Solutions

MATH 242 . S2016 TEST 1 (Chapter 1) First, Last Name________________
(print)

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I understand that if I do not show a detailed work/give written explanations supporting my answer for each problem I will not get credit for the problem even if my answer is correct.
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6 points.

1. According to the Torricelli's law of draining the time rate of change of volume V of water in a draining tank is proportional to the square root of the depth, y, of water in the tank. If the rate of change of volume is equal to -3.36 ft^3/min when the depth $y = 9$ ft, write down the differential equation for the situation described. *Do not solve the equation.*

$$\frac{dV}{dt} = k\sqrt{y}$$

$$-3.36 = k\sqrt{9}$$

$$k = -1.12$$

$$\frac{dV}{dt} = -1.12\sqrt{y}$$

12 points each

2. Find the position function $x(t)$ of a moving particle with the given acceleration $a(t) = \dfrac{1}{\sqrt{t+4}}$, initial position $x(0) = 1$, and initial velocity $v(0) = -1$.

$$v(t) = \int \frac{dt}{\sqrt{t+4}} + C_1 = 2\sqrt{t+4} + C_1$$

$$-1 = 2\sqrt{4} + C_1 \qquad C_1 = -5$$

$$v(t) = 2\sqrt{t+4} - 5$$

$$x(t) = 2\int \sqrt{t+4}\,dt - \int 5\,dt + C_2 \qquad C_2 = 1 - \frac{32}{3} = -\frac{29}{3}$$

$$x(t) = \frac{4}{3}(t+4)^{\frac{3}{2}} - 5t + C_2 \qquad \boxed{x(t) = \frac{4}{3}(t+4)^{\frac{3}{2}} - 5t - \frac{29}{3}}$$

position function

$$1 = \frac{4}{3}\cdot 2^3 + C_2$$

$$1 = \frac{32}{3} + C_2$$

3. Find the EXPLICIT form of the general solution and the particular solution. Prime denotes derivative with respect to x.

$xy' - y = 2x^2y, \ y(1) = 1$ Separable DE

$xy' = y(2x^2+1)$

$\int \frac{dy}{y} = \int \frac{2x^2+1}{x} dx + c = \int (2x + \frac{1}{x}) dx + c$

$\ln y = x^2 + \ln x + c$

$y = e^{x^2 + \ln x + c} = e^{x^2} \cdot e^{\ln x} \cdot e^{c} = \underline{c\, x\, e^{x^2}}$ - general sol.

$1 = C \cdot 1 e^{1}$

$C = \frac{1}{e} = e^{-1}$

$y(x) = e^{-1} x e^{x^2}$

$\underline{y(x) = x e^{x^2-1}}$ - particular sol.

4. When plant fertilizer is dissolved in water, the amount of fertilizer, A, that remains undissolved after t minutes satisfies the differential equation

$\frac{dA}{dt} = -kA, \ k > 0.$

If 25% of the fertilizer dissolves after 1 min, how long does it take for half of the fertilizer to dissolve?

$\int \frac{dA}{A} = \int -k\,dt + c$

$\ln A = -kt + c$

$\underline{A = c e^{-kt} \text{ if } t = 0 \ A = C = A_0}, \ A = A_0 e^{-kt}$

when $t = 1$ min: $A = .75 A_0$

$0.75 \cancel{A_0} = \cancel{A_0} e^{-k \cdot 1}$

$0.75 = e^{-k}$

$\ln 0.75 = -k$

$k \approx 0.29$

$\underline{A = A_0 e^{-0.29t}}$ gen. solution

When $A = 0.5 A_0$ $t = ?$

$0.5 \cancel{A_0} = \cancel{A_0} e^{-0.29t}$

$t = \frac{\ln 0.5}{-0.29} \approx \underline{2.4 \text{ min}}$

5. Verify, that the given equation is exact and find the general solution:
$(2xy^3 + e^x)dx + (3x^2y^2 + \sin y)dy = 0$

$M(x,y) = 2xy^3 + e^x$ $\quad\quad N(x,y) = 3x^2y^2 + \sin y$

$\frac{\partial M}{\partial y} = 6xy^2$ $\quad\quad \frac{\partial N}{\partial x} = 6xy^2$ $\quad$ DE is exact

$F(x,y) = \int (2xy^3 + e^x)dx + g(y)$

$F(x,y) = x^2y^3 + e^x + g(y)$

$\frac{\partial F}{\partial y} = \cancel{3x^2y^2} + g'(y) = \cancel{3x^2y^2} + \sin y$

$\frac{dg}{dy} = \sin y$

$g(y) = \int \sin y\, dy + C = -\cos y + C$

$\underline{F(x,y) = x^2y^3 + e^x - \cos y + C}$

6. Find the explicit general and the particular solution of the differential equation. Prime denotes derivative with respect to x.
$3xy' + y = 12x,\ y(1) = 0$

$y' + \frac{1}{3x}y = 4$ linear 1st order DE.

$\rho(x) = e^{\int \frac{dx}{3x}} = e^{\frac{1}{3}\ln x} = x^{\frac{1}{3}}$

$yx^{\frac{1}{3}} = \int 4x^{\frac{1}{3}}dx + C$

$yx^{\frac{1}{3}} = 4 \cdot \frac{3}{4} x^{\frac{4}{3}} + C$

$\underline{y = 3x + Cx^{-\frac{1}{3}}}$ general sol.

$0 = 3 + C$

$C = -3$

$y(x) = 3x - \frac{3}{x^{1/3}} = \underline{3(x - x^{-\frac{1}{3}})}$ particular sol.

7. Find the general solution of the differential equation. Prime denotes derivative with respect to x.

$(x + y)y' = 1$ $\qquad$ $x+y$ is a linear form: $y' = F(x+y)$

$y' = \frac{1}{x+y}$, let $p = x+y$, $y = p - x$, $y' = p' - 1$

$p' - 1 = \frac{1}{p}$, $p' = \frac{1}{p} + 1 = \frac{1+p}{p}$

$\frac{dp}{dx} = \frac{1+p}{p}$ separable d.e.

$\int \frac{p\,dp}{1+p} = \int dx + C = x + C$

$\int \frac{p+1-1}{1+p}\,dp = \int \frac{\cancel{p+1}}{\cancel{1+p}}\,dp - \int \frac{1}{1+p}\,dp = p - \ln(1+p)$

$p - \ln(1+p) = x + C$

$\cancel{x} + y - \ln(1 + x + y) = \cancel{x} + C$

$y = \ln(x+y+1) + C$ general solution.

8. Find the explicit general solution of the differential equation. Prime denotes derivative with respect to x.

$xy'' + 2y' = 6x$ $\qquad$ - reducible 2^{nd} order, y is missing.

$p = y'$, $y'' = p'$

$xp' + 2p = 6x$

$p' + \frac{2}{x}p = 6$ linear 1^{st} order

$f(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$px^2 = \int 6x^2\,dx + C_1 = 2x^3 + C_1$

$p = 2x + \frac{C_1}{x^2} \Rightarrow \frac{dy}{dx} = 2x + \frac{C_1}{x^2}$, $\int dy = \int \left(2x + \frac{C_1}{x^2}\right)dx + C_2$

$y(x) = x^2 + \frac{C_1}{x} + C_2$ general solution