ABSTRACT ALGEBRA I: HOMEWORK 7 SOLUTIONS

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Problem (Section 3.6, Problem 18)**.** Let S be an infinite set. Let H be the set of all elements $\sigma \in \mathrm{Sym}(S)$ such that $\sigma(x) = x$ for all but finitely many x. Prove that H is a subgroup of $\mathrm{Sym}(S)$.

Proof. Let $\sigma, \tau \in H$. Let $A \subset S$ be the set of elements not fixed by σ, and $B \subset S$ the set of elements not fixed by τ. By assumption, A and B are finite sets, so $A \cup B$ is a finite set. Let $x \notin A \cup B$. Then $\sigma(\tau(x)) = \sigma(x) = x$, so $\sigma\tau$ leaves all but at most finitely elements of S fixed, i.e. $\sigma\tau \in H$.

Now observe that if $y \notin A$, then $\sigma(y) = y$, so $\sigma^{-1}(y) = y$. Thus $\sigma^{-1} \in H$, and H is a subgroup. □

Problem (Section 3.6, Problem 19)**.** The center of a group is the set of all elements that commute with every other element of the group. That is, $Z(G) = \{x \in G \mid xg = gx, \forall\, g \in G\}$. Show that if $n \geq 3$, then the center of S_n is trivial.

Proof. Let $\sigma \in S_n$. If $\sigma \neq e$, then $\sigma(i) = j$ for some $i \neq j$. Let $k \neq i, j$ (such a k must exist since $n \geq 3$), and let $\tau = (i\ k)$. Then

$$\sigma(\tau(i)) = \sigma(k) \neq j = \tau(j) = \tau(\sigma(i)).$$

Since for every non-identity element of S_n we have found an element which does not commute with it, the center of S_n cannot contain any non-identity element. Since the center always contains the identity element, we have $Z(S_n) = \{e\}$. □

Problem (Section 3.6, Problem 20)**.** Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Find the smallest subgroup of D_n that contains a^2 and b. Hint: Consider two cases, depending on whether n is odd or even.

Proof. Let n be odd. Then $(a^2)^{\frac{n+1}{2}} = a^{n+1} = a$. By definition, $\{a, b\}$ is a generating set of D_n. Since the subgroup generated by a^2 and b contains a and b, it contains all of D_n.

Now let n be even. Then a^2 has order $n/2$, b has order 2, and $ba^2 = a^{-1}ba = a^{-2}b = (a^2)^{-1}b$. Since these properties define the dihedral group, we have $\langle a^2, b\rangle$ is isomorphic to $D_{n/2}$. □

Problem (Section 3.6, Problem 21)**.** Find the center of the dihedral group D_n. Hint: Consider two cases, depending on whether n is odd or even.

Proof. Let n be odd. Observe that $ba^k = a^{-k}b \neq a^k b$, so $b \notin Z(D_n)$ and $a^k \notin Z(D_n)$. Similarly, $ba^k \notin Z(D_n)$, since

$$aba^k = aa^{-k}b = a^{1-k}b = ba^{k-1} \neq ba^{k+1} = ba^k a.$$

Thus $Z(D_n) \cong \{e\}$.

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If n is even, then $a^{-k}b = a^k b$ if and only if $k = n/2$, so by the same argument as before, $b \notin Z(D_n)$ and $a^k \notin Z(D_n)$ for $k \neq n/2$. As above, $ba^k \notin Z(D_n)$. Observe that

$$ba^i(a^{n/2}) = ba^{i+n/2} = a^{-i-n/2}b = a^{n/2}a^{-i}b = a^{n/2}ba^i,$$

so $a^{n/2} \in Z(G)$. Thus $Z(G) \cong \{e, a^{n/2}\}$. $\square$