T (solid) alongside $e^{2\sigma t}$ (dashed)
Right: mean growth rate $\sigma_m = \ln G(T)/(2T)$

Analogous system: KH stabilized by density gradients



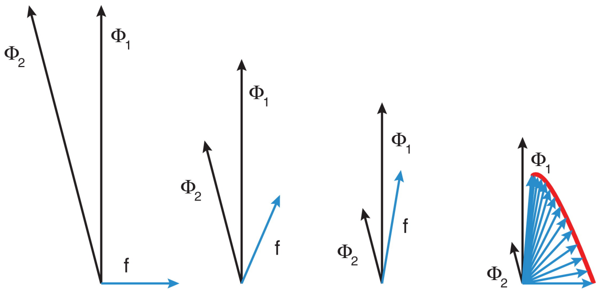
(Kaminski *et al.* 2017)

Nonlinear evolution of buoyancy (isosurface) and vorticity (color) for linearly-optimal perturbation at $\mathrm{Ri} = 0.4$

$\rightarrow$ **Significant mixing, turbulence, despite linear stability**

Non-normality drives transient growth

Nonmodal growth: characteristic feature of **non-normal systems**



Schmid PJ. 2007.
Annu. Rev. Fluid Mech. 39:129–62

Consider $\dot{x} = Ax$, for 2×2 matrix A with eigenvectors Φ_1, Φ_2, eigenvalues σ_1, σ_2

Even if $\sigma_1, \sigma_2 < 0$, amplitude $||f||^2 = \langle f, f \rangle$ of disturbance can transiently grow if $\langle \Phi_1, \Phi_2 \rangle \neq 0$ *(sensitive to choice of inner product!)*

$\rightarrow$ **This is what drives growth in KH-stable systems**

This work: test whether significant growth, mixing also possible when stabilizing effect is magnetic tension, not density stratification

Other signatures of non-normality: ε-pseudospectra

Transient growth is a result of the eigenvectors f_j of operator A being non-orthogonal $\leftrightarrow$ A is non-normal

Another consequence of non-normality: eigenvalues σ of A are sensitive to perturbations to A, i.e., $A \to A + \varepsilon\tilde{A}$ changes σ significantly for small ε

- $\rightarrow$ Observed incidentally in MHD KH by Fraser *et al.* PoP 2021: σ highly sensitive to slight changes to the driving flow & field
- $\rightarrow$ This work: quantify by calculating the ε-**pseudospectrum**: the set of eigenvalues σ_ε achievable via $O(\varepsilon)$ perturbations $A \to A + \varepsilon\tilde{A}$

Calculate ε-pseudospectra with Eigentools

Eigentools[1]: Python package for studying eigenvalue problems, built on Dedalus[2]

Key feature for this work: calculate ε-pseudospectra of **generalized** eigenvalue problems $Lx = \sigma Mx$ with **user-specified norms** (e.g. energy norm), following Embree & Keeler[3]

[1]Oishi *et al.* JOSS 2021; `github.com/DedalusProject/eigentools`
[2]Burns *et al.* PRR 2020; `dedalus-project.org`
[3]Embree & Keeler SIAM J. Matrix Anal. Appl. (2017)

System setup: incompressible MHD, 2D for now

Start from the incompressible MHD equations:

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\nabla p + \frac{1}{c}\mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{v},$$

$$\frac{\partial}{\partial t}\mathbf{B} - \nabla\times(\mathbf{v}\times\mathbf{B}) = \eta\nabla^2\mathbf{B},$$

$$\nabla\cdot\mathbf{v} = 0, \qquad \nabla\cdot\mathbf{B} = 0$$

Consider background flow $\mathbf{V}_0 = U_0\tanh(z/d)\hat{\mathbf{x}}$ and uniform field $B_0\hat{\mathbf{x}}$

- Perturb and linearize: $\mathbf{v} = \mathbf{V}_0 + \mathbf{v}_1$ and $\mathbf{v}_1\cdot\nabla\mathbf{v}_1 \to 0$
- Express as streamfunction ϕ & flux function ψ: $\mathbf{v}_1 = \hat{\mathbf{y}}\times\nabla\phi$, $\mathbf{B}_1 = \hat{\mathbf{y}}\times\nabla\psi$
- Non-dimensionalize in terms of U_0, d, and B_0

System setup: incompressible MHD, 2D for now

Resulting equations are:

$$\frac{\partial}{\partial t}\nabla^2\phi = \frac{d^2U(z)}{dz^2}\frac{\partial\phi}{\partial x} - U(z)\frac{\partial}{\partial x}\nabla^2\phi + \frac{1}{M_A^2}\frac{\partial}{\partial x}\nabla^2\psi + \frac{1}{\mathrm{Re}}\nabla^4\phi,$$

$$\frac{\partial}{\partial t}\psi = \frac{\partial\phi}{\partial x} - U(z)\frac{\partial\psi}{\partial x} + \frac{1}{\mathrm{Rm}}\nabla^2\psi$$

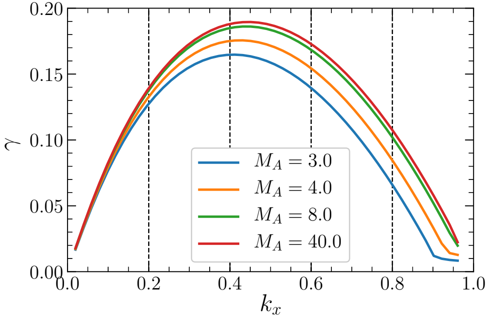
Where $U(z) = \tanh(z)$, $M_A = U_0/v_A \propto U_0/B_0$, $\mathrm{Re} = U_0 d/\nu$, $\mathrm{Rm} = U_0 d/\eta$

Important part: they're of the form

$$\frac{\partial}{\partial t}\mathcal{M}\vec{f} = \mathcal{L}\vec{f}$$

$\Rightarrow$ Normal-mode ansatz $\vec{f} \sim \vec{f}_j e^{ik_x x+\sigma_j t}$ yields generalized eigenvalue problem for eigenvalue σ_j, eigenmode $\vec{f}_j$

MHD KH: stabilized by magnetic tension



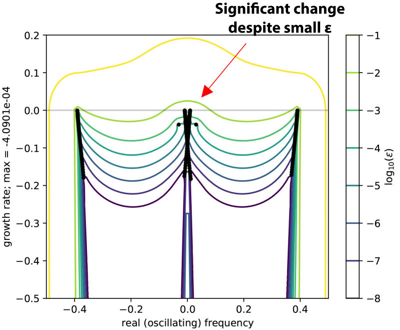
Recall KH is stabilized if flow is sub-Alfvénic, i.e., $M_A \lesssim 1$ (threshold here is $M_{A,\text{crit}} \approx 1.1$)

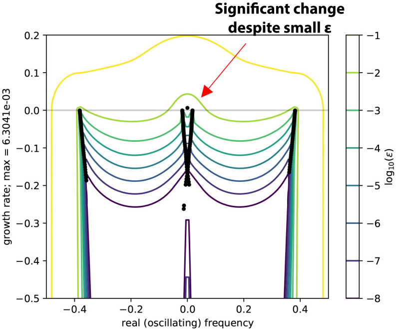
Otherwise, unstable for $0 < k_x \lesssim 1$

Following Kaminski *et al.* 2014, 2017, look for signatures of non-normality near marginal stability threshold $M_A \approx M_{A,\text{crit}}$

ε-pseudospectra for MHD KH: signatures of non-normality

Dots: spectrum σ (i.e. growth rate & frequency) for barely-stable (left, $M_A = 1.05$) and barely-unstable (right, $M_A = 1.1$) cases

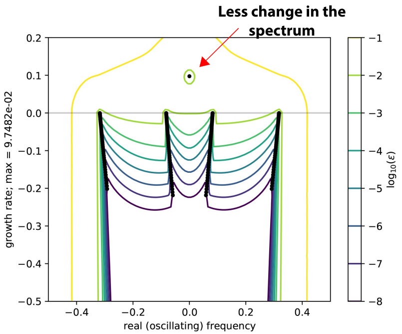




Colored lines show the ε-pseudospectrum: the change in the spectrum due to $O(\varepsilon)$ changes to the system

Significant changes despite small $\varepsilon \Rightarrow$ non-modal growth likely in stable systems!

Non-normality significantly less pronounced away from marginal stability



Increasing M_A (reducing stabilizing magnetic field strength) rapidly reduces non-normality

$\Rightarrow$ Little non-modal growth expected for $M_A \gg M_{A,\text{crit}}$

Conclusions, Future Work

Conclusions thus far:

- ε-pseudospectra show signatures of non-normality in MHD KH near marginal stability, as in stratified KH
- Expect significant transient growth, sensitivity to perturbations to the linear operator even in stable regime

Next steps:

- Calculate linear optimal perturbations
- How does non-normality for 3D perturbations compare to 2D case?
- Calculate nonlinear optimal perturbations with direct adjoint looping
- *Does this effect belong in "parasitic saturation" models?*