

1. Let $f(n) = n^3 - 2n^2 + n$ and $g(n) = -n^2$ show whether $f(n) = \Omega(g(n))$ is true (or) false justify your answer.

1.) * express the growth rates:

$$f(n) = n^3 - 2n^2 + n$$

$$g(n) = -n^2.$$

* dominat terms:

for large n , the dominant terms in $f(n)$ is n^3 because it grows faster than the terms.

* the dominate term $g(n)$ is $-n^2$.

compare $f(n)$ and $g(n)$:

* As n grows large, n^3 will dominate $-2n^2 + n$

* thus $f(n) \approx n^3$ to large n .

* we need to show that $f(n) > c \cdot (-n^2)$ for some $c > 0$ and sufficiently large n .

choose a suitable constant c :

* let's choose $c = 1$. then the inequality becomes:

$$n^3 - 2n^2 + n \geq -n^2$$

simplify the inequality:

* for large n , n^3 will dominate $-n^2 + n$.

* Hence, $n^3 - 2n^2 + n$ is positive for sufficiently large n .

$$f(n) = \Omega(g(n))$$

$f(n) = n^3 - 2n^2 + n$ is indeed $\Omega(g(n) = -n^2)$.

determine whether $h(n) = n \log n + n$ is in $\Theta(n \log n)$ prove a rigorous proof for your conclusion.

express the functions:

$$h(n) = n \log n + n$$

$$g(n) = n \log n.$$

dominant terms:

* for large n , $n \log n$ dominates n

* Hence, $h(n) \approx n \log n$ for large n .

from the inequalities:

* we need to show there exist positive constants c_1, c_2 and n_0 such that for all $n \geq n_0$: $c_1 \cdot n \log n \leq n \log n + n \leq c_2 \cdot n \log n$.

prove the lower bound:

* $n \log n$ is always less than or equal to $n \log n + n$, because n is positive for $n > 0$.

* therefore, $c_1 = 1$ works and we have

$$n \log n \leq n \log n + n.$$

prove the upper bound:

* to show $n \log n + n \leq c_2 \cdot n \log n$ for large n , we write the inequality as: $n \log n + n \leq c_2 \cdot n \log n$.

* divide both sides by n :

$$\log n + 1 \leq c_2 \log n.$$

* we can choose $c_2 = 2$. then for large n :

$$\log n + 1 \leq 2 \log n.$$

combine the bounds:

we have shown that

$$n \log n \leq n \log n + n \leq 2n \log n.$$

this holds for all $n \geq n_0$.

$$\therefore h(n) = n \log n + n = \theta(n \log n).$$

3. solve the following recurrence relation and find the order of growth for solutions.

$$T(n) = 4T(n/2) + n^2, T(1) = 1.$$

Master's theorem:

* $f(n) = O(n^c)$ where $c < \log_b a$ then $T(n) = O(n^{\log_b a})$

* $f(n) = O(n^{\log_b a})$, then $T(n) = O(n^{\log_b a} \log n)$

* $f(n) = \Omega(n^c)$ where $c > \log_b a$, and if $a f(\frac{n}{b}) \leq K f(n)$ for some $K < 1$ and sufficiently large n , then $T(n) = O(f(n))$.

Apply:

* calculate $\log_b a$:

$$\log_b a = \log_2 4 = 2$$

* compare $f(n)$ with $n^{\log_b a}$.

$$f(n) = n^2$$

$$n^{\log_b a} = n^2$$

Since $f(n) = n^2$ and $n^2 = n^{\log_b a}$, we are in case 2 of the master theorem, which states:

if $f(n) = O(n^{\log_b a})$, then $T(n) = O(n^{\log_b a} \log n)$.

$$T(n) = 4T(n/2) + n^2 \text{ is } T(n) = O(n^2 \log n)$$

4. Given an array of $[4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7]$ integers.

Find the maximum and minimum product that can be obtained by multiplying two integers from the array.

a) sort the array:

* sorting help us easily find the largest and smallest elements.

* sorted array

$$[-9, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]$$

maximum product:

* two largest positive numbers in the sorted array 10 and 11

$$10 \times 11 = 110$$

* the two most negative numbers are -9 and -8

$$(-9) \times (-8) = 72$$

* so, the maximum product is 110.

* the two smallest numbers are -9 and -8.

$$(-9) \times (-8) = 72$$

* the most negative product can also be obtained by multiplying the smallest negative number -9 and the largest number 11.

$$(-9) \times 11 = -99$$

* so, the minimum product is -99.

conclusion:

maximum product: 110

minimum product: -99.

demonstrates: binary search method to search key = 23, from the array $\text{arr}[] = \{1, 2, 5, 8, 12, 16, 23, 38, 56, 72, 9\}$

Initial state:

$$\text{low} = 0$$

$$\text{high} = 9$$

* first iteration:

$$\text{calculate mid} = \left\lfloor \frac{0+9}{2} \right\rfloor = 4$$

* compare $\text{arr}[\text{mid}]$ with key:

$$\text{arr}[4] = 16$$

since $16 < 23$, set $\text{low} = 5$.

$$\text{calculate mid} = \frac{5+9}{2} = 7$$

* compare $\text{arr}[\text{mid}]$ with key.

$$\text{arr}[7] = 56$$

since $56 > 23$, set $\text{high} = 6$.

third iteration:

$$\text{mid} = \frac{5+6}{2} = 5$$

* compare $\text{arr}[\text{mid}]$ with key:

$$\text{arr}[5] = 23$$

since $23 == 23$.

* key element is found at index 5.