Hamiltonian Cycles and the Knight's Tour

Anne-Marie Freudenthal

Outline

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What is graph theory?

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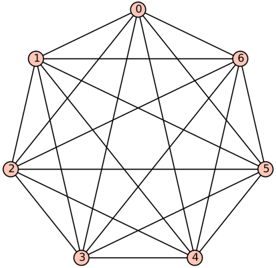

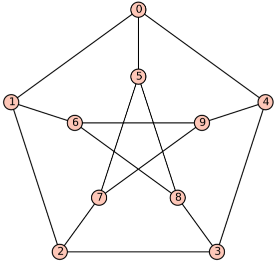

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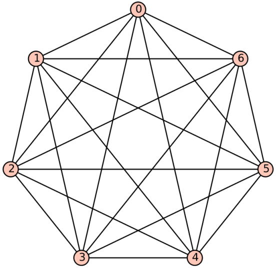

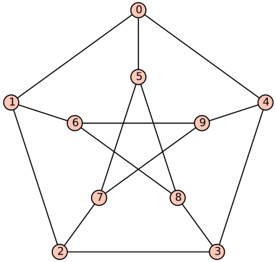

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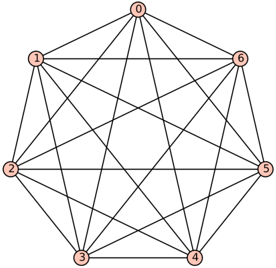

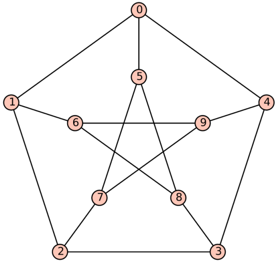

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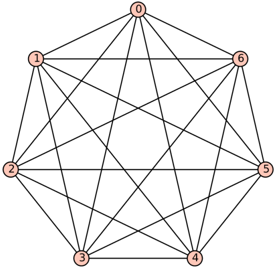

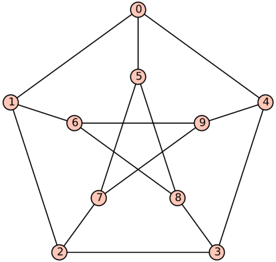

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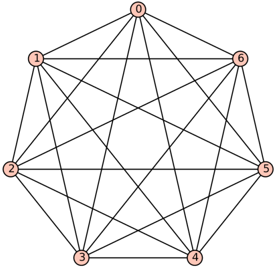

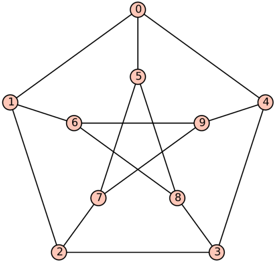

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Why is graph theory important?

Because it's fun! And there are many applications.

Applications of Graph Theory

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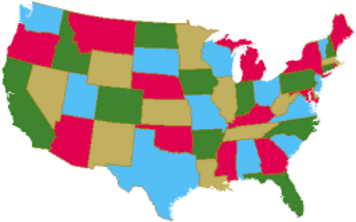
Coloring Maps

Every geographical map can be colored in with just four colors.



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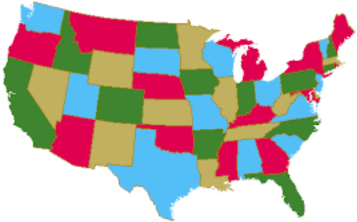

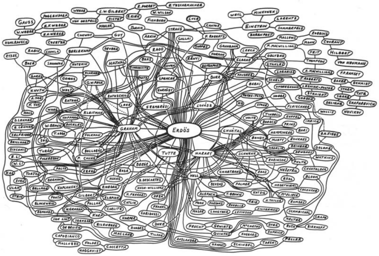

Interactions between different parties

Links between webpages, social graphs, Erdös number.

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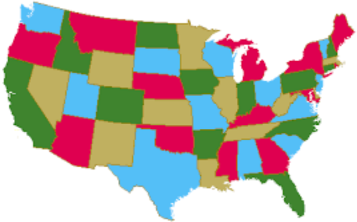

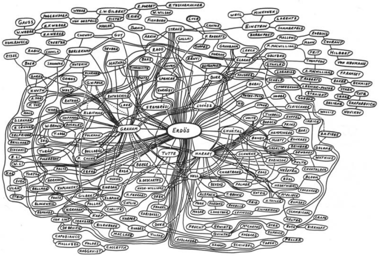

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Route Finding

Graph theory can be used to help determine the most efficient way to get from one place to another.

Some Definitions

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Hamiltonian Cycle: A cycle which visits every vertex in the graph.

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Theorem: If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph G has at most $|S|$ components.

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Theorem: If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph G has at most $|S|$ components.

That is to say: If you remove a set of vertices S, then if G has a Hamiltonian cycle, it will have at most same number of components left over as there were vertices in S.

Application of Hamiltonian Cycles

Traveling Salesman
The salesman needs to travel efficiently from city to city. How does he plan his route?

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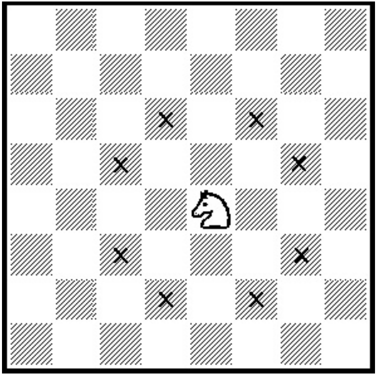
Knight's Tour
Moving the knight around chessboards of different sizes to see if he can move to every square exactly once and return to his starting point.

Knight's Tour

You're a knight. The queen has threatened to behead you unless you slay the bad guys on every square of a chessboard as quickly as possible.

What's the best way to get this done?

Knight Movement



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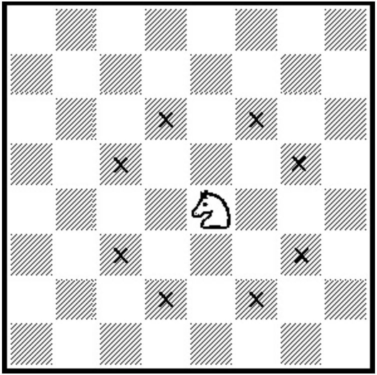
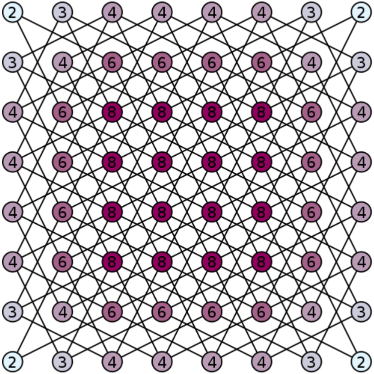


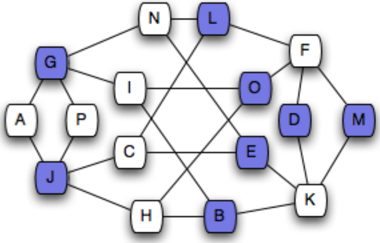


Some Proofs

Assume the chessboard is *mxn* in size.

Proposition: When *m* and *n* are both odd, the board will not have a knight's tour.
Proof: We know that the knight can only move from black to white or white to black. Any closed tour will need to visit an equal number of white and black squares, but because the total size of the board is odd, there can't be an equal number of black and white squares. A closed tour cannot be constructed on a board with odd size.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P



More Proofs

Proposition: When $m = 1$ or 2, a knights tour does not exist.
Proof: For $m = 1$ and $m = 2$ it's obvious that there isn't enough room for the knight to move around.

Proposition: When $m = 3$ and $n = 6$, or 8, there is no knight's tour.
Proof: For each case a set S of vertices can be chosen, which, when deleted from the graph, leaves more than $|S|$ components.

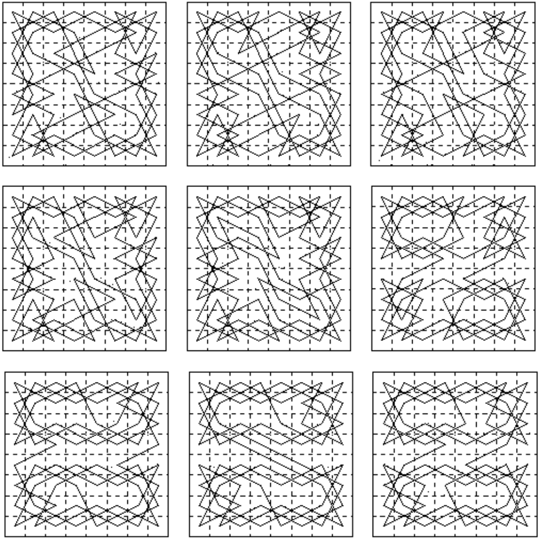
When $m = 4$

Proposition: When $m = 4$, no matter what n equals, there is no knight's tour.
Proof: Same idea! Choose a set S of vertices which, when deleted from the graph, leave more than $|S|$ components. Using 4x6 as a base case, by induction, any chessboard of size 4xn will not have a knight's tour.

Schwenk's Theorem: An mxn chessboard with $m \leq n$ has a knight's tour unless:

a) m and n are both odd;
b) $m = 1$, 2, or 4;
c) $m = 3$ and $n = 4$, 6, or 8.

8x8 chessboard



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Images

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