

Marine Population Dynamics

OCEA265

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Alexa would like to gratefully acknowledge the following sources:

- Alan Hastings' book, "Population Biology: Concepts and Models"
- Hal Caswell's book, "Matrix Population Models"
- Marissa Baskett's ESP121 course notes (UC Davis)
- Dave Tilman's UCSB course ESM201, for which Alexa was a TA
- chatGPT, which generated a lot of the Quarto code for the math.

Welcome

Introduce yourselves:

- Your preferred name
- What you do at UCSC (your position, research, etc.)

Introduction to population modeling

What models are

Reflect individually: What do you think models are? What comes to mind with that word?

Classes of models:

- Static (e.g., linear regression) *vs.* dynamic (time is a variable; differential or difference equations describe system evolution over time)
- Discrete *vs.* continuous (is any biological data ever in continuous time?)
- Statistical / phenomenological *vs.* theoretical / mechanistic / process-based
- Stochastic (incorporates randomness) *vs.* deterministic (get the same outcome every time)

Structure and content of this class

Structure and content

- The course textbook is Population Biology: Concepts and Methods (Hastings, 2nd edition).
- We meet weekly to work through the math of biological models by hand. I suggest taking notes in a notebook or using a tablet and pencil. You can also generate notes like the ones I'm using with LaTeX (see overleaf.com) or Quarto (see posit.co). Classes will be a mix of lecture and in-class activities, with lots of breaks.
- Problem sets ask you to explore fitting the models we learned to data. You can do this in any software you like. This course is intentionally software-agnostic to allow you to practice skills in whatever language or interface you use in your own research.
- Review grading policy, the rest of the [syllabus](#), and introduce the [course repo on GitHub](#).

On math anxiety

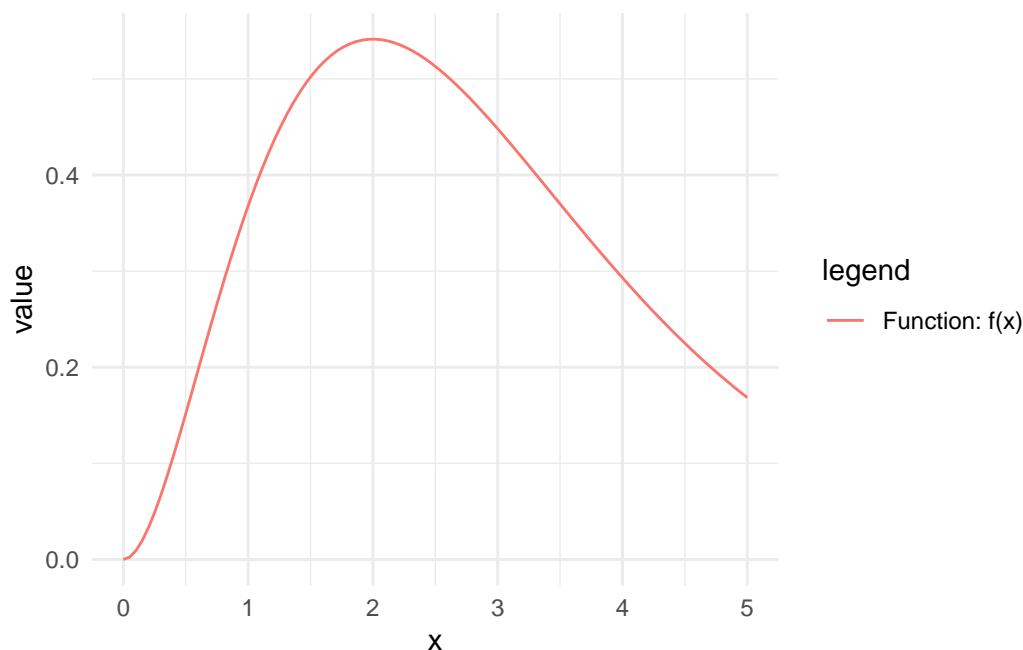
- “A feeling of tension, apprehension, or fear that interferes with math performance” (Ashcroft, 2002)
- Math can initiate physiological responses including increased heart rate and cortisol levels, and neural activations analogous to those from physical pain (Ramirez *et al.*, 2018)
- These responses are also associated with lower math performance, probably because the person is dividing their working memory between the math problem and anxiety-associated thoughts (e.g., about the consequences of failure) (Ramirez *et al.*, 2018)
- Math anxiety is extremely common in higher education and is not a binary classification; many students have some degree of math anxiety (Ramirez *et al.*, 2018)
- Socioenvironmental factors are a primary driver of math anxiety, including home experiences (parental involvement in schooling), exposure to negative framings (“math is confusing”, “I always hated math”), negative classroom experiences (often associated with teachers who also have math anxiety) or assessment outcomes, and students’ “self-concept” (Ramirez *et al.*, 2018)

Calculus refresher

(Drafted by chatGPT, edited by Alexa)

Differentiation

Let's say we have some function that looks like this:



We don't need to do anything to know what the value of $f(x)$ is at any x on this graph; we can already figure that out.

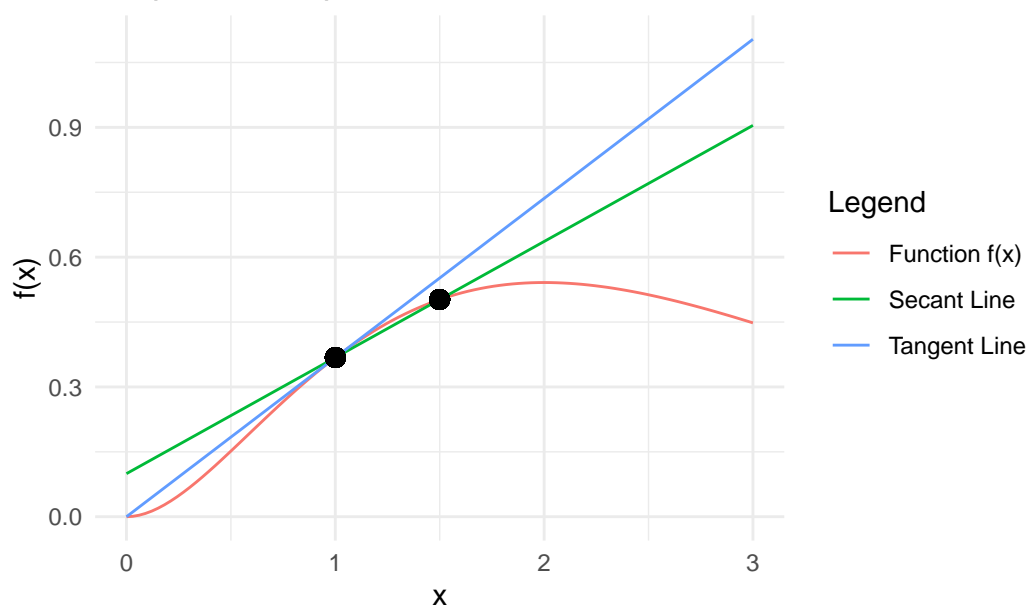
However, we don't currently know what the **rate of change** is in $f(x)$ at any x on this graph. To figure that out, we have to take the **derivative**, via “differentiation”. Differentiation describes the rate of change of a function. If $f(x)$ represents a population size over time, its derivative measures the instantaneous rate of change.

We describe this mathematically using the concept of **limits**. These let us calculate the derivative at any x as the slope of a line from x to $x + \Delta x$. We define the derivative as the limit of this slope as Δx goes to 0.

$$\frac{d}{dx}f(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Let's unpack limits a little more. Limits allow us to compute the derivative at any point x by considering the slope of a line connecting x to $x + \Delta x$. The derivative is then defined as the limit of this slope as Δx approaches zero. Let's explore this graphically with a secant line (a line that intersects a curve at two distinct points) and a tangent line (which touches a curve at just one point, and does not cross it).

Graphical Interpretation of Limits and Derivatives



This is a good time to remind you that this course has zero memorization! Don't worry if these terms are unfamiliar or don't stick.

Limits help us graphically understand differentiation, but they aren't always convenient for computation or calculation, because we calculate them one x at a time. Thankfully rules exist to derive a general formula to compute the slope of a function at any x in the function's domain. We're going to hold off on the derivatives refresher though, until it actually comes up in class; knowing the intuition behind what a derivative is will suffice for today.

Integration

Integration finds the total accumulation of a quantity. The integral of $f(x)$ over an interval $[a, b]$ is:

$$\int_a^b f(x) dx$$

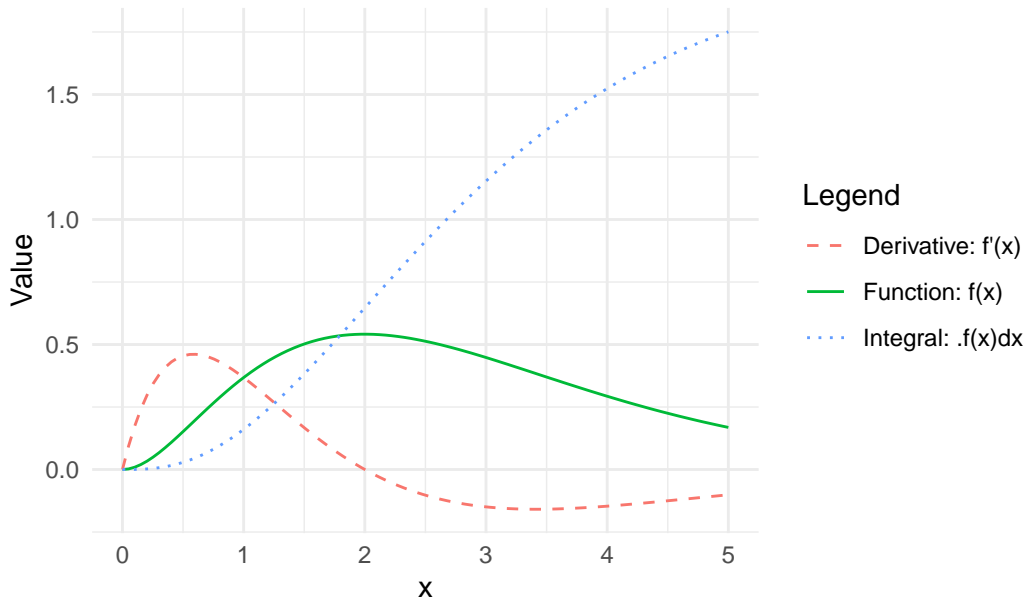
Common Integrals

- Power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- Exponential function: $\int e^x dx = e^x + C$
- Logarithmic function: $\int \frac{1}{x} dx = \ln |x| + C$

Graphical interpretation

To visualize differentiation and integration, consider the function $f(x) = x^2e^{-x}$. The derivative represents the slope at each point, while the integral gives the accumulated area under the curve.

Graphical Interpretation of Differentiation and Integration



BREAK

Density-independent population growth

(Hastings pp1-16)

This is the simplest population model we can write down. It assumes that:

- The rate of births is proportional to the number of individuals present.
- The rate of deaths is proportional to the number of individuals present.

Discuss: What are the implications for this model at small population sizes? At large population sizes?

Here and forever, the population size is going to be denoted by N .

The discrete time case (geometric growth)

This model has separate time steps.

When would we want to use a model like this?

- Non-overlapping generations (e.g., seasonal insects)
- Synchronous and distinct reproductive events (e.g., salmon, annual plants)

For now, let's ignore organisms that live for more than one time step (i.e., no age structure), and just think about something whose entire life cycle happens between t and $t + 1$. Note that we often think of t as a year in discrete-time models, but it is actually in units of *a single generation*. This could be days, weeks, months, seasons, or many years.

All discrete-time models can be stated as:

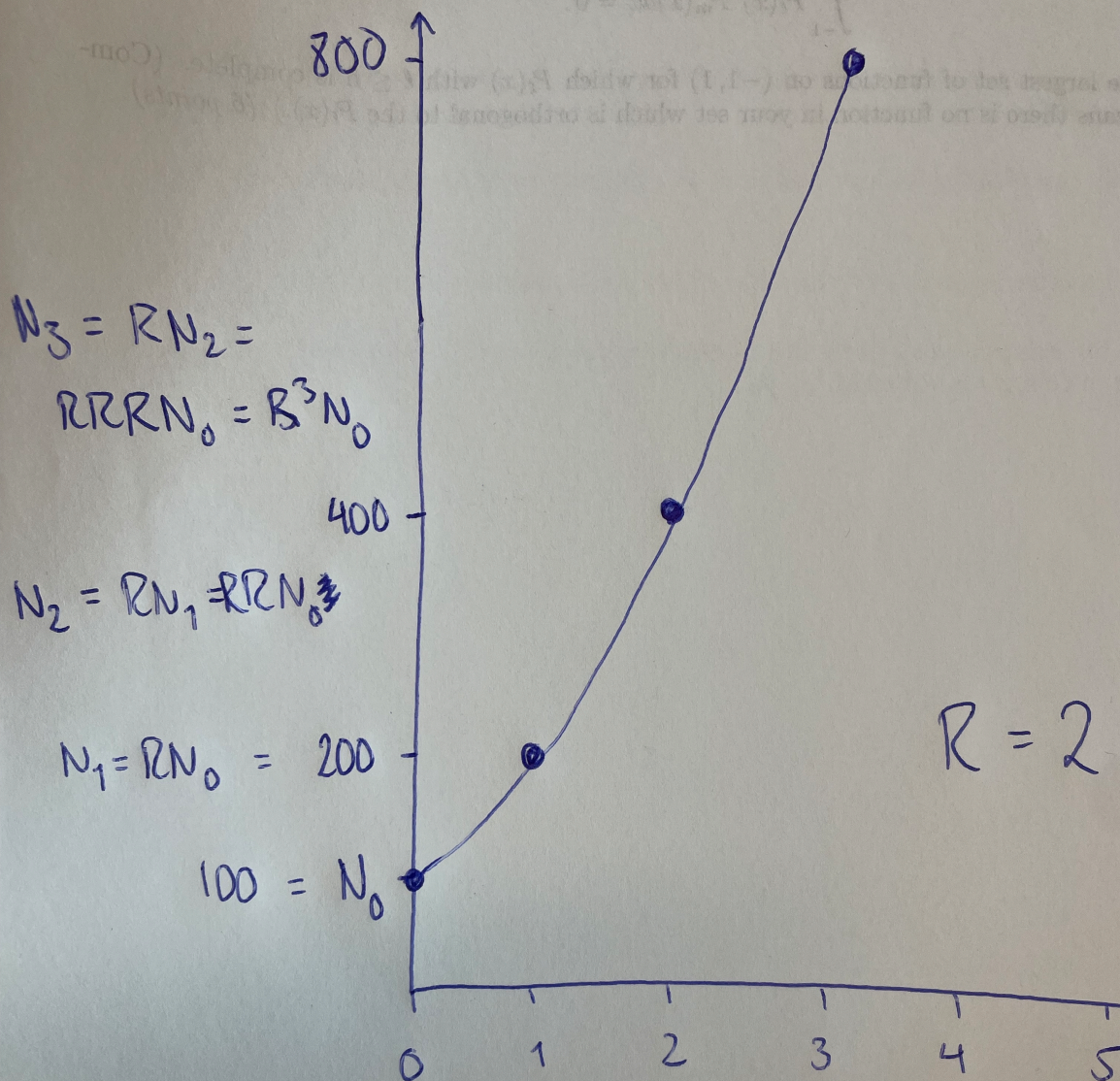
$$N_{t+1} = F(N_t)$$

We'll spend a lot of this class talking about all the forms F can take!

The simplest form is a dimensionless coefficient:

$$N_{t+1} = RN_t$$

“Big R ” represents the chance of survival *on average* per offspring. You can define $R = B - M$ where B is the average *per capita* birth rate and M is the average *per capita* death rate (mortality). Let's plot out the implications of this model for population dynamics, starting at $t = 0$ with an initial population size (defined as N_0) of 100 and set $R = 2$, with t on the x-axis and N on the y-axis. (This is a really common plot that we'll draw a zillion times in this course.)



$$N_1 = 2 \cdot 100 = 200$$

This implies a general way to calculate population size at any time step if we know the initial population size N_0 and the average chance of survival per offspring, R :

$$N_t = R^t N_0$$

Based on the example above, what does this imply for population change over time if R is relatively large?

What if $R = 1$?

Draw it out—the population stays constant; for any t , if $R = 1$, then $N_t = 1 \times N_0 = N_0$.

What happens if R is less than 1? **Ask the students to collectively plot** this graph (t vs. x) for $R = 0.5$ and $N_0 = 1000$. We call this “geometric decay” (the case above was “geometric growth”).

$N_t = R^t N_0$ is a **geometric population model**. Remember, this is a discrete-time model assuming that births and deaths are both proportional to N .

This model implies that:

- If $R > 1$, populations will grow unbounded.
- If $R < 1$, populations will go extinct.

Under what circumstances are those reasonable assumptions and implications in nature?

The continuous time case (exponential growth)

Let's deal with one of those statements that is not always realistic: that generations are non-overlapping and reproduction is a discrete event. In many populations, generations are overlapping and reproduction is happening continuously.

Can you think of some examples?

- Most microbes, including plankton
- People

This necessitates a **continuous time model**. Because this isn't broken into time steps, rather than talking about the difference between N_t and N_{t+1} , we'll phrase this model in terms of dN/dt . This is defined as the **instantaneous rate of population change**.

The simplest model of continuous-time population change is:

$$\frac{dN}{dt} = rN$$

There's a new term here: r . "Little r " is also defined as the per-capita average birth rate b minus the per-capita average death rate m , but note that these are all lowercase letters: they're instantaneous rates. (Before they were also averaged over time within the generation t ; that's no longer true here.) Little r is also called the *intrinsic rate of increase* or *intrinsic growth rate*.

This is a really important concept in population biology so let's spend a little more time on the intuition behind r . If dN/dt is in units of population size / time, then dividing both sides of the equation above by N gives us $r = dN/Ndt$. Notice two things: r is a *per capita* rate (N is on the bottom of the equation), and the units of r itself are $1/time$. So, r is telling us about the per-capita average population change over a really, really, small—*instantaneous!*—time step.

It would be useful to have an explicit formula for $N(t)$, and $dN/dt = rN$ is simple enough that we can actually solve it for N . Let's integrate this. Assume the initial population size is ($N(0) = N_0$), and we want to find the population size at time (T), which we'll call ($N(T)$).

First, we'll *separate the variables*—put t on one side and N on the other.

$$\frac{dN}{dt} = rN$$

Divide both sides by N (assuming $N > 0$):

$$\frac{1}{N} \frac{dN}{dt} = r$$

Now multiply both sides by dt :

$$\frac{1}{N} dN = r dt$$

Now that we've separated the variables we can integrate both sides. Remember our very brief calculus refresher? What were the common integrals, again?

- Power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- Exponential function: $\int e^x dx = e^x + C$
- Logarithmic function: $\int \frac{1}{x} dx = \ln|x| + C$

So if we want to integrate both sides of this equation:

$$\int_{t=0}^{t=T} \frac{1}{N} dN = \int_{t=0}^{t=T} r dt$$

We can use the *logarithmic function* to integrate the left-hand side:

$$[\ln N(t)]_{t=0}^{t=T} = \ln N(T) - \ln N_0$$

On the right-hand side, we evaluate the integral of the constant r with respect to dt (which integrates to a constant, rt) at both values of t . This yields:

$$[rt]_{t=0}^{t=T} = rT - 0 = rT$$

(Why is the integral of any constant (like r) times dt just that same constant times t ? Again, let's think back to what led us to having $r dt$ in the derivative to begin with. We're asking, "What function has a derivative equal to a constant r ?" And the answer to that is, it's a straight line with slope r , so the equation for that line is rt . We'll unpack this more on your homework.)

Putting the two sides back together:

$$\ln N(T) - \ln N_0 = rT$$

Now we want to get $N(T)$ on its own. Recall that $\ln a - \ln b = \ln(a/b)$. In other words, the difference of two logarithms is the same as the logarithm of a quotient. So we can rewrite the above as,

$$\ln\left(\frac{N(T)}{N_0}\right) = rT$$

Now that we have only one logarithm we can exponentiate both sides:

$$\frac{N(T)}{N_0} = e^{rT}$$

or

$$N(T) = N_0 e^{rT}$$

This is the equation for exponential growth in continuous time. We don't actually need "big T" here, since it can be evaluated at any value of t , so it's usually written as $N(t) = N_0 e^{rt}$.

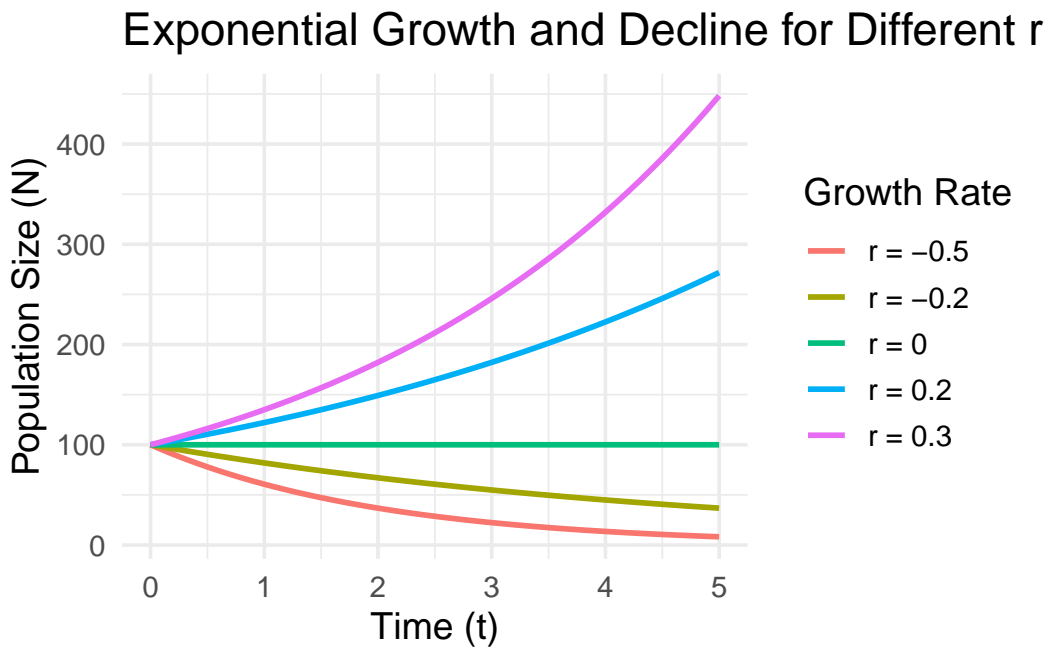
BREAK

Relationship between continuous and discrete time (r and R)

Since it's a little harder to do exponents in your head, we won't manually calculate the population size in an exponential growth model for different values of r the way we did for the discrete time case, with the exception of $r = 0$. What does that evaluate to?

$$N(t) = N_0 e^0 = N_0$$

Here is the exponential growth curve for some other values of r .



Let's think back to how this model connects to the discrete-time case. To recap, we started thinking about rates of population change in discrete time:

$$N_{t+1} = RN_t$$

And in continuous time:

$$\frac{dN}{dt} = rN$$

In both cases, we wanted to get a general equation for N . We were able to get this algebraically for the discrete time case:

$$N_{t+1} = N_0 R^t$$

And we integrated the continuous-time case to get:

$$N(t) = N_0 e^{rt}$$

What do these equations—and N and r —have to do with each other *mathematically*? We saw graphically that they result in the same curve shapes, which we called “geometric” in the discrete case and “exponential” in the continuous case.

Question: For what values of R and r respectively do the discrete- and continuous-time models lead to stable population sizes over time?

We can think of e^r and R as being analogous to one another. If we compare the discrete and continuous time equations, we see that they can essentially be substituted for one another. But remember that r is **not the same** as R . To convert instantaneous to time-averaged per capita population change, we need to integrate, which is what e^r does.

In reality, which set of math you use (continuous vs discrete time) depends partly on the system you’re studying and partly on mathematical convenience.

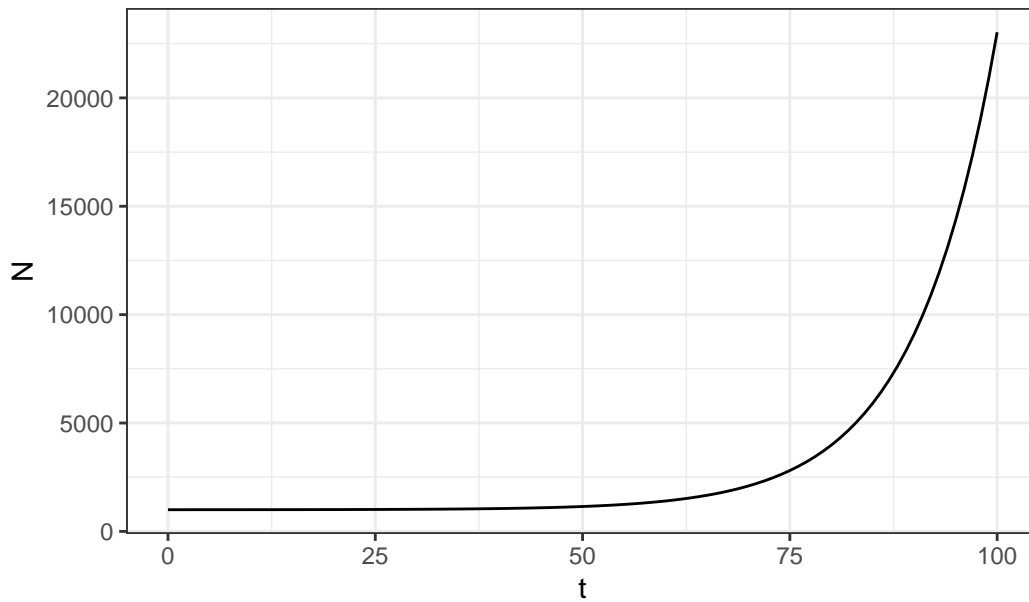
BREAK

Per capita population change in continuous-time exponential growth

We’ve been talking about several quantities: Population size N and its change (dN/dt), and per capita population growth rate (R or r). I want to build your intuition for the relationships between these quantities in the continuous-time exponential model.

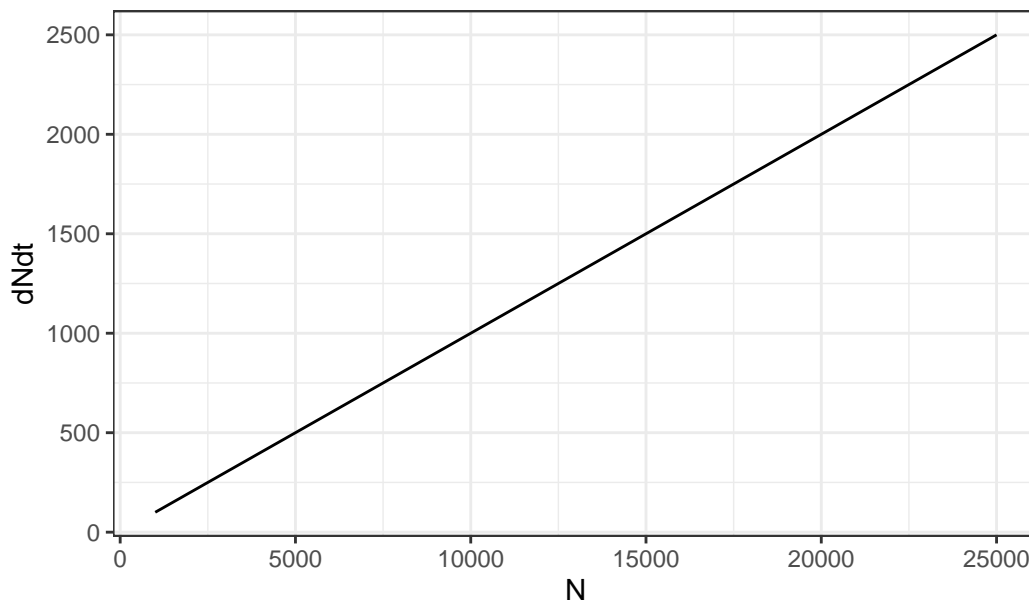
Here’s a graph of N vs t :

Exponential growth in continuous time, $r = 0.1$



Now, let's plot the derivative of this curve:

Population rate of change vs. population size, $r = 0.1$



Talk through the intuition behind this graph—that the population-level rate of change is proportional to the population size. (This is true in both continuous and discrete time.) This was obvious mathematically because we knew $dN/dt = rN$, which is the equation for a line of

slope r , but it isn't always obvious graphically. If we divide both sides of that equation by N , we would get a flat line, $r = dN/Ndt$.

Density-dependent population growth

Hastings pp81-95

Logistic model

Exponential (or geometric) growth are rare in nature. **Discuss:** Why?

The growth of natural populations cannot continue forever. Eventually population growth rates must slow down. **Discuss:** Can we think of examples of this in nature? At what population sizes was the population growth rate (dN/dt) fast *vs.* slow? Sketch N *vs.* t on the board.

Over and over we see a pattern of fast growth when things are rare (i.e., populations are small) and slowing growth as populations get bigger—think of diseases and invasive species.

Reflect individually: Many hypotheses have been proposed for what regulates the growth of populations.

- Changing weather and climate
- Resources, such as food supply
- Interactions with other species: predation, parasitism, competition
- Populations regulate themselves through behavior (territoriality, cannibalism)

The simplest case is one of intrinsic factors (e.g., running out of resources), so we'll start with that one. Earlier we discussed the simplest possible (continuous-time) population model,

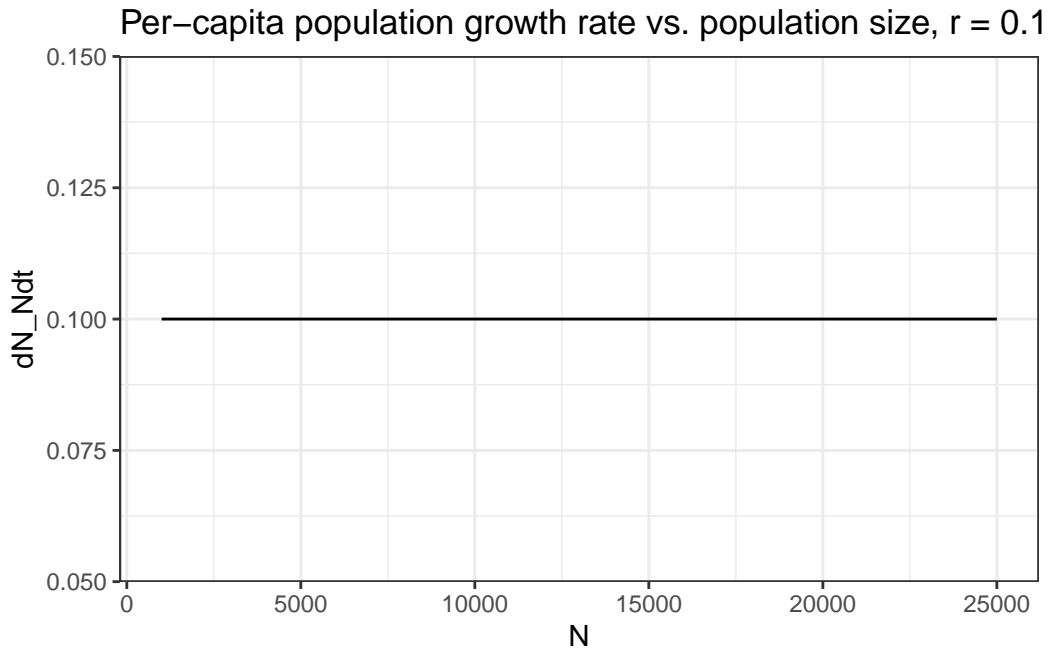
$$\frac{dN}{dt} = rN$$

Now rather than a constant per-capita growth rate r we introduce a function on the right-hand side,

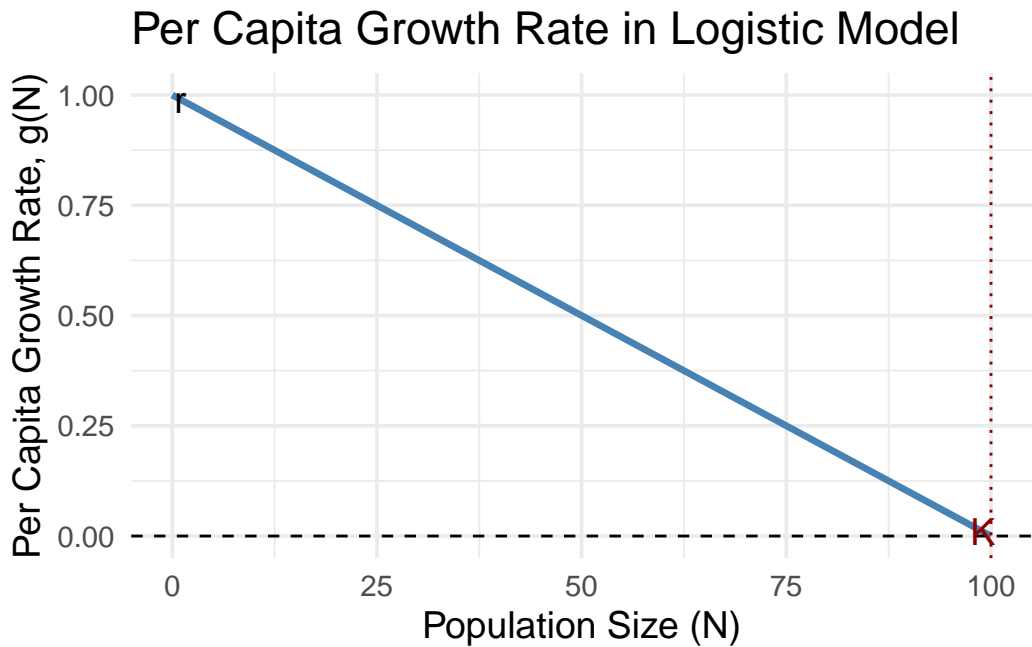
$$\frac{dN}{dt} = N g(N)$$

Now the population growth rate is not simply proportional to population size. $g(N)$ still represents the per-capita growth rate, but *it is no longer a constant*. Because this growth rate is now dependent in some way on N , this is called a **density-dependent model**. Exponential growth is a **density-independent model** because r is not dependent on N . Recall that in

the exponential growth model, we said that if we plotted N vs. dN/Ndt , we would get a straight line?



The simplest possible extension to this is that the *per capita population growth rate*, $g(N)$ or dN/Ndt , goes down with N . This suggests that when the population size is higher, the average number of offspring per individual goes down.

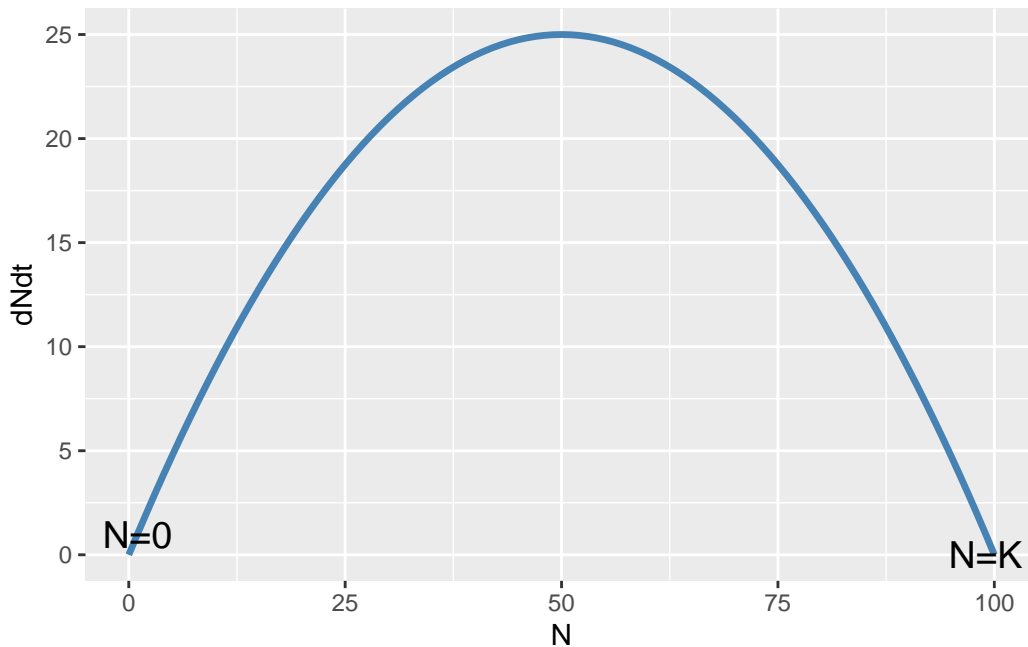


The equation describing this line is,

$$g(N) = r(1 - \frac{N}{K})$$

This is the same r as before, but now—as we said previously—the per-capita population growth rate is a function of density as well, and this quantity K . K is a core ecological concept called **carrying capacity**. The intuition behind K is that it's the largest population size the ecosystem can sustain without per-capita population growth becoming negative, i.e., deaths outweighing births.

Before we solve the equation above let's try to intuitively reconstruct the relationship between N and dN/dt —i.e., to multiply the equation for the line above by N . Recall from the exponential model that we are trying to understand three sets of relationships: dN/Ndt vs. N (plotted above), dN/dt vs. N (which we're about to figure out), and N vs. t (which we sketched earlier on the board). **Discuss:** Where are the points when population size is not changing? What is dN/dt at those values? What do we think it is between those values?



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Because $g(N) = dN/Ndt$, if $g(N) = r(1 - \frac{N}{K})$, then the equation for the curve above simply multiplies both sides by N :

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

This is the logistic population rate of change in continuous time. We'll pick up next week with stability analyses, integration of the above equation, and discrete-time logistic growth.

Density-dependent population models, continued

Hastings pp81-95

Let's start by putting on the board what we learned last week:

Geometric growth in discrete time— $N_{t+1} = RN$, from which we calculated that $N = N_0 R^t$, and exponential growth in continuous time— $dN/dt = rN$, which we integrated to find that $N(t) = N_0 e^{rt}$. **What is the per capita growth rate in each of these models?** Answer: in discrete time, it's $R = N_{t+1}/N$, and in continuous time, it's $r = dN/Ndt$. Recall these are both constants! This is basically the only model in which per capita growth is a constant.

We addressed that assumption by making per capita growth conditional on population size, such that $dN/Ndt = r(1 - \frac{N}{K})$. If we multiply both sides by N to get the population growth rate, we get $dN/dt = rN(1 - N/K)$. This is called the logistic model in continuous time. Integrating this equation leads us to:

$$N(t) = \frac{N_0 e^{rt}}{1 + N_0(e^{rt} - 1)/K}$$

If you want to challenge yourself to walk through this integral, it's on p84-85 of the textbook. Before we go any further, let's draw N vs. t , dN/dt vs. N , and dN/Ndt vs. N on the board.

Stability and equilibrium analysis

In this class, our goal is always to connect math to populations. How well does the logistic model describe populations? In some cases, it may describe (and predict!) them quantitatively. But often we don't have the ability to fit this model well, or it fits the data we have but as population dynamics unfold it turns out it wasn't very predictive. **What are some ways that we might think a logistic model is fitting well and then learn later we were wrong?**

This model may be most useful in making *qualitative* predictions. These are more robust (meaning that they won't change with changes to the model parameters, unlike quantitative predictions). One great way to do this is through equilibrium analysis.

Equilibria are points where the population is *stable*. Mathematically, what does that mean in terms of the quantities we have been calculating and plotting?

If we want to know at what size the population is stable, i.e. $dN/dt = 0$, for any model, we can just set the equation for dN/dt equal to zero, and figure out at what population sizes (N) it solves to 0. Let's try that for the exponential model first:

$$\frac{dN}{dt} = rN = 0$$

This model only has one equilibrium state, when $N = 0$. Does it make sense that an exponential-growth model has no stable population size once $N > 0$?

Let's do the logistic model next:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = 0$$

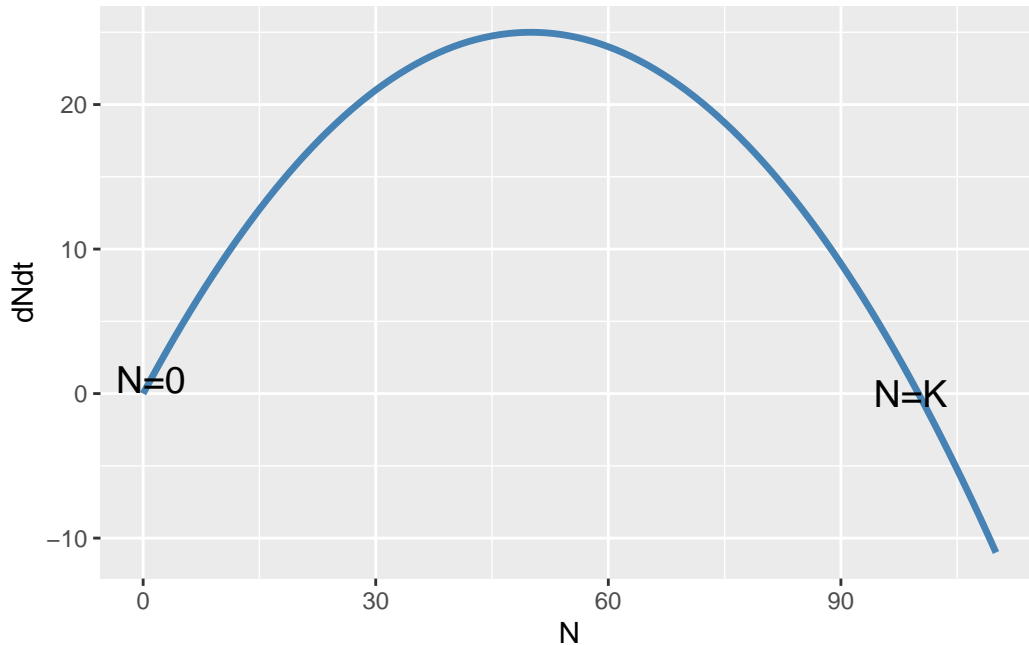
This equation has two solutions, $\hat{N} = K$ and $\hat{N} = 0$. (It's always worth checking if $N = 0$ is a solution; it usually is, because extinct populations are stable!)

In addition to finding this algebraically, we could have figured it out graphically. How? By graphing dN/dt vs. N and looking for the population sizes at which growth rates are zero.

We're going through this whole exercise to build more intuition about the qualitative predictions of these models. Last week we talked about how the logistic model, qualitatively, predicts rapid *per capita* growth when populations are small, and a leveling-off when populations are big. Equilibrium analysis gives us a different kind of intuition: it helps us understand when populations *are likely to remain where they are*.

Equilibrium analysis has a mathematical meaning: it's when the derivative goes to zero, meaning the system is not changing. But this can happen for a blink of an eye or for a long time. Whether we reach a **stable equilibrium** affects the qualitative behavior we expect. Intuitively, what do you think is the difference between a stable and an unstable equilibrium?

To find out whether equilibria are stable or unstable, we need to understand the behavior of solutions near the equilibrium point. The most intuitive way to do this is graphically. Returning to our graph:



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If the population is close to $\hat{N} = 0$, what happens to dN/dt ? Does that bring it closer to or further from the equilibrium? [Draw arrows showing that as N increases, dN/dt increases, so small perturbations will drive the system away from that equilibrium, meaning it is unstable.]

If the population is close to $\hat{N} = K$, what happens to dN/dt ? Does that bring it closer to or further from the equilibrium? [Draw arrows showing that as N increases, dN/dt decreases, and as N decreases, dN/dt increases, so small perturbations will drive the system toward that equilibrium, meaning it is stable.]

This conclusion—that the equilibrium $\hat{N} = 0$ is an unstable one, and that $\hat{N} = K$ is a stable equilibrium, is a feature of many models with density-dependence—not just the logistic function. This happens whenever we have the qualitative pattern that *per capita* growth is high when populations are rare, and when we have feedbacks to slow population growth when populations are abundant (i.e., “negative” density dependence.)

Local stability analysis: the graphical explanation

Above we mathematically found equilibria by setting the derivative function $dN/dt = 0$ and solving it. But that didn’t tell us whether the equilibrium was stable or unstable. We can also do that mathematically, but it takes a bit more work. To find an equilibrium we have to set the function, say $G(N)$, equal to 0. To find if the equilibrium is stable, we have to

understand the *slope* of the function near that equilibrium—specifically, whether the slope is negative (stable) or positive (unstable). Keep in mind that these are slope values *at an intercept*, because we’re already at an equilibrium, which means by definition that $G(N) = 0$. The reason that a negative-slope function crossing the intercept creates a stable equilibrium is that, if N goes down a little (below \hat{N}), suddenly the function is positive and brings the population back up. Conversely, if N goes up a little, suddenly the function is negative and brings N back down. The inverse is true for a positive-slope function crossing the intercept (and creating an unstable equilibrium).

Local stability analysis: the mathematical explanation (optional)

The more mathematical way to do this is to actually calculate the slope of the derivative function $G'(N)$ and ask whether it is negative or positive at different equilibria. We can do this for any population model in two steps (finding the equilibria and then testing them for stability). We’re going to walk through this in great detail here because you can build a lot of intuition about stability from the logistic model, which is relatively straightforward. In future models you may not do all the math but this will help you understand the relationship between the population model and the stability of its equilibria.

We begin, again, by setting the derivative equal to 0—for convenience we’re going to call it $G(N)$ and say that the values of N that give equilibria (i.e., $G(N) = 0$) are \hat{N} . So, we’re looking for:

$$G(\hat{N}) = 0$$

We already identified the equilibria for the logistic model earlier ($\hat{N} = 0$ and $\hat{N} = K$). Next, we want to understand the dynamics of $G(N)$ when it is close to \hat{N} . We use a similar approach as we do for derivation: we think about the value of $G(N)$ very close but not exactly at \hat{N} , and try to understand its behavior as we approach \hat{N} . Let’s define a new quantity n , defined as the *deviation from the equilibrium in units of N* : $n + \hat{N} = N$.

We want to understand how n changes with time. That sounds like a derivative! Let’s differentiate both sides of that equation, $n + \hat{N} = N$. Since \hat{N} is a constant, the left- and right-hand sides of the equation above have the same derivative:

$$dn/dt = dN/dt$$

And recall that since we defined $G(N) = dN/dt$ and $x + \hat{N} = N$,

$$dn/dt = G(N) = G(n + \hat{N})$$

We really just want to understand the dynamics of $G(N)$ when $N = x + \hat{N}$. We can approximate the value of the population size equation, $G(N)$, at this value using a *Taylor series*. A Taylor series allows us to approximate any function at a single point. As Hastings (p55) says, “any function $f(x)$ can be approximated near the value x_0 using a *Taylor series* as $f(x_0)$ plus the difference between x and x_0 multiplied by how much the function f changes as its argument, x , changes:”

$$f(x) \approx f(x_0) + (x - x_0) \left. \frac{df}{dx} \right|_{x=x_0}$$

In our applications, $f(x_0)$ is always an equilibrium point that corresponds to a known value of N . Let’s try to re-write the equation for the Taylor series above in our use case.

$$G(\hat{N} + n) \approx G(\hat{N}) + n \left. \frac{dG}{dN} \right|_{N=\hat{N}}$$

In words, this equation is telling us that the rate of population growth a little bit away from the equilibrium value can be approximated by the rate of population growth at the equilibrium value plus an extra term, calculated by multiplying the distance from equilibrium by how fast the function is changing when it is close to the equilibrium value.

Let’s substitute in the actual equation for the rate of population change in the logistic model:

$$G(N) = rN \left(1 - \frac{N}{K}\right)$$

Let’s deal with the easiest part of our Taylor series first. We know that at \hat{N} , the population is at equilibrium, so $dN/dt = G(N) = 0$. (You can also convince yourself of that by substituting 0 into the logistic growth equation above.) This simplifies our Taylor series down to:

$$G(\hat{N} + n) \approx n \left. \frac{dG}{dN} \right|_{N=\hat{N}}$$

Next we’ll figure out that weird term on the right. What is dG/dN ? Well, that sounds a lot like the derivative of a function called $G(N)$ with respect to N . No problem! We can find the derivative of that function, $rN(1 - N/K)$. Let’s rearrange it to look more like a polynomial:

$$G(N) = rN - \frac{rN^2}{K}$$

Now we can totally take the derivative of this. The derivative of rN with respect to N is just r , since r is a constant. To calculate the second term, let’s separate out N^2 : $d/dN(N^2 \times R/K)$. The derivative of N^2 is just $2N$, so that term becomes $2rN/K$.

$$G'(N) = dG/dN = r - \frac{2rN}{K}$$

Now, we're ready to calculate the Taylor approximation at both equilibria. Let's substitute the equation we just derived for dG/dN :

$$G(\hat{N} + n) \approx n \left. \frac{dG}{dN} \right|_{N=\hat{N}} \approx n \left(r - \frac{2rN}{K} \right) \Big|_{N=\hat{N}}$$

Let's start with $\hat{N} = 0$. In this case, we get:

$$G(\hat{N} + n) \approx rn$$

This is positive, implying it is an **unstable equilibrium**. Next, we want $\hat{N} = K$:

$$G(\hat{N} + n) \approx n \left(r - \frac{2rK}{K} \right) = n(r - 2r) = n(-r) = -rn$$

This is negative, implying it is a **stable equilibrium**.

Local stability analysis: the mathematical shortcut

You can generalize this procedure of mathematical stability analysis by simply saying that we want to calculate the slope of $G'(N)$ at a small deviation from equilibrium, $n = N - \hat{N}$. We can approximate $dN/dt \approx \lambda n$, where

$$\lambda = \left. \frac{dG}{dN} \right|_{N=\hat{N}}$$

If λ is positive the equilibrium is unstable, and it's stable if negative. In other words, all you need to do is get the equation for the derivative of the population growth function and evaluate it at \hat{N} . (Hastings, 91: "The rate of return to the equilibrium, or the rate at which the system moves away from the equilibrium, is determined by λ .") The reason this works is the Taylor expansion we walked through, but in practice you'd mostly use the equation above—or the graphical interpretation—to evaluate stability.

Here's how that would have looked for the logistic model:

$$G(N) = dN/dt = rN(1 - N/K)$$

Restating the equation as $rN - rN^2/K$ and taking the derivative:

$$G'(N) = dG/dN = r - \frac{2rN}{K}$$

Plugging in 0 for N gives you $G'(N) = r$, and plugging in K gives you $G'(N) = -r$.

Extensions to harvested populations

One of the remarkable things about population modeling is that it really underpins a lot of how we manage ocean resources—from marine spatial planning (we'll get to that in the metapopulations lecture) to fisheries management.

Discuss: What would sustainable fisheries mean in quantitative terms? What do we want the terms we've learned about so far to be or not be? Are we trying to maximize or minimize something? If so, what quantity?

Maximum sustainable yield in a logistic model (constant harvest)

Let's assume we have a fish population that actually follows the logistic growth model. If we imagine subtracting a set quota of fish every year, H , then

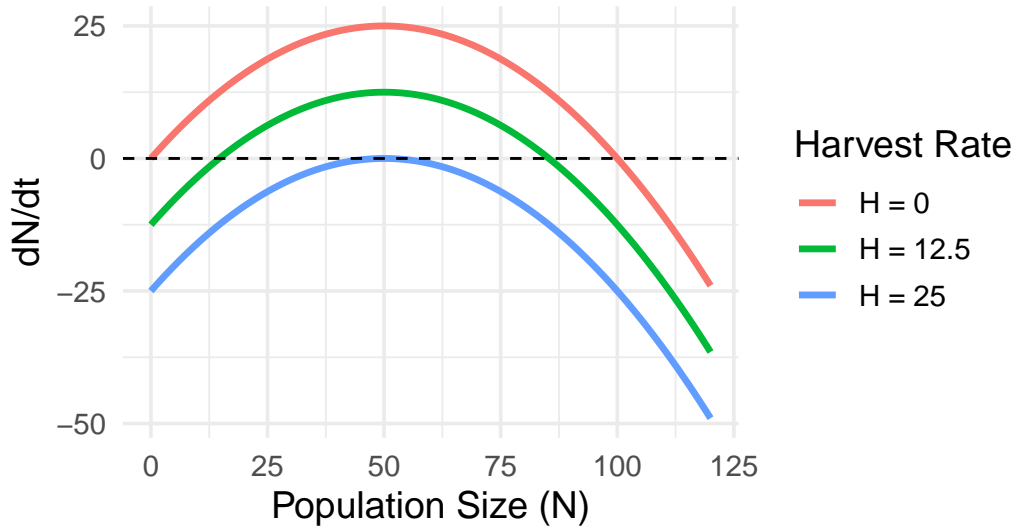
$$\frac{dN}{dt} = rN(1 - N/K) - H$$

How would harvest affect our graph of dN/dt vs. N ?

Start out by drawing the curve with no H and then a moderate H . Discuss the equilibrium points and whether they are stable or unstable. Then add the line for $H = H_{max}$. May want to draw this with H bringing up the x-axis rather than bringing the curve down.

Logistic Growth With Varying Harvest Rates

$H_{\max} = 25$ occurs at $N = K/2 = 50$



Is the equilibrium at $H = H_{\max}$ stable or unstable? Let's use our rule of thumb previously to figure this out.

$$G'(N) = dG/dN = r - \frac{2rN}{K}$$

Because $H_{\max} = K/2$, we can state:

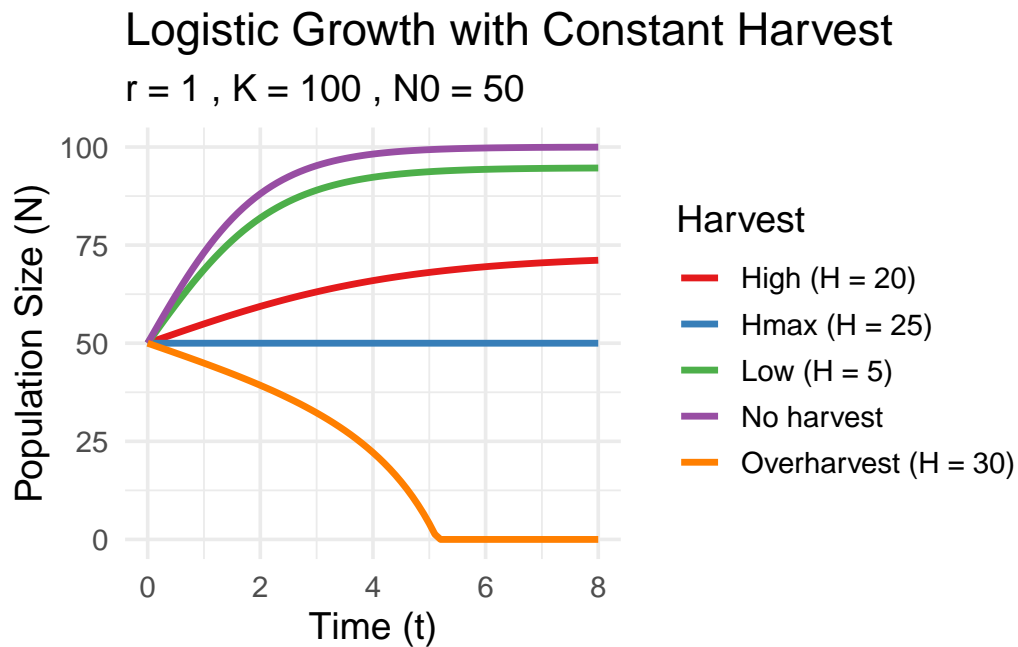
$$r - \frac{2rN}{K} = r - \frac{2r \times K/2}{K} = r - r = 0$$

Zero! We didn't have a rule of thumb for this above. We know that at this point, the first derivative of population size dN/dt or $F'(N)$ is zero (because we're at the x-intercept), and the second derivative dG/dN or $G'(N)$ or $F''(N)$ is... also zero—meaning the slope is flat. In math this is called a “degenerate case” because the system is neither classically stable or unstable. But we can figure out the answer graphically. Let's call $H_{\max} = 25$ fish/year and $K = 100$. Then at H_{\max} , $N = 50$. What happens if we slightly overshoot—meaning we harvest 26 fish by accident? Then dN/dt becomes negative—in other words, the population's natural rate of increase is less than H_{\max} . What happens if we harvest 24 fish? That essentially shifts the curve upwards by 1—so you are back in the land of two equilibria, a stable one at higher N and an unstable one at lower N . This is a *saddle-node bifurcation*: it is stable on one side, and unstable on the other. (We can also see this graphically at the top of the parabola.)

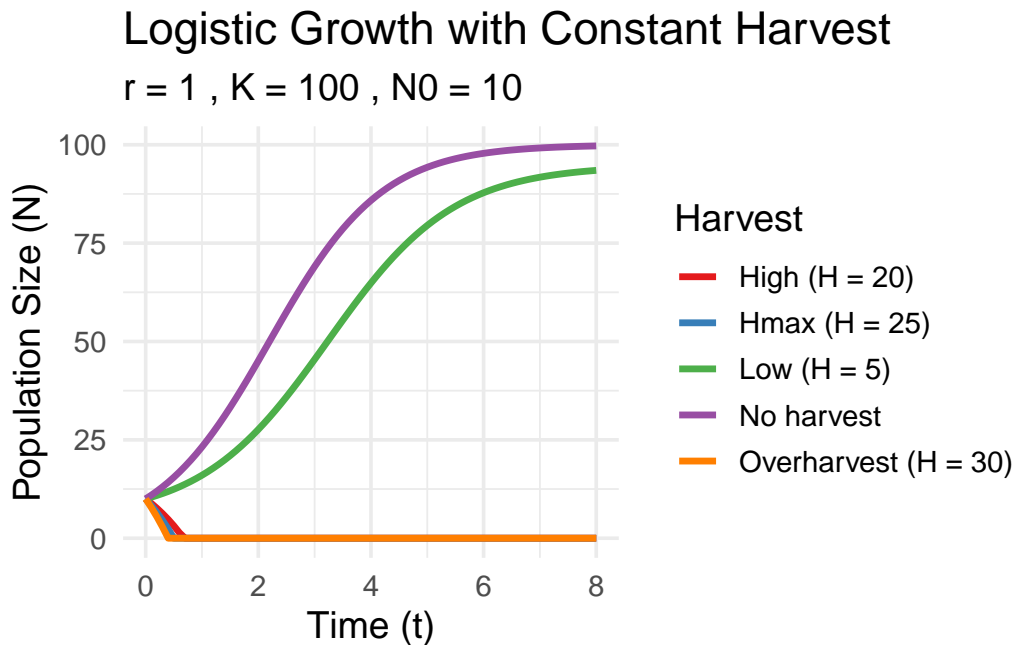
This point is an *optimal harvest level from an economic standpoint* because it maximizes revenue by balancing production (of the fish stock) with returns (of harvest). There is no

mathematical solution that yields greater yearly harvest. However, **it is not stable**, and is extremely sensitive to any level of overharvesting. Roughgarden and Smith (1996) wrote, “Maintaining the stock at the optimal size... is like balancing a marble on top of a dome.” **What are some other reasons why this might not be the most realistic or wise approach to fisheries management?**

To emphasize this point, here are some plots of N vs. t in a logistic-growth, constant-harvest scenario with different levels of H . Let's start by assuming we are already at $K/2$ (i.e., $N_0 = K/2$):



What if we draw the same curves with $N_0 = 0$?



Alternatives to MSY management (constant effort)

Are there more robust ways to achieve sustainable yields than setting harvest quotas equal to the maximum dN/dt ? An alternative to a constant harvest rate (measured in fish per unit time) is a constant effort model (still measured in units of fish per time, but now a dynamic equation). Here, harvest is proportional to population size and **effort** (the intensity of fishing, e.g., hours on the water, number of nets deployed, etc.)

Instead of

$$\frac{dN}{dt} = rN(1 - N/K) - H$$

Where H was a constant rate, we now have:

$$\frac{dN}{dt} = rN(1 - N/K) - qEN$$

Where N is the population size, q is a catchability coefficient (more on this later), and E is effort. Wait—but shouldn't this still be a rate? Let's work through the units and intuition of these different parameters.

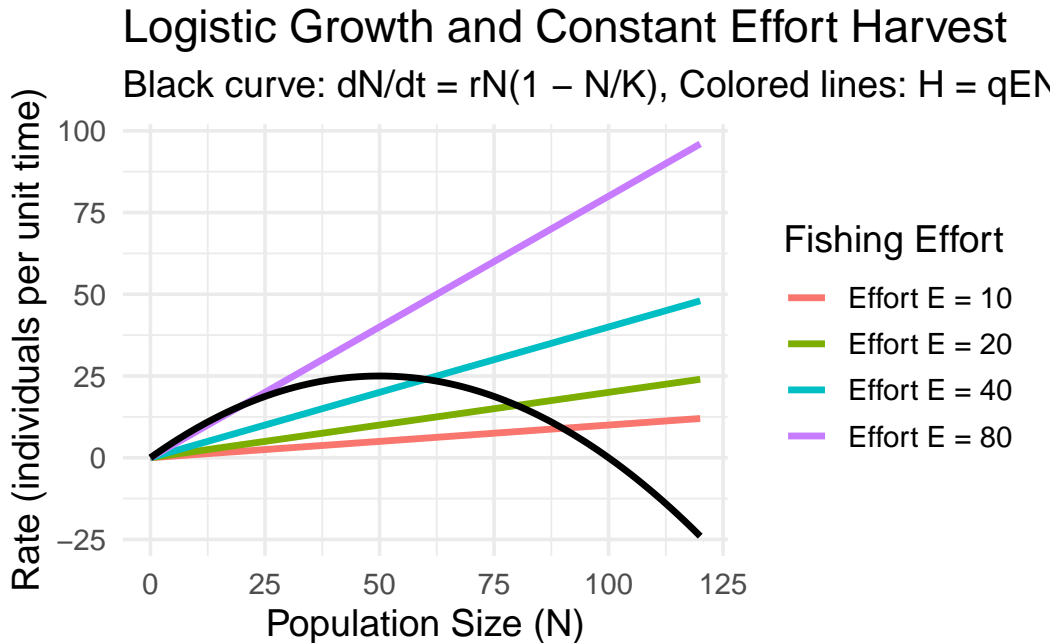
The unit of N is individuals. The unit of E is effort; something like boats, days, permits, traps, etc. For this to end up in the same units of individuals per time, what are the units of q ?

q is measured in $1 / (\text{effort} \times \text{individuals} \times \text{time})$. Here, it's a scaling constant that translates the fishing effort into the number of fish that will actually be caught.

So q is a constant here. Since this is a “constant effort” model, E is also a constant! It tells us that regardless of how many fish are in the ocean, the fishing industry will do the same thing to try to catch them. **Is this a more or less realistic model than the constant harvest model, above?**

But because N is not a constant, harvest here (qEN) is now a dynamic equation. **What form would you expect this equation to take?**

Let's draw it on top of our now-familiar curve for dN/dt vs. N in the logistic model. **For moderate levels of E , where are the equilibria? Which are stable and which are not?**



For moderate levels of effort, aside from the trivial equilibrium of $N = 0$ (no fish = no growth), there is only one other equilibrium, and it is stable. If population size exceeds harvest, then the population will grow up to the point where the diagonal crosses the curved line. If harvest exceeds population growth (i.e., if you go further up the constant effort line), then population size will go down, and effort will also go down. Let's explore this mathematically by writing the equation for this model and setting it equal to 0.

$$0 = rN(1 - N/K) - qEN = rN - rN^2/K - qEN = N(r - rN/K - qE) = N(r(1 - N/K) - qE)$$

What are the solutions to this equilibrium? As we said above there is one where $N = 0$. To find the other, we start with

$$qE = r(1 - N/K)$$

and solve for N:

$$qE/r = 1 - N/K$$

$$N/K = 1 - qE/r$$

$$\hat{N} = K(1 - qE/r)$$

When does this yield a stable equilibrium? Well, let's take the derivative of (dN/dt) and plug in our equilibrium value, \hat{N} .

$$G(N) = rN(1 - N/K) - qEN = N(r - qE) - N^2(r/K)$$

$$G'(N) = r - qE - 2N(r/K)$$

Plugging in \hat{N} ,

$$G'(\hat{N}) = r - qE - 2(K(1 - \frac{qE}{r}))(\frac{r}{K}) = r - qE - 2r(1 - \frac{qE}{r}) = r - qE - 2r + 2qE = qE - r$$

Great! But now ... is $qE - r$ positive or negative? **What is the intuition behind the difference between these two numbers?** Recall that r is the intrinsic growth rate or the maximum per capita growth rate, which dN/Ndt approaches when N is small. qE is in the same units and represents the harvest rate per capita.

So, under what circumstances is this a stable equilibrium? It's when $qE < r$, meaning that the per capita harvest rate never exceeds the population's intrinsic growth rate.

Let's go back to our graph of constant effort and draw a really high effort rate. **Does this ever have a stable equilibrium?** You can see that it does not graphically, because it never intersects the growth function when it has a negative slope (meaning that $G'(\hat{N})$ would be negative).

Discuss: What else might cause overfishing or collapse of a harvested population that isn't captured here?

Other forms of density dependence in continuous time

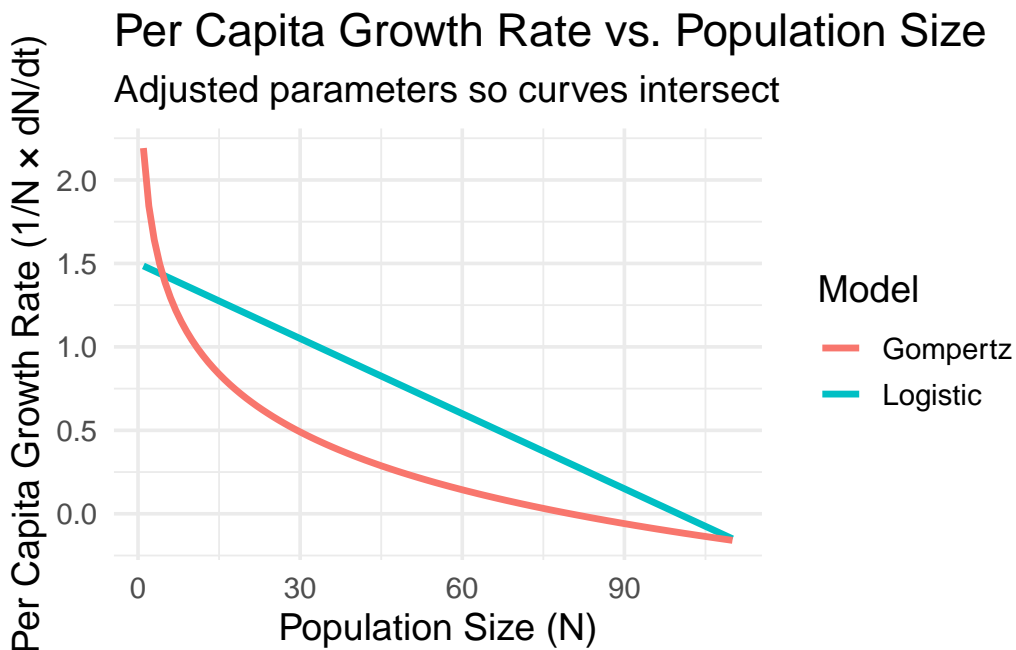
We've been working a lot with plots of dN/dt vs. N . But recall that we usually think about three plots: N vs. t , dN/dt vs. N , and dN/Ndt vs. N . We're going to revisit the latter. Recall that this was a straight, horizontal line in the exponential, continuous-time model, and it was a straight, sloping line in the logistic, continuous-time model. This represents an assumption that density dependence is linear: i.e., that as N increases, the density-dependent term $(1-N/K)$ decreases linearly from 1 (at $N=0$) to 0 (at $N=K$). This implies that per capita growth slows down at a constant rate as populations increase.

Gompertz model

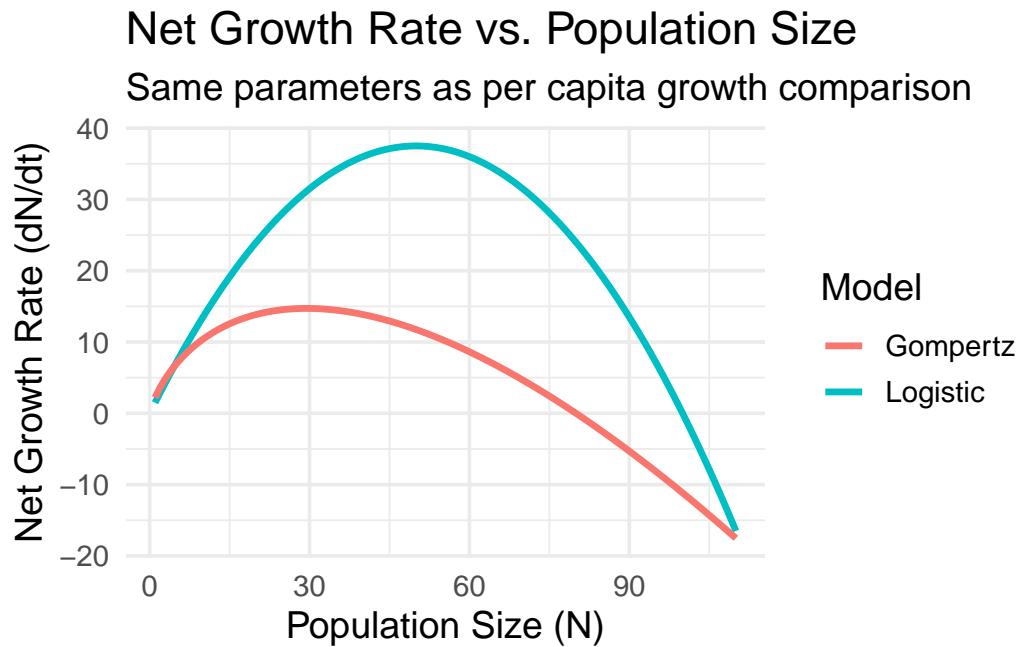
This is an easy thing to modify, because in reality, we might expect this relationship to be nonlinear: maybe per capita growth is not super sensitive at low N , and then drops steeply at high N . Let's try this equation instead:

$$dN/Ndt = r \ln(K/N)$$

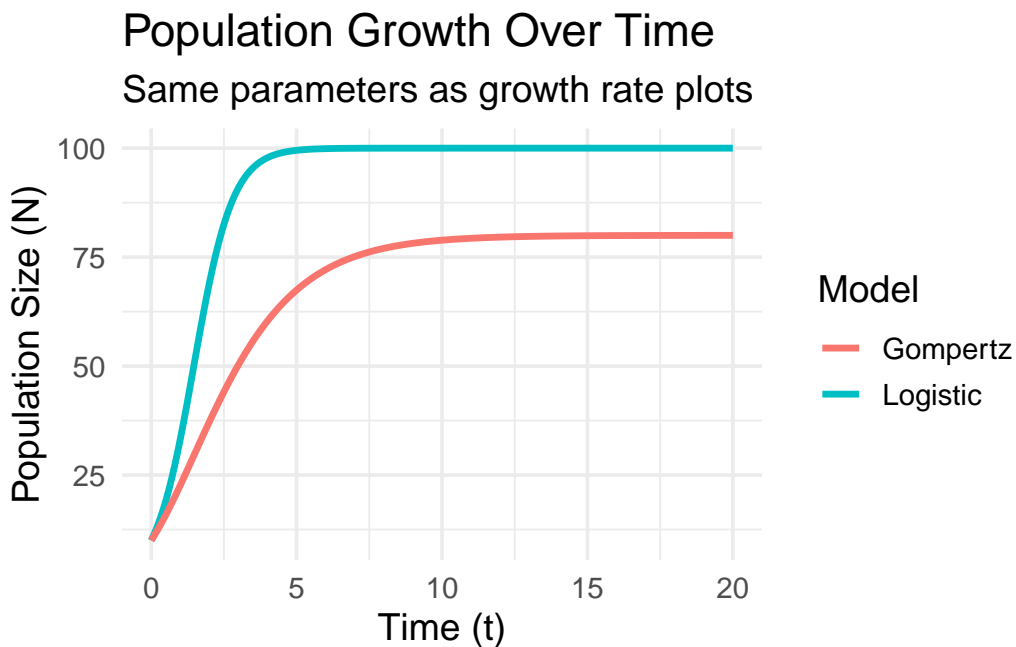
(Because $\ln(A/B) = -\ln(B/A)$, you could also restate this as $-r \ln(N/K)$, if you want to compare it more directly to the slope we had before with K in the denominator.)



We said earlier that the Gompertz model was going to start off with gradual population growth and then show steeper density dependence at high N . Let's plot dN/dt vs. N to explore this.



And finally, N vs. t .



Note that the shape of these curves relative to one another is totally dependent on the parameter values. One K may be higher than the other, depending on parameters. The thing to remember about the Gompertz model is basically just that the relationship between per capita growth rate and population size is nonlinear, which may fit some data and some models better.

If you work in fisheries and you hear someone mention a Gompertz model, just understand it is a density-dependent, continuous-time model that has a slightly different formulation than the logistic.

Positive density dependence

All of the density dependence we've been talking about so far is technically called *negative density dependence*. It's confusing to remember which is which, but think of it this way: is the relationship between population size and average offspring (per-capita population growth rate, dN/Ndt) positive or negative in the logistic model?

There's another way that density can interact with per-capita population growth rate. Think about abalone in California. All seven of our abalone species are endangered (six are critically endangered). These were massively depleted by humans because they are really easy to access and harvest. Disease, warming, heatwaves, and kelp loss have also negatively impacted these species.

Abalone are broadcast spawners. They have separate sexes, so the eggs and sperm need to meet in the water. If there are too few of them, this won't happen and we'll see a *recruitment failure*—when reproduction fails to lead to a new generation of an ecologically meaningful size.

How would this affect the plot of N vs. dN/Ndt in the logistic growth model?

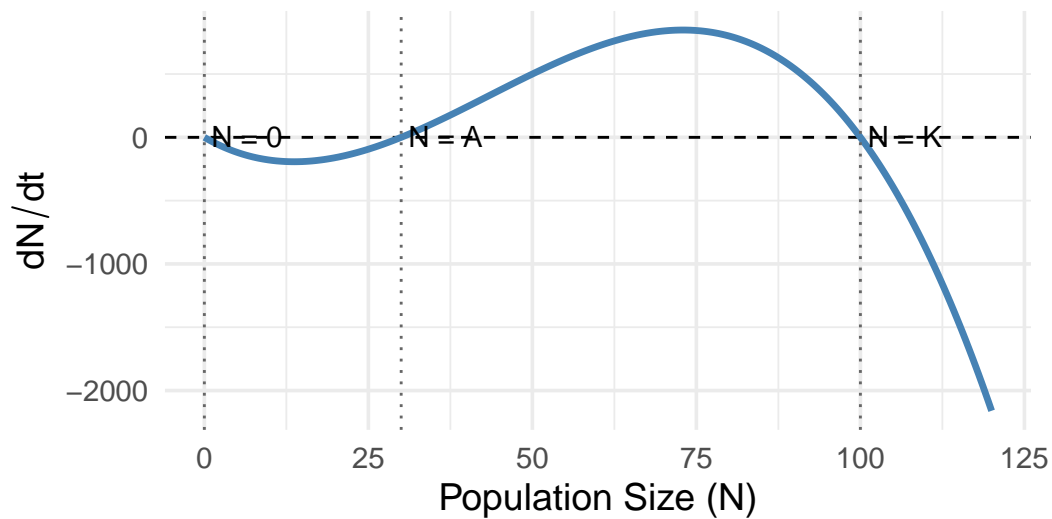
We can model this effect by introducing a new parameter into the *continuous-time* logistic growth equation, A (this stands for **Allee effect**, another word for this form of positive density dependence.) Then the population growth rate is given by

$$dN/dt = rN(N - A)(1 - N/K)$$

Let's start by finding the equilibria of this model. Because it's a product of three terms, we can say from looking at it that it will be equal to zero if any one of the three terms is zero. The first equilibrium is when $rN = 0$; because r is a constant, that means $\hat{N} = 0$. The second is when $N - A = 0$, so we also have $\hat{N} = A$. Finally, if $(1 - N/K) = 0$, we get $\hat{N} = K$.

To figure out if these equilibria are stable, let's graph the equation above.

Growth Rate with Allee Effect: $\frac{dN}{dt} = rN(N - A)$

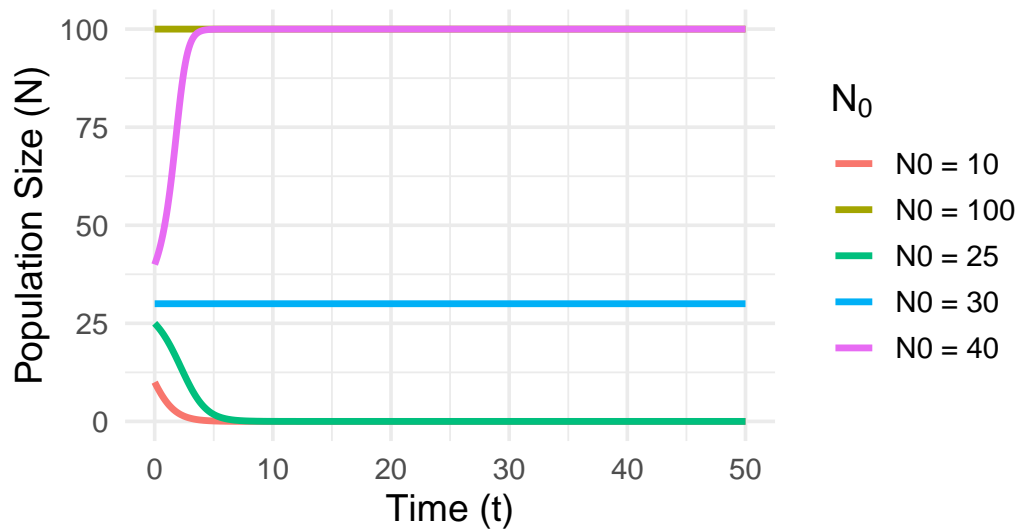


Graphically, which equilibria are stable and which are unstable?

Take a couple minutes to think and draw: What do you think the graph of N vs t will look like for a few different values of N_0 ?

Population Trajectories under Allee Effect

Different initial population sizes relative to Allee threshold



Time lags and density dependence

One criticism that would be very valid of all the models we have discussed so far is that they are too “neat.” You all plotted data of an actual system this week, and saw that it is almost certainly much more dynamic than these smooth curves. The textbook goes into more detail about criticism of the logistic model, which mostly centers around it being very difficult to quantitatively prove, partly because it’s hard to know if a stable equilibrium has been reached when looking at real data, and partly because our ability to estimate the exact parameters (of any density-dependent model—logistic, Gompertz, etc.) is limited.

Delayed logistic

(Hastings pp 92-96) Most of the rest of the course will be adding different forms of complexity to better approximate real systems. Probably the next simplest case is a population that does not reach a single stable equilibrium, but instead oscillates around K . This is common in microbes and insects, especially in laboratory studies.

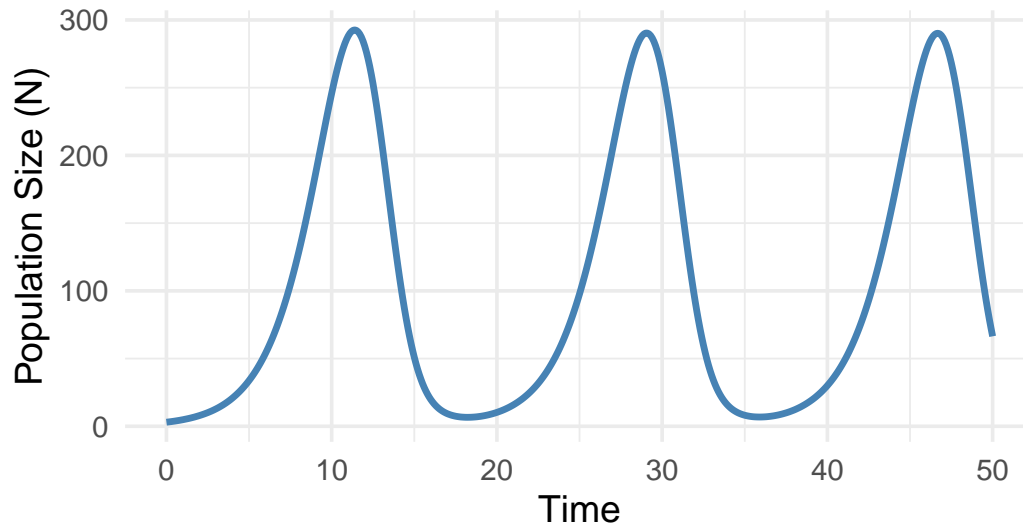
The analogy in the textbook is a thermostat in a room: even though the mean air temperature is at a given setting, in practice the continuous temperature probably repeatedly dips and warms. We can model this relatively simply by introducing a time delay in the logistic model. This implies that there is a lag between when population size changes, and when population growth rate changes. We call that lag T , and define

$$dN/dt = rN_t(1 - N_{t-T}/K)$$

You can see here that the effect of density dependence ($1-N/K$) is lagged T amount behind t .

Delayed Logistic Population Growth

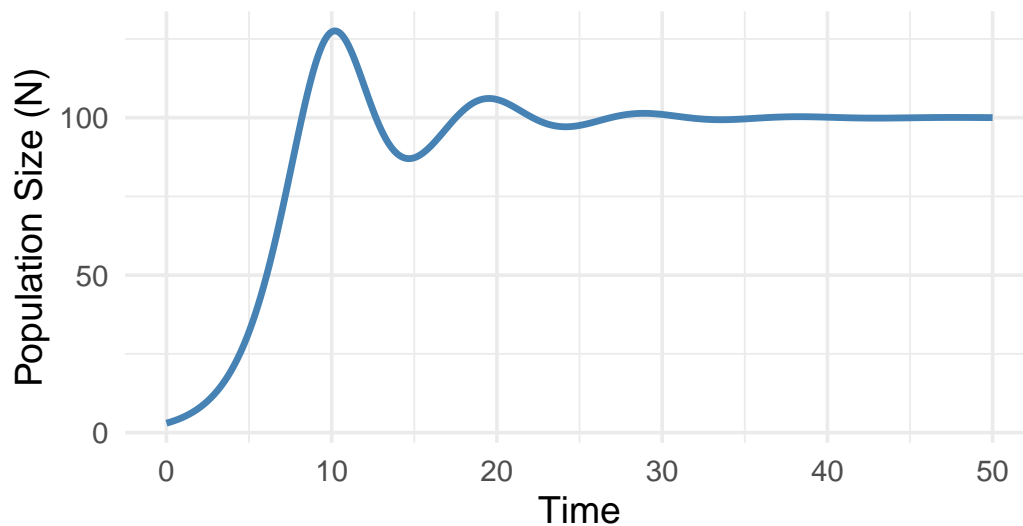
Delay $T = 4$, $r = 0.5$



This is a really fun equation to play around with, because you can see that the predicted dynamics change with r and T (and, specifically, their product; you get oscillations when $rT > \pi/2 \approx 1.57$, and sustained oscillations at even greater values.) If I shrink T I get damped oscillations:

Delayed Logistic Population Growth

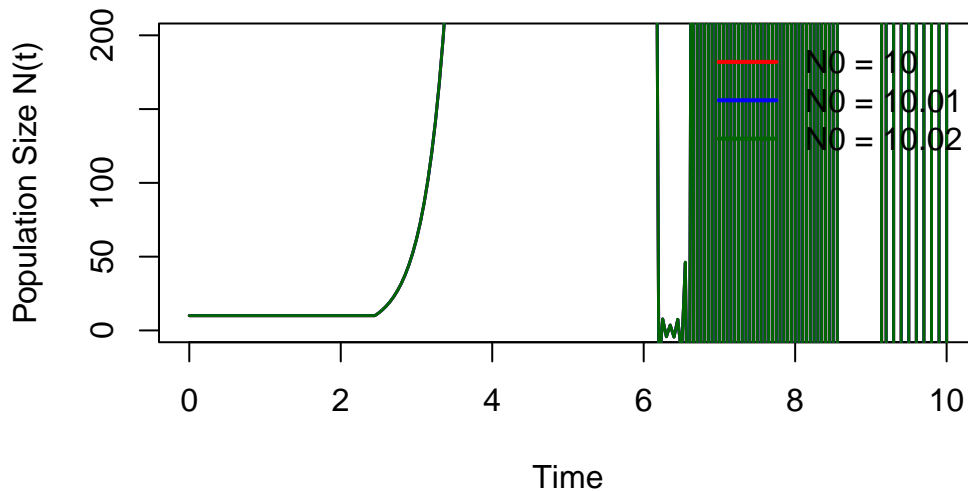
Delay $T = 2$, $r = 0.5$



And if rT is large I get outcomes that are highly sensitive to initial conditions—even though we haven't introduced any randomness or stochasticity! All we've done is slightly change

initial conditions or parameter values, and run a deterministic simple model, but one with time lags.

Chaos in Delayed Logistic Model ($r = 4$, $T = 2.5$)



Have you ever heard of something where tiny changes in initial conditions lead to wildly different outcomes? People call this the *butterfly effect*: a butterfly flaps its wings and some time later there's a tornado. Mathematically, this means the system is **chaotic**. Even though the system is deterministic, not stochastic, tiny changes in some variables can lead to huge differences.

The delayed logistic model can approximate some, but not all, population dynamics. What species do you think it would describe well? Which would it describe poorly?

Crucially, it is missing:

- The effect of different generations and histories of reproductive success (i.e., age structure)
- More complex lagged effects
- The effects of seasonal reproduction

Density dependence in discrete time

(Hastings 96-101)

Let's return to discrete time and explore the dynamics of some discrete-time models that also include density dependence. The first equation we wrote down in this class was:

$$N_{t+1} = F(N_t)$$

And I promised you that we'd spend a lot of the class talking about the forms that F can take. The first form we looked at was the simplest,

$$F(N_t) = RN_t$$

$$N_t = N_0 R^t$$

Where the way you get from population size at time t to population size at time $t+1$ is just multiplying by a constant, R . Let's write down some options with density dependence. **Discuss:** Qualitatively, what would a discrete-time model for density dependence look like? What would be on the right-hand side of $F(N_t)$?

Ricker model

We're starting with this model because we know all the parameters in the equation, although the equation itself is new to us:

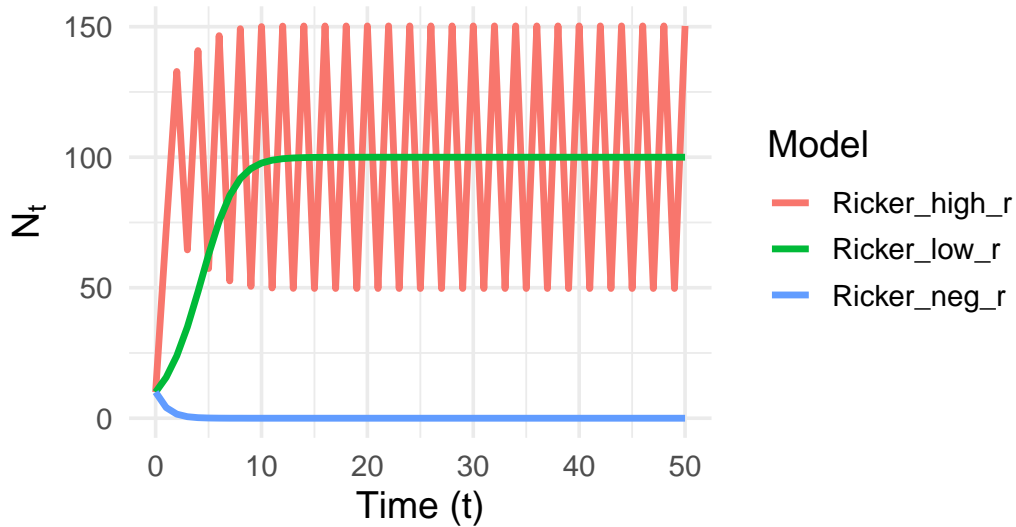
$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}$$

Recall how we discussed above that for a discrete-time model to have density dependence, in this equation, N_t would have to be multiplied by something that isn't a constant (and is, in fact, a function of N)? Here we go! And notice how similar this is to the continuous-time logistic model. **Why might we exponentiate that term? When you raise e to a power, can it ever be negative?**

Most people start with graphs of N_t vs. N_{t+1} but I always like to look at a population trend image so we'll do that first.

Ricker Model Population Trajectories

Initial $N_0 = 10$, $K = 100$



First return plots

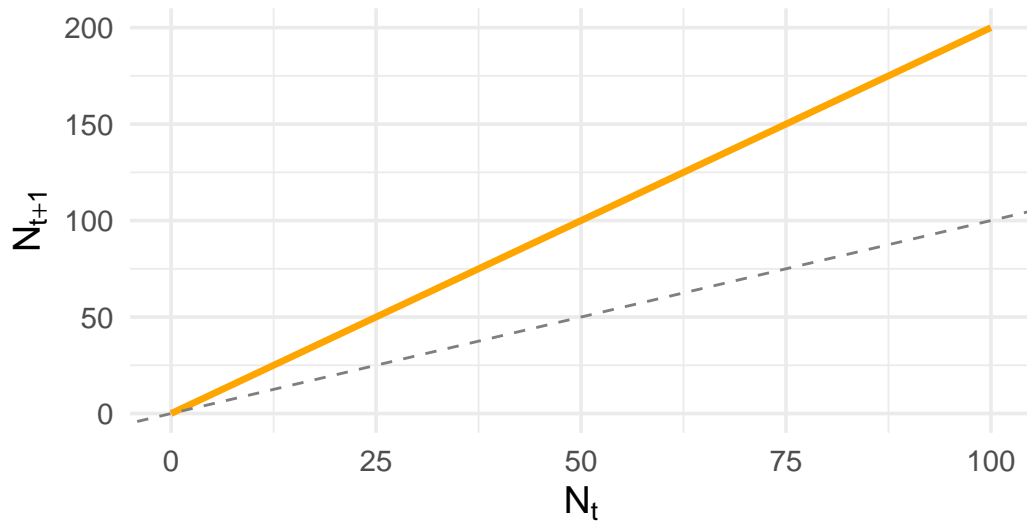
We've been looking at three classes of graphs, but today we're going to add in a fourth one: N_t vs. N_{t+1} . These are called “first return” or “next-generation” plots, and they're really common in discrete-time models. This shows how the population at time t determines the population at $t + 1$; in other words, given its current value, where will the population go? To orient ourselves to these figures let's quickly plot one for geometric growth, $N_{t+1} = RN_t$ (or $F(N) = R^t N_0$). **Take a few minutes and try to draw this figure yourself.**

What graph did we see much earlier in the class that looks like this one? This plot is highly analogous to dN/dt vs. N in the exponential growth continuous time case. They both describe how the current population size affects the rate of change, and they both give us the direction and speed of population change.

Take a moment to think: Where are the equilibria of this model? How would you know? Here, we can find equilibria graphically when the function crosses the “identity” (1:1) line. **Does the geometric growth model have any equilibria?**

Geometric Growth: $N[t+1]$ vs $N[t]$

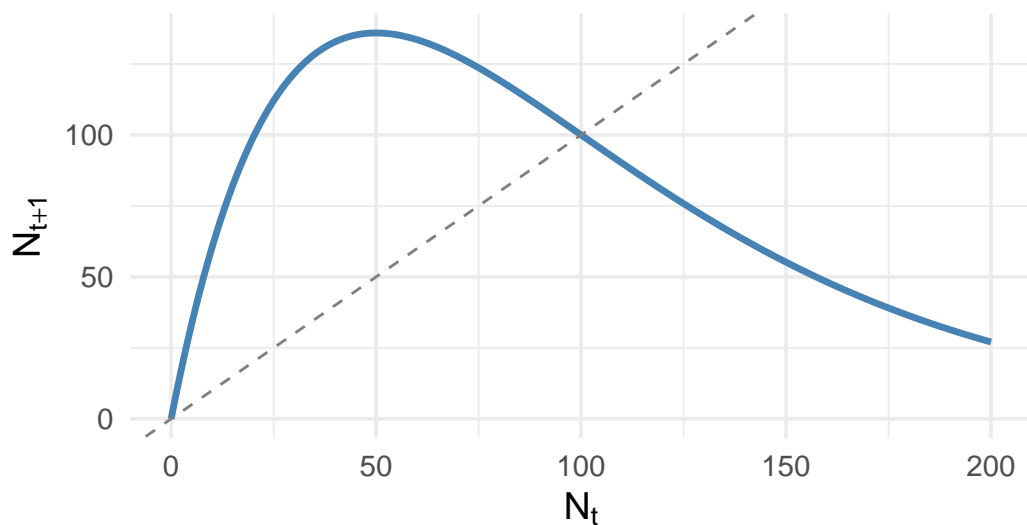
$$R = 2$$



Keeping in mind all the properties of the dN/dt vs. N plot of the logistic continuous-time model above (the upside-down parabola), let's graph the first-return plot for the Ricker model.

Ricker Model: Next Generation vs. Current Population

$$r = 2, K = 100$$



From the Ricker model formula, what is its growth rate when rare?

Now let's ask all of our usual questions: does this have any equilibria? What does it mean for population size, per capita growth, etc.?

We solved for equilibria above by setting $dN/dt = 0$. Here, we want to set $N_t = N_{t+1} = \hat{N}$.

$$\hat{N} = \hat{N}e^{r(1-\frac{\hat{N}}{K})}$$

This has a trivial equilibrium at $\hat{N} = 0$. But dividing both sides by \hat{N} and taking the log, we get:

$$0 = r(1 - \hat{N}/K)$$

This simplifies to $\hat{N} = K$. What other model had equilibria at population sizes of 0 and K? If I were better at drawing you could see that on the graph above as well. Your next question, now that we have found the equilibrium for the Ricker model, might be: But is it stable? Even though this is a discrete-time model, you can still do this by taking the derivative of this equation:

$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}$$

And asking about its behavior at \hat{N} . Let's take the derivative of this equation. We're going to take advantage of the **product rule**, which says that for any u and v , $dG/dN = u'v + uv'$. In other words, when we want the derivative of a product, we take the derivative of each, multiply it by the other, and sum them. Here, $u' = 1$ (the derivative of N with respect to N). v' is:

$$v' = \frac{d}{dN} e^{r(1-N/K)}$$

How do we take this derivative? Hopefully this is triggering some vague memories from calculus. You need to remember two things. First, the *chain rule*, which applies to composite functions—like $f(x) = g(h(x))$. Then, $f'(x) = g'(h(x)) \times h'(x)$. In other words you multiply the derivative of the outer function (keeping the inside intact) by the derivative of the inner function. Let's work through this for an exponential function, $f(x) = e^{g(x)}$.

The second thing you need to remember is that the derivative of e to the something is just ... itself! $d/dx e^x = e^x$. So, we can separate this into:

$$v' = \frac{d}{dN} e^{r(1-N/K)} = e^{r(1-N/K)} \times \frac{d}{dN} r(1 - N/K)$$

Now we just have to take the derivative of $r(1 - N/K)$, which we've done before. We multiply it out into $r - rN/K$, and when taking the derivative with respect to N , this turns into $-r/K$. So our whole v' turns into $-r/K \times e^{r(1-N/K)}$. Recall that we started with $G(N) = N_t e^{r(1-\frac{N_t}{K})}$ and said that $dG/dN = u'v + uv'$ (the product rule). So putting it all together (recall that here, $u' = 1$), we get:

$$dG/dN = e^{r(1-N/K)} - r/K e^{r(1-N/K)} \times N = e^{r(1-N/K)}(1 - rN/K)$$

This is the derivative of the Ricker model in discrete time. Remember, we did all this to understand whether the equilibrium $\hat{N} = K$ is stable or unstable. Let's substitute that in. $e^{r(1-N/K)}$ turns into 1, and so does N/K . So we get that $dG/dN = 1 - r$.

Stability analysis of discrete-time models

In continuous time, what was our stability criterion with respect to this second derivative at the equilibrium value? Stability in continuous time is defined by the sign of the derivative near the equilibrium. If it's negative, it's stabilizing, and if it's positive, it's destabilizing. Think back to our plots of dN/dt vs. N ; if the function crosses the $dN/dt = 0$ line (meaning that, at least momentarily, population is at equilibrium) with a positive slope, the dynamics pull the system away from that point. The reverse is true if it crosses with a negative slope.

The mathematical rule of thumb

But now think back to our first-return plot. When did we define an equilibrium? When $N_{t+1} = N_t$. Previously we were interested in $dG/dN = 0$, meaning when functions crossed the x-intercept. Now we're interested in when functions cross the identity line, meaning that $G'(N) = 1$. This is also the criterion for stability in *all discrete-time models, not just the Ricker model*: if $|f'(\hat{N})| < 1$.

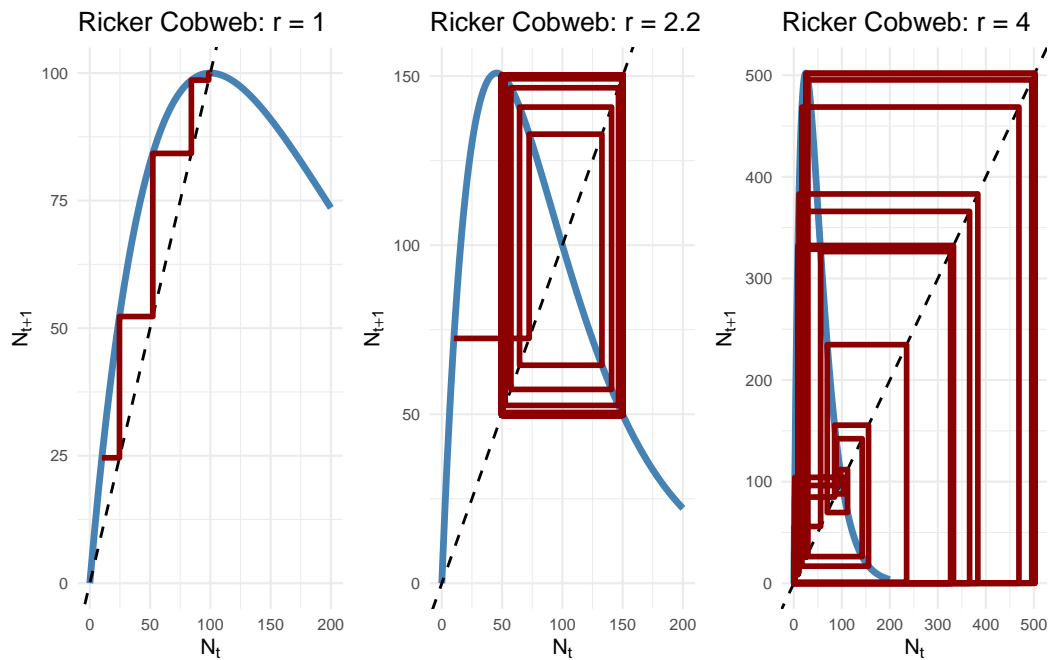
Revisiting the second derivative of the Ricker model at $\hat{N} = K$, $dG/dN = 1 - r$: What values does r have to take on for this to be stable? In other words, for what r is $|1 - r| < 1$? Answer: r must be between 0 and 2.

The graphical explanation

Let's try to build some graphical intuition for stability in discrete-time models, which have more complex behavior than continuous-time ones. We're again going to trace population dynamics from one moment to the next, but this time in the first-return plot. Let's start with a value of N a little above 0. Then we're going to:

- Go vertically up to the curve. This gives us the next population size (at the x-value we picked, what's the y-axis value?)
- Now let's re-consider that value to be our new x. How do we get from y to x on this plot? Go horizontally toward the 1:1 (identity) line. That translates the y you found (last time step's N_{t+1}) into an x (this time-step's N_t).
- Repeat!

Drawing plots is getting harder in this class, but let's start with a low value of r , which means the peak of the Ricker curve is on the identity line. (Because this curve has a long tail we're going to stop drawing the axes on the same scales.) We know where the equilibria are: the question is, does the dynamics of the system pull us toward or away those two points?



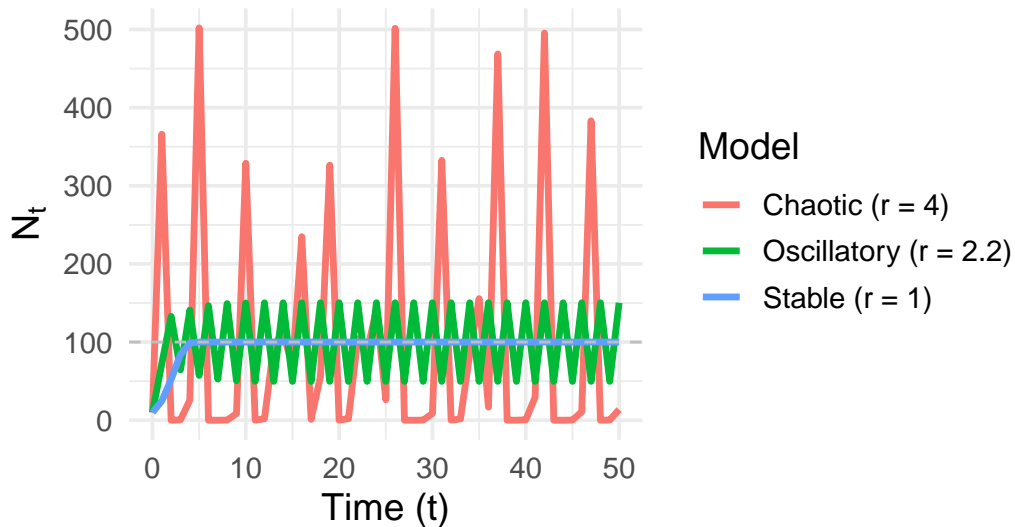
We can see that $\hat{N} = 0$ is always unstable because we are always drawing away from it, and the equilibrium $\hat{N} = K$ is very stable for low r . Let's draw out these diagrams for higher values of r . How on Earth do we interpret these cobweb diagrams? The second one eventually exhibits regular bouncing between the same population sizes. **What did we call that behavior earlier?** The third one never stabilizes and goes all over the place. **What did we call that behavior earlier?**

We know that the Ricker model equilibrium at $\hat{N} = K$ is stable for $0 < r < 2$. It turns out that r controls the steepness of the N_{t+1} vs N plot near the equilibrium point. (Remember that the second derivative was just $1 - r$? That is the slope of the N_{t+1} vs N plot at the equilibrium! It only depends on r .) Thus, if r is small (and likely stable), the curve is not very steep when it crosses the identity line. We could have actually guessed whether the plots above would lead to stability without doing the cobweb diagrams!

To check our intuition, let's graph the exact same parameter values in N vs t space:

Population Trajectories for the Ricker Model

$K = 100$, $N_0 = 10$



This is also consistent with our mathematical condition for stability in the Ricker model above. When r gets too large, we see sustained oscillations and then chaos.

Reflect and then share: The only thing varying between these curves is r . Why would the value of r , biologically speaking, have such a big impact on population dynamics? Why would larger values of r lead to more complex behavior in a discrete-time model? What are some real-world populations that might exhibit this behavior?

Chaos in ecology

That simple models in discrete time could yield chaotic dynamics at realistic values of r was a surprising discovery (see May, 1974). We now suspect that many systems have chaotic dynamics, particularly plankton systems (lots to read about this if you're interested.) **Reflect and then share:** If chaos is common in simple population models, what does that mean for the predictability of the real world?

Age and stage structure

Discuss: Up to this point we've been assuming all individuals in a population are the same. What are some ways in which that might not be true? What might vary among individuals, and how might that variation affect population dynamics?

The loggerhead sea turtle scenario

Loggerhead sea turtles, *Caretta caretta*, are large marine reptiles found in the Atlantic, Pacific, and Indian Oceans as well as the Mediterranean. This species has a complex life cycle with very different behavior and reproductive activity by life stage. Females mate every 2-3 years and then nest on beaches close to where they themselves hatched. They do not tend to their eggs, which hatch after several months. Hatchlings then make their way to the water, where they disperse into the open ocean. As they grow larger they often reappear in coastal habitats, frequenting lagoons and estuaries. Females do not reach sexual maturity until they are around 35 years old. Loggerheads can live more than 80 years and are often over a meter long.

As you can see, modeling loggerheads as a population of identical individuals who all reproduce at the same rate is probably a bad assumption. Also, loggerhead populations are generally declining worldwide. Their different life stages face very different threats. Adults are often killed by boat strikes, fishing gear entanglements, or marine debris. Habitat degradation is common in coastal areas, which is particularly harmful to species like loggerheads that return to the beaches where they hatched. Turtle sex is determined by the temperature during incubation, so climate change will likely alter the sex ratio of this species. Human activity on or near beaches poses a risk to eggs (which can be destroyed easily) and hatchlings (which may not reach the water, especially if they are disoriented by artificial light).

Matrix refresher

A **matrix** is a rectangular array of numbers. For example, a 2×3 matrix has 2 rows and 3 columns:

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 0 & 5 \end{bmatrix}$$

We usually denote a matrix with uppercase letters (e.g., A , B), and its elements with lowercase letters and subscripts, like a_{ij} for the element in row i and column j .

- A matrix with m rows and n columns is called an $m \times n$ **matrix** (rows always go first!)
- A one-dimensional matrix is called a **vector** (this should sound familiar from computing)
- A **column vector** is a matrix with size $n \times 1$
- A **row vector** is $1 \times n$

For example, a population vector tracking three age classes might look like:

$$\mathbf{n} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \end{bmatrix}$$

Here, we aren't saying anything about *how to get from the matrix to a single number*. If we wanted to state **n** as the sum of the three age classes here we would just do that. Rather we're saying that this matrix *holds* entries for each age class. We're going to use matrices to track age-structured populations, because they're a tidy way to hold information about multiple ages (or stages) without transforming it (like summing it or averaging it, which we don't necessarily want to do—which is why we're moving away from calculating a single N).

Now, let's refresh matrix operations. You can add or subtract matrices to/from one another **only if they have the same dimensions**. You add corresponding elements (matrix elements with the same position):

If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Then:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Subtraction works the same way:

$$A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

The above is called **matrix addition or subtraction**. Mathematically, you aren't supposed to just add or subtract scalars (single values); i.e., $A+5$ is an illegal operation because a matrix is an array of numbers and a scalar is not. You can do this operation in lots of programming languages though, and in practice it typically adds the scalar elementwise (separately to every element of the matrix, i.e., adding 5 to every value in A.)

Matrix multiplication is a little more complicated, so we'll split this into three parts: *matrix-scalar*, *matrix-vector*, and *matrix-matrix multiplication*. You can multiply a matrix by a single number (scalar) by multiplying **every entry** in the matrix:

If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad k = 5,$$

then:

$$kA = 5 \cdot A = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 2 \\ 5 \cdot 3 & 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

This operation scales the entire matrix up or down. In this class, we'll be mostly using *matrix-vector multiplication*. If A is a 3×3 matrix and \mathbf{n} is a 3×1 column vector:

$$A \cdot \mathbf{n} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} n_0 \\ n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} a_{00}n_0 + a_{01}n_1 + a_{02}n_2 \\ a_{10}n_0 + a_{11}n_1 + a_{12}n_2 \\ a_{20}n_0 + a_{21}n_1 + a_{22}n_2 \end{bmatrix}$$

Each new entry is a **linear combination** of the population across age classes. In other words, for square matrices, we take the i th element of the column matrix (i.e., i th row) and multiply each entry in the i th *column* of the matrix by that value. Then we *add all those values together*. For n -by- n square matrices multiplied by a n -by-1 column vector, after you do the addition for each element, you get back a n -by-1 matrix.

Age structure

Hastings pp16-20, Caswell 8-18

If your goal was to conserve loggerhead turtles, in addition to chronicling their population dynamics enough to know they are declining, your next most-pressing question would probably be: **which of these life stage-specific threats is driving population decline, or would be most valuable to protect to help the species?**

Loggerhead sea turtles have a pretty complex life cycle so we're going to introduce these models with a simpler example and come back to turtles later. To make this as easy as possible, we're going to assume that per capita birth rate and survival rate are not affected by the number of individuals (*no density dependence*). However, they ARE affected by age.

In the discrete-time, geometric growth case, how did we calculate R from counts of individuals observed from one time step to the next?

In our simple example, let's assume that:

- Our hypothetical organism lives for up to three years. We're going to call the individuals in their first year of life (0-364 days old) the "age 0 class", individuals in their second year of life the "age 1 class" (just like humans: you are 1 until you turn 2), and so on. All age classes can reproduce, but their rates may differ. No individuals survive past 3, so there is no age class 3.
- The population has two sexes, but we are only tracking and quantifying females. (This is a not-uncommon approach in population dynamics.)

- Age 0 individuals have, on average, b_0 offspring per year (per capita). Age 1 individuals have, on average, b_1 offspring per year (per capita). Age 2 individuals have, on average, b_2 offspring per year (per capita).
- s_0 is the probability that a 0-year-old survives to become a 1-year-old. s_1 is the probability that a 1-year-old survives to become a 2-year-old. (We don't need to specify a survival rate for age 2; why?)
- N_t is the total number of individuals at time t , and it is a single value (scalar). $N_t = \sum n(t)$, where $n_t = n_0(t) + n_1(t)$. Note that these are lowercase because they are *not* total population sizes and/or because they are vectors—none of them are the same as “big N ”.

$n_0(t)$ is the number of offspring of 0-year-olds in year t plus the number of offspring of 1-year-olds in year t . How would we calculate this, i.e., turn this statement into an equation?

$n_1(t)$ is the number of 0-year-olds from time $t-1$ that survive to time t . How would we calculate this?

$n_2(t)$ is the number of 1-year-olds from time $t-1$ that survive to time t . How would we calculate this?

Conventionally we put $t+1$ on the left and t on the right (rather than $t-1$ and t). We could also denote these $n_{0,t}$ but we put t in parentheses because having everything in the subscript gets awkward when we are in $t+1$ land and beyond.

$$n_0(t+1) = n_0(t)b_0 + n_1(t)b_1 + n_2(t)b_2$$

$$n_1(t+1) = n_0(t)s_0$$

$$n_2(t+1) = n_1(t)s_1$$

Work out what the population size is at $t+1, t+2, t+3, t+4$ if the population at N_t has (A) 10 age 0 individuals, no age 1 individuals, and 5 age 2 individuals or (B) no age 0 individuals, 10 age 1 individuals, no age 2 individuals. Hint: A table with rows for the age classes and columns for the time steps will help.

The equations above can be written, much more conveniently, in matrix form. As a group, try writing these down in matrix form, with population sizes at $t+1$ on one side and population sizes at t on the other.

We start by just spelling out the vectors on both sides:

$$\begin{bmatrix} n_0(t+1) \\ n_1(t+1) \\ n_2(t+1) \end{bmatrix} = \begin{bmatrix} (n_0(t)b_0 + n_1(t)b_1 + n_2(t)b_2) \\ n_0(t)s_0 \\ n_1(t)s_1 \end{bmatrix}$$

We want to factor the right-hand side into a matrix multiplied by something. If we factor out the vector of population sizes, we get:

$$\begin{bmatrix} n_0(t+1) \\ n_1(t+1) \\ n_2(t+1) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & b_2 \\ s_0 & 0 & 0 \\ 0 & s_1 & 0 \end{bmatrix} \begin{bmatrix} n_0(t) \\ n_1(t) \\ n_2(t) \end{bmatrix}$$

The above is a convenient way to summarize the age-structured population dynamics. We can summarize this even more compactly by defining the matrix in the middle as \mathbf{L} , which stands for Leslie matrix (one of the first people to utilize this model in ecology). Then we can state,

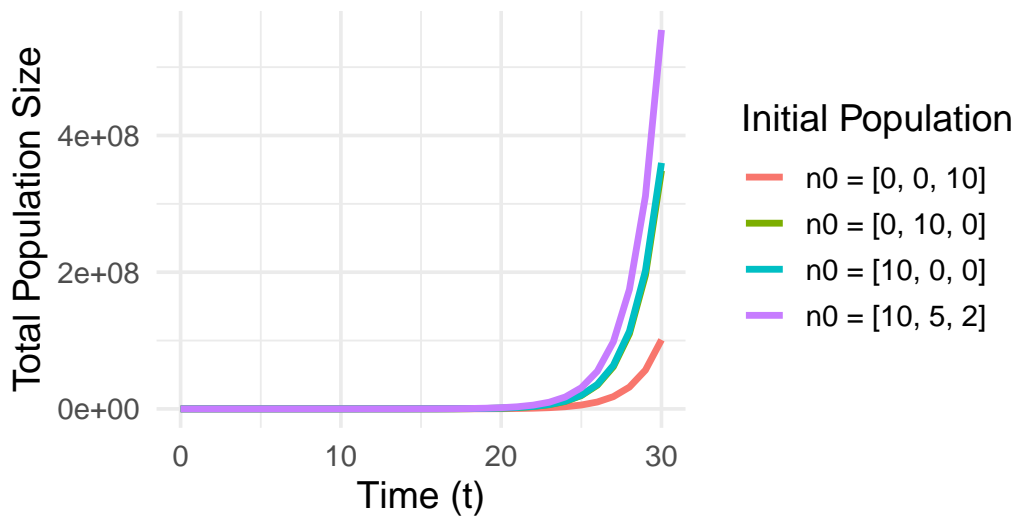
$$n(t+1) = \mathbf{L}n(t)$$

Matrix population models are generally classified by the form that \mathbf{L} takes, and this is the simplest one—where \mathbf{L} never changes. Nonetheless this sets us up to ask all kinds of questions about population dynamics! Your first one might be, will this age-structured population still exhibit exponential growth? It *looks* like it should, but recall that \mathbf{L} is a matrix—not a scalar—and n is a vector. This model DOES tell us that *birth and death rates* are constant, but does that itself lead to exponential growth?

Matrix models are so popular in population dynamics because they can be used to make projections of abundance across ages. Let's look at some examples.

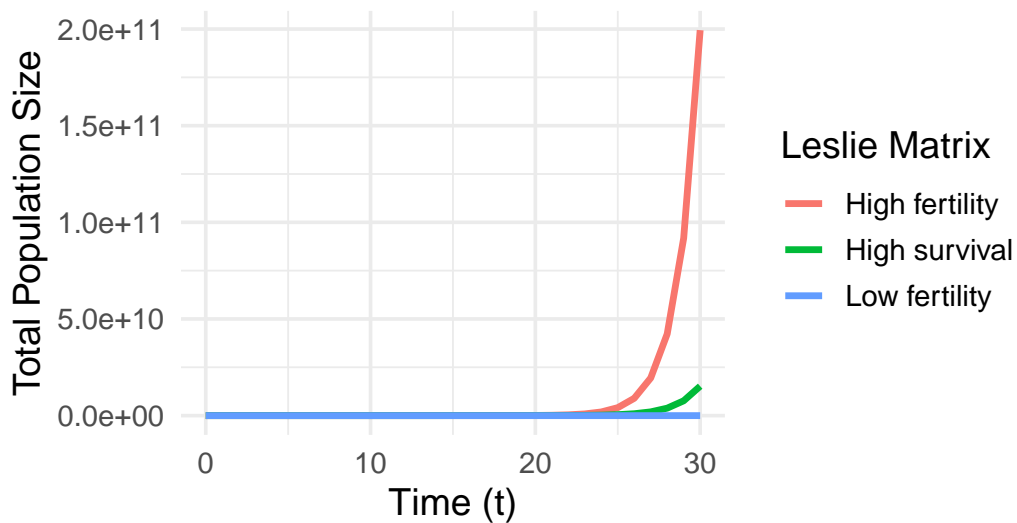
Population Size Over Time

Matrix model: $\mathbf{n}(t+1) = \mathbf{L} \times \mathbf{n}(t)$



Population Size Over Time for Different Les

All simulations use $\mathbf{n}_0 = [10, 5, 2]$



Several observations here. First—yes, this model does lead to exponential growth, even with age structure. However, both the initial population sizes and the Leslie matrix parameters affect the rate of exponential growth. Let's think more about the properties of this matrix:

$$\begin{bmatrix} n_0(t+1) \\ n_1(t+1) \\ n_2(t+1) \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & b_2 \\ s_0 & 0 & 0 \\ 0 & s_1 & 0 \end{bmatrix} \begin{bmatrix} n_0(t) \\ n_1(t) \\ n_2(t) \end{bmatrix}$$

A few things to notice about the Leslie matrix:

- Birth rates, or fecundity, are always in the first row.
- The subdiagonal contains survival fractions, or probabilities.
- The other entries are all zero.
- n-by-n Leslie matrices have n eigenvalues and n eigenvectors.

These things hold for **all age-structured population models, regardless of the number of age classes, or whether we make b or s conditional on other stuff.** (They are not true for stage-structured models, which we'll get to later.) What additional stuff could b or s be conditional on? They could be driven by the environment,

$$n(t+1) = \mathbf{L}_t n(t)$$

Or by the population itself (i.e., density dependence).

$$n(t+1) = \mathbf{L}_n n(t)$$

You can imagine turning some or all of the elements of \mathbf{L} into functions of population size and the environment. You could certainly make an argument that both birth and survival have a density dependence, either positive (Allee effects) or negative (logistic growth). We're not going to go that far in this class. Matrix population modeling is a whole class and field unto itself, adhering closely to Hal Caswell's phenomenal book on the topic. It also requires linear algebra techniques that not everyone came into this class with. But I very highly recommend that book if you are interested in learning more. We are going to unpack a few more properties of the simplest (exponential growth) model, and then discuss stage structure.

One thing we might want to do, similar to when we studied geometric growth in discrete time without age structure, is get a general function for $n(t)$ as a function of the initial population size $n(0)$. This allows us to explore age structure mathematically without stepping through every generation one at a time. Without age structure, we had $N(t+1) = RN_t$ and solved this to find that $N(t) = R^t N_0$. Similarly, here, we have $n(t+1) = Ln(t)$ and can deduce that $n(t) = \mathbf{L}^t n(0)$. (Convince yourself of this by looking at the equations above.)

From here, we want a simple expression for $\mathbf{L}^t n(0)$ in terms of the parameters of the Leslie matrix that we can use to understand this model better. In particular, for large values of t, it would be nice to not have to figure out what L is raised to a large exponent. To simplify this, we will now take advantage of a property of n-by-n Leslie matrices, which is that they have n eigenvalues and eigenvectors. We will denote these λ and \vec{n} . If this doesn't sound familiar to

you from linear algebra, or even if it does, here's how they are defined: for a n-by-n matrix \mathbf{M} , there are at most n (and for Leslie matrices, *exactly* n) sets of scalars and vectors for which the following is true:

$$\mathbf{M}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$$

Remember, we can't just divide both sides by $\vec{\mathbf{v}}$, which is a column vector. In this class we won't get into actually *finding* eigenvalues and eigenvectors, but we will use them to gain inference into age- and stage-structured models.

Here, you need to know a little trick of eigenvectors. For a set of eigenvectors, e.g., $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, you can state any arbitrary vector \mathbf{x} as $\mathbf{x} = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2$, where a_1, a_2 are constants you can solve for computationally.

We care about this rule because it makes the equation above, $n(t) = \mathbf{L}^t n(0)$, much more tractable. Let's call the eigenvectors of \mathbf{L} $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ and the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Recalling that $n(0) = [n_0 + n_1 + n_2]$, we can say that

$$n(0) = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + a_3\vec{\mathbf{v}}_3$$

Substituting this in gives us

$$n(t) = \mathbf{L}^t n(0) = \mathbf{L}^t(a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + a_3\vec{\mathbf{v}}_3) = \mathbf{L}^t a_1\vec{\mathbf{v}}_1 + \mathbf{L}^t a_2\vec{\mathbf{v}}_2 + \mathbf{L}^t a_3\vec{\mathbf{v}}_3$$

And recall that above we defined the eigenvalue as the λ for which $\mathbf{M} \vec{\mathbf{v}} = \lambda \vec{\mathbf{v}}$? We can therefore substitute:

$$n(t) = \mathbf{L}^t n(0) = a_1\lambda_1^t\vec{\mathbf{v}}_1 + a_2\lambda_2^t\vec{\mathbf{v}}_2 + a_3\lambda_3^t\vec{\mathbf{v}}_3$$

Now, all of a sudden, we have the entire right-hand side of this equation expressed in terms of constants except for t. All we need to do is raise λ to t. **As t gets large, what single value in this equation drives the population size?**

It's the biggest $|\lambda|$ —the **dominant eigenvalue**, which is often denoted λ_1 . As time goes on, $n(t) \approx a_1\lambda_1^t\vec{\mathbf{v}}_1$.

Stable age distribution

Stage structure

After all this, we still aren't quite ready to build a model of the loggerhead sea turtle. That's because we've been talking about age structure, but with turtles (and many species), what really matters is *stage structure*—how many individuals are at each life stage. These aren't the same thing. Recall how turtles can be reproductive adults for decades, but their habitat use changes a lot early in life? Not only are life stages important, but turtles stay at them for multiple years, and different durations depending on the stage. We could build a giant matrix for every possible age of turtles, but we also often don't *know* their age, only their stage. (this is true for many surveyed populations).

Stochasticity

Dispersal and spatial population dynamic models / metapopulations

Interacting populations

Coexistence

Applications of population modeling to management and research

Other topics to add:

- R^*
- Alternative stable states and hysteresis
- Resilience, multiple stressors, ball-and-cup theory