# Inflation and GDP Dynamics in Production Networks: A Sufficient Statistics Approach*

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February 28, 2022

*preliminary and incomplete draft*

**Abstract**

We provide closed-form solutions for inflation and GDP *dynamics* in multi-sector New Keynesian economies with arbitrary input-output linkages. In particular, we show that a sufficient statistic for dynamics of sectoral prices is the *principal square root* of the Leontief matrix that is appropriately adjusted for the sectoral frequencies of price adjustments. We measure this sufficient statistic for the U.S. and find a significant quantitative role for production networks in the propagation of monetary and sectoral supply shocks. In response to monetary shocks, input-output linkages significantly increase the persistence of inflation and lead to a GDP response that is more than three times the response in an economy with a horizontal production network. In response to a negative supply shock in the "computers and electronics industry," input-output linkages propagate this shock by increasing the cost of downstream sectors, which acts like a markup shock to these sectors. These spillover effects generate a bigger and more persistent increase in aggregate inflation relative to an economy with a horizontal production network, which leads to about a three times bigger contraction in aggregate GDP. We show that a monetary policy response which fully offsets the inflationary effect of this negative sectoral supply shock on impact generates an aggregate GDP contraction that is more than twice as large as the case with no such policy response.

*JEL Codes:* E32, E52, C67
*Key Words:* Production networks; Multi-sector model; Sufficient statistics; Inflation dynamics; Real effects of monetary policy; Sectoral shocks

*We thank Jennifer La'O, Iván Werning, and Mike Woodford for helpful comments. We are grateful to Raphael Schoenle for sharing data on the sectoral frequencies of price adjustments. We also thank Edson Wu for excellent research assistance.

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# 1 Introduction

Multi-sector models with production networks have been widely used to study positive and normative questions in macroeconomics, especially in static environments. Yet, some of the most pressing questions about inflation in the wake of recent macroeconomic events, such as supply chain disruptions, require dynamic predictions. For instance, how long do we expect high inflation to last in response to supply chain disruptions and monetary policy shocks?

This paper presents a dynamic model where production across several sectors has input-output linkages, and firms make forward-looking nominal pricing decisions under staggered pricing-setting. We do not restrict the input-output linkages and allow for heterogeneity in frequencies of price adjustments and household expenditure shares across sectors and model both aggregate and sector-specific shocks. In this environment, we provide analytical results and sufficient statistics on how the dynamics of sectoral and aggregate inflation rates are affected by production networks in response to aggregate and sectoral shocks, which plays a crucial role in the eventual propagation to aggregate GDP. We show, both analytically and quantitatively, how the interaction of sticky prices with a dynamic setting provides new insights into the transmission of shocks in economies with arbitrary input-output linkages.

On the analytical side, we provide closed-form results on dynamic responses of inflation and GDP to sectoral and aggregate shocks, where, in particular, we show theoretically how input-output linkages affect the persistence of macroeconomic variables. First, we show that the equilibrium sectoral prices are fully characterized by a system of differential equations that involve sectoral price gaps (the deviation of sectoral prices from their counterfactual flexible price benchmarks). Second, we show that a Leontief matrix appropriately adjusted for sectoral price adjustments frequencies governs the role of these sectoral price gaps in influencing inflation dynamics. This characterization leads to our main theoretical result: the sufficient statistics for the dynamic responses of inflation and GDP to sectoral and aggregate shocks are the *principal square root* of this frequency adjusted Leontief matrix and the vector of household's expenditure shares of sectoral goods.

Building on this main theoretical result, we then show several other substantive analytical results related to the real effects of monetary policy and the spillover effects of shocks that originate in one sector and how they permeate through the economy over time. First, we provide an analytical result on monetary non-neutrality in this economy: The cumulative impulse response of GDP to an aggregate monetary policy shock is fully determined by the principal square root of the frequency-adjusted Leontief matrix.

Next, we provide an analytical result on the effects of persistent, but transitory, sectoral shocks in this economy. We show that the principle square root of the frequency adjusted Leontief matrix also governs the impulse response functions of aggregate inflation and GDP to these shocks once they have propagated through the *inverse* Leontief matrix as in static models without nominal rigidities.

Our analytical solutions, therefore, shed light the on two separate roles of the Leontief matrix in the propagation of sectoral shocks: while the inverse Leontief matrix governs the propagation of sectoral shocks through the network at any given time—as in static models—the frequency adjusted Leontief matrix governs their propagation across sectors *through* time and thus directly affects the persistence of inflation and GDP responses to these shocks.[1] Accordingly, the endogenous transition dynamics of the production network economy are captured by the frequency-adjusted Leontief matrix.

To study the quantitative importance of production networks in governing the dynamic response of the U.S. economy to monetary and sectoral supply shocks, we use input-output tables as well as data on consumption shares and frequencies of price adjustments across different sectors in the U.S. to construct our sufficient statistics. Comparing our calibrated U.S. economy with a counterfactual horizontal economy with no input-output linkages, we first show that taking input-output linkages into account leads to a more persistent aggregate inflation response to the monetary policy shock. Next, input-output linkages also lead to 3.45 times higher real effects of the monetary policy shock (as measured by the cumulative impulse response of GDP) than in a counterfactual horizontal economy. We interpret such larger effects as coming through strategic complementarities in pricing decisions introduced by production networks, which increase aggregate inflation persistence by slowing down price adjustment across sectors, leading to more significant real effects.

Next, motivated by recent supply chain disruptions during the pandemic, we simulate the effects of a negative supply shock in the "computers and electronics industry." Compared to the effects predicted purely based on the expenditure share of this industry, in our calibrated U.S. economy, we find that such a negative sectoral supply shock leads to a much bigger and more persistent increase in aggregate inflation. In the presence of input-output linkages, a negative TFP shock to this sector propagates downstream through the network by increasing the input prices of downstream sectors, leading to a higher aggregate inflation response on impact. More importantly, since the increase in input prices is endogenously persistent due to price stickiness, an initial rise in the price of this sector slowly permeates downstream and causes a significantly more persistent aggregate inflation response relative to the economy with no input-output linkages.

Associated with this effect on aggregate inflation is a contraction in aggregate GDP, which is also quite persistent over time. There is a negative effect on aggregate GDP because, for sectors that increase their prices without directly facing a negative TFP shock, demand declines, and in the aggregate, value-added production falls. Put another way, with nominal expenditures constant as monetary policy has not changed, higher inflation leads to lower GDP. Such spillover effects on other sectors mean that the negative supply shock in this sector propagates as a *markup shock* in *all* downstream sectors and leads to a more significant effect on aggregate inflation together with an aggregate GDP contraction.

To illustrate further how sectoral TFP shocks manifest as markup shocks in the rest of the economy,

[1]Thus, in the flexible price version of our model, where there are no endogenous dynamics, or in the steady-state of our model, consistent with previous studies, the inverse Leontief matrix fully characterizes the effects of sectoral shocks.

we compare our calibrated economy with a horizontal economy with no input-output linkages and show that the negative effect on aggregate GDP of this sectoral shock is lower compared to our calibrated economy, as now there is no mechanism that leads to a propagation of the negative sectoral shock as a markup shock to other sectors. In particular, the cumulative impulse response of GDP to the negative sectoral shock is 2.73 times larger in our baseline economy. This shows the role played by production networks in amplifying the negative aggregate GDP effects of negative sectoral shocks by affecting aggregate inflation dynamics.

Having shown how a negative TFP shock in the computers and electronics sector propagates like a markup shock above, we then consider a monetary policy response to this sectoral shock. In particular, we model a case where money supply contracts precisely by the amount necessary to fully stabilize the inflationary effects of the shock on impact. We find that such a policy is non-trivially contractionary over time for aggregate GDP as the cumulative impulse response of GDP is 2.43 times lower than the case where the monetary policy does not respond to this shock.

**Related Literature.** Our paper is related to several papers in the literature. Carvalho, Lee, and Park (2021); La'O and Tahbaz-Salehi (2021); Rubbo (2020); Pasten, Schoenle, and Weber (2020) recently introduce input-output linkages in models with heterogeneous degrees of nominal rigidities across sectors and are closely related to our paper. Our main contribution to this strand of the literature is to provide an analytical characterization of sectoral inflation dynamics together with sufficient statistics for the dynamics of inflation and GDP to both aggregate and sectoral shocks. Moreover, in addition to studying the effects of an aggregate monetary policy shock, we also study the effects of a sectoral supply shock, which is motivated by recent supply chain issues during the pandemic. These analytical characterizations then inform our quantitative results, as they show how the persistence of aggregate inflation gets affected by production networks, which has important implications for effects on aggregate GDP.

Our results in connecting the effects of input-output linkages to real effects of monetary policy shocks are related to the original insight of Basu (1995) in a model of round-about production technologies. Basu (1995) showed that when the final good produced by all firms in the economy is also used as input for production, such an input-output linkage increases the real effects of monetary policy by introducing strategic complementarities in price-setting decisions.[2] In more recent work, La'O and Tahbaz-Salehi (2021) show that this insight extends to economies with arbitrary input-output linkages in a static framework. We build on these insights and show how the dynamic effects of strategic complementarities due to arbitrary input-output linkages are summarized by a sufficient statistic that is the principal square root of a frequency-adjusted Leontief matrix.

In deriving sufficient statistics for real effects of monetary policy shocks in sticky-price models

[2] Woodford (2003) contains a comprehensive discussion of various sources of strategic complementarities in New Keynesian models, such as round-about production networks, segmented labor markets, and non-CES demand that lead to variable markups, which can all contribute to higher real effects of monetary policy shocks.

with strategic complementarities, our paper connects to recent work by Wang and Werning (2021) and Alvarez, Lippi, and Souganidis (2022). Our main contribution to this strand of the literature is that we consider a multi-sector New Keynesian model with input-output linkages. Our environment is, however, simpler on other dimensions: it does not model oligopolistic behavior within a sector (as in Wang and Werning, 2021) or feature menu costs (as in Alvarez, Lippi, and Souganidis, 2022). Our sufficient statistic, which is the principal square root of the frequency-adjusted Leontief matrix, is in close correspondence to, and complements, the ones in Wang and Werning (2021); Alvarez, Lippi, and Souganidis (2022), as they all share an underlying transmission mechanism based on strategic complementarities in pricing decisions.

Finally, there is by now a rich literature in static settings that considers various formulations of exogenous production networks in macroeconomic models. For example, Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), Baqaee and Farhi (2020), and Bigio and La'O (2020) are important contributions and Carvalho (2014); Carvalho and Tahbaz-Salehi (2019) are comprehensive surveys of the literature. These papers study how sectoral shocks propagate to the aggregate economy to cause business cycles and how (if at all) they affect aggregate total factor productivity or the labor wedge. Our sectoral shock results are related to these ideas, but we focus on how they affect the dynamics of aggregate inflation and, thereby, the response of GDP in an economy with nominal pricing frictions. Yet other papers, such as Taschereau-Dumouchel (2020), consider endogenous production networks and study phenomena such as cascades. We use exogenous production networks, thereby using a simpler setting, but we study a dynamic model with sticky prices.[3]

**Outline.** The paper is organized as follows. Section 2 presents the general framework and the environment of our model. Section 3 includes our main theoretical results and derives sufficient statistics for the responses of sectoral and aggregate prices and aggregate GDP. Section 4 constructs our sufficient statistics using U.S. data and presents quantitative results on inflation and GDP responses to monetary and sectoral TFP shocks. Section 5 concludes.

# 2 Model

## 2.1 Environment

Time is continuous and is indexed by $t \in \mathbb{R}_+$. The economy consists of a representative household, monetary and fiscal authorities and $n$ industries indexed by $i \in [n] \equiv \{1, \dots, n\}$ with input-output linkages. Each industry $i \in [n]$ consists of a continuum of monopolistically competitive producers and a competitive final good producer with a CES production function.[4]

[3] A recent application of production networks in a real model with dynamics is Liu and Tsyvinski (2021), which derives analytical results for the dynamics of the real variables in a model with adjustment costs on inputs. We instead consider an economy with nominal rigidities but no adjustment cost. The model environments are therefore inherently different.

[4] Assuming the existence of this final good producer in each industry, which has no value-added to the economy, is convenient as it allows the industry to produce a unified good that is purchased by other industries. See La'O and Tahbaz-Salehi (2021) for a similar assumption.

**Household.** The representative household demands the final goods produced by each industry, supplies labor in a competitive market, and holds money as well as nominal bonds with nominal yield $i_t$. Household's instantaneous preferences over aggregate consumption $C$, and labor supply $L$ is given by $U(C) - V(L)$, where $U$ and $V$ are both strictly increasing, $U$ is strictly concave, $V$ is strictly convex, and they both satisfy Inada conditions. Formally, given initial bond and money holdings $\{B_{0^-}, M_{0^-}\}$, the household solves

$$\max_{\{(C_{i,t})_{i\in[n]}, L_t, B_t, M_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \left[U(C_t) - V(L_t)\right] \mathrm{d}t \tag{1}$$

$$\text{subject to} \quad \sum_{i\in[n]} P_{i,t}C_{i,t} + \dot{B}_t + \dot{M}_t \leq W_t L_t + i_t B_t + M_t + \text{Profits}_t + T_t \tag{2}$$

$$C_t \equiv \Phi(C_{1,t}, \ldots, C_{n,t}) \tag{3}$$

$$\sum_{i\in[n]} P_{i,t}C_{i,t} \leq M_t \tag{4}$$

Here, $C_t$ is an aggregate consumption index that is a function of the household's consumption bundle from all industries $(C_{i,t})_{i\in[n]}$ and is defined by the aggregator function $\Phi(.)$ that is homogenous of degree one, and increasing in each of its arguments. Moreover, $L_t$ is the household's labor supply at wage $W_t$, $P_{i,t}$ is the price of the final good of sector $i$, $\text{Profits}_t$ is the aggregate profits of all monopolistic firms in the economy rebated to the household and $T_t$ is a lump-sum transfer, possibly zero, used by the government to finance taxes or subsidies on firms. Finally, similar to La'O and Tahbaz-Salehi (2021), we model demand for money with a cash-in-advance constraint in Equation (4) where the velocity of money per unit of time is constant and normalized to 1. This approach significantly simplifies the aggregate dynamics in the model.[5]

**Monetary and Fiscal Policy.** We assume that the monetary authority controls the supply of money over time, $(M_t^s)_{t\geq 0}$, where we will later model a monetary shock an *unexpected* one-time increase in $M_t^s$, which is a common approach utilized in the literature (e.g., Golosov and Lucas, 2007; Alvarez, Le Bihan, and Lippi, 2016; Wang and Werning, 2021).

Finally, we allow for the fiscal authority to tax (subsidize) intermediate firms' sales in every sector $i$ at a deterministic but possibly time-varying rate $\tau_{i,t}$. These wedges can be used to alleviate aggregate and relative distortions from market power and sticky prices or be set to zero. We assume these taxes or subsidies are lump sum transferred to the household at every instant and nominal bonds are at zero net supply.

**Final Good Producers.** Every industry $i$ has a competitive final good producer that buys from a continuum of intermediate firms, indexed by $ij$, $j \in [0, 1]$ in the sector and produces a final sectoral good with a CES production function with elasticity of substitution $\sigma_i > 1$. Formally, this producer's

[5] An alternative approach that would yield a similar money demand function is use money in the utility function. See, e.g., Wang and Werning (2021); Alvarez, Lippi, and Souganidis (2022).

problem, at any given point in time, is

$$\max_{(Y^d_{ij,t})_{j\in[0,1]}} P_{i,t}Y_{i,t} - \int_0^1 P_{ij,t}Y^d_{ij,t}\mathrm{d}j \quad \text{subject to} \quad Y_{i,t} = \left[\int_0^1 (Y^d_{ij,t})^{1-\sigma_i^{-1}}\mathrm{d}j\right]^{\frac{1}{1-\sigma_i^{-1}}} \tag{5}$$

where $Y^d_{ij,t}$ is the final good producer's demand for variety $ij$, $Y_{i,t}$ is its total production of the final good, $P_{i,t}$ is the price of $i$'s final good—which is taken as given by the producer—and $P_{ij,t}$ is the variety $ij$'s price at time $t$. It follows that the final producer's demand for variety $ij$ is:

$$Y^d_{ij,t} = \mathcal{D}(P_{ij,t}/P_{i,t}; Y_{i,t}) \equiv Y_{i,t}\left(\frac{P_{ij,t}}{P_{i,t}}\right)^{-\sigma_i} \quad \text{where} \quad P_{i,t} = \left[\int_0^1 P_{ij,t}^{1-\sigma_i}\mathrm{d}j\right]^{\frac{1}{1-\sigma_i}} \tag{6}$$

Since the final good producer is a price-taker that produces with a constant returns to scale production function, it has zero value added to the economy and its existence is merely a point of convenience for aggregation: it allows industries to have a unified good, which is then the input to all intermediate firms in other industries that use $i$'s output for production.

**Intermediate Goods Producers.** Every industry $i$ has a unit measure of intermediate goods producers, indexed by $ij$, $j \in [0,1]$ that supply to their industry's final good producer and use labor and other final goods as inputs. More precisely, an intermediate producer $ij$'s production function is given by

$$Y^s_{ij,t} = Z_{i,t}F_i(L_{ij,t}, X_{ij,1,t}, \ldots, X_{ij,n,t}) \tag{7}$$

where $Z_{i,t}$ is a Hicks-neutral sector-specific productivity level that is deterministic but possibly changes over time, $L_{ij,t}$ is the firm's labor demand from the competitive labor market and $X_{ij,k,t}$ is the firm's demand for the final good of sector $k$. Moreover, the function $F_i : \mathbb{R}^n \to \mathbb{R}$ is homogenous of degree one and satisfies proper Inada conditions so that demand for all inputs are strictly positive at all prices. Thus, the firm's total cost for producing output $Y$, given the vector of aggregate wage $W_t$ and all sectoral prices, $\mathbf{P}_t \equiv (W_t, P_{i,t})_{i\in[n]}$, is:

$$\begin{aligned}\mathcal{C}_i(Y; \mathbf{P}_t, Z_{i,t}) \equiv & \min_{(L_{ij,t}, X_{ij,k,t})_{k\in[n]}} W_tL_{ij,t} + \sum_{k\in[n]} P_{k,t}X_{ij,k,t} \\ & \text{subject to} \quad Z_{i,t}F_i(L_{ij,t}, X_{ij,1,t}, \ldots, X_{ij,n,t}) \geq Y\end{aligned} \tag{8}$$

Intermediate goods producers are monopolistically competitive and set their prices under a Calvo-type sticky prices friction, where the opportunities for changing prices are i.i.d. across all firms and arrive according to Poisson processes with intensity $\theta_i$. Given the cost function $\mathcal{C}_i(Y^s_{ij,t}; \mathbf{P}_t, Z_{i,t})$ in Equation (8) and its demand $Y^d_{ij,t}$ from the final goods producer in Equation (6), a firm $ij$ that has received the opportunity to change its price at time $t$ chooses its *reset price*, which we denote by $P^{\#}_{ij,t}$, to maximize the expected net present value of its profits until the next price change taking into account that they will have to meet the implied demand at each point in time in between the two price changes:[6]

$$P^{\#}_{ij,t} \equiv \arg\max_{P_{ij,t}} \int_0^\infty \theta_i e^{-(\theta_{i,t} h + \int_0^h i_{t+s}\mathrm{d}s)} \left[(1-\tau_i) P_{ij,t} \mathcal{D}(P_{ij,t}/P_{i,t+h}; Y_{i,t+h}) - \mathcal{C}_i(Y^s_{ij,t+h}; \mathbf{P}_{t+h}, Z_{i,t+h})\right] \mathrm{d}h$$

$$\text{subject to} \quad Y^s_{ij,t+h} \geq \mathcal{D}(P_{ij,t}/P_{i,t+h}; Y_{i,t+h}), \quad \forall h \geq 0 \tag{9}$$

where $\theta_i e^{-\theta_i h}$ is the density of time until next price change (captured here by $h$), $e^{-\int_0^h i_{t+h}\mathrm{d}s}$ is the discount rate based on nominal rates for profits at time $t+h$, and $\tau_{i,t}$ is the constant tax rate on intermediate firms' sales in sector $i$. Note that the only source of dynamic considerations for the firms is stickiness in prices. Were prices flexible, maximizing the net present value of profits for the firms would be equivalent to maximizing the static profits within every instant $t$. Let us define the firm's *desired* price, denoted by $P^*_{ij,t}$, as the price that the firm would choose under such flexible prices. Then, $P^*_{ij,t}$ solves:

$$P^*_{ij,t} \equiv \arg\max_{P_{ij,t}} (1-\tau_{i,t}) P_{ij,t} \mathcal{D}(P_{ij,t}/P_{i,t}; Y_{i,t}) - \mathcal{C}_i(Y^s_{ij,t}; \mathbf{P}_t, Z_{i,t}) \tag{10}$$

$$\text{subject to} \quad Y^s_{ij,t} \geq \mathcal{D}(P_{ij,t}/P_{i,t}; Y_{i,t})$$

### 2.2 Equilibrium Definition

Having specified the actions and objectives of all the agents, we now formally define the equilibrium of this economy given the allocation of goods and the set of prices.

**Definition 1.** A **sticky price equilibrium** for this economy is

(a) an allocation for the household, $\mathcal{A}_h = \{(C_{i,t})_{i\in[n]}, C_t, L_t, B_t, M_t\}_{t\geq 0} \cup \{B_{0^-}, M_{0^-}\}$,

(b) an allocation for all firms $\mathcal{A}_f = \{(Y_{i,t}, Y^d_{ij,t}, Y^s_{ij,t}, L_{ij,t}, X_{ij,k,t})_{i\in[n],j\in[0,1]}\}_{t\geq 0}$,

(c) a set of monetary and fiscal policies $\mathcal{A}_g = \{(M^s_t, T_t, \tau_{1,t}, \ldots, \tau_{n,t})_{t\geq 0}\}$,

(d) and a set of prices $\mathcal{P} = \{(P_{i,t}, P_{ij,t})_{i\in[n],j\in[0,1]}, W_t, P_t, i_t\}_{t\geq 0} \cup \{(P_{ij,0^-})_{i\in[n],j\in[0,1]}\}$

such that

1. given $\mathcal{P}$ and $\mathcal{A}_g$, $\mathcal{A}_h$ solves the household's problem in Equation (1),
2. given $\mathcal{P}$ and $\mathcal{A}_g$, $\mathcal{A}_f$ solves the final goods producers problems in Equation (5), intermediate goods producers' cost minimization in Equation (8) and their pricing problem in Equation (9),
3. labor, money, bonds and final sectoral goods markets clear and government budget constraint is satisfied:

$$M_t = M^s_t, \quad B_t = 0, \quad L_t = \sum_{i\in[n]} \int_0^1 L_{ij,t}\mathrm{d}j, \quad \sum_{i\in[n]} \int_0^1 (1-\tau_i) P_{ij,t} Y_{ij,t}\mathrm{d}j = T_t \quad \forall t \geq 0 \tag{11}$$

$$Y_{k,t} = C_{k,t} + \sum_{i\in[n]} \int_0^1 X_{ij,k,t}\mathrm{d}j \quad \forall k \in [n], \quad \forall t \geq 0 \tag{12}$$

Furthermore, to understand how the stickiness of prices will affect and distort the equilibrium allocations, we will make comparisons between the equilibrium defined above and its *flexible-price*

[6] This is the common New Keynesian assumption that while prices are fixed, firms produce enough to meet demand. See Woodford (2003) or Galí (2015) for discussions of this assumption.

analog, formally defined below.

**Definition 2.** A **flexible price equilibrium** is an equilibrium defined similar to Definition 1 with the only difference that intermediate goods producers' prices solve the flexible price problems specified in Equation (10) instead of the sticky price problem in Equation (9).

Finally, since we have defined our economy without any aggregate or sectoral shocks, we will pay specific attention to *stationary* equilibria, which we define below.

**Definition 3.** A **stationary equilibrium** for this economy is an equilibrium as in Definition 1 or Definition 2 with the additional requirement that all the allocative variables in the household's allocation in $\mathcal{A}_h$ and the sectoral production of final good producers $(Y_{i,t})_{i\in[n]}$ as well as the distributions of the allocative variables for intermediate good producers in $\mathcal{A}_i$ are constant over time.[7]

# 3 Theoretical Results

This section presents our main theoretical results and derives our sufficient statistics for inflation and GDP dynamics for small perturbations around a stationary efficient equilibrium.

## 3.1 Log-Linearized Approximation of Optimality Conditions

We start by deriving log-linear approximations to optimality conditions of the model presented in Section 2. These optimality conditions are discussed in Appendix B. We derive these log-linear approximations around a stationary equilibrium that is efficient—i.e., all exogenous variables are constant over time and taxes are set to fully offset firms' market power. Moreover, following Golosov and Lucas (2007), we assume that household's preferences are such that $U(C) = \log(C)$ and $V(L) = L$, which simplify our analytical representations significantly. Finally, notation-wise, small letters in this section correspond to logs of their corresponding variables in capitalized letters in Section 2.

**Firms.** For any given sector $i$, three price indices summarize the behavior of sectoral prices: a *desired* price, $p^*_{i,t}$, which is the firms' optimal price were prices flexible; a *reset* price, $p^{\#}_{i,t}$, which is the optimal price of price-setting firms at $t$ under price stickiness; and an aggregate *sectoral* price, $p_{i,t}$, which is the average price of all the firms in sector $i$ at time $t$.

Let us start with desired prices, which depend on the marginal cost of firms in sector $i$, $mc_{i,t}$, and on a wedge, $\omega_{i,t}$, which can denote deviations in markups or taxes. With input-output linkages and labor as the only production factor, $mc_{i,t}$ depends on the aggregate wage, $w_t$, the aggregate sectoral prices, and a Hicks-neutral productivity measure, $z_{i,t}$. Formally, with all prices denoted in logs:

$$p^*_{i,t} \equiv \omega_{i,t} + mc_{i,t}, \quad mc_{i,t} \equiv \alpha_i w_t + \textstyle\sum_{k\in[n]} a_{ik} p_{k,t} - z_{i,t}, \quad \omega_{i,t} \equiv \log\left(\frac{\sigma_i}{\sigma_i - 1} \times \frac{1}{1-\tau_{i,t}}\right) \tag{13}$$

[7] Note that the production and input demands of individual intermediate goods producers do not need to be time-invariant in the stationary equilibrium, but their distributions do.

where $\alpha_i > 0, a_{ik} \geq 0, \forall k$ and $\alpha_i + \sum_{k\in[n]} a_{ik} = 1$, denoting constant returns to scale in production, and

$$\mathbf{\Omega} \in \mathbb{R}^{n\times n} \quad \text{where} \quad [\mathbf{\Omega}]_{ik} = a_{ik} \tag{14}$$

corresponds to the *cost-based input-output matrix* under firms' optimal expenditure shares.

Given desired prices, firms' optimal reset price in sector $i$ is an average of all the *future* desired prices, weighted by the probability density of the time between price changes.

$$p^{\#}_{i,t} = (\rho + \theta_i) \int_0^\infty e^{-(\rho+\theta_i)h} p^*_{i,t+h} \mathrm{d}h \tag{15}$$

where $\theta_i$ is the frequency of price changs in sector $i$ and $\rho > 0$ corresponds to the steady-state interest rate (the representative household's discount rate). Finally, given sector $i$'s initial aggregate price $p_{i,t}$ is an average of the *past* reset prices and initial price $p_{0-}$ (left limit of prices at $t = 0$), weighted by the density of time between price changes:

$$p_{i,t} = \theta_i \int_0^t e^{-\theta_i h} p^{\#}_{i,t-h} \mathrm{d}h + e^{-\theta_i t} p_{i,0-} \tag{16}$$

**Aggregate Price and GDP.** The household's compensated demand for sectoral goods defines the aggregate price index, $p_t$, as an average of sectoral prices, weighted by the household's expenditure shares:

$$p_t = \textstyle\sum_{i\in[n]} \beta_i p_{i,t}, \quad \text{with} \quad \sum_{i\in[n]} \beta_i = 1 \tag{17}$$

where $\boldsymbol{\beta} = (\beta_i)_{i\in[n]} \in \mathbb{R}^n$ denotes expenditure shares from sectoral goods in the point of approximation (stationary efficient allocation) that sum up to one. given the preferences in Golosov and Lucas (2007), household's total expenditures clear the money market at a given path of money supply, $(m^s_t)_{t\geq 0}$, and labor supply is fully elastic so that GDP, $y_t$, is equal to the real wage:

$$y_t = w_t - p_t \quad \text{(fully elastic labor supply)} \qquad m^s_t = p_t + y_t \quad \text{(money supply = demand)} \tag{18}$$

**Equilibrium.** Given our log-linear approximation, it is worth to reiterate our definition of the equilibrium in the previous section in this context.

**Definition 4.** Given a path for the primitives, $(\boldsymbol{\omega}_t, \boldsymbol{z}_t, m^s_t)_{t\geq 0}$, a **sticky price equilibrium** for the log-linearized economy is a path for GDP, wage and prices, $\vartheta \equiv \{y_t, w_t, p_t, (p^*_{i,t}, p^{\#}_{i,t}, p_{i,t})_{i\in[n]}\}_{t\geq 0}$, such that given a vector of initial sectoral prices, $\mathbf{p}_{0-} = (p_{i,0-})_{i\in[n]}$, $\vartheta$ solves the optimality conditions in Equations (13) and (15) to (18). Finally, an equilibrium is **stationary** if the real GDP and relative sectoral prices are constant over time.

**Flexible Prices and the Flexible Price Level of GDP.** Consider a counterfactual economy where prices are flexible so that all sectoral prices are equal to *desired* prices in Equation (13). Letting $\mathbf{p}^f_t \in \mathbb{R}^n$ denote such prices, we have:

$$\mathbf{p}^f_t = w_t \boldsymbol{\alpha} + \mathbf{\Omega} \mathbf{p}^f_t + \boldsymbol{\omega}_t - \boldsymbol{z}_t \quad \Rightarrow \quad \mathbf{p}^f_t = m^s_t \mathbf{1} + \mathbf{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t) \tag{19}$$

where $\boldsymbol{\alpha} \equiv (\alpha_i)_{i\in[n]} \in \mathbb{R}^n$, $\mathbf{1}$ is the vector of ones in $\mathbb{R}^n$, and $\boldsymbol{\Psi} \equiv (\mathbf{I} - \boldsymbol{\Omega})^{-1}$ is the infamous *inverse* Leontief matrix.[8] We can then define the *flexible price GDP*, $y_t^f$, as the level of output that prevails in this counterfactual economy. Using Equations (18) and (19), we arrive at:

$$y_t^f = m_t^s - \boldsymbol{\beta}^\intercal \mathbf{p}_t^f = \boldsymbol{\lambda}^\intercal (\boldsymbol{z}_t - \boldsymbol{\omega}_t), \qquad \boldsymbol{\lambda} \equiv \boldsymbol{\Psi}^\intercal \boldsymbol{\beta} \tag{20}$$

where the vector $\boldsymbol{\lambda}$ is known as the Domar weights under the efficient allocation—i.e., the ratio of total sales of all sector relative to the household's total nominal expenditures.[9] Therefore, Equation (20) echoes Hulten's theorem (Hulten, 1978): up to a first-order approximation *around the efficient allocation*, log-changes in the aggregate TFP is equal to Domar-weighted log-changes in the sectoral productivities.[10]

**Evolution of Prices and Sectoral Phillips Curves with Sticky Prices.** Given a vector of initial prices, $\mathbf{p}_{0^-}$, let $\mathbf{p}_t \equiv (p_{i,t})_{i\in[n]}$, $\mathbf{p}_t^\# \equiv (p_{i,t}^\#)_{i\in[n]}$ and $\mathbf{p}_t^* \equiv (p_{i,t}^*)_{i\in[n]}$ denote the vectors of sectoral aggregate, reset and desired prices, respectively. Then, an immediate implication of Equation (13) is that we can write desired prices in the following vector form:

$$\mathbf{p}_t^* = (\mathbf{I} - \boldsymbol{\Omega})\mathbf{1} w_t + \boldsymbol{\Omega} \mathbf{p}_t + \boldsymbol{\omega}_t - \boldsymbol{z}_t \tag{21}$$

This equation hints at La'O and Tahbaz-Salehi (2021)'s insight as it shows how the input-output matrix $\boldsymbol{\Omega}$ plays the role of a matrix of *strategic complementarities* across the economy for the vector of sectoral prices.

Similarly, we can write reset and aggregate sectoral prices in Equations (15) and (16) in vector form. It follows that these prices uniquely solve the following systems of differential equation:

$$\vec{\boldsymbol{\pi}}_t^\# \equiv {}^{\mathrm{d}\mathbf{p}_t^\#}/_{\mathrm{d}t} = (\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^\# - \mathbf{p}_t^*), \quad \text{with boundary condition} \quad \lim_{t\to\infty} e^{-(\rho\mathbf{I}+\boldsymbol{\Theta})t}\mathbf{p}_t^\# = 0, \tag{22}$$

$$\vec{\boldsymbol{\pi}}_t \equiv {}^{\mathrm{d}\mathbf{p}_t}/_{\mathrm{d}t} = \boldsymbol{\Theta}(\mathbf{p}_t^\# - \mathbf{p}_t), \quad \text{with boundary condition} \quad \mathbf{p}_0 = \mathbf{p}_{0^-} \tag{23}$$

where $\boldsymbol{\Theta} = \mathrm{diag}(\theta_i) \in \mathbb{R}^n$ is the diagonal matrix whose $i$'th diagonal entry is the frequnecy of price adjustments in sector $i$, and the boundary conditions on the right hand side are chosen so that the solution of the differential equations coincides with Equations (15) and (16).[11] With one further step, we can combine these two equations to a differential equation just in terms of $\mathbf{p}_t$.

[8] The inverse Leontief matrix exists because the spectral radius of $\boldsymbol{\Omega}$ is strictly less than one by the assumption that $0 < \alpha_i \le 1, \forall i$ (see the discussion below Remark 1 for more details). Also, in deriving Equation (19) we have utilized the fact that $\boldsymbol{\alpha} = (\mathbf{I} - \boldsymbol{\Omega})\mathbf{1}$.

[9] Under inefficient allocations, Domar weights can be defined either based on costs or sales of industries, each of which represent different roles in aggregation (Baqaee and Farhi, 2020). Under the efficient allocation, however, the two are the same as all firms make zero profits with constant returns to scale and no distortions.

[10] Equation (16) shows that the changes in log-GDP are the same as the change in aggregate TFP because, with log preferences on consumption, the income and substitution effects on labor supply fully offset one another and labor does not change. To see this, recall that in the efficient economy with no nominal rigidities $W_t/P_t = V'(L_t)/U'(C_t) = C_t/L_t$. Combining these equations and assuming $U(C) = \log(C)$, we arrive at $V'(L_t)L_t = 1$, meaning that labor supply is fixed and independent of productivity. Thus, all changes in GDP are due to changes in aggregate TFP.

[11] Throughout this draft, we frequently use exponential function of square matrices, defined by its corresponding power series: $\forall \mathbf{A} \in \mathbb{R}^{n\times n}$, $e^{\mathbf{A}} \equiv \sum_{k=0}^{\infty} \mathbf{A}^k/k!$, which is well-defined because these power series are always convergent.

**Proposition 1.** Given $\mathbf{p}_{0-}$, sectoral prices evolve according to the following **sectoral Phillips curves**:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\pi}}_t = \rho\vec{\boldsymbol{\pi}}_t - \boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})(\mathbf{I}-\boldsymbol{\Omega})(\mathbf{p}_t^f - \mathbf{p}_t) \tag{24}$$

Equation (24) corresponds to the economy's sectoral Phillips curves because it relates the sectoral inflation rates to the deviations of these prices from their flexible counterparts, which move one to one with *sectoral consumption gaps* when labor supply is fully elastic.[12] This observation gives a unique interpretation to the matrix $\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})(\mathbf{I}-\boldsymbol{\Omega})$, which we summarize in the following remark.

**Remark 1.** The matrix $\boldsymbol{\Gamma} \equiv \boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})(\mathbf{I}-\boldsymbol{\Omega})$, denoted as the **frequency-adjusted Leontief matrix** is the **slope of sectoral Phillips curves** in matrix form, which is uniquely determined by the Leontief matrix, $\mathbf{I}-\boldsymbol{\Omega}$, adjusted by a quadratic form of price adjustment frequencies, $\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})$.

Intuitively, dynamics of prices in a production network should depend on how fast prices adjust to shocks in each sector (here captured by $\boldsymbol{\Theta}$) and how shocks propagate through the input-output linkages (captured by the Leontief matrix). Proposition 1 and Remark 1 formalize this intuition and show that the exact form through which these two mechanisms interact is summarized by a particular combination that is captured by $\boldsymbol{\Gamma}$, which rescales every row of the Leontief matrix by the squared frequency of price changes in the corresponding sector.

Due to its direct correspondance to the slopes of sectoral Phillips curves, the frequency-adjusted Leontief matrix, and as we will see, its principal square root, are intimately connected to dynamics of output and inflation and plays a fundamental role in our analysis. To briefly discuss the existence and properties of these matrices it is useful to note that under the assumption that $\boldsymbol{\Omega}$ is positive and has row sums strictly less than one, $\mathbf{I}-\boldsymbol{\Omega}$ is a nonsingular $M$-matrix—which implies that the *inverse* Leontief matrix, defined as $\boldsymbol{\Psi} \equiv (\mathbf{I}-\boldsymbol{\Omega})^{-1}$, exists, has positive entries, and all of its eigenvalues have positive real parts (see, e.g., Carvalho and Tahbaz-Salehi, 2019, p. 639). It is straightforward to show that $\boldsymbol{\Gamma}$ is also an $M$-matrix.[13] Moreover, what is perhaps less commonly known but is crucial to our analysis is that every nonsingular $M$-matrix has exactly one $M$-matrix as its square root (Alefeld and Schneider, 1982, Theorem 5), which is also its principal square root. Formally, we have the following lemma.

**Lemma 1.** Let $\sqrt{\boldsymbol{\Gamma}}$ denote the principal square root of the frequency-adjusted Leontief matrix. Then, $\sqrt{\boldsymbol{\Gamma}}$ is an $M$-matrix. In particular, all the eigenvalues of $\sqrt{\boldsymbol{\Gamma}}$ have positive real parts.

A second observation about Equation (24) is that it is a system of second-order differential equations in $\mathbf{p}_t$ that with its two boundary conditions—$\mathbf{p}_0 = \mathbf{p}_{0-}$ and non-explosive prices—uniquely pins down

[12] To see this, let $\mathbf{c}_t$ and $\mathbf{c}_t^f$ denote the log-consumption vectors from sectoral goods in the flexible and sticky price economies, respectively. Then, given that in both economies, nominal demand of the household is equal to money supply by Equation (18), we have:

$$m^s\mathbf{1} = \mathbf{p}_t^f + \mathbf{c}_t^f + \log(\boldsymbol{\beta}) = \mathbf{p}_t + \mathbf{c}_t + \log(\boldsymbol{\beta}) \qquad \Rightarrow \qquad \mathbf{p}_t^f - \mathbf{p}_t = \mathbf{c}_t - \mathbf{c}_t^f \tag{25}$$

[13] By Theorem 2.3 in (Berman and Plemmons, 1994, p. 134, condition $N_{38}$), $\boldsymbol{\Gamma}$ is an $M$-matrix if it is inverse-positive; i.e., $\boldsymbol{\Gamma}^{-1}$ exists and $\boldsymbol{\Gamma}^{-1} \geq 0$ elementwise. Since $\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})$ is invertible because $\theta_i > 0, \forall i$, and $\mathbf{I}-\boldsymbol{\Omega}$ is also invertible with $(\mathbf{I}-\boldsymbol{\Omega})^{-1} = \sum_{n=0}^{\infty}\boldsymbol{\Omega}^n$, $\boldsymbol{\Gamma}^{-1}$ exists and is the infinite sum of positive matrices: $\boldsymbol{\Gamma}^{-1} = \sum_{n=0}^{\infty}\boldsymbol{\Omega}^n(\rho\mathbf{I}+\boldsymbol{\Theta})^{-1}\boldsymbol{\Theta}^{-1} \geq 0$.

the path of sectoral prices as a function of the flexible price path $(\mathbf{p}_t^f)_{t\geq 0}$. This leads to the following remark.

**Remark 2.** All primitives $(\boldsymbol{\omega}_t, \boldsymbol{z}_t, m_t^s)_{t\geq 0}$ affect dynamics of prices *only* through flexible prices, $(\mathbf{p}_t^f)_{t\geq 0}$.

The observation in Remark 2 demonstrates the power of expressing inflation dynamics in terms of *sectoral price gaps* relative to a counterfactual equilibrium with flexible prices because it implies that solving for the dynamics of prices for a given path of $\mathbf{p}_t^f$ is equivalent to having characterized impulse response functions of *all* the prices in the economy to all three types of TFP, markup, and monetary shocks in a unified framework. For an arbitrary path of flexible prices, the following Proposition characterizes these dynamics and shows that $\sqrt{\mathbf{\Gamma}}$ is a sufficient statistic for how the vector of sectoral prices evolve over time.

**Proposition 2.** Suppose $\mathbf{p}_t^f$ is piece-wise continuous and is bounded,[14] and let $\rho = 0$.[15] Then, given $\mathbf{p}_t^f$ and a vector of initial prices $\mathbf{p}_{0^-}$, $\sqrt{\mathbf{\Gamma}}$ is a sufficient statistic for dynamics of prices and the unique non-explosive solution to Equation (24) is given by:[16]

$$\mathbf{p}_t = \underbrace{e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{p}_{0^-} + \sqrt{\mathbf{\Gamma}}e^{-\sqrt{\mathbf{\Gamma}}t}\int_0^t \sinh(\sqrt{\mathbf{\Gamma}}h)\mathbf{p}_h^f \mathrm{d}h}_{\text{inertial effect of past prices due to stickiness}} + \underbrace{\sqrt{\mathbf{\Gamma}}\sinh(\sqrt{\mathbf{\Gamma}}t)\int_t^\infty e^{-\sqrt{\mathbf{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h}_{\text{forward looking effect of future prices}} \tag{26}$$

While Proposition 2 characterizes the dynamics of prices for any piece-wise continuous and bounded path of $\mathbf{p}_t^f$, we are particularly interested in characterizing impulse response functions of prices, inflation and GDP to monetary, TFP and markup (wedge) shocks. We would like our characterization to be general enough to capture two sets of properties. First, we would like to capture both permanent changes (for monetary policy shocks) and persistent but transitory changes (for TFP or markups shocks) in flexible prices. Second, we want the path for $\mathbf{p}_t^f$ to be general enough to allow for heterogeneous shocks across sectors so that we can capture both sectoral and aggregate shocks.

To this end, let us consider an economy in its steady-state at $t = 0^-$ (left limit at $t = 0$) meaning that prices are at their flexible level: $\mathbf{p}_{0^-} = \mathbf{p}_{0^-}^f = m_{0^-}^s\mathbf{1} + \mathbf{\Psi}(\boldsymbol{\omega}_{0^-} - \boldsymbol{z}_{0^-})$. For this economy, consider the following paths for money supply, productivities and wedges:

$$m_t^s = m_{0^-} + \delta_m, \quad \forall t \geq 0, \qquad \boldsymbol{\omega}_t - \boldsymbol{z}_t = \boldsymbol{\omega}_{0^-} - \boldsymbol{z}_{0^-} + e^{-\mathbf{\Phi}t}\boldsymbol{\delta}_z, \quad \forall t \geq 0 \tag{27}$$

where $\delta_m \in \mathbb{R}$ captures a permanent change in money supply—with a positive $\delta_m$ denoting an expansionary monetary shock—and $\boldsymbol{\delta}_z \in \mathbb{R}^n$ is a vector of sectoral shocks to TFP or wedges that decay

[14] In deterministic environments like ours, piece-wise continuity guarantees that the $\mathbf{p}_t^f$ is Riemann integrable and is without significant loss of generality our unexpected shocks will introduce only a finite number of jumps in flexible prices. In stochastic environments this assumption would need to be adapted for stochastic integrals. Moreover, the boundedness assumption is also without loss of generality because we assume zero trend inflation and this assumption guarantees existence of the Laplace transform which we use in the proof of this proposition. With trend inflation, boundedness should be replaced with $\mathbf{p}_t^f$ being of an exponential order.

[15] The assumption of $\rho = 0$ is not necessary for analytical tractability but simplifies the analytical representation of the solution significantly. Moreover, given the small value of this parameter in a model that is calibrated to the long-run interest rates at a monthly frequency ($\rho = 0.96^{(1/12)} \approx 0.001$), it is without a significant loss of generality.

[16] The hyperbolic sine of a square matrix $\mathbf{A}$ is defined as $\sinh(\mathbf{A}) \equiv (e^{\mathbf{A}} - e^{-\mathbf{A}})/2$.

at the rate $\mathbf{\Phi} \in \mathbb{R}^{n\times n}$—with positive $\boldsymbol{\delta}_z$ denoting a vector negative TFP or a positive wedge shocks to sectors. Here, the matrix $\mathbf{\Phi}$ is a positive diagonal matrix whose diagonal entries capture the decay rates of the TFP/wedge shocks to all sectors. Combining these paths, we arrive at the following expression for the dynamics of flexible prices:

$$\mathbf{p}_t^f = \mathbf{p}_{0-}^f + \delta_m \mathbf{1} + \mathbf{\Psi} e^{-\mathbf{\Phi} t}\boldsymbol{\delta}_z \tag{28}$$

Note that this formulation for $\mathbf{p}_t^f$ satisfies all the conditions stated above: while $\delta_m$ captures monetary shocks, different combinations of $\boldsymbol{\delta}_z$ and $\mathbf{\Phi}$ capture arbitrarily persistent shocks to either aggregate or sectoral TFP/wedges. The following Theorem derives the dynamics of sectoral prices for this path.

**Proposition 3.** Suppose $\mathbf{p}_t^f$ follows the dynamics specified in Equation (28). Let $\rho = 0$ and assume $\mathbf{\Phi}^2$ and $\mathbf{\Gamma} = \mathbf{\Theta}^2(\mathbf{I} - \mathbf{\Omega})$ have no common eigenvalues.[17] Then, the dynamics of sectoral inflation rates, $\vec{\boldsymbol{\pi}}_t$, are uniquely determined by the principal square root $\sqrt{\mathbf{\Gamma}}$ and a matrix $\mathbf{A}$:

$$\mathbf{p}_t = \mathbf{p}_{0-}^f + \underbrace{(\mathbf{I} - e^{-\sqrt{\mathbf{\Gamma}}t})\mathbf{1}\delta_m}_{\text{response to monetary shock}} + \underbrace{\mathbf{A}e^{-\mathbf{\Phi}t}\boldsymbol{\delta}_z - e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{A}\boldsymbol{\delta}_z}_{\text{response to TFP/wedge shock(s)}} \tag{29}$$

where $\mathbf{A}$ is the unique solution to the Sylvester equation $\mathbf{\Gamma}\mathbf{A} - \mathbf{A}\mathbf{\Phi}^2 = \mathbf{\Theta}^2$.

Having shown how sectoral prices evolve when aggregate and sectoral shocks hit the economy, we now present the solution in the following Collorary for aggregate GDP and inflation, which are a focus of the paper as they illustrate the key aggregate implications.

**Corollary 1.** GDP and aggregate inflation dynamics are given by

$$y_t = y_{0-}^f + \underbrace{\boldsymbol{\beta}^\intercal e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{1}\delta_m}_{\text{response to monetary shock}} + \underbrace{\boldsymbol{\beta}^\intercal(e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{A} - \mathbf{A}e^{-\mathbf{\Phi}t})\boldsymbol{\delta}_z}_{\text{response to TFP/wedge shock(s)}} \tag{30}$$

$$\pi_t = \underbrace{\boldsymbol{\beta}^\intercal(\sqrt{\mathbf{\Gamma}}e^{-\sqrt{\mathbf{\Gamma}}t})\mathbf{1}\delta_m}_{\text{response to monetary shock}} + \underbrace{\boldsymbol{\beta}^\intercal(\sqrt{\mathbf{\Gamma}}e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{A} - \mathbf{A}e^{-\mathbf{\Phi}t}\mathbf{\Phi})\boldsymbol{\delta}_z}_{\text{response to TFP/wedge shock(s)}} \tag{31}$$

Next, as a summary statistic for GDP effects, we present in the Corollary below the solution for the cumulative impulse response (CIR) of GDP. This is the main object of interest in the literature on sufficient statistics for non-neutrality of monetary policy shocks. Here, we provide such a result for both monetary policy shock as well as sectoral TFP shocks.

**Corollary 2.** The cumulative impulse response (CIR) of GDP is given by

$$\mathrm{CIR}_y(\delta_m, \boldsymbol{\delta}_z) \equiv \int_0^\infty (y_t - y_{0-}^f)\mathrm{d}t = \underbrace{\boldsymbol{\beta}^\intercal\sqrt{\mathbf{\Gamma}}^{-1}\mathbf{1}\delta_m}_{\text{response to monetary shock}} + \underbrace{\boldsymbol{\beta}^\intercal(\sqrt{\mathbf{\Gamma}}^{-1}\mathbf{A} - \mathbf{A}\mathbf{\Phi}^{-1})\boldsymbol{\delta}_z}_{\text{response to TFP/wedge shock(s)}} \tag{32}$$

[17] This is not a very restrictive assumption but allows for a tremendous amount of tractability in our analysis by ruling out issues that arise from repeated eigenvalues. To see why this assumption is not very restrictive, we can think of $\mathbf{\Gamma}$, which we will measure in the data, as being drawn from a distribution that is absolutely continuous with respect to the Lebesgue measure—i.e., it has a density with respect to this measure (by Radon-Nikodym Theorem). Then the eigenvalues of $\mathbf{\Gamma}$ are almost surely different from the eigenvalues of $\mathbf{\Phi}^2$.

Finally, to compare our results with the production networks literature in static settings, as well as for a reference point, we present in the Corollary below the solution for the cumulative impulse response (CIR) of GDP in a counterfactual case of fully flexible prices. In such an environment, there are no internal dynamics in the model and the CIR of GDP is given by the Domar weights, as in the literature. We note that the Domar weights depend on the inverse Leontief matrix, while our model solutions above in the case of sticky prices depend on the Leontief matrix (appropriately adjusted for frequency of price adjustment) for transition dynamics.

**Corollary 3.** The cumulative impulse response (CIR) of the flexible prices GDP to sectoral TFP shocks $\boldsymbol{\delta}_z$ is given by

$$\mathrm{CIR}_y^f(\boldsymbol{\delta}_z) \equiv \int_0^\infty (y_t^f - y_{0-}^f)\mathrm{d}t = -\underbrace{\boldsymbol{\lambda}^\intercal}_{\text{Domar weights}} \times \underbrace{\boldsymbol{\Phi}^{-1}\boldsymbol{\delta}_z}_{\text{cumulative TFP response}} \tag{33}$$

To see how Equation (33) follows from Equation (32), note that the CIR of flexible price GDP is the limit of CIR of GDP in the sticky price economy when $\theta_i \to \infty, \forall i \in [n]$—i.e., frequencies of price adjustments are arbitrarily large. In this case, $\sqrt{\boldsymbol{\Gamma}}^{-1} \to \mathbf{0}$ (as $\boldsymbol{\Gamma}^{-1} = (\mathbf{I} - \boldsymbol{\Omega})^{-1}\boldsymbol{\Theta}^{-2}$) and, more importantly, $\mathbf{A}$ tends to the inverse Leontief matrix, $\boldsymbol{\Psi}$.[18] Equation (33) then follows from the fact that $\boldsymbol{\beta}^\intercal\mathbf{A} \to \boldsymbol{\beta}^\intercal\boldsymbol{\Psi} = \boldsymbol{\lambda}$.

# 4 Quantitative results

We now present quantitative results on dynamic responses of inflation and GDP to aggregate and sectoral shocks, using U.S. data to construct our sufficient statistics. We then do counterfactual experiments to show the role of various model ingredients that affect the propagation of shocks.

**Sufficient Statistics Construction From Data.** Proposition 3 shows that the sufficient statistics for inflation and output dynamics in response to shocks in our model are the frequency-adjusted Leontief matrix, as given by $\boldsymbol{\Gamma} \equiv \boldsymbol{\Theta}^2(\mathbf{I} - \boldsymbol{\Omega})$, and the vector of consumption expenditure shares across sectors, as given by $\boldsymbol{\beta}$. Here, we briefly describe here how we construct $\boldsymbol{\Gamma}$ and $\boldsymbol{\beta}$ using detailed sectoral U.S. data. Further details are in the Appendix.

First, we use the input-output (IO) tables from the BEA to construct the input-output linkages across sectors, given by the matrix $\boldsymbol{\Omega}$; the consumption expenditure shares across sectors, given by the vector $\boldsymbol{\beta}$; and the sectoral labor shares, given by the vector $\boldsymbol{\alpha}$. We construct these objects using the IO tables from 2019 at the summary-level disaggregation, excluding the government sectors. This leads to 66 sectors in our sample. In particular, to construct $\boldsymbol{\Omega}$, we use both the make and use IO tables. The use IO table also provides data on compensation of employees, which we use to construct the sectoral labor shares $\boldsymbol{\alpha}$. Moreover, we also construct the consumption expenditure shares across sectors, $\boldsymbol{\beta}$, using the use IO table, where the consumption share for a given sector is given by the personal consumption

[18] To see this, note that $A$ is the solution of the Sylvester equation $\boldsymbol{\Gamma}\mathbf{A} - \mathbf{A}\boldsymbol{\Phi}^2 = \boldsymbol{\Theta}^2$. Multiplying this equation by $\boldsymbol{\Theta}^{-2}$ from left we have: $(\mathbf{I} - \boldsymbol{\Omega})\mathbf{A} - \boldsymbol{\Theta}^{-2}\mathbf{A}\boldsymbol{\Phi}^2 = \mathbf{I}$. Taking the limit as $\boldsymbol{\Theta} \to \infty$ we have $\mathbf{A} \to \boldsymbol{\Psi}$.

expenditure on that sector over total personal consumption expenditure.

Next, we construct the diagonal matrix $\mathbf{\Theta}^2$, whose diagonal elements are the squared frequency of price adjustment in each sector, using data on 341 sectors from Pasten, Schoenle, and Weber (2020). We match data from Pasten, Schoenle, and Weber (2020) on frequency of price changes with the 2002 concordance table between the IO industry codes and the NAICS codes. Then, we match the resulting table with the 2012 concordance table between the IO industry codes and the NAICS codes. The last step is needed to get the link between the frequency of price adjustment at the detail level disaggregation, which is a finer disaggregation, and the summary level disaggregation, which is what we use in the paper.[19] The weighted average frequency of price changes across sectors is 0.185 (0.204), before (after) our continuous time transformation.[20]

**Dynamic Responses to a Monetary Policy Shock.** For our calibrated economy, in Figure 1, we show impulse responses of aggregate inflation and GDP to an expansionary monetary policy shock. The shock size is chosen such that it leads to a 1 percent increase in inflation on impact.

After increasing by 1 percent on impact, inflation slowly goes back to steady-state, as endogenous state variables evolve over time and input-output linkages and differential price stickiness across sectors slow down the inflation adjustment. More importantly, there are substantial real effects on GDP of this shock, as seen by the large initial effect on GDP of around 10 percent. Critically, these effects on GDP are persistent and decay slowly, and the cumulated impulse response of GDP is about 130 percent.

To put these magnitudes in context, as well as to illustrate the roles of model ingredients that lead to such substantial real effects, we now do various counterfactual experiments. In these counterfactuals, we keep the initial impact on inflation the same at 1 percent.[21] In Figure 2, we compare our calibrated baseline economy to a counterfactual horizontal economy. This counterfactual economy thus does not feature any input-output linkages and labor is the only input in production. The cumulated impulse response of GDP is 3.45 times larger in our baseline economy, which shows the role played by production networks in amplifying the real effects of monetary policy shocks. Strategic complementarity in price setting that arises through input-output linkages, as we pointed out while discussing the analytical results, is the driving force for this result.

In addition to input-output linkages, another source that amplifies the real effects of monetary policy in our model is heterogenous price stickiness across sectors, as pointed out by Carvalho (2006) in a New Keynesian model without production networks. To investigate the role of this channel, in Figure 3, we compare our calibrated baseline economy to a counterfactual economy that has homogenous price stickiness across sectors. We calibrate the common frequency of price changes in this economy to

[19] We linked the frequency of price adjustment data with the 2002 concordance table first because Pasten, Schoenle, and Weber (2020) used the IO tables for 2002.

[20] As we work in continuous time, we calculate the continuous time counterpart of the frequency of price adjustment in the data. Thus, let fpa be the frequency of price adjustment in Pasten, Schoenle, and Weber (2020). Then, the frequency of price adjustment used in this paper is $\theta = -\log(1 - fpa)$.

[21] The monetary policy shock size is therefore different across the baseline and the counterfactual cases. The cumulated impulse response of aggregate inflation corresponds to the monetary policy shock size in our model.

be the same as the weighted average of the frequency of price changes across sectors in our baseline economy.[22] This economy therefore, still features input-output linkages, and through that, strategic complementarity in price setting. The cumulated impulse response of GDP is 1.862 times larger in our baseline economy, which shows that heterogeneity in price stickiness across sectors does play a quantitatively important role in magnifying monetary non-neutrality. The importance of this channel however, is not as high as that of input-output linkages that arise through our modelling of production networks.

Finally, shutting down both channels, in Figure 4, we compare our calibrated baseline economy to a counterfactual horizontal economy that also has homogenous price stickiness across sectors. This economy can be considered a textbook multi-sector New Keynesian model. The results show that compared to this economy, the cumulated impulse response of GDP is 5.179 times larger in our baseline economy.[23]

**Dynamic Responses to a Sectoral Supply Shock.** Motivated by supply chain issues during the ongoing pandemic, we now consider implications of a negative supply (TFP) shock in the computers and electronics sector. For our calibrated economy, in Figure 5 we show impulse responses of aggregate inflation, aggregate GDP, and sectoral inflation to a negative TFP shock in the computers and electronics sector. The shock size is chosen such that it leads to a 1 percent increase in inflation on impact in that sector, as shown in Figure 5. The average duration of the sectoral shock is 6 months in this experiment.

Given that the consumption expenditure share of this sector is 0.00663, in a basic model with no input-output linkages, we would expect the impact effect on aggregate inflation to be 0.00663. Moreover, in such an economy, we would expect the aggregate inflationary impact to last for about 7 months, the same duration for which there is inflationary effect on sectoral inflation.[24] Instead, what we see in Figure 5 is that the initial impact effect is almost double that and the inflationary impact lasts longer, for 10 months.[25] The reason aggregate inflation increases by more and more persistently is that input-output linkages mean that sectors that are not directly hit by the negative supply shock see an increase in prices for their inputs that is provided by the computer and electronics sector. Moreover, this then has a ripple effect on input prices throughout the economy. This effect induces many sectors to increase prices for their goods, leading to higher sectoral inflation, and which shows up in higher aggregate inflation.

Associated with this effect on aggregate inflation is a contraction in aggregate GDP, which is

[22]As we mentioned above in the data description, the weighted average frequency of price adjustment is 0.2048.

[23]Note that even in this textbook type multi-sector New Keynesian model, inflation effects are persistent because our modelling of monetary policy introduces an endogenous state variable in the model. This is a standard approach in the literature on sufficient statistics of monetary policy shocks, but is a different approach than assuming a Taylor rule where the interest rate feedback coefficient is on inflation.

[24]The frequency of price adjustment in this sector is 0.0928, which is lower than the weighted average frequency of price adjustment across sectors of 0.2048.

[25]Inflation eventually goes negative, but that is by construction, as in our model, in response to transitory shocks, the cumulated impulse response of aggregate inflation is zero.

quite persistent over time. The cumulated impulse response of GDP is -1.169 percentage, as shown in Figure 5. In this sense, the negative sectoral supply shock acts like an aggregate markup shock, leading on an aggregate inflationary effect that goes together with a contraction in aggregate GDP. The reason there is a negative effect on aggregate GDP is that for sectors that increase prices without having directly faced a negative TFP shock, demand for their goods declines and sectoral GDP falle. Then in the aggregate, value-added production also declines. Put another way, with money supply constant, higher aggregate inflation leads to lower GDP.

To further show clearly these input-output mechanisms that lead to such spillover effects on sectoral inflation and GDP, Figure 6, we show how the impact effect on sectoral inflation and CIR of sectoral GDP correlate with the input share of the computers and electronics sector in various sectors in the economy. [26] As is clear, higher the input share of the computers and electronics sector, higher the secotral inflation and more negative the CIR of sectoral GDP.

Morever, to show that the increased duration of aggregate inflationary effects and the aggregate contraction are not a numerical artifact, in Figure 7, we show impulse responses when the average duration of the sectoral shock is 12 months. As is clear, now as well, while inflationary effects in the sector last for 11 months, inflationary effects on the aggregate are longer, for 15 months. Moreover, the cumulative impulse response of GDP is now proportionately bigger, at -2.513 percentage.

Next, like with the aggregate monetary policy shock, to put magnitudes in context and to illustrate the roles of model ingredients that lead to the magnified aggregate inflationary effect, we now do a counterfactual experiment. In this experiment, we keep the initial impact on sectoral inflation the same, at 1 percent, and consider a counterfactual horizontal economy. In Figure 8, we compare our calibrated baseline economy to this counterfactual horizontal economy. In this horizontal economy, aggregate inflation increases by less on impact and it is also less persistent. That is, as can be seen, aggregate inflation just simply follows the path of sectoral inflation of this sector. The reason is that without input-output linkages, sectoral inflation dynamics become de-coupled across sectors. Moreover, note how secotral inflation dynamic are indistinguishable between the two economies, and yet aggregate inflation dynamics are quite different between them.[27]

As a result, the negative effect on aggregate GDP is lower in the horizontal economy, as now there is no mechanism that leads to a propagation of the negative sectoral shock as an aggregate markup shock. In particular, as Figure 8 shows, the cumulated impulse response of GDP to the negative sectoral shock is 2.73 times larger in our baseline economy, which shows the role played by production networks in amplifying the aggregate GDP effects of negative sectoral shocks by affecting aggregate inflation dynamics.

[26] To focus on the spillover effects, we exclude the computers and electronics sector itself from this Figure.

[27] The sectoral inflation dynamics could be different in theory, depending on the input-output linkages, but for this sector, the input-output linkages clearly have a second-order effect in practice in this numerical example. Moreover, sectoral GDP responses are also extremely close between the counterfactual horizontal and baseline economies.

**Effects of Monetary Policy Response to a Negative Sectoral Supply Shock.** Having shown how a negative TFP shock in the computers and electronics sector propagates like an aggregate markup shock above, we now consider a monetary policy response. In particular, we consider a case where money supply contracts exactly by the amount necessary to stabilize aggregate inflation fully on impact. This policy experiment is motivated by ongoing policy discussions on how and if monetary policy should respond to ongoing inflation pressures that are evident in the aggregate data. Figure 9 shows that such a policy response would be non-trivially contractionary for aggregate GDP. In particular, the cumulated impulse response of GDP is roughly 2 times higher than the case in Figure 5 where monetary policy does not respond in this manner to this shock.

## 5 Conclusion

We provide sufficient statistics for inflation and GDP *dynamics* in multi-sector dynamic New Keynesian economies with input-output linkages. We show that the sufficient statistic for these dynamic responses is the *principal square root* of the Leontief matrix appropriately adjusted for the sectoral frequencies of price adjustments.

We construct this sufficient statistic using data from input-output tables and frequencies of price adjustments across sectors in the U.S. In quantitative experiments on this calibrated economy, we find a significant role for production networks in the propagation of aggregate monetary and sectoral TFP shocks. First, monetary shocks lead to effects on GDP that are thrice as large, relative to a baseline multi-sector economy with a horizontal production network. Second, in response to a negative supply shock in the "computers and electronics industry," input-output linkages lead to a much bigger and more persistent increase in aggregate inflation than the increase predicted purely based on the expenditure share of this industry. It also leads to a greater aggregate output contraction compared to a horizontal economy. Negative supply shock in this sector thus manifests itself as an aggregate markup shock as it leads to aggregate inflation together with an aggregate GDP contraction.

In future work, we plan to extend our framework and analysis in several directions. For instance, it will be interesting to study welfare and optimal policy implications in our model. We also plan to extend the model to capture another important source of dynamics, through endogenous capital accumulation, to further develop the framework for business cycle analysis.

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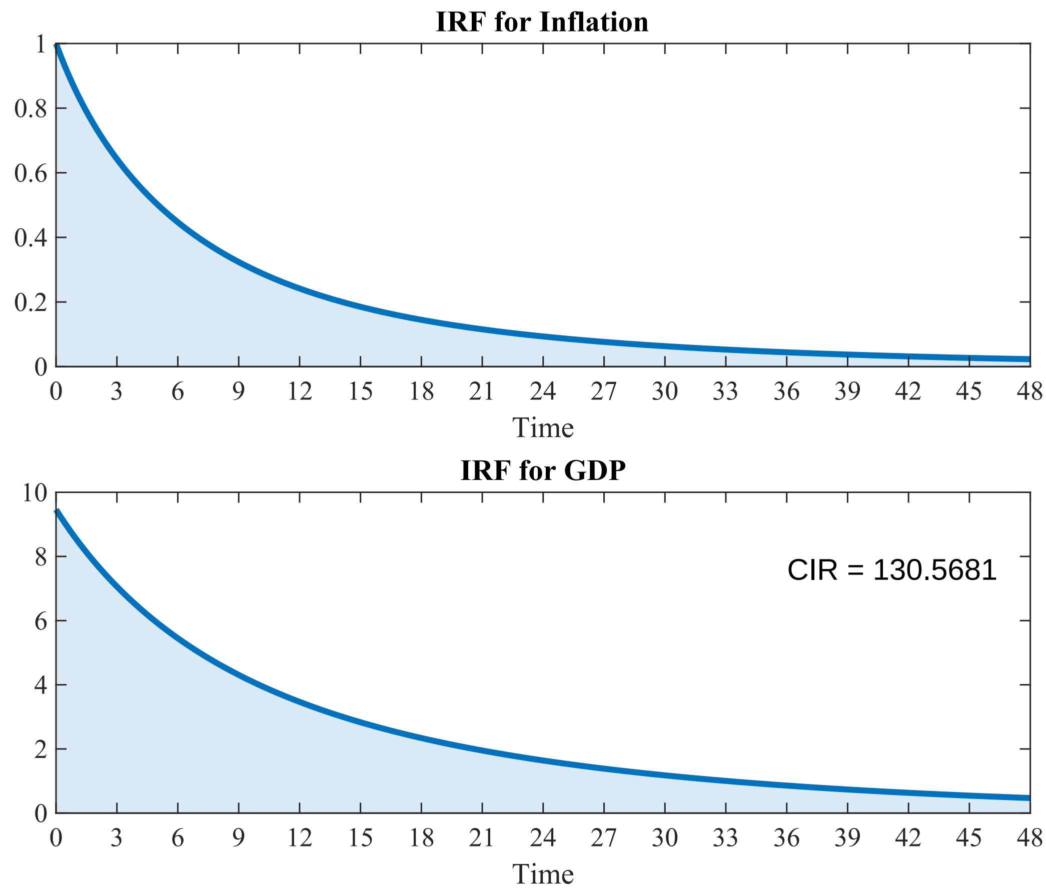
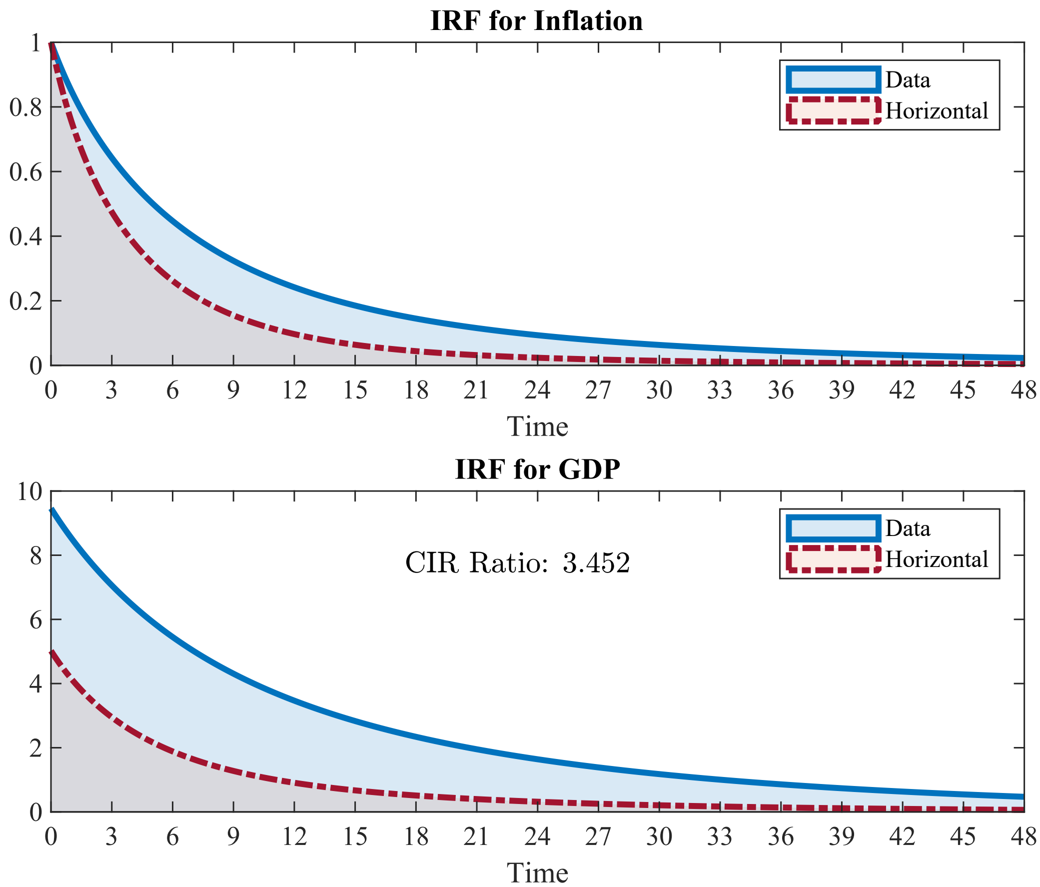
## 6 Figures

Figure 1: Impulse response functions to a monetary policy shock



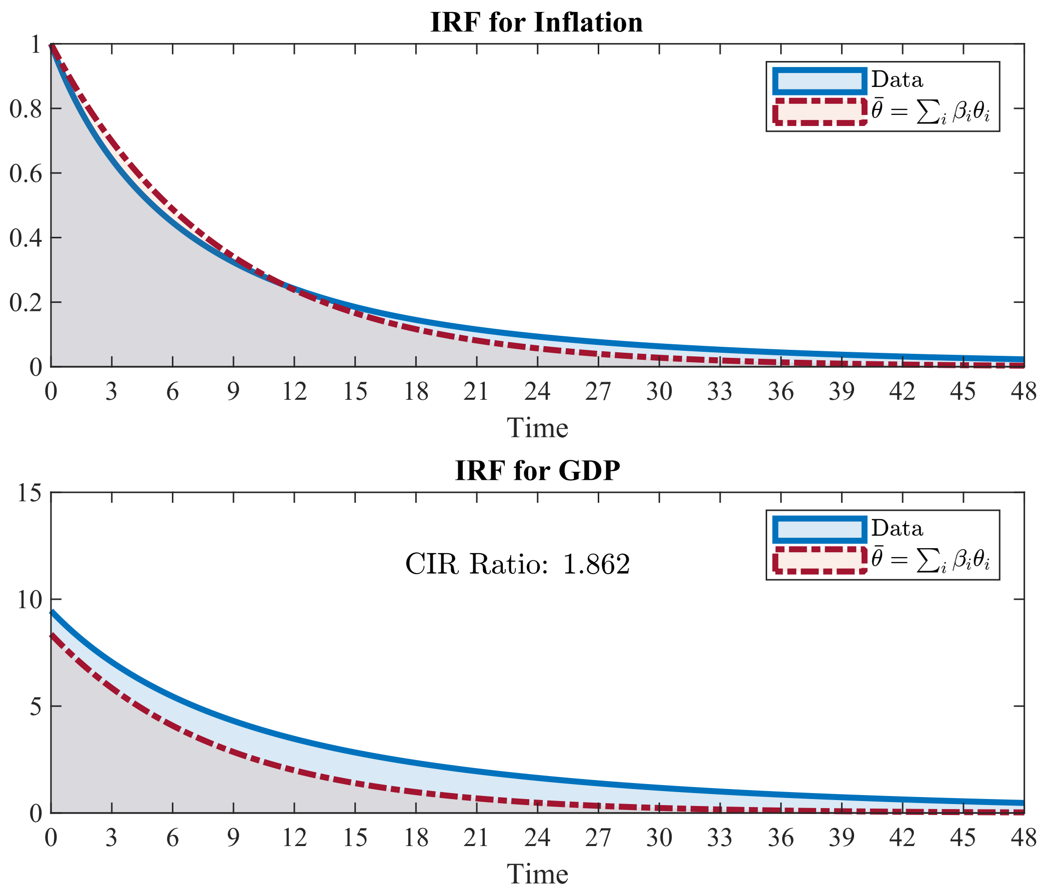

*Notes:* This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 2: Impulse response functions to a monetary policy shock in two economies



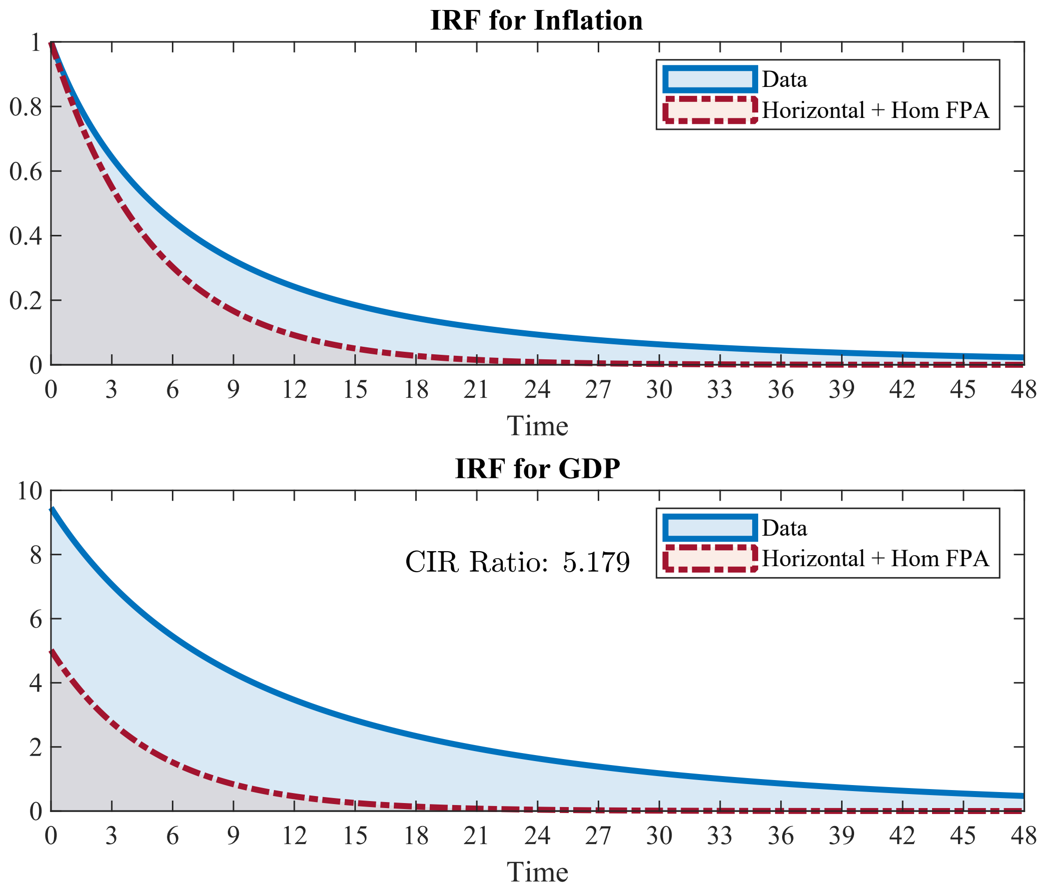

*Notes:* This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy that has production networks with an economy that has a horizonal production structure where only labor is used as an input for production. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 3: Impulse response functions to a monetary policy shock in two economies



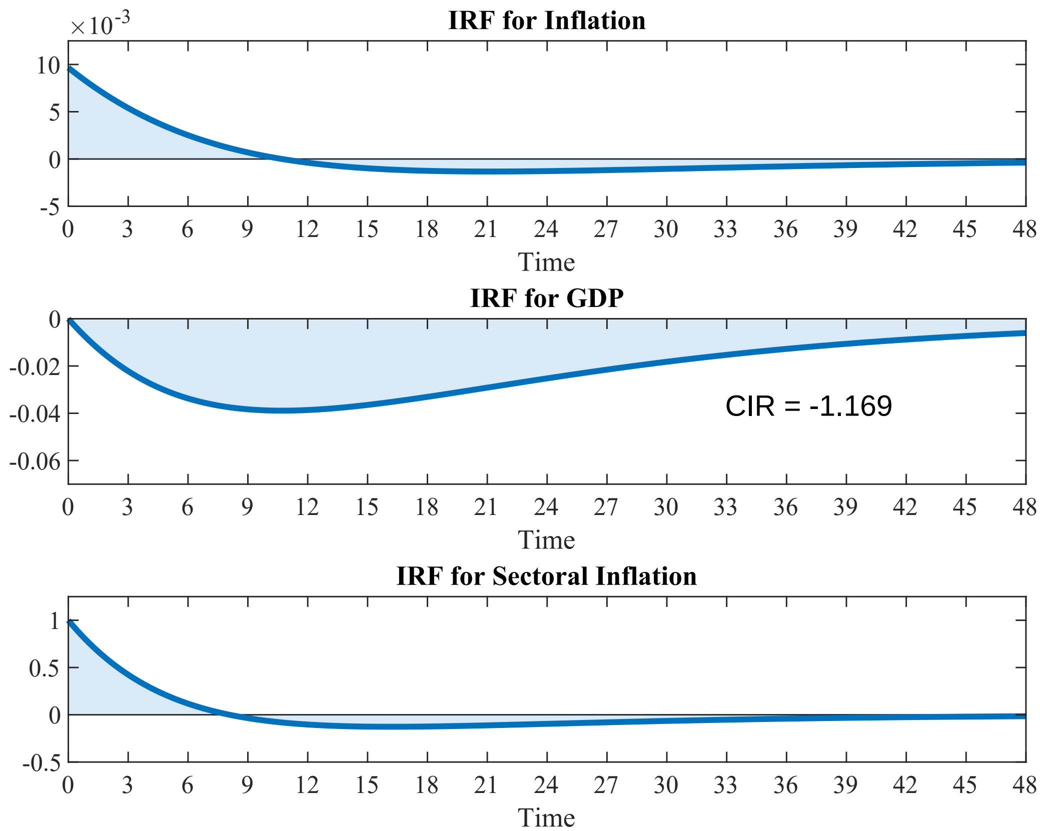

*Notes:* This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy that has heterogeneous price stickiness across sectors with an economy that has homogeneous price stickiness across sectors. The homogeneous price adjustment frequency is calibrated to be the weighted average of the price adjustment frequencies across sectors. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 4: Impulse response functions to a monetary policy shock in two economies



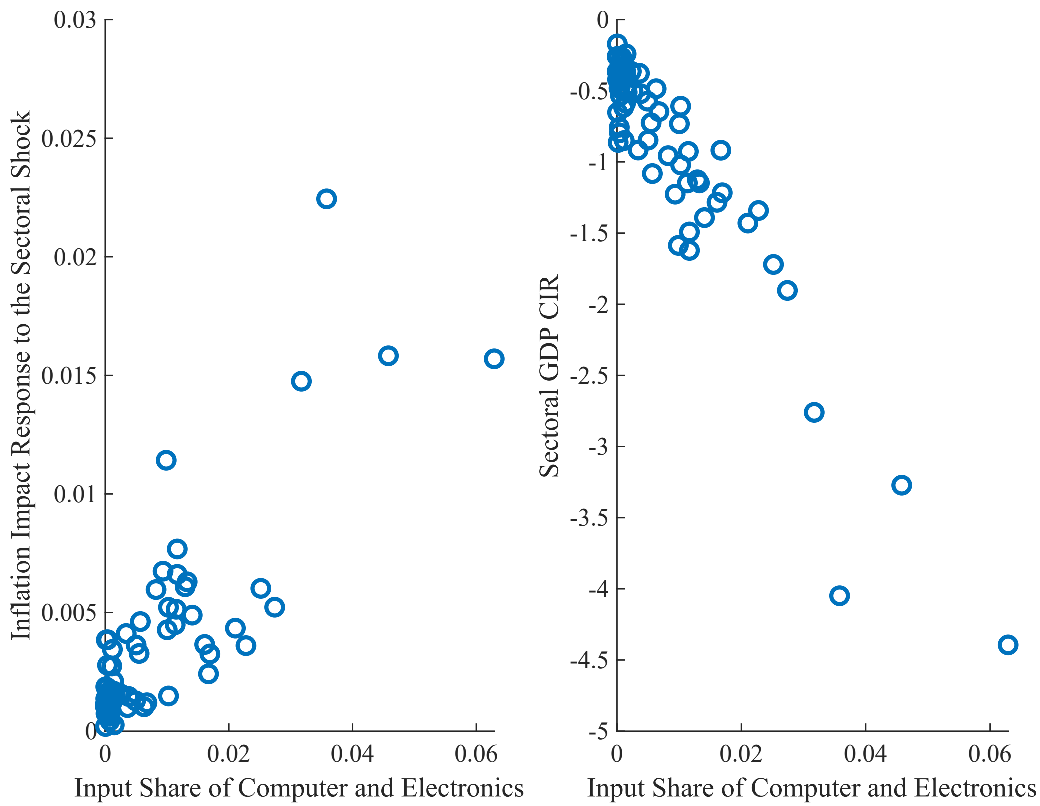

*Notes:* This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy that has production networks and heterogeneous price stickiness across sectors with an economy that has both a horizonal production structure where only labor is used as an input for production as well as homogeneous price stickiness across sectors. The homogeneous price adjustment frequency is calibrated to be the weighted average of the price adjustment frequencies across sectors. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 5: Impulse response functions to a sectoral TFP shock



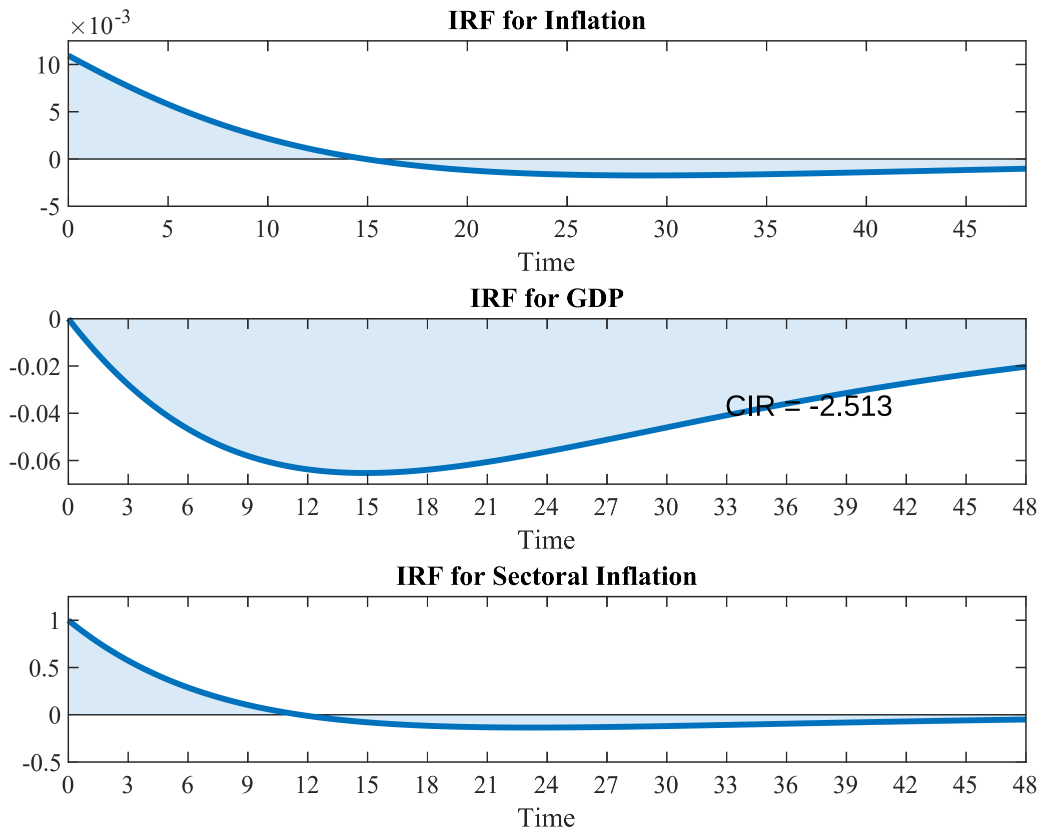

*Notes:* This figure plots the impulse response functions for aggregate inflation, GDP, and sectoral inflation to a negative sectoral TFP shock that generates a one percentage increase in sectoral inflation on impact. The sectoral shock is in the "computers and electronics industry" and the average duration of the shock is six months. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 6: Relationship between input share and response of inflation and GDP of other sectors



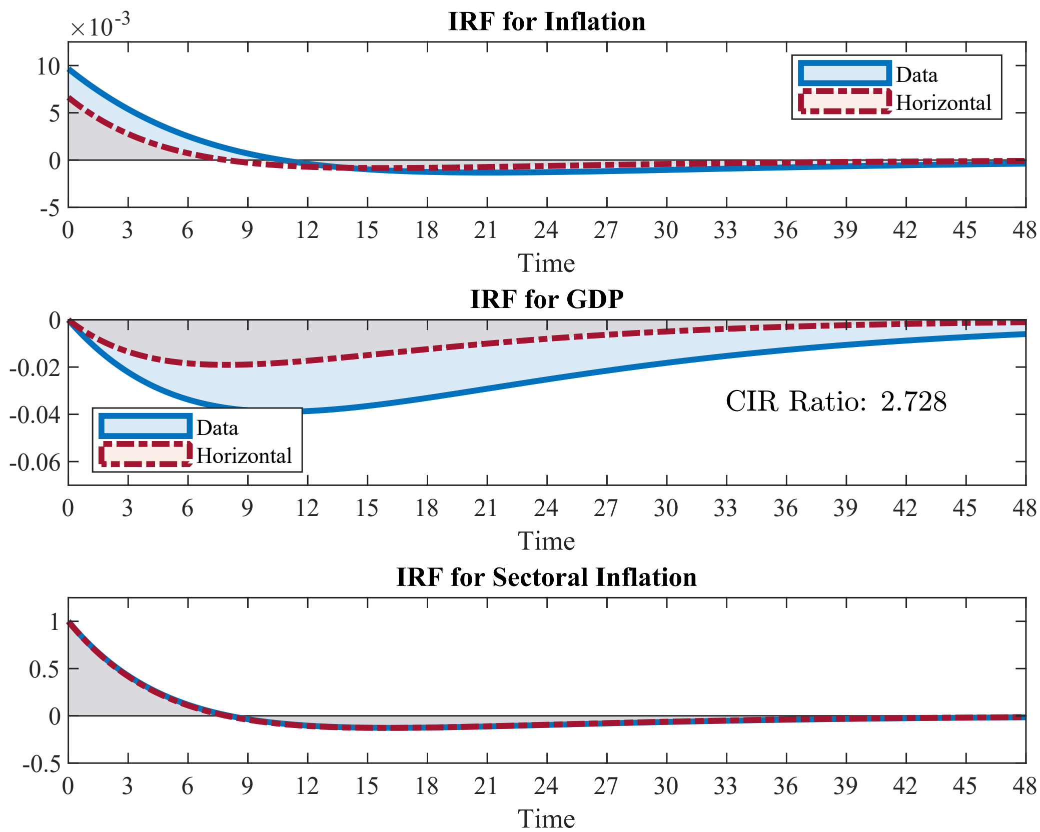

*Notes:* This figure plots how the impact response of inflation and CIR of GDP of other sectors depend on the input share of the "computers and electronics industry" in those sectors. The shock considered is a negative sectoral TFP shock in the "computers and electronics industry" that generates a one percentage increase in that sector's inflation on impact. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 7: Impulse response functions to a sectoral TFP shock



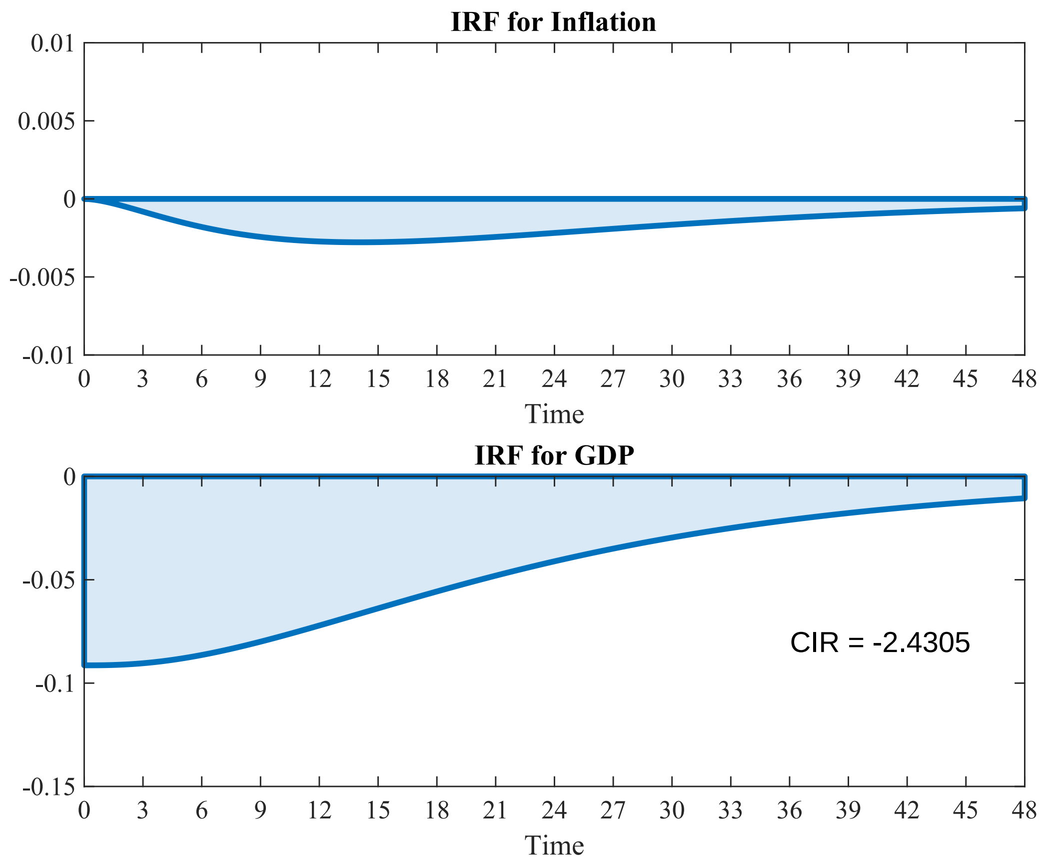

*Notes:* This figure plots the impulse response functions for aggregate inflation, GDP, and sectoral inflation to a negative sectoral TFP shock that generates a one percentage increase in sectoral inflation on impact. The sectoral shock is in the "computers and electronics industry" and the average duration of the shock is twelve months. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 8: Impulse response functions to a sectoral TFP shock in two economies




*Notes:* This figure plots the impulse response functions for aggregate inflation, GDP, and sectoral inflation to a negative sectoral TFP shock that generates a one percentage increase in sectoral inflation on impact. The sectoral shock is in the "computers and electronics industry" and the average duration of the shock is six months. It compares our baseline economy that has production networks with an economy that has a horizonal production structure where only labor is used as an input for production. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

Figure 9: Impulse response functions to a sectoral TFP shock together with a monetary policy response to offset infaltion




*Notes:* This figure plots the impulse response functions for aggregate inflation, GDP, and sectoral inflation to a joint shock: a negative sectoral TFP shock that generates a one percentage increase in sectoral inflation on impact together with a monetary policy that responds with a contractionary shock to fully offset the aggregate inflation on impact. The sectoral shock is in the "computers and electronics industry" and the average duration of the shock is six months. The calibration of the model is at a montly frequency. CIR denotes the cumulative impulse response.

# Appendices

# A Proofs

## A.1 Proof of Proposition 1

Differentiating Equation (23) with respect to time and substituting Equation (22) we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\pi}}_t = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{p}_t &= \boldsymbol{\Theta}(\vec{\boldsymbol{\pi}}_t^{\#} - \vec{\boldsymbol{\pi}}_t) = \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^{\#} - \mathbf{p}_t^*) - \boldsymbol{\Theta}\vec{\boldsymbol{\pi}}_t \\ &= \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t - \mathbf{p}_t^*) + \underbrace{\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^{\#} - \mathbf{p}_t) - \boldsymbol{\Theta}\vec{\boldsymbol{\pi}}_t}_{=\rho\vec{\boldsymbol{\pi}}_t \text{ by Equation (23)}}\end{aligned} \tag{A.1}$$

Now using the definition of $\mathbf{p}_t^*$ from Equation (13) observe that:

$$\mathbf{p}_t - \mathbf{p}_t^* = \mathbf{p}_t - \boldsymbol{\omega}_t + \boldsymbol{z}_t - m_t^s\boldsymbol{\alpha} + \boldsymbol{\Omega}\mathbf{p}_t = -(\mathbf{I} - \boldsymbol{\Omega})(\underbrace{m_t^s\mathbf{1} + \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)}_{=\mathbf{p}_t^f \text{ by Equation (19)}} - \mathbf{p}_t) \tag{A.2}$$

Combining Equations (A.1) and (A.2) gives us the desired result. ∎

## A.2 Proof of Lemma 1

Since $\boldsymbol{\Gamma}$ is a nonsingular $M$-matrix, it satisfies the assumptions of Theorem 5 in Alefeld and Schneider (1982) which states that $\boldsymbol{\Gamma}$ has a unique square root matrix that is also an $M$-matrix. Let us denote this square root by $\sqrt{\boldsymbol{\Gamma}}$. Since the real parts of all the eigenvalues of a $M$-matrix are non-negative, $\sqrt{\boldsymbol{\Gamma}}$ is also the principal square root of $\boldsymbol{\Gamma}$. ∎

## A.3 Proof of Proposition 2

For $\rho = 0$, the differential equation in Equation (24) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\pi}}_t = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{p}_t = \boldsymbol{\Gamma}(\mathbf{p}_t - \mathbf{p}_t^f) \tag{A.3}$$

Since $\mathbf{p}_t^f$ is piece-wise continuous and bounded, it has a Laplace transform for any $s \geq 0$. Let $\mathbf{P}^f(s) = \mathcal{L}_s(\mathbf{p}_t^f) \equiv \int_0^\infty e^{-st}\mathbf{p}_t^f \mathrm{d}t$ denote the Laplace transform of $\mathbf{p}_t^f$. Similarly, let $\mathbf{P}(s) = \mathcal{L}_s(\mathbf{p}_t)$ denote the Laplace transform of $\mathbf{p}_t$. Then, applying the Laplace transform to the differential equation above, we have:

$$\mathbf{P}(s) = (s^2\mathbf{I} - \boldsymbol{\Gamma})^{-1}(s\mathbf{p}_{0+} + \vec{\boldsymbol{\pi}}_{0+}) - (s^2\mathbf{I} - \boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\mathbf{P}^f(s) \tag{A.4}$$

Thus,

$$\mathbf{p}_t = \sqrt{\boldsymbol{\Gamma}}^{-1}\sinh(\sqrt{\boldsymbol{\Gamma}}t)\vec{\boldsymbol{\pi}}_{0+} + \cosh(\sqrt{\boldsymbol{\Gamma}}t)\mathbf{p}_{0+} - \mathcal{L}_t^{-1}\left[(s^2\mathbf{I} - \boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\mathbf{P}^f(s)\right] \tag{A.5}$$

where $\mathbf{c}_0$ and $\mathbf{c}_1$ are vectors in $\mathbf{R}^n$ and are appropriate linear transformations of $\mathbf{p}_{0+}$ and $\vec{\boldsymbol{\pi}}_{0+}$. Moreover, the last terms is the inverse Laplace transform of the product of $(s^2\mathbf{I} - \boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}$ and $\mathbf{P}^f(s)$. Since the inverse Laplace transform of a product is the convolution of inverse Laplace of individual functions, we have:

$$\mathcal{L}_t^{-1}\left[(s^2\mathbf{I}-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\mathbf{P}^f(s)\right] = \int_0^t \mathcal{L}_{t-h}^{-1}\left[(s^2\mathbf{I}-\boldsymbol{\Gamma})^{-1}\boldsymbol{\Gamma}\right]\mathbf{p}_h^f \mathrm{d}h$$
$$= \sqrt{\boldsymbol{\Gamma}}\int_0^t \sinh(\sqrt{\boldsymbol{\Gamma}}(t-h))\mathbf{p}_h^f \mathrm{d}h \tag{A.6}$$

Combining Equations (A.5) and (A.6) and using the definitions of $\sinh(.)$ and $\cosh(.)$ we arrive at

$$\mathbf{p}_t = \frac{1}{2}e^{\sqrt{\boldsymbol{\Gamma}}t}\left[\sqrt{\boldsymbol{\Gamma}}^{-1}\vec{\boldsymbol{\pi}}_{0+} + \mathbf{p}_{0+} - \sqrt{\boldsymbol{\Gamma}}\int_0^t e^{-\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h\right]$$
$$-\frac{1}{2}e^{-\sqrt{\boldsymbol{\Gamma}}t}\left[\sqrt{\boldsymbol{\Gamma}}^{-1}\vec{\boldsymbol{\pi}}_{0+} - \mathbf{p}_{0+} - \sqrt{\boldsymbol{\Gamma}}\int_0^t e^{\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h\right] \tag{A.7}$$

Now, in terms of boundary conditions $\mathbf{p}_t$ satisfies the following two: (1) it is continuous at $t=0$, since the probability of price change opportunities arriving at a short interval around any point is arbitrarily small—i.e., $\mathbf{p}_{0+} = \mathbf{p}_{0-}$ because no firm changes their price exactly at $t=0$ as it is a measure zero event, (2) we are looking for the solution in which prices are non-explosive; in fact bounded because $\mathbf{p}_t^f$ is bounded. So the term multiplying $e^{\sqrt{\boldsymbol{\Gamma}}t}$ has to be zero as $t\to\infty$ and we have:

$$\sqrt{\boldsymbol{\Gamma}}^{-1}\vec{\boldsymbol{\pi}}_{0+} + \mathbf{p}_{0-} = \sqrt{\boldsymbol{\Gamma}}\int_0^\infty e^{-\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h \tag{A.8}$$

Plugging these boundary conditions into the solution we have:

$$\mathbf{p}_t = e^{-\sqrt{\boldsymbol{\Gamma}}t}\mathbf{p}_{0-} + \frac{\sqrt{\boldsymbol{\Gamma}}}{2}e^{\sqrt{\boldsymbol{\Gamma}}t}\int_t^\infty e^{-\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h - \frac{\sqrt{\boldsymbol{\Gamma}}}{2}e^{-\sqrt{\boldsymbol{\Gamma}}t}\int_0^\infty e^{-\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h + \frac{\sqrt{\boldsymbol{\Gamma}}}{2}e^{-\sqrt{\boldsymbol{\Gamma}}t}\int_0^t e^{\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h$$
$$= e^{-\sqrt{\boldsymbol{\Gamma}}t}\mathbf{p}_{0-} + \sqrt{\boldsymbol{\Gamma}}e^{-\sqrt{\boldsymbol{\Gamma}}t}\int_0^t \sinh(\sqrt{\boldsymbol{\Gamma}}h)\mathbf{p}_h^f \mathrm{d}h + \sqrt{\boldsymbol{\Gamma}}\sinh(\sqrt{\boldsymbol{\Gamma}}t)\int_t^\infty e^{-\sqrt{\boldsymbol{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h \tag{A.9}$$

■

### A.4 Proof of Proposition 3

This can be derived from Proposition 2 by solving explicitly for the integrals in Equation (26) but it is more convenient to guess a particular solution for the differential equation in Equation (24) for the particular path of flexible prices specified in Equation (28): with $\rho=0$, a path of non-explosive prices, $\mathbf{p}_t$, is uniquely characterized by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{\pi}}_t = \tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{p}_t = -\boldsymbol{\Gamma}(\mathbf{p}_{0-} + \delta_m\mathbf{1} + \boldsymbol{\Psi}e^{-\boldsymbol{\Phi}t}\boldsymbol{\delta}_z - \mathbf{p}_t) \qquad \text{with} \qquad \mathbf{p}_0 = \mathbf{p}_{0-}^f \tag{A.10}$$

Noting that this is a system of non-homogenous differential equations, the *general* solution to this system can be written as $\mathbf{p}_t = \mathbf{p}_t^p + \mathbf{p}_t^g$, where $\mathbf{p}_t^p$ is a particular solution to the non-homogenous system of differential equations above and $\mathbf{p}_t^g$ is the general solution to the homogenous system, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{p}_t^g = \boldsymbol{\Gamma}\mathbf{p}_t^g$. To obtain the solution we start with the guess that a candidate for the particular solution is

$$\mathbf{p}_t^p = \mathbf{p}_{0-}^f + \delta_m\mathbf{1} + \mathbf{A}e^{-\boldsymbol{\Phi}t}\boldsymbol{\delta}_z \tag{A.11}$$

for some $\mathbf{A}\in\mathbb{R}^{n\times n}$. Plugging this into Equation (A.10) we obtain $(\boldsymbol{\Gamma}\mathbf{A} - \mathbf{A}\boldsymbol{\Phi}^2 - \boldsymbol{\Gamma}\boldsymbol{\Psi})e^{-\boldsymbol{\Phi}t}\boldsymbol{\delta}_z = 0$. Since we want this equation to hold for any $t\geq 0$ and any $\boldsymbol{\delta}_z$, and noting that $\boldsymbol{\Gamma}\boldsymbol{\Psi} = \boldsymbol{\Theta}^2(\mathbf{I}-\boldsymbol{\Omega})(\mathbf{I}-\boldsymbol{\Omega})^{-1} = \boldsymbol{\Theta}^2$, it follows that our guess is verified when $\mathbf{A}$ is the solution to the Sylvester equation

$$\mathbf{\Gamma}\mathbf{A} - \mathbf{A}\mathbf{\Phi}^2 = \mathbf{\Theta}^2 \tag{A.12}$$

which is unique because we assumed that $\mathbf{\Gamma}$ and $\mathbf{\Phi}^2$ do not have any common eigenvalues (see, e.g., Horn and Johnson, 2012, Theorem 2.4.4.1). As for the general solution, $\mathbf{p}_t^g$, one can solve this differential equation by the method of undetermined coefficients for second-order matrix differential equations (see Apostol, 1975). In particular, one can easily confirm that such a solution has the form:

$$\mathbf{p}_t^g = \sum_{k=0}^{\infty} \frac{\mathbf{\Gamma}^k t^{2k}}{(2k)!}\mathbf{c}_0 + \sum_{k=0}^{\infty} \frac{\mathbf{\Gamma}^k t^{2k+1}}{(2k+1)!}\mathbf{c}_1 \tag{A.13}$$

whose domain of convergence in $t$ includes our time domain $[0, \infty)$ and $\mathbf{c}_0, \mathbf{c}_1$ are constant vectors in $\mathbb{R}^n$. Now, letting $\sqrt{\mathbf{\Gamma}}$ denote the principal square root of $\mathbf{\Gamma}$, which exists and is a non-singular $M$-matrix by Lemma 1, we can write the equation above as

$$\mathbf{p}_t^g = \underbrace{\sum_{k=0}^{\infty} \frac{(\sqrt{\mathbf{\Gamma}}t)^k}{k!}}_{=e^{\sqrt{\mathbf{\Gamma}}t}} (\underbrace{\frac{\mathbf{c}_0 + \sqrt{\mathbf{\Gamma}}^{-1}\mathbf{c}_1}{2}}_{\equiv\tilde{\mathbf{c}}_0}) + \underbrace{\sum_{k=0}^{\infty} \frac{(-\sqrt{\mathbf{\Gamma}}t)^k}{k!}}_{=e^{-\sqrt{\mathbf{\Gamma}}t}} (\underbrace{\frac{\mathbf{c}_0 - \sqrt{\mathbf{\Gamma}}^{-1}\mathbf{c}_1}{2}}_{\equiv\tilde{\mathbf{c}}_1}) \tag{A.14}$$

Thus, the general solution to the non-homogenous system is given by

$$\mathbf{p}_t = \mathbf{p}_t^p + \mathbf{p}_t^g = \mathbf{p}_{0-}^f + \delta_m \mathbf{1} + \mathbf{A}e^{-\mathbf{\Phi}t}\boldsymbol{\delta}_z + e^{\sqrt{\mathbf{\Gamma}}t}\tilde{\mathbf{c}}_0 + e^{-\sqrt{\mathbf{\Gamma}}t}\tilde{\mathbf{c}}_1 \tag{A.15}$$

Now, to determine the constant vectors $\tilde{\mathbf{c}}_0, \tilde{\mathbf{c}}_1$, we have the two sets of boundary conditions. (1) $\mathbf{p}_0 = \mathbf{p}_{0-}^f$ (notice with positive and finite frequencies of price changes, no firm gets an opportunity to change their prices at instant zero so the left and right limits are the same). (2) With zero trend inflation (which is the assumption here), prices converge to a steady-state level as $t \to \infty$—i.e., the price function is non-explosive over time. The second set of boundary conditions immediately imply $\tilde{\mathbf{c}}_0 = 0$ because all of the eigenvalues of $\mathbf{\Gamma}$ have strictly positive real parts by Lemma 1. The first set of boundary conditions imply: $\tilde{\mathbf{c}}_0 = -\delta_m \mathbf{1} - \mathbf{A}\boldsymbol{\delta}_z$. Thus,

$$\mathbf{p}_t = \mathbf{p}_{0-}^f + \delta_m (\mathbf{I} - e^{-\sqrt{\mathbf{\Gamma}}t})\mathbf{1} + \mathbf{A}e^{-\mathbf{\Phi}t}\boldsymbol{\delta}_z - e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{A}\boldsymbol{\delta}_z \tag{A.16}$$

■

### A.5 Proof of Corollary 1

By Equation (18):

$$\begin{aligned} y_t = m_t^s - p_t &= m_{0-}^s + \delta_m - \boldsymbol{\beta}^\intercal \mathbf{p}_t \\ &= \underbrace{m_{0-}^s - \boldsymbol{\beta}^\intercal \mathbf{p}_{0-}^f}_{\equiv y^f \text{ (steady-state output)}} + \delta_m \boldsymbol{\beta}^\intercal e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{1} - \boldsymbol{\beta}^\intercal \mathbf{A}e^{-\mathbf{\Phi}t}\boldsymbol{\delta}_z + \boldsymbol{\beta}^\intercal e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{A}\boldsymbol{\delta}_z \end{aligned} \tag{A.17}$$

For inflation, we start by differentiating Equation (29) with respect to time to get the impulse response function of sectoral inflation rates:

$$\vec{\boldsymbol{\pi}}_t = \delta_m(\sqrt{\boldsymbol{\Gamma}}e^{-\sqrt{\boldsymbol{\Gamma}}t})\mathbf{1} + (\sqrt{\boldsymbol{\Gamma}}e^{-\sqrt{\boldsymbol{\Gamma}}t}\mathbf{A} - \mathbf{A}e^{-\boldsymbol{\Phi}t}\boldsymbol{\Phi})\boldsymbol{\delta}_z \tag{A.18}$$

Now, since $\pi_t = \boldsymbol{\beta}^\intercal \vec{\boldsymbol{\pi}}_t$, multiplying the equation above by $\boldsymbol{\beta}^\intercal$ from left gives us the desired result. ■

# B Derivations of Optimality Conditions in the Model

We now turn to charachterizing the flexible- and sticky-price stationary equilibrium of this economy.

## B.1 Households' Optimality Conditions

**Demand for Sectoral Goods.** We can decompose the household's consumption problem into two stages, where for a *given* level of $C_t$ the household minimizes her expenditure on sectoral goods (compensated demand) and then decides on the optimal level of $C_t$ as a function of life-time income (uncompensated demand). The compensated demand of the household for sectoral goods given the vector of sectoral prices $\mathbf{P}_t = (P_{1,t}, \ldots, P_{n,t})$ gives us the expenditure function:

$$\begin{aligned}\mathcal{E}(C_t; \mathbf{P}_t) &\equiv \min_{C_{1,t},\ldots,C_{n,t}} \sum_{i\in[n]} P_{i,t}C_{i,t} \quad \text{subject to} \quad \Phi(C_{1,t},\ldots,C_{n,t}) \geq C_t \\ &= P_tC_t, \quad P_t \equiv \mathcal{E}(1, \mathbf{P}_t)\end{aligned} \tag{B.1}$$

where the second line follows from the first degree homogeneity of the function $\Phi(.)$ and $P_t$ is the cost of a *unit* of $C_t$ and, or in short, the price of $C_t$. Note that due to first degree homogeneity of $\Phi(.)$, $P_t$ does not depend on household's choices and is *just* a function of the sectoral prices, $\mathbf{P}_t$. Applying Shephard's lemma, we obtain that the household's expenditure share of sectoral good $i$ is proportional to the elasticity of the expenditure fuction with respect to the price of $i$:

$$P_{i,t}C^*_{i,t} = \beta_i(\mathbf{P}_t) \times P_tC_t \quad \text{where} \quad \beta_i(\mathbf{P}_t) \equiv \frac{\partial \log(\mathcal{E}(C_t, \mathbf{P}_t))}{\partial \log(P_{i,t})} \tag{B.2}$$

It is important to note that due to the first degree homogeneity of the expenditure function these elasticities are independent of aggregate consumption $C_t$ and only depend on sectoral prices, $\mathbf{P}_t$. Moreover, it is easy to verify that they are also a homongeous of degree zero in these prices so that the vector of household's expenditure shares, denoted by $\boldsymbol{\beta}_t \in \mathbb{R}^n$, can be written as a function of sectoral prices relative to wage:

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}(\mathbf{P}_t/W_t) \tag{B.3}$$

Notably, a vector of constant expenditure shares corresponds to $\Phi(.)$ being a Cobb-Douglas aggregator where sectoral goods are neither complements nor substitutes.

**Household's Other Decisions.** Given the household's expenditure function and the aggregate price index $P_t$ in Equation (B.1), it is straightforward to derive the labor supply and Euler equations for bonds:

$$\underbrace{\gamma(C_t)\frac{\dot{C}_t}{C_t}}_{\text{marginal loss from saving}} = \underbrace{i_t - \rho - \frac{\dot{P}_t}{P_t}}_{\text{marginal gain from saving}} \quad \text{where} \quad \underbrace{\gamma(C_t) \equiv -\frac{U''(C_t)C_t}{U'(C_t)}}_{\text{inverse elasticity of intertemporal substitution}} \tag{B.4}$$

$$\underbrace{\frac{V'(L_t)}{U'(C_t)}}_{\text{MRS}_{LC}} = \underbrace{\frac{W_t}{P_t}}_{\text{real wage}} \quad \Rightarrow \quad \psi(L_t)\frac{\dot{L}_t}{L_t} + \gamma(C_t)\frac{\dot{C}_t}{C_t} = \frac{\dot{W}_t}{W_t} - \frac{\dot{P}_t}{P_t} \quad \text{where} \quad \underbrace{\psi(L_t) \equiv \frac{V''(L_t)L_t}{V'(L_t)}}_{\text{inverse Frisch elasticity of labor supply}} \tag{B.5}$$

Moreover, as long as the interest rate $i_t > 0$, which we will confirm is the case in the stationary equilibria as well as in small enough neighborhoods around them, the cash-in-advance constraint binds. Combined with the money supply rule, this implies that nominal demand grows at the same rate as money supply:

$$\frac{\dot{P}_t}{P_t} + \frac{\dot{C}_t}{C_t} = \frac{\dot{M}_t}{M_t} = \mu \tag{B.6}$$

Note that by combining Equations (B.4) to (B.6) we can write the growth rate of wages as well as the nominal interest rates as a function of consumption and labor supply growths:

$$\frac{\dot{W}_t}{W_t} = \mu + \psi(L_t)\frac{\dot{L}_t}{L_t} + (\gamma(C_t) - 1)\frac{\dot{C}_t}{C_t}, \qquad i_t = \rho + \mu + (\gamma(C_t) - 1)\frac{\dot{C}_t}{C_t} \tag{B.7}$$

As shown and utilized by Golosov and Lucas (2007) and more recenty by Wang and Werning (2021), a convenient set of preferences that simplify these conditions tremendously are $U(C_t) = \log(C_t)$ and $V(L_t) = L_t$ which imply $\gamma(C_t) = 1$ and $\psi(L_t) = 0$. Plugging these elasticities into Equation (B.7), we can see how these preferences simplifiy aggregate dynamics by setting wage growth to the constant rate of $\mu$ and interest rates to a constant rate at $\rho + \mu$.

### B.2 Firms' Optimality Conditions

**Cost Minimization and the Input-Output Matrices.** We start by characterizing firms' expenditure shares on inputs by first solving their expenditure minimization problems. Since expenditure minimization is a static decision within every period, our characterization of these expenditure shares closely follow Bigio and La'O (2020); Baqaee and Farhi (2020), and we refer the reader to these papers for more detailed treatments.

Let us start with the observation that the firms' cost function in Equation (8), given the wage $W_t$ and sectoral prices $\mathbf{P}_t = (P_{i,t})_{i\in[n]}$, is homogenous of degree one in production:

$$\begin{aligned} \mathcal{C}_i(Y^s_{ij,t}; W_t, \mathbf{P}_t) &= \min_{L_{jk,t},(X_{ij,k,t})_{k\in[n]}} W_t L_{ij,t} + \sum_{k\in[n]} P_{k,t}X_{ij,k,t} \quad \text{subject to} \quad Z_i F_i(L_{ij,t}, (X_{ij,k,t})_{k\in[n]}) \geq Y^s_{ij,t} \\ &= \mathsf{MC}_i(W_t, \mathbf{P}_t) \times Y^s_{ij,t}, \quad \mathsf{MC}_i(W_t, \mathbf{P}_t) \equiv \mathcal{C}_i(1; W_t, \mathbf{P}_t) \end{aligned} \tag{B.8}$$

where the second line follows from the first degree homogeneity of the production function $Z_iF_i(.)$ and $\mathsf{MC}_i(W_t, \mathbf{P}_t)$ is the cost of producing a *unit* of output, or in short, the firm's marginal cost of production. Note that due to the first degree homogeneity of the production function, marginal costs are independent of level of production and depend only on the sector's production function and input prices. Applying Shephard's lemma and re-arranging firms' optimal demand for inputs gives us the result that firms' expenditure share of any input is the elasticity of the cost function with respect to that input:

$$W_t L^*_{ij,t} = \alpha_i(W_t, \mathbf{P}_t) \times \mathsf{MC}_i(W_t, \mathbf{P}_t) Y^s_{ij,t}, \quad P_{k,t} X^*_{ij,k,t} = a_{ik}(W_t, \mathbf{P}_t) \times \mathsf{MC}_i(W_t, \mathbf{P}_t) Y^s_{ij,t}, \quad \forall k \in [n] \tag{B.9}$$

where $\alpha_i(W_t, \mathbf{P}_t)$ and $a_{ik}(W_t, \mathbf{P}_t)$ are the elasticities of the sector $i$'s cost function with respect to labor and sector $k$'s final good respectively:

$$\alpha_i(W_t, \mathbf{P}_t) \equiv \frac{\partial \log(\mathcal{C}_i(Y; W_t, \mathbf{P}_t))}{\partial \log(W_t)}, \qquad a_{ik}(W_t, \mathbf{P}_t) \equiv \frac{\partial \log(\mathcal{C}_i(Y; W_t, \mathbf{P}_t))}{\partial \log(P_{k,t})} \quad \forall k \in [n] \tag{B.10}$$

with the property that $\alpha_i(W_t, \mathbf{P}_t) + \sum_{k \in [n]} a_{ik}(W_t, \mathbf{P}_t) = 1$. It is important to note that the first degree homogeneity of the cost function in Equation (8) also implies that these elasticities are only functions of the aggregate wage and sectoral prices. It is also well-known that these elasticities are directly related to the *cost-based* input-output matrix, denoted by $\mathbf{\Omega}_t \in \mathbb{R}^{n \times n}$, and the labor share vector, denoted by $\boldsymbol{\alpha}_t \in \mathbb{R}^n$:

$$[\mathbf{\Omega}_t]_{i,k} \equiv \frac{\text{total expenditure of sector } i \text{ on sector } k}{\text{total expenditure on inputs in sector } i} = a_{ik}(W_t, \mathbf{P}_t), \quad \forall (i,k) \in [n]^2 \tag{B.11}$$

$$[\boldsymbol{\alpha}_t]_i \equiv \frac{\text{total expenditure of sector } i \text{ on labor}}{\text{total expenditure on inputs in sector } i} = \alpha_i(W_t, \mathbf{P}_t), \quad \forall i \in [n] \tag{B.12}$$

where the second equality holds *only* under firms' optimal expenditure shares and follows from integrating Equation (B.9). Since these elasticities are also homogenous of degree zero in the price vector $(W_t, \mathbf{P}_t)$, Equations (B.11) and (B.12) imply that in *any equilibrium*, the cost-based input-output matrix and the vector of sectoral labor shares are only a function of the sectoral prices relative to the nominal wage; i.e.,

$$\mathbf{\Omega}_t = \mathbf{\Omega}(\mathbf{P}_t/W_t) = [a_{ik}(1, \mathbf{P}_t/W_t)], \qquad \boldsymbol{\alpha}_t = \boldsymbol{\alpha}(\mathbf{P}_t/W_t) = [\alpha_i(1, \mathbf{P}_t/W_t)] \tag{B.13}$$

A notable example is Cobb-Douglas production functions, which imply constant elasticities for the cost function—because inputs are neither substitutes nor complements—and lead to a constant input-output matrix and constant vector of labor shares over time.

**Optimal Prices.** Having characterized firms' cost functions, we now derive the optimal *desired prices*, $P^*_{ij,t}$, in Equation (10) and *reset prices*, $P^{\#}_{ij,t}$ in Equation (B.15). It follows that the optimal desired price is a markup over the marginal cost of production and proporitional to the wedge introduced through taxes/subsidies:

$$P^*_{ij,t} = P^*_{i,t} \equiv \underbrace{\frac{1}{1-\tau_i}}_{\text{tax/subsidy wedge}} \times \underbrace{\frac{\sigma_i}{\sigma_i - 1}}_{\text{markup}} \times \underbrace{\mathsf{MC}_i(W_t, \mathbf{P}_t)}_{\text{marginal cost}} \tag{B.14}$$

It is then straightforward to show that the firms' optimal reset prices are a weighted average of all future desired prices in industry $i$:

$$P^{\#}_{ij,t} = P^{\#}_{i,t} \equiv \underbrace{\int_0^\infty \overbrace{\frac{e^{-(\theta_i h + \int_0^h i_{t+s}\mathrm{d}s)} Y_{i,t+h} P^{\sigma_i}_{i,t+h}}{\int_0^\infty e^{-(\theta_i h + \int_0^h i_{t+s}\mathrm{d}s)} Y_{i,t+h} P^{\sigma_i}_{i,t+h} \mathrm{d}h}}^{\text{weight (density) on } P^*_{i,t+h}} \times P^*_{i,t+h} \mathrm{d}h}_{\text{weighted average of all future desired prices}} \tag{B.15}$$

Given this reset price, we can then calculate the aggregate price of sector $i$ from Equation (6) as:

$$P^{1-\sigma_i}_{i,t} = \int_0^1 P^{1-\sigma_i}_{ij,t} \mathrm{d}j = \theta_i \int_0^t e^{-\theta_i h} (P^{\#}_{i,t-h})^{1-\sigma_i} \mathrm{d}h + e^{-\theta_i t} \int_0^1 P^{1-\sigma_i}_{ij,0^-} \mathrm{d}j \tag{B.16}$$

where the second equality follows from the observation that at time $t$ the density of firms that reset their prices $h$ periods ago to $P^{\#}_{i,t}$ is governed by the exponential distribution of time between price changes and is equal to $\theta_i e^{-\theta_i h}$.

### B.3 Market Clearing and Total Value Added

Define the sales-based Domar weight of sector $i \in [n]$ at time $t$ as the ratio of the final producer's sales relative to the households total expenditure on consumption:

$$\lambda_{i,t} \equiv P_{k,t} Y_{k,t} / (P_t C_t) \tag{B.17}$$

Now, substituting optimal consumption of the household from sector $k \in [n]$ in Equation (B.2) and optimal demand of firms for the final good of sector $k \in [n]$ in Equation (B.9) into the market clearing condition for final good of sector $k$ and dividing by household's total expenditure, we get

$$\lambda_{k,t} = \beta_i(\mathbf{P}_t/W_t) + \textstyle\sum_{i \in [n]} a_{ik}(1, \mathbf{P}_t/W_t) \lambda_{i,t} \Delta_{i,t} / \mu_{i,t} \tag{B.18}$$

where $\mu_{i,t} \equiv P_{i,t}/\mathsf{MC}_i(\mathbf{P}_t, W_t)$ is the markup of sector $i$ and $\Delta_{i,t}$ is the well-known measure of price dispersion in the New Keynesian literature defined as

$$\Delta_{i,t} = \int_0^1 (P_{ij,t}/P_{i,t})^{-\sigma_i} \mathrm{d}j \geq 1 \tag{B.19}$$

Where the inequality follows from applying Jensen's inequality to the definion of the aggregate price index $P_{i,t}$.[28] Thus, letting $\boldsymbol{\lambda}_t \equiv (\lambda_{i,t})_{i \in [n]}$ denote the vector of sales-based domar weights at time $t$ across sectors and $\boldsymbol{\mathcal{M}}_t \equiv \mathrm{diag}(\mu_{i,t}/\Delta_{i,t})$ as the diagonal matrix whose $i$'th diagonal entry is the price dispersion adjusted markup wedge of sector $i$, we can write Equation (B.18) in the following matrix form:

$$\boldsymbol{\lambda}_t = (\mathbf{I} - \boldsymbol{\Omega}_t^\intercal \boldsymbol{\mathcal{M}}_t^{-1})^{-1} \boldsymbol{\beta}_t \tag{B.20}$$

[28]Note that $1 = [\int_0^1 (P_{ij,t}/P_{i,t})^{1-\sigma_i} \mathrm{d}j]^{\frac{\sigma_i}{\sigma_i - 1}} \mathrm{d}j = [\int_0^1 ((P_{ij,t}/P_i, t)^{-\sigma_i})^{\frac{\sigma_i - 1}{\sigma_i}} \mathrm{d}j]^{\frac{\sigma_i}{\sigma_i - 1}} \mathrm{d}j \leq \int_0^1 (P_{i,t}/P_t)^{-\sigma_i} \mathrm{d}j.$

Finally, substituting firms labor demand into the labor market clearing condition, we arrive at the following expression for the labor share:

$$\frac{W_t L_t}{P_t C_t} = \boldsymbol{\alpha}_t^{\intercal} \boldsymbol{\mathcal{M}}_t^{-1} \boldsymbol{\lambda}_t \tag{B.21}$$

# C Data and Calibration

Proposition 3 shows that the sufficient statistics for inflation and output dynamics in response to shocks in our model are the frequency-adjusted Leontief matrix, as given by $\mathbf{\Gamma} \equiv \mathbf{\Theta}^2(\mathbf{I} - \mathbf{\Omega})$, and the consumption expenditure shares across sectors, as given by the vector $\boldsymbol{\beta}$. We now describe in detail how we construct $\mathbf{\Gamma}$ and $\boldsymbol{\beta}$ using detailed sectoral US data.

First, we use the IO tables from the BEA to construct the input-output linkages across sectors, given by the matrix $\mathbf{\Omega}$; the consumption expenditure shares across sectors, given by the vector $\boldsymbol{\beta}$; and the sectoral labor shares, given by the vector $\boldsymbol{\alpha}$. We construct these objects using the tables from 2019 at the summary-level disaggregation, excluding the government sectors, which implies 66 sectors in our sample. In particular, to construct $\mathbf{\Omega}$ we use both the make and use input-output tables. The make table shows the value of the production of goods by industries. Each row represents an industry and the columns for that row represent the commodities produced by this industry. Therefore, given a row, adding up its columns gives the value of the total production of the sector associated with this row. The use table shows the value of each commodity used by industry or by final use. Each column represents an industry and the rows for that column represent the commodities used by this industry. Figure C.1 presents the matrix $\mathbf{\Omega}$ we construct from the data, in a heat-map version.

Next, the use IO table shows the components of value added used by a industry. In particular, it provides data on compensation of employees, which is also used to construct the sectoral labor shares $\boldsymbol{\alpha}$. Figure C.2 shows the distribution of labor share across sectors in our data. Moreover, we also construct the consumption expenditure shares across sectors $\boldsymbol{\beta}$ using the use IO table, where the consumption share for a given sector is given by the share of the personal consumption expenditure on that sector over the total personal consumption expenditure. Figure C.3 presents the distribution of consumption expenditure share across sectors in our data.

For the final component, we construct the diagonal matrix $\mathbf{\Theta}^2$, whose diagonal elements are the squared frequency of price adjustment in each sector using data on 341 sectors from Pasten, Schoenle, and Weber (2020). We match data from Pasten, Schoenle, and Weber (2020) on frequency of price changes with the 2002 concordance table between IO industry codes and the related 2002 NAICS codes. Then, we match the resulting table with the 2012 concordance table between IO industry codes and the related 2012 NAICS codes. The last step is performed in order to get the link between the frequency of price adjustment at the detail level disaggregation, which is a finer disaggregation, and the summary level disaggregation, which is what we use in the paper.[29] In order to aggregate the frequency

[29] We linked the frequency of price adjustment data with the 2002 concordance table first because Pasten, Schoenle, and

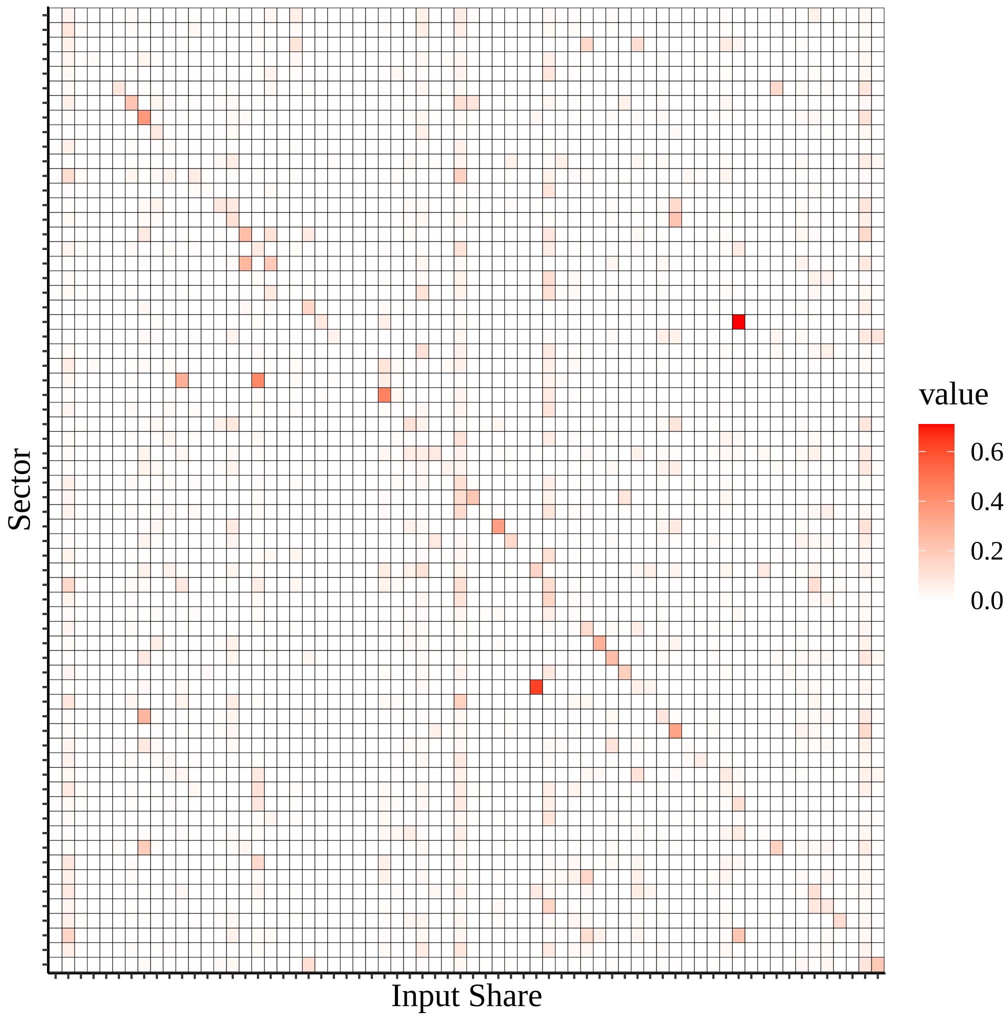



Figure C.1: U.S. sectoral input-output matrix (heat map) in 2019

*Notes:* This figure presents the sectoral input-output matrix in a heat map version, using data from the make and use input-otput tables produced by the BEA in 2019. The industry classification is at the summary-level disaggregation, for a total of 66 sectors.

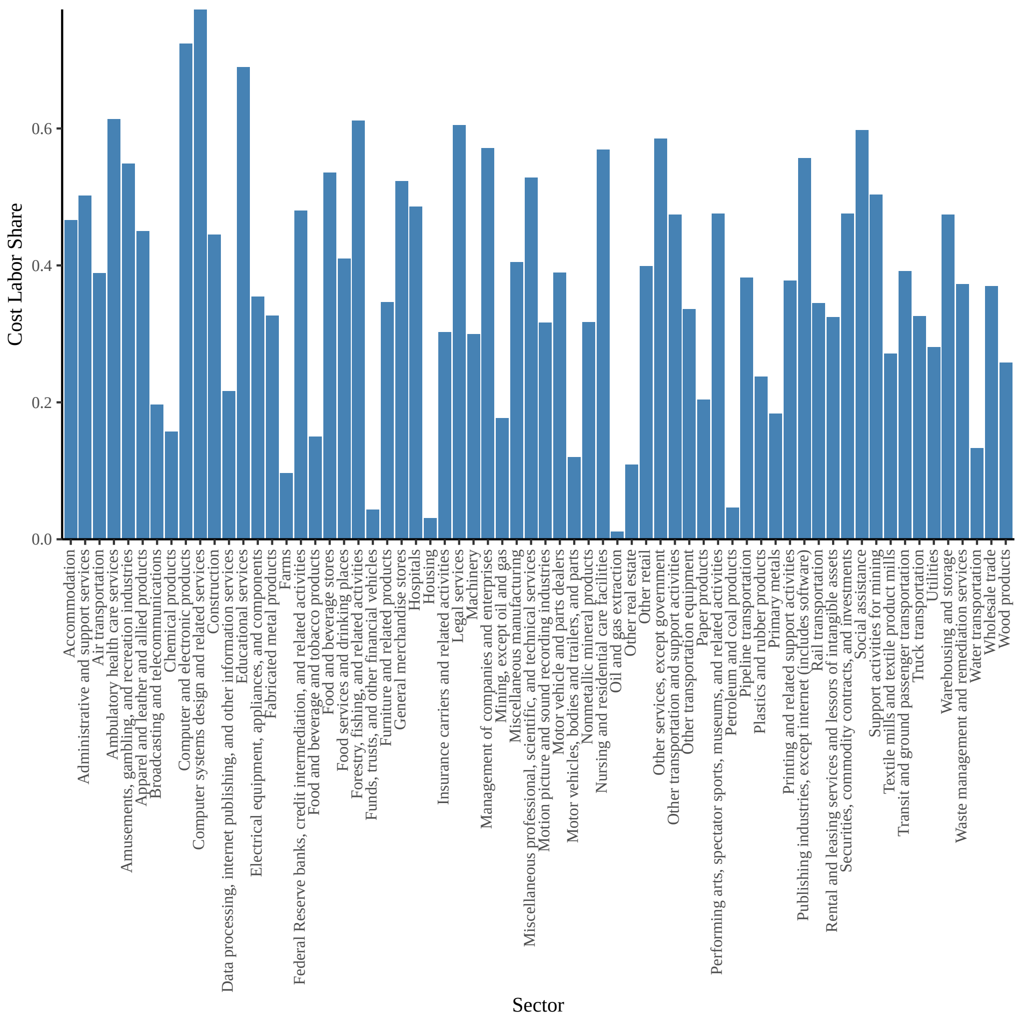



Figure C.2: U.S. sectoral labor share in 2019

*Notes:* This figure presents the sectoral labor share, using compensation of employees data from the use input-otput tables produced by the BEA in 2019. The industry classification is at the summary-level disaggregation, for a total of 66 sectors.

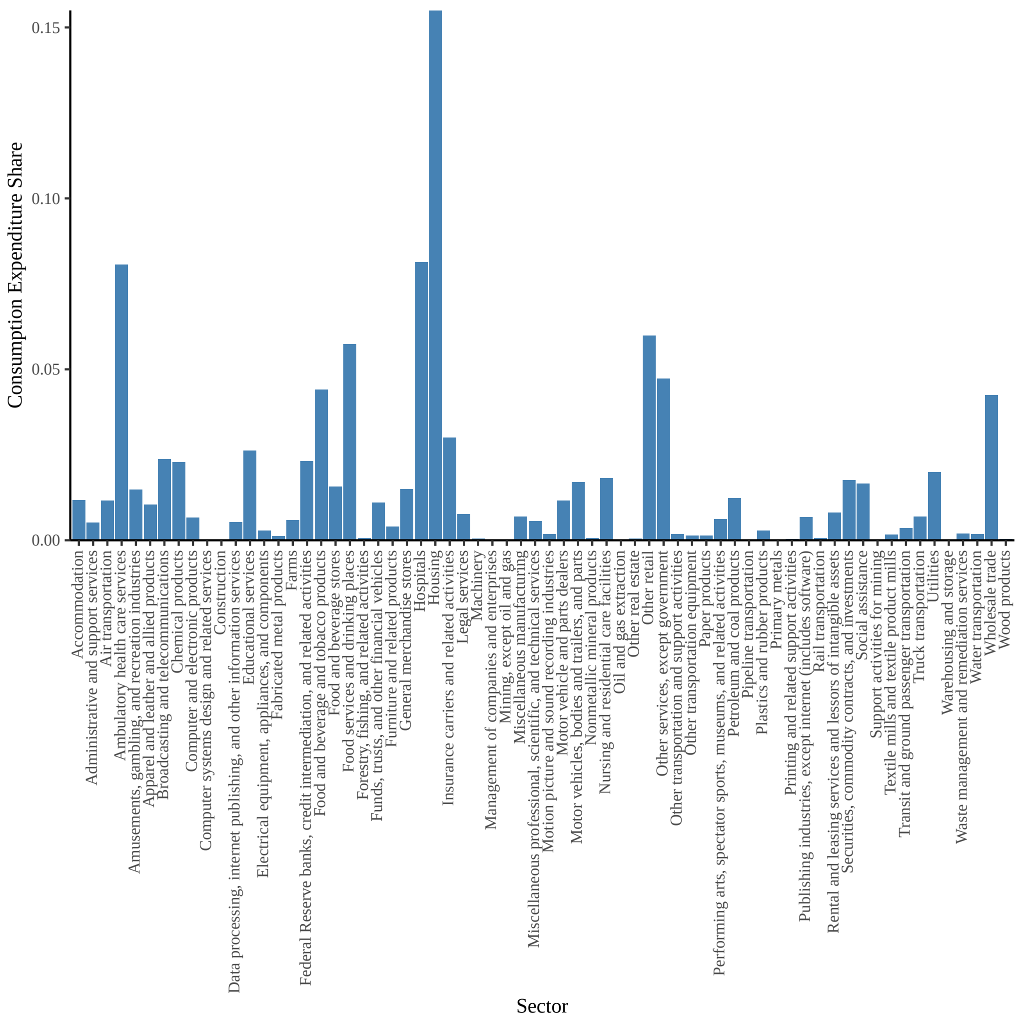



Figure C.3: U.S. sectoral consumption expenditure share in 2019

*Notes:* This figure presents the sectoral consumption expenditure share, using data from the use input-otput tables produced by the BEA in 2019. The industry classification is at the summary-level disaggregation, for a total of 66 sectors.

of price adjustment from the detail level to the summary level, we took the average frequency of price adjustment across detail level industries within a given summary level.

This procedure gives us the frequency of price adjustment for 50 sectors. For the 16 sectors that we were not able to calculate the frequency of price adjustment in this way, we impute their value using the average frequency of price adjustment across sectors in the data. The weighted average frequency of price changes across sectors is 0.185 (0.204), before (after) the continuous time transformation.[30] Figure C.4 presents the distribution of frequency of price adjustment across sectors in our data.

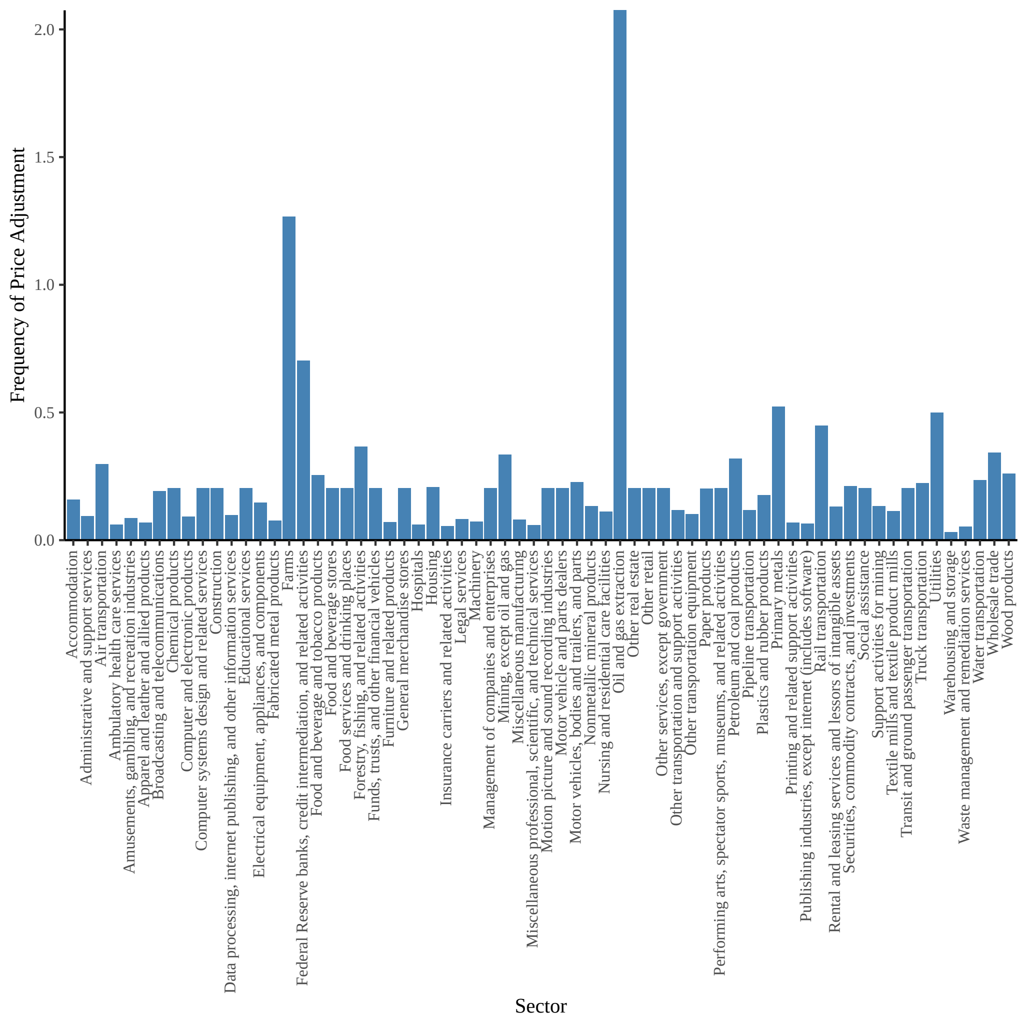



Figure C.4: U.S. sectoral frequency of price adjustment in 2002

*Notes:* This figure presents sectoral frequency of price adjustment in 2002, using data from Pasten, Schoenle, and Weber (2020). The industry classification is at the summary-level disaggregation, for a total of 66 sectors.

Weber (2020) used the tables for 2002.

[30] As we work in continuous time, we calculate its continuous time counterpart. Thus, let fpa be the frequency of price adjustment in Pasten, Schoenle, and Weber (2020). Then, the frequency of price adjustment used in this paper is $\theta = -\log(1 - fpa)$.