CSCE 355
Assignment 1 Solutions

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1. Let $A \coloneqq \{1,2,3\}$ and $B \coloneqq \{2,4,6,8\}$. Along with their cardinality, give the value of the following expressions:

(a) $A \cup B$

Solution

$A \cap B = \{1,2,3,4,6,8\}$

$|A \cap B| = 6$

(b) $A \cap B$

Solution

$A \cap B = \{2\}$

$|A \cap B| = 1$

(c) $A - B$

Solution

$A \cap B = \{1,3\}$

$|A \cap B| = 2$

(d) $A \triangle B$

Solution

$A \cap B = \{1,3,4,6,8\}$

$|A \cap B| = 5$

(e) $A \times B$

Solution

$$\begin{aligned} A \times B = \{&(1,2),(1,4),(1,6),(1,8),\\ &(2,2),(2,4),(2,6),(2,8),\\ &(3,2),(3,4),(3,6),(3,8)\} \end{aligned}$$

$|A \cap B| = 12$

(f) 2^B

Solution

$$\begin{aligned} A \times B = \{&\emptyset,\{2\},\{4\},\{6\},\{8\},\\ &\{2,4\},\{2,6\},\{2,8\},\\ &\{4,6\},\{4,8\},\{6,8\},\\ &\{2,4,6\},\{2,4,8\},\{2,6,8\},\\ &\{4,6,8\},\{2,4,6,7\}\} \end{aligned}$$

$|A \cap B| = 16$

2. True or false: $2^{\emptyset} = \emptyset$. Explain.

Solution

False

$$2^{\emptyset} = \{\emptyset\} \neq \emptyset$$

3. Using just the numeral 1, braces, and commas, write the set $2^{2^{\{1\}}}$ in "long hand".

Solution

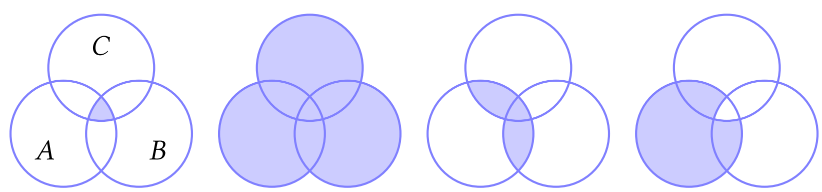
$$2^{\{1\}} = \{\emptyset, \{1\}\}$$

Setting $\dot{0} \coloneqq \emptyset, \dot{1} = \{1\}$

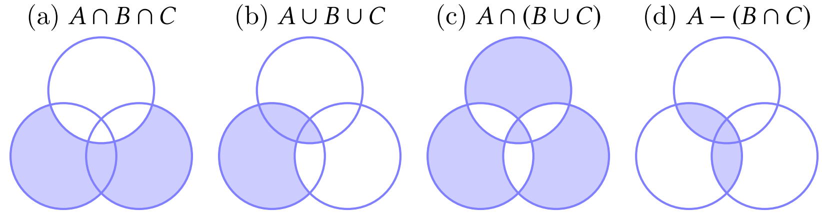
$$\begin{aligned} 2^{2^{\{1\}}} &= 2^{\{\dot{0}, \dot{1}\}} = \{\emptyset, \{\dot{0}\}, \{\dot{1}\}, \{\dot{0}, \dot{1}\}\} \\ &= \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\} \end{aligned}$$

4. Draw and fill in a Venn diagram to illustrate each of the expressions below.

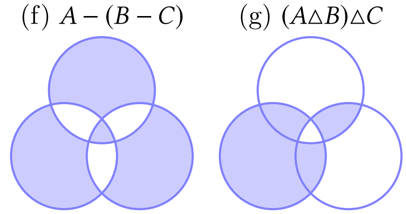
Solution



(a) $A \cap B \cap C$ (b) $A \cup B \cup C$ (c) $A \cap (B \cup C)$ (d) $A - (B \cap C)$



(e) $(A \cup B) - C$ (f) $A - (B - C)$ (g) $(A \triangle B) \triangle C$ (h) $(A \cap B) \cup (A \cap C)$



(i) $A \triangle (B \triangle C)$ (j) $A \triangle (B \cap C)$

5. What set theoretic identities are shown by the last problem?

Solution

(a) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. Suppose A and B are finite sets with $|A| = m$ and $|B| = n$, for some natural numbers m and n. You don't know anything else about the sets A and B. In terms of m and n, what can you say for sure about the following:

   (a) $|A \cup B|$

   Solution

   $$\max_{i \in \{m,n\}} i \leq |A \cup B| \leq m + n$$

   (b) $|A \cap B|$

   Solution

   $$0 \leq |A \cap B| \leq \min_{i \in \{m,n\}} i$$

   (c) $|A - B|$

   Solution

   $$0 \leq |A - B| \leq m$$

Relations

Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8\}$.

7. Let $R, S \subseteq A \times B$ be given by

$$R \coloneqq \{(1,4), (1,8), (2,2), (2,4), (3,2), (3,8)\}$$
$$S \coloneqq \{(1,2), (1,6), (2,8), (3,4), (3,8)\}$$

(a) Give R^{-1} and S^{-1}

Solution

$$R^{-1} \coloneqq \{(2,2), (2,3), (4,1), (4,2), (8,1), (8,3)\}$$
$$S^{-1} \coloneqq \{(2,1), (4,3), (6,1), (8,2), (8,3)\}$$

(b) Give $S^{-1} \circ R$ and $S \circ R^{-1}$

Solution

$S^{-1} \circ R \coloneqq \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,3)\}$

$S \circ R^{-1} \coloneqq \{(2,4), (2,8), (4,2), (4,6), (4,8), (8,2), (8,4), (8,6), (8,8)\}$

(c) Give $R \circ S^{-1} \circ R$

Solution

$R \circ S^{-1} \circ R \coloneqq \{(1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (3,2), (3,4), (3,8)\}$

8. Give an example of a nonempty binary relation on A that is symmetric and transitive but not reflexive.

Solution

$$R = \emptyset$$

Trivially symmetric and transitive, but not reflexive.

9. How many binary relations on A are there that are both reflexive and symmetric?

Solution

$$R_1 = \{(1,1),(2,2),(3,3)\}$$
$$R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$$
$$R_3 = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$$
$$R_4 = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$$
$$R_5 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3)\}$$
$$R_6 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$$
$$R_7 = \{(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$
$$R_8 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

10. How many equivalence relations are there on B?

Solution

$$15$$

(Try listing the partitions)

Functions

Let $A = \{1,2,3\}$ and $B = \{2,4,6,8\}$.

11. Give an example of a one-to-one function $f : A \to B$. How many such functions are there?

Solution

$$f = \{(1,2),(2,4),(3,6)\}$$

There are 24 injective functions mapping A to B.

12. Let $g : A \to B$ be defined by $g(x) = 2x$, for all $x \in A$. Is g one-to-one? Is g onto? Explain.

Solution

The graph of g is given by

$$g = \{(1,2),(2,4),(3,6)\}$$

This is injective, not surjective: $(\nexists x \in A)\ g(x) = 8$

13. Let $h : \mathbb{N} \to \mathbb{N}$ be defined by $h(x) = x^2 - 4x + 4$ for all $x \in \mathbb{N}$. Is h one-to-one? Onto? What is $h(A)$?

Solution

Claim. H is not injective (Specifically, $(\exists x, y)\ h(x) = h(y)$)

Proof. h(1) = h(3) = 1 □

H is not surjective.

Proof. Note that for $x \geq 2$, h is monotonic increasing.

$$h(0) = 4 \qquad h(3) = 1$$
$$h(1) = 1 \qquad h(4) = 4$$
$$h(2) = 0$$
$$\Rightarrow (\nexists x \in \mathbb{N})\ h(x) = 3$$

□

$$h(A) = \{0, 1\}$$