Inflation and GDP Dynamics in Production Networks: A Sufficient Statistics Approach*

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Abstract

We derive closed-form solutions and sufficient statistics for inflation and GDP dynamics in multi-sector New Keynesian economies with arbitrary input-output linkages. Analytically, we show how (1) production linkages amplify inflation and GDP persistence in response to monetary and sectoral shocks and (2) monetary policies that stabilize price indices or the GDP gap affect shock propagation. Quantitatively, sectors with large input-output adjusted price stickiness have disproportionate effects relative to their GDP shares: The three sectors with the highest contribution to the persistence of aggregate inflation have GDP shares of around zero but explain 16% of monetary non-neutrality.

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1 Introduction

Recent supply chain disruptions have underscored the importance of how production linkages impact the *dynamics* of sectoral prices, aggregate inflation, and GDP. For instance, monetary policymakers have been grappling with whether shocks to sectoral prices, e.g., oil or semiconductors, played any role in the rise of aggregate inflation, and if so, whether these effects have been persistent. In this paper, we answer the following question: In an economy with sticky prices and production networks, what determines each sector's contribution to the *persistence* and the magnitude of sectoral prices, aggregate inflation, and GDP responses to shocks?

In a dynamic multi-sector model, we analytically characterize how arbitrary input-output linkages interact with staggered heterogeneous sticky prices to amplify the persistence and the magnitude of inflation and GDP responses to monetary and sectoral shocks. These effects are quantitatively large. In the case of monetary shocks, production linkages of the U.S. economy quadruple monetary non-neutrality and double the half-life of the consumer price index (CPI) inflation response, underscoring the significant lagged effects of monetary policy (Friedman, 1961).

In the case of sectoral shocks, we first show that in the absence of endogenous monetary responses, inflation in an upstream but flexible price sector such as the Oil and Gas Extraction industry has a high but transitory pass-through to aggregate inflation. In contrast, inflation in an upstream but stickier sector such as the Semiconductor Manufacturing Machinery industry has persistent spillover effects on aggregate inflation with large GDP gap effects. Next, we show how monetary policies that endogenously respond to such shocks and aim to stabilize aggregate variables such as CPI inflation or GDP gap affect their pass-through. For instance, stabilizing aggregate inflation in response to Oil shocks *contracts* GDP *gap* significantly. Overall, our findings show that the responses of aggregate inflation and GDP gap to such endogenous monetary policies interact non-trivially with their sectoral origins and our theory analytically unveils these interactions.

We derive these results in a production network economy with multiple sectors. Each sector contains a continuum of monopolistically competitive intermediate goods firms which use labor and goods from other sectors to produce with sector-specific production functions subject to sectoral productivity shocks. These firms also make staggered forward-looking pricing decisions, where price changes arrive at sector-specific Poisson rates as in Calvo (1983). A competitive producer in each sector aggregates these intermediate products into a final sectoral good and sells it for household consumption and for intermediate input use across sectors. Importantly, we do not restrict or impose any symmetries across sectors in terms of price change frequencies or input-output linkages. In our benchmark, monetary policy controls nominal GDP.[1] In this framework, we derive closed-form solutions for the local dynamics of the model around an efficient steady state.

Our first result is that the local dynamics of this model, in response to any arbitrary path of shocks, is summarized by a system of second-order differential equations, which can be interpreted as the economy's sectoral Phillips curves. Importantly, while such equations are generally necessary conditions for equilibrium, we show (1) that in our benchmark economy, these are both *necessary* and *sufficient* for characterizing the path of sectoral prices and (2) how particular adjustments of these sectoral Phillips curves under endogenous monetary policy preserves their *sufficiency*. Relying on this sufficiency condition, we show that *all* model parameters affect the dynamics of the model exclusively through a novel adjustment of the Leontief matrix that takes the duration of price spells across sectors into account. The explicit solution to this system reveals that the sufficient statistic for the dynamics of all model variables in response to any path of shocks is the principal square root of the duration-adjusted Leontief (PRDL) matrix. Intuitively, a particular interaction of price stickiness and input-output linkages fully pins down the model IRFs, all of which decay exponentially at the rate of the PRDL matrix.

Two observations immediately follow from this result. (1) *Monetary shocks* have distortionary and asymmetric effects on *relative* sectoral prices, governed by the eigen-decomposition of the PRDL matrix: All else equal, sectors that spend more on stickier suppliers have more persistent responses and disproportionally affect the persistence of aggregate inflation. (2) The input-output matrix has a dual role in the propagation of *sectoral shocks*. First, consistent with insights from static models, input-output linkages amplify the effects of sectoral shocks through the inverse Leontief matrix and increase the pass-through of these shocks on impact. Second, a novel dynamic force amplifies this total pass-through by increasing the persistence of IRFs. Importantly, this second force is independent of the role of the inverse Leontief matrix. Instead, it stems from the precise interaction of input-output linkages with staggered price changes through the PRDL matrix. We show that these two separate forces accumulate: more input-output linkages amplify static propagation through the inverse Leontief matrix and create dynamic effects

[1] In extensions, we also study rules that aim to stabilize arbitrary price indices as well as a Taylor rule.

that last longer through the PRDL matrix.

Having established the importance of the PRDL matrix in governing the dynamics of the log-linearized model, we next derive a series of new analytical results that shed light on the economic forces encoded by this matrix (through its eigendecomposition). We use perturbation theory to approximate the eigenvalues and eigenvectors of the PRDL matrix based on the underlying model parameters.[2] This approach allows us to prove three key and novel results on how input-output linkages amplify (1) the persistence of inflation response to monetary shocks in all sectors, (2) the degree of monetary non-neutrality, and (3) the pass-through of sectoral inflation to aggregate inflation. These analytical results uncover how stickiness trickles to downstream sectors. In particular, sectors with large input-output adjusted price spell durations play a disproportionate role (relative to their expenditure shares) in amplifying monetary non-neutrality and inflation persistence.

Using input-output tables, price adjustment frequencies, and consumption shares, we construct our sufficient statistics for the U.S. and quantify the importance of production networks for propagation of shocks. In the case of monetary shocks, we find that production linkages quadruple the cumulative response of GDP and double the half-life of the consumer price index (CPI) inflation response. Underneath these aggregate responses, we identify a rich distribution of sectoral responses, with few sectors disproportionately affecting monetary non-neutrality and inflation persistence. This exercise highlights how distortions in the distribution of relative prices can lead to a persistent aggregate inflation response that is driven by more flexible sectors in the short run but by stickier sectors in the long run, with the network amplifying these interconnections. To illustrate this last point, in a counterfactual exercise, we find that dropping the top three sectors with the largest input-output adjusted price spell durations reduces monetary non-neutrality by 16 percent, even though the combined (direct) GDP share of these three sectors is approximately zero.

We then quantify the pass-through of sectoral shocks to aggregate inflation *on impact*. To do so, we consider idiosyncratic sectoral shocks that increase the inflation of their corresponding sector by 1 percent. We then measure the spillover pass-through of this shock as its impact on aggregate inflation *minus* the direct effect coming from the expenditure share of its sector (so that in the absence of production linkages, this pass-through is zero). While we provide comprehensive rankings of sectors, we use two industries that have been salient recently as informative examples of our analysis: the Oil

[2] We later verify that this approximation is remarkably accurate for the input-output matrix in the U.S.

and Gas Extraction industry and the Semiconductor Manufacturing Machinery industry. We find that the Oil and Gas Extraction industry is among the top sectors that have a large spillover pass-through to aggregate inflation on impact, due to its role as an input to many sectors.

Next, we quantify the effects of these sectoral shocks on the *persistence* of aggregate inflation response. Relying on our perturbed eigenvalues, we show that the key determinant of these effects is an input-output adjusted duration of price spells within these sectors. To provide concrete examples, this adjusted duration in Oil and Gas Extraction industry is relatively small due to its high price flexibility. Thus, a shock to this sector does not lead to persistent aggregate inflation effects. In contrast, the Semiconductor Manufacturing Machinery industry has very persistent aggregate inflation effects because its adjusted duration is relatively larger. Moreover, to connect these persistent responses with the real effects of sectoral shocks, we also show that sectoral shocks that cause more persistent inflation responses also lead to greater GDP gap effects.

Finally, having established these analytical and quantitative results on the *separate* roles of monetary and sectoral shocks, we study the propagation of sectoral shocks when monetary policy endogenously responds to neutralize their inflationary effects. This is non-trivial because while in benchmark New Keynesian (NK) models inflationary pressures are determined by the slope of the *aggregate* Phillips curve (elasticity of inflation to output gap), multisector economies with input-output linkages can no longer be summarized by one elasticity and the whole distribution of relative prices, and how they are affected by policy, jointly determine the path of sectoral and aggregate prices. Nonetheless, our contribution here is to show that the path of prices under such endogenous policies; i.e., CPI inflation stabilization or GDP gap stabilization, are still fully characterized by proper adjustments of the PRDL matrix that take the effects of these policies into account. This observation also provides a new perspective into understanding how such policies interact with sectoral origins of shocks: For instance, in the case of GDP gap stabilization, we show that propagation under such a policy is identical to that of an economy with a richer production network in which monetary policy keeps interest rates constant. The key driving force for this result is that endogenous policy creates further interactions across the economy by relating the prices of different sectors to each other through the objective of policy, which can be captured by adjusting the PRDL matrix to take such interactions into account. The upshot is that the PRDL matrix remains a sufficient statistic for the dynamics of the model under these policies.

We then show that these adjustments of the PRDL matrix are crucial for understanding the real effects of these policies. For instance, we find that stabilizing aggregate inflation in response to Oil TFP shocks *contracts* the GDP gap significantly due to the indirect effects of this policy on other sectors.[3] Accordingly, to avoid such large contractionary effects, a GDP gap stabilization policy lets the inflationary effects of Oil shocks to pass-through almost completely to aggregate inflation. This is in contrast to inflation originating in sectors with large network-adjusted price stickiness. For instance, stabilizing aggregate inflation conditional on an inflationary TFP shock to the Semiconductor Manufacturing Machinery industry is not very costly in terms of GDP gap. This is because while this industry is also an input to many sectors similar to the Oil industry, it has a much higher duration-adjusted price stickiness relative to its downstream sectors. Thus, the contractionary effects of stabilizing aggregate inflation are also smaller because sectoral inflation in that sector does not distort relative prices as much.

Related Literature. Our paper is related to the recent work studying multi-sector NK models with production networks and relative price distortions (La'O and Tahbaz-Salehi, 2022, Rubbo, 2023, Lorenzoni and Werning, 2023a,b).[4] In a similar framework, our main contribution is to provide analytical characterization of shock propagation and inflation dynamics; in particular, by using spectral approximation methods. Specifically, we provide analytical expressions for (1) propagation of *sectoral* shocks and their impact on aggregates, (2) impact of *monetary* shocks, particularly focusing on how they distort relative prices leading to disproportionate effects of few sectors on inflation and GDP dynamics, and (3) how endogenous stabilization monetary policies interact with the sectoral origins of inflationary shocks and shape the dynamics of aggregate inflation and GDP. Methodologically, our use of spectral methods to analyze shock propagation in production networks with sticky prices is novel and follows Liu and Tsyvinski (2021) who were the first to use spectral methods in analyzing the dynamics of a real production

[3] This result is in contrast to the standard NK model where stabilizing TFP-driven inflation is expansionary to the output gap because sticky prices do not increase as much as they would under flexible prices. In the extreme case when monetary policy fully stabilizes TFP-driven inflation in those models, the output gap is also stabilized.

[4] More broadly, a large sticky price literature discusses the role of relative price distortions in aggregate inflation. Ball and Mankiw (1995) show how this role arises due to sectoral heterogeneity in the size of shocks. Heterogeneity in price stickiness across sectors in a multi-sector model was also emphasized in Woodford (2003) and Ruge-Murcia and Wolman (2022). Moreover, a similar channel exists in multi-country models (Benigno (2004); Gali and Monacelli (2008)) as well as in models with both sticky prices and wages (Erceg, Henderson, and Levin (2000); Blanchard and Gali (2007);Gali (2008)). We discuss more precisely the connections of some of these papers with our model and results later.

network economy with adjustment costs. Moreover, we contribute further by applying a new spectral approximation method to sticky price economies with production networks that yields explicit expressions for how individual sectors' characteristics contribute to the dynamics of GDP and prices.

Our analytical results on the real effects of monetary shocks are related to two broader strands of the literature. First, they connect to Carvalho (2006) and Nakamura and Steinsson (2010) which showed that heterogeneous price stickiness amplifies monetary non-neutrality. Second, our findings on how production linkages amplify real effects of monetary shocks build on the insights of Blanchard (1983), Basu (1995) and more recently La'O and Tahbaz-Salehi (2022) which showed that such amplification stems from strategic complementarities introduced by production networks. More recently, Carvalho, Lee, and Park (2021), Pasten, Schoenle, and Weber (2020), Woodford (2021), and Ghassibe (2021) study the transmission of monetary shocks in specific production networks.[5] Our contribution is to study a multi-sector NK model with *unrestricted* input-output linkages.

Our results on the propagation of sectoral shocks in models with production networks build on a rich literature, mostly in settings without nominal rigidities. Long and Plosser (1983), Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), Jones (2013) are important contributions and Carvalho (2014), Carvalho and Tahbaz-Salehi (2019) provide comprehensive surveys of the literature. In more recent work, Guerrieri, Lorenzoni, Straub, and Werning (2020) characterize how supply shocks to a sector can lead to aggregate contractions, and Minton and Wheaton (2023) provide empirical support for the dynamic propagation of price stickiness through production networks. Our contribution is to characterize the forces that determine the propagation of monetary and sectoral shocks under nominal rigidities in dynamic settings with production networks.[6]

Finally, our paper also contributes to the literature on aggregate inflation persistence and the long and variable lags of monetary policy. We show both analytically and numerically how aggregate inflation persistence is the driver of aggregate GDP dynamics in our model, which relates it to the important early work on this topic by Fuhrer and Moore (1995); Fuhrer (2010); and Fuhrer (2018).[7] Moreover, by emphasizing how production net-

[5] This also relates our work to Wang and Werning (2021) and Alvarez, Lippi, and Souganidis (2022), which derive similar statistics with oligopolies and menu costs featuring strategic complementarities, but not production networks.

[6] Other papers, such as Taschereau-Dumouchel (2020), consider endogenous production networks in real models. We use exogenous production networks, but we study a dynamic model with sticky prices.

[7] See also Dittmar, Gavin, and Kydland (2005); Guerrieri (2006); Sbordone (2007); Benati (2008); Kurozumi and Zandweghe (2023); and Gallegos (2023) for contributions on the drivers of endogenous inflation

works increase the persistence of aggregate inflation following a monetary policy shock, we connect to papers focused on long and variable lags in monetary policy transmission, starting with the venerable work of Friedman (1961).[8]

2 Model

2.1. Environment

Time is continuous and is indexed by $t \in \mathbb{R}_+$. The economy consists of a representative household, monetary and fiscal authorities, and n sectors with input-output linkages. In each sector $i \in [n] \equiv \{1, 2, \dots, n\}$, a unit measure of monopolistically competitive firms use labor and goods from all sectors to produce and supply to a competitive final good producer within the same industry. These final goods are sold to the household and other industries.

Household. The representative household demands the final goods produced by each industry, supplies labor in a competitive market, and holds nominal bonds with nominal yield i_t. Household's preferences over consumption C and labor supply L is $U(C) - V(L)$, where U and V are strictly increasing with Inada conditions, and $U''(.) < 0$, $V''(.) > 0$. Household solves:

$$\max_{\{(C_{i,t})_{i\in[n]}, L_t, B_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t}[U(C_t) - V(L_t)]\mathrm{d}t \tag{1}$$

$$s.t. \quad \sum_{i\in[n]} P_{i,t}C_{i,t} + \dot{B}_t \leq W_t L_t + i_t B_t + \text{Profits}_t - T_t, \qquad C_t \equiv \Phi(C_{1,t}, \dots, C_{n,t}) \tag{2}$$

Here, $\Phi(.)$ defines the consumption index C_t over the household's consumption from sectors $(C_{i,t})_{i\in[n]}$. It is degree one homogeneous, strictly increasing in each $C_{i,t}$, satisfying Inada conditions. L_t is labor supply at wage W_t, $P_{i,t}$ is sector i's final good price, B_t is demand for nominal bonds, Profits_t denote all firms' profits rebated to the household, and T_t is a lump-sum tax.

Monetary and Fiscal Policy. For our baseline, we assume monetary authority directly controls the path of nominal GDP, $\{M_t \equiv P_t C_t\}_{t\geq 0}$, where P_t is the consumer price index (CPI).[9] In Section 4, we study the more general case of policies with endogenous feedback such as those that aim to stabilize a particular price index or the GDP gap. Moreover, a Taylor rule extension is in Section 5.2. The fiscal authority taxes or subsidizes intermediate

persistence.

[8]Other more recent papers on this topic include Bryan and Gavin (1994); Kilponen and Leitemo (2011).

[9]Such policy can be implemented by a cash-in-advance constraint (e.g. La'O and Tahbaz-Salehi, 2022), money in utility (e.g. Golosov and Lucas, 2007) or NGDP growth targeting (e.g. Afrouzi and Yang, 2019).

firms' sales in each sector i at a possibly time-varying rate $\tau_{i,t}$, lump-sum transferred back to the household. A *wedge shock* to sector i is an *unexpected* disturbance in that sector's taxes.

Final Good Producers. A competitive final good producer in each industry i buys from a continuum of intermediate firms in its sector, indexed by $ij: j \in [0,1]$, and produces a final sectoral good using a CES production function. The profit maximization problem of this firm is:

$$\max_{(Y^d_{ij,t})_{j\in[0,1]}} P_{i,t}Y_{i,t} - \int_0^1 P_{ij,t}Y^d_{ij,t}\mathrm{d}j \quad s.t. \quad Y_{i,t} = \left[\int_0^1 (Y^d_{ij,t})^{1-\sigma_i^{-1}}\mathrm{d}j\right]^{\frac{1}{1-\sigma_i^{-1}}} \tag{3}$$

where $Y^d_{ij,t}$ is the producer's demand for variety ij at price $P_{ij,t}$, $Y_{i,t}$ is its production at price $P_{i,t}$, and $\sigma_i > 1$ is the substitution elasticity across varieties in i. Thus, demand for variety ij is:

$$Y^d_{ij,t} = \mathscr{D}(P_{ij,t}/P_{i,t}; Y_{i,t}) \equiv Y_{i,t}\left(\frac{P_{ij,t}}{P_{i,t}}\right)^{-\sigma_i} \quad \text{where} \quad P_{i,t} = \left[\int_0^1 P_{ij,t}^{1-\sigma_i}\mathrm{d}j\right]^{\frac{1}{1-\sigma_i}} \tag{4}$$

Final good producers define a unified good for each industry and have zero value added due to being competitive and constant returns to scale (CRS) production.

Intermediate Goods Producers. The intermediate good producer ij uses labor as well as the sectoral goods as inputs and produces with the following CRS production function:

$$Y^s_{ij,t} = Z_{i,t}F_i(L_{ij,t}, X_{ij,1,t}, \dots, X_{ij,n,t}) \tag{5}$$

where $Z_{i,t}$ is sector i's Hicks-neutral productivity, $L_{ij,t}$ is firm ij's labor demand, and $X_{ij,k,t}$ is its demand for sector k's final good. The function F_i is strictly increasing in all arguments with Inada conditions. The firm's total cost for producing output Y, given $\mathbf{P}_t \equiv (W_t, P_{i,t})_{i\in[n]}$, is:

$$\mathscr{C}_i(Y;\mathbf{P}_t, Z_{i,t}) \equiv \min_{L_{ij,t}, X_{ij,k,t}} W_tL_{ij,t} + \sum_{k\in[n]} P_{k,t}X_{ij,k,t} \quad s.t. \quad Z_{i,t}F_i(L_{ij,t}, X_{ij,1,t}, \dots, X_{ij,n,t}) \geq Y \tag{6}$$

In each sector i, firms set their prices under a Calvo friction, where i.i.d. price change opportunities arrive at Poisson rates θ_i. Given its cost in Equation (6) and its demand in Equation (4), a firm ij that has the opportunity to change its price at time t chooses its *reset price*, denoted by $P^{\#}_{ij,t}$, to maximize the expected net present value of its profits until the next price change. Formally, $P^{\#}_{ij,t}$ solves:

$$\max_{P_{ij,t}} \int_0^\infty \theta_i e^{-(\theta_i h + \int_0^h i_{t+s}\mathrm{d}s)}\left[(1-\tau_{i,t+h})P_{ij,t}\mathscr{D}(P_{ij,t}/P_{i,t+h}; Y_{i,t+h}) - \mathscr{C}_i(Y^s_{ij,t+h};\mathbf{P}_{t+h}, Z_{i,t+h})\right]\mathrm{d}h$$

$$s.t. \quad Y^s_{ij,t+h} \geq \mathscr{D}(P_{ij,t}/P_{i,t+h}; Y_{i,t+h}), \quad \forall h \geq 0 \tag{7}$$

where $\theta_i e^{-\theta_i h}$ is the duration density of the next price change, $e^{-\int_0^h i_{t+h} \mathrm{d}s}$ is the discount rate based on nominal rates, and $\tau_{i,t}$ is the tax/subsidy rate on sales. Were prices flexible, maximizing net present value of profits would be equivalent to choosing *desired* prices, denoted by $P^*_{ij,t}$, that maximized firms' static profits within every instant. $P^*_{ij,t}$ solves:

$$\max_{P_{ij,t}} (1-\tau_{i,t}) P_{ij,t} \mathscr{D}(P_{ij,t}/P_{i,t}; Y_{i,t}) - \mathscr{C}_i(Y^s_{ij,t}; \mathbf{P}_t, Z_{i,t}) \quad s.t. \quad Y^s_{ij,t} \geq \mathscr{D}(P_{ij,t}/P_{i,t}; Y_{i,t}) \tag{8}$$

Equilibrium Definition. An equilibrium is a set of allocations for households and firms, monetary and fiscal policies, and prices such that: (1) given prices and policies, the allocations are optimal for households and firms, and (2) markets clear. A precise definition is in Appendix C.

2.2. Log-Linearized Approximation

We log-linearize this economy around an efficient zero inflation steady-state, derivations of which are in Appendix S.M.1. For our baseline analysis, we use Golosov and Lucas (2007)'s preferences, $U(C) - V(L) = \log(C) - L$, which simplifies the analytical expressions. In Section 5.1, we consider a more general specification with partially elastic labor supply. Going forward, small letters denote the log deviations of their corresponding variables from their steady-state values.

Sectoral Prices. While prices are staggered within sectors, the Calvo assumption implies that we can fully characterize aggregate sectoral prices by desired and reset prices. First, desired prices are equal to firms' marginal costs, $(mc_{i,t})_{i\in[n]}$, up to a wedge that captures markups or other distortions, $(\omega_{i,t})_{i\in[n]}$. With input-output linkages, $mc_{i,t}$ depends on the aggregate wage, w_t, sectoral prices, $(p_{k,t})_{k\in[n]}$, and the sectoral productivity, $z_{i,t}$:

$$p^*_{i,t} \equiv \omega_{i,t} + mc_{i,t}, \quad mc_{i,t} \equiv \alpha_i w_t + \textstyle\sum_{k\in[n]} a_{ik} p_{k,t} - z_{i,t}, \quad \omega_{i,t} \equiv \log\left(\frac{\sigma_i}{\sigma_i - 1} \times \frac{1}{1-\tau_{i,t}}\right) \tag{9}$$

where α_i and $a_{i,k}$ are sector i's firms' labor share and expenditure share on sector k's final good in the steady-state, respectively. Thus, the steady-state *input-output matrix* is $\mathbf{A} \equiv [a_{ik}] \in \mathbb{R}^{n\times n}$.

Second, the reset price in sector i is the average of all *future* desired prices, discounted at rate ρ and the probability density of the time between price changes, $e^{-(\rho+\theta_i)h}$:

$$p^\#_{i,t} = (\rho + \theta_i) \int_0^\infty e^{-(\rho+\theta_i)h} p^*_{i,t+h} \mathrm{d}h \tag{10}$$

Finally, given sector i's initial aggregate price at $t = 0$, $p_{i,0^-}$, the *aggregate* sectoral price

$p_{i,t}$ is an average of the *past* reset prices, weighted by the density of time between price changes:

$$p_{i,t} = \theta_i \int_0^t e^{-\theta_i h} p^{\#}_{i,t-h} \mathrm{d}h + e^{-\theta_i t} p_{i,0^-} \tag{11}$$

Aggregate Price and GDP. The household's demand for goods defines the aggregate Consumer Price Index (CPI) as the expenditure share weighted average of sectoral prices:

$$p_t = \sum_{i \in [n]} \beta_i p_{i,t}, \quad \text{with} \quad \sum_{i \in [n]} \beta_i = 1 \tag{12}$$

where $\boldsymbol{\beta} = (\beta_i)_{i \in [n]}$ is the vector of the household's expenditure shares in the efficient steady-state.

The aggregate GDP, y_t, is equal to aggregate consumption and is given by the difference between the nominal GDP, m_t, and the CPI, p_t: $y_t \equiv m_t - p_t$. Fully elastic labor supply implies that the wage is equal to nominal demand:

$$w_t = p_t + y_t = m_t \quad \text{(fully elastic labor supply)} \tag{13}$$

Equilibrium in the Approximated Economy. Given a bounded path for $(\boldsymbol{\omega}_t, \boldsymbol{z}_t, m_t)_{t \geq 0}$, an equilibrium is a path for GDP, wage and prices, $\vartheta \equiv \{y_t, w_t, p_t, (p^*_{i,t}, p^{\#}_{i,t}, p_{i,t})_{i \in [n]}\}_{t \geq 0}$, such that given a vector of initial sectoral prices, $\mathbf{p}_{0^-} = (p_{i,0^-})_{i \in [n]}$, ϑ solves Equations (9) to (13).

Flexible Prices and GDP. Consider a counterfactual economy where all prices are flexible. By Equation (9), we can derive *flexible prices* of this economy, denoted by $\mathbf{p}^f_t \in \mathbb{R}^n$, as:

$$\mathbf{p}^f_t = w_t \boldsymbol{\alpha} + \mathbf{A}\mathbf{p}^f_t + \boldsymbol{\omega}_t - \boldsymbol{z}_t \quad \Longrightarrow \quad \mathbf{p}^f_t = m_t \mathbf{1} + \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t) \tag{14}$$

where $\boldsymbol{\alpha} \equiv (\alpha_i)_{i \in [n]}$ contains labor shares, $\mathbf{1}$ is the vector of ones, and $\boldsymbol{\Psi} \equiv (\mathbf{I} - \mathbf{A})^{-1}$ is the inverse Leontief matrix. A key observation is that $\mathbf{p}^f_t$ is only a function of exogenous shocks and model parameters. We can also derive the *flexible price GDP*, y^f_t, in this counterfactual economy as:

$$y^f_t = m_t - \boldsymbol{\beta}^\top \mathbf{p}^f_t = \underbrace{\boldsymbol{\lambda}^\top \boldsymbol{z}_t}_{\text{aggregate TFP}} - \underbrace{\boldsymbol{\lambda}^\top \boldsymbol{\omega}_t}_{\text{labor wedge}}, \qquad \boldsymbol{\lambda} \equiv \left(\tfrac{P_i Y_i}{PC}\right)_{i \in [n]} = \boldsymbol{\Psi}^\top \boldsymbol{\beta} \tag{15}$$

where $\boldsymbol{\lambda}$ is the vector of Domar weights in the steady state.[10] Equation (15) shows that two terms determine flexible GDP around the efficient steady-state up to first order: (1)

[10]The Domar weight of a sector i, λ_i, is the ratio of its total sales to the household's total nominal expenditures. Baqaee and Farhi (2020) emphasize the distinction between cost-based and sales-based input-output matrices and Domar weights. In an efficient equilibrium, like the one we linearize around, the two are the same.

the aggregate TFP, which is the Domar-weighted sectoral productivities (Hulten, 1978), (2) the labor wedge due to distortions, which is the Domar-weighted wedges across sectors (Bigio and La'O, 2020).

3 Sufficient Statistics

Here, we solve sectoral price dynamics in closed form and derive our sufficient statistics results. We then measure these sufficient statistics for the U.S. economy and provide quantitative results on aggregate and sectoral shocks. All proofs are included in Appendix B.

3.1. Dynamics of Prices

Let $\mathbf{p}_t \equiv (p_{i,t})_{i\in[n]}$, $\mathbf{p}_t^\# \equiv (p_{i,t}^\#)_{i\in[n]}$ and $\mathbf{p}_t^* \equiv (p_{i,t}^*)_{i\in[n]}$ be the vectors of sectoral aggregate, reset and desired prices, respectively. Using Equations (9) and (14):[11]

$$\mathbf{p}_t^* = (\mathbf{I}-\mathbf{A})\mathbf{p}_t^f + \mathbf{A}\mathbf{p}_t \tag{16}$$

where $\mathbf{p}_t^f$ is the vector of flexible equilibrium prices in Equation (14). Equation (16) shows that firms' desired prices across sectors is a convex combination of *exogenous* flexible equilibrium prices and *endogenous* sectoral prices in the sticky price economy, with the input-output matrix $\mathbf{A}$ fully capturing the *strategic complementarities* induced by production linkages across the economy (Blanchard, 1983, Basu, 1995, La'O and Tahbaz-Salehi, 2022).

Accordingly, reset and sectoral prices in Equations (10) and (11) solve:

$$\boldsymbol{\pi}_t^\# \equiv \dot{\mathbf{p}}_t^\# = (\rho\mathbf{I}+\boldsymbol{\Theta})(\mathbf{p}_t^\# - \mathbf{p}_t^*), \qquad \text{forward-looking with} \qquad \lim_{t\to\infty} e^{-(\rho\mathbf{I}+\boldsymbol{\Theta})t}\mathbf{p}_t^\# = 0, \tag{17}$$

$$\boldsymbol{\pi}_t \equiv \dot{\mathbf{p}}_t = \boldsymbol{\Theta}(\mathbf{p}_t^\# - \mathbf{p}_t), \qquad \text{backward-looking with} \qquad \mathbf{p}_0 = \mathbf{p}_{0^-} \tag{18}$$

Here, $\boldsymbol{\pi}_t^\#$ and $\boldsymbol{\pi}_t$ are the *inflation rates* in reset and aggregate prices across sectors, respectively. $\boldsymbol{\Theta} = \operatorname{diag}(\theta_i) \in \mathbb{R}^{n\times n}$ is a diagonal matrix, with its i'th diagonal entry representing the frequency of price adjustments in sector i.[12] The memorylessness of the Poisson price adjustments (Calvo assumption) allows us to represent this system only in terms of sectoral prices, $\mathbf{p}_t$:

Proposition 1. *Given a vector of initial prices $\mathbf{p}_0 = \mathbf{p}_{0^-}$, the following set of differential*

[11] Using $\boldsymbol{\alpha} = (\mathbf{I}-\mathbf{A})\mathbf{1}$, the vector form of Equation (9) is $\mathbf{p}_t^* = (\mathbf{I}-\mathbf{A})(\mathbf{1}w_t + \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \mathbf{z}_t)) + \mathbf{A}\mathbf{p}_t$.

[12] In this draft, we frequently use the exponential function of square matrices, defined by its corresponding power series: $\forall \mathbf{X} \in \mathbb{R}^{n\times n}$, $e^{\mathbf{X}} \equiv \sum_{k=0}^{\infty} \mathbf{X}^k / k!$, which is well-defined because these series always converge.

equations are necessary *and* sufficient *for the non-explosive dynamics of sectoral prices:*

$$\dot{\boldsymbol{\pi}}_t = \rho \boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{p}_t - \mathbf{p}_t^f), \qquad \mathbf{p}_0 = \mathbf{p}_{0^-} \text{ given } \& \lim_{t\to\infty} \|\mathbf{p}_t - \mathbf{p}_t^f\| < \infty. \quad (19)$$

We discuss the main implications of Proposition 1 in the following four remarks.

Remark 1. *Equation (19) represents the* sectoral Phillips curves *of this economy in vector form, linking changes in inflation to the* gap *between prices and their counterparts in a flexible economy. The matrix* $\boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})$*—the Leontief matrix,* $\mathbf{I} - \mathbf{A}$*, adjusted by a quadratic form of price adjustment frequencies,* $\boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})$*—encodes the slopes of these Phillips curves.*

Equation (19) differs from the usual representations of Phillips curves featuring output gap. Such an equivalent representation exists for Equation (19), which we discuss in detail in Section 4. However, we start with the representation above because it is the clearest way to demonstrate the following remarks and derive our analytical results.

Remark 2. *Sectoral Phillips curves, with boundary conditions* $\mathbf{p}_0 = \mathbf{p}_{0^-}$ *and non-explosive prices, uniquely pin down the path of sectoral prices for a given path* $(\mathbf{p}_t^f)_{t\geq 0}$.

The key to this observation is that the only endogenous variables in the system of second-order differential equations in Equation (19) are nominal prices and their inflation rates, $\mathbf{p}_t$ and $\boldsymbol{\pi}_t$, with $\mathbf{p}_t^f$ acting as an *exogenous* forcing term. Intuitively, nominal prices in the sticky price economy should adjust towards their flexible levels, $\mathbf{p}_t^f$. This is formalized in Equation (19), where inflation in sectoral prices depends solely on the time series of nominal price gaps, $\mathbf{p}_t - \mathbf{p}_t^f$.

Remark 3. *All shocks* $(\boldsymbol{\omega}_t, \boldsymbol{z}_t, m_t)_{t\geq 0}$ *affect price dynamics* only *through* $(\mathbf{p}_t^f)_{t\geq 0}$.

Remark 3 demonstrates the power of expressing inflation dynamics in terms of *nominal price gaps*. It implies that solving for the dynamics of prices for a given path of $\mathbf{p}_t^f$ is equivalent to characterizing impulse response functions of *all* the prices in the economy to all three types shocks–TFP, markup/wedge, and monetary–in a unified framework.

Remark 4. *All parameters affect the dynamics of sectoral prices* only *through the adjusted Leontief matrix* $\boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})$*, and the household's discount rate,* ρ.

Intuitively, the dynamics of prices in a production network depend on the frequency of price adjustments ($\boldsymbol{\Theta}$) and how these shocks propagate through input-output linkages (the Leontief matrix). Proposition 1 formally shows how these two mechanisms interact through the matrix $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ and the parameter ρ, which also shows up independently through $\rho\boldsymbol{\pi}_t$ in Equation (19).[13]

Since $\mathbf{p}_t^f$ is an exogenous forcing term to the system of differential equations in Equation (19), we can fully solve this system and derive an analytical expression for the evolution of sectoral prices as a function of the time-path of $\mathbf{p}_t^f$. To do so, we first define our *duration-adjusted Leontief matrix*, which plays a key role in our analytical results:

$$\mathbf{\Gamma} \equiv \mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A}) + \frac{\rho^2}{4}\mathbf{I} \tag{20}$$

Note that $\mathbf{\Gamma}$ is the slope matrix $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ from Remark 1 shifted by $\rho^2/4$ times the identity matrix, which captures the independent role of ρ in price dynamics through the term $\rho\boldsymbol{\pi}_t$ in Equation (19).[14] Moreover, as we will see below, the solution to the differential equations in Proposition 1 depends on the principal square root of $\mathbf{\Gamma}$, denoted by $\sqrt{\mathbf{\Gamma}}$, defined such that

$$\mathbf{\Gamma} = (\sqrt{\mathbf{\Gamma}})^2 = \sqrt{\mathbf{\Gamma}}\cdot\sqrt{\mathbf{\Gamma}}, \qquad \lambda \in \operatorname{eig}(\sqrt{\mathbf{\Gamma}}) \implies \operatorname{Re}(\lambda) > 0 \tag{21}$$

Here "$\cdot$" denotes standard matrix multiplication. In general, an $n \times n$ matrix can have between 0 and 2^n square roots.[15] The *principal* square root of a matrix, if exists, is the one whose eigenvalues have positive real parts, which is the verbal description of the second part of Equation (21). Thus, to confirm that $\sqrt{\mathbf{\Gamma}}$ is well-defined, we show its existence and uniqueness in the following Lemma:

Lemma 1. *The principal square root of* $\mathbf{\Gamma}$, *denoted by* $\sqrt{\mathbf{\Gamma}}$, *exists and is unique; the matrices* $\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I}$ *and* $\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I}$ *are positive stable—i.e., all their eigenvalues have positive real parts.*

In short, the proof of Lemma 1 relies on showing that $\mathbf{\Gamma}$ is an M-matrix and applies Theorem 5 in Alefeld and Schneider (1982) to show the existence and uniqueness of the principal square root $\sqrt{\mathbf{\Gamma}}$ under the assumptions that all sectors have strictly positive labor shares and price adjustment frequencies. We can now state the analytical solution to the dynamics of sectoral prices:

Proposition 2. *Suppose* $\mathbf{p}_t^f$ *is piece-wise continuous and bounded.*[16] *Then, given* $\mathbf{p}_t^f$, *a*

[13] Moreover, note that substitution elasticities across different inputs have no impact on price dynamics at the first order. This is due to the flatness of the marginal cost function with respect to inputs at the optimum by Shephard's Lemma (see, e.g., Baqaee and Farhi, 2020).

[14] When $\rho \downarrow 0$, the second term disappears and we obtain $\mathbf{\Gamma} = \mathbf{\Theta}^2(\mathbf{I}-\mathbf{A})$.

[15] For instance, $I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has 2^2 square roots: $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$ but the matrix $M_2 \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has none.

[16] In perfect foresight linear models, piece-wise continuity ensures that $\mathbf{p}_t^f$ is Riemann integrable with

vector of initial prices $\mathbf{p}_{0^-}$*, and the parameter* ρ*, the* principal square root of the duration-adjusted Leontief (PRDL) matrix, $\sqrt{\mathbf{\Gamma}}$*, is a sufficient statistic for sectoral prices' dynamics:*[17]

$$\begin{aligned}\mathbf{p}_t = \overbrace{e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{p}_{0^-} + (\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\int_0^t \frac{e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h} - e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}}{2}\mathbf{p}_h^f \mathrm{d}h}^{\textit{inertial effect of past prices due to stickiness}} \\ \underbrace{+ (\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\frac{e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}}{2}\int_t^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h}_{\textit{forward looking effect of future prices}}\end{aligned} \tag{23}$$

Drawing on Remarks 1 to 4, Proposition 2 presents the analytical solution for dynamics of *all* sectoral prices. This solution specifically highlights the interplay between the *forward-looking nature of pricing decisions* and *the backward-looking nature of aggregation*, Equations (17) and (18), which are governed by $\sqrt{\mathbf{\Gamma}}$ (rather than $\mathbf{\Gamma}$ itself) due to the dual forward- and backward-looking nature of prices: While firms take the future path of $\mathbf{p}_t^f$ into account when setting prices, aggregate prices also depend on the past path of $\mathbf{p}_t^f$ due to the persistence of stickiness over time.

3.2. Impulse Response Functions (IRFs)

Using Proposition 2, we can obtain IRFs by plugging in specific paths for $\mathbf{p}_t^f$ implied by shocks. Consider the economy in its steady state at $t = 0^-$ (left limit at $t = 0$), so that exogenous variables $(\boldsymbol{z}_t, \boldsymbol{\omega}_t, m_t) = (\boldsymbol{z}_{0^-}, \boldsymbol{\omega}_{0^-}, m_{0^-})$ for $t \uparrow 0$ and all prices are at their flexible level: $\mathbf{p}_{0^-} - \mathbf{p}_{0^-}^f = 0$.

3.2.1. Monetary Shocks. An expansionary monetary shock is a one-time unexpected but permanent increase in nominal GDP: $m_t = m_{0^-} + \delta_m, \forall t \geq 0$ where δ_m denotes the shock size. The implied path for $\mathbf{p}_t^f$ is $\mathbf{p}_t^f = \mathbf{p}_{0^-}^f + \delta_m \mathbf{1}$, where $\mathbf{1}$ is a vector of ones.

Proposition 3. *The IRFs of sectoral prices,* $\mathbf{p}_t$*; CPI inflation,* $\pi_t = \boldsymbol{\beta}^\intercal \boldsymbol{\pi}_t$*; GDP,* y_t*; and GDP gap,* $\tilde{y}_t \equiv y_t - y_t^f$ *to an expansionary monetary shock are given by:*

$$\tfrac{\partial}{\partial \delta_m}\mathbf{p}_t = (\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\mathbf{1}, \qquad \tfrac{\partial}{\partial \delta_m}\pi_t = \boldsymbol{\beta}^\intercal(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{1}$$

$$\tfrac{\partial}{\partial \delta_m}y_t = \tfrac{\partial}{\partial \delta_m}\tilde{y}_t = \boldsymbol{\beta}^\intercal e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{1}$$

unexpected shocks introducing countable jumps in flexible prices. The boundedness assumption is not restrictive with zero trend inflation. With trend inflation, boundedness is replaced with exponential order.

[17] In the limit of $\rho \downarrow 0$, the general expression of the proposition simplifies to:

$$\mathbf{p}_t = e^{-\sqrt{\mathbf{\Gamma}}t}\mathbf{p}_{0^-} + \sqrt{\mathbf{\Gamma}}e^{-\sqrt{\mathbf{\Gamma}}t}\int_0^t \sinh(\sqrt{\mathbf{\Gamma}}t)\mathbf{p}_h^f \mathrm{d}h + \sqrt{\mathbf{\Gamma}}\sinh(\sqrt{\mathbf{\Gamma}}t)\int_t^\infty e^{-\sqrt{\mathbf{\Gamma}}h}\mathbf{p}_h^f \mathrm{d}h \tag{22}$$

where hyperbolic sine of a square matrix $\mathbf{X}$ is defined as $\sinh(\mathbf{X}) \equiv (e^{\mathbf{X}} - e^{-\mathbf{X}})/2$.

Proposition 3 shows: (1) The only relevant objects for the sectoral price, inflation, and GDP dynamics are $\sqrt{\mathbf{\Gamma}}$ and expenditure shares $\boldsymbol{\beta}$. Thus, we can compute these IRFs for the input-output structure of the U.S. economy once we construct $\sqrt{\mathbf{\Gamma}}$ and the expenditure shares $\boldsymbol{\beta}$ from the data. (2) Although *relative* sectoral prices converge back to the steady state in the long run, the aggregate monetary shock distorts these relative prices on the transition path. These distortions are also captured by $\sqrt{\mathbf{\Gamma}}$, which can be constructed using data. (3) $\sqrt{\mathbf{\Gamma}}$ also captures the degree of monetary non-neutrality in the economy since GDP response to a monetary shock is zero in the flexible economy. We see this in the cumulative impulse response (CIR) of GDP, obtained by integrating the area under its impulse response function:

$$\mathrm{CIR}_{\tilde{y},m} \equiv \int_0^\infty \frac{\partial}{\partial \delta_m} \tilde{y}_t \mathrm{d}t = \boldsymbol{\beta}^\intercal \left(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}\right)^{-1} \mathbf{1} \tag{24}$$

3.2.2. TFP and Wedge Shocks. How do sectoral prices, CPI and GDP respond to sectoral TFP/wedge shocks? To answer this question, consider the following shock to any sector i:

$$\omega_{i,t} - z_{i,t} = \omega_{i,0^-} - z_{i,0^-} + e^{-\phi_i t}\delta_z^i, \quad \forall t \geq 0 \tag{25}$$

Here, a positive δ_z^i captures a negative TFP or a positive wedge shock to sector i that decays at the rate $\phi_i > 0$. We note that $\phi_i \downarrow 0$ would correspond to a permanent TFP/wedge shock in the limit while a positive ϕ_i denotes a temporary disturbance that disappears at rate ϕ_i. The implied path for $\mathbf{p}_t^f$, given such a shock, is $\mathbf{p}_t^f = \mathbf{p}_{0^-}^f + e^{-\phi_i t}\delta_z^i \mathbf{\Psi}\mathbf{e}_i$, where $\mathbf{\Psi}$ is the inverse Leontief matrix and $\mathbf{e}_i$ is the i'th standard basis vector. Economically, $\mathbf{\Psi}\mathbf{e}_i$ is a measure of sector i's *upstreamness* as it measures how much sector i, directly and indirectly, supplies to other sectors.

Proposition 4. *Suppose* $\phi_i \notin eig(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})$ *and let* $\mathbf{X}_i \equiv (\mathbf{\Gamma} - (\frac{\rho}{2} + \phi_i)^2\mathbf{I})^{-1}(\mathbf{\Gamma} - \frac{\rho^2}{4}\mathbf{I})$.[18] *Then, the IRFs of sectoral prices,* $\mathbf{p}_t$*; CPI inflation,* $\pi_t = \boldsymbol{\beta}^\intercal \boldsymbol{\pi}_t$*; GDP,* y_t*; and GDP gap,* $\tilde{y}_t = y_t - y_t^f$*, to a TFP/wedge shock in sector i are given by:*

$$\frac{\partial}{\partial \delta_z^i}\mathbf{p}_t = \mathbf{X}_i(e^{-\phi_i t}\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t})\mathbf{\Psi}\mathbf{e}_i, \qquad \frac{\partial}{\partial \delta_z^i}\pi_t = \boldsymbol{\beta}^\intercal \mathbf{X}_i((\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} - \phi_i e^{-\phi_i t}\mathbf{I})\mathbf{\Psi}\mathbf{e}_i$$

$$\frac{\partial}{\partial \delta_z^i} y_t = \boldsymbol{\beta}^\intercal \mathbf{X}_i(e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} - e^{-\phi_i t}\mathbf{I})\mathbf{\Psi}\mathbf{e}_i, \qquad \frac{\partial}{\partial \delta_z^i}\tilde{y}_t = \boldsymbol{\beta}^\intercal(\mathbf{X}_i e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} + (\mathbf{I} - \mathbf{X}_i)e^{-\phi_i t})\mathbf{\Psi}\mathbf{e}_i$$

The most important observation from Proposition 4 is that, aside from the exogenous dynamics introduced by the shock ($e^{-\phi_i t}$), all endogenous dynamics are captured by $e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t}$. This is best illustrated in the limiting case when the shock is almost permanent

[18] Assuming $\phi_i \notin \mathrm{eig}(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})$; i.e., ϕ_i is not an eigenvalue of the $\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}$ is a technical assumption that simplifies analytical derivations by guaranteeing that $\mathbf{X}_i$ is invertible, but it is without much loss of generality: A limit of IRFs can be taken and is valid when $\phi_i \to x \in \mathrm{eig}(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})$.

$(\phi_i \downarrow 0)$:

$$\tfrac{\partial}{\partial \delta_z^i}\pi_t|_{\phi_i \downarrow 0} = \boldsymbol{\beta}^\top(\sqrt{\boldsymbol{\Gamma}} - \tfrac{\rho}{2}\mathbf{I})e^{-(\sqrt{\boldsymbol{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\boldsymbol{\Psi}\mathbf{e}_i, \qquad \tfrac{\partial}{\partial \delta_z^i}\tilde{y}_t|_{\phi_i \downarrow 0} = \boldsymbol{\beta}^\top e^{-(\sqrt{\boldsymbol{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\boldsymbol{\Psi}\mathbf{e}_i \tag{26}$$

This observation uncovers *two* separate roles of the Leontief matrix in the dynamic economy.

Remark 5. *The inverse Leontief matrix, $\boldsymbol{\Psi}$, determines the* static *propagation of TFP/wedge shocks by passing them through the network ($\mathbf{e}_i \to \boldsymbol{\Psi}\mathbf{e}_i$). The principal square root, $\sqrt{\boldsymbol{\Gamma}}$, determines the* dynamic *propagation of these shocks over time ($\boldsymbol{\Psi}\mathbf{e}_i \to e^{-(\sqrt{\boldsymbol{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\boldsymbol{\Psi}\mathbf{e}_i$).*

Moreover, in response to TFP/wedge shocks, the GDP response combines both the response under flexible prices and the response of the GDP gap under sticky prices. To separate these, we decompose the CIR of GDP to its two components:

$$\mathrm{CIR}_{y,z_i} \equiv \int_0^\infty \tfrac{\partial}{\partial \delta_z^i} y_t \mathrm{d}t = \underbrace{-\phi_i^{-1}\lambda_i}_{\substack{\mathrm{CIR}_{y^f,z_i} \equiv \text{Flexible GDP Response} \\ \text{(Domar-weighted cumulative TFP)}}} + \underbrace{\boldsymbol{\beta}^\top((\phi_i + \tfrac{\rho}{2})\mathbf{I} + \sqrt{\boldsymbol{\Gamma}})^{-1}\boldsymbol{\Psi}\mathbf{e}_i}_{\mathrm{CIR}_{\tilde{y},z_i} \equiv \text{Cumulative GDP Gap Response}} \tag{27}$$

This decomposition provides intuition for the limiting case when $\phi_i \to 0$. In this case, the flexible GDP CIR explodes because, with a permanent shock to TFP, the economy diverges from the initial steady-state (which is why we are only considering the case when $\phi_i \to 0$ and not $\phi_i = 0$). However, the GDP gap CIR is not explosive in this limit as the effects of sticky prices are only temporary deviations from the flexible price response:

$$\mathrm{CIR}_{\tilde{y},z^i}|_{\phi_i \downarrow 0} = \boldsymbol{\beta}^\top(\tfrac{\rho}{2}\mathbf{I} + \sqrt{\boldsymbol{\Gamma}})^{-1}\boldsymbol{\Psi}\mathbf{e}_i \tag{28}$$

Equations (24) and (28) illustrate a more general takeaway in the context of permanent shocks. They show that the total effect of a monetary or sectoral shock on the cumulative response of GDP gap is a combination of two forces, where the interaction is captured by the inner product of two vectors: (1) A vector that captures the pass-through of the shock to flexible prices ($\mathbf{1}$ for monetary shocks and $\boldsymbol{\Psi}\mathbf{e}_i$ for TFP/wedge shocks as seen from Equation (14)), and (2) A second vector that captures the dynamic propagation of shocks which is *independent* of whether the shock is a monetary or sectoral shock. Instead, it only depends on the expenditure share vector and the PRDL matrix ($\boldsymbol{\beta}^\top(\frac{\rho}{2}\mathbf{I} + \sqrt{\boldsymbol{\Gamma}})^{-1}$). This is the dynamic force that converts the static pass-through of the shock to its endogenous dynamic propagation through the terms involving $e^{-(\sqrt{\boldsymbol{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}$ in Propositions 3 and 4. Accordingly, $\sqrt{\boldsymbol{\Gamma}} - \frac{\rho}{2}\mathbf{I}$ connects the persistence of inflation response to the shocks' total effects on the GDP gap. Next, we study the economic interpretation of this matrix.

3.3. Perturbation Around Disconnected Economies

We have shown that $\sqrt{\mathbf{\Gamma}}$ encodes all the economic forces that shape the endogenous dynamics of the log-linearized model, including price and GDP. But what is its economic interpretation? In principle, we could use the Jordan decomposition of $\sqrt{\mathbf{\Gamma}}$ to conduct a spectral analysis, but this approach does not provide economic intuition. Suppose $\sqrt{\mathbf{\Gamma}}$ is diagonalizable so that there exists a diagonal $\mathbf{D} = \text{diag}(d_1, \ldots, d_n)$, and an invertible matrix $\mathbf{P}$ such that $\sqrt{\mathbf{\Gamma}} = \mathbf{PDP}^{-1}$. Recall from Proposition 3 that the impulse responses of GDP (gap) and inflation to monetary shocks are determined by $\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I} = \mathbf{P}(\mathbf{D} - \frac{\rho}{2}\mathbf{I})\mathbf{P}^{-1}$,[19] which would imply that GDP and inflation responses to a monetary shock are

$$\tfrac{\partial}{\partial \delta_m} \tilde{y}_t = \boldsymbol{\beta}^\top e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t}\mathbf{1} = \sum_{i=1}^{n} w_i e^{-(d_i - \frac{\rho}{2})t}, \tag{29}$$

$$\tfrac{\partial}{\partial \delta_m} \pi_t = \boldsymbol{\beta}^\top (\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}) e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t}\mathbf{1} = \sum_{i=1}^{n} d_i w_i e^{-(d_i - \frac{\rho}{2})t}, \quad w_i \equiv \boldsymbol{\beta}^\top \mathbf{P}\mathbf{e}_i \mathbf{e}_i^\top \mathbf{P}^{-1}\mathbf{1} \tag{30}$$

The problem with this representation is that it is unclear how the structure of the economy is reflected in the eigenvalues $\{d_i\}$ and coefficients $\{w_i\}$.

The key idea here is to approximate the dynamics of the log-linearized economy with an *arbitrary* input-output matrix by perturbing it around "disconnected" economies, whose eigendecomposition has a clear economic interpretation. We do not use this approximation in the quantitative results presented in Section 3.4 below but derive it here to provide intuition.

Definition 1. *A* ***disconnected*** *economy is characterized by a diagonal input-output matrix.*

Figure 1a depicts disconnected economies. These are multi-sector economies with heterogeneous price stickiness where sectors only use their own output in production.

Eigendecomposition of Disconnected Economies. Disconnected economies are useful benchmarks because for each sector i, the corresponding decay rate in Equation (29), $d_i - \frac{\rho}{2}$, is given by:

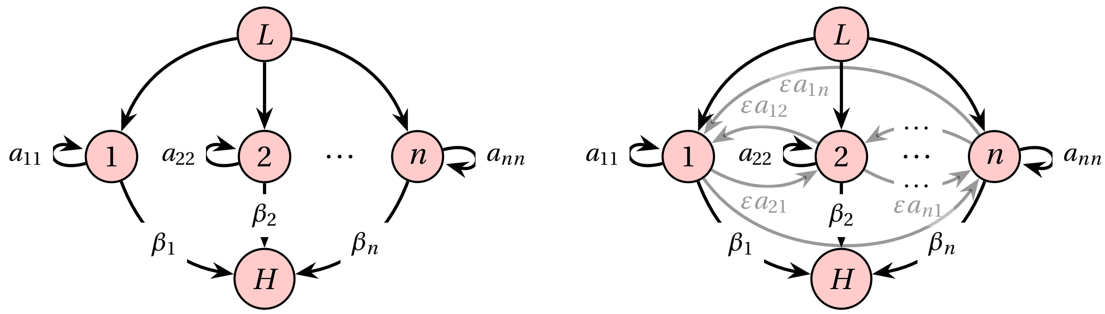
$$d_i - \frac{\rho}{2} = \xi_i \equiv \sqrt{\theta_i(\rho + \theta_i)(1 - a_{ii}) + \frac{\rho^2}{4}} - \frac{\rho}{2} > 0 \tag{31}$$

We interpret ξ_i's as the *adjusted* frequencies of their corresponding sectors due to the following properties: (a) Each ξ_i is independent of frequencies and labor share of other sectors, and depends only on sector i's frequency, θ_i, own input share a_{ii}, and the

[19] Here $\mathbf{D} - \frac{\rho}{2}\mathbf{I} = \text{diag}(d_i - \frac{\rho}{2})$ is a diagonal matrix whose non-zero entries are the eigenvalues of $\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}$. Note that by Lemma 1, the latter is a positive stable matrix so $\text{Re}(d_i - \frac{\rho}{2}) > 0$ for all $i \in [n]$.

Figure 1: Perturbation around Disconnected Economies

(a) n-Sector Disconnected Economies (b) Perturbation towards $\mathbf{A} = [a_{ij}]$



Notes: Figure 1a draws the structure of disconnected economies where sectors operate independently but are allowed to use their own output in roundabout production. Figure 1b shows our parameterized perturbation of an arbitrary input-output matrix $\mathbf{A}$ around its disconnected structure: the perturbation is given by keeping a sector's own input shares from their output fixed, and only adding their input from other sectors proportional to an $\varepsilon > 0$.

discount factor ρ. (b) With $a_{ii} = 0$, ξ_i is equal to the frequency of sector i, θ_i. (c) With $a_{ii} > 0$, ξ_i decreases with ρ as well as a_{ii}, and increases with θ_i.

Recall that since the arrival of price change opportunities in each sector i is a Poisson process, the expected duration between price changes is the inverse of arrival frequency and is equal to θ_i^{-1}. Thus, we can also interpret ξ_i^{-1} as the *adjusted* duration of price change frequency of sector i that takes the input share from its own output into account.

Moreover, to obtain the eigendecomposition in Equation (29), we observe that the corresponding weight w_i is the household's expenditure share for that sector, giving:

$$\frac{\partial}{\partial \delta_m}\tilde{y}_t = \sum_{i=1}^{n} \beta_i e^{-\xi_i t}, \qquad \frac{\partial}{\partial \delta_m}\pi_t = \sum_{i=1}^{n} \beta_i \xi_i e^{-\xi_i t} \tag{32}$$

Note that these expressions are now interpretable as both d_i's and w_i's are now explicitly stated in terms of model parameters; e.g., GDP response is the expenditure-weighted average of exponential functions, each decaying at the rate of the sector's *adjusted* frequency. Moreover, integrating the GDP gap response in Equation (32), we obtain:

$$\mathrm{CIR}_{\tilde{y},m}\big|_{\varepsilon=0} \equiv \int_0^\infty \frac{\partial}{\partial \delta_m}\tilde{y}_t \,\mathrm{d}t = \textstyle\sum_{i=1}^{n} \frac{\beta_i}{\xi_i} \tag{33}$$

Equation (33) connects two separate insights about monetary non-neutrality in one framework. First, when $a_{ii} = 0, \forall i \in [n]$ (so that $\xi_i = \theta_i$), it shows that in a pure multisector economy, monetary non-neutrality is the expenditure weighted average of price spell durations. Since these durations are convex functions of the frequencies, applying Jensen's inequality shows that heterogeneity in frequencies amplifies monetary non-neutrality (Carvalho, 2006, Nakamura and Steinsson, 2010).

Second, when $n = 1$ but $a_{11} \neq 0$, what determines monetary non-neutrality is no longer the duration of price spells but their duration adjusted by the input share of that sector from its total output. Since $a_{11} > 0$, we can see that fixing the frequency, a higher input share from this final product—i.e. "roundabout" production—amplifies monetary non-neutrality (Basu, 1995). This is because when firms use the sector's output as an input for production, the sectoral price of the good feeds back into firms' marginal costs. But since the sectoral price is sticky, this feedback affects the pricing decision of the firms that get the opportunity to change their price: Even when firms get to reset their price, they now reset it by less than before, taking into account the stickiness of their marginal costs over time. This dynamic consideration of the price setters also explains the role of ρ in the adjusted duration. With higher ρ the price-setters are less forward-looking and the short-term stickiness of the sectoral price has a larger effect on their pricing decision.

Third, in the more general case when $n > 1$ and $a_{ii} \neq 0$, Equation (33) extends these insights and shows that, even in a disconnected economy, monetary non-neutrality depends on the duration of price spells adjusted for the input-output structure of an economy. In particular, it delivers the novel result that even when all sectors have the same frequency, heterogeneity in these adjusted frequencies amplifies monetary non-neutrality.[20] Thus, roundabout production within the sector increases the effective *stickiness* of the sectoral price precisely in a manner that is captured by the adjusted frequency ξ_i, so much so that a multisector economy with roundabout production in its sectors is equivalent to a multisector economy with *no* roundabout production, whose price adjustment frequencies are equal to the adjusted frequencies ξ_i's.

Eigen-perturbation around Disconnected Economies. Now, consider an *arbitrary n*-sector economy with frequency matrix $\boldsymbol{\Theta} = \mathrm{diag}(\theta_1, \ldots, \theta_n)$ and input-output matrix $\mathbf{A} = [a_{ij}]$, and define the corresponding disconnected economy as $\mathbf{A}_D \equiv \mathrm{diag}(a_{11}, \ldots, a_{nn})$. Thus, we can write the duration-adjusted Leontief matrix $\boldsymbol{\Gamma} = \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A}) + \frac{\rho^2}{4}\mathbf{I}$ as the sum of the one in the disconnected economy $\boldsymbol{\Gamma}_D = \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A}_D) + \frac{\rho^2}{4}\mathbf{I}$ and the

[20] This follows neither from Carvalho (2006) nor Basu (1995). The latter is a one-sector economy and thus has no predictions for multisector economies, while the former predicts that a multisector economy with the same frequency across sectors implies the same degree of monetary non-neutrality as a one-sector economy with that frequency.

off-diagonal matrix $\mathbf{\Gamma}_R$:

$$\mathbf{\Gamma} = \mathbf{\Gamma}_D + \mathbf{\Gamma}_R, \quad \text{with} \quad \mathbf{\Gamma}_R \equiv \mathbf{\Theta}(\rho\mathbf{I} + \mathbf{\Theta})(\mathbf{A}_D - \mathbf{A}) \tag{34}$$

This is a classic exercise in perturbation theory where we replace $\mathbf{\Gamma}$ with $\mathbf{\Gamma}(\varepsilon) = \mathbf{\Gamma}_D + \varepsilon\mathbf{\Gamma}_R$ for some $\varepsilon > 0$ and express the eigenvalues and eigenvectors as power series in ε (see, e.g., Kato 1995, ch. 2 or Bender and Orszag 1999, p. 350). The economic interpretation is that we move from the disconnected economy, $\mathbf{A}_D$, towards the arbitrary economy, $\mathbf{A}$, in proportion to ε, as shown in Figure 1b. Notably, $\varepsilon = 0$ corresponds to the disconnected economy and $\varepsilon = 1$ corresponds to the arbitrary economy $\mathbf{A}$.

Generally, eigenvalues and eigenvectors of $\mathbf{\Gamma}(\varepsilon)$ do not need to be differentiable in ε, especially for non-symmetric matrices as in our case. However, assuming that eigenvalues of $\mathbf{\Gamma}_D$ are distinct (i.e., sectors of the disconnected economy have distinct adjusted frequencies),[21] we obtain the following Lemma that relies on Theorems 1 and 2 in Greenbaum, Li, and Overton (2020).

Lemma 2. *Suppose ξ_i's, as defined in Equation (31), are distinct for all $i \in [n]$. Let $(d_i(\varepsilon), \mathbf{v}_i(\varepsilon))$ be an eigenvalue/eigenvector pair for the principal square root of the perturbed economy, $\sqrt{\mathbf{\Gamma}(\varepsilon)}$. Then,*

$$d_i(\varepsilon) = \xi_i + \frac{\rho}{2} + \mathscr{O}(\|\varepsilon\|^2) \quad \mathbf{v}_i(\varepsilon) = \mathbf{e}_i + \varepsilon \times \sum_{j\neq i}\left[\frac{a_{ji}}{1-a_{jj}} \times \frac{\xi_j^2+\rho\xi_j}{(\xi_i+\xi_j+\rho)(\xi_j-\xi_i)}\right]\mathbf{e}_j + \mathscr{O}(\|\varepsilon\|^2) \tag{35}$$

Lemma 2 is useful because it links the mathematical properties of $\sqrt{\mathbf{\Gamma}}$ to its economic properties. It shows that up to first-order in ε, the eigenvalues of $\sqrt{\mathbf{\Gamma}}$ are the same as the disconnected economy; i.e. $\frac{\partial}{\partial\varepsilon} d_i(\varepsilon)|_{\varepsilon=0} = 0$. Importantly, note that in theory, this perturbation does not have to be accurate for $\varepsilon = 1$. But as we plot in Figure S.M.1 in the Appendix, it is a *remarkably accurate approximation* for the eigenvalues of the measured $\sqrt{\mathbf{\Gamma}}$ for the U.S. economy.

3.3.1. Aggregate and Sectoral Effects of Monetary Shocks. We now discuss analytically how monetary shocks propagate in our approximate economy. We first present the results for sectoral inflation and then aggregate these responses to obtain the effects on CPI inflation and GDP.

Proposition 5 ***(Sectoral Inflation Responses).*** *Suppose ξ_i's are distinct. The impulse*

[21] This is a fairly weak assumption because ξ_i's are almost surely distinct if the distributions of $\mathbf{\Theta}$ and $\mathbf{A}$ in the data are drawn from distributions with densities with respect to the Lebesgue measure. In other words, the event that two sectors have the same adjusted frequencies in the data has zero probability.

response of inflation in sector $i \in [n]$ to a monetary shock is:

$$\frac{\partial}{\partial \delta_m}\pi_{i,t} = \underbrace{\xi_i e^{-\xi_i t}}_{\textit{disconnected baseline}} \underbrace{+\varepsilon \times \xi_i \times \sum_{j \neq i} \frac{a_{ij}}{1-a_{ii}} \times \frac{\xi_i+\rho}{\xi_i+\xi_j+\rho} \times \frac{\xi_j e^{-\xi_j t} - \xi_i e^{-\xi_i t}}{\xi_i - \xi_j}}_{\textit{first order effect of the network}} + \mathcal{O}(\|\varepsilon\|^2) \tag{36}$$

Equation (36) shows that introducing production linkages creates spillover effects on the inflation of sector i through all of its suppliers, captured by the term labeled the "first order effect of the network." It is straightforward to verify that these first-order effects are negative initially but turn positive after some t. Intuitively, since i's suppliers have sticky prices, increasing production linkages (higher ε) leads to an initial dampening of the inflation response in sector i to a monetary shock. However, since money is neutral in the long run, this dampened response has to be compensated for in terms of inflation in the long run, which implies that inflation in sector i is more persistent with higher ε.[22] The following corollary shows how these sectoral effects translate into the response of aggregate inflation to monetary shocks.

Proposition 6 ***(Impact and Asymptotic Inflation Response).*** *Input-output linkages dampen CPI inflation response to a monetary shock on impact*

$$\underbrace{\frac{\partial}{\partial \varepsilon}\left[\frac{\partial}{\partial \delta_m}\pi_0\right]\Big|_{\varepsilon=0}}_{\partial\textit{impact response}/\partial\varepsilon} = -\sum_{i=1}^{n} \beta_i \xi_i \sum_{j \neq i} \frac{a_{ij}}{1-a_{ii}} \times \frac{\xi_i+\rho}{\xi_i+\xi_j+\rho} < 0 \tag{37}$$

but amplify its persistence; letting $\iota \equiv \arg\min_i\{\xi_i\}$ denote the sector with the lowest *adjusted frequency, we have:*

$$\underbrace{\frac{\partial}{\partial \varepsilon}\left[\frac{\partial}{\partial \delta_m}\pi_t|_{t\to\infty}\right]\Big|_{\varepsilon=0}}_{\partial\textit{asymptotic response}/\partial\varepsilon} \sim \sum_{j \neq \iota} \left(\beta_j \xi_j \frac{a_{j\iota}(\xi_j+\rho)}{1-a_{jj}} + \beta_\iota \xi_\iota \frac{a_{\iota j}(\xi_\iota+\rho)}{1-a_{\iota\iota}}\right) \frac{\xi_\iota e^{-\xi_\iota t}}{(\xi_j-\xi_\iota)(\xi_\iota+\xi_j+\rho)} > 0 \tag{38}$$

Finally, we show in the next proposition that this increase in the persistence of inflationary responses corresponds to an increase in monetary non-neutrality.

Proposition 7 ***(Monetary Non-Neutrality).*** *Input-output linkages amplify monetary non-*

[22] While we have derived this analytical result when monetary policy follows a nominal GDP rule, as we show later, it also holds numerically when monetary policy follows a Taylor rule (Figure 8). What is essential for this result is just long-run monetary neutrality, which requires all relative prices to return to levels independent of monetary policy. Thus, regardless of what monetary policy does, sectors that adjust their prices faster in response to a monetary shock, should have more transient responses and vice versa.

neutrality measured by the CIR of GDP to a monetary shock.

$$CIR_{\tilde{y},\delta_m} = \sum_{i=1}^{n} \underbrace{\beta_i \xi_i^{-1}}_{\substack{\text{direct effect}\\ \text{of sector } i}} + \varepsilon \sum_{i=1}^{n} \underbrace{\overbrace{\xi_i^{-1}}^{1} \times \sum_{j\neq i}^{n} \overbrace{a_{ji}}^{2} \times \overbrace{\frac{\beta_j}{1-a_{jj}}}^{3} \times \overbrace{\frac{\xi_j+\rho}{\xi_i+\xi_j+\rho}}^{4}}_{\substack{\text{first-order indirect effect of}\\ \text{sector } i \text{ through network } \geq 0}} + \underbrace{\mathcal{O}(\|\varepsilon\|^2)}_{\substack{\text{higher-order}\\ \text{effects}}} \tag{39}$$

Equation (39) shows how monetary non-neutrality varies with ε around the disconnected economy. First, the term labeled the "direct effect of sector i" corresponds to the expression in Equation (33) and its ensuing discussion, where the contribution of each sector to monetary non-neutrality is its expenditure weighted *adjusted* duration. Beyond this direct effect, each sector i also contributes to monetary non-neutrality through all of its downstream firms, the first-order terms of which are labeled 1 – 4.

For economic interpretation of these terms, note that, intuitively, input-output linkages amplify monetary non-neutrality through a sector i by propagating its price stickiness to its downstream firms. Thus, the first important factor on how much monetary non-neutrality will increase through i (indirectly) should depend on the adjusted duration of sector i's own price spells, which is what 1 captures. Given this adjusted duration, to capture the total first-order indirect effects of a sector i on monetary non-neutrality, we then need to sum over its immediate downstream sectors, captured by $\sum_{j\neq i}$ in Equation (39). For each downstream sector j, then we need to take into account the exposure of that sector to sector i, captured by its expenditure share a_{ji} in 2. Moreover, we need to take into account sector j's own centrality in affecting GDP, which is captured by its Domar weight in the disconnected economy, which we have labeled 3.[23]

Finally, the term under 4 captures the dynamic adjustment based on the relative adjusted duration of the upstream sector i to downstream sector j. When the adjusted duration of price spells in the upstream sector i is relatively small compared to that of the downstream sector j, then firms in j are not very responsive to the price changes of supplier i, so the indirect effect of sector i through sector j is muted. Alternatively, when sector j is more flexible relative to its supplier i, then i's indirect effect through j is amplified because prices in j would have been more responsive to monetary shocks were it not for the stickiness in their marginal costs through i. Moreover, to see why ρ appears in this term, note that these effects are anticipatory in the sense that firms adjust the size of their response by taking into account the duration of their own price spell relative to

[23]The Domar weight of any sector j in the disconnected economy is $\beta_j/(1-a_{jj})$.

their upstream sector. Thus, the more myopic the firms are ($\rho \uparrow$), the less we expect 4 to affect the pass-through. Indeed, we observe that 4 goes to unity as $\rho \to \infty$.

Thus, with more input-output linkages, monetary non-neutrality becomes larger through the interaction of these four forces. We use these findings in our quantitative analysis below in identifying sectors that have disproportionate effects in the propagation of monetary shocks.

3.3.2. Aggregate Effects of Sectoral Shocks. We now characterize the pass-through of sectoral inflation to aggregate CPI inflation. The experiment is to consider a negative sectoral TFP shock to sector i that raises the inflation rate in that sector by 1 percent on impact. Our goal is to characterize how much aggregate CPI inflation rises in response to this sectoral shock, and how this pass-through is affected by the network. The following proposition presents this pass-through for the impact response of inflation. The full expression for the dynamic response of inflation is available, but more complicated and is only included in the proof of the proposition.

Proposition 8 ***(Pass-through of Sectoral to Aggregate Inflation).*** *Input-output linkages amplify the pass-through of sectoral inflation rates to aggregate CPI inflation.*

$$\frac{\partial \pi_0}{\partial \pi_{i,0}}\Big|_{\delta_z^i} = \underbrace{\beta_i}_{\text{direct pass-through}} \underbrace{+\varepsilon \sum_{j \neq i} \overbrace{a_{ji}}^{1} \times \overbrace{\frac{\beta_j}{1-a_{jj}}}^{2} \times \overbrace{\frac{\xi_j}{\phi_i + \xi_j + \rho}}^{3} \times \overbrace{\frac{\xi_j + \rho}{\xi_i + \xi_j + \rho}}^{4}}_{\text{first-order indirect pass-through via network}} + \underbrace{\mathcal{O}(\|\varepsilon\|^2)}_{\text{higher-order effects}} \tag{40}$$

Equation (40) relates the pass-through of sectoral inflation rate in sector i to aggregate inflation *conditional* on a negative TFP shock to sector i. The first term on the right-hand side is the direct pass-through of sectoral inflation to aggregate inflation: a one percent inflation in sector i directly feeds to inflation proportional to the expenditure share of the sector, denoted by β_i. The second term, which itself consists of four components, labeled by 1 – 4, captures the first-order *indirect* pass-through of sectoral inflation to aggregate inflation through the network.

The indirect effect can be understood as follows: an inflationary shock in sector i, up to first-order, propagates through its buyers. Thus, we need to sum over all the other sectors that purchase from i. When considering a buyer $j \neq i$, the impact of i's inflationary shock on the economy through j is proportional to j's expenditure share on i, 1, and j's own Domar weight in the baseline economy, 2. These two components jointly determine the potency of i's shock on j and resemble results from static models.

The next two terms, however, capture dynamic considerations. The term labeled 3 accounts for the fact that if the duration of the shock to i, ϕ_i^{-1}, is small compared to the adjusted duration of price spells in the downstream sector j, ξ_j^{-1}, then the shock's pass-through via j is weakened. This occurs because stickier downstream sectors, measured by their adjusted duration ξ_j^{-1}, are less responsive to a transient shock because they anticipate it will dissipate relatively faster than prices in their sector will adjust. The term under 4 captures a similar effect, but relative to the adjusted duration of price spells in the upstream sector i itself. When the adjusted duration of price spells in the upstream sector i is relatively small compared to that of the downstream sector j, then firms in j are not very responsive to the price changes of supplier i since they anticipate those prices will readjust faster than their own prices. Finally, note that since 3 and 4 are both anticipatory effects, their strength should depend on the discount rate ρ, similar to the discussion below Proposition 7.

3.4. Measurement and Quantitative Implications

In this section, we measure the sufficient statistics implied by the model for the U.S. and study the dynamic responses of inflation and GDP using the statistics.

3.4.1. Sufficient Statistics Construction From Data. Propositions 2 to 4 show that the sufficient statistics for inflation and GDP dynamics are the PRDL matrix, $\sqrt{\mathbf{\Gamma}}$, and the expenditure shares vector, $\boldsymbol{\beta}$. We use the make and use input-output (IO) tables from 2012, made available by the BEA, to construct the input-output matrix $\mathbf{A}$; the consumption expenditure share vector $\boldsymbol{\beta}$; and the sectoral labor shares vector $\boldsymbol{\alpha}$. We construct them at the detailed disaggregation level, which, excluding the government sectors, leads to 393 sectors. Figure S.M.2 shows the heatmap of the matrix $\mathbf{A}$ that we construct from the data. Moreover, we construct the diagonal matrix $\mathbf{\Theta}^2$, whose diagonal elements are the squared frequency of price adjustments in these sectors, using data on 341 sectors from Pasten, Schoenle, and Weber (2020). A detailed description is provided in Appendix S.M.3.

3.4.2. Dynamic Aggregate Responses to a Monetary Policy Shock. Panel A of Figure 4 shows impulse responses of aggregate inflation and GDP to an expansionary monetary policy shock in our calibrated economy. The size of this shock is normalized so that inflation responds by 1 percent on impact, after which it slowly goes back to its steady state level at zero. The persistence of this convergence is governed by our measured $\sqrt{\mathbf{\Gamma}}$, with a half-life of around 6 months. Moreover, the shock has substantial real effects. GDP rises by around 10 percent on impact and decays slowly back to zero. The cumulative response of GDP is about 132 percent.

To illustrate the roles of various model ingredients that lead to such substantial real effects, we consider the following counterfactual experiments. In these counterfactuals, the initial impact on inflation is always at 1 percent.[24] In Panel B of Figure 4, we compare our calibrated economy to a horizontal economy, where we set $\mathbf{A} = \mathbf{0}$ while keeping $\boldsymbol{\Theta}$ the same as before. Thus, this economy features no input-output linkages but has the same price change frequencies. The cumulative impulse response of GDP is 4.1 times larger in our baseline economy. Strategic complementarity in price setting that arises through input-output linkages, as we pointed out in the discussion below Equation (16), is the driving force for this result. This in turn leads to a more persistent inflation response, which amplifies GDP response both on impact and over time. These results quantify our results for inflation persistence and monetary non-neutrality in Propositions 6 and 7.

In addition to input-output linkages, another source that amplifies the real effects of monetary policy is heterogenous price stickiness across sectors, as discussed below Equation (33) and Proposition 7. To investigate the role of this channel, in Panel C of Figure 4, we compare our calibrated baseline economy to an economy with homogeneous frequencies, which keeps $\mathbf{A}$ the same as before but sets $\boldsymbol{\Theta} = \bar{\theta}\mathbf{I}$. We calibrate the frequency of price changes in this economy to be the same as the expenditure-weighted average of the frequency of price changes across sectors in our baseline economy—i.e., $\bar{\theta} \equiv \beta_i \theta_i$. Note that this economy still features the same input-output linkages, and through that, strategic complementarities in price setting. The cumulative impulse of GDP is 2.4 times larger in our baseline economy, which shows that heterogeneity in price stickiness across sectors does play a quantitatively important role in magnifying monetary non-neutrality. The quantitative importance of this channel, however, is not as high as that of input-output linkages.

Finally, shutting down both channels, in Panel D of Figure 4, we compare our calibrated baseline economy to a horizontal economy with homogeneous price stickiness across sectors ($\mathbf{A} = \mathbf{0}, \boldsymbol{\Theta} = \bar{\theta}\mathbf{I}$). The cumulative impulse response of GDP is 6.9 times larger in our baseline economy.[25] This total effect is approximately equal to the sum of the two

[24] The monetary policy shock size is therefore different across the baseline and the counterfactual cases. Recall that the cumulated impulse response of aggregate inflation corresponds to the monetary policy shock size in our model. Keeping the initial impact on aggregate inflation the same across various model specifications brings out the crucial role played by the persistence of inflation.

[25] Note that even in this textbook type multisector New Keynesian model, inflation effects are persistent

separate counterfactual effects shown above.[26]

3.4.3. Heterogeneous Sectoral Inflation Responses to a Monetary Policy Shock. Underlying the aggregate inflation response to the monetary shock discussed above is a distribution of sectoral inflation responses. In Figure 5, we show impulse responses of some selected sectors' inflation to an expansionary monetary policy shock. Sectoral inflation responses differ significantly both in terms of the impact response and the persistence. Moreover, since relative prices need to go back to the same steady state, nominal prices all rise by the same amount in the long run; therefore, sectors where inflation responds by a larger amount initially have more short-lived responses. In particular, Figure 5 shows that sectoral inflation in the Oil and Gas Extraction industry is high in the initial periods but dissipates fast, while sectoral inflation in the Semiconductor Manufacturing Machinery industry responds by a small amount initially but is persistently positive over time. For completeness, Table 4 provides a ranking of the top twenty sectors by their initial sectoral inflation response while Table 5 provides a ranking of the top twenty sectors by the half-life of their sectoral inflation response.

For interpretation, we turn to Proposition 5 and its discussion, where we showed that inflation in sectors with more flexible prices and less input-output linkages respond more strongly initially. Specifically, Equation (36) showed that the relevant statistic for impact sectoral inflation response (evaluated at $t=0$) is $\xi_i - \varepsilon \sum_{j\neq i} \frac{\xi_i a_{ij}}{1-a_{ii}} \frac{\xi_i+\rho}{\xi_i+\xi_j+\rho}$. Panel A of Figure 2a shows the correlation between the actual ranks of sectors and the ranks predicted from this statistic. The approximated statistic accounts extremely well for the exact numerical results. Moreover, as mentioned above, sectors where inflation responds more initially tend to have short-lived responses. Panel B of Figure 2a shows the correlation between actual ranks of sectors given by half-life of sectoral inflation response and the ranks predicted from the impact response statistic. The correlation is negative.

3.4.4. Sectoral Origins of Aggregate Inflation and GDP Dynamics. Motivated by supply chain issues, commodity price increases, and persistent aggregate inflation in the U.S. recently, we now study aggregate implications of sectoral shocks. Specifically, we compute

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because our modeling of monetary policy preserves an endogenous state variable. This is a standard approach in the literature on sufficient statistics of monetary policy shocks, but is a different approach than assuming a Taylor rule where the interest rate feedback coefficient is on inflation. We show results from this case later.

[26]These counterfactual experiments are related to, but different from, the ones in Pasten, Schoenle, and Weber (2020) as we compare the production network economy with a horizontal economy and in all our experiments, recalibrate the shock size to lead to a 1 percent impact effect on aggregate inflation.

sectoral shocks that lead to a 1 percent increase in sectoral inflation and then study the pass-through of such sectoral inflation increases on aggregate inflation. The average duration of the sectoral shocks is 6 months.[27]

We start by identifying sectors that lead to a high on-impact response of aggregate inflation in Table 1. We provide a ranking of the top twenty sectors by their initial effect on aggregate inflation, where we remove the effect coming from the size of the sector. This metric, therefore, provides an evaluation of the spillover of sectoral inflation to aggregate inflation due to input-output linkages for in the absence of such linkages, this pass-through metric would be zero for all sectors.[28] As one example, the Oil and Gas Extraction industry ranks very high in Table 1. As we showed analytically in Proposition 8, sectors that serve as input to other sectors and have more input-output adjusted sticky prices cause greater spillover to aggregate inflation. Specifically, in Equation (40) we showed that the relevant statistic for this impact pass-through on aggregate inflation is $\sum_{j\neq i} a_{ji} \frac{\beta_j}{1-a_{jj}} \frac{\xi_j}{\phi_i+\xi_j+\rho} \frac{\xi_j+\rho}{\xi_i+\xi_j+\rho}$. Panel A of Figure 2b shows the correlation between the actual ranks of sectors and the ranks predicted from this statistic. The approximated statistic accounts well for the exact numerical results, thereby providing an economic interpretation to the rankings.

We next identify sectors that lead to persistent aggregate inflation dynamics when sectoral inflation increases by 1 percent. Table 2 provides a ranking of the top twenty sectors by the half-life of the aggregate inflation response. One clear pattern emerges: Sectors with more sticky prices lead to persistent aggregate inflation dynamics when sectoral shocks cause a rise in sectoral inflation. Semiconductor Manufacturing Machinery industry is one sector that ranks high in Table 2. These results highlight that identifying which sectors are the main sources of persistent aggregate inflation dynamics is critical because those persistent effects translate to larger aggregate GDP gap effects. We discussed this link and the theoretical reasons behind it in the discussion below Equation (28).

To make this clear quantitatively, in Panel B of Figure 2b, we show that the cumulative impulse response of aggregate GDP gap is very tightly correlated with the half-life of aggregate inflation. Here, we compute the ratio of the cumulative impulse response of GDP to the cumulative impulse response of GDP under flexible prices for a unit sectoral shock. The size of the sectoral shocks are thus the same in this experiment. This implies

[27]We interpret these sectoral shocks as negative supply shocks. Note that while the average duration of the sectoral shock is the same across all sectors, the size of the sectoral shock is different in this exercise as we calibrate the size such that sectoral inflation increases by 1 percent across all sectors.

[28]We are thus capturing what are sometimes called second-round effects of sectoral inflation increases.

that it is precisely the shocks to sectors that are the sources of persistent aggregate inflation dynamics that will have a bigger impact on the real macroeconomy.

3.4.5. A Spectral Analysis of Aggregate Inflation Persistence. So far, we have highlighted the critical role played by the persistence of aggregate inflation in driving macroeconomic dynamics. In particular, for monetary shocks, we showed in Section 3.4.2 that model features which increase the persistence of aggregate inflation lead to higher monetary non-neutrality. We now investigate further the origins of aggregate inflation persistence by identifying which sectors play a key role in propagating monetary policy shocks in the longer run. In terms of long-run dynamics, given our analytical solution, the smallest eigenvalues of $\sqrt{\mathbf{\Gamma}} \equiv \sqrt{\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})+\frac{\rho^2}{4}\mathbf{I}}$ play the dominant role.

Theoretically, eigenvalues as such depend on the whole network and might not be intimately connected to any particular sector. To make such a connection, we turn to Lemma 2, which showed that these eigenvalues are given by $d_i = \sqrt{\theta_i(\rho+\theta_i)(1-a_{ii})+\frac{\rho^2}{4}} + \mathcal{O}(\|\varepsilon\|^2)$. To measure the accuracy of this approximation, in Table 3, we sort the eigenvalues of $\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I} \equiv \sqrt{\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})+\frac{\rho^2}{4}\mathbf{I}}-\frac{\rho}{2}\mathbf{I}$ together with $\sqrt{\theta_i(\rho+\theta_i)(1-a_{ii})+\frac{\rho^2}{4}}-\frac{\rho}{2}$ for several industries. The eigenvalues are extremely close across these two cases, thus helping us identify sectors that are associated with the smallest eigenvalues. Figure S.M.1 shows that this extremely close association holds across the full range of eigenvalues. This remarkable accuracy stems from the feature that the diagonal entries of the input-output matrix are large relative to its off-diagonal entries. Accordingly, an application of Gershgorin circle theorem delivers a visual and complementary interpretation of why this approximation is so accurate.

To show the aggregate implications of shocks to these sectors with the lowest eigenvalues, we do a counterfactual exercise by dropping the three sectors with the smallest eigenvalues and recomputing the impulse responses of inflation and GDP.[29] Just dropping these three sectors leads to a noticeable change in dynamics, with the cumulative IRF of real GDP in the calibrated economy higher by around 16 percent.[30] These results show that a few sectors play a very influential role in driving monetary non-neutrality in the

[29] In this exercise, we recompute the counterfactual input-output matrix by moving the share of these dropped sectors (as inputs) to the labor share. Moreover, these sectors correspond closely to sectors that have the highest half-life of sectoral inflation to a monetary shock.

[30] Two of these sectors have a zero sectoral share in aggregate GDP while the third one has an extremely small sectoral share of 0.0015 percent. As such, in a disconnected economy, dropping them would not have affected the response of aggregate GDP. That is, the "direct effect of sector i" term in Equation (39) would be zero for these sectors.

economy as they determine the persistence of aggregate inflation. To show this clearly, in Figure 6 we plot the impulse responses of inflation and GDP to a monetary shock for both our calibrated and counterfactual economies. They depict that over the longer horizon, inflation response is lower in the counterfactual economy and this difference in dynamics gets reflected in a lower response of real GDP throughout. We had highlighted this critical role of sectors with low $\xi_i = \sqrt{\theta_i(\rho+\theta_i)(1-a_{ii}) + \frac{\rho^2}{4}} - \frac{\rho}{2}$ in driving monetary non-neutrality analytically in Proposition 7 and results here are the quantitative counterpart to those insights.

4 Propagation with Endogenous Monetary Policy Responses

So far, we have examined the economy's responses to monetary policy and sectoral TFP or wedge shocks separately; i.e., we have used a framework where monetary policy did not respond to the impact of TFP or wedge shocks on sectoral prices. In this section, we extend our analysis to study policies where monetary policy endogenously responds to sectoral shocks (e.g., it stabilizes aggregate GDP gap or aggregate inflation). We show that all these changes can be captured by appropriately adjusting the duration-adjusted Leontief matrix, $\mathbf{\Gamma}$. Thus, the mechanisms highlighted in the previous section remain the main drivers of the dynamics of the model variables, but with the use of the newly adjusted PRDL matrix.

To illustrate this point most clearly, in Section 4.1, we first consider the specific case of aggregate GDP gap targeting and discuss how such a policy alters the propagation of shocks through an adjustment of the matrix $\mathbf{\Gamma}$. This case is of particular interest for three reasons: First, it connects naturally with the usual representation of Phillips curves in New Keynesian models that involve GDP gaps. Second, it has the feature that it naturally puts more weight on sectors with stickier prices and thus, as shown by previous research, it approximates optimal policy pretty closely.[31] Third, as we show below, it is equivalent to an alternative economy with a different production network where the monetary policy does *not* respond to shocks endogenously, which allows us to characterize rigorously the proper adjustment of the $\mathbf{\Gamma}$ matrix.

Section 4.2 then considers the general case of stabilization policies that target a

[31] La'O and Tahbaz-Salehi (2022), Rubbo (2023) make this case for production network economies with sticky prices. See also Woodford (2003) and Gali (2008) for similar results in models without production networks. In particular, (Woodford, 2003, page 442) and its ensuing discussion provides clear intuition for why such policies are nearly optimal as they naturally put more weight on more sticky price sectors (see, also, Aoki, 2001).

weighted average of sectoral inflation rates and derives the appropriate adjustment of the matrix $\boldsymbol{\Gamma}$ when monetary policy stabilizes such an arbitrary price index. In particular, CPI inflation targeting is a specific case of such policies, where these weights correspond to the household expenditure shares.

4.1. Phillips Curve with GDP Gap Representation and GDP Gap Stabilization Policy

In Proposition 1, we derived sectoral Phillips curves in terms of inflation and nominal price gaps and discussed how this representation delivers analytical results for general paths of money supply and sectoral shocks. To study endogenous monetary policy responses, however, it is useful to relate our results to conventional representations of Phillips curves which involve GDP gaps, and which are combined with real wage gaps in sticky-price and sticky-wage models (e.g., Woodford, 2003, Gali, 2008) and with relative price gaps in multi-sector models (e.g., Aoki, 2001, Benigno, 2004).

To this end, consider the sectoral Phillips curves in Proposition 1 and recall from its proof that Equation (19) is always a necessary condition for the equilibrium. However, as we show below, this equation is no longer a sufficient condition for the equilibrium path of sectoral prices when monetary policy endogenously responds to sectoral shocks. This is intuitive in the presence of endogenous monetary policy because the path of sectoral prices should, and will, also depend on the *stance* of monetary policy when it is endogenous.

Since our focus here is on a monetary policy which stabilizes the GDP gap of this economy, we re-write Equation (19) in terms of the GDP gap. Define the relative sectoral prices, $\mathbf{q}_t \equiv \mathbf{p}_t - p_t\mathbf{1}$, as the vector of sectoral prices relative to the CPI price index in log form. Then, let $\mathbf{q}_t^f \equiv \mathbf{p}_t^f - p_t^f\mathbf{1} = (\boldsymbol{\Psi} - \mathbf{1}\boldsymbol{\lambda}^\top)(\boldsymbol{\omega}_t - \boldsymbol{z}_t)$ (as implied by Equation (14)) denote the same object in the flexible price economy, and let $\tilde{y}_t \equiv y_t - y_t^f$ denote the GDP gap. We then observe that,

Lemma 3. *The sectoral Phillips curves in Equation (19) can be re-written as:*

$$\dot{\boldsymbol{\pi}}_t = \rho\boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{q}_t - \mathbf{q}_t^f) - \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})\boldsymbol{\alpha}\tilde{y}_t \tag{41}$$

Equation (41) shows that the nominal price gaps can be decomposed into *relative* price gaps and a term that involves the aggregate GDP gap. We can also now derive the usual representation of the *aggregate* Phillips curve in terms of the GDP gap as follows :

$$\dot{\pi}_t = \rho\pi_t + \boldsymbol{\beta}^\top\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{q}_t - \mathbf{q}_t^f) - \boldsymbol{\beta}^\top\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})\boldsymbol{\alpha}\tilde{y}_t \tag{42}$$

after we multiply Equation (41) by the expenditure shares of households from different sectors. Thus, we see that in a network economy, relative price distortions affect inflation dynamics independently of the GDP gap. This is in contrast to standard one-sector economies in which the output gap summarizes the deviation of the allocations in the sticky economy from the one in the flexible economy. A multi-sector economy with n sectors has instead n gaps. GDP gap is one of those and the other $n-1$ are represented by the deviation of relative prices from their flexible price counterparts in Equation (41).[32]

For a given path of sectoral shocks and initial level of prices, now suppose that monetary policy does all it takes to stabilize the GDP gap, such that $\tilde{y}_t = 0$ for all t.[33] Then, Equation (41) implies that the dynamics of sectoral prices is governed by the following system of differential equations:

$$\dot{\boldsymbol{\pi}}_t = \rho\boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{q}_t - \mathbf{q}_t^f) \quad \text{where} \quad \mathbf{q}_t = (\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\intercal)\mathbf{p}_t \tag{43}$$

Notice that since we have now incorporated the stance of monetary police into Equation (43), along with the boundary conditions $\mathbf{p}_0 = \mathbf{p}_{0^-}$, it is a system of differential equations where the only endogenous variables are the sectoral prices, and relative price gaps in the flexible economy $(\mathbf{q}_t^f)_{t\geq 0}$ is an exogenous forcing term to the system that is fully determined by the path of sectoral shocks. Thus, Equation (43) is now also sufficient for characterizing the equilibrium path of sectoral prices:

Corollary 1. *Given initial prices $\mathbf{p}_0 = \mathbf{p}_{0^-}$, Equation (43) is necessary and sufficient for the path of non-explosive sectoral prices under a monetary policy that sets $\tilde{y}_t = 0, \forall\, t \geq 0$.*

While Corollary 1 confirms the sufficiency of Equation (43) for the equilibrium path of sectoral prices, it is not immediately clear how to solve this system of differential equations. To this end, we derive an equivalence result between this economy with the endogenous monetary policy that stabilizes the GDP gap and an alternative economy that satisfies the assumptions of the previous section, where monetary policy was exogenous.

Proposition 9. *For a given $\mathbf{p}_0 = \mathbf{p}_{0^-}$ and path of sectoral shocks $(\boldsymbol{\omega}_t, \boldsymbol{z}_t)_{t\geq 0}$, the dynamics of sectoral prices under a monetary policy that fully stabilizes the GDP gap is* identical *to the dynamics of sectoral prices in a counterfactual economy where monetary policy is exogenously fixed at $m_t = 0, \forall\, t \geq 0$, but where the input-output matrix is $\mathbf{A}_{\boldsymbol{\beta}} \equiv \mathbf{A} + \boldsymbol{\alpha}\boldsymbol{\beta}^\intercal$.*

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[32] The same point holds for the *slope* of the Phillips curve as well. In one-sector economies, the slope of the Phillips curve is a sufficient statistic for the impact of demand-driven shocks on aggregate inflation. However, in a multi-sector economy, multiple gaps affect the economy simultaneously and the impact of shocks depend on the bilateral interactions of all these gaps, which is why the sufficient statistic in our setting is the $n \times n$ adjusted Leontief matrix.

[33] In the next subsection, we show the feasibility of such a policy and for now take feasibility as given.

Proposition 9 is critical because it allows us to apply Propositions 3 and 4—which were derived for exogenous monetary policies—to study the dynamics of sectoral prices under an endogenous monetary policy that stabilizes the GDP gap; it shows that to do so, we only need to modify the $\mathbf{\Gamma}$ matrix as if the production network was $\mathbf{A}_{\boldsymbol{\beta}} = \mathbf{A} + \boldsymbol{\alpha}\boldsymbol{\beta}^{\top}$. Thus, let us define this adjusted matrix as:

$$\bar{\mathbf{\Gamma}}_{\boldsymbol{\beta}} \equiv \mathbf{\Theta}(\rho\mathbf{I} + \mathbf{\Theta})(\mathbf{I} - \mathbf{A}_{\boldsymbol{\beta}}) + \frac{\rho^2}{4}\mathbf{I} \tag{44}$$

We can then apply Propositions 3 and 4 to obtain the implied IRFs of the model variables under a monetary policy that stabilizes the GDP gap given this adjusted $\bar{\mathbf{\Gamma}}_{\boldsymbol{\beta}}$ matrix.

The ensuing question then is how propagation of sectoral shocks under $\bar{\mathbf{\Gamma}}_{\boldsymbol{\beta}}$ differs from propagation under $\mathbf{\Gamma}$. We will provide numerical results for the calibrated economy below for this comparison, but to build intuition on how this policy affects propagation, consider the case of disconnected economies and note that a disconnected economy is no longer characterized by a diagonal network. Instead, $\bar{\mathbf{A}} = \mathbf{A}_D + \boldsymbol{\alpha}\boldsymbol{\beta}^{\top}$, where $\mathbf{A}_D$ is the diagonal input-output matrix of the disconnected economy as in the previous section. Note that even though there are no true input-output linkages in a disconnected economy *across* sectors, the GDP gap stabilization policy induces a *de facto* input-output structure, where it is as if sector i's input share of sector $j \neq i$'s output is $\alpha_i \beta_j$. The intuition is that monetary policy that stabilizes the GDP gap in response to a specific sector's shock also affects the marginal costs of all other sectors, thereby creating a pass-through of one sector's shocks to other sectors.[34] However, this is not the only effect: By increasing the "as if" input-output linkages, GDP gap stabilization also increases the degree of strategic complementarities in price setting. Thus, it also leads to higher price stickiness.[35] More precisely, by relying on this equivalence result, we can conclude from Propositions 5, 6 and 8 that the more input-output linkages *induced by this policy* amplify *price stickiness* and *inflation pass-through* across sectors.

Building on these insights, we next consider the more general case of stabilization policies that target an arbitrary weighted average of sectoral inflation rates.

[34] This is a general feature of stabilization policies that target a weighted average of sectoral inflation rates, as we will show in Section 4.2.

[35] In fact, one can think of any stabilization policy as one that makes a *certain* price index perfectly sticky, so that it does not respond to any shocks.

4.2. Theoretical Results for Generalized Stabilization Policies

Consider a general environment where the central bank is concerned with the stabilization of a particular price index, $p_{\boldsymbol{\eta},t} \equiv \boldsymbol{\eta}^\intercal \mathbf{p}_t$, where $\boldsymbol{\eta}$ is an arbitrary vector of weights. For instance, this could be the CPI index ($\boldsymbol{\eta} = \boldsymbol{\beta}$), or as we will derive later, some other price index whose stabilization will result in stabilization of the GDP gap. Since we are not restricting ourselves in this section to a specific $\boldsymbol{\eta}$, we refer to such a policy as a generalized stabilization policy associated with $\boldsymbol{\eta}$.

Monetary Rules for Stabilization Policies. To characterize the dynamics of the model variables under such a policy, we start with our result in Proposition 1. Reviewing the proof of that result, we see that its validity does not rely on the monetary instrument, m_t, being exogenous and the dynamics of sectoral prices still satisfy

$$\dot{\boldsymbol{\pi}}_t = \rho \boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{p}_t - m_t \mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)), \quad \text{with} \quad \mathbf{p}_0 = \mathbf{p}_{0^-} \text{ given.} \tag{45}$$

where now we have substituted $\mathbf{p}_t^f = m_t \mathbf{1} + \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)$ directly in the sectoral Phillips curves because $\mathbf{p}_t^f$ is now a nominal quantity that depends on the monetary policy in the sticky price economy and no longer has the interpretation of the flexible price level. Moreover, we can then multiply Equation (45) with $\boldsymbol{\eta}^\intercal$ from the left to obtain the Phillips curve for the price index $p_{\boldsymbol{\eta},t}$ as:

$$\dot{\pi}_{\boldsymbol{\eta},t} = \rho \pi_{\boldsymbol{\eta},t} + \boldsymbol{\eta}^\intercal \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{p}_t - m_t \mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)) \tag{46}$$

We want to characterize a monetary rule for m_t that stabilizes $p_{\boldsymbol{\eta},t}$. But note that if $p_{\boldsymbol{\eta},t}$ is indeed stabilized, it has to be that $\pi_{\boldsymbol{\eta},t} = \dot{\pi}_{\boldsymbol{\eta},t} = 0$ for all t. Plugging these into Equation (46) we obtain that a *necessary condition* for stabilization of $\pi_{\boldsymbol{\eta},t}$ is that:

$$m_t = \tilde{\boldsymbol{\eta}}^\intercal (\mathbf{p}_t - \tilde{\mathbf{p}}_t^f), \qquad \tilde{\boldsymbol{\eta}} \equiv \frac{(\mathbf{I} - \mathbf{A}^\intercal)(\rho \mathbf{I} + \boldsymbol{\Theta})\boldsymbol{\Theta}\boldsymbol{\eta}}{\boldsymbol{\eta}^\intercal \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})\mathbf{1}}, \qquad \tilde{\mathbf{p}}_t^f \equiv \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t) \tag{47}$$

Equation (47) is a price targeting rule that sets the monetary instrument, m_t, equal to a particular weighted average—according to $\tilde{\boldsymbol{\eta}}$—of sectoral prices, $\mathbf{p}_t$, relative to their counterparts in the flexible economy, where $\tilde{\mathbf{p}}_t^f$ is now defined as the vector of sectoral prices in the flexible economy *relative* to nominal demand. It is important to observe that since $\tilde{\mathbf{p}}_t^f$ is now a real object and as classical dichotomy holds in the flexible economy, it is independent of monetary policy.

Moreover, Equation (47) is a necessary condition in the sense that if the price index $p_{\boldsymbol{\eta},t}$ is stabilized then m_t should satisfy this rule. The following lemma shows that this rule is also sufficient for the stabilization of $p_{\boldsymbol{\eta},t}$, under the boundary conditions specified in the previous section.

Proposition 10. *Under boundary conditions $\mathbf{p}_0 = \mathbf{p}_{0^-}$ and non-explosive prices, the monetary rule specified in Equation (47) is necessary and sufficient for the stabilization of the price index $p_{\boldsymbol{\eta},t}$.*

Proposition 10 thus shows that the monetary rule in Equation (47) is both necessary and sufficient for the stabilization of $p_{\boldsymbol{\eta},t}$. We next show that these rules lead to a particular adjustment of the $\boldsymbol{\Gamma}$ matrix as we saw for the case of GDP gap stabilization.

Adjustment of PRDL Matrices under Stabilization Policies. Having established that Equation (47) is both necessary and sufficient for the stabilization of the target price index $p_{\boldsymbol{\eta},t}$, we now turn to characterize the evolution of sectoral prices and GDP (gap) under such a policy. To do so, we simply need to substitute the monetary rule in Equation (47) into Equation (45), which yields the following dynamics for sectoral prices:

$$\dot{\boldsymbol{\pi}}_t = \rho \boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A}_{\tilde{\boldsymbol{\eta}}})(\mathbf{p}_t - \tilde{\mathbf{p}}_t^f), \qquad \mathbf{A}_{\tilde{\boldsymbol{\eta}}} \equiv \mathbf{A} + \boldsymbol{\alpha}\tilde{\boldsymbol{\eta}}^\top \tag{48}$$

Notice how this equation now resembles Equation (19), but with an adjusted matrix that incorporates $\mathbf{I} - \mathbf{1}\tilde{\boldsymbol{\eta}}^\top$, and satisfies the property that $\tilde{\mathbf{p}}_t^f$ is an *exogenous* forcing term that only depends on the path of shocks. Thus, Equation (48) satisfies all ensuing remarks of Proposition 1 in Remarks 1 to 4 but with the adjusted matrix $\mathbf{A}_{\tilde{\boldsymbol{\eta}}}$ instead of the input-output matrix $\mathbf{A}$. Because this equation is now necessary and sufficient for the dynamics of sectoral prices (given proper boundary conditions), we can see that these dynamics are fully governed by the adjusted matrix $\bar{\boldsymbol{\Gamma}}_{\tilde{\boldsymbol{\eta}}}$:

$$\bar{\boldsymbol{\Gamma}}_{\tilde{\boldsymbol{\eta}}} \equiv \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A}_{\tilde{\boldsymbol{\eta}}}) + \frac{\rho^2}{4}\mathbf{I} \tag{49}$$

In fact, Proposition 9 is just a special case of this result, where $\tilde{\boldsymbol{\eta}} = \boldsymbol{\beta}$, with the only difference being that in the general case where $\boldsymbol{\eta}$ can be any vector of weights, the adjusted matrix $\mathbf{A}_{\tilde{\boldsymbol{\eta}}}$ is no longer necessarily equivalent to another input-output matrix under our original assumptions. The reason is that the vector $\tilde{\boldsymbol{\eta}}$ can have negative entries depending on the weights in the original $\boldsymbol{\eta}$, which can lead to negative entries in $\mathbf{A}_{\tilde{\boldsymbol{\eta}}}$. Therefore, the more general case of arbitrary $\boldsymbol{\eta}$ can resemble an equivalence relation similar to Proposition 9 as long as $\mathbf{A}_{\tilde{\boldsymbol{\eta}}}$ has positive entries, which will guarantee that it is the input-output network of some economy.[36]

[36] Note that this also requires the row sums of $\mathbf{A}_{\tilde{\boldsymbol{\eta}}}$ to be less than one so that the production function can be constant returns to scale. This is, however, always satisfied, as we can confirm that the row sums are always equal to 1 for any $\boldsymbol{\eta}$.

However, note that such an equivalence is not necessary for characterizing the solution of the system above, and as long as $\bar{\mathbf{\Gamma}}_{\tilde{\boldsymbol{\eta}}}$ has a principal square root; i.e., a positive stable matrix $\sqrt{\bar{\mathbf{\Gamma}}_{\tilde{\boldsymbol{\eta}}}}$ exists such that $(\sqrt{\bar{\mathbf{\Gamma}}_{\tilde{\boldsymbol{\eta}}}})^2 = \bar{\mathbf{\Gamma}}_{\tilde{\boldsymbol{\eta}}}$, then the proof of Proposition 2 goes through, and Propositions 3 and 4 hold.

In particular, to connect this to our example of GDP gap stabilization, note that if $\boldsymbol{\eta} = \mathbf{\Theta}^{-1}(\rho\mathbf{I} + \mathbf{\Theta})^{-1}\mathbf{\Psi}^\top\boldsymbol{\beta}$ then $\tilde{\boldsymbol{\eta}} = \boldsymbol{\beta}$ and Equation (48) becomes identical to Equation (43). Thus, the monetary rule in Equation (47) with $\boldsymbol{\eta} = \mathbf{\Theta}^{-1}(\rho\mathbf{I} + \mathbf{\Theta})^{-1}\boldsymbol{\lambda}$ (where $\boldsymbol{\lambda} \equiv \mathbf{\Psi}^\top\boldsymbol{\beta}$ is the steady state vector of Domar weights) stabilizes the GDP gap. Thus, stabilizing this special price index is equivalent to GDP gap stabilization, as pointed out by Rubbo (2023); who shows that while the aggregate Phillips curve in multi-sector economies with production networks involves terms other than the GDP gap, there always exists a composite price index whose corresponding Phillips curve only includes inflation in that price index and the GDP gap, and thus refers to this price as the "divine coincidence index."[37]

Another special case that is of interest is when monetary policy stabilizes the CPI index, where $\boldsymbol{\eta} = \boldsymbol{\beta}$; i.e., when monetary policy weighs sectors by their household expenditure share. This is in contrast to GDP gap stabilization, where $\boldsymbol{\eta} = \mathbf{\Theta}^{-1}(\rho\mathbf{I} + \mathbf{\Theta})^{-1}\boldsymbol{\lambda}$; and we see that GDP gap stabilization policy puts more relative weight on sectors with *higher price stickiness* and *higher Domar weights*. This explains why GDP gap stabilization closely approximates the *optimal* price index, as characterized by La'O and Tahbaz-Salehi (2022) and Rubbo (2023) (but does not coincide with it).

4.3. Quantitative Results with Endogenous Monetary Policy Responses

We now present quantitative results that are counterparts to our theoretical discussions above by showing how different monetary policy responses can alter the transmission of sectoral shocks. For this exercise, we choose the Oil and Gas Extraction and Semiconductor Manufacturing Machinery industries as representatives of upstream industries with high and low adjusted price change frequencies, respectively, and use sectoral shocks to them as our numerical examples.

We start by examining CPI inflation stabilization policies in response to sectoral

[37] Special cases of this result in two sector economies with heterogeneous price stickiness as well as models with sticky prices and sticky wages are derived previously in (Woodford, 2003, page 442) and (Gali, 2008, Equation (33) and the discussion on page 137), respectively. To see Rubbo (2023)'s point in our framework, define the divine coincidence price index as $p_t^{DC} \equiv \boldsymbol{\beta}^\top\mathbf{\Psi}\mathbf{\Theta}^{-1}(\rho\mathbf{I}+\mathbf{\Theta})^{-1}\mathbf{p}_t/(\boldsymbol{\beta}^\top\mathbf{\Psi}\mathbf{\Theta}^{-1}(\rho\mathbf{I}+\mathbf{\Theta})^{-1}\mathbf{1})$. Using Equation (41), inflation in this price index is

$$\dot{\pi}_t^{DC} = \rho\pi_t^{DC} - \frac{1}{\boldsymbol{\beta}^\top\mathbf{\Psi}\mathbf{\Theta}^{-1}(\rho\mathbf{I}+\mathbf{\Theta})^{-1}\mathbf{1}}\tilde{y}_t, \qquad \boldsymbol{\beta}^\top\mathbf{\Psi}\mathbf{\Theta}^{-1}(\rho\mathbf{I}+\mathbf{\Theta})^{-1}\mathbf{1} = \sum_{i\in[n]}\frac{\lambda_i}{\theta_i(\rho+\theta_i)} \tag{50}$$

shocks. In Figure 3a, we plot impulse responses for the two sectoral shocks under (a) our baseline monetary policy, which stabilizes nominal rates,[38] and (b) under a policy that fully stabilizes aggregate inflation. The sectoral shocks are calibrated to lead to a 1 percent increase in sectoral inflation under the baseline monetary policy specification.

As discussed in Section 3.4.4, under baseline policy, sectoral inflation in the Oil and Gas Extraction industry passes through substantially on impact to aggregate inflation, but the effects are transient. The key result we want to highlight here is that if monetary policy responds by stabilizing aggregate inflation driven by a shock to the Oil and Gas Extraction industry, it creates a large negative GDP gap. In fact, this policy is so contractionary that it leads the GDP in the economy to fall below the GDP under flexible prices![39] In sharp contrast, stabilizing aggregate inflation due to a shock to the Semiconductor Manufacturing Machinery industry is not nearly as contractionary in terms of aggregate GDP. The reason is that the Oil and Gas Extraction industry has a relatively short adjusted price spell duration, and as such, rises in sectoral inflation in that sector do not cause a large dispersion in relative prices *when* policy does not respond to it. However, when monetary policy does respond by stabilizing aggregate inflation it creates large relative price gaps that lead to a negative aggregate GDP gap.[40]

We conclude this section by considering the consequences of a GDP gap stabilization policy in response to sectoral shocks. In Figure 3b, we plot impulse responses for the two sectoral shocks under (a) our baseline monetary policy, which stabilizes nominal rates, and (b) under a policy that fully stabilizes the aggregate GDP gap. Given our discussion above of Figure 3a we then naturally see that here, compared to baseline policy, aggregate inflation is slightly lower initially (but still positive), for a shock to the Oil and Gas Extraction industry while it is much lower initially (and negative), for a shock to the Semiconductor Manufacturing Machinery industry. Thus, a monetary policy rule that stabilizes the aggregate GDP gap allows substantial pass-through of the sectoral shock to aggregate inflation if it originates in a sector such as the Oil and Gas Extraction industry. This is consistent with the theoretical observation in Section 4.2 that GDP gap

[38] Note that with Golosov and Lucas (2007) preferences, as our baseline monetary policy fixes nominal GDP at some m, it also fixes nominal interest rates at ρ.

[39] This is a direct consequence of network spillover effects as such responses are not possible in one sector NK models.

[40] To illustrate this clearly, Figure S.M.3 repeats this exercise in a model with a homogeneous frequency of price adjustment across sectors while Figure S.M.4 does so in a model where the Oil and Gas Extraction industry has the same price duration as the Semiconductor Manufacturing Machinery industry and we see that responding to inflation originating in this industry does not lead to a negative GDP gap in either case.

stabilization policy puts more weight on stickier sectors and allows a higher aggregate inflation pass-through for shocks to more flexible price sectors.

5 Extensions

We now present some key extensions of our theoretical and quantitative results.

5.1. General Labor Supply Elasticity

So far, we used preferences that imply an infinite Frisch elasticity of labor supply. Our solution techniques, analytical results, and quantitative insights do not, however, depend on this simplification. In Appendix S.M.2.1, we present the details of the model with a general labor supply elasticity and present here the counterpart of Proposition 1 with $\rho \downarrow 0$:

$$\dot{\boldsymbol{\pi}}_t = \boldsymbol{\Gamma}(\mathbf{I} + \psi \mathbf{1}\boldsymbol{\beta}^\intercal)(\mathbf{p}_t - \mathbf{p}_t^f), \qquad \mathbf{p}_t^f \equiv m_t \mathbf{1} - \boldsymbol{\Psi} \boldsymbol{z}_t + (\boldsymbol{\Psi} - \frac{\psi}{1+\psi}\mathbf{1}\boldsymbol{\lambda}^\intercal)\boldsymbol{\omega}_t \tag{51}$$

where ψ is the inverse Frisch elasticity of labor supply. We can then extend Propositions 3 and 4 to this case by replacing $\boldsymbol{\Gamma}$ with $\boldsymbol{\Gamma}_\psi \equiv \boldsymbol{\Gamma}(\mathbf{I} + \psi \mathbf{1}\boldsymbol{\beta}^\intercal)$ and adjusting for $\mathbf{p}_t^f$ as above. In particular, the impulse responses for monetary and sectoral productivity shocks only change through $\boldsymbol{\Gamma}_\psi$. The impulse responses for sectoral wedge shocks, however, also need to be adjusted through $\mathbf{p}_t^f$.

In Figure S.M.5 we show impulse responses of aggregate inflation and GDP to an expansionary monetary policy shock when the Frisch elasticity is calibrated at 2. Since a finite Frisch elasticity introduces aggregate strategic substitutability, it reduces the persistence of inflation and thereby, the extent of monetary non-neutrality. This calibration also does not alter our quantitative results on the various forces that drive monetary non-neutrality, as shown in Figure S.M.6 - Figure S.M.8.[41]

5.2. Taylor Rule as Monetary Policy Rule

We now model monetary policy as following a Taylor rule. Our derivations generalize to using such a rule, the details of which are in Appendix S.M.2.2.[42] First, the counterpart of Proposition 1 with $\rho \to 0$ and a Taylor rule, $i_t = \phi_\pi \boldsymbol{\beta}^\intercal \boldsymbol{\pi}_t + v_t$—where v_t captures

[41] Figure S.M.9 shows that the distribution of sectoral inflation response after a monetary policy shock also depicts the same patterns as in Section 3.4.2.

[42] We need to impose boundary conditions that ensure that inflation and relative sectoral prices are stationary and for solving the resulting set of equilibrium system of equations, we use a Schur decomposition.

deviations from the rule—is:[43]

$$\ddot{\boldsymbol{\pi}}_t = \boldsymbol{\Gamma}(\mathbf{I} - \phi_\pi \mathbf{1}\boldsymbol{\beta}^\top)(\boldsymbol{\pi}_t - \boldsymbol{\pi}_t^f), \qquad (\mathbf{I} - \phi_\pi \mathbf{1}\boldsymbol{\beta}^\top)\boldsymbol{\pi}_t^f \equiv \mathbf{1}\,v_t - \boldsymbol{\Psi}(\dot{\boldsymbol{z}}_t - \dot{\boldsymbol{\omega}}_t) \tag{52}$$

In this representation, $\boldsymbol{\pi}_t^f$ is the sectoral inflation rate that would have prevailed in a flexible price economy with the same Taylor rule and is exogenous to the system of differential equations. We can see that this equation differs from our Proposition 1 in two aspects. First, it is a second-order differential equation in $\boldsymbol{\pi}_t$ rather than in prices. This is because, with an inflation-targeting Taylor rule, the economy is no longer price stationary, similar to one-sector New Keynesian models. Second, the dynamics of the second-order differential equations are governed by $\boldsymbol{\Gamma}$, now adjusted for the endogenous response of monetary policy through the Taylor rule: $\boldsymbol{\Gamma}_{\phi,\pi} \equiv \boldsymbol{\Gamma}(\mathbf{I} - \phi_\pi \mathbf{1}\boldsymbol{\beta}^\top)$.

A Taylor rule in terms of inflation makes sticky price models forward-looking and thus the source of persistence is exogenous. In our baseline calibration, fixing the Taylor rule coefficient at the standard value of $\phi_\pi = 1.5$, we introduce persistent shocks to the Taylor rule. We then calibrate the size and persistence of the shocks to generate a response of aggregate inflation that matches the aggregate inflation response in our nominal GDP rule economy of Section 3.4.2. Figure 7 shows the impulse responses of aggregate inflation and GDP to an expansionary monetary policy shock. The monetary non-neutrality, by design, is essentially the same as in Section 3.4.2.

Given this calibrated Taylor rule economy, we investigate the various forces that drive monetary non-neutrality, which are presented in Figure S.M.10 - Figure S.M.12. Overall, these results are consistent with our main conclusion that both production networks and heterogenous price stickiness play a quantitatively important role in amplifying monetary non-neutrality. We note that the amplification coming from them jointly, compared to the horizontal economy with homogeneous price stickiness across sectors, is a bit smaller than in Section 3.4.2.[44] The reason is that in this economy, persistent

[43] In Appendix S.M.2.2, we derive a general version for $\rho \neq 0$ and a Taylor rule that targets other price indices.

[44] In addition, compared to the results in Section 3.4.2, production networks and heterogeneous price stickiness play a similar role quantitatively. In these counterfactuals, we keep the monetary policy shock and persistence the same as the baseline calibration. The reason is that with the Taylor rule as a monetary policy rule, inflation becomes forward-looking in the model and as such, differences in model features show up as affecting the level response of inflation, and not the persistence. We thus will not fix the impact response of inflation across various counterfactual exercises. For intuition, in the one-sector model with the Taylor rule, the slope of the Phillips curve that incorporates strategic complementarity only affects the impact response of inflation. In using a Taylor rule with persistent shocks, our counterfactual exercises here are more closely related to the ones in Pasten, Schoenle, and Weber (2020), but nevertheless there are differences as we compare the production network economy with a horizontal economy.

dynamics in inflation come about through persistence in the monetary policy shock itself, which increases monetary non-neutrality even in the basic multi-sector economy.[45]

6 Conclusion

In this paper, we derive closed-form solutions for inflation and GDP dynamics in multi-sector New Keynesian economies with arbitrary production networks. In a series of new analytical results, we isolate the precise interactions of production linkages and price stickiness across sectors and show how stickiness trickles to downstream sectors through the input-output network. We show that these amplification results are quantitatively significant. For instance, the three sectors with the most contribution to the persistence of aggregate inflation have a combined consumption share of around zero and yet, they explain around 16% of the GDP response to monetary shocks. Finally, we explore how endogenous monetary policy leads to significantly different implications for aggregate variables depending on the sectoral source of inflation.

Our framework presents new avenues for future research. As an example, a model with state-dependent pricing, due to fixed costs of changing prices, could lead to new insights. It will also be interesting to study welfare and optimal policy implications in our model with various shocks that help match historical sectoral inflation dynamics well.

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[45]Finally, Figure 8 shows that the distribution of sectoral inflation response after a monetary policy shock depicts the same patterns as in Section 3.4.2.

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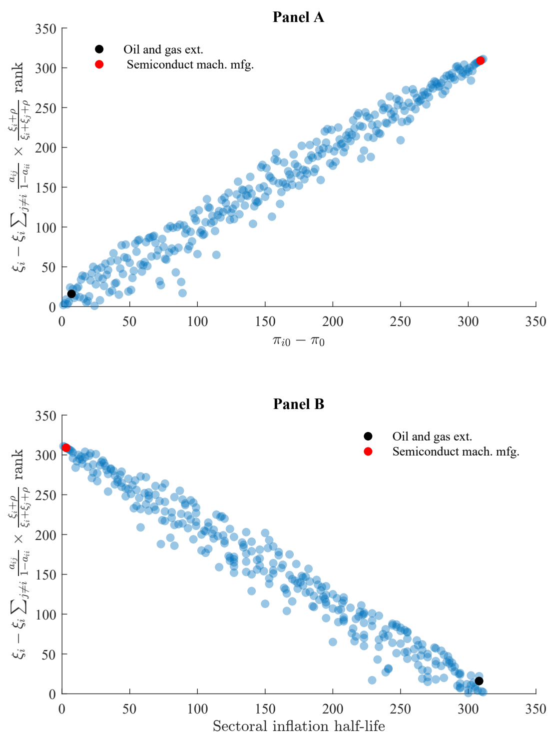
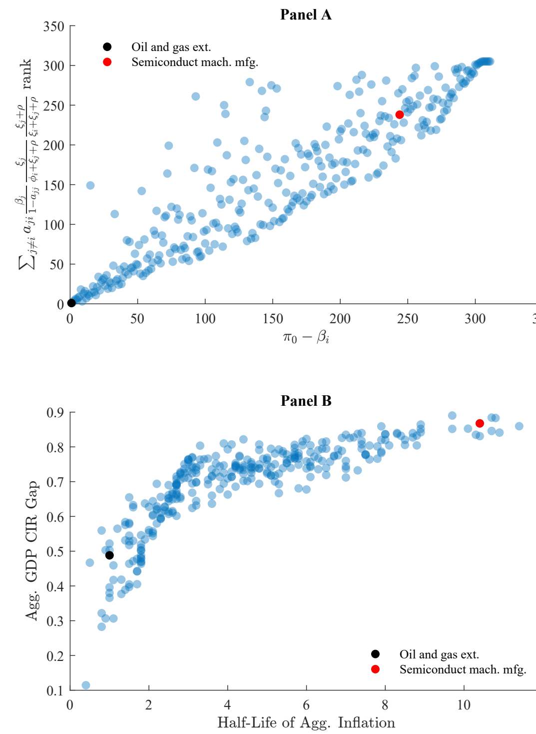
Figure 2: Rank correlations across sectors

(a) Correlation of actual ranks of sectors and ranks using an approximated statistic for sectoral inflation response to a monetary policy shock



Notes: This figure plots the actual ranks and ranks using an approximated statistic for sectoral inflation response to a monetary policy shock that generates a one percentage increase in aggregate inflation on impact. Panel A plots sectoral inflation impact response while Panel B plots the sectoral inflation half-life. Each dot in the figure represents a sector. The calibration of the model is at a monthly frequency.

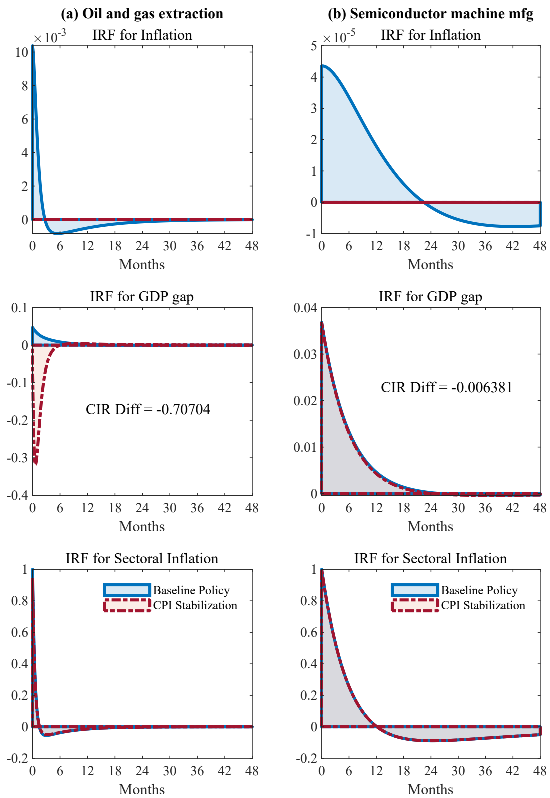
(b) Aggregate inflation and GDP dynamics following sectoral shocks



Notes: Panel A of the figure plots actual ranks of sectors and ranks using an approximated statistic for aggregate inflation impact response after a sectoral shock increases sectoral inflation by one percentage on impact. Panel B of the figure plots how aggregate GDP gap and half-life of aggregate inflation are correlated when a unit sectoral TFP shock hits the economy. Average duration of the sectoral shocks is six months. Each dot in the figure represents a sector. The calibration of the model is at a monthly frequency.

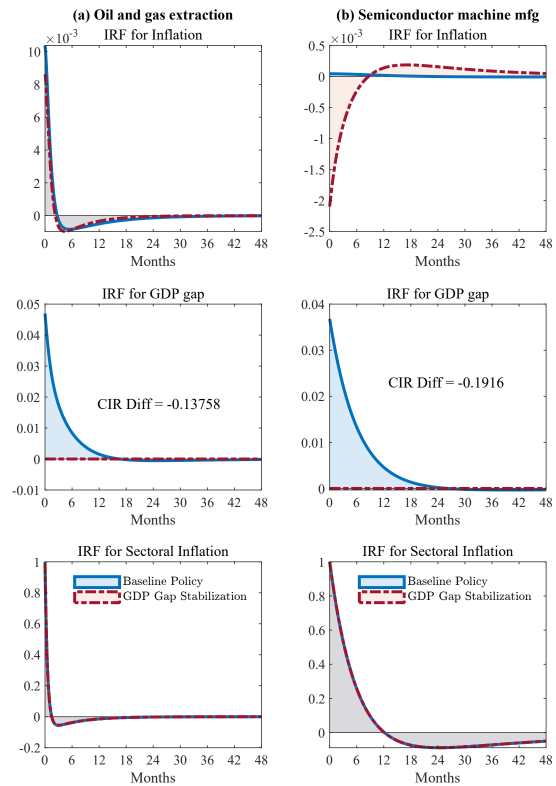
Figure 3: Dynamics following sectoral shocks under different policies

(a) Baseline policy vs. CPI stabilization



Notes: This figure plots the impulse response functions for inflation, gdp gap, and sectoral inflation to a sectoral shock that increases sectoral inflation by one percent on impact in the baseline policy economy. It compares the baseline policy economy with an economy where monetary policy stabilizes aggregate inflation. Panel A: Oil and gas extraction. Panel B: Semiconductor machine manufacturing. The calibration of the model is at a monthly frequency.

(b) Baseline policy vs. GDP gap stabilization



Notes: This figure plots the impulse response functions for inflation, gdp gap, and sectoral inflation to a sectoral shock that increases sectoral inflation by one percent on impact in the baseline policy economy. It compares the baseline policy economy with an economy where monetary policy stabilizes the aggregate GDP gap. Panel A: Oil and gas extraction. Panel B: Semiconductor machine manufacturing. The calibration of the model is at a monthly frequency.

Table 1: Ranking of industries by pass-through to aggregate inflation after a sectoral shock

Industry	Agg. Inflation Impact Resp.
Oil and gas extraction	9.542×10^{-3}
Insurance agencies, brokerages, and related act...	8.392×10^{-3}
Employment services	6.014×10^{-3}
Legal services	5.688×10^{-3}
Management consulting services	5.633×10^{-3}
Advertising, public relations, and related serv...	5.011×10^{-3}
Accounting, tax preparation, bookkeeping, and p...	4.981×10^{-3}
Architectural, engineering, and related services	4.968×10^{-3}
Warehousing and storage	4.964×10^{-3}
Electric power generation, transmission, and di...	3.833×10^{-3}
Services to buildings and dwellings	3.696×10^{-3}
Monetary authorities and depository credit inte...	3.628×10^{-3}
Scenic and sightseeing transportation and suppo...	3.413×10^{-3}
Securities and commodity contracts intermediati...	3.354×10^{-3}
Other support activities for mining	3.236×10^{-3}
Truck transportation	3.187×10^{-3}

Notes: Ranking of industries by aggregate inflation impact response when a sectoral shock leads to an increase in 1% in the shocked sector's inflation on impact. Average duration of the sectoral shock is 6 months.

Table 2: Ranking of industries by half-life of aggregate inflation repsonse after a sectoral shock

Industry	Agg. Inflation Half Life
Packaging machinery manufacturing	1.140×10^{1}
Miscellaneous nonmetallic mineral products	1.090×10^{1}
Coating, engraving, heat treating and allied ac...	1.080×10^{1}
Industrial process furnace and oven manufacturing	1.070×10^{1}
All other forging, stamping, and sintering	1.070×10^{1}
Semiconductor machinery manufacturing	1.040×10^{1}
Printing ink manufacturing	1.040×10^{1}
Speed changer, industrial high-speed drive, and...	1.030×10^{1}
Machine shops	1.010×10^{1}
Insurance agencies, brokerages, and related act...	9.700
Turned product and screw, nut, and bolt manufac...	9.700
Fluid power process machinery	8.900
Other communications equipment manufacturing	8.900
Electricity and signal testing instruments manu...	8.900
Industrial and commercial fan and blower and ai...	8.800
Relay and industrial control manufacturing	8.800

Notes: Ranking of industries by half-life of aggregate inflation response when a sectoral shock that leads to an increase in 1% in the shocked sector's inflation on impact. Average duration of the sectoral shock is 6 months.

Table 3: Comparison of eigenvalues of the calibrated economy with eigenvalues of the disconnected economy associated with specific industries

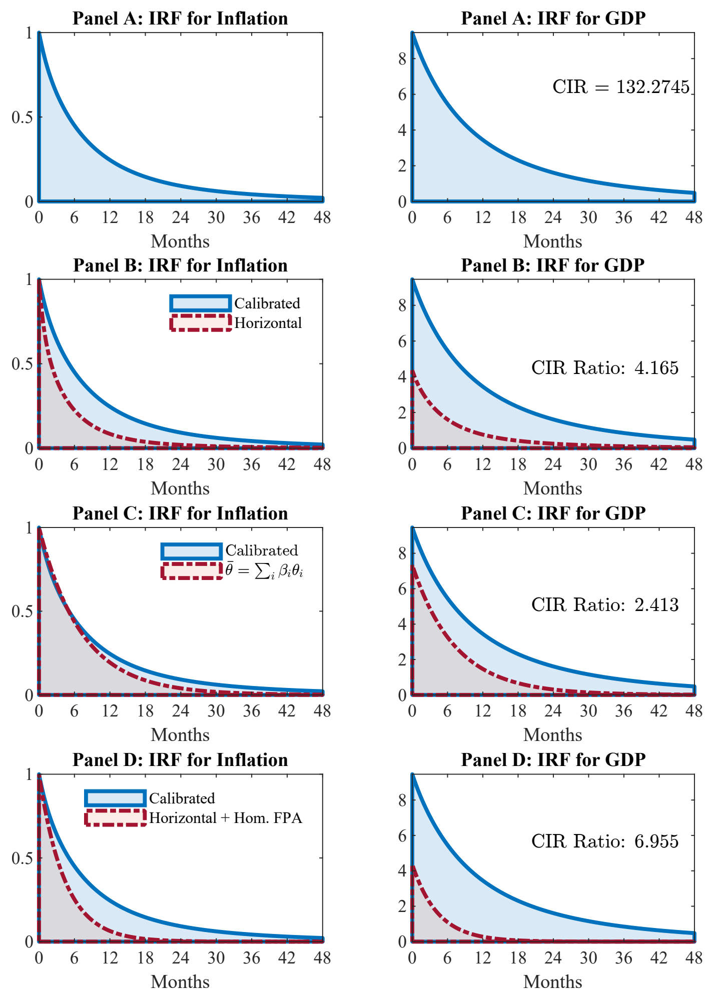
Industry	θ_i	$\sqrt{\theta_i(\rho+\theta_i)(1-a_{ii})+\rho^2/4}$ $-\rho/2$	Eigenvalue $\sqrt{\Gamma}-\frac{\rho}{2}\mathbf{I}$
Insurance agencies, brokerages, and related act...	3.559×10^{-2}	2.240×10^{-2}	2.214×10^{-2}
Coating, engraving, heat treating and allied ac...	2.780×10^{-2}	2.743×10^{-2}	2.731×10^{-2}
Warehousing and storage	3.241×10^{-2}	3.062×10^{-2}	3.052×10^{-2}
Semiconductor machinery manufacturing	3.400×10^{-2}	3.283×10^{-2}	3.283×10^{-2}
Flavoring syrup and concentrate manufacturing	3.890×10^{-2}	3.845×10^{-2}	3.840×10^{-2}
Packaging machinery manufacturing	4.067×10^{-2}	3.932×10^{-2}	3.931×10^{-2}
Showcase, partition, shelving, and locker manuf...	3.977×10^{-2}	3.933×10^{-2}	3.932×10^{-2}
Machine shops	4.432×10^{-2}	4.349×10^{-2}	4.276×10^{-2}
Watch, clock, and other measuring and controlli...	4.393×10^{-2}	4.368×10^{-2}	4.359×10^{-2}
Other communications equipment manufacturing	4.415×10^{-2}	4.394×10^{-2}	4.391×10^{-2}
Turned product and screw, nut, and bolt manufac...	4.499×10^{-2}	4.421×10^{-2}	4.431×10^{-2}
Electricity and signal testing instruments manu...	4.808×10^{-2}	4.457×10^{-2}	4.457×10^{-2}
Broadcast and wireless communications equipment	5.367×10^{-2}	4.512×10^{-2}	4.510×10^{-2}
Fluid power process machinery	4.716×10^{-2}	4.584×10^{-2}	4.580×10^{-2}
Optical instrument and lens manufacturing	4.820×10^{-2}	4.612×10^{-2}	4.601×10^{-2}
Other aircraft parts and auxiliary equipment ma...	5.171×10^{-2}	4.630×10^{-2}	4.610×10^{-2}
All other miscellaneous manufacturing	4.751×10^{-2}	4.632×10^{-2}	4.627×10^{-2}
Miscellaneous nonmetallic mineral products	4.912×10^{-2}	4.633×10^{-2}	4.632×10^{-2}
Cutlery and handtool manufacturing	4.778×10^{-2}	4.775×10^{-2}	4.770×10^{-2}
Analytical laboratory instrument manufacturing	4.835×10^{-2}	4.809×10^{-2}	4.811×10^{-2}

Notes: The actual eigenvalues of the calibrated economy are compared with eigenvalues of the counterfactual disconnected economy. In the disconnected economy, the eigenvalues are associated with specific industries, which are given in the first column.

APPENDIX
(FOR ONLINE PUBLICATION)

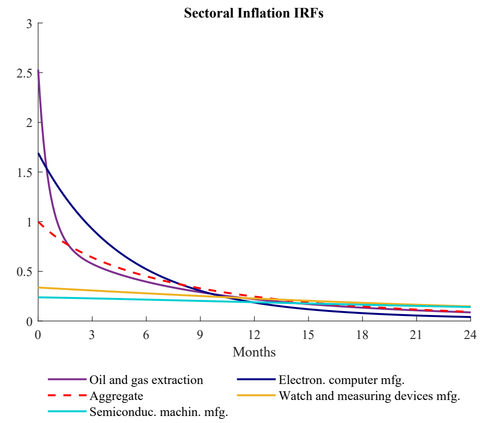
A Additional Figures and Tables

Figure 4: Impulse response functions to a monetary policy shock



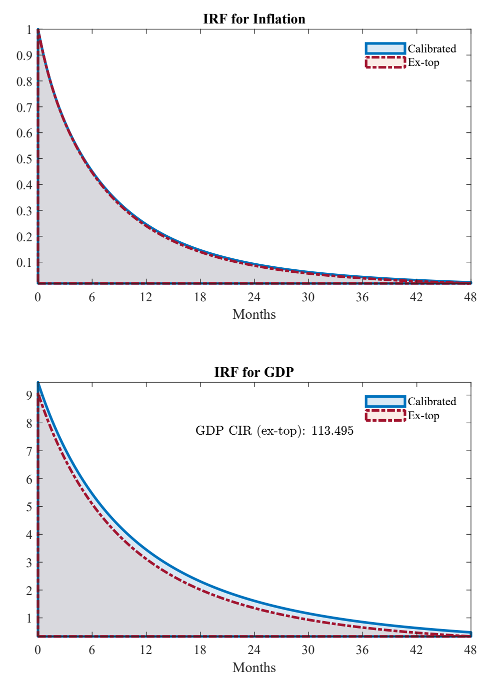
Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. The different panels show the results from the baseline calibrated economy (Panel A) as well as various counterfactual economies (Panels B, C, and D). CIR denotes the cumulative impulse response. CIR Ratio denotes the ratio of CIR of the baseline economy to the counterfactual economy. The calibration of the model is at a monthly frequency.

Figure 5: Sectoral inflation response to a monetary policy shock



Notes: This figure plots the impulse response functions for aggregate inflation and sectoral inflation to a monetary shock that generates a one percentage increase in aggregate inflation on impact. The aggregate inflation response is shown in dashed lines. The calibration of the model is at a monthly frequency.

Figure 6: Impulse response functions to a monetary policy shock in two economies



Notes: This figure plots the inflation and GDP responses after a monetary policy shock that generates a one percent increase in inflation on impact in the baseline economy and in a counterfactual economy where the top-3 sectors by lowest eigenvalues (in the disconnected economy) are excluded. CIR denotes the cumulative impulse response. The calibration of the model is at a monthly frequency.

Table 4: Ranking of industries by inflation impact after a monetary policy shock

Industry	Inflation Impact Resp.
Alumina refining and primary aluminum production	3.681
Other crop farming	2.593
Monetary authorities and depository credit inte...	2.533
Dairy cattle and milk production	2.025
Animal production, except cattle and poultry an...	1.959
Wholesale electronic markets and agents and bro...	1.736
Oil and gas extraction	1.530
Automobile manufacturing	1.464
Natural gas distribution	1.287
Copper, nickel, lead, and zinc mining	1.241
Fishing, hunting and trapping	1.210
Rail transportation	1.054
Nonferrous Metal (except Aluminum) Smelting and...	9.940×10^{-1}
Professional and commercial equipment and supplies	9.670×10^{-1}
Machinery, equipment, and supplies	8.513×10^{-1}
Poultry processing	8.361×10^{-1}
Electric lamp bulb and part manufacturing	8.241×10^{-1}
Poultry and egg production	8.057×10^{-1}
Fluid milk and butter manufacturing	8.023×10^{-1}
Petrochemical manufacturing	7.985×10^{-1}

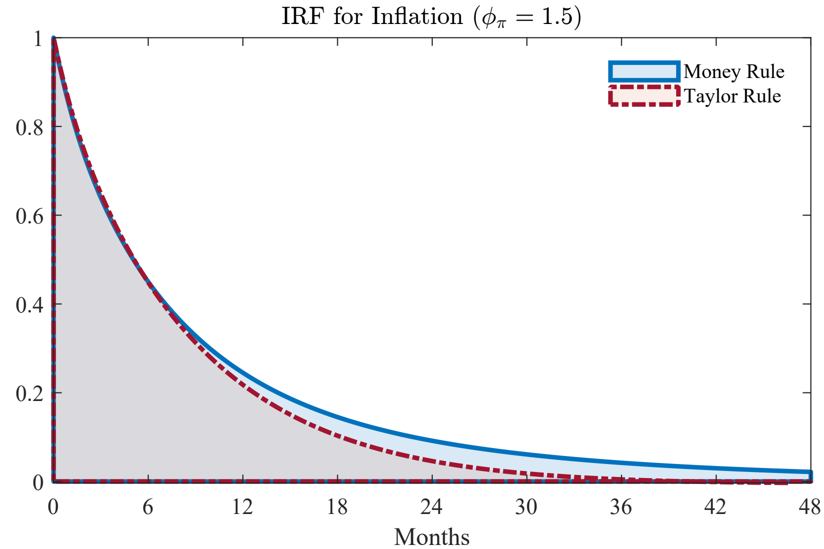
Notes: Ranking of industries by their inflation response on impact after a monetary policy shock that generates a one percent increase in aggregate inflation on impact in the baseline economy. The inflation impact response is measured as the difference between the sectoral inflation IRF and the aggregate inflation IRF on impact.

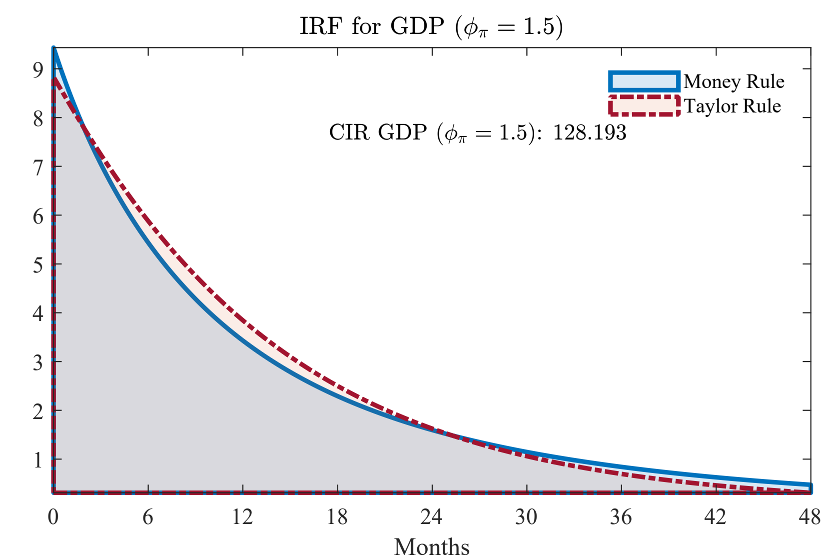
Table 5: Ranking of industries by inflation half-life after a monetary policy shock

Industry	Inflation Half-Life
Insurance agencies, brokerages, and related act...	3.420×10^{1}
Coating, engraving, heat treating and allied ac...	3.360×10^{1}
Semiconductor machinery manufacturing	3.020×10^{1}
Warehousing and storage	2.930×10^{1}
Packaging machinery manufacturing	2.550×10^{1}
Flavoring syrup and concentrate manufacturing	2.540×10^{1}
Showcase, partition, shelving, and locker manuf...	2.450×10^{1}
Turned product and screw, nut, and bolt manufac...	2.400×10^{1}
Toilet preparation manufacturing	2.400×10^{1}
Breakfast cereal manufacturing	2.330×10^{1}
Other engine equipment manufacturing	2.280×10^{1}
Other industrial machinery manufacturing	2.270×10^{1}
Miscellaneous nonmetallic mineral products	2.250×10^{1}
Fluid power process machinery	2.220×10^{1}
All other miscellaneous manufacturing	2.180×10^{1}
Cut stone and stone product manufacturing	2.180×10^{1}
Electricity and signal testing instruments manu...	2.140×10^{1}
Other aircraft parts and auxiliary equipment ma...	2.140×10^{1}
Metal crown, closure, and other metal stamping ...	2.120×10^{1}
Machine shops	2.110×10^{1}

Notes: Ranking of industries by their half-lives after a monetary policy shock that generates a one percent increase in aggregate inflation on impact in the baseline economy.

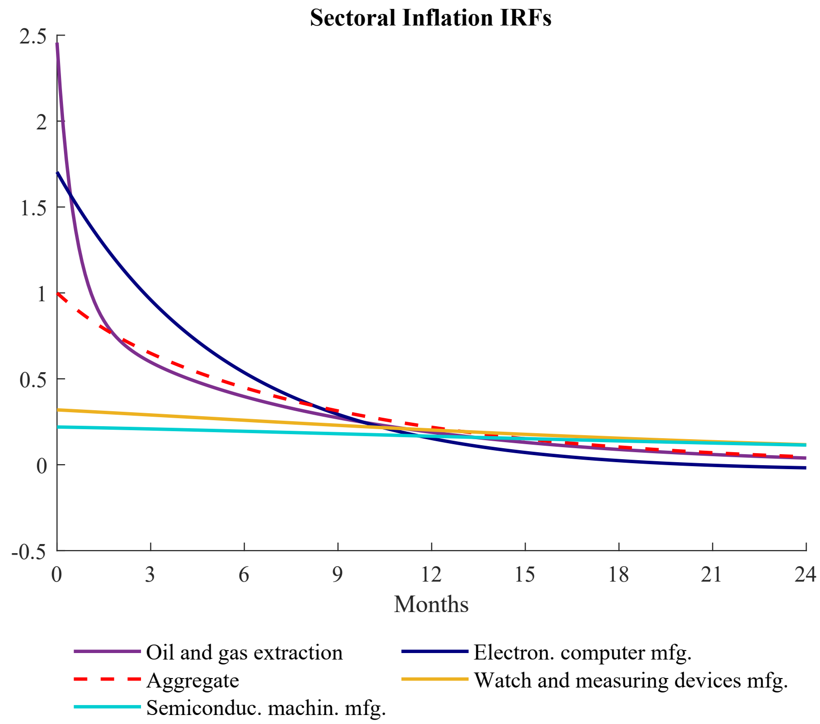
Figure 7: Impulse response functions to a monetary policy shock in two economies





Notes: The figure compares the impulse responses for inflation and GDP to a monetary shock in the nominal GDP rule economy and the Taylor rule economy. The initial shock size and its persistence in the Taylor rule economy are calibrated to match: 1) aggregate inflation response as one percentage on impact; 2) half-life of aggregate inflation the same as in the nominal GDP rule economy. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration fixes the feedback parameter on the Taylor rule to $\phi_\pi = 1.5$.

Figure 8: Inflation response to a monetary policy shock under a Taylor rule



Notes: This figure plots the impulse response functions for aggregate inflation and sectoral inflation to a monetary shock that generates a one percentage increase in aggregate inflation on impact. The calibration of the model is at a monthly frequency. The aggregate inflation response is shown in dashed lines. The calibration fixes the feedback parameter on the Taylor rule to $\phi_\pi = 1.5$.

B Proofs

B.1. Proof of Proposition 1

Necessity: Differentiating Equation (18) with respect to time and substituting Equation (17) we arrive at

$$\begin{aligned}\dot{\boldsymbol{\pi}}_t = \ddot{\mathbf{p}}_t = \boldsymbol{\Theta}(\boldsymbol{\pi}_t^{\#} - \boldsymbol{\pi}_t) &= \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^{\#} - \mathbf{p}_t^{*}) - \boldsymbol{\Theta}\boldsymbol{\pi}_t \\ &= \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t - \mathbf{p}_t^{*}) + \underbrace{\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^{\#} - \mathbf{p}_t) - \boldsymbol{\Theta}\boldsymbol{\pi}_t}_{=\rho\boldsymbol{\pi}_t \text{ by Equation (18)}}\end{aligned} \tag{53}$$

Now using the definition of $\mathbf{p}_t^{*}$ from Equation (9) observe that:

$$\mathbf{p}_t - \mathbf{p}_t^{*} = \mathbf{p}_t - \boldsymbol{\omega}_t + \boldsymbol{z}_t - m_t\boldsymbol{\alpha} + \mathbf{A}\mathbf{p}_t = -(\mathbf{I} - \mathbf{A})(\underbrace{m_t\mathbf{1} + \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)}_{=\mathbf{p}_t^{f} \text{ by Equation (14)}} - \mathbf{p}_t) \tag{54}$$

Combining Equations (53) and (54) gives us the desired result:

$$\dot{\boldsymbol{\pi}}_t = +\rho\boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{p}_t - \mathbf{p}_t^{f}) \tag{55}$$

Sufficiency: Since the paths for $(m_t, \boldsymbol{\omega}_t, \boldsymbol{z}_t)_{t\geq 0}$ are exogenously given in the baseline model, it follows that $\mathbf{p}_t^{f}$ is also exogenously given. Therefore, Equation (55) is an n-dimensional second-order linear differential equation in the vector of sectoral prices $\mathbf{p}_t$, with an *exogenous* force term $\mathbf{p}_t^{f}$, and boundary conditions $\mathbf{p}_0 = \mathbf{p}_{0^-}$ as well as non-explosive prices (saddle-path stability) as $t \to \infty$. Thus, the solution to this differential equation is unique (conditional on existence given the exogenous time paths of $(m_t, \boldsymbol{\omega}_t, \boldsymbol{z}_t)_{t\geq 0}$) and thus characterizes the dynamics of non-explosive sectoral prices. ■

B.2. Proof of Lemma 1

In this proof, we utilize several properties of non-singular M-matrices, which are a subset of Z-matrices. First, a Z-matrix is a matrix with real entries all of whose off-diagonal entries are either zero or negative(Berman and Plemmons, 1994, p. 132). Second, Theorem 2.3 in Berman and Plemmons (1994) provides a number of equivalent statements for when a Z-matrix is a non-singular M-matrix. In particular, it establishes that "a Z-matrix $\mathbf{C}$ is a non-singular M-matrix" if "$\mathbf{C}$ is positive stable; that is, the real part of each eigenvalue of $\mathbf{C}$ is positive" (Condition G_{20}), or "it is inverse positive; i.e., $\mathbf{C}^{-1}$ exists and $\mathbf{C} \geq 0$" (Condition N_{38}). Third, Theorem 5 in Alefeld and Schneider (1982) states that every non-singular M-matrix has exactly one square root that is also an M-matrix.

We now show that the matrix $\boldsymbol{\Gamma} - \frac{\rho^2}{2}\mathbf{I} = \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})$ is a non-singular M-matrices. To see this, first, note that $\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})$ is a Z-matrix as its ij'th entry for $i \neq j$ is

$-\theta_i(\theta_i+\rho)a_{ij} \le 0$. Second, note that $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ is inverse positive. In particular, since $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})$ is invertible because $\theta_i > 0$, we have:

$$(\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A}))^{-1} = (\mathbf{I}-\mathbf{A})^{-1}(\rho\mathbf{I}+\mathbf{\Theta})^{-1}\mathbf{\Theta}^{-1} = \sum_{k=0}^{\infty}\mathbf{A}^k(\rho\mathbf{I}+\mathbf{\Theta})^{-1}\mathbf{\Theta}^{-1} \tag{56}$$

where the last equality follows from the Neumann series representation of $(\mathbf{I}-\mathbf{A})^{-1}$, which is convergent because the spectral radius of $\mathbf{A}$ is strictly less than one (Carvalho and Tahbaz-Salehi, 2019, p. 638).[46] Since $\mathbf{A} \ge 0$ and $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta}) \ge 0$, it follows that $(\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A}))^{-1}$ is the sum of positive matrices and is itself positive. Thus, $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ is both a Z-matrix and inverse positive, which by condition N_{38} of Theorem 2.3 in Berman and Plemmons (1994) implies that it is a non-singular M-matrix. Condition G_{20} then implies that all the eigenvalues of $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ have positive real parts.

Now, consider the matrix $\mathbf{\Gamma} = \mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A}) + \frac{\rho^2}{4}\mathbf{I}$. First, since $\mathbf{\Gamma}$ and $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ share the same off-diagonal entries, it follows that $\mathbf{\Gamma}$ is also a Z-matrix. Second, suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})$ with eigenvector $\mathbf{v}$. It follows that $\lambda + \frac{\rho^2}{4}$ is an eigenvector of $\mathbf{\Gamma}$ with eigenvector $\mathbf{v}$:

$$\mathbf{\Theta}(\rho+\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})\mathbf{v} = \lambda\mathbf{v} \iff \mathbf{\Gamma}\mathbf{v} = (\mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A}) + \tfrac{\rho^2}{4}\mathbf{I})\mathbf{v} = (\lambda + \tfrac{\rho^2}{4})\mathbf{v} \tag{57}$$

So $\mathbf{\Gamma}$ is a Z-matrix the real part of whose eigenvalues are at least as large as $\rho^2/4 > 0$, and thus are positive. Then, by condition G_{20} of Theorem 2.3 in Berman and Plemmons (1994), $\mathbf{\Gamma}$ is a non-singular M-matrix. Since $\mathbf{\Gamma}$ is a non-singular M-matrix, it satisfies the assumptions of Theorem 5 in Alefeld and Schneider (1982) which states that $\mathbf{\Gamma}$ has exactly one square root matrix that is also an M-matrix. Let us denote this square root by $\sqrt{\mathbf{\Gamma}}$. Since the real parts of all the eigenvalues of a M-matrix are non-negative, $\sqrt{\mathbf{\Gamma}}$ is also the principal square root of $\mathbf{\Gamma}$.

Finally, to see that last part of the Lemma, suppose $\zeta_r + i\zeta_i \in \mathbb{C}$ is an eigenvalue for $\sqrt{\mathbf{\Gamma}}$ associated with an eigenvector $\mathbf{v}$. Since $\sqrt{\mathbf{\Gamma}}$ is a non-singular M-matrix, we know that $\zeta_r \ge 0$. Also,

$$\sqrt{\mathbf{\Gamma}}\mathbf{v} = \zeta\mathbf{v} \implies (\sqrt{\mathbf{\Gamma}} \pm \tfrac{\rho}{2}\mathbf{I})\mathbf{v} = (\zeta \pm \tfrac{\rho}{2}\mathbf{I})\mathbf{v} \quad \text{and} \quad \mathbf{\Gamma}\mathbf{v} = \zeta^2\mathbf{v} \tag{58}$$

So, we make the observation that $\zeta + \frac{\rho}{2}$ and $\zeta - \frac{\rho}{2}$ are eigenvalues of $\sqrt{\mathbf{\Gamma}} + \frac{\rho}{2}\mathbf{I}$ and $\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}$, respectively, and that $\zeta^2 = \zeta_r^2 - \zeta_i^2 + 2i\zeta_r\zeta_i$ is an eigenvalue for $\mathbf{\Gamma}$. But as we showed above,

[46] This follows from the fact that all sectors have strictly positive labor share so that the row sums of the matrix $\mathbf{A}$ are strictly less than 1.

real parts of eigenvalues of $\mathbf{\Gamma}$ are at least as large as $\rho^2/4$, so

$$\zeta_r^2 - \zeta_i^2 \geq \frac{\rho^2}{4} \implies \zeta_r^2 \geq \frac{\rho^2}{4} \implies \zeta_r \geq \frac{\rho}{2} \tag{59}$$

where the last implication follows from the fact that ζ_r is the positive square root of ζ_r^2. Moreover, we can show that $\zeta_r > \frac{\rho}{2}$ (i.e., the inequality is strict). To see this, note that if $\zeta_i = 0$ then $\zeta^2 = \zeta_r^2 \in \mathbb{R}$ is an eigenvalue of $\mathbf{\Gamma}$ which is a non-singular matrix and cannot have a zero eigenvalue, so it has to be that $\zeta_r^2 > 0, \zeta_r \geq 0 \implies \zeta_r > 0$; alternatively if $\zeta_i \neq 0$ then we have $\zeta_r \geq 0, \zeta_r^2 - \zeta_i^2 \geq \frac{\rho^2}{4} \implies \zeta_r^2 > \frac{\rho^2}{4} \implies \zeta_r > \frac{\rho}{2}$. Therefore, we observe that

$$\mathrm{Re}(\zeta - \frac{\rho}{2}) = \zeta_r - \frac{\rho}{2} > 0, \quad \text{and} \quad \mathrm{Re}(\zeta + \frac{\rho}{2}) = \zeta_r + \frac{\rho}{2} > \rho > 0 \tag{60}$$

meaning that all eigenvalues of $\sqrt{\mathbf{\Gamma}} + \frac{\rho}{2}\mathbf{I}$ and $\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}$ have positive real parts. Combined with the fact that these matrices are both Z-matrices (because $\sqrt{\mathbf{\Gamma}}$ is an M-matrix and shares the same off-diagonal entries with both these matrices), we conclude from condition G_{20} of Theorem 2.3 in Berman and Plemmons (1994) that $\sqrt{\mathbf{\Gamma}} + \frac{\rho}{2}\mathbf{I}$ and $\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}$ are non-singular M-matrices or equivalently are positive stable. ■

B.3. Proof of Proposition 2

Given the definition of the matrix $\mathbf{\Gamma} = \mathbf{\Theta}(\rho\mathbf{I} + \mathbf{\Theta})(\mathbf{I} - \mathbf{A}) + \frac{\rho^2}{4}\mathbf{I}$ and the fact that $\boldsymbol{\pi}_t = \dot{\mathbf{p}}_t$, the differential equation in Equation (19) is

$$\ddot{\mathbf{p}}_t = \rho\dot{\mathbf{p}}_t + (\mathbf{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{p}_t - \mathbf{p}_t^f) \tag{61}$$

Since $\mathbf{p}_t^f$ is piece-wise continuous and bounded, it has a Laplace transform for any $s \geq 0$. Let $\mathbf{P}^f(s) = \mathscr{L}_s(\mathbf{p}_t^f) \equiv \int_0^\infty e^{-st}\mathbf{p}_t^f \mathrm{d}t$ denote the Laplace transform of $\mathbf{p}_t^f$. Similarly, let $\mathbf{P}(s) = \mathscr{L}_s(\mathbf{p}_t)$ denote the Laplace transform of $\mathbf{p}_t$. Then, applying the Laplace transform to the differential equation above, we have:

$$\mathbf{P}(s) = ((s - \tfrac{\rho}{2})^2\mathbf{I} - \mathbf{\Gamma})^{-1}(\boldsymbol{\pi}_0 + (s - \rho)\mathbf{p}_0) - ((s - \tfrac{\rho}{2})^2\mathbf{I} - \mathbf{\Gamma})^{-1}(\mathbf{\Gamma} - \tfrac{\rho^2}{4}\mathbf{I})\mathbf{P}^f(s) \tag{62}$$

where $\mathbf{p}_0$ and $\boldsymbol{\pi}_0$ are initial values for the sectoral price and inflation rates at $t = 0$ that are determined by the two boundary conditions of the system, which we use and discuss later in the proof. Now, let $\sqrt{\mathbf{\Gamma}}$ denote the principal square root of $\mathbf{\Gamma}$ as in Lemma 1. Then, we can factor

$$((s - \tfrac{\rho}{2})^2\mathbf{I} - \mathbf{\Gamma})^{-1} = (s\mathbf{I} - (\sqrt{\mathbf{\Gamma}} + \tfrac{\rho}{2}\mathbf{I}))^{-1}(s\mathbf{I} + (\sqrt{\mathbf{\Gamma}} - \tfrac{\rho}{2})\mathbf{I})^{-1} \tag{63}$$

and write the above Laplace transform as:

$$\mathbf{P}(s) = (s\mathbf{I} + (\sqrt{\mathbf{\Gamma}} - \tfrac{\rho}{2}\mathbf{I}))^{-1}\mathbf{c}_0 + (s\mathbf{I} - (\sqrt{\mathbf{\Gamma}} + \tfrac{\rho}{2})\mathbf{I})^{-1}\mathbf{c}_1$$

$$+\frac{1}{2}\sqrt{\mathbf{\Gamma}}^{-1}(\mathbf{\Gamma}-\tfrac{\rho^2}{4}\mathbf{I})\left[(s\mathbf{I}+(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho}{2}\mathbf{I}))^{-1}-(s\mathbf{I}-(\sqrt{\mathbf{\Gamma}}+\tfrac{\rho}{2})\mathbf{I})^{-1}\right]\mathbf{P}^f(s) \tag{64}$$

where $\mathbf{c}_0$ and $\mathbf{c}_1$ are vectors in $\mathbf{R}^n$ and are appropriate linear transformations of $\mathbf{p}_0$ and $\boldsymbol{\pi}_0$. Applying the inverse Laplace transform to the above equation, and noting that the for any matrix $\mathbf{X}$, $\mathscr{L}_t^{-1}[(s\mathbf{I}+\mathbf{X})^{-1}]=e^{-\mathbf{X}t}$, we have:

$$\begin{aligned}\mathbf{p}_t &= e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{c}_0+e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}\mathbf{c}_1\\ &+\frac{1}{2}(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\mathscr{L}_t^{-1}\left[\left((s\mathbf{I}+(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho}{2}\mathbf{I}))^{-1}-(s\mathbf{I}-(\sqrt{\mathbf{\Gamma}}+\tfrac{\rho}{2})\mathbf{I})^{-1}\right)\mathbf{P}^f(s)\right]\end{aligned} \tag{65}$$

Since the inverse Laplace transform of a product is the convolution of inverse Laplace of individual functions, we can write the inverse Laplace transform in the last term as:

$$\begin{aligned}&\mathscr{L}_t^{-1}\left[\left((s\mathbf{I}+(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho}{2}\mathbf{I}))^{-1}-(s\mathbf{I}-(\sqrt{\mathbf{\Gamma}}+\tfrac{\rho}{2})\mathbf{I})^{-1}\right)\mathbf{P}^f(s)\right]\\ &=\int_0^t\mathscr{L}_{t-h}^{-1}\left[(s\mathbf{I}+(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho}{2}\mathbf{I}))^{-1}-(s\mathbf{I}-(\sqrt{\mathbf{\Gamma}}+\tfrac{\rho}{2})\mathbf{I})^{-1}\right]\mathbf{p}_h^f\mathrm{d}h\\ &\int_0^t\left(e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})(t-h)}-e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})(t-h)}\right)\mathbf{p}_h^f\mathrm{d}h\end{aligned} \tag{66}$$

Combining Equations (65) and (66), we arrive at

$$\begin{aligned}\mathbf{p}_t &= e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\left[\mathbf{c}_0+\frac{1}{2}(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_0^t e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f\mathrm{d}h\right]\\ &+e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}\left[\mathbf{c}_1-\frac{1}{2}(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_0^t e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f\mathrm{d}h\right]\end{aligned} \tag{67}$$

Now, in terms of boundary conditions $\mathbf{p}_t$ satisfies the following two: (1) it is continuous at $t=0$, since the probability of price change opportunities arriving at a short interval around any point is arbitrarily small—i.e., $\mathbf{p}_0=\mathbf{p}_{0^-}$—because no firm changes their price exactly at $t=0$ as it is a measure zero event, (2) we are looking for the solution in which prices are non-explosive; in fact bounded because $\mathbf{p}_t^f$ is bounded.

From the first boundary condition, we get:

$$\mathbf{c}_0+\mathbf{c}_1=\mathbf{p}_{0^-} \tag{68}$$

Now, recall from Lemma 1 that $\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I}$ and $\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I}$ are both positive stable. Thus, $e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}$ explodes as time goes to infinity, while $e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}$ goes to zero with $t\to\infty$ (because $-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})$ is negative stable). Therefore, for the solution to be stable, the term multiplying $e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}$ has to be zero as $t\to\infty$ and we have:

$$\lim_{t\to\infty}\left[\frac{1}{2}(\sqrt{\mathbf{\Gamma}}-\frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_0^t e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f\mathrm{d}h\right]=\mathbf{c}_1 \tag{69}$$

which implies that

$$\mathbf{c}_1 = \frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h$$
$$\implies \mathbf{c}_0 = \mathbf{p}_{0^-} - \mathbf{c}_1 = \mathbf{p}_{0^-} - \frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h$$

Plugging these boundary conditions into the solution we have:

$$\mathbf{p}_t = e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\left[\mathbf{p}_{0^-} + \frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left(\int_0^t e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h - \int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h\right)\right]$$
$$+ e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}\left[\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_t^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h\right] \tag{70}$$

Regrouping terms, we obtain the expression of interest:

$$\mathbf{p}_t = e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{p}_{0^-} + (\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\int_0^t \frac{e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h} - e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}}{2}\mathbf{p}_h^f \mathrm{d}h$$
$$+ (\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\frac{e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}}{2}\int_t^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathbf{p}_h^f \mathrm{d}h \tag{71}$$

■

B.4. Proof of Propositions 3 and 4

First, note that we can combine all the shocks in both propositions into a single path for $\mathbf{p}_t^f$ as:

$$\mathbf{p}_t^f = \mathbf{p}_{0^-} + \delta_m \mathbf{1} + \mathbf{\Psi}\sum_{i=1}^n \delta_z^i e^{-\phi_i t}\mathbf{e}_i \tag{72}$$

where $\mathbf{p}_{0^-}$ are the steady-state prices before shocks, δ_m is the monetary shock, $\mathbf{e}_i$ is the i'th standard basis vector, and δ_z^i is the TFP/wedge shock to sector i that decays at rate $\phi_i > 0$. We can then plug this path into Proposition 2 to derive the response of the economy to all of these shocks. In particular, since we have log-linearized the model, the response of the economy to this aggregated path is simply the sum of the impulse responses to individual shocks. So, using Equation (70), we observe that the IRF of $\mathbf{p}_t$ with respect to the monetary shock δ_m is

$$\frac{\partial}{\partial \delta_m}\mathbf{p}_t = e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\left[\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left(\int_0^t e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h}\frac{\partial}{\partial \delta_m}\mathbf{p}_h^f \mathrm{d}h - \int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\frac{\partial}{\partial \delta_m}\mathbf{p}_h^f \mathrm{d}h\right)\right]$$
$$+ e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}\left[\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \frac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_t^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\frac{\partial}{\partial \delta_m}\mathbf{p}_h^f \mathrm{d}h\right] \tag{73}$$

Noting that $\frac{\partial}{\partial \delta_m}\mathbf{p}^f_h = \mathbf{1}, \forall h \geq 0$, this becomes:

$$
\begin{aligned}
&\frac{\partial}{\partial \delta_m}\mathbf{p}_t \\
=&\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left[e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\left(\int_0^t e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h}\mathrm{d}h - \int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathrm{d}h\right) + e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}\int_t^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\mathrm{d}h\right]\mathbf{1} \\
=&\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left[\left((\sqrt{\mathbf{\Gamma}}-\tfrac{\rho}{2}\mathbf{I})^{-1}(\mathbf{I}-e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}) - (\sqrt{\mathbf{\Gamma}}+\tfrac{\rho}{2}\mathbf{I})^{-1}e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\right) + (\sqrt{\mathbf{\Gamma}}+\tfrac{\rho}{2}\mathbf{I})^{-1}\right]\mathbf{1} \\
=&\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})(\mathbf{\Gamma}-\tfrac{\rho^2}{4}\mathbf{I})^{-1}\left[2\sqrt{\mathbf{\Gamma}} - 2\sqrt{\mathbf{\Gamma}}e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\right]\mathbf{1} \\
=&(\mathbf{I}-e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\mathbf{1}
\end{aligned} \tag{74}
$$

which is the response of sectoral prices to a monetary shock in Proposition 3. Moreover, note that in our calculations above, we have used the fact that the matrices $\mathbf{\Gamma}$, $\sqrt{\mathbf{\Gamma}}$, and $\sqrt{\mathbf{\Gamma}} \pm \frac{\rho}{2}\mathbf{I}$ all commute with one another because they share the same basis (we will continue using this property in the rest of the proof as well). Now, to get the rest of the IRFs in that proposition, note that

$$\tfrac{\partial}{\partial \delta_m}\pi_t = \tfrac{\partial}{\partial \delta_m}\tfrac{\partial}{\partial t}\boldsymbol{\beta}^\intercal \mathbf{p}_t = \boldsymbol{\beta}^\intercal(\sqrt{\mathbf{\Gamma}}-\tfrac{\rho}{2}\mathbf{I})e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{1} \tag{75}$$

$$\tfrac{\partial}{\partial \delta_m}y_t = \tfrac{\partial}{\partial \delta_m}(m_t - \boldsymbol{\beta}^\intercal\mathbf{p}_t) = \boldsymbol{\beta}^\intercal e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{1} \tag{76}$$

$$\tfrac{\partial}{\partial \delta_m}\tilde{y}_t = \tfrac{\partial}{\partial \delta_m}(y_t - y_t^f) = \tfrac{\partial}{\partial \delta_m}\boldsymbol{\beta}^\intercal(\mathbf{p}_t^f - \mathbf{p}_t) = \boldsymbol{\beta}^\intercal e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\mathbf{1} \tag{77}$$

Similarly, using Equation (70), we observe that the IRF of $\mathbf{p}_t$ with respect to a TFP/wedge shock to sector i, δ_z^i, is

$$
\begin{aligned}
\frac{\partial}{\partial \delta_z^i}\mathbf{p}_t = & e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\left[\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left(\int_0^t e^{(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})h}\frac{\partial}{\partial \delta_z^i}\mathbf{p}_h^f\mathrm{d}h - \int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\frac{\partial}{\partial \delta_z^i}\mathbf{p}_h^f\mathrm{d}h\right)\right] \\
&+ e^{(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})t}\left[\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\int_t^\infty e^{-(\sqrt{\mathbf{\Gamma}}+\frac{\rho}{2}\mathbf{I})h}\frac{\partial}{\partial \delta_z^i}\mathbf{p}_h^f\mathrm{d}h\right]
\end{aligned} \tag{78}
$$

Noting that $\frac{\partial}{\partial \delta_z^i}\mathbf{p}_h^f = e^{-\phi_i h}\mathbf{\Psi}\mathbf{e}_i, \forall h \geq 0$, this becomes:

$$
\begin{aligned}
&\frac{\partial}{\partial \delta_z^i}\mathbf{p}_t \\
=&\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left[e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t}\int_0^t e^{(\sqrt{\mathbf{\Gamma}}-(\phi_i+\frac{\rho}{2})\mathbf{I})h}\mathrm{d}h + (e^{-\phi_i t}\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\int_0^\infty e^{-(\sqrt{\mathbf{\Gamma}}+(\phi_i+\frac{\rho}{2})\mathbf{I})h}\mathrm{d}h\right]\mathbf{\Psi}\mathbf{e}_i \\
=&\frac{1}{2}(\sqrt{\mathbf{\Gamma}} - \tfrac{\rho^2}{4}\sqrt{\mathbf{\Gamma}}^{-1})\left[(\sqrt{\mathbf{\Gamma}}-(\phi_i+\tfrac{\rho}{2})\mathbf{I})^{-1} + (\sqrt{\mathbf{\Gamma}}+(\phi_i+\tfrac{\rho}{2})\mathbf{I})^{-1}\right](e^{-\phi_i t}\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\mathbf{\Psi}\mathbf{e}_i \\
=&(\mathbf{\Gamma}-\tfrac{\rho^2}{4}\mathbf{I})(\mathbf{\Gamma}-(\phi_i+\tfrac{\rho}{2}\mathbf{I})^{-1})(e^{-\phi_i t}\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\mathbf{\Psi}\mathbf{e}_i \\
=&(\mathbf{I}-(\phi_i^2+\rho\phi_i)(\mathbf{\Gamma}-\tfrac{\rho^2}{4}\mathbf{I})^{-1})^{-1}(e^{-\phi_i t}\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\mathbf{\Psi}\mathbf{e}_i = \mathbf{X}_i(e^{-\phi_i t}\mathbf{I} - e^{-(\sqrt{\mathbf{\Gamma}}-\frac{\rho}{2}\mathbf{I})t})\mathbf{\Psi}\mathbf{e}_i
\end{aligned} \tag{79}
$$

which is the response of sectoral prices to a TFP/wedge shock in Proposition 4 with

$$\mathbf{X}_i = \left(\boldsymbol{\Gamma} - (\tfrac{\rho}{2} + \phi_i)^2 \mathbf{I}\right)^{-1} (\boldsymbol{\Gamma} - \tfrac{\rho^2}{4}\mathbf{I}) \tag{80}$$

Note that in deriving this expression we have assumed that the matrices $\sqrt{\boldsymbol{\Gamma}} - (\phi_i + \frac{\rho}{2})\mathbf{I}$ and $\sqrt{\boldsymbol{\Gamma}} + (\phi_i + \frac{\rho}{2})\mathbf{I}$ are invertible. To see why this is true, recall from the proof of Lemma 1 that matrices $\sqrt{\boldsymbol{\Gamma}} \pm \frac{\rho}{2}\mathbf{I}$ are positive stable, so their eigenvalues have positive real parts. Now, note that for any $\lambda \in \operatorname{eig}(\sqrt{\boldsymbol{\Gamma}} + (\phi_i + \frac{\rho}{2}))$, we have:

$$\operatorname{Re}(\lambda) = \operatorname{Re}(\operatorname{eig}(\sqrt{\boldsymbol{\Gamma}})) + \phi_i + \frac{\rho}{2} > \rho + \phi_i > 0 \tag{81}$$

which guarantees that $\sqrt{\boldsymbol{\Gamma}} + (\phi_i + \frac{\rho}{2})\mathbf{I}$ is invertible. Similarly, for any $\lambda \in \operatorname{eig}(\sqrt{\boldsymbol{\Gamma}} - (\phi_i + \frac{\rho}{2}))$, we have:

$$\operatorname{Re}(\lambda) = \operatorname{Re}(\operatorname{eig}(\sqrt{\boldsymbol{\Gamma}} - \frac{\rho}{2}\mathbf{I})) - \phi_i \neq 0 \tag{82}$$

where $\neq$ follows from the assumption of the proposition that $\phi_i \notin \operatorname{eig}(\sqrt{\boldsymbol{\Gamma}} - \frac{\rho}{2}\mathbf{I})$. So $\sqrt{\boldsymbol{\Gamma}} - (\phi_i + \frac{\rho}{2})\mathbf{I}$ is also invertible as all of its eigenvalues have non-zero real parts.

Now, to get the rest of the IRFs in that proposition, note that

$$\frac{\partial}{\partial \delta_z^i}\pi_t = \frac{\partial}{\partial \delta_z^i}\frac{\partial}{\partial t}\boldsymbol{\beta}^\top \mathbf{p}_t = \boldsymbol{\beta}^\top \mathbf{X}_i ((\sqrt{\boldsymbol{\Gamma}} - \tfrac{\rho}{2}\mathbf{I}) e^{-(\sqrt{\boldsymbol{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} - \phi_i e^{-\phi_i t}\mathbf{I})\boldsymbol{\Psi}\mathbf{e}_i \tag{83}$$

$$\frac{\partial}{\partial \delta_z^i} y_t = \frac{\partial}{\partial \delta_z^i}(m_t - \boldsymbol{\beta}^\top \mathbf{p}_t) = \boldsymbol{\beta}^\top \mathbf{X}_i (e^{-(\sqrt{\boldsymbol{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} - e^{-\phi_i t}\mathbf{I})\boldsymbol{\Psi}\mathbf{e}_i \tag{84}$$

$$\frac{\partial}{\partial \delta_z^i}\tilde{y}_t = \frac{\partial}{\partial \delta_z^i}(y_t - y_t^f) = \boldsymbol{\beta}^\top (\mathbf{X}_i e^{-(\sqrt{\boldsymbol{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} + (\mathbf{I} - \mathbf{X}_i) e^{-\phi_i t})\boldsymbol{\Psi}\mathbf{e}_i \tag{85}$$

B.5. Proof of Lemma 2

Consider the matrix $\boldsymbol{\Gamma}(\varepsilon) = \boldsymbol{\Gamma}_D + \varepsilon\boldsymbol{\Gamma}_R$ as defined in the main text, where $\boldsymbol{\Gamma}_D \equiv \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A}_D) + \frac{\rho^2}{4}\mathbf{I}$ is the diagonal matrix corresponding to the duration-adjusted Leontief matrix of the disconnected economy and $\boldsymbol{\Gamma}_R \equiv \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{A}_D - \mathbf{A})$ where $\mathbf{A}_D \equiv \operatorname{diag}(\mathbf{A})$ is the diagonal matrix whose diagonal entries agree with the diagonal entries of $\mathbf{A}$.

We divide this proof into several consecutive but short steps:

Step 1. ($\boldsymbol{\Gamma}(\varepsilon)$ is Diagonalizable for Small ε) In this step, we show that there exists $\bar{\varepsilon} > 0$ such that for all ε with $|\varepsilon| < \bar{\varepsilon}$, $\boldsymbol{\Gamma}(\varepsilon)$ is diagonalizable and all the eigenvalues of $\boldsymbol{\Gamma}(\varepsilon)$, denoted by the set $\{\Delta_i(\varepsilon)\}_{i\in[n]}$, are *strictly positive*, *real*, and *distinct*.

To show these, we first show that $\boldsymbol{\Gamma}(\varepsilon)$ is diagonalizable for $|\varepsilon| < \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$. To see this, we apply the Gershgorin circle theorem, which states that every eigenvalue of a square complex matrix lies within at least one of the Gershgorin discs, each centered at a diagonal entry with the radius that is equal to the sum of absolute values of non-diagonal entries of its corresponding row (Lax, 2007, p. 323). To see this, define

$$R_i \equiv \sum_{j \neq i} |\mathbf{\Gamma}_{R,ij}|$$

as the i'th row sum of the entries in $\mathbf{\Gamma}_R$ and let $\bar{R} = \max_i\{R_i\}$. Note that if $\bar{R} = 0$ then $\mathbf{\Gamma}_R = 0 \implies \mathbf{\Gamma}(\varepsilon) = \mathbf{\Gamma}_D$ which is trivially diagonal(izable) for any $\varepsilon \in \mathbb{R}$. Suppose now that $\bar{R} > 0$ instead and define

$$\bar{\varepsilon} \equiv \frac{1}{2}\frac{\min\{|x_i - x_j| : i, j \in [n]\}}{\bar{R}}$$

where $\{x_i, x_j\} \subset \{0, \mathbf{\Gamma}_{D,ii}\}_{i\in[n]}$ (i.e., the numerator is the minimal distance between any two eigenvalues of $\mathbf{\Gamma}_D$ from one another or zero). Note that $\bar{\varepsilon} > 0$ because we have assumed all $\xi_i = \sqrt{\theta_i(\rho+\theta_i)(1-a_{ii}) + \frac{\rho^2}{4}} - \frac{\rho}{2}$ are distinct and positive, which means that $\mathbf{\Gamma}_{D,ii} = \theta_i(\rho+\theta_i)(1-a_{ii}) + \frac{\rho^2}{4}$'s are distinct from one another *and* distinct from zero. Now, let $|\varepsilon| < \bar{\varepsilon}$ and observe that all Gershgorin discs, $\{z \in \mathbb{C} : |z - \mathbf{\Gamma}_{D,ii}| \leq \sum_{j\neq i} |\mathbf{\Gamma}_i(\varepsilon)| = |\varepsilon| R_i\}$, are disjoint by construction of $\bar{\varepsilon}$. Gershgorin circle theorem then implies that each of these discs contain exactly one eigenvalue of $\mathbf{\Gamma}(\varepsilon)$ (see, e.g., Lax, 2007, p. 324). Thus, for $|\varepsilon| < \bar{\varepsilon}$, all eigenvalues of $\mathbf{\Gamma}(\varepsilon)$ are non-zero and distinct. The latter is a sufficient condition for $\mathbf{\Gamma}(\varepsilon)$ to be diagonalizable (see, e.g., Peterson, 2014, Theorem 2.5.10, p. 167). It also follows that all of these eigenvalues are real: Since $\mathbf{\Gamma}(\varepsilon)$ is a real matrix, all of its complex eigenvalues should come in conjugate pairs (see, e.g., Peterson, 2014, p. 124). Thus, if $\mathbf{\Gamma}(\varepsilon)$ has any complex eigenvalues, then there are at least two of them that share the same real part; so they both must fall within the same Gershgorin disc, which is a contradiction. Thus, letting $\Delta_i(\varepsilon)$ denote that the eigenvalue in the i'th Gershgorin disc, we know that when $|\varepsilon| < \bar{\varepsilon}$, $\Delta_i(\varepsilon), \forall i$ is a *real, strictly positive* number.

Step 2. (Existence of Principal Square Root) We have shown earlier in Lemma 1 that when $\mathbf{\Gamma}$ is an M-matrix, its principal square root exists and is unique. However, in this Lemma, $\mathbf{\Gamma}(\varepsilon)$ is not an M-matrix when $\varepsilon < 0$ and even though we are only interested in the case of $\varepsilon > 0$, we show the existence and uniqueness of the principal square root of $\mathbf{\Gamma}(\varepsilon)$ for $|\varepsilon| < \bar{\varepsilon}$ here to be able to differentiate it later at $\varepsilon = 0$.

Since $\mathbf{\Gamma}(\varepsilon)$ is diagonalizable for $|\varepsilon| < \bar{\varepsilon}$, there exists an invertible matrix $\mathbf{V}(\varepsilon)$ and a diagonal matrix $\mathbf{\Delta}(\varepsilon) = \mathrm{diag}(\Delta_1(\varepsilon), \ldots, \Delta_n(\varepsilon))$ such that

$$\mathbf{\Gamma}(\varepsilon) = \mathbf{V}(\varepsilon)\mathbf{\Delta}(\varepsilon)\mathbf{V}(\varepsilon)^{-1} \tag{86}$$

Since, by the previous step, each $\Delta_i(\varepsilon)$ is a real, strictly positive number, we can define $\mathbf{\Delta}(\varepsilon)^{1/2} = \mathrm{diag}(\sqrt{\Delta_1(\varepsilon)}, \ldots, \sqrt{\Delta_n(\varepsilon)})$ and note that the matrix $\sqrt{\mathbf{\Gamma}(\varepsilon)} \equiv \mathbf{V}(\varepsilon)\mathbf{\Delta}(\varepsilon)^{1/2}\mathbf{V}(\varepsilon)^{-1}$ is a principal square root for $\mathbf{\Gamma}(\varepsilon)$ because all of its eigenvalues have positive real parts and:

$$(\sqrt{\mathbf{\Gamma}(\varepsilon)})^2 = \mathbf{V}(\varepsilon)\mathbf{\Delta}(\varepsilon)^{1/2}\mathbf{V}(\varepsilon)^{-1}\mathbf{V}(\varepsilon)\mathbf{\Delta}(\varepsilon)^{1/2}\mathbf{V}(\varepsilon)^{-1} = \mathbf{V}(\varepsilon)\mathbf{\Delta}(\varepsilon)\mathbf{V}(\varepsilon)^{-1} = \mathbf{\Gamma}(\varepsilon) \tag{87}$$

Thus, a principal square root for $\mathbf{\Gamma}(\varepsilon)$ exists when $|\varepsilon| < \bar{\varepsilon}$. We next show that this matrix is unique.

Step 3. (Uniqueness of Principal Square Root) Let $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ denote a principal square root of $\mathbf{\Gamma}(\varepsilon)$ so that $(\sqrt{\mathbf{\Gamma}(\varepsilon)})^2 = \mathbf{\Gamma}(\varepsilon)$. We show that $(\Delta_i(\varepsilon), \mathbf{v}_i(\varepsilon))$ is an eigenvalue/eigenvector pair for $\mathbf{\Gamma}(\varepsilon)$ if and only if $(\sqrt{\Delta_i(\varepsilon)}, \mathbf{v}_i(\varepsilon))$ is an eigenvalue/eigenvector pair for $\sqrt{\mathbf{\Gamma}(\varepsilon)}$. Thus, with $\mathbf{V}(\varepsilon) \equiv [\mathbf{v}_1(\varepsilon), \ldots, \mathbf{v}_n(\varepsilon)]$, $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ has the unique representation of $\sqrt{\mathbf{\Gamma}(\varepsilon)} = \mathbf{V}(\varepsilon)\mathbf{\Delta}(\varepsilon)^{1/2}\mathbf{V}(\varepsilon)^{-1}$.

($\Longrightarrow$) Suppose $\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon) = \sqrt{\Delta_i(\varepsilon)}\mathbf{v}_i(\varepsilon)$. Multiply by $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ from left to get

$$\mathbf{\Gamma}(\varepsilon)\mathbf{v}_i(\varepsilon) = \sqrt{\Delta_i(\varepsilon)}\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon) = \Delta_i(\varepsilon)\mathbf{v}_i(\varepsilon)$$

so $(\Delta_i(\varepsilon), \mathbf{v}_i(\varepsilon))$ is an eigenvalue/eigenvector pair for $\mathbf{\Gamma}(\varepsilon)$.

($\Longleftarrow$) Suppose $\mathbf{\Gamma}(\varepsilon)\mathbf{v}_i(\varepsilon) = \Delta_i(\varepsilon)\mathbf{v}_i(\varepsilon)$. Multiply this by $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ from left to get

$$\mathbf{\Gamma}(\varepsilon) \times \left(\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon)\right) = \Delta_i(\varepsilon) \times \left(\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon)\right)$$

which implies that the vector $\mathbf{z}_i(\varepsilon) \equiv \sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon)$ is also an eigenvector for $\mathbf{\Gamma}(\varepsilon)$ associated with the eigenvalue $\Delta_i(\varepsilon)$. But since all $\Delta_i(\varepsilon)$'s are distinct, all of the eigenspaces of $\mathbf{\Gamma}(\varepsilon)$ are one-dimensional so it has to be that $\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon) \in \text{span}\{\mathbf{v}_i(\varepsilon)\}$; i.e., there exists a scalar $d_i(\varepsilon)$ such that

$$\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon) = d_i(\varepsilon)\mathbf{v}_i(\varepsilon)$$

Note that this immediately implies that $\mathbf{v}_i(\varepsilon)$ is also an eigenvector for $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ now associated with $d_i(\varepsilon)$ as its corresponding eigenvalue. Multiplying the equation above by $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ from left, we get

$$\mathbf{\Gamma}(\varepsilon)\mathbf{v}_i(\varepsilon) = d_i(\varepsilon)\sqrt{\mathbf{\Gamma}(\varepsilon)}\mathbf{v}_i(\varepsilon) = d_i(\varepsilon)^2\mathbf{v}_i(\varepsilon)$$

So $d_i(\varepsilon)^2$ is the eigenvalue of $\mathbf{\Gamma}(\varepsilon)$ associated with $\mathbf{v}_i(\varepsilon)$; i.e,

$$d_i(\varepsilon)^2 = \Delta_i(\varepsilon) \implies d_i(\varepsilon) = \sqrt{\Delta_i(\varepsilon)} \in \mathbb{R}_{++}$$

where we $d_i(\varepsilon)$ is the positive root $\sqrt{\Delta_i(\varepsilon)}$ because $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ is the principal square root of $\mathbf{\Gamma}(\varepsilon)$.

Step 4. (Perturbed Eigenvalues/Eigenvectors of $\mathbf{\Gamma}(\varepsilon)$) In this step we rely on Theorems 1 and 2 in Greenbaum, Li, and Overton (2020), which characterize perturbation results that can be applied to eigenvectors and eigenvalues of the matrix $\mathbf{\Gamma}(\varepsilon)$. Moreover, since Step 3 implies that both $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ and $\mathbf{\Gamma}(\varepsilon)$ have the same eigenvectors, it immediately follows that the perturbed eigenvectors of $\mathbf{\Gamma}(\varepsilon)$ are also perturbed eigenvectors of $\sqrt{\mathbf{\Gamma}(\varepsilon)}$. As for the eigenvalues of $\sqrt{\mathbf{\Gamma}(\varepsilon)}$, suppose $\Delta_i(\varepsilon) \in \operatorname{eig}(\mathbf{\Gamma}(\varepsilon))$ has the following Taylor expansion by Theorem 1 of Greenbaum, Li, and Overton (2020):

$$\Delta_i(\varepsilon) = \Delta_i(0) + \Delta_i'(0)\varepsilon + \mathcal{O}(\|\varepsilon\|^2)$$

Then $d_i(\varepsilon) = \sqrt{\Delta_i(\varepsilon)}$, which is an eigenvalue of $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ by the previous steps, has the following Taylor expansion:

$$d_i(\varepsilon) = \sqrt{\Delta_i(0)} + \frac{\Delta_i'(0)}{2\sqrt{\Delta_i(0)}}\varepsilon + \mathcal{O}(\|\varepsilon\|^2)$$

The latter follows from the fact that $d_i'(0) = \frac{\Delta_i'(0)}{2\sqrt{\Delta_i(0)}}$ and it is a well-defined approximation because $\Delta_i(0) = \theta_i(\rho+\theta_i)(1-a_{ii}) + \frac{\rho^2}{4} > 0, \forall i$ so that the denominator is non-zero.

Thus, in order to characterize the perturbed eigenvalues and eigenvectors of $\sqrt{\mathbf{\Gamma}(\varepsilon)}$ we simply need the perturbed eigenvalues and eigenvectors of $\mathbf{\Gamma}(\varepsilon)$. These are characterized by the formulas in Theorems 1 and 2 of Greenbaum, Li, and Overton (2020), but for completeness, we re-derive these perturbations below: Consider the pair $(\Delta_i(\varepsilon), \mathbf{v}_i(\varepsilon))$ such that

$$\mathbf{\Gamma}(\varepsilon)\mathbf{v}_i(\varepsilon) = \Delta_i(\varepsilon)\mathbf{v}_i(\varepsilon)$$

and differentiate with respect to ε and evaluate at $\varepsilon = 0$ to get

$$\mathbf{\Gamma}(0)\mathbf{v}_i'(0) + \mathbf{\Gamma}'(0)\mathbf{v}_i(0) = \Delta_i'(0)\mathbf{v}_i(0) + \Delta_i(0)\mathbf{v}_i'(0)$$

Note that $\mathbf{\Gamma}(0) = \mathbf{\Gamma}_D$ is diagonal so $\Delta_i(0) = \mathbf{\Gamma}_{D,ii}$ and $\mathbf{v}_i(0) = \mathbf{e}_i$ where $\mathbf{e}_i$ is the i'th standard basis vector. Also, note that $\mathbf{\Gamma}'(0) = \partial_\varepsilon[\mathbf{\Gamma}_D + \varepsilon\mathbf{\Gamma}_R]_{\varepsilon=0} = \mathbf{\Gamma}_R$. Substituting these into the equation above, we get

$$\mathbf{\Gamma}_D\mathbf{v}_i'(0) + \mathbf{\Gamma}_R\mathbf{e}_i = \Delta_i'(0)\mathbf{e}_i + \Gamma_{D,ii}\mathbf{v}_i'(0)$$

Multiplying this equation by $\mathbf{e}_j'$ from the left for $j \in \{1,\ldots,n\}$ we have:

$$\begin{aligned}
&j = i \implies \Delta_i'(0) = [\mathbf{\Gamma}_R]_{ii} = 0\\
&j \neq i \implies [\mathbf{v}_i'(0)]_j = \frac{[\mathbf{\Gamma}_R]_{ji}}{\mathbf{\Gamma}_{D,ii} - \mathbf{\Gamma}_{D,jj}} = \frac{-\theta_j(\rho+\theta_j)a_{ji}}{\theta_i(\rho+\theta_i)(1-a_{ii}) - \theta_j(\rho+\theta_j)(1-a_{jj})}
\end{aligned}$$

Now recall that for all i

$$\xi_i \equiv \sqrt{\theta_i(\rho+\theta_i)(1-a_{ii})+\frac{\rho^2}{4}}-\frac{\rho}{2}$$
$$\implies \xi_i^2+\rho\xi_i=\theta_i(\rho+\theta_i)(1-a_{ii})$$

so

$$[\mathbf{v}_i'(0)]_j=\frac{a_{ji}}{1-a_{jj}}\times\frac{\xi_j^2+\rho\xi_j}{(\xi_i+\xi_j+\rho)(\xi_j-\xi_i)}\forall j\neq i$$

Finally, $[\mathbf{v}_i'(0)]_i=\mathbf{e}_i^{\top}\mathbf{v}_i'(0)=0,\forall i\in[n]$ because $\mathbf{v}_i'(0)$ is orthogonal to $\mathbf{e}_i$ (the eigenvector associated with $\Delta_i(\varepsilon)$ at $\varepsilon=0$) for all i (see Theorem 2 in Greenbaum, Li, and Overton, 2020).

Step 5. (Perturbed Eigenvalues/Eigenvectors of $\sqrt{\boldsymbol{\Gamma}(\varepsilon)}$) Having derived the Taylor expansions for a pair of eigenvalues and eigenvectors of $\boldsymbol{\Gamma}(\varepsilon)$ following Greenbaum, Li, and Overton (2020) in Step 4, denoted by $(\Delta_i(\varepsilon),\mathbf{v}_i(\varepsilon))$, we can now use the result in Step 3 to retrieve the implied perturbations for corresponding eigenvalues and eigenvectors of $\sqrt{\boldsymbol{\Gamma}(\varepsilon)}$, denoted by $(d_i(\varepsilon)=\sqrt{\Delta_i(\varepsilon)},\mathbf{v}_i(\varepsilon))$:

$$\begin{aligned}d_i(\varepsilon)&=\underbrace{\sqrt{\Delta_i(0)}}_{=\sqrt{\boldsymbol{\Gamma}_{D,ii}}}+\underbrace{\frac{\Delta_i'(0)}{2\sqrt{\Delta_i(0)}}}_{=0}\varepsilon+\mathcal{O}(\|\varepsilon\|^2)\\&=\sqrt{\theta_i(\theta_i+\rho)(1-a_{ii})+\frac{\rho^2}{4}}+\mathcal{O}(\|\varepsilon\|^2)\\&=\xi_i+\frac{\rho}{2}+\mathcal{O}(\|\varepsilon\|^2)\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_i(\varepsilon)&=\mathbf{v}_i(0)+\mathbf{v}_i'(0)\varepsilon+\mathcal{O}(\|\varepsilon\|^2)\\&=\mathbf{e}_i+\varepsilon\times\sum_{j\neq i}\frac{a_{ji}}{1-a_{jj}}\times\frac{\xi_j^2+\rho\xi_j}{(\xi_i+\xi_j+\rho)(\xi_j-\xi_i)}\mathbf{e}_j+\mathcal{O}(\|\varepsilon\|^2)\end{aligned}$$

■

B.6. Proof of Proposition 5

For a given $\boldsymbol{\Gamma}(\varepsilon)$, recall form Proposition 3 that the IRF of sectoral inflation rates to a monetary shock is

$$\tfrac{\partial}{\partial\delta_m}\boldsymbol{\pi}_t=\left(\sqrt{\boldsymbol{\Gamma}(\varepsilon)}-\tfrac{\rho}{2}\mathbf{I}\right)e^{-(\sqrt{\boldsymbol{\Gamma}(\varepsilon)}-\frac{\rho}{2}\mathbf{I})t}\mathbf{1} \tag{88}$$

Using the eigendecomposition and approximation from Lemma 2, $\sqrt{\boldsymbol{\Gamma}(\varepsilon)} = \mathbf{V}(\varepsilon)\boldsymbol{\Delta}(\varepsilon)^{1/2}\mathbf{V}(\varepsilon)^{-1}$, and letting $\mathbf{D}(\varepsilon) \equiv \boldsymbol{\Delta}(\varepsilon)^{1/2} - \frac{\rho}{2}\mathbf{I}$, we have the Taylor expansion around $\varepsilon = 0$:

$$\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t = \left[\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t\right]_{\varepsilon=0} + \varepsilon \times \frac{\partial}{\partial \varepsilon}\left[\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t\right]_{\varepsilon=0} + \mathcal{O}(\|\varepsilon\|^2) \tag{89}$$

where

$$\left[\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t\right]_{\varepsilon=0} = \mathbf{D}(0)e^{-\mathbf{D}(0)t}\mathbf{1} \tag{90}$$

and

$$\frac{\partial}{\partial \varepsilon}\left[\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t\right]_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\left[\mathbf{V}(\varepsilon)\mathbf{D}(\varepsilon)e^{-\mathbf{D}(\varepsilon)t}\mathbf{V}(\varepsilon)^{-1}\mathbf{1}\right]_{\varepsilon=0} \tag{91}$$

$$= \mathbf{V}'(0)\mathbf{D}(0)e^{-\mathbf{D}(0)t}\mathbf{1} - \mathbf{D}(0)e^{-\mathbf{D}(0)t}\mathbf{V}'(0)\mathbf{1} \tag{92}$$

Now, note $\mathbf{D}(0) = \operatorname{diag}(\xi_1, \ldots, \xi_n)$ and that the IRF of inflation in sector i is the i'th element of this vector, so that:

$$\frac{\partial}{\partial \delta_m}\pi_{i,t} = \mathbf{e}_i^\top\left(\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t\right) = \xi_i e^{-\xi_i t} + \varepsilon\sum_{j\neq i}[\mathbf{V}'(0)]_{ij}(\xi_j e^{-\xi_j t} - \xi_i e^{-\xi_i t}) + \mathcal{O}(\|\varepsilon\|^2) \tag{93}$$

$$= \xi_i e^{-\xi_i t} + \varepsilon\xi_i\sum_{j\neq i}\frac{\xi_i+\rho}{\xi_i+\xi_j+\rho}\frac{a_{ij}}{1-a_{ii}}\frac{\xi_j e^{-\xi_j t} - \xi_i e^{-\xi_i t}}{\xi_i - \xi_j} + \mathcal{O}(\|\varepsilon\|^2) \tag{94}$$

where we have used the expression for $[\mathbf{V}'(0)]_{ij} = \frac{\partial}{\partial \varepsilon}\left[\mathbf{v}_j(\varepsilon)\right]_i\big|_{\varepsilon=0}$ form the proof of Lemma 2.

■

B.7. Proof of Proposition 6

Noting that the IRF for CPI inflation is the expenditure share weighted average of sectoral inflation rates, $\frac{\partial}{\partial \delta_m}\pi_t = \boldsymbol{\beta}^\top\frac{\partial}{\partial \delta_m}\boldsymbol{\pi}_t$, we can use the result from Proposition 5 to write:

$$\frac{\partial}{\partial \delta_m}\pi_t = \sum_{i=1}^n \beta_i\left[\xi_i e^{-\xi_i t} + \varepsilon\sum_{j\neq i}\frac{\xi_i^2 + \rho\xi_i}{(\xi_i - \xi_j)(\xi_i + \xi_j + \rho)}\frac{a_{ij}}{1 - a_{ii}}(\xi_j e^{-\xi_j t} - \xi_i e^{-\xi_i t})\right] + \mathcal{O}(\|\varepsilon\|^2) \tag{95}$$

Evaluating this at $t = 0$, differentaiting with respect to ε, and letting $\varepsilon = 0$ gives the impact response in the Proposition:

$$\frac{\partial\frac{\partial}{\partial \delta_m}\pi_0}{\partial \varepsilon}\Big|_{\varepsilon=0} = -\sum_{i=1}^n \beta_i\xi_i\left[\sum_{j\neq i}\frac{\xi_i+\rho}{\xi_i+\xi_j+\rho}\frac{a_{ij}}{1-a_{ii}}\right] < 0 \tag{96}$$

which is strictly negative as long as some a_{ij} is not zero. Now, to get the asymptotic responses, let $\iota \equiv \arg\min_i\{\xi_i\}$, divide $\frac{\partial}{\partial \delta_m}\pi_t$ in Equation (95) by $e^{-\xi_\iota t}$ and take the limit as

$t \to \infty$:

$$\lim_{t\to\infty} \frac{\frac{\partial}{\partial \delta_m}\pi_t}{e^{-\xi_\iota t}} = \beta_\iota \xi_\iota + \varepsilon \beta_\iota \sum_{j\neq\iota} \frac{\xi_\iota^2 + \rho\xi_\iota}{(\xi_j - \xi_\iota)(\xi_j + \xi_\iota + \rho)} \frac{a_{\iota j}}{1-a_{\iota\iota}} \xi_\iota + \varepsilon \sum_{j\neq\iota} \beta_j \frac{\xi_j^2 + \rho\xi_j}{(\xi_j - \xi_\iota)(\xi_j + \xi_\iota + \rho)} \frac{a_{j\iota}}{1-a_{jj}} \xi_\iota + \mathcal{O}(\|\varepsilon\|^2) \tag{97}$$

$$= \beta_\iota \xi_\iota + \varepsilon \sum_{j\neq i} \left[\beta_\iota \xi_\iota \frac{a_{\iota j}(\xi_\iota + \rho)}{1-a_{\iota\iota}} + \beta_j \xi_j \frac{a_{j\iota}(\xi_j + \rho)}{1-a_{jj}} \right] \frac{\xi_\iota}{(\xi_j - \xi_\iota)(\xi_j + \xi_\iota + \rho)} \tag{98}$$

Differentiating this with respect to ε and setting $\varepsilon = 0$ we have:

$$\frac{\partial \frac{\partial}{\partial \delta_m}\pi_t}{\partial \varepsilon}\Big|_{\varepsilon=0} \sim \sum_{j\neq i} \left[\beta_\iota \xi_\iota \frac{a_{\iota j}(\xi_\iota+\rho)}{1-a_{\iota\iota}} + \beta_j \xi_j \frac{a_{j\iota}(\xi_j+\rho)}{1-a_{jj}} \right] \frac{\xi_\iota e^{-\xi_\iota t}}{(\xi_j-\xi_\iota)(\xi_j+\xi_\iota+\rho)} > 0 \tag{99}$$

which is strictly positive as long as $a_{\iota j}$ or $a_{j\iota}$ are not all zero. ■

B.8. Proof of Proposition 7

For a given $\mathbf{\Gamma}(\varepsilon)$, recall from Equation (24) that the CIR of GDP (gap) to a monetary shock is given by $\boldsymbol{\beta}^\top(\sqrt{\mathbf{\Gamma}(\varepsilon)} - \frac{\rho}{2}\mathbf{I})^{-1}\mathbf{1}$. Using the approximation from Lemma 2, we have

$$\boldsymbol{\beta}^\top \sqrt{\mathbf{\Gamma}(\varepsilon)}^{-1}\mathbf{1} = \boldsymbol{\beta}^\top(\sqrt{\mathbf{\Gamma}_D} - \frac{\rho}{2}\mathbf{I})^{-1}\mathbf{1} + \varepsilon \boldsymbol{\beta}^\top[\mathbf{V}'(0)(\sqrt{\mathbf{\Gamma}_D} - \frac{\rho}{2}\mathbf{I})^{-1} - (\sqrt{\mathbf{\Gamma}_D} - \frac{\rho}{2}\mathbf{I})^{-1}\mathbf{V}'(0)]\mathbf{1} + \mathcal{O}(\|\varepsilon\|^2) \tag{100}$$

Now note that for $i \neq j$:

$$[\mathbf{V}'(0)(\sqrt{\mathbf{\Gamma}_D} - \frac{\rho}{2}\mathbf{I})^{-1} - (\sqrt{\mathbf{\Gamma}_D} - \frac{\rho}{2}\mathbf{I})^{-1}\mathbf{V}'(0)]_{ji} = [\mathbf{V}'(0)]_{ji}(\frac{1}{\xi_i} - \frac{1}{\xi_j}) \tag{101}$$

$$= \frac{a_{ji}}{1-a_{jj}} \frac{\xi_j^2 + \rho\xi_j}{(\xi_j + \xi_i + \rho)(\xi_j - \xi_i)} \frac{\xi_j - \xi_i}{\xi_j \xi_i} \tag{102}$$

$$= \frac{a_{ji}}{1-a_{jj}} \frac{\xi_j + \rho}{\xi_i + \xi_j + \rho} \frac{1}{\xi_i} \tag{103}$$

Thus,

$$\boldsymbol{\beta}^\top \sqrt{\mathbf{\Gamma}(\varepsilon)}^{-1}\mathbf{1} = \sum_{i=1}^n \beta_i \xi_i^{-1} + \varepsilon \sum_{i=1}^n \xi_i^{-1} \sum_{j\neq i} \beta_j \frac{a_{ji}}{1-a_{jj}} \frac{\xi_j + \rho}{\xi_i + \xi_j + \rho} + \mathcal{O}(\|\varepsilon\|^2) \tag{104}$$

which concludes the proof. ■

B.9. Proof of Proposition 8

Recall from Proposition 4 that

$$\frac{\partial}{\partial \delta_z^i}\boldsymbol{\pi}_t = \mathbf{X}_i \left((\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I}) e^{-(\sqrt{\mathbf{\Gamma}} - \frac{\rho}{2}\mathbf{I})t} - \phi_i e^{-\phi_i t}\mathbf{I} \right) \mathbf{\Psi} \mathbf{e}_i$$

where $\mathbf{X}_i = \left(\boldsymbol{\Gamma} - (\frac{\rho}{2}+\phi_i)^2\mathbf{I}\right)^{-1}(\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})$. Evaluating the impulse response above at $t=0$ and plugging in the expression for $\mathbf{X}_i$, we have

$$
\begin{aligned}
\frac{\partial}{\partial \delta_z^i}\boldsymbol{\pi}_0 &= \mathbf{X}_i^{-1}\left(\sqrt{\boldsymbol{\Gamma}} - (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)\boldsymbol{\Psi}\mathbf{e}_i \\
&= \left(\boldsymbol{\Gamma} - (\frac{\rho^2}{4}+\phi_i^2+\rho\phi_i)\mathbf{I}\right)^{-1}\left(\sqrt{\boldsymbol{\Gamma}} - (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)(\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})\boldsymbol{\Psi}\mathbf{e}_i \\
&= \left(\boldsymbol{\Gamma} - (\phi_i+\frac{\rho}{2})^2\mathbf{I}\right)^{-1}\left(\sqrt{\boldsymbol{\Gamma}} - (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)(\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})\boldsymbol{\Psi}\mathbf{e}_i \\
&= \left(\sqrt{\boldsymbol{\Gamma}} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})\mathbf{e}_i
\end{aligned}
$$

Thus, the pass-through expression is

$$
\begin{aligned}
\frac{\partial \pi_0}{\partial \pi_{i,0}}\Big|_{\delta_z^i} \equiv \frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}} &= \frac{\boldsymbol{\beta}^\intercal\left(\sqrt{\boldsymbol{\Gamma}} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})\mathbf{e}_i}{\mathbf{e}_i^\intercal\left(\sqrt{\boldsymbol{\Gamma}} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})\mathbf{e}_i} \\
&= \frac{\boldsymbol{\beta}^\intercal\left(\sqrt{\boldsymbol{\Gamma}} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\mathbf{e}_i}{\mathbf{e}_i^\intercal\left(\sqrt{\boldsymbol{\Gamma}} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\mathbf{e}_i}
\end{aligned}
$$

Now letting $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\varepsilon)$ we have the following the Taylor expansion:

$$
\frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}} = \left[\frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}}\right]_{\varepsilon=0} + \frac{\partial}{\partial\varepsilon}\left[\frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}}\right]_{\varepsilon=0} \times \varepsilon + \mathcal{O}(\|\varepsilon\|^2)
$$

where

$$
\begin{aligned}
\left[\frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}}\right]_{\varepsilon=0} &= \frac{\boldsymbol{\beta}^\intercal\left(\sqrt{\boldsymbol{\Gamma}_D} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\mathbf{e}_i}{\mathbf{e}_i^\intercal\left(\sqrt{\boldsymbol{\Gamma}_D} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\mathbf{e}_i} \\
&= \beta_i
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon}\left[\frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}}\right]_{\varepsilon=0} &= -\frac{(\boldsymbol{\beta}^\intercal - \beta_i\mathbf{e}_i^\intercal)\left(\sqrt{\boldsymbol{\Gamma}_D} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\left[\frac{\partial}{\partial\varepsilon}\sqrt{\boldsymbol{\Gamma}(\varepsilon)}\right]_{\varepsilon=0}\left(\sqrt{\boldsymbol{\Gamma}_D} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\mathbf{e}_i}{(\xi_i+\rho+\phi_i)^{-1}} \\
&= -\frac{\boldsymbol{\beta}^\intercal\left(\sqrt{\boldsymbol{\Gamma}_D} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}(\mathbf{V}'(0)\sqrt{\boldsymbol{\Gamma}_D} - \sqrt{\boldsymbol{\Gamma}_D}\mathbf{V}'(0))\left(\sqrt{\boldsymbol{\Gamma}_D} + (\frac{\rho}{2}+\phi_i)\mathbf{I}\right)^{-1}\mathbf{e}_i}{(\xi_i+\rho+\phi_i)^{-1}} \\
&= -\frac{\sum_{j\neq i}\frac{\beta_j}{(\xi_i+\rho+\phi_i)(\xi_j+\rho+\phi_i)}\frac{a_{ji}}{1-a_{jj}}\frac{\xi_j^2+\rho\xi_j}{(\xi_j+\xi_i+\rho)(\xi_j-\xi_i)}(\xi_i-\xi_j)}{(\xi_i+\rho+\phi_i)^{-1}} \\
&= \sum_{j\neq i} a_{ji}\frac{\beta_j}{1-a_{jj}}\frac{\xi_j}{\phi_i+\xi_j+\rho}\frac{\xi_j+\rho}{\xi_i+\xi_j+\rho}
\end{aligned}
$$

Thus,

$$\frac{\frac{\partial}{\partial \delta_z^i}\pi_0}{\frac{\partial}{\partial \delta_z^i}\pi_{i,0}} = \beta_i + \varepsilon \times \sum_{j\neq i} a_{ji}\frac{\beta_j}{1-a_{jj}}\frac{\xi_j}{\phi_i+\xi_j+\rho}\frac{\xi_j+\rho}{\xi_i+\xi_j+\rho} + \mathcal{O}(\|\varepsilon\|^2)$$

■

B.10. Proof of Lemma 3

Recall from Equation (19) that

$$\dot{\boldsymbol{\pi}}_t = \rho\boldsymbol{\pi}_t + \underbrace{\boldsymbol{\Theta}(\rho\mathbf{I}+\boldsymbol{\Theta})(\mathbf{I}-\mathbf{A})}_{\boldsymbol{\Gamma}-\frac{\rho^2}{4}\mathbf{I}}(\mathbf{p}_t - \mathbf{p}_t^f)$$

where $\mathbf{p}_t^f \equiv w_t\mathbf{1} + \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)$ from Equation (14) (Note, however, that with endogenous monetary policy, $\mathbf{p}_t^f$ is a nominal quantity that also depends on the stance of monetary policy and is no longer exogenous to the system).

Recall also from the household's intra-temporal Euler equation that with infinite Frisch elasticity $w_t = p_t + y_t = \boldsymbol{\beta}^\top\mathbf{p}_t + y_t$. Finally, recall from Equation (15) that output in the flexible economy is given by $y_t^f = \boldsymbol{\beta}^\top\boldsymbol{\Psi}(\boldsymbol{z}_t - \boldsymbol{\omega}_t)$. From the last two equations, we observe that

$$\begin{aligned}
\mathbf{p}_t - \mathbf{p}_t^f &= (\mathbf{I}-\mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t + (\boldsymbol{\beta}^\top\mathbf{p}_t - w_t)\mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)\\
&= (\mathbf{I}-\mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t - y_t\mathbf{1} + y_t^f\mathbf{1} - y_t^f\mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)\\
&= (\mathbf{I}-\mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t - \tilde{y}_t\mathbf{1} - (\mathbf{I}-1\boldsymbol{\beta}^\top)\boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)\\
&= \underbrace{(\mathbf{I}-\mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t}_{\mathbf{q}_t=\mathbf{p}_t-\mathbf{1}\boldsymbol{\beta}^\top\mathbf{p}_t} - \underbrace{(\mathbf{I}-\mathbf{1}\boldsymbol{\beta}^\top)\boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)}_{\mathbf{q}_t^f=\mathbf{p}_t^f-\mathbf{1}\boldsymbol{\beta}^\top\mathbf{p}_t^f} - \tilde{y}_t\mathbf{1}\\
&= (\mathbf{q}_t - \mathbf{q}_t^f) - \tilde{y}_t\mathbf{1}
\end{aligned}$$

Plugging this into the sectoral Phillips curves we have:

$$\dot{\boldsymbol{\pi}}_t = \rho\boldsymbol{\pi}_t + (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{q}_t - \mathbf{q}_t^f) - (\boldsymbol{\Gamma}\mathbf{1} - \frac{\rho^2}{4}\mathbf{1})\tilde{y}_t$$

■

B.11. Proof of Corollary 1

Necessity: Recall from Lemma 3 that the sectoral Phillips curve is given by

$$\dot{\boldsymbol{\pi}}_t = \rho\boldsymbol{\pi}_t + (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{q}_t - \mathbf{q}_t^f) - (\boldsymbol{\Gamma}\mathbf{1} - \frac{\rho^2}{4}\mathbf{1})\tilde{y}_t \tag{105}$$

Thus, if monetary policy aims to stabilize the GDP gap—i.e., $\tilde{y}_t = 0$ for all $t \geq 0$—then it is necessary for sectoral prices to satisfy the above equation when $\tilde{y}_t = 0, \forall t \geq 0$:

$$\dot{\boldsymbol{\pi}}_t = \rho \boldsymbol{\pi}_t + \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(\mathbf{q}_t - \mathbf{q}_t^f) \tag{106}$$

Sufficiency: To show sufficiency, note that $\mathbf{q}_t - \mathbf{q}_t^f = (\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t - \mathbf{q}_t^f$ where

$$\mathbf{q}_t^f = (\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top)\boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t) \tag{107}$$

which is an exogenous function of productivities and wedges across sectors. Thus, noting that $\boldsymbol{\pi}_t = \dot{\mathbf{p}}_t$, we can write the necessary condition above as

$$\ddot{\mathbf{p}}_t = \rho \dot{\mathbf{p}}_t + (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t - (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})\mathbf{q}_t^f \tag{108}$$

which is a system of second-order differential equations with an exogenous force term $\mathbf{q}_t^f$. Thus, given the boundary conditions $\mathbf{p}_0 = \mathbf{p}_{0^-}$ and non-explosive prices, this system of differential equations solely characterizes the dynamics of sectoral prices and is thus sufficient for their dynamics. ■

B.12. Proof of Proposition 9

Recall from Corollary 1 that the following sectoral Phillips curves are necessary and sufficient for dynamics of sectoral prices when monetary policy fully stabilizes the GDP gap:

$$\ddot{\mathbf{p}}_t = \rho \dot{\mathbf{p}}_t + (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top)\mathbf{p}_t - (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})\mathbf{q}_t^f \tag{109}$$

where $\mathbf{q}_t^f = (\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top)\boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)$. Now, let $\tilde{\mathbf{p}}_t^f \equiv \boldsymbol{\Psi}(\boldsymbol{\omega}_t - \boldsymbol{z}_t)$ and note that, per Equation (14), this corresponds to the flexible prices in an economy where monetary policy exogenously sets $m_t = 0$. Thus, the above system of differential equations can be written as

$$\ddot{\mathbf{p}}_t = \rho \dot{\mathbf{p}}_t + (\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top)(\mathbf{p}_t - \tilde{\mathbf{p}}_t^f) \tag{110}$$

Furthermore, note that

$$(\boldsymbol{\Gamma} - \frac{\rho^2}{4}\mathbf{I})(\mathbf{I} - \mathbf{1}\boldsymbol{\beta}^\top) = \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A})(I - \mathbf{1}\boldsymbol{\beta}^\top) = \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A} - \boldsymbol{\alpha}\boldsymbol{\beta}^\top) \tag{111}$$

where the last step follows from the fact that $(\mathbf{I} - \mathbf{A})\mathbf{1} = \boldsymbol{\alpha}$. Thus, Equation (110) can be re-written as

$$\ddot{\mathbf{p}}_t = \rho \dot{\mathbf{p}}_t + \boldsymbol{\Theta}(\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{I} - \mathbf{A}_{\boldsymbol{\beta}})(\mathbf{p}_t - \tilde{\mathbf{p}}_t^f) \tag{112}$$

where $\mathbf{A}_{\boldsymbol{\beta}} = \mathbf{A} + \boldsymbol{\alpha}\boldsymbol{\beta}^\top$. Now, comparing Equation (112) with the system of differential equations in Proposition 1, we see that the sectoral prices of the economy with GDP gap stabilization are equivalent to the sectoral prices of an economy with the adjusted network $\mathbf{A}_{\boldsymbol{\beta}}$ where monetary policy exogenously fixes $m_t = 0, \forall t \geq 0$. ■

B.13. Proof of Proposition 10

Necessity follows immediately from the observation that if $p_{\boldsymbol{\eta},t}$ is stabilized at $p_{\boldsymbol{\eta},0^-}$, then $\pi_{\boldsymbol{\eta},t} = \dot{\pi}_{\boldsymbol{\eta},t} = 0$ for all $t \geq 0$. To show sufficiency, we need to show that if m_t is chosen according to Equation (47) then $\pi_{\boldsymbol{\eta},t} = \dot{\pi}_{\boldsymbol{\eta},t} = 0$ for all t. To this end, note that if m_t is chosen according to Equation (47) then Equation (46) reads:

$$\dot{\pi}_{\boldsymbol{\eta},t} = \rho\pi_{\boldsymbol{\eta},t} \tag{113}$$

which is a second order differential equation in $p_{\boldsymbol{\eta},t}$ with the general solution:

$$p_{\boldsymbol{\eta},t} = K_0 e^{\rho t} + K_1 \tag{114}$$

for constants K_0 and K_1 that are to be determined under the boundary conditions that $p_{\boldsymbol{\eta},0} = p_{\boldsymbol{\eta},0^-}$ and non-explosive prices. Since $\rho > 0$, non-explosive prices imply $K_0 = 0$, and since K_1 is then determined as $K_1 = p_{\boldsymbol{\eta},0^-}$. Thus, $p_{\boldsymbol{\eta},t} = p_{\boldsymbol{\eta},0^-}$ for all $t \geq 0$. ■

C Equilibrium Definition

Here, we precisely define the equilibrium concept used in this paper.

Definition 2. *A **sticky price equilibrium** for this economy is*

(a) *an allocation for the household,* $\mathscr{A}_h = \{(C_{i,t})_{i\in[n]}, C_t, L_t, B_t\}_{t\geq 0} \cup \{B_{0^-}\}$,

(b) *an allocation for all firms* $\mathscr{A}_f = \{(Y_{i,t}, Y^d_{ij,t}, Y^s_{ij,t}, L_{ij,t}, X_{ij,k,t})_{i\in[n], j\in[0,1]}\}_{t\geq 0}$,

(c) *a set of monetary and fiscal policies* $\mathscr{A}_g = \{(M^s_t, T_t, \tau_{1,t}, \ldots, \tau_{n,t})_{t\geq 0}\}$,

(d) *and a set of prices* $\mathscr{P} = \{(P_{i,t}, P_{ij,t})_{i\in[n], j\in[0,1]}, W_t, P_t, i_t\}_{t\geq 0} \cup \{(P_{ij,0^-})_{i\in[n], j\in[0,1]}\}$

such that

1. *given* $\mathscr{P}$ *and* $\mathscr{A}_g$, $\mathscr{A}_h$ *solves the household's problem in Equation (1),*
2. *given* $\mathscr{P}$ *and* $\mathscr{A}_g$, $\mathscr{A}_f$ *solves the final goods producers problems in Equation (3), intermediate goods producers' cost minimization in Equation (6) and their pricing problem in Equation (7),*
3. *labor, money, bonds and final sectoral goods markets clear and government budget*

constraint is satisfied:

$$M_t = M_t^s, \quad B_t = 0, \quad L_t = \sum_{i\in[n]} \int_0^1 L_{ij,t} \mathrm{d}j, \quad \sum_{i\in[n]} \int_0^1 \tau_{i,t} P_{ij,t} Y_{ij,t} \mathrm{d}j = -T_t \quad \forall t \geq 0 \tag{115}$$

$$Y_{k,t} = C_{k,t} + \sum_{i\in[n]} \int_0^1 X_{ij,k,t} \mathrm{d}j \quad \forall k \in [n], \quad \forall t \geq 0 \tag{116}$$

Furthermore, to understand how the stickiness of prices will affect and distort the equilibrium allocations, we will make comparisons between the equilibrium defined above and its *flexible-price* analog, formally defined below.

Definition 3. *A **flexible price equilibrium** is an equilibrium defined similarly to Definition 2 with the only difference that intermediate goods producers' prices solve the flexible price problems specified in Equation (8) instead of the sticky price problem in Equation (7).*

Finally, since we have defined our economy without any aggregate or sectoral shocks, we will pay specific attention to *stationary* equilibria, which we define below.

Definition 4. *A **stationary equilibrium** for this economy is an equilibrium as in Definition 2 or Definition 3 with the additional requirement that all the allocative variables in the household's allocation in $\mathscr{A}_h$ and the sectoral production of final good producers $(Y_{i,t})_{i\in[n]}$ as well as the distributions of the allocative variables for intermediate good producers in $\mathscr{A}_i$ are constant over time.*[47]

[47] Note that the production and input demands of individual intermediate goods producers do not need to be time-invariant in the stationary equilibrium, but their distributions do.

Supplemental Materials

Inflation and GDP Dynamics in Production Networks: A Sufficient Statistics Approach

By Hassan Afrouzi and Saroj Bhattarai

S.M.1 Derivations of Optimality Conditions in the Model

Here, we characterize the flexible- and sticky-price stationary equilibria of this economy.

S.M.1.1. Households' Optimality Conditions

We can decompose the household's consumption problem into two stages, where for a *given* level of C_t the household minimizes her expenditure on sectoral goods (compensated demand) and then decides on the optimal level of C_t as a function of lifetime income (uncompensated demand). The compensated demand of the household for sectoral goods given the vector of sectoral prices $\mathbf{P}_t = (P_{1,t}, \ldots, P_{n,t})$ gives us the expenditure function:

$$\begin{aligned}\mathscr{E}(C_t, \mathbf{P}_t) &\equiv \min_{C_{1,t},\ldots,C_{n,t}} \sum_{i\in[n]} P_{i,t}C_{i,t} \quad \text{subject to} \quad \Phi(C_{1,t},\ldots,C_{n,t}) \geq C_t \\ &= P_t C_t, \quad P_t \equiv \mathscr{E}(1, \mathbf{P}_t) \end{aligned} \tag{S.M.1}$$

where the second line follows from the first-degree homogeneity of the function $\Phi(.)$ and P_t is the cost of a *unit* of C_t and, or in short, the price of C_t. Note that due to the first-degree homogeneity of $\Phi(.)$, P_t does not depend on household's choices and is *just* a function of the sectoral prices, $\mathbf{P}_t$. Applying Shephard's lemma, we obtain that the household's expenditure share of sectoral good i is proportional to the elasticity of the expenditure function with respect to the price of i:

$$P_{i,t}C^*_{i,t} = \beta_i(\mathbf{P}_t) \times P_t C_t \quad \text{where} \quad \beta_i(\mathbf{P}_t) \equiv \frac{\partial \ln(\mathscr{E}(C_t, \mathbf{P}_t))}{\partial \ln(P_{i,t})} \tag{S.M.2}$$

It is important to note that due to the first-degree homogeneity of the expenditure function, these elasticities are independent of aggregate consumption C_t and only depend on sectoral prices, $\mathbf{P}_t$. Moreover, it is easy to verify that they are also a homogeneous of degree zero in these prices so that the vector of household's expenditure shares, denoted by $\boldsymbol{\beta}_t \in \mathbb{R}^n$, can be written as a function of sectoral prices relative to wage:

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}(\mathbf{P}_t/W_t) \tag{S.M.3}$$

Notably, a vector of constant expenditure shares corresponds to $\Phi(.)$ being a Cobb-Douglas aggregator where sectoral goods are neither complements nor substitutes.

Given the household's expenditure function and the aggregate price index P_t in Equation (S.M.1), it is straightforward to derive the labor supply and Euler equations for bonds:

$$\underbrace{\gamma(C_t)\frac{\dot{C}_t}{C_t}}_{\text{marginal loss from saving}} = \underbrace{i_t - \rho - \frac{\dot{P}_t}{P_t}}_{\text{marginal gain from saving}} \quad \text{where} \quad \underbrace{\gamma(C_t) \equiv -\frac{U''(C_t)C_t}{U'(C_t)}}_{\text{inverse elasticity of intertemporal substitution}} \tag{S.M.4}$$

$$\underbrace{\frac{V'(L_t)}{U'(C_t)}}_{\text{MRS}_{LC}} = \underbrace{\frac{W_t}{P_t}}_{\text{real wage}} \implies \psi(L_t)\frac{\dot{L}_t}{L_t} + \gamma(C_t)\frac{\dot{C}_t}{C_t} = \frac{\dot{W}_t}{W_t} - \frac{\dot{P}_t}{P_t} \quad \text{where} \quad \underbrace{\psi(L_t) \equiv \frac{V''(L_t)L_t}{V'(L_t)}}_{\text{inverse Frisch elasticity of labor supply}} \tag{S.M.5}$$

Moreover, given a path of $\{M_t\}_{t\geq 0}$ set by monetary policy where $M_t = P_t C_t$ is the nominal GDP, we have:

$$\frac{\dot{P}_t}{P_t} + \frac{\dot{C}_t}{C_t} = \frac{\dot{M}_t}{M_t} \tag{S.M.6}$$

Note that by combining Equations (S.M.4) to (S.M.6) we can write the growth rate of wages as well as the nominal interest rates as a function of consumption and labor supply growths:

$$\frac{\dot{W}_t}{W_t} = \mu + \psi(L_t)\frac{\dot{L}_t}{L_t} + (\gamma(C_t)-1)\frac{\dot{C}_t}{C_t}, \qquad i_t = \rho + \frac{\dot{M}_t}{M_t} + (\gamma(C_t)-1)\frac{\dot{C}_t}{C_t} \tag{S.M.7}$$

Given Golosov and Lucas (2007) preferences $U(C_t) = \log(C_t)$ and $V(L_t) = L_t$ which imply $\gamma(C_t) = 1$ and $\psi(L_t) = 0$. Plugging these elasticities into the Euler equations above we can see how these preference simplify aggregate dynamics by relating interest rates to nominal GDP growth and nominal wages equal to nominal GDP:

$$i_t = \rho + \dot{M}_t/M_t, \qquad W_t = M_t$$

S.M.1.2. Firms' Cost Minimization and Input-Output Matrices

We start by characterizing firms' expenditure shares on inputs by first solving their expenditure minimization problems. Since expenditure minimization is a static decision within every period, our characterization of these expenditure shares closely follow Bigio and La'O (2020), Baqaee and Farhi (2020), and we refer the reader to these papers for more detailed treatments.

Let us start with the observation that the firms' cost function in Equation (6), given the wage and sectoral prices $\mathbf{P}_t = (W_t, P_{i,t})_{i\in[n]}$,[48] is homogeneous of degree one in

[48]Note that previously in characterizing the expenditure shares of the households, we defined $\mathbf{P}_t$ as the vector of sectoral prices. Here, without loss of generality and with a slight abuse of notation, we are augmenting this vector with the wage W_t.

production:

$$\mathscr{C}_i(Y^s_{ij,t};\mathbf{P}_t,Z_{i,t}) = \min_{L_{jk,t},(X_{ij,k,t})_{k\in[n]}} W_t L_{ij,t} + \sum_{k\in[n]} P_{k,t}X_{ij,k,t} \quad \text{subject to} \quad Z_{i,t}F_i(L_{ij,t},(X_{ij,k,t})_{k\in[n]}) \geq Y^s_{ij,t}$$
$$= \mathsf{MC}_i(\mathbf{P}_t,Z_{i,t}) \times Y^s_{ij,t}, \quad \mathsf{MC}_i(\mathbf{P}_t,Z_{i,t}) \equiv \mathscr{C}_i(1;\mathbf{P}_t,1)/Z_{i,t} \tag{S.M.8}$$

where the second line follows from the first-degree homogeneity of the production function $Z_iF_i(.)$ and $\mathsf{MC}_i(\mathbf{P}_t,Z_{i,t})$ is the cost of producing a *unit* of output, or in short, the firm's marginal cost of production. Note that due to the first-degree homogeneity of the production function, marginal costs are independent of the level of production and depend only on the sector's production function and input prices. Applying Shephard's lemma and re-arranging firms' optimal demand for inputs gives us the result that firms' expenditure share of any input is the elasticity of the cost function with respect to that input:

$$W_tL^*_{ij,t} = \alpha_i(\mathbf{P}_t)\times\mathsf{MC}_i(\mathbf{P}_t,Z_{i,t})Y^s_{ij,t}, \quad P_{k,t}X^*_{ij,k,t} = a_{ik}(\mathbf{P}_t)\times\mathsf{MC}_i(\mathbf{P}_t,Z_{i,t})Y^s_{ij,t}, \quad \forall k\in[n] \tag{S.M.9}$$

where $\alpha_i(\mathbf{P}_t)$ and $a_{ik}(\mathbf{P}_t)$ are the elasticities of the sector i's cost function with respect to labor and sector k's final good respectively:

$$\alpha_i(\mathbf{P}_t) \equiv \frac{\partial\ln(\mathscr{C}_i(Y;\mathbf{P}_t,1)/Z_{i,t})}{\partial\ln(W_t)}, \qquad a_{ik}(\mathbf{P}_t) \equiv \frac{\partial\ln(\mathscr{C}_i(Y;\mathbf{P}_t,1)/Z_{i,t})}{\partial\ln(P_{k,t})} \quad \forall k\in[n] \tag{S.M.10}$$

with the property that $\alpha_i(\mathbf{P}_t)+\sum_{k\in[n]}a_{ik}(\mathbf{P}_t)=1$. It is important to note that the first-degree homogeneity of the cost function in Equation (6) also implies that these elasticities are only functions of the aggregate wage and sectoral prices. It is also well-known that these elasticities are directly related to the *cost-based* input-output matrix, denoted by $\mathbf{A}_t\in\mathbb{R}^{n\times n}$, and the labor share vector, denoted by $\boldsymbol{\alpha}_t\in\mathbb{R}^n$:

$$[\mathbf{A}_t]_{i,k} \equiv \frac{\text{total expenditure of sector } i \text{ on sector } k}{\text{total expenditure on inputs in sector } i} = a_{ik}(\mathbf{P}_t), \quad \forall (i,k)\in[n]^2 \tag{S.M.11}$$

$$[\boldsymbol{\alpha}_t]_i \equiv \frac{\text{total expenditure of sector } i \text{ on labor}}{\text{total expenditure on inputs in sector } i} = \alpha_i(\mathbf{P}_t), \quad \forall i\in[n] \tag{S.M.12}$$

where the second equality holds *only* under firms' optimal expenditure shares and follows from integrating Equation (S.M.9). Since these elasticities are also homogeneous of degree zero in the price vector $\mathbf{P}_t$, Equations (S.M.11) and (S.M.12) imply that in *any equilibrium*, the cost-based input-output matrix and the vector of sectoral labor shares are only a function of the sectoral prices relative to the nominal wage; i.e.,

$$\mathbf{A}_t = \mathbf{A}(\mathbf{P}_t/W_t) = [a_{ik}(\mathbf{P}_t/W_t)], \qquad \boldsymbol{\alpha}_t = \boldsymbol{\alpha}(\mathbf{P}_t/W_t) = [\alpha_i(\mathbf{P}_t/W_t)] \tag{S.M.13}$$

A notable example is Cobb-Douglas production functions, which imply constant elasticities for the cost function—because inputs are neither substitutes nor complements—and lead to a constant input-output matrix and constant vector of labor shares over time.

S.M.1.3. Firms' Optimal Prices

Having characterized firms' cost functions, we now derive the optimal *desired prices*, $P^*_{ij,t}$, in Equation (8) and *reset prices*, $P^{\#}_{ij,t}$ in Equation (S.M.15). It follows that the optimal desired price is a markup over the marginal cost of production and proportional to the wedge introduced through taxes/subsidies:

$$P^*_{ij,t} = P^*_{i,t} \equiv \underbrace{\frac{1}{1-\tau_i}}_{\text{tax/subsidy wedge}} \times \underbrace{\frac{\sigma_i}{\sigma_i - 1}}_{\text{markup}} \times \underbrace{\frac{\mathsf{MC}_i(\mathbf{P}_t, 1)}{Z_{i,t}}}_{\text{marginal cost}} \tag{S.M.14}$$

It is then straightforward to show that the firms' optimal reset prices are a weighted average of all future desired prices in industry i:

$$P^{\#}_{ij,t} = P^{\#}_{i,t} \equiv \underbrace{\int_0^\infty \frac{\overbrace{e^{-(\theta_i h + \int_0^h i_{t+s} \mathrm{d}s)} Y_{i,t+h} P^{\sigma_i}_{i,t+h}}^{\text{weight (density) on } P^*_{i,t+h}}}{\int_0^\infty e^{-(\theta_i h + \int_0^h i_{t+s} \mathrm{d}s)} Y_{i,t+h} P^{\sigma_i}_{i,t+h} \mathrm{d}h} \times P^*_{i,t+h} \mathrm{d}h}_{\text{weighted average of all future desired prices}} \tag{S.M.15}$$

Given this reset price, we can then calculate the aggregate price of sector i from Equation (4) as:

$$P^{1-\sigma_i}_{i,t} = \int_0^1 P^{1-\sigma_i}_{ij,t} \mathrm{d}j = \theta_i \int_0^t e^{-\theta_i h} (P^{\#}_{i,t-h})^{1-\sigma_i} \mathrm{d}h + e^{-\theta_i t} \underbrace{\int_0^1 P^{1-\sigma_i}_{ij,0^-} \mathrm{d}j}_{=P^{1-\sigma_i}_{i,0^-}} \tag{S.M.16}$$

where the second equality follows from the observation that at time t the density of firms that reset their prices h periods ago to $P^{\#}_{i,t}$ is governed by the exponential distribution of time between price changes and is equal to $\theta_i e^{-\theta_i h}$.

S.M.1.4. Market Clearing and Total Value Added

Define the sales-based Domar weight of sector $i \in [n]$ at time t as the ratio of the final producer's sales relative to the household total expenditure on consumption:

$$\lambda_{i,t} \equiv P_{i,t} Y_{i,t} / (P_t C_t) \tag{S.M.17}$$

Now, substituting optimal consumption of the household from sector $k \in [n]$ in Equation (S.M.2) and optimal demand of firms for the final good of sector $k \in [n]$ in Equation (S.M.9) into the market clearing condition for final good of sector k and dividing by household's total expenditure, we get

$$\lambda_{k,t} = \beta_i(\mathbf{P}_t/W_t) + \textstyle\sum_{i\in[n]} a_{ik}(1,\mathbf{P}_t/W_t)\lambda_{i,t}\Delta_{i,t}/\mu_{i,t} \tag{S.M.18}$$

where $\mu_{i,t} \equiv P_{i,t}/\mathsf{MC}_i(\mathbf{P}_t, W_t)$ is the markup of sector i and $\Delta_{i,t}$ is the well-known measure of price dispersion in the New Keynesian literature defined as

$$\Delta_{i,t} = \int_0^1 (P_{ij,t}/P_{i,t})^{-\sigma_i} \mathrm{d}j \geq 1 \tag{S.M.19}$$

Where the inequality follows from applying Jensen's inequality to the definion of the aggregate price index $P_{i,t}$.[49] Thus, letting $\boldsymbol{\lambda}_t \equiv (\lambda_{i,t})_{i\in[n]}$ denote the vector of sales-based domar weights at time t across sectors and $\mathcal{M}_t \equiv \mathrm{diag}(\mu_{i,t}/\Delta_{i,t})$ as the diagonal matrix whose i'th diagonal entry is the price dispersion adjusted markup wedge of sector i, we can write Equation (S.M.18) in the following matrix form:

$$\boldsymbol{\lambda}_t = (\mathbf{I} - \mathbf{A}_t^\top \mathcal{M}_t^{-1})^{-1} \boldsymbol{\beta}_t \tag{S.M.20}$$

Finally, substituting firms labor demand into the labor market clearing condition, we arrive at the following expression for the labor share:

$$\frac{W_t L_t}{P_t C_t} = \boldsymbol{\alpha}_t^\top \mathcal{M}_t^{-1} \boldsymbol{\lambda}_t \tag{S.M.21}$$

S.M.1.5. Efficient Steady State

We log-linearize the model around an efficient steady state where the rate of growth in money supply is zero ($\mu = 0$), and fiscal policy sets distortionary subsidies such that in each sector prices are equal to marginal costs. It is straightforward to verify that the allocation that prevails under these assumptions coincide with the first-best allocations chosen by a social planner—hence justifying the term efficient steady state. This is a standard result in New Keynesian models and we refer the reader to La'O and Tahbaz-Salehi (2022) for its characterization in network economies with multiple sectors.

Here, we characterize this steady state. To implement the efficient steady-state, fiscal policy sets taxes to undo distortions arising from monopolistic competition so that $\tau_i = -\frac{1}{\sigma_i - 1}, \forall i \in [n]$

$$P_{ij}^* = \mathsf{MC}_i(\mathbf{P}, Z_i), \forall j \in [0,1], i \in [n] \tag{S.M.22}$$

where $\mathsf{MC}_i \equiv \mathscr{C}_i(1; \mathbf{P}, Z_i)$ is the marginal cost of sector $i \in [n]$ at the efficient stationary

[49]Note that $1 = [\int_0^1 (P_{ij,t}/P_{i,t})^{1-\sigma_i} \mathrm{d}j]^{\frac{\sigma_i}{\sigma_i - 1}} \mathrm{d}j = [\int_0^1 \left((P_{ij,t}/P_i, t)^{-\sigma_i}\right)^{\frac{\sigma_i - 1}{\sigma_i}} \mathrm{d}j]^{\frac{\sigma_i}{\sigma_i - 1}} \mathrm{d}j \leq \int_0^1 (P_{i,t}/P_t)^{-\sigma_i} \mathrm{d}j$.

equilibrium, which is given by Equation (S.M.8) evaluated at $(\mathbf{P}, Z_i)$. From the firm's cost minimization problem at the stationary equilibrium, we also get the demand for labor and intermediate inputs

$$W L_{ij} = \alpha_i(\mathbf{P}) \times \mathrm{MC}_i(\mathbf{P}, Z_i)\, Y_{ij}^s \tag{S.M.23}$$

$$P_k X_{ij,k} = a_{ik}(\mathbf{P}) \times \mathrm{MC}_i(\mathbf{P}, Z_i)\, Y_{ij}^s,\ \forall k \in [n] \tag{S.M.24}$$

where $\alpha_i(\mathbf{P})$ and $a_{ik}(\mathbf{P})$ are the elasticities of the sector i's cost function with respect to labor and sector k's final good at the stationary equilibrium, respectively:

$$\alpha_i(\mathbf{P}) \equiv \frac{\partial \ln(\mathscr{C}_i(Y; \mathbf{P}, Z_i))}{\partial \ln(W)}, \quad a_{ik}(\mathbf{P}) \equiv \frac{\partial \ln(\mathscr{C}_i(Y; \mathbf{P}, Z_i))}{\partial \ln(P_k)},\ \forall k \in [n] \tag{S.M.25}$$

Note that these elasticities are only functions of prices because the cost function is homogeneous of degree one in production Y and homogeneous of degree -1 in productivity Z_i, so the partial derivatives of the log cost function do not vary with Y or Z_i. Moreover, since the cost function is homogeneous of degree 1 in the vector $\mathbf{P}$, these elasticities are homogeneous of degree zero in $\mathbf{P}$ so that they will not change if we normalize all prices in $\mathbf{P}$ by a constant wage W. Then, the cost-based input-output matrix and the sectoral labor shares at the efficient stationary equilibrium can be written in terms of these relative prices and are given by

$$\mathbf{A} = \mathbf{A}(\mathbf{P}/W) = [a_{ik}(1, \mathbf{P}/W)], \qquad \boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{P}/W) = [\alpha_i(1, \mathbf{P}/W)] \tag{S.M.26}$$

where we used the observation that the cost-based input-output matrix and the vector of sectoral labor shares are only a function of the sectoral prices relative to the nominal wage. From the representative retailer's optimality conditions and the monopolistically competitive firm's optimal price, the aggregate sectoral price is

$$P_i/W = \left(\int_0^1 \left(P_{ij}^*/W \right)^{1-\sigma_i} \mathrm{d}j \right)^{\frac{1}{1-\sigma_i}} = \left(\int_0^1 \left(\mathrm{MC}_i(\mathbf{P}, Z_i)/W \right)^{1-\sigma_i} \mathrm{d}j \right)^{\frac{1}{1-\sigma_i}} = \mathrm{MC}_i(\mathbf{P}/W, Z_i),\ \forall i \in [n]\ \forall j \in [0,1] \tag{S.M.27}$$

where the last equality uses the first-degree homogeneity of the marginal cost function with respect to $\mathbf{P}$. Now, let $\tilde{\mathbf{p}} \equiv (\ln(P_i/W))_{i\in[n]}$ denote the vector of log of the sectoral prices relative to the wage in the steady-state. With slight abuse of notation, also let $e^{\tilde{\mathbf{p}}} \equiv (P_i/W)_{i\in[n]}$. Then, writing Equation (S.M.27) in terms of $\tilde{\mathbf{p}}$ gives:

$$\tilde{\mathbf{p}} = f(\tilde{\mathbf{p}}) \equiv \left(\ln(\mathrm{MC}_i(e^{\tilde{\mathbf{p}}}, Z_i)) \right)_{i\in[n]} \tag{S.M.28}$$

Note that function $f(.) : \mathbb{R}^n \to \mathbb{R}^n$ depends only on log relative prices and the steady-state values of productivity across sectors that are exogenous to the model. Thus, we see that relative prices in the steady state are fully pinned down by the structure of the marginal cost functions and the steady-state values of productivities. Moreover, these relative prices are a fixed point of the function $f(.)$. Furthermore, note that the Jacobian of the $f(x)$ function is the input-output matrix evaluated at the implied relative prices by x, which we will refer to as $\mathbf{A}(x)$. Note that the spectral radius of this Jacobian, denoted by $\rho(\mathbf{A}(x))$, is strictly less than one under the assumption of CRS production functions and the fact that production functions satisfy Inada conditions. With a slightly stronger assumption that $\sup_x \rho(\mathbf{A}(x)) < 1-\epsilon$, for some however infinitesimal $\epsilon > 0$, it is straightforward to show that the function $f(.)$ is a contraction mapping in $\mathbf{R}^n$ and has a unique fixed point according to Banach fixed point theorem.[50] So, there exists a unique set of $\tilde{\mathbf{p}}^* \in \mathbf{R}^n$ such that

$$\tilde{\mathbf{p}}^* = f(\tilde{\mathbf{p}}^*) \tag{S.M.29}$$

Moreover, recall that in the equilibrium with Golosov and Lucas (2007) preferences, $W = M$, where M is the nominal demand of the economy that is fixed by the central bank. So, given this nominal anchor, nominal sectoral prices are given by

$$\mathbf{P}^*/W = e^{\tilde{\mathbf{p}}^*} \implies \mathbf{P}^* = Me^{\tilde{\mathbf{p}}^*} \tag{S.M.30}$$

Thus, having solved for P_i^* for every sector i, we have

$$P_{ij}^* = P_i^* \implies Y_{ij}^d = Y_i, \ \forall i \in [n], \ \forall j \in [0,1] \tag{S.M.31}$$

So we only need to solve for quantities Y_i^* and C_i^*. To get these, first, recall that

$$P_i C_i = \beta_i(\mathbf{P}) \times PC = \beta_i(\mathbf{P}^*/\mathbf{w}) \times M \tag{S.M.32}$$

where

$$\beta_i(\mathbf{P}) \equiv \frac{\partial \ln(\mathscr{E}(C;\mathbf{P}))}{\partial \ln(P_i)} \tag{S.M.33}$$

with $\mathscr{E}(C;\mathbf{P})$ is the expenditure function in the stationary equilibrium, which is fully pinned down by the shape of the aggregator function Φ. Thus,

$$C_i^* = \frac{\beta_i(\mathbf{P}^*)M}{P_i^*}, \quad \forall i \in [n] \tag{S.M.34}$$

[50]For instance, in a Cobb-Douglas, it is straightforward to verify that such an $\epsilon > 0$ exists as long as all sectors have a positive labor share, which follows from the Inada conditions.

Finally, note that

$$Y_i^* = \frac{M\lambda_i^*}{P_i^*} \tag{S.M.35}$$

where λ_i^* is the Domar weight of sector i in the steady state. Note that, by Equation (S.M.20), these Domar weights are given by the vector of prices as

$$(\lambda_i^*)_{i\in[n]} = \boldsymbol{\lambda} = (\mathbf{I} - \mathbf{A}(\mathbf{P}^*)^\intercal)^{-1}\boldsymbol{\beta}(\mathbf{P}^*) \tag{S.M.36}$$

and

$$C^* = W^*/P^* = M^*/P^*, \quad P^* = \mathscr{E}(1;\mathbf{P}^*) \tag{S.M.37}$$

Finally, other variables of the model are implied by these prices and quantities:

$$i^* = \rho, \quad L^* = P^*C^*/M = 1 \tag{S.M.38}$$

where the second equation follows from Equation (S.M.26) evaluated in the steady state.

S.M.1.6. Log-linearization

Let small letters denote the log deviations of their corresponding variables from their stationary equilibrium values. That is, $x_t \equiv \ln(X_t/X^*)$.

Desired Prices. Taking the FOC for desired prices in Equation (8), we obtain:

$$P_{i,t}^* = \frac{\sigma_i}{\sigma_i - 1}\frac{1}{1-\tau_{i,t}}\mathsf{MC}_{i,t} \tag{S.M.39}$$

Letting $\omega_{i,t} \equiv \ln(\frac{\sigma_i}{\sigma_i-1}\frac{1}{1-\tau_{i,t}})$, first note that the value of $\omega_{i,t}$ in the efficient steady state is 0, and second, we have

$$p_{i,t}^* = \omega_{i,t} + mc_{i,t} \tag{S.M.40}$$

Marginal Cost. Recall from Equation (S.M.8) that the marginal cost of a firm in sector i is equal to their average cost due to constant returns to scale and is defined by their cost minimization problem

$$\mathsf{MC}_i(\mathbf{P}_t, Z_{i,t}) = \min_{L_{jk,t},(X_{ij,k,t})_{k\in[n]}} W_t L_{ij,t} + \sum_{k\in[n]} P_{k,t}X_{ij,k,t} \tag{S.M.41}$$

$$\text{subject to} \quad Z_{i,t}F_i(L_{ij,t},(X_{ij,k,t})_{k\in[n]}) \geq 1 \tag{S.M.42}$$

which also holds in the efficient steady state. Now, log-linearizing this equation around the efficient steady state, we have:

$$mc_{i,t} \approx \frac{\partial \ln(\mathsf{MC}_i^*)}{\partial \ln(W^*)}w_t + \sum_{k\in[n]}\frac{\partial \ln(\mathsf{MC}_i^*)}{\partial \ln(P_k^*)}p_{k,t} - z_{i,t} \tag{S.M.43}$$

$$= \alpha_i w_t + \sum_{k\in[n]} a_{ik} p_{k,t} - z_{i,t}, \quad \forall i \in [n] \tag{S.M.44}$$

where $(\alpha_i, a_{ik})_{k\in[n]}$ in the second line are the elasticities of marginal cost with respect to wage and prices in the steady state, respectively. Applying the envelope theorem to the cost minimization problem (Shephard's Lemma), we can see that α_i is the labor share of firms in sector i and a_{ik} is their expenditure share on intermediate input k, under steady state prices. Finally, note that under Golosov and Lucas (2007) preferences $w_t = p_t + c_t = m_t$ so that

$$mc_{i,t} = \alpha_i m_t + \sum_{k\in[n]} a_{ik} p_{k,t} - z_{i,t}, \quad \forall i \in [n] \tag{S.M.45}$$

Reset Prices. Consider the derivation of optimal reset prices in Equation (S.M.15) and let

$$\Xi_{i,t,h} \equiv \frac{e^{-(\theta_i h + \int_0^h i_{t+s} \mathrm{d}s)} Y_{i,t+h} P_{i,t+h}^{\sigma_i}}{\int_0^\infty e^{-(\theta_i h + \int_0^h i_{t+s} \mathrm{d}s)} Y_{i,t+h} P_{i,t+h}^{\sigma_i} \mathrm{d}h} \tag{S.M.46}$$

Note that at any given t and i, by definition, $\int_0^\infty \Xi_{i,t,h} \mathrm{d}i = 1$. Moreover, given this notation, we can re-write Equation (S.M.15) as

$$P_{ij,t}^{\#} = P_{i,t}^{\#} = \int_0^\infty \Xi_{i,t,h} P_{i,t+h}^{*} \mathrm{d}h \tag{S.M.47}$$

Log-linearizing this, we obtain that (up to first order deviations):

$$p_{i,t}^{\#} \approx \int_0^\infty \Xi_{i,h}^{*} p_{i,t+h}^{*} \mathrm{d}h + \int_0^\infty (\Xi_{i,t,h} - \Xi_{i,h}^{*}) \mathrm{d}h \tag{S.M.48}$$

but note that the second integral is zero because both $\Xi_{i,t,h}$ and $\Xi_{i,h}^{*}$ integrate to 1. Moreover, note that the value of $\Xi_{i,t,h}$ in the steady state is given by

$$\Xi_{i,h}^{*} = \frac{e^{-(\theta_i + i^*)h} Y_i^* P_i^{*\sigma_i}}{\int_0^\infty e^{-(\theta_i + i^*)h} Y_i^* P_i^{*\sigma_i} \mathrm{d}h} = (\theta_i + \rho) e^{-(\theta_i + \rho)h} \tag{S.M.49}$$

where we have used the fact that $i^* = \rho$ in the efficient steady state. Hence, we have that

$$p_{i,t}^{\#} \approx (\theta_i + \rho) \int_0^\infty e^{-(\rho + \theta_i)h} p_{i,t+h}^{*} \mathrm{d}h \tag{S.M.50}$$

Aggregate Sectoral Prices. Recall from Equation (S.M.16) that the aggregate sectoral price of sector i is the generalized mean of past reset prices and the initial sectoral price at time 0^-, weighted by the density of time between price changes:

$$P_{i,t} = \left[\int_0^t \theta_i e^{-\theta_i h} (P_{i,t-h}^{\#})^{1-\sigma_i} \mathrm{d}h + e^{-\theta_i t} P_{i,0^-}^{1-\sigma_i} \right]^{\frac{1}{1-\sigma_i}}, \quad \forall i \in [n] \tag{S.M.51}$$

Log-linearizing this gives:

$$p_{i,t} \approx \theta_i \int_0^t e^{-\theta_i h} p^{\#}_{i,t-h} \mathrm{d}h + e^{-\theta_i t} p_{i,0^-} \tag{S.M.52}$$

Consumer Price Index. Recall from Equation (S.M.1) that the consumer price index, denoted by P_t, is given by:

$$P_t = \mathscr{E}(1, \mathbf{P}_t) \tag{S.M.53}$$

where $\mathscr{E}$ is the expenditure function and $\mathbf{P}_t$ is the vector of sectoral prices at time t. Log-linearizing this gives:

$$p_t \approx \sum_{i\in[n]} \frac{\partial \ln(\mathscr{E}(1,\mathbf{P}^*))}{\partial \ln(P_i^*)} p_{i,t} = \sum_{i\in[n]} \beta_i p_{i,t} \tag{S.M.54}$$

where β_i after the second equality is the elasticity of the price index with respect to the price of sector i. Applying Shephard's Lemma, we obtain that β_i is the consumption expenditure share of the household on the final good of sector i.

Aggregate GDP and GDP gap. Under our benchmark where monetary policy directly sets the aggregate nominal GDP, in log deviations, aggregate real GDP is simply the difference between log nominal GDP and log aggregate price:

$$y_t = m_t - p_t \tag{S.M.55}$$

Moreover, if the nominal aggregate GDP set by monetary policy is the same across the flexible and sticky price economies (e.g. when monetary policy does not respond to endogenous prices or quantities), then the output gap is given by the nominal CPI gap:

$$m_t = p_t + y_t = p_t^f + y_t^f \implies \tilde{y}_t = y_t - y_t^f = p_t^f - p_t, \qquad p_t^f = m_t + \boldsymbol{\lambda}^\intercal(\boldsymbol{\omega}_t - \boldsymbol{z}_t) \tag{S.M.56}$$

where the expression for p_t^f is coming from Equation (14). Given these, the interest rates are then determined passively as a function of these allocations from the Euler equation. In particular, with Golosov and Lucas (2007) preferences and a fixed nominal GDP over time, interest rates are simply equal to ρ on the equilibrium path.

Beyond our benchmark economy, however, e.g. when preferences deviate from Golosov and Lucas (2007) or monetary policy is endogenous like in the case of a Taylor rule, the GDP and sectoral prices are jointly determined by the endogenous monetary policy and the log-linearized Euler equation of the household. We derive the expressions for these cases in our extensions in Appendix S.M.2.1 and Appendix S.M.2.2.

The Labor Share Equation and the Aggregate Production Function. Recall from Equation (S.M.26) that the aggregate labor share of this economy is given by:

$$\frac{W_t L_t}{P_t C_t} = \boldsymbol{\alpha}_t^\top \mathcal{M}_t^{-1} \boldsymbol{\lambda}_t \tag{S.M.57}$$

$$= \mathbf{1}^\top (\mathbf{I} - \mathbf{A}_t^\top) \mathcal{M}_t^{-1} (\mathbf{I} - \mathbf{A}_t^\top \mathcal{M}_t^{-1})^{-1} \boldsymbol{\beta}_t \tag{S.M.58}$$

$$= \mathbf{1}^\top (\mathbf{I} + (\mathcal{M}_t - \mathbf{I}) \boldsymbol{\Psi}_t^\top)^{-1} \boldsymbol{\beta}_t \tag{S.M.59}$$

where $\boldsymbol{\Psi}_t \equiv (\mathbf{I} - \mathbf{A}_t)^{-1}$ is the inverse Leontief matrix, $\mathcal{M}_t \equiv \mathrm{diag}({}^{\mu_{i,t}}/_{\Delta_{i,t}})$ is the diagonal matrix whose i'th diagonal entry is the price dispersion adjusted markup wedge of sector i. First note that the value of the labor share in the efficient steady state is 1 because in this steady state $\mathcal{M} = \mathbf{I}$ (net markups are zero and there is no price dispersion), so:

$$\frac{W^* L^*}{P^* C^*} = \mathbf{1}^\top \boldsymbol{\beta} = 1 \tag{S.M.60}$$

where the second equality follows from the fact that expenditure shares in $\boldsymbol{\beta}$ sum to 1. Moreover, note that:

$$\mathcal{M}_t - \mathbf{I} = \mathrm{diag}(p_{i,t} - mc_{i,t}) - \mathrm{diag}(\Delta_{i,t}) + \mathcal{O}\left(\|\mathcal{M}_t - \mathbf{I}\|^2\right) \tag{S.M.61}$$

But note that $mc_{i,t}$, up to first-order, is itself a function of prices as we showed above. Moreover, it is straightforward to verify that price dispersion $\Delta_{i,t}$ is of second order in prices in sector i (see, e.g., Gali, 2008, p. 63). Thus, letting $\hat{\mathbf{p}} \equiv (p_{ij,t})_{i\in[n], j\in[0,1]}$, we obtain:

$$\mathcal{M}_t - \mathbf{I} = \mathrm{diag}(p_{i,t} - mc_{i,t}) + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.62}$$

Therefore, noting that $\boldsymbol{\Psi}_t$ is also changing over time only as a function of prices (Since the input-output matrix is determined by prices as in Equation (S.M.11)):

$$(\mathbf{I} + (\mathcal{M}_t - \mathbf{I})\boldsymbol{\Psi}_t^\top)^{-1} = \mathbf{I} - \mathrm{diag}(p_{i,t} - mc_{i,t})\boldsymbol{\Psi}^\top + \underbrace{(\mathcal{M} - I)}_{=0}(\boldsymbol{\Psi}_t^\top - \boldsymbol{\Psi}^\top) + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.63}$$

Hence, noting also that $\boldsymbol{\beta}_t$ only depends on time through prices, we obtain:

$$\mathbf{1}^\top (\mathbf{I} + (\mathcal{M}_t - \mathbf{I})\boldsymbol{\Psi}_t^\top)^{-1} \boldsymbol{\beta}_t = 1 - \boldsymbol{\beta}^\top \boldsymbol{\Psi}(\mathbf{p}_t - \mathbf{mc}_t) + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.64}$$

$$= 1 - \boldsymbol{\beta}^\top \boldsymbol{\Psi}(\mathbf{I} - \mathbf{A})(\mathbf{p}_t - w_t \mathbf{1}) - \boldsymbol{\beta}^\top \boldsymbol{\Psi} \boldsymbol{z}_t + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.65}$$

$$= 1 - p_t + w_t - \boldsymbol{\lambda}^\top \boldsymbol{z}_t + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.66}$$

where $\boldsymbol{\lambda}$ is the vector of Domar weights in the efficient steady state. Finally. note that in log deviations the labor share equation can be written as

$$w_t + l_t - p_t - c_t = \ln(1 - p_t + w_t - \boldsymbol{\lambda}^\top \boldsymbol{z}_t + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right)) \tag{S.M.67}$$

$$= -p_t + w_t - \boldsymbol{\lambda}^\intercal \boldsymbol{z}_t + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.68}$$

Therefore, we obtain the following log-linear aggregate production function:

$$c_t = \boldsymbol{\lambda}^\intercal \boldsymbol{z}_t + l_t + \mathcal{O}\left(\|\hat{\mathbf{p}}\|^2\right) \tag{S.M.69}$$

S.M.2 Derivations for Extensions

S.M.2.1. Derivations for Finite Frisch Elasticity

For general labor supply elasticity, let $U(C) = \ln C$ and $V(L) = \frac{L^{1+\psi}}{1+\psi}$. Under these preferences, the agent's intra-temporal first-order condition becomes

$$\frac{W_t}{P_t} = C_t L_t^{\psi} \tag{S.M.70}$$

With its log-linearized version being

$$w_t - p_t = c_t + \psi l_t \tag{S.M.71}$$

Using $m_t = p_t + c_t$ aggregate GDP, we get

$$w_t = m_t + \psi l_t \tag{S.M.72}$$

Since, in this benchmark, m_t is exogenous and the same across both flexible and sticky economies, doing the same at the flexible price equilibrium, and taking differences we have

$$(w_t - w_t^f) - (p_t - p_t^f) = (c_t - c_t^f) + \psi(l_t - l_t^f) \tag{S.M.73}$$

$$w_t - w_t^f = \psi(l_t - l_t^f) \tag{S.M.74}$$

Thus,

$$m_t = p_t + c_t = p_t^f + c_t^f \implies c_t - c_t^f = -(p_t - p_t^f) \tag{S.M.75}$$

Moreover, from the aggregate production function in Equation (S.M.69) we have that up to first order $c_t = \boldsymbol{\lambda}^\intercal \boldsymbol{z}_t + l_t$, which implies that

$$c_t - c_t^f = l_t - l_t^f \tag{S.M.76}$$

Combining the last three equations, we have:

$$w_t - w_t^f = \psi(l_t - l_t^f) = \psi(c_t - c_t^f) = -\psi(p_t - p_t^f) \tag{S.M.77}$$

Now, consider the equation for the desired prices, while adding and subtracting $(\mathbf{I}-\mathbf{A})\mathbf{1}w_t^f$. We obtain:

$$\mathbf{p}_t^* = (\mathbf{I}-\mathbf{A})\mathbf{1}w_t - (\mathbf{I}-\mathbf{A})\mathbf{1}w_t^f + (\mathbf{I}-\mathbf{A})\mathbf{1}w_t^f + \mathbf{A}\mathbf{p}_t - \boldsymbol{z}_t + \boldsymbol{\omega}_t \tag{S.M.78}$$

First, recall that the flexible price equilibrium is given by this equation when $\mathbf{p}_t^* = \mathbf{p}_t = \mathbf{p}_t^f$:

$$\begin{aligned}\mathbf{p}_t^f &= (\mathbf{I}-\mathbf{A})\mathbf{1}w_t^f + \mathbf{A}\mathbf{p}_t^f - \boldsymbol{z}_t + \boldsymbol{\omega}_t && \text{(S.M.79)}\\ &= w_t^f\mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{z}_t - \boldsymbol{\omega}_t) && \text{(S.M.80)}\end{aligned}$$

now multiplying by $\boldsymbol{\beta}^\top$ we have:

$$\begin{aligned} & p_t^f = w_t^f - \boldsymbol{\lambda}^\top(\boldsymbol{z}_t - \boldsymbol{\omega}_t) = m_t + \psi l_t^f - \boldsymbol{\lambda}^\top(\boldsymbol{z}_t - \boldsymbol{\omega}_t) && \text{(S.M.81)}\\ \Longrightarrow\ & c_t^f = m_t - p_t^f = -\psi l_t^f + \boldsymbol{\lambda}^\top(\boldsymbol{z}_t - \boldsymbol{\omega}_t) && \text{(S.M.82)}\\ \Longrightarrow\ & \boldsymbol{\lambda}^\top\boldsymbol{z}_t + l_t^f = -\psi l_t^f + \boldsymbol{\lambda}^\top(\boldsymbol{z}_t - \boldsymbol{\omega}_t) && \text{(S.M.83)}\\ \Longrightarrow\ & l_t^f = -\frac{1}{1+\psi}\boldsymbol{\lambda}^\top\boldsymbol{\omega}_t && \text{(S.M.84)}\\ \Longrightarrow\ & w_t^f = m_t + \psi l_t^f = m_t - \frac{\psi}{1+\psi}\boldsymbol{\lambda}^\top\boldsymbol{\omega}_t && \text{(S.M.85)}\end{aligned}$$

so that

$$\mathbf{p}_t^f = w_t^f\mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{z}_t - \boldsymbol{\omega}_t) = m_t\mathbf{1} - \boldsymbol{\Psi}\boldsymbol{z}_t + (\boldsymbol{\Psi} - \frac{\psi}{1+\psi}\mathbf{1}\boldsymbol{\lambda}^\top)\boldsymbol{\omega}_t \tag{S.M.86}$$

Moreover, using Equation (S.M.78) and Equation (S.M.77), we can re-write Equation (S.M.80) as:

$$\begin{aligned}\mathbf{p}_t^* - \mathbf{p}_t &= (\mathbf{I}-\mathbf{A})\left((w_t - w_t^f)\mathbf{1} + w_t^f\mathbf{1} - \boldsymbol{\Psi}(\boldsymbol{z}_t - \boldsymbol{\omega}_t) - \mathbf{p}_t\right) && \text{(S.M.87)}\\ &= (\mathbf{I}-\mathbf{A})(\mathbf{I} + \psi\mathbf{1}\boldsymbol{\beta}^\top)(\mathbf{p}_t^f - \mathbf{p}_t) && \text{(S.M.88)}\end{aligned}$$

Now, recall from Equations (17) and (18) that $\boldsymbol{\pi}_t = \dot{\mathbf{p}}_t = \boldsymbol{\Theta}(\mathbf{p}_t^\# - \mathbf{p}_t)$ and $\boldsymbol{\pi}_t^\# = (\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^\# - \mathbf{p}_t^*)$, which still hold in this economy because they are implied by firm side optimality of aggregation conditions. Differentiating Equation (18) with respect to time and using and Equation (S.M.87) we obtain:

$$\begin{aligned}\dot{\boldsymbol{\pi}}_t = \ddot{\mathbf{p}}_t &= \boldsymbol{\Theta}(\boldsymbol{\pi}_t^\# - \boldsymbol{\pi}_t) = \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^\# - \mathbf{p}_t^*) - \boldsymbol{\Theta}\boldsymbol{\pi}_t \\ &= \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t - \mathbf{p}_t^*) + \underbrace{\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^\# - \mathbf{p}_t) - \boldsymbol{\Theta}\boldsymbol{\pi}_t}_{=\rho\boldsymbol{\pi}_t \text{ by Equation (18)}} && \text{(S.M.89)}\\ &= \rho\boldsymbol{\pi}_t - \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I}-\mathbf{A})(\mathbf{I} + \psi\mathbf{1}\boldsymbol{\beta}^\top)(\mathbf{p}_t^f - \mathbf{p}_t) && \text{(S.M.90)}\end{aligned}$$

S.M.2.2. Equilibrium with Taylor Rule

Here, we consider a version of our benchmark economy where the monetary authority, instead of directly fixing the nominal GDP, sets the nominal interest rate according to a Taylor rule of the type:

$$i_t = \boldsymbol{\eta}^\top \boldsymbol{\pi}_t + v_t \tag{S.M.91}$$

where, if $\boldsymbol{\eta} = \phi_\pi \boldsymbol{\beta}$ as in the main text, the Taylor rule targets the CPI inflation, but note that the central bank can target any weighted sum of sectoral inflation rates (e.g. ignoring energy and food prices, which leads to targeting core inflation). Moreover, v_t constitutes deviations from the rule, with its path over time given under perfect foresight. Usually, in the monetary literature, v_t is assumed to be an AR(1) process under perfect foresight.

It is important to note that the main difference between this economy and our benchmark is that, here, the implied nominal GDP is endogenous (as opposed to being exogenous and the same across the flexible and sticky economies), and the Taylor rule creates feedback between prices and nominal GDP. In other words, in our benchmark economy, monetary policy exogenously determined the size of nominal GDP and the supply side frictions determined its divide between nominal prices and quantities. But now, the size of the nominal GDP itself depends on these frictions. Note, however, that under Golosov and Lucas (2007) preferences, the intratemporal condition $w_t = m_t$ still holds, but now $m_t \neq m_t^f$. Moreover, with these preferences, the intertemporal Euler equation is $\dot{y}_t = i_t - \pi_t$ or, moving things around, $i_t = \dot{m}_t = \pi_t + \dot{y}_t$. Combining the Euler equation and the Taylor rule above, we arrive at:

$$i_t = \dot{m}_t = \boldsymbol{\eta}^\top \boldsymbol{\pi}_t + v_t \tag{S.M.92}$$

Notice how the *level* of nominal GDP is no longer pinned down when monetary policy targets inflation through a Taylor rule. Instead, only the *growth* rate of nominal GDP is determined by the path of interest rates. This raises the usual issues with determinacy that we discuss below.

To solve for inflation rates in this economy, we start from Equations (17) and (18), which still hold in the Taylor rule economy because they purely depend on firm side optimization and aggregation:

$$\boldsymbol{\pi}_t = \boldsymbol{\Theta}(\mathbf{p}_t^\# - \mathbf{p}_t), \qquad \boldsymbol{\pi}_t^\# = (\rho \mathbf{I} + \boldsymbol{\Theta})(\mathbf{p}_t^\# - \mathbf{p}_t^*)$$

where

$$\mathbf{p}_t^* = (\mathbf{I}-\mathbf{A})\mathbf{1}m_t + \mathbf{A}\mathbf{p}_t - \boldsymbol{z}_t + \boldsymbol{\omega}_t \tag{S.M.93}$$

Differentiating this last equation with respect to time and using Equation (S.M.92), we obtain:

$$\dot{\mathbf{p}}_t^* - \boldsymbol{\pi}_t = (\mathbf{I}-\mathbf{A})\mathbf{1}\dot{m}_t - (\mathbf{I}-\mathbf{A})\boldsymbol{\pi}_t - \dot{\boldsymbol{z}}_t + \dot{\boldsymbol{\omega}}_t = (\mathbf{I}-\mathbf{A})(\mathbf{1}\boldsymbol{\eta}^\intercal - \mathbf{I})\boldsymbol{\pi}_t - \dot{\boldsymbol{z}}_t + \dot{\boldsymbol{\omega}}_t + (\mathbf{I}-\mathbf{A})\mathbf{1}\upsilon_t \tag{S.M.94}$$

Note that again, we can define the flexible economy inflation rate as the rate that arises when $\boldsymbol{\pi}_t^* = \boldsymbol{\pi}_t = \boldsymbol{\pi}_t^f$, which gives:

$$(\mathbf{I} - \mathbf{1}\boldsymbol{\eta}^\intercal)\boldsymbol{\pi}_t^f = \mathbf{1}\upsilon_t - \boldsymbol{\Psi}(\dot{\boldsymbol{z}}_t - \dot{\boldsymbol{\omega}}_t) \tag{S.M.95}$$

Note that all the terms in this expression are exogenous, which implies that $\boldsymbol{\pi}_t^f$ is exogenous. Combined with Equation (S.M.94), this gives:

$$\dot{\mathbf{p}}_t^* - \boldsymbol{\pi}_t = (\mathbf{I}-\mathbf{A})(\mathbf{I} - \mathbf{1}\boldsymbol{\eta}^\intercal)(\boldsymbol{\pi}_t^f - \boldsymbol{\pi}_t) \tag{S.M.96}$$

Now, differentiate Equation (18) with respect to time twice, Equation (17) once, and use Equation (S.M.96) to get

$$\ddot{\boldsymbol{\pi}}_t = \boldsymbol{\Theta}(\dot{\boldsymbol{\pi}}_t^\# - \dot{\boldsymbol{\pi}}_t) = \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\boldsymbol{\pi}_t^\# - \dot{\mathbf{p}}_t^*) - \boldsymbol{\Theta}\dot{\boldsymbol{\pi}}_t \tag{S.M.97}$$

$$= -\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\dot{\mathbf{p}}_t^* - \boldsymbol{\pi}_t) \underbrace{+\boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\boldsymbol{\pi}_t^\# - \boldsymbol{\pi}_t) - \boldsymbol{\Theta}\dot{\boldsymbol{\pi}}_t}_{=\rho\dot{\boldsymbol{\pi}}_t} \tag{S.M.98}$$

$$\Longrightarrow \ddot{\boldsymbol{\pi}}_t = \rho\dot{\boldsymbol{\pi}}_t + \boldsymbol{\Theta}(\rho\mathbf{I} + \boldsymbol{\Theta})(\mathbf{I}-\mathbf{A})(\mathbf{I} - \mathbf{1}\boldsymbol{\eta}^\intercal)(\boldsymbol{\pi}_t - \boldsymbol{\pi}_t^f) \tag{S.M.99}$$

Note that Equation (S.M.99) is a system of second-order differential equations in the vector of sectoral inflation rate, $\boldsymbol{\pi}_t$, with $\boldsymbol{\pi}_t^f$ acting as an exogenous force term to the system. Thus, the equilibrium is a solution to this system of differential equations. But note that by introducing the Taylor rule, we have to discuss a new set of boundary conditions for the system. In particular, we need to characterize what determinacy requires in this system (recall that even in the one-sector NK model, the solution to the model can be indeterminate—i.e. the system can have multiple non-explosive equilibria—if the Taylor principle is not satisfied).

To obtain a particular solution to the system of differential equations above, we need $2n$ boundary conditions. Of those, n of them are given by the non-explosiveness of the solution as before. Moreover, of all the non-explosive solutions, the equilibrium requires that the solution be such that relative prices go back to their steady-state values; i.e.,

$$\lim_{t\to\infty} p_{i,t} - p_{i,0^-} = p_{j,t} - p_{j,0^-}, \quad \forall i \in [n], \forall j \neq i, j \in [n] \tag{S.M.100}$$

which defines another $n-1$ set of boundary conditions. Therefore, non-explosive prices plus the requirement that relative prices go back to their steady state values gives us $2n-1$ boundary conditions. The last boundary condition is given by the extension of the Taylor principle to this network economy. This essentially requires that the matrix $\Gamma_\eta \equiv \mathbf{\Theta}(\rho\mathbf{I}+\mathbf{\Theta})(\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{1}\boldsymbol{\eta}^\top)$ has a negative eigenvalue so that one cannot construct more than one non-explosive solution that converges back to the same steady state. Notice that in one sector economies, where η is scalar,

$$\mathbf{\Gamma}_\eta < 0 \iff \eta > 1 \tag{S.M.101}$$

which is exactly the Taylor principle. Take, for instance, the case where $\pi_t^f = 0$ in such an economy. Then, with $\eta > 1$, the only non-explosive solution that converges to the steady state is $\pi_t = 0$ (unless we accept oscillatory solutions, but they do not converge back to the steady state). But with $0 < \eta < 1$, we have a continuum of non-explosive convergent solutions $\pi_t = C_0 e^{-\eta t}$ for any $C_0 \in \mathbb{R}$, which is the same as indeterminacy.

As for the general system above, given these boundary conditions, we can solve for the equilibrium given any path of shocks. Specifically, in our numerical exercises, when we solve this model, we numerically verify that $\mathbf{\Gamma}_\eta$ has exactly one negative eigenvalue and proceed to solve for the equilibrium using Schur decomposition and imposing the relevant boundary conditions.

S.M.3 Data Appendix

Propositions 2 to 4 show that the sufficient statistics for inflation and output dynamics in response to shocks in our model are the duration-adjusted Leontief matrix, $\mathbf{\Gamma} = \mathbf{\Theta}^2(\mathbf{I}-\mathbf{A})$, and the consumption expenditure shares across sectors, given by $\boldsymbol{\beta}$. We now describe in detail how we construct $\mathbf{\Gamma}$ and $\boldsymbol{\beta}$ using detailed sectoral US data.

First, we use the input-output (IO) tables from the BEA to construct the input-output linkages across sectors,[51] given by the matrix $\mathbf{A}$; the consumption expenditure shares across sectors, given by the vector $\boldsymbol{\beta}$; and the sectoral labor shares, given by the vector $\boldsymbol{\alpha}$. In particular, to construct $\mathbf{A}$, we use both the "make" and "use" IO tables.[52] The "use" IO

[51] We construct industry-by-industry IO tables. We use industry and sector interchangeably.

[52] The "make" table is a matrix of industries on the rows and commodities on the columns that gives the value of each commodity on the column produced by the industry on the rows. The "use" table is a matrix of

table also provides data on the compensation of employees, which is used to construct the sectoral labor shares $\boldsymbol{\alpha}$. Moreover, the "use" IO table also provides data on personal consumption expenditure, which is used to construct the consumption expenditure shares across sectors, $\boldsymbol{\beta}$. Figure S.M.2 presents the matrix $\mathbf{A}$ we construct from the data, in a heat-map version.

Next, we construct the diagonal matrix $\boldsymbol{\Theta}^2$, whose diagonal elements are the squared frequency of price adjustment in each sector, using data on 341 sectors from Pasten, Schoenle, and Weber (2020). First, we match data from Pasten, Schoenle, and Weber (2020) on the frequency of price changes with the 2002 concordance table between IO industry codes and the 2002 NAICS codes. Then, we match these codes with the 2012 concordance table between IO industry codes and 2012 NAICS codes. The last step is performed in order to get the frequency of price adjustment data for sectors in the 2012 IO table.

S.M.3.1. Constructing the Input-Output Matrix

In this subsection, we describe how we use the "Make" and "Use" matrices to get the cost-based industry-by-industry input-output table. Specifically, we use the 2012 "Make" table after redefinitions and the 2012 "Use" table after redefinitions in producers' value.

Recall that the "Make" table is a matrix of Industry-by-Commodity. Given a row, each column shows the values of each commodity produced by the industry in the row. The "Use" table is a matrix of Commodity-by-Industry. Given a column, each row shows the value of each commodity used by the industry (or final use) in the column. In order to create an industry-by-industry IO table, we combine both. We follow the Handbook of Input-Output Table Compilation and Analysis from the UN (United Nations Department of Economic and Social Affairs, 1999) and Concepts and Methods of the United States Input-Output Accounts from the BEA (Horowitz and Planting, 2009). We exclude the government sector, Scraps, Used and secondhand goods, Noncomparable imports, and Rest of the world adjustment.[53]

It is important to note that an industry can produce many commodities. Although

commodities on the rows and industries on the columns that gives the value of each commodity on the row that was used by each industry in the column. We combine both matrices to give an industry-by-industry IO matrix.

[53] Baqaee and Farhi (2020) also exclude these sectors. Besides them, we exclude Customs duties, which is an industry with zero commodity use and zero compensation of employees. After excluding these industries and commodities, we end up having 392 commodities and 393 industries. The industry that does not have a corresponding commodity with the same code is 'Secondary smelting and alloying of aluminum', with code 331314

each industry may have its own primary product,[54] an industry can produce more products in addition to its primary ones. These are shown in the "Make" table. Besides that, each industry has its own use of commodities to produce its output. This is shown in the "Use" table. As a result, there is a distinction between industries and commodities, as a given commodity can be produced by different industries while industries can produce different commodities.

In our model, we consider a log-linearization of the economy around an efficient steady state. This implies that in the steady state, the wedges are equal to zero for all sectors and the revenue-based and the cost-based input-output matrices are the same. In the data, these are not the same and we need to take into account the wedge between revenue and cost when calculating the object of interest in our model - the cost-based input-output matrix.

Input-Output Matrix (A) and Labor Shares ($\boldsymbol{\alpha}$). From the "Use" table from the BEA, a given column j gives:

$$\begin{aligned}\text{Total Industry Output}_j =& \text{Total Intermediate}_j \\ &+\text{Compensation of Employees}_j \\ &+\text{Taxes on production and imports, less subsidies}_j \\ &+\text{Gross operating surplus}_j\end{aligned}$$

where Total Intermediate$_j$ is the sum of the dollar amount of each commodity used by industry j. The total cost is given by

$$\text{Total Industry Cost}_j = \text{Total Intermediate}_j + \text{Compensation of Employees}_j$$

Therefore,

$$\underbrace{P_j Y_j}_{\text{Total Industry Output}} = \underbrace{(1+\omega_j)}_{\text{Wedge}} \underbrace{\left(\sum_i P_i X_{ji} + W L_j\right)}_{\text{Total Industry Cost}}$$

where we implicitly assume that the wedge is attributed to taxes and gross operating surplus. That is

$(1+\omega_j)$

[54] According to the BEA, 'each commodity is assigned the code of the industry in which the commodity is the primary product'.

$$\equiv \frac{\text{Total Intermediate}_j + \text{Compensation of Employees}_j + \text{Taxes}_j + \text{Gross Operating Surplus}_j}{\text{Total Intermediate}_j + \text{Compensation of Employees}_j} \tag{S.M.102}$$

Let $\mathrm{diag}(1+\omega)$ be the diagonal matrix in which each j-th diagonal is the wedge in industry j. We calculate the cost-based IO matrix by first calculating the revenue-based IO matrix and then, using these wedges, recovering the cost-based IO matrix. First, we calculate the revenue-based IO matrix. Let $\mathbf{U}_{(N_C+1)\times N_I}$ be the "Use" matrix (commodity-by-industry) that gives for each cell u_{ij} the dollar value of commodity i used in the production of industry j and in the last row the compensation of employees. Let $\mathbf{M}_{N_I\times N_C}$ be the "Make" matrix (industry-by-commodity) that gives for each cell m_{ij} the dollar value of commodity j produced by i. Let $\mathbf{g}_{N_I\times 1}$ be the vector of industry total output and $\mathbf{q}_{N_C\times 1}$ be the vector of commodity output, where N_C is the number of commodities and N_I is the number of industries. Then, define the following matrices

$$\mathbf{B} = \mathbf{U} \times \mathrm{diag}(\mathbf{g})^{-1} \tag{S.M.103}$$

$$\mathbf{D} = \mathbf{M} \times \mathrm{diag}(\mathbf{q})^{-1} \tag{S.M.104}$$

where $\mathrm{diag}(\mathbf{g})$ is the diagonal matrix of vector $\mathbf{g}$ and $\mathrm{diag}(\mathbf{q})$ is the diagonal matrix of vector $\mathbf{q}$. The matrix $\mathbf{D}$ is a market share matrix. Its entry d_{ij} gives the market share of industry i in the production of commodity j. The matrix $\mathbf{B}$ is a direct input matrix. Its entry b_{ij} gives the dollar amount share of commodity i in the output of industry j. Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_I^{N_C\times N_I} \\ \tilde{\boldsymbol{\alpha}}^{\intercal}_{1\times N_I} \end{bmatrix} \tag{S.M.105}$$

where $\mathbf{B}_I$ is the part of $\mathbf{B}$ that includes all intermediate inputs and industries and $\tilde{\boldsymbol{\alpha}}'$ includes the labor share in each industry's output. Then, the revenue-based industry-by-industry IO matrix is given by

$$\tilde{\mathbf{A}} = (\mathbf{D}\mathbf{B}_I)^{\intercal} \tag{S.M.106}$$

To go from the revenue-based IO matrix to the cost-based IO matrix, first recall that

$$[\tilde{\mathbf{A}}]_{ij} = \frac{P_j X_{ij}}{P_i Y_i} \tag{S.M.107}$$

where $P_j X_{ij}$ is the expenditure of industry i on industry j, $P_i Y_i$ is the revenue of the industry i. The cost-based IO matrix is given by

$$\mathbf{A} = \left[a_{ij}\right]_{i\in[n], j\in[n]}, \; a_{ij} = \frac{P_j X_{ij}}{\mathscr{C}_i} \tag{S.M.108}$$

where $\mathscr{C}_i \equiv \sum_k P_k X_{ik} + W L_i$ is the total cost of industry i. Since $P_i Y_i = (1+\omega_i)\mathscr{C}_i$, we have that

$$a_{ij} = \frac{P_j X_{ij}}{\mathscr{C}_i} = \frac{P_i Y_i}{\mathscr{C}_i}\frac{\mathscr{C}_i}{P_i Y_i}\frac{P_j X_{ij}}{\mathscr{C}_i} = (1+\omega_i)[\tilde{\mathbf{A}}]_{ij} \tag{S.M.109}$$

Hence, as in Baqaee and Farhi (2020)

$$\mathbf{A} = \mathrm{diag}(1+\omega_i)\tilde{\mathbf{A}} \tag{S.M.110}$$

To go from the revenue-based labor share to the cost-based labor share, first recall that

$$\tilde{\alpha}_i = \frac{W L_i}{P_i Y_i} \tag{S.M.111}$$

where $W L_i$ is the compensation of employees of the industry i. The cost-based labor share is given by

$$\boldsymbol{\alpha} = (\alpha_i)_{i\in[n]}, \; \alpha_i = \frac{W L_i}{\mathscr{C}_i} \tag{S.M.112}$$

Using similar arguments as above, we have that

$$\alpha_i = \frac{W L_i}{\mathscr{C}_i} = \frac{P_i Y_i}{\mathscr{C}_i}\frac{\mathscr{C}_i}{P_i Y_i}\frac{W L_i}{\mathscr{C}_i} = (1+\omega_i)\tilde{\alpha}_i \tag{S.M.113}$$

and the cost-based labor shares are given by

$$\boldsymbol{\alpha} = \mathrm{diag}(1+\omega_i)\tilde{\boldsymbol{\alpha}} \tag{S.M.114}$$

Alternatively, instead of using the vector of industry outputs $\mathbf{g}$, we can calculate the cost-based IO matrix using the vector of industry costs (total intermediate + compensation of employees) $\tilde{\mathbf{g}}$. In this case, define $\tilde{\mathbf{B}} \equiv \mathbf{U}\mathrm{diag}(\tilde{\mathbf{g}})^{-1}$, where, similar to above, $\tilde{\mathbf{B}}$ is composed of $\tilde{\mathbf{B}}_I$ that includes all intermediate inputs and industries and $\boldsymbol{\alpha}^\intercal$ which is the vector with compensation of employees for each industry. Given this decomposition, we can construct the corresponding cost-based IO matrix as $\mathbf{A} = (\mathbf{D}\tilde{\mathbf{B}}_I)^\intercal$. Note that this gives the same cost-based IO matrix as above.

Consumption Share ($\boldsymbol{\beta}$). The "Use" table gives the Personal Consumption Expenditures on each commodity. Since we are working with an industry-by-industry IO matrix, we need to calculate an industry consumption share vector. In order to do that, let C_i be the consumption dollar amount of commodity i, and $\mathbf{c}$ be the vector of the consumption dollar amount of all commodities in the economy. Then, the vector that contains the dollar equivalent consumption amount of each industry is given by **Dc**:

$$\mathbf{Dc} = \begin{bmatrix} \sum_j d_{1j} c_j \\ \sum_j d_{2j} c_j \\ \vdots \\ \sum_j d_{nj} c_j \end{bmatrix} \tag{S.M.115}$$

Recall that d_{ij} gives the market share of industry i in the production of commodity j. Therefore, $d_{ij} c_j$ is the amount in dollars spent by households on commodity j produced by i. Then, the sum over j gives the total expenditure in dollars of households on commodities produced by industry i. That is, the total expenditure in dollars of households on industry i. Then,

$$\boldsymbol{\beta} = \frac{\mathbf{Dc}}{\mathbf{1}'\mathbf{Dc}} \tag{S.M.116}$$

S.M.3.2. Constructing the Frequency of Price Adjustment Matrix

To get the 2012 detail-level industry frequency of price adjustments from the 2002 detail-level industry frequency of price adjustments, we had to manually match them. There were five cases in which industries could fall:

1. Industries with exact matching: the 2002 detail-level industry exactly correspond to the 2012 detail-level industry. In these cases, we use the 2002 detail-level industry frequency of price adjustment as the 2012 detail-level industry frequency of price adjustment. E.g.: Poultry and egg production (2002 IO Code: 112300, 2002 NAICS Code: 1123; 2012 IO Code: 112300; 2012 NAICS Code: 1123).
2. Industries with close matching: the 2002 detail-level industry closely correspond to the 2012 detail-level industry. In these cases, we use the 2002 detail-level industry frequency of price adjustment as the 2012 detail-level industry frequency of price adjustment. E.g.: In 2012 there is Metal crown, closure, and other metal stamping (except automotive) (2012 IO Code: 332119, 2012 NAICS Code: 332119). In 2002, there is Crown and closure manufacturing and metal stamping (2002 IO Code: 33211B, 2002 NAICS Code: 332115-6).
3. Industry present in 2002, but not in 2012: these are detail-level industries that were present in 2002, but not in 2012. These are 2002 industries that seem to be put into a coarser industry in 2012. We match the 2002 industries with the coarser 2012 industry. If there are more than one 2002 industry that are associated with

the coarser industry in 2012 with frequency of price adjustment data, we use their average frequency of price adjustment as the 2012 industry frequency of price adjustment. E.g.: Other crop farming (2012 IO Code: 111900, 2012 NAICS Code: 1119). In 2002, there were three industries for which we have data on frequency of price adjustment, that seem to belong to that industry: All other crop farming (2002 IO Code: 1119B0; 2002 NAICS Code: 11194, 111992, 111998), Tobacco farming (2002 IO Code: 111910, 2002 NAICS Code: 11191), Cotton farming (2002 IO Code: 111920, 2002 NAICS Code: 11192). We take the average of these industries' frequency of price adjustment and use as the Other crop farming frequency of price adjustment.

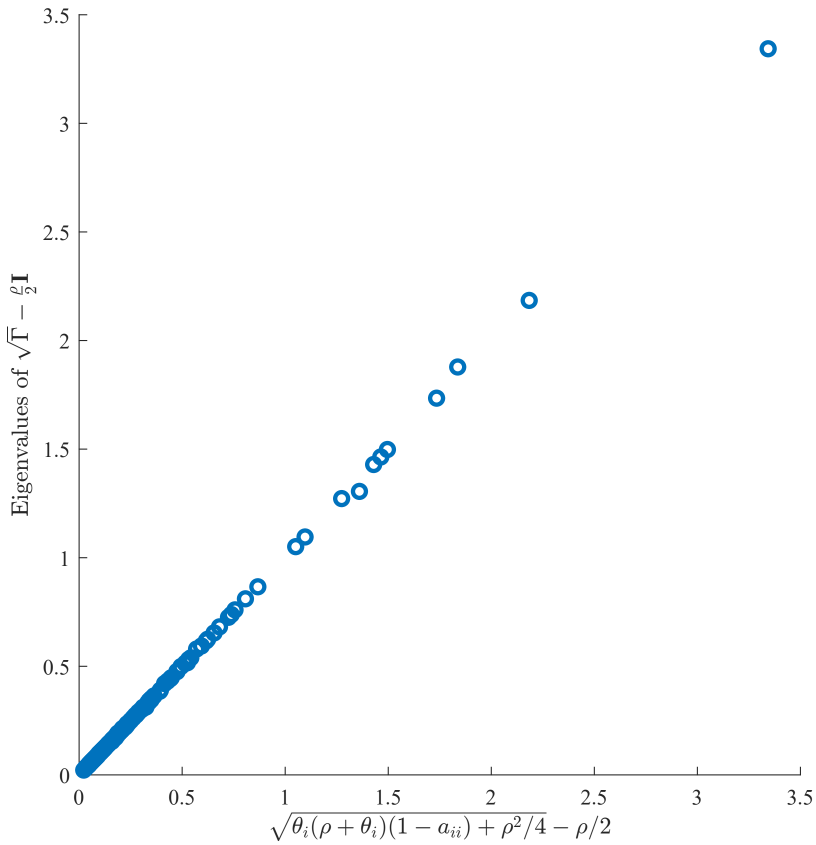
4. Industry present in 2012, but not in 2002: these are detail-level industries that were present in 2012, but not in 2002. These are industries in 2012 that seem to be put into a coarser industry in 2002. In these cases, we use the 2002 coarser industry frequency of price adjustment to impute the 2012 finer industry frequency of price adjustment. E.g.: In 2002, retail trade was a single industry (2002 IO Code: 4A0000; 2002 NAICS Code: 44, 45). In 2012, within retail trade, there were Motor vehicle and parts dealers (2012 IO Code: 441000, 2012 NAICS Code: 441), Food and beverage stores (2012 IO Code: 445000, 2012 NAICS Code: 445), General merchandise stores (2012 IO Code: 452000, 2012 NAICS Code: 452), Building material and garden equipment and supplies dealers (2012 IO Code: 444000, 2012 NAICS Code: 444), Health and personal care stores (2012 IO Code: 446000, 2012 NAICS Code: 446), Gasoline stations (2012 IO Code: 447000, 2012 NAICS Code: 447), Clothing and clothing accessories stores (2012 IO Code: 448000, 2012 NAICS Code: 448), Nonstore retailers (2012 IO Code: 454000, 2012 NAICS Code: 454), All other retail (2012 IO Code: 4B0000, 2012 NAICS Code: 442, 443, 451, 453). For all these 2012 industries, we impute their frequency of price adjustment with the 2002 Retail Trade value.
5. Industry present in 2012, but not in 2002 without correspondence: these are 2012 detail-level industries for which there was no correspondent 2002 detail-level industry. In these cases, we impute their frequency of price adjustment with the average frequency of price adjustment among industries with data. E.g.: Motion picture and video industries (2012 IO Code: 512100; 2012 NAICS Code: 5121).

For the industries in cases three, four and five, a concordance table is available upon request. The average frequency of price adjustment among sectors with data is given by 0.171. Its continuous counterpart is 0.1875. This is the value that is used to impute the sectors that are present in 2012, but not in 2002 without any correspondence in the simulations.[55] Finally, the consumption weighted average frequency of price adjustment is given by $\bar{\theta} = \sum_i \beta_i \theta_i$, where β_i is sector's i consumption share, θ_i its frequency of price adjustment. This is the value that is used for the counterfactual economy in which we set a homogeneous frequency of price adjustment.

[55] Its value is given by $-\ln(1-0.171)$

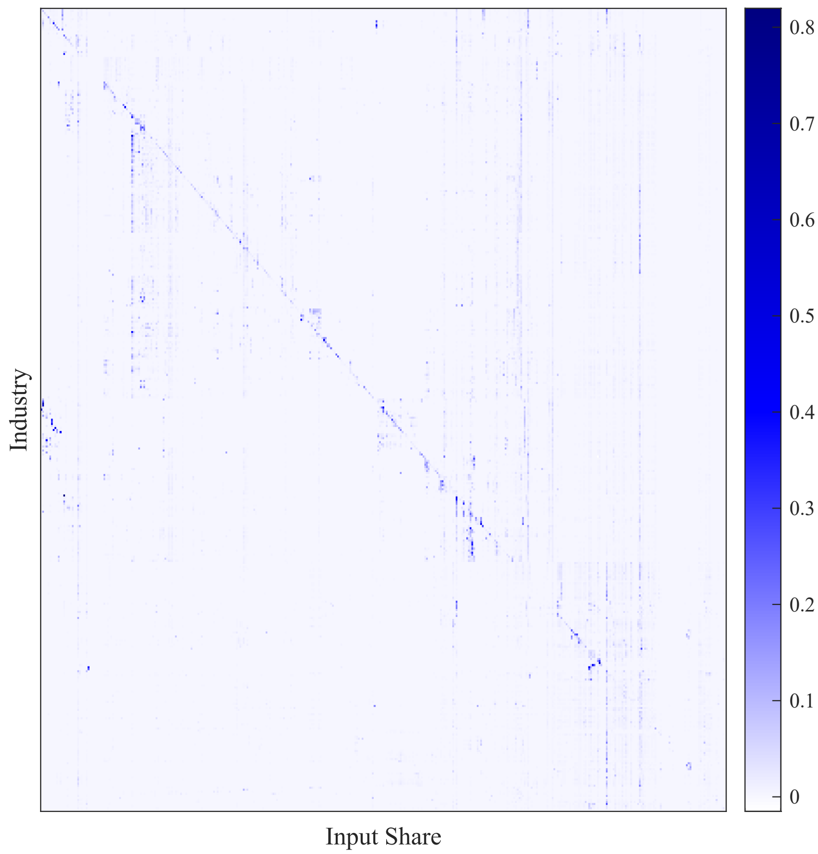
S.M.4 Additional Figures and Tables

Figure S.M.1: Relationship between exact and approximate eigenvalues



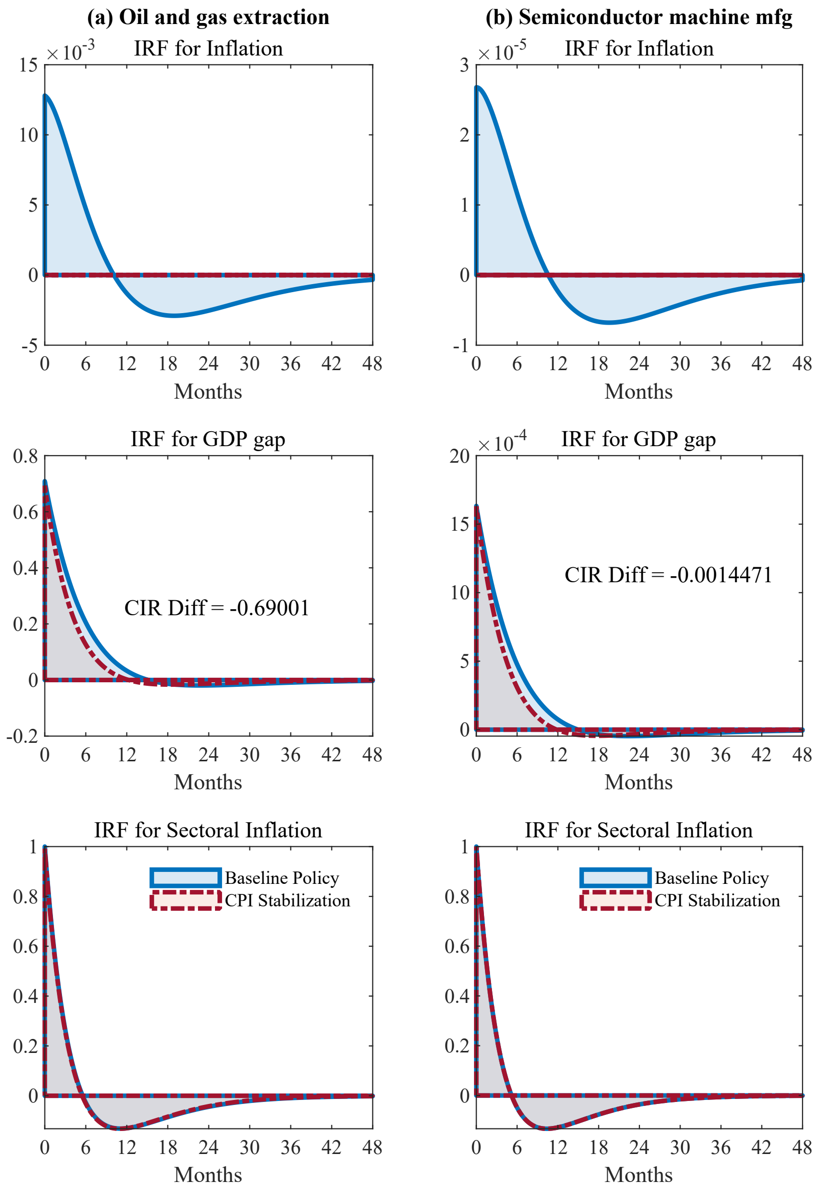
Notes: This figure plots the relationship between the eigenvalues in the diagonal economy and the eigenvalues in the baseline calibrated economy

Figure S.M.2: U.S. sectoral input-output matrix (heat map) in 2012



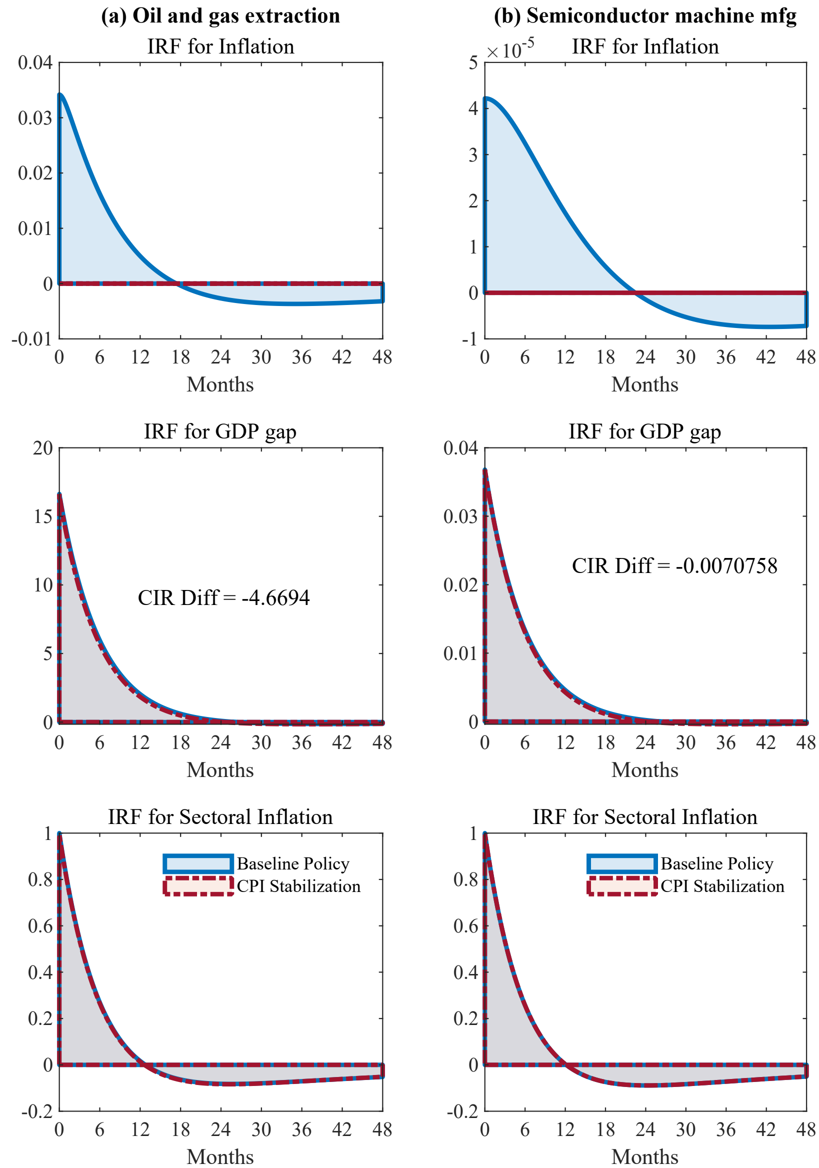
Notes: This figure presents the sectoral input-output matrix in a heat map version, using data from the make and use input-otput tables produced by the BEA in 2012. The industry classification is at the detail-level disaggregation, for a total of 393 sectors.

Figure S.M.3: Dynamics following sectoral shocks in a homogeneous frequency of price adjustment economy



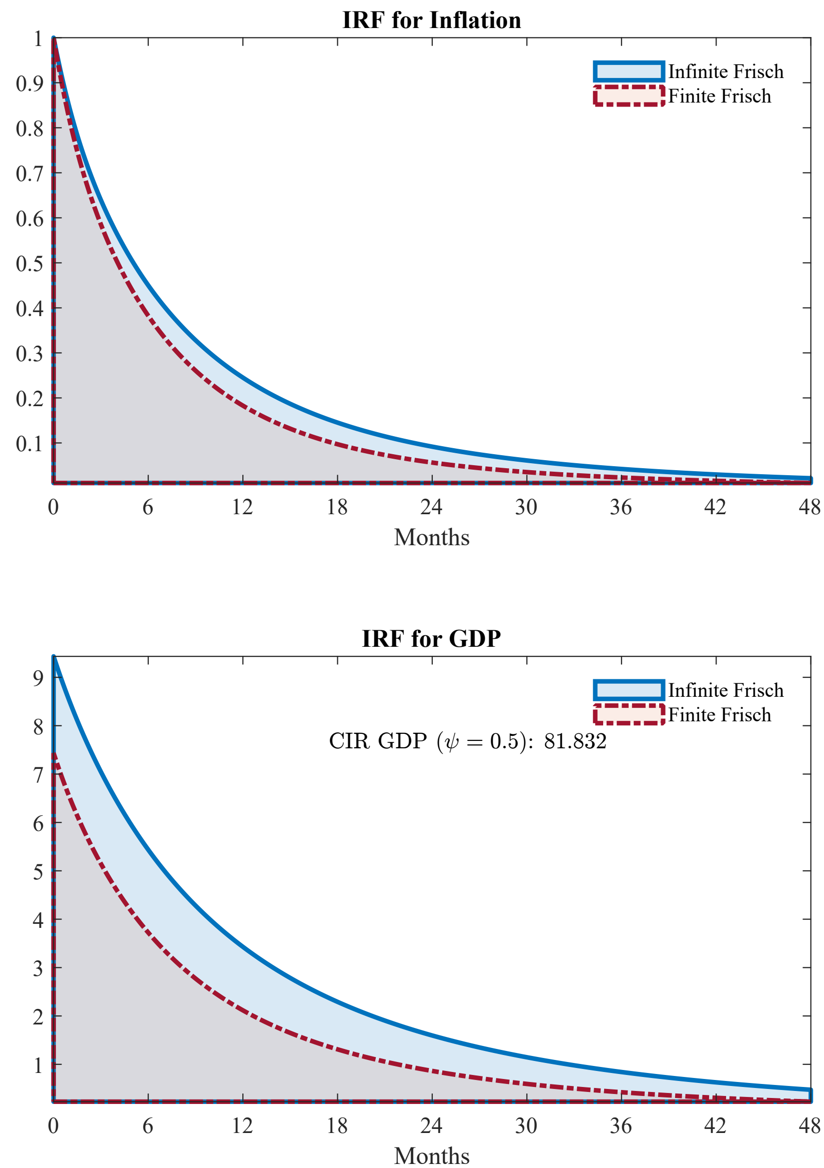
Notes: This figure plots the impulse response functions for inflation, gdp gap, and sectoral inflation to a sectoral shock that increases sectoral inflation by one percent on impact in the baseline policy economy. It compares the baseline policy economy with an economy where monetary policy stabilizes aggregate inflation. Panel A: Oil and gas extraction. Panel B: Semiconductor machine manufacturing. This calibration imposes a homogeneous frequency of price adjustment across sectors.

Figure S.M.4: Dynamics following sectoral shocks in a counterfactual economy



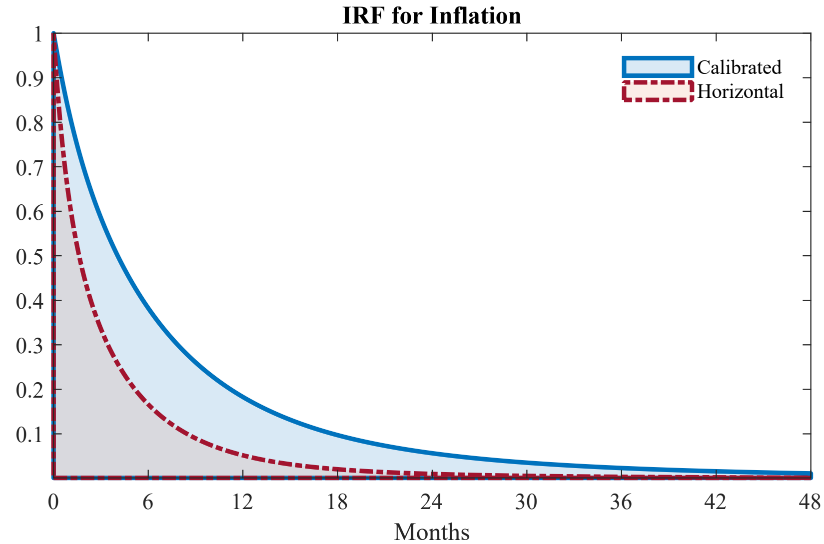
Notes: This figure plots the impulse response functions for inflation, gdp gap, and sectoral inflation to a sectoral shock that increases sectoral inflation by one percent on impact in the baseline policy economy. It compares the baseline policy economy with an economy where monetary policy stabilizes aggregate inflation. Panel A: Oil and gas extraction. Panel B: Semiconductor machine manufacturing. This calibration is the same as the baseline, except that we assume that the 'oil and gas extraction' frequency of price adjustment is the same as the 'semiconductor machinery mfg' frequency of price adjustment. That is, $\theta_{\text{oil and gas extraction}} = \theta_{\text{semiconductor machinery mfg}} = 0.0340$.

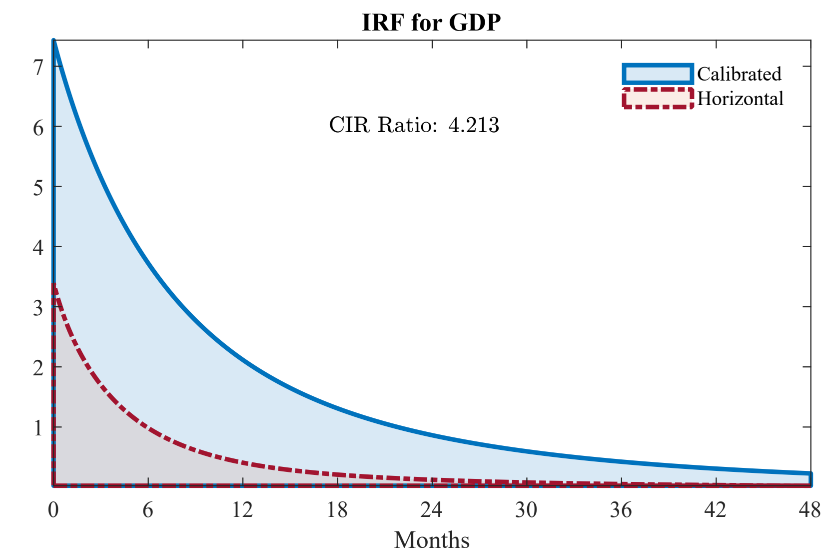
Figure S.M.5: Impulse response functions to a monetary policy shock in two economies



Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration uses a (finite) Frisch elasticity of 2.

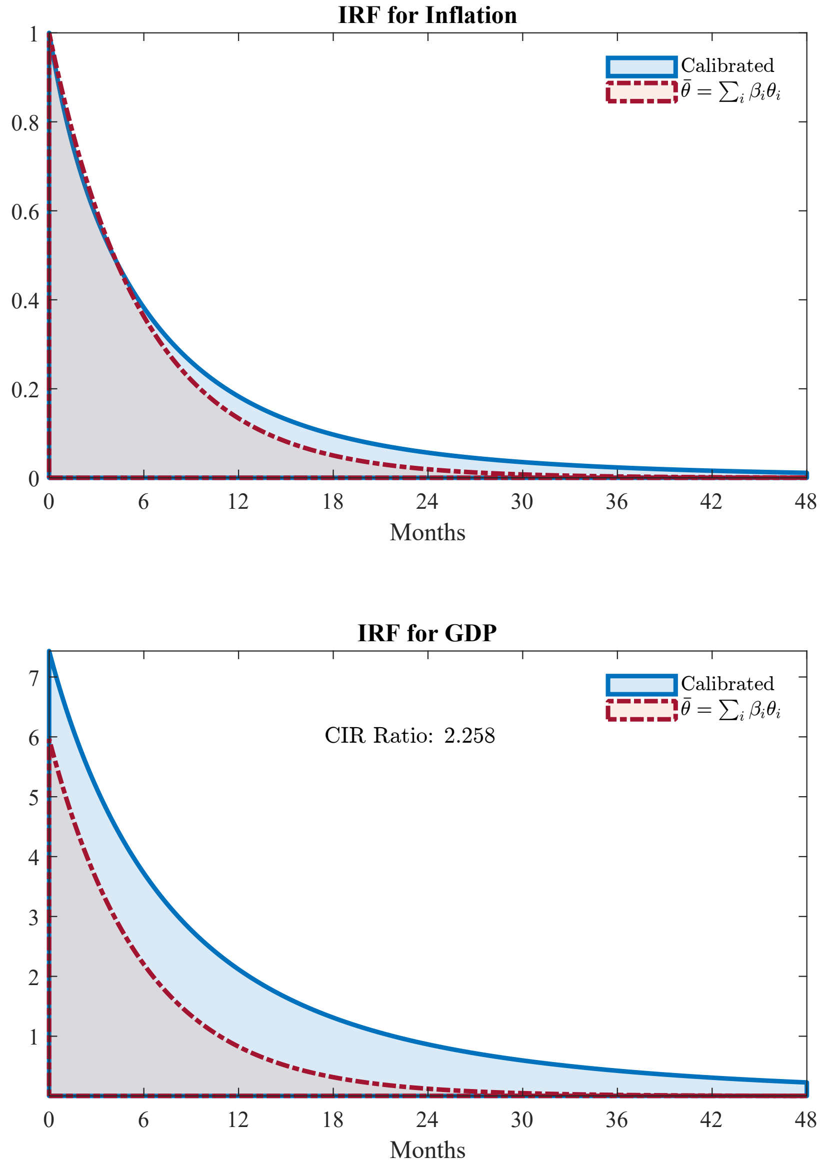
Figure S.M.6: Impulse response functions to a monetary policy shock in two economies





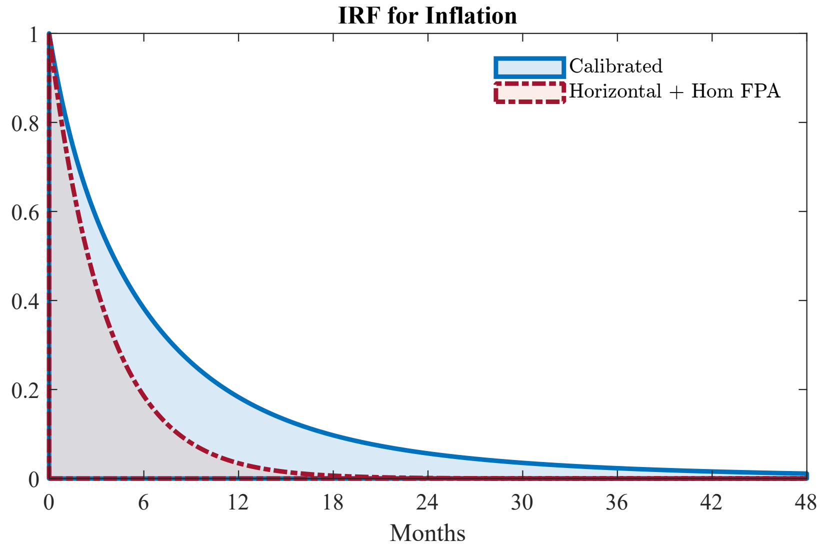
Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy that has production networks with an economy that has a horizontal production structure where only labor is used as an input for production. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration uses a (finite) Frisch elasticity of 2.

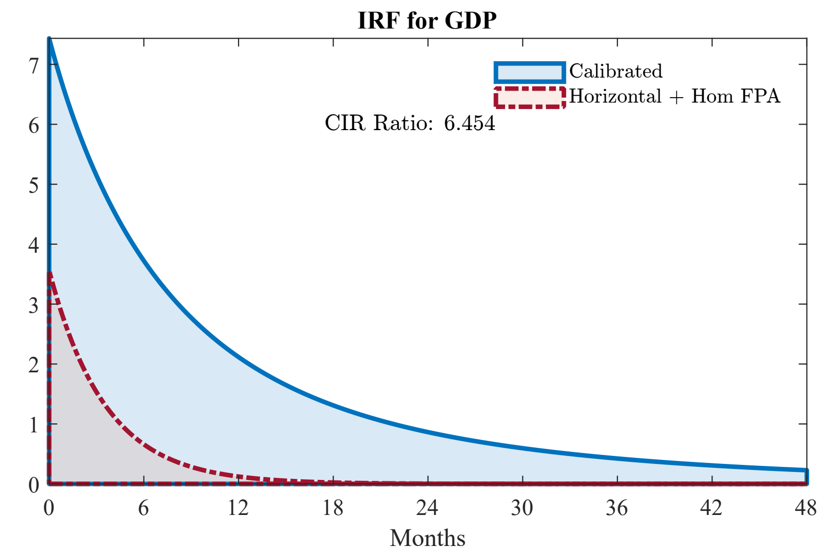
Figure S.M.7: Impulse response functions to a monetary policy shock in two economies



Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy with production networks and heterogeneous price stickiness across sectors with an economy that has homogeneous price stickiness across sectors. The homogeneous price adjustment frequency is calibrated to be the weighted average of the price adjustment frequencies across sectors. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration uses a (finite) Frisch elasticity of 2.

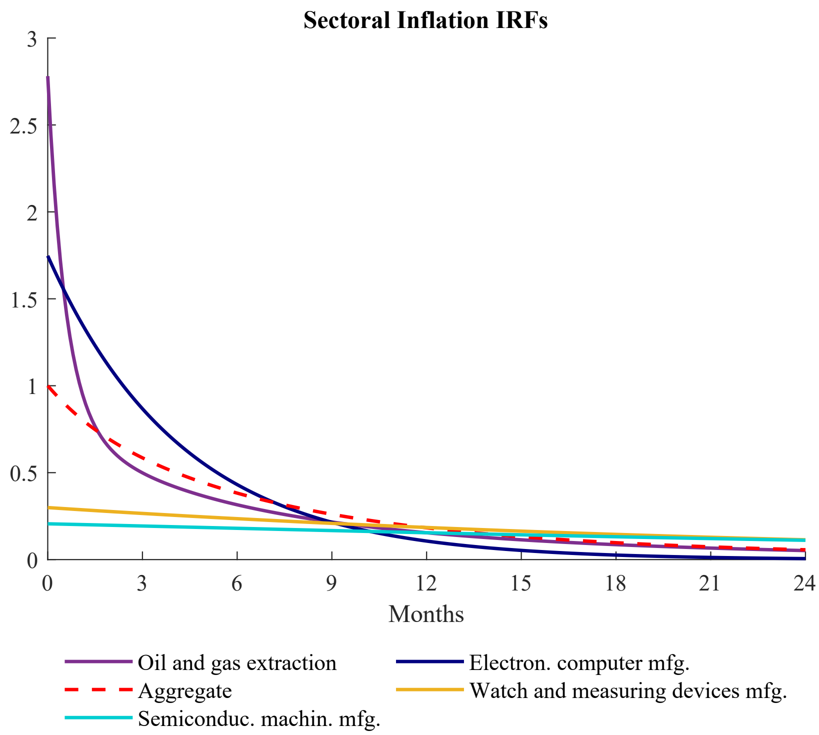
Figure S.M.8: Impulse response functions to a monetary policy shock in two economies





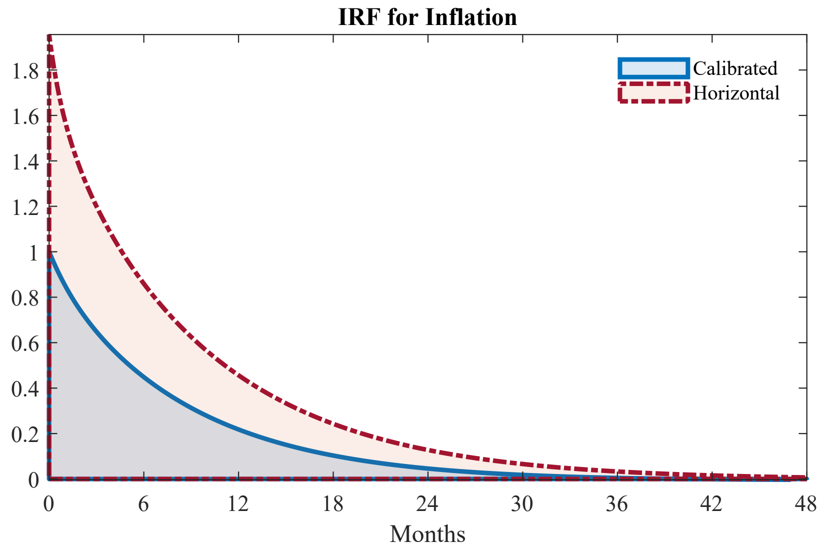
Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy with production networks and heterogeneous price stickiness across sectors with an economy that has both a horizontal production structure where only labor is used as an input for production as well as homogeneous price stickiness across sectors. The homogeneous price adjustment frequency is calibrated to be the weighted average of the price adjustment frequencies across sectors. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration uses a (finite) Frisch elasticity of 2.

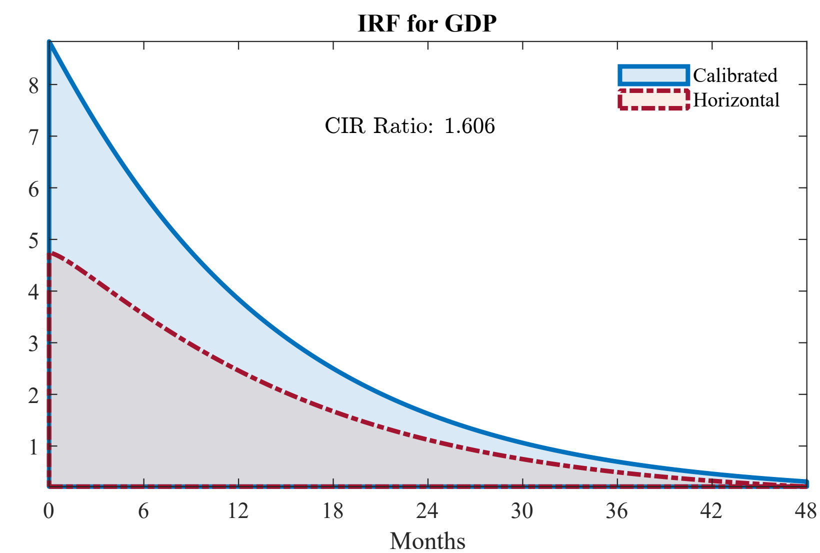
Figure S.M.9: Inflation response to a monetary policy shock



Notes: This figure plots the impulse response functions for aggregate inflation and sectoral inflation to a monetary shock that generates a one percentage increase in aggregate inflation on impact. The calibration of the model is at a monthly frequency. The aggregate inflation response is shown in dashed lines. The calibration uses a (finite) Frisch elasticity of 2.

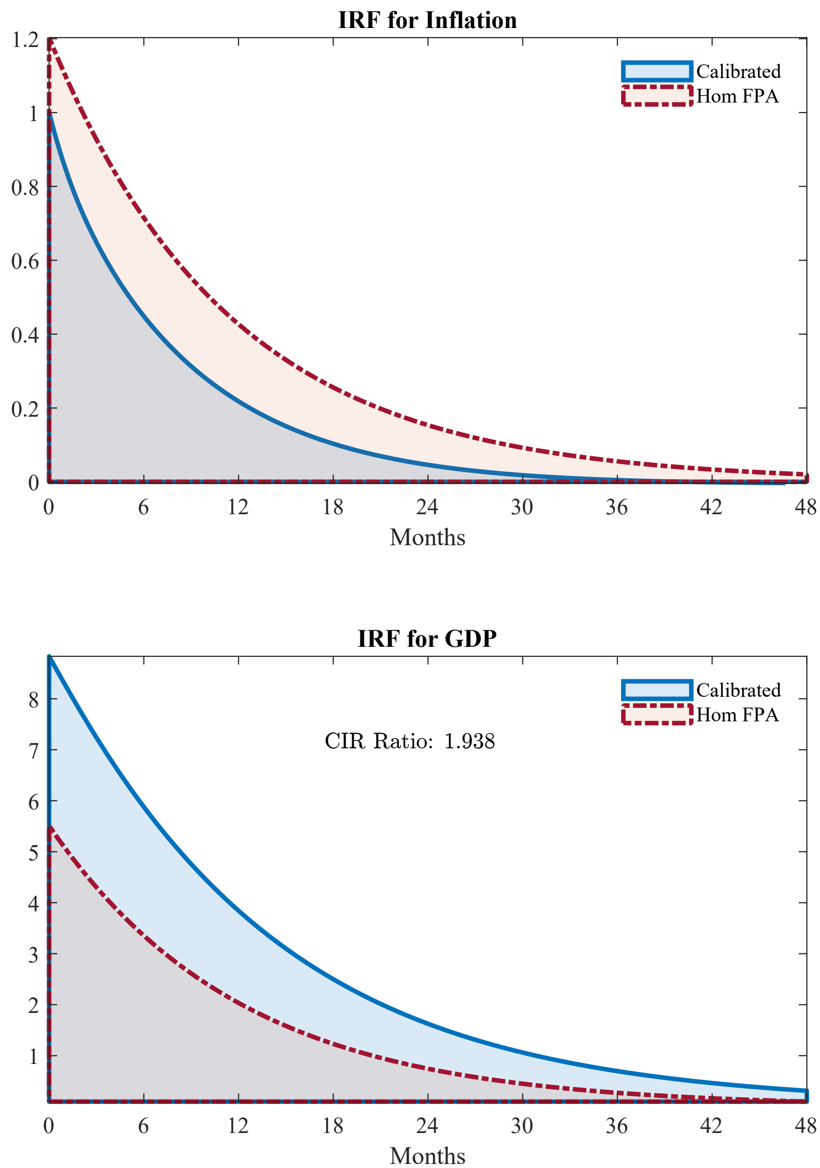
Figure S.M.10: Impulse response functions to a monetary policy shock under a Taylor rule





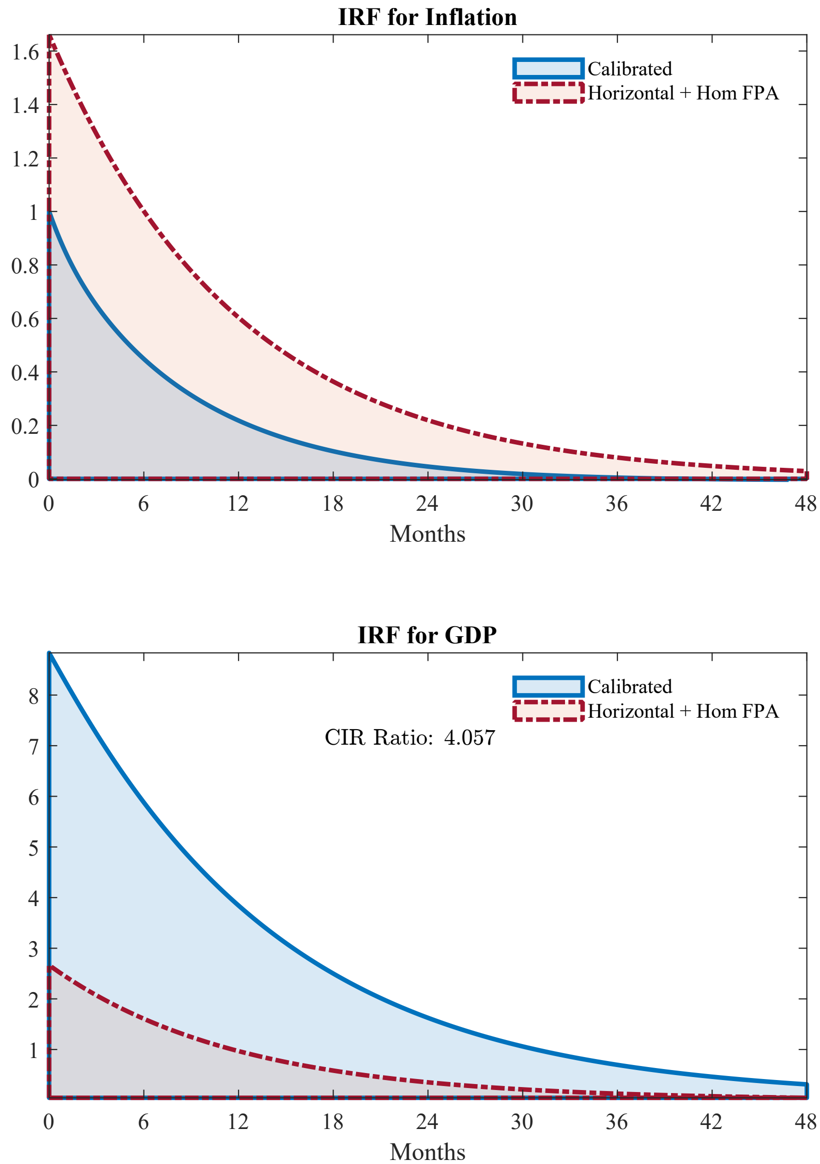
Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock. It compares our baseline Taylor rule economy that has production networks with an economy that has a horizontal production structure where only labor is used as an input for production. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration fixes the feedback parameter on the Taylor rule to $\phi_\pi = 1.5$. The monetary shock size and persistence are the same across the two economies.

Figure S.M.11: Impulse response functions to a monetary policy shock under a Taylor rule



Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock. It compares our baseline Taylor rule economy that has heterogeneous price stickiness across sectors with an economy that has homogeneous price stickiness across sectors. The homogeneous price adjustment frequency is calibrated to be the weighted average of the price adjustment frequencies across sectors. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response. The calibration fixes the feedback parameter on the Taylor rule to $\phi_\pi = 1.5$. The monetary shock size and persistence are the same across the two economies.

Figure S.M.12: Impulse response functions to a monetary policy shock under a Taylor rule



Notes: This figure plots the impulse response functions for inflation and GDP to a monetary shock that generates a one percentage increase in inflation on impact. It compares our baseline economy with production networks and heterogeneous price stickiness across sectors with an economy that has both a horizontal production structure where only labor is used as an input for production as well as homogeneous price stickiness across sectors. The homogeneous price adjustment frequency is calibrated to be the weighted average of the price adjustment frequencies across sectors. The calibration of the model is at a monthly frequency. CIR denotes the cumulative impulse response.The calibration fixes the feedback parameter on the Taylor rule to $\phi_\pi = 1.5$. The monetary shock size and persistence are the same across the two economies.