Practice Final Exam Solutions.

1. $x^2y'' + 3xy' = 2$ reducible second order

$p = y'$, $p' = y''$

$x^2p' + 3xp = 2$

$p' + \frac{3}{x}p = \frac{2}{x^2}$

$\rho(x) = e^{\int \frac{3}{x}dx} = e^{\ln x^3} = x^3$

$px^3 = 2\int \frac{x^3}{x^2}dx + C_1 = x^2 + C_1$

$p = \frac{1}{x} + \frac{C_1}{x^3}$ $\qquad \frac{dy}{dx} = \frac{1}{x} + \frac{C_1}{x^3}$

$y = \ln x - \frac{C_1}{2x^2} + C_2 = \underline{\underline{\ln x + Ax^{-2} + B}}.$

2. $xy^2 + 3y^2 - x^2y' = 0$

$y^2(x+3) = x^2y'$ Separable

$\int \frac{dy}{y^2} = \int \frac{x+3}{x^2}dx + C$, $-\frac{1}{y} = \ln x - \frac{3}{x} + C$

$\frac{1}{y} = \frac{3}{x} - \frac{x\ln x}{x} + \frac{Cx}{x}$, $y = \underline{\underline{\frac{x}{3 - x\ln x + Cx}}}$

3. $\frac{dT}{dt} = -K(T-A)$ $\qquad T(0) = 38°F$, $A = 350°F$

$T(120) = 150$

$\frac{dT}{dt} = -K(T-350) = K(350-T)$

$\int \frac{dT}{T-350} = \int -Kdt + C$, $\ln|T-350| = -Kt + C$

$T - 350 = Ce^{-Kt}$

$T = 350 + Ce^{-Kt}$

$38 = 350 + C$

$C = -312$

$T = 350 - 312e^{-Kt}$

$150 = 350 - 312e^{-K \cdot 120}$

$K = \frac{\ln(200/312)}{-120} = +0.0037$

$\underline{T = 350 - 312e^{-0.0037t}}$

12 ②

#3 (Cont.) $170 = 350 - 312e^{-0.0037t}$

$t = \dfrac{\ln(180/312)}{-0.0037} \approx 148.66 \text{ min} = 2.5 \text{ hrs.}$

$2\text{pm} + 2.5 \text{ hrs} = \underline{\underline{4:30 \text{ pm}}}$.

4. $xy' = 2y + x^3\cos x, \quad y(\pi) = 1$

$xy' - 2y = x^3 \cos x$

$y' - \frac{2}{x}y = x^2\cos x$ linear 1st order.

$\rho(x) = e^{-\int \frac{2}{x}dx} = e^{\ln x^{-2}} = x^{-2}$

$yx^{-2} = \int x^2\cos x \cdot x^{-2}\,dx + C$

$yx^{-2} = \sin x + C \qquad y = x^2(\sin x + C)$

$1 = \pi^2(\sin\pi + C), \quad 1 = \pi^2 C, \quad \underline{C = \frac{1}{\pi^2}}$

$$\underline{\underline{y = x^2\left(\sin x + \frac{1}{\pi^2}\right)}}$$

5. $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0.$

$\frac{\partial}{\partial x}(2y + xe^{xy}) = e^{xy} + xye^{xy}, \quad \frac{\partial}{\partial y}(1 + ye^{xy}) = e^{xy} + xye^{xy}$

exact eq.

$\frac{\partial F}{\partial x} = 1 + ye^{xy} \qquad F(x,y) = x + y\cdot\frac{1}{y}e^{xy} + g(y)$

$\frac{\partial F}{\partial y} = \cancel{xe^{xy}} + g'(y) = 2y + \cancel{xe^{xy}} \qquad g(y) = y^2 + C$

$$\underline{\underline{F(x,y) = x + e^{xy} + y^2 + C}}$$

6. $\frac{dP}{dt} = KP(90 - P) \qquad P(0) = 60, \quad P(50) = ?$

$\int \frac{dP}{P(90-P)} = \int K\,dt + C$

$\frac{1}{90}\left(\ln P - \ln(90 - P)\right) = kt + C$

$\frac{A}{P} + \frac{B}{90-P} = \frac{90A - AP + BP}{P(90-P)}$

$90A = 1 \qquad B - A = 0$

$A = \frac{1}{90} \qquad A = B \qquad B = \frac{1}{90}$

12 #6 (cont.)

$$\ln \frac{P}{90-P} = 90kt + C$$

$$\frac{P}{90-P} = Ce^{90kt}, \quad P = 90Ce^{90kt} - PCe^{90kt}$$

$$P(1+Ce^{90kt}) = 90Ce^{90kt}, \quad P = \frac{90Ce^{90kt}}{1+Ce^{90kt}}$$

$$60 = \frac{90C}{1+C}, \quad 60 + 60C = 90C, \quad 30C = 60$$

$$C = 2.$$

$$P = \frac{180e^{90kt}}{1+2e^{90kt}} = \frac{180}{e^{-90kt}+2}, \quad t = 50 \text{ in } 2000$$

$$P = \frac{180}{e^{-90\cdot 0.0002\cdot 50}+2} = \underline{74.80} \text{ million in 2000}$$

7. $y^{(4)} + 18y'' + 81y = 0.$ $\quad r^4 + 18r^2 + 81 = 0$

$(r^2+9)^2 = 0$

$y = A\cos 3x + B\sin 3x +$ $\quad r^2 = -9, \; r = \pm 3i, \; k = 2$

$+ x(C\cos 3x + D\sin 3x)$ repeated complex roots.

$$\underline{y = (A+Cx)\cos 3x + (B+Dx)\sin 3x}$$

8. $10x'' + 9x' + 2x = 0 \quad x(0) = 0, \quad x'(0) = 5.$

a) $10r^2 + 9r + 2 = 0$

$$r = \frac{-9 \pm \sqrt{81-80}}{20} = \frac{-9 \pm 1}{20} = -\frac{1}{2}, -\frac{2}{5}$$

$x = C_1 e^{-\frac{1}{2}t} + C_2 e^{-\frac{2}{5}t}$ $\quad C_1 + C_2 = 0, \; C_1 = -C_2$

$x' = -\frac{1}{2}C_1 e^{-\frac{1}{2}t} - \frac{2}{5}C_2 e^{-\frac{2}{5}t}$ $\quad -\frac{1}{2}C_1 - \frac{2}{5}C_2 = 5$

$-\frac{1}{2}C_1 + \frac{2}{5}C_1 = 5$

$\underline{x(t) = -50e^{-t/2} + 50e^{-2t/5}}$ $\quad -5C_1 + 4C_1 = 50$

$C_1 = -50, \; C_2 = 50.$

12

#8 (Cont.)

b) $x' = 0$, $\quad 25e^{-t/2} - 20e^{-2t/5} = 0$

$25e^{-t/2} = 20e^{-2t/5}$, $\quad \frac{5}{4} = e^{-\frac{2t}{5} + \frac{t}{2}}$ $\qquad -\frac{4}{10} + \frac{5}{10} = \frac{1}{10}$

$t = \dfrac{\ln(5/4)}{0.1} \approx 2.23$ sec.

$x = -50e^{-2.23/2} + 50e^{-2\cdot 2.23/5} = \underline{\underline{4.096}}$ max. distance.

#9. See next page.

#10. $5I'' + 200I' + \frac{1}{0.001} I = 1000 \cos 10t$

$I'' + 40I' + 200 I = 200 \cos 10t$.

$I_{sp} = I_p$

$I_p = A\cos 10t + B \sin 10t$

$I_p' = -10A\sin 10t + 10B\cos 10t$

$I_p'' = -100A\cos 10t - 100B \sin 10t$

$-100A\cos 10t - \underline{100B\sin 10t} - \underline{400A\sin 10t} + 400B\cos 10t +$
$+ 200A\cos 10t + \underline{200B\sin 10t} = 200\cos 10t$

$-100B - 400A + 200B = 0 \qquad 400A = 100B,\ B = 4A.$

$-100A + 400B + 200A = 200, \qquad 4B + A = 2$
$16A + A = 2$
$A = \frac{2}{17}$

$B = \frac{8}{17}$

$$\underline{\underline{I_{sp} = \frac{2}{17}\cos 10t + \frac{8}{17}\sin 10t}}$$

\#9. $x'' + 6x' + 13x = 10\sin 5t \quad x(0) = x'(0) = 0.$

$r^2 + 6r + 13 = 0$

$r = \dfrac{-6 \pm \sqrt{36-52}}{2} = \dfrac{-6 \pm 4i}{2} = -3 \pm 2i$

$X(t) = e^{-3t}(A\cos 2t + B\sin 2t)$

$x_p = C\cos 5t + D\sin 5t \qquad x_p'' = -25C\cos 5t - 25D\sin 5t$

$x_p' = -5C\sin 5t + 5D\cos 5t$

$-25C\cos 5t - 25D\sin 5t - 30C\sin 5t + 30D\cos 5t + 13C\cos 5t + 13D\sin 5t = 10\sin 5t.$

$-25C + 30D + 13C = 0 \quad | \quad -12C + 30D = 0$

$-25D - 30C + 13D = 10 \quad | \quad -30C - 12D = 10$

$-2C + 5D = 0 \qquad 5D = 2C \qquad D = \frac{2}{5}C$

$-15C - 6D = 5 \qquad -15C - \frac{12}{5}C = 5, \quad -\frac{75}{5}C - \frac{12}{5}C = 5$

$-\frac{87}{5}C = 5 \qquad C = -\frac{25}{87}$

$D = -\frac{25}{87} \cdot \frac{2}{5} = -\frac{10}{87} = D$

$C\sqrt{\frac{25^2}{87^2} + \frac{10^2}{87^2}} = \frac{\sqrt{(5^2)^2 + 5^2 \cdot 4}}{87} = \frac{5\sqrt{29}}{87} = \frac{5\sqrt{29}}{3 \cdot 29} = \frac{5}{3\sqrt{29}}$

$x_p = -\frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$

$x(t) = e^{3t}(A\cos 2t + B\sin 2t) - \frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$

$x(0) = 0 \qquad A - \frac{25}{87} = 0, \quad A = \frac{25}{87}$

$x'(t) = -3e^{-3t}(A\cos 2t + B\sin 2t) + e^{-3t}(-2A\sin 2t + 2B\cos 2t)$

$+ \frac{125}{87}\sin 5t - \frac{50}{87}\cos 5t.$

$x'(0) = 0 \qquad -3A + 2B = \frac{50}{87} \qquad 2B = \frac{50}{87} + 3 \cdot \frac{25}{87} = \frac{50+75}{87}$

$B = \frac{125}{2 \cdot 87} = \frac{125}{174}$

$x(t)_{Tr} = e^{-3t}\left(\frac{25}{87}\cos 2t + \frac{125}{174}\sin 2t\right) = e^{-3t}\frac{25}{6\sqrt{29}}\cos(2t - 1.19)$

$C = \sqrt{\frac{25^2}{87^2} + \frac{125^2}{2^2 \cdot 87^2}} = \sqrt{\frac{25^2 \cdot 2^2 + 125^2}{2^2 \cdot 87^2}} = \frac{\sqrt{25^2 \cdot 2^2 + 5^2 \cdot 25^2}}{2 \cdot 87} = \frac{25}{174}\sqrt{29}$

$= \frac{25\sqrt{29}}{6 \cdot 29} = \frac{25}{6\sqrt{29}}$

$x_{sp}(t) = \frac{5}{3\sqrt{29}}\cos(5t - 5.90)$

#11. $x'' - 6x' + 8x = 2 \qquad x(0) = x'(0) = 0.$

$$s^2 X(s) - 6s X(s) + 8X(s) = \frac{2}{s}$$

$$X(s)(s^2 - 6s + 8) = \frac{2}{s}$$

$$X(s) = \frac{2}{s(s-4)(s-2)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-2}$$

$$A(s-4)(s-2) + Bs(s-2) + Cs(s-4) = 2$$

$s = 4$: $\quad 8B = 2, \quad B = \frac{1}{4}$

$s = 2$: $\quad -4C = 2 \quad C = -\frac{1}{4}$

$s = 0$ $\quad 8A = 2 \quad A = \frac{1}{4}$

$$X(s) = \frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s-4} - \frac{1}{4}\frac{1}{s-2}$$

$$\mathcal{L}^{-1}(X(s)) = \frac{1}{4} + \frac{1}{4}e^{4t} - \frac{1}{4}e^{2t}$$

$$x(t) = \frac{1}{4}(1 + e^{4t} - e^{2t})$$

#12. $t^3 x^{(3)} - 2t^2 x'' + 3t x' + 5x = \ln t.$

$x_1 = x, \quad x_1' = x_2 = x', \quad x_2' = x_3 = x'', \quad x_3' = x^{(3)}$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ t^3 x_3' = -5x_1 - 3t x_2 + 2t^2 x_3 + \ln t \end{cases}$$

#13. $\begin{cases} x' + 2y' = 4x + 5y \\ 2x' - y' = 3x \end{cases}$ $\qquad x(0) = 1, \quad y(0) = -1.$

$$\begin{array}{rl} 1^{st}: & -2x' - 4y' = -8x - 10y \\ 2^{nd}: + & 2x' - y' = 3x \\ \hline & -5y' = -5x - 10y \end{array}$$

$$y' = x + 2y \qquad \underline{x = y' - 2y}, \quad \underline{x' = y'' - 2y'}$$

1^{st}: $y'' - 2y' + 2y' = 4y' - 8y + 5y$

$$y'' - 4y' + 3y = 0 \qquad (r-3)(r-1) = 0$$

$$r^2 - 4r + 3 = 0 \qquad r = 3, 1$$

$$\underline{y(t) = c_1 e^{3t} + c_2 e^{t}} \qquad c_1 + c_2 = -1$$

$$y'(t) = 3c_1 e^{3t} + c_2 e^{t}$$

$$x(t) = 3c_1 e^{3t} + c_2 e^{t} - 2c_1 e^{3t} - 2c_2 e^{t}$$

$$\underline{x(t) = c_1 e^{3t} - c_2 e^{t}} \qquad c_1 - c_2 = 1$$

$$\begin{array}{rl} & c_1 + c_2 = -1 \\ + & c_1 - c_2 = 1 \\ \hline & 2c_1 = 0 \end{array}$$

$$c_1 = 0, \quad c_2 = -1$$

$$\underline{\underline{x(t) = e^{t}}}, \quad \underline{\underline{y(t) = -e^{t}}}.$$