Heaven's light is our guide"

Rajshahi University of Engineering & Technology
Department of Computer Science & Engineering

Network Security

Course No. : 305

Chapter 8: Graphs

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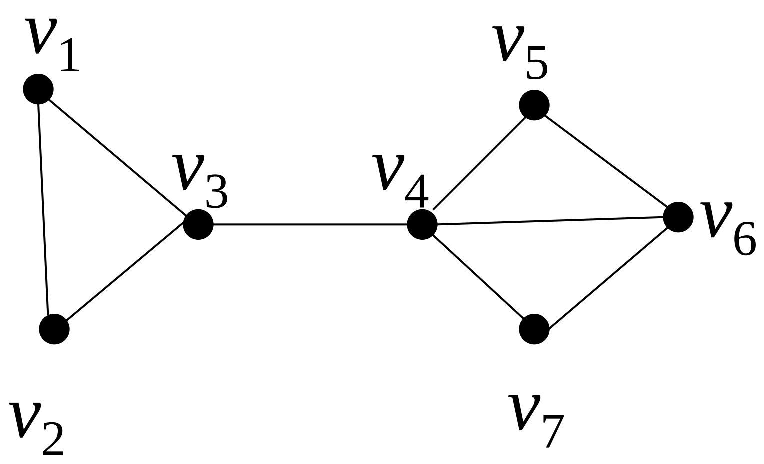
8.1 Graphs and Graphs Models

8.1 Graphs and Graphs Models

- **DEFINITION 1(Graph):**
  - ✓ A graph G=(V,E)
  - ✓ consists of:
    - V , a nonempty set of vertices, points or nodes of G
    - E , a set of edges of G
    - Each edge has either one or two vertices associated with it, called endpoints.
    - An edge is said to connect its endpoints.
  - ✓ Example:



$G=(V, E)$, where

$V=\{v_1,v_2,\ldots, v_7\}$

$E=\{\{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}$
$\{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}$
$\{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\}\}$

8.1 Graphs and Graphs Models

- The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set is called an ***infinite graph.***
- A graph with a finite vertex set is called a ***finite graph.***
- **Simple graph:**
  - ✓ Graph in which each edge connects two different vertices and
  - ✓ No two edges connect the same pair of vertices.
- **Multi graph:**
  May have multiple edges connecting the same vertices.
- **Pseudo graph:**
  Like a multi graph, but edges connecting a node to itself are allowed.
- **Trivial graph:**
  The finite graph with one vertex and no edges that is a single point is called trivial graph. Example:

- **Undirected graphs:**
  Edges are not directed.

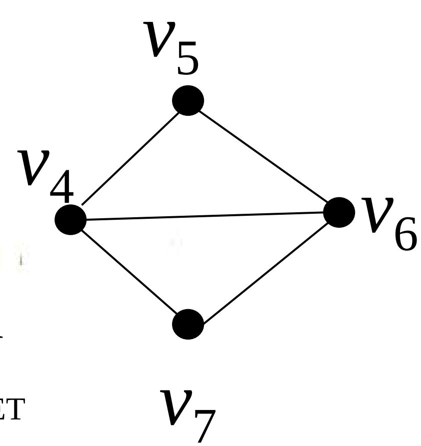


Figure 2: Simple and undirected graph

8.1 Graphs and Graphs Models

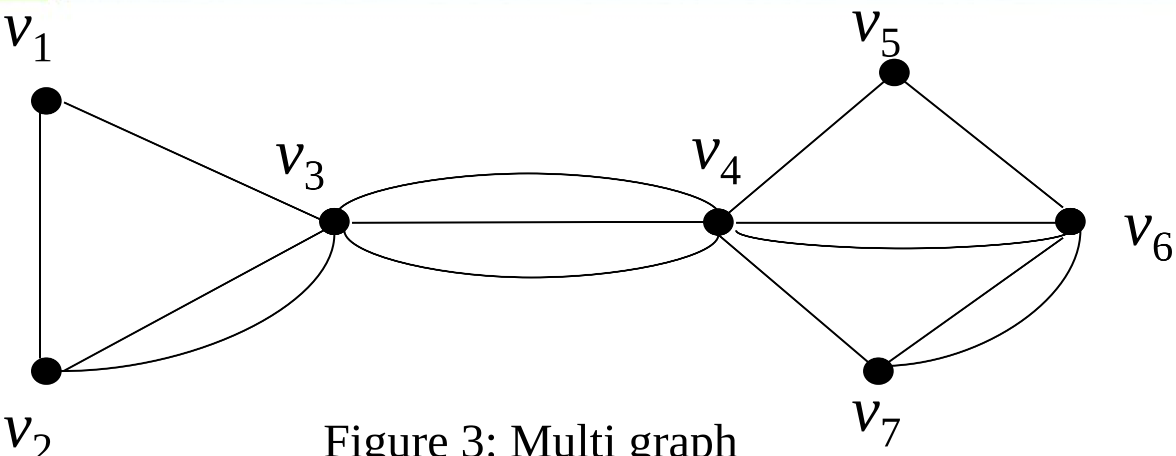


Figure 3: Multi graph

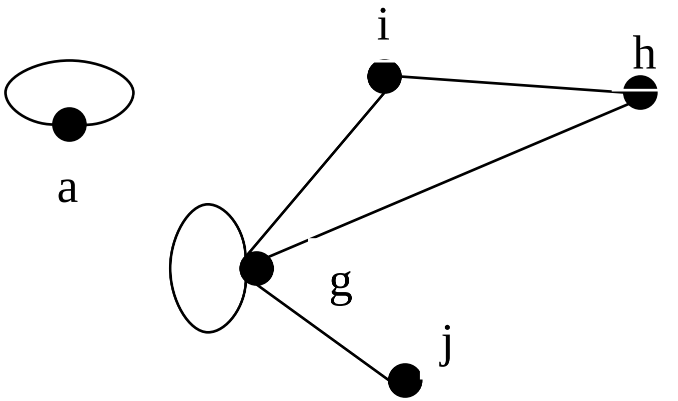
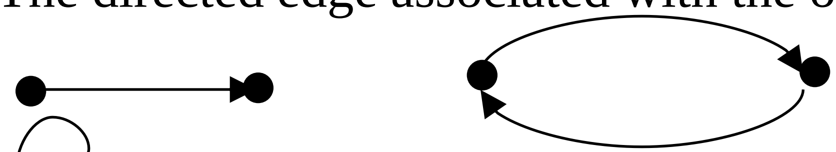
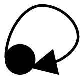


Figure 4: Pseudo graph

- **Directed graph:**
  - ✓ A *directed graph* (*V*,*E*) consists of a set of vertices *V* and a set of directed edged *E*.
  - ✓ Each directed edges is associated with an ordered pair of vertices.
  - ✓ The directed edge associated with the ordered pair.



  - ✓  is allowed in a directed graph
- **Mixed graph:**

  Graph with both directed and undirected edge.

8.1 Graphs and Graphs Models

- **Graph Terminology:**

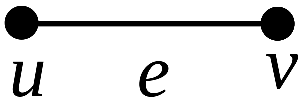
Type	Edges	Multiple Edges	Loops
(simple) graph	undirecteded ge: {u,v}	✗	✗
Multigraph		✓	✗
Pseudograph		✓	✓
Directed graph	directed edge: (u,v)	✗	✓
Directed multi graph		✓	✓

- **Graph models:**
  1) Influence graph
  2) Hollywood graph
  3) Round robin tournaments
  4) Call graph
  5) The web graph
  6) Roadmaps

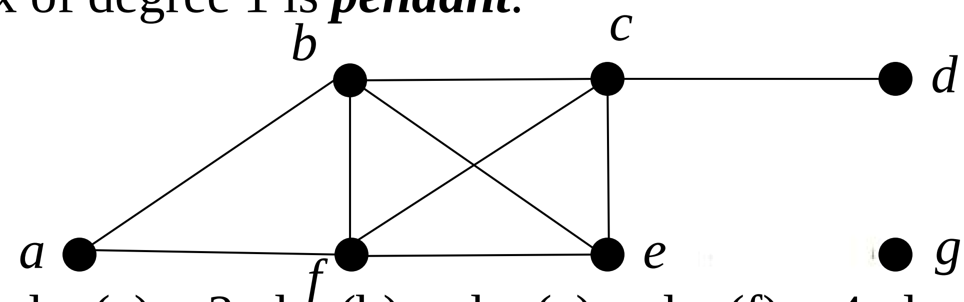
8.2 Graph Terminology and Special Types of Graphs

8.2 Graph Terminology and Special Types of Graphs

- **Definition – 1(adjacent):**
  - ✓ G = (v,e) : undirected graph
  - ✓ if u --e-- v ==>
  - ✓ u and v are ***adjacent*** (or neighbors)
  - ✓ Edge e is ***incident*** with u (& v).
  - ✓ e connects u and v.
  - ✓ u and v are ***end points*** of e.

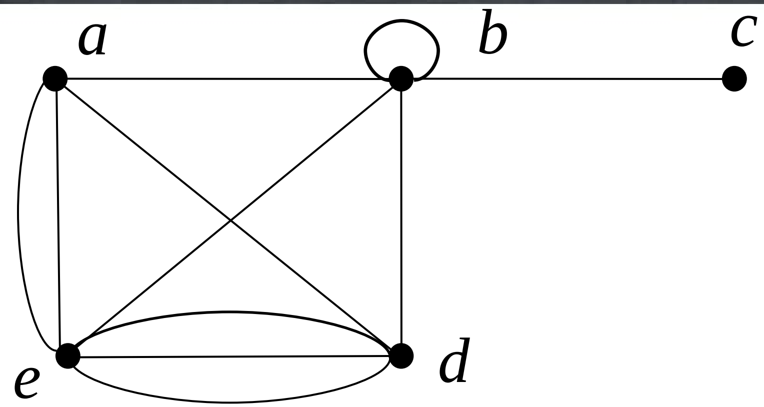


- **Definition – 2(degree):**
  - ✓ Let G be an undirected graph, v ∈ V a vertex.
  - ✓ The degree of v, deg(v), is its number of incident edges. (Except that any self-loops are counted twice.)
  - ✓ A vertex with degree 0 is ***isolated***.
  - ✓ A vertex of degree 1 is ***pendant***.



❖ In the figure deg(a) = 2, deg(b) = deg(c) = deg(f) = 4, deg(e) = 3, deg(d) = 1 and deg(g) = 0.

8.2 Graph Terminology and Special Types of Graphs



- In the figure deg(a)=4, deg(b)=6, deg(*c*)=1, deg(*d*)=5, deg(*e*)=6
- **Theorem 1:**

  Let G be an undirected (simple, multi-, or pseudo-) graph with vertex set V and edge set E. Then

  - ❖ Note that this applies even if multiple edges and loops are present.
- **Example 3.** How many edges are there in a graph with 10 vertices each of degree six?

  Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2e = 60$. Therefore, $e = 30$.

8.2 Graph Terminology and Special Types of Graphs

- **Theorem 2:**

  The number of vertices in a graph with odd degree is even.

  Proof: V_1 and V_2 be the set of vertices of even degree and odd degree.

  $$2e = \sum_{\in V} \deg(V) = \sum_{V \in V_1} \deg(V) + \sum_{V \in V_2} \deg(V)$$

  deg(v) is even for € V1, the first term in right-hand side of equality is even. Hence the second term in the sum is also even. Because all the terms in this sum are odd. Thus there are an even number of vertices of odd degree.

- **Definition-3:**

  Let G be a directed (possibly multi-) graph, and let e be an edge of G that is (or maps to) (u,v). Then we say:

  - ✓ u is ***adjacent*** to v, v is adjacent from u
  - ✓ e comes from u, e goes to v.
  - ✓ e connects u to v, e goes from u to v
  - ✓ the ***initial vertex*** of e is u
  - ✓ the ***terminal vertex*** of e is v

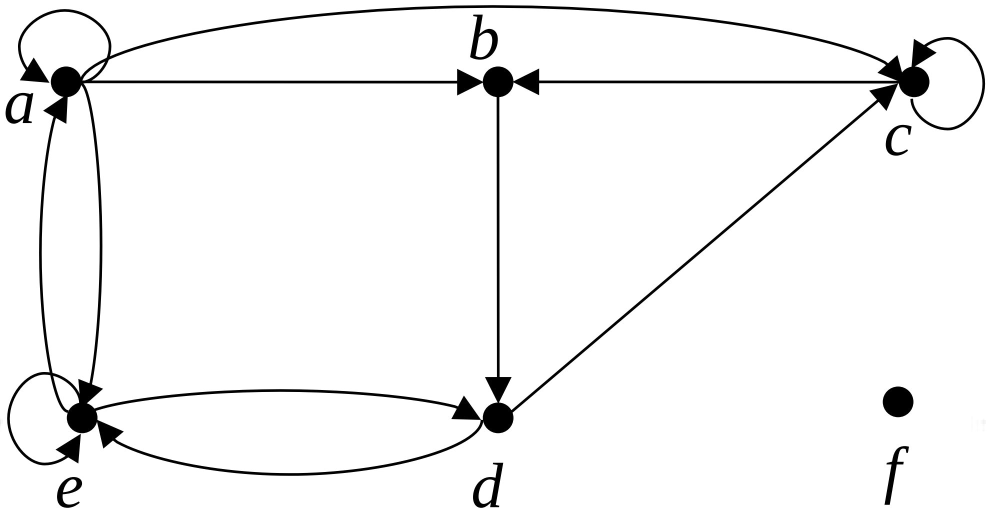
8.2 Graph Terminology and Special Types of Graphs

- **Definition-4:**

  Let G be a directed graph, v a vertex of G.

  - ✓ The ***in-degree*** of v, $\deg^-(v)$, is the number of edges going to v or the number of edges with as their terminal vertex.
  - ✓ The ***out-degree*** of v, $\deg^+(v)$, is the number of edges coming from v or the number of edges with as their initial vertex.
  - ✓ The ***degree*** of v, $\deg(v) = \deg^-(v) + \deg^+(v)$, is the sum of v's in-degree and out-degree.

- **Example 4:** Find the in-degree and out-degree of each vertex in the graph G with directed edges.



$\deg^-(a)=2$, $\deg^+(a)=4$
$\deg^-(b)=2$, $\deg^+(b)=1$
$\deg^-(c)=3$, $\deg^+(c)=2$
$\deg^-(d)=2$, $\deg^+(d)=2$
$\deg^-(e)=3$, $\deg^+(e)=3$
$\deg^-(f)=0$, $\deg^+(f)=0$

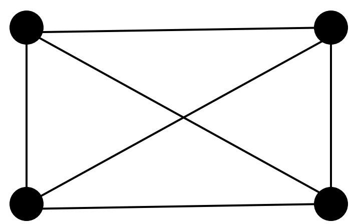
8.2 Graph Terminology and Special Types of Graphs

- **Theorem 3:**

  Let G = (V , E) b e a graph with directed edges. Then

$$\sum_{v\in V} \deg^{-}(v) = \sum_{v\in V} \deg^{+}(v) = |E|$$

- A simple graph $G=(V, E)$ is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if $\deg(v)=n$, $\forall v \in V$

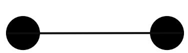


is 3-regular.

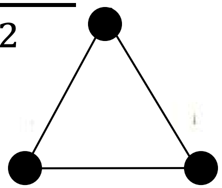
<u>Some Special Simple Graph</u>

- **Complete Graph:**
  - ✓ The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.
  - ✓ K_n is (n-1)-regular, $|V(K_n)|=n$, edge = $\frac{n(n-1)}{2}$

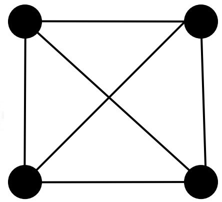
K_1



K_2



K_3

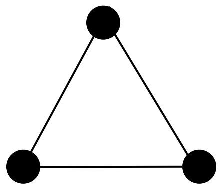


K_4

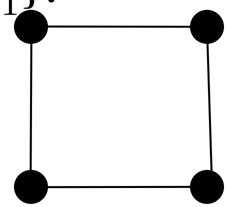
8.2 Graph Terminology and Special Types of Graphs

- **Cycles:**

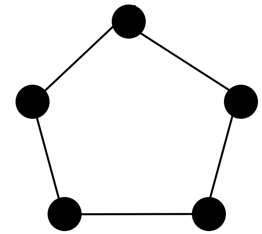
  The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}$, $\{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}$.



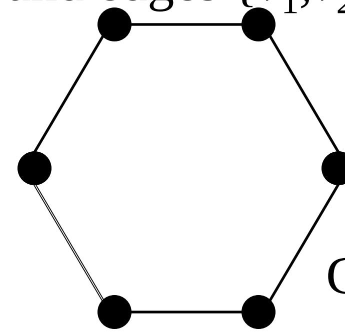
$\boldsymbol{C_3}$



$\boldsymbol{C_4}$



$\boldsymbol{C_5}$

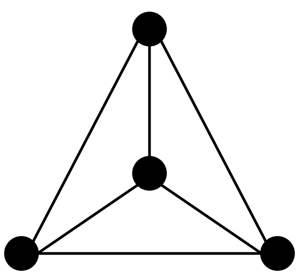


C_6

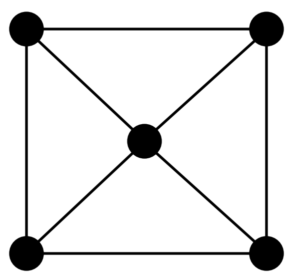
  ❖ **Note** C_n is 2-regular, $|V(C_n)| = n$, $|E(C_n)| = n$

- **Wheels:**

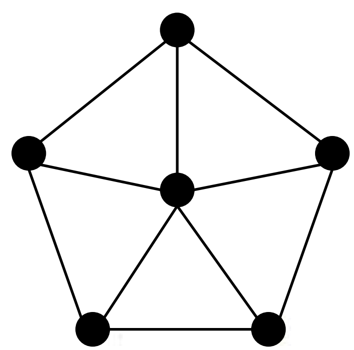
  For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{hub}, v_1\}, \{v_{hub}, v_2\}, \ldots,$ $\{v_{hub}, v_n\}\}$.



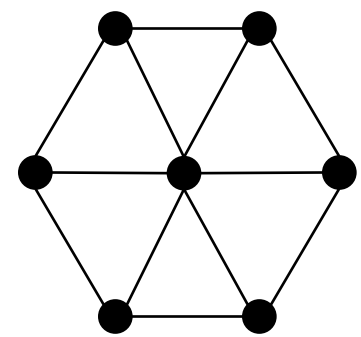
W_3



W_4



W_5



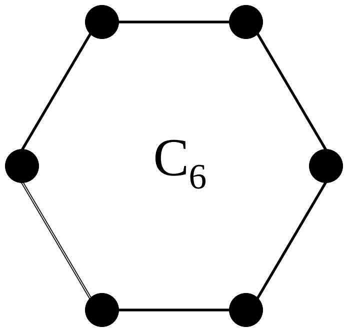
W_6

8.2 Graph Terminology and Special Types of Graphs

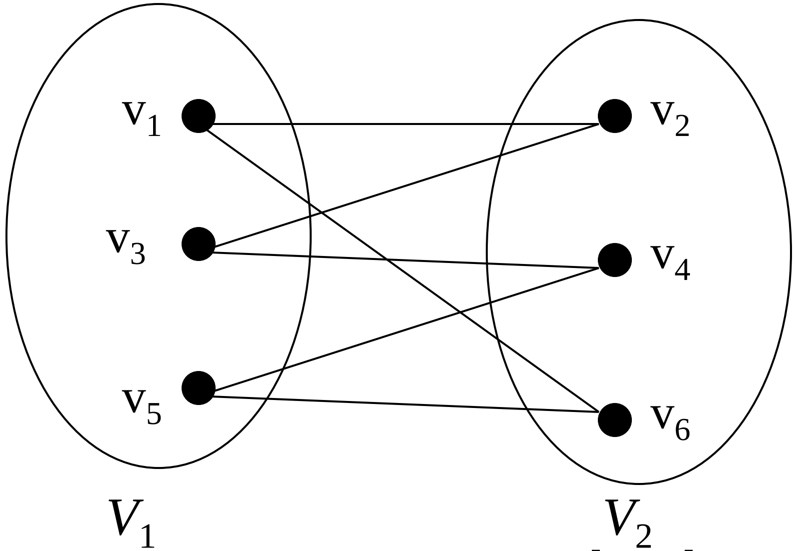
- **Bipartite Graphs:**

  A simple graph $G=(V,E)$ is called ***bipartite*** if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \emptyset$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.

- **EXAMPLE 9:**

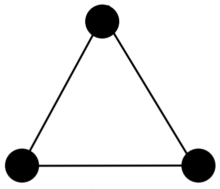


C_6 is bipartite. Because



- **Example:** Is C_3 bipartite?

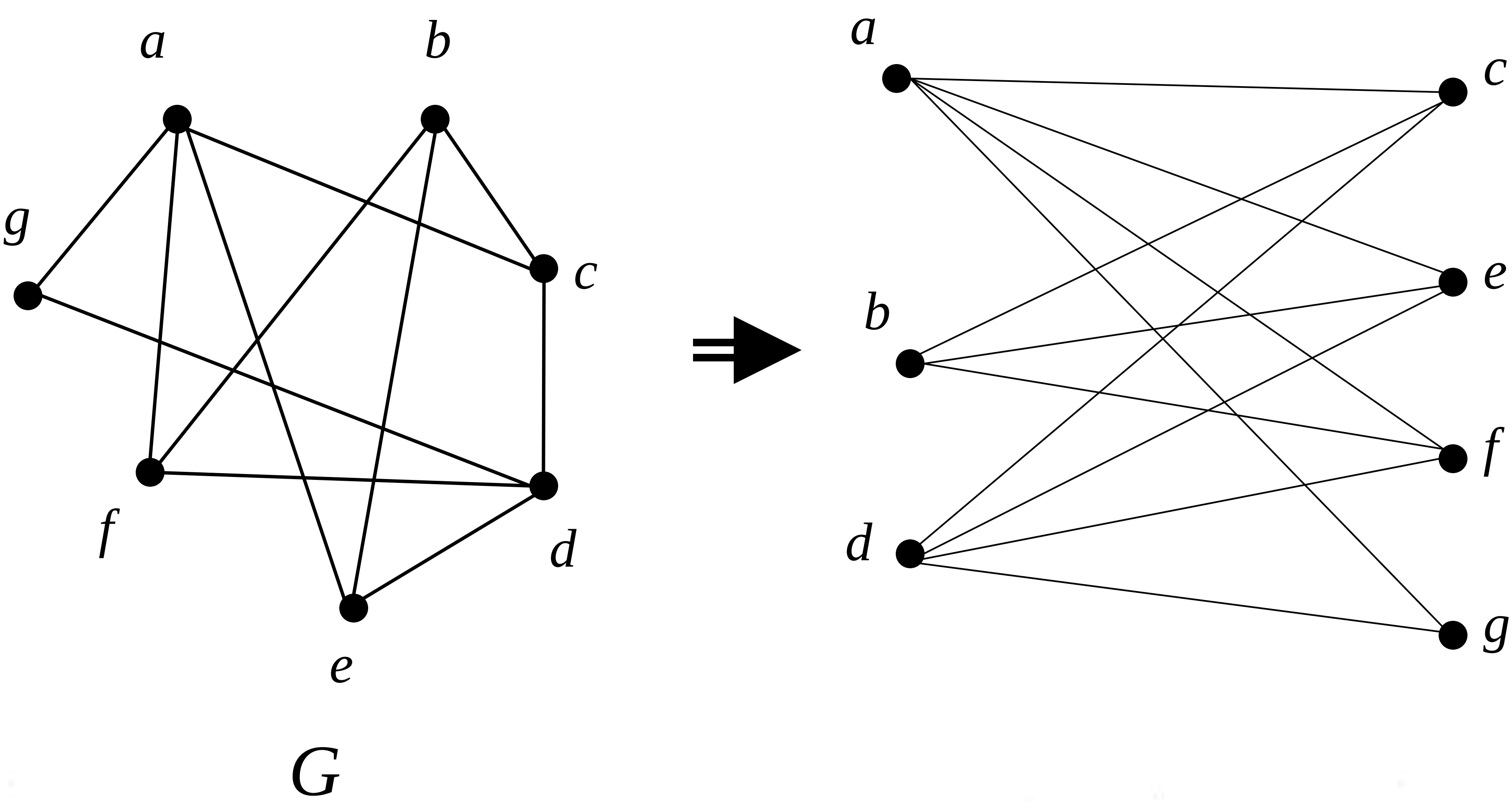
  No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.



$\boldsymbol{C_3}$

8.2 Graph Terminology and Special Types of Graphs

- **EXAMPLE 11:** Is the graph G bipartite?

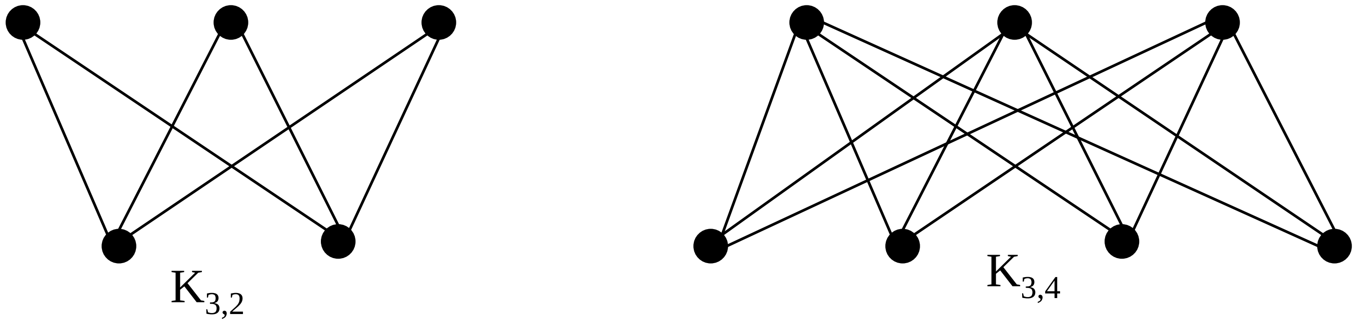


8.2 Graph Terminology and Special Types of Graphs

- **THEOREM 4:**

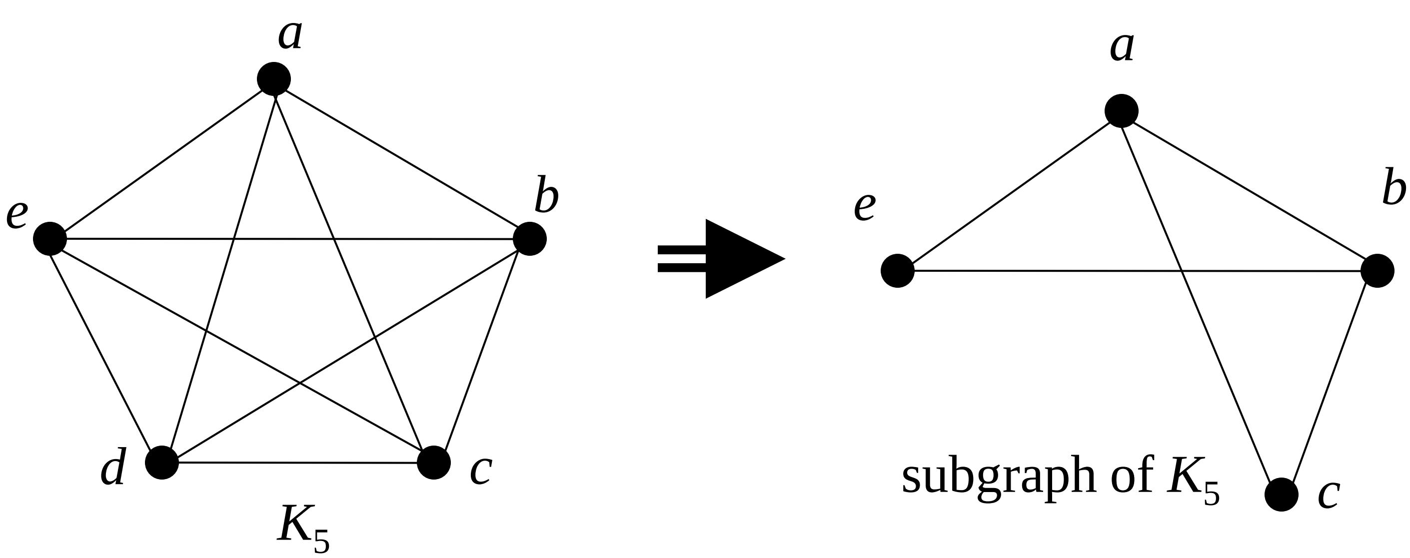
  A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

- **Complete bipartite graph:**
  - ✓ The ***complete bipartite*** graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively.
  - ✓ Two vertices are connected if and only if they are in different subsets.



8.2 Graph Terminology and Special Types of Graphs

- **Definition 6:**
  A ***subgraph*** of a graph $G=(V, E)$ is a graph $H=(W, F)$ where $W \subseteq V$ and $F \subseteq E$.
- **Example 14:** A subgraph of K_5

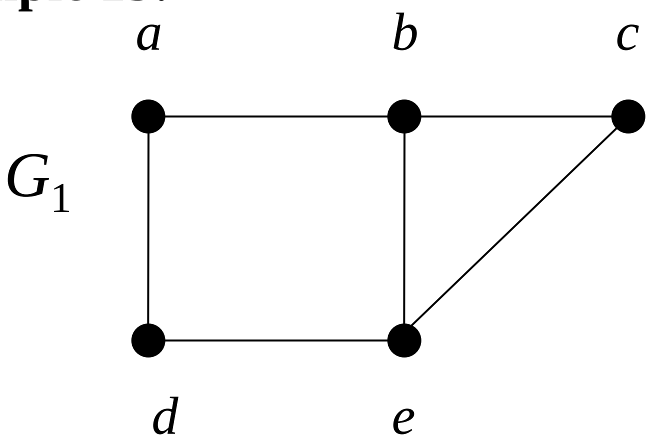


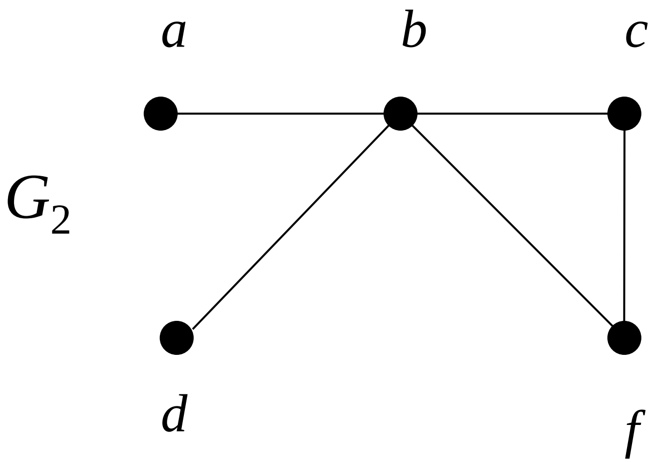
8.2 Graph Terminology and Special Types of Graphs

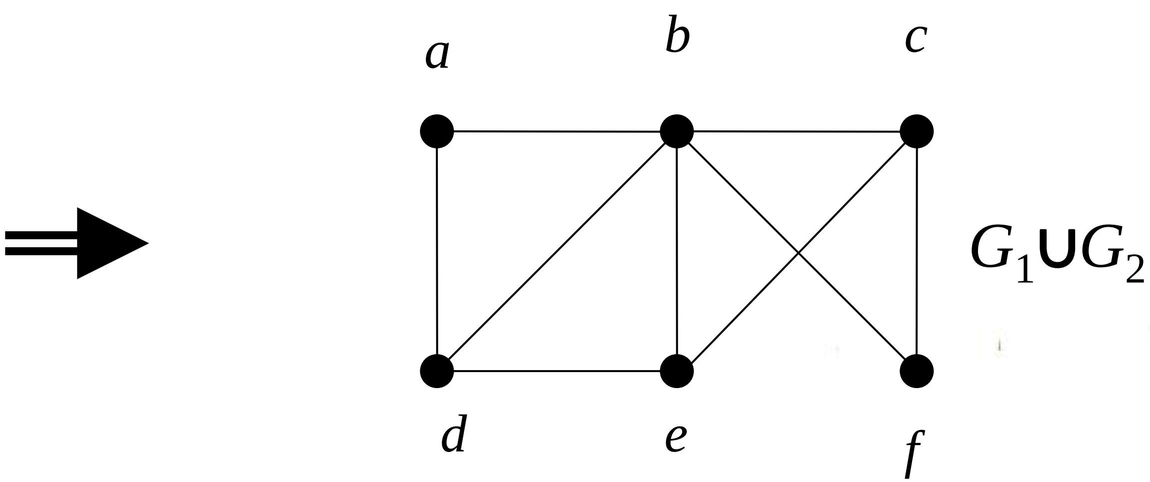
- **Definition 7:**

  The ***union*** of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$

- **Example 15:**



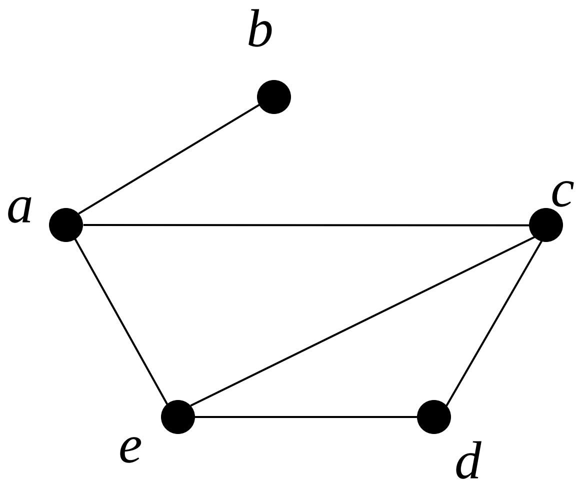




8.3 Representing Graphs and Graph Isomorphism

8.3 Representing Graphs and Graph Isomorphism

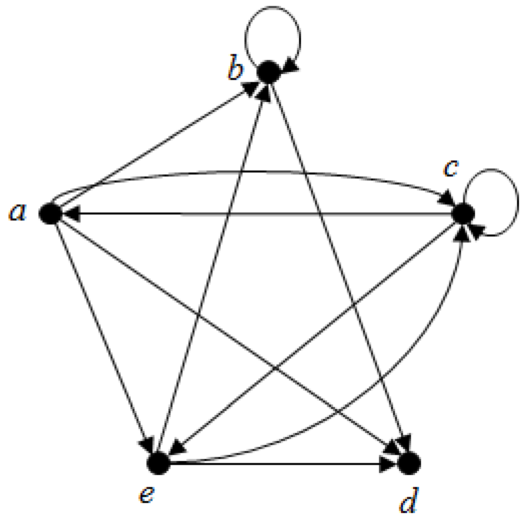
- **Adjacency list:**
  - ✓ A table with 1 row per vertex, listing its adjacent vertices.
  - ✓ Which specify the vertices that are adjacent to each vertex of the graph.
  - ✓ In directed graph listing the terminal nodes of each edge incident from that node.
- **Example 1:** Use adjacency lists to describe the simple graph given below.



Vertex	Adjacent Vertices
a	*b,c,e*
b	*a*
c	*a,d,e*
d	*c,e*
e	*a,c,d*

8.3 Representing Graphs and Graph Isomorphism

- **Example 2:** Represent the directed graph by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

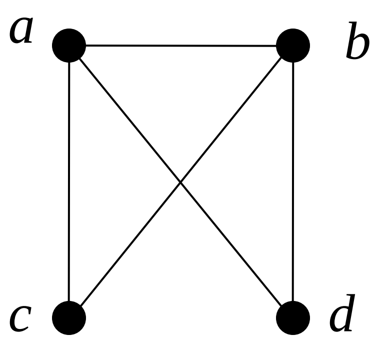


Initial vertex	Terminal vertices
a	*b,c,d,e*
b	*b,d*
c	*a,c,e*
d	
e	*b,c,d*

- **Adjacency Matrices:**
  - ✓ $G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$.
  - ✓ A matrix A is called the adjacency matrix of G if $A=[a_{ij}]_{n\times n}$, where $a_{ij} = \begin{cases} 1, & \text{if } \{v_i,v_j\}\in E, \\ 0, & \text{otherwise.} \end{cases}$

8.3 Representing Graphs and Graph Isomorphism

- **Example 3:** Use an adjacency matrix to represent the graph shown in Figure.

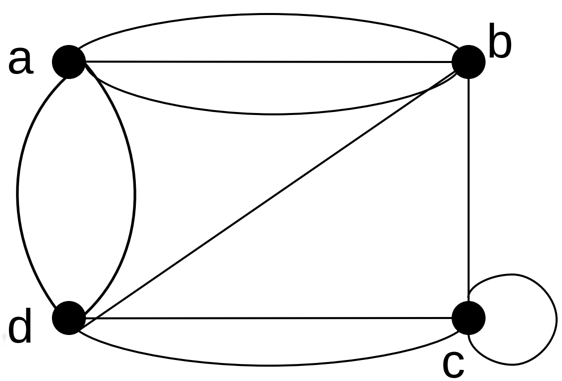


$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array}\!\!\begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note:

1. There are n! different adjacency matrices for a graph with n vertices.
2. The adjacency matrix of an undirected graph is symmetric.

- **Example 5:** Use an adjacency matrix to represent the pseudograph.



$$\Longrightarrow \qquad A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

8.3 Representing Graphs and Graph Isomorphism

- **Example 6:**

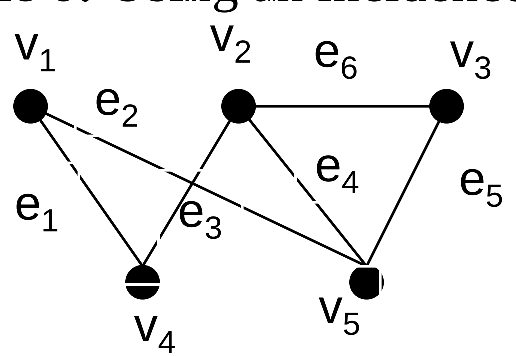
  If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

$$a_{ij} = \begin{cases} 1 & \text{, if} \\ 0 & \text{, otherwise} \end{cases}$$

- **Incidence Matrices:**
  - ✓ Let G = (V, E) be an undirected graph. The vertices and edges of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$, respectively.
  - ✓ The ***incidence matrix*** of G with respect to this listing of the vertices and edges is the $n \times m$ zero-one matrix with 1 as its (i, j) entry when edge e_j is incident with v_i, and 0 otherwise.

$$m_{ij} = \begin{cases} 1 & \text{, if edge } e_j \text{ is incident with } v_i \\ 0 & \text{, otherwise} \end{cases}$$

- **Example 6:** Using an incidence matrix, represent the following undirected graph:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

8.3 Representing Graphs and Graph Isomorphism

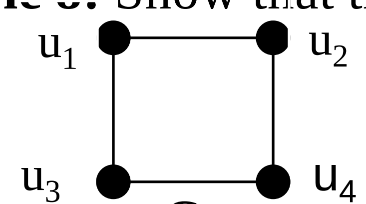
Isomorphism of Graph

- **Definition 1:**

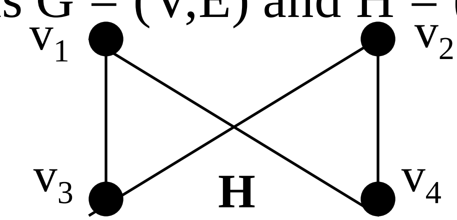
  The simple graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are ***isomorphic*** if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V_1. Such a function f is called an ***isomorphism***.

- **Necessary condition for $G_1 = (V_1,E_1)$ to be isomorphic to $G_2 = (V_2,E_2)$:**
  - ✓ $V_1 = V_2$, and $E_1 = E_2$
  - ✓ The number of vertices with degree n is the same in both graphs.
  - ✓ For every proper subgraph g of one graph, there is a proper subgraph of the other graph that is isomorphic to g..

- **Example 8:** Show that the graphs G = (V,E) and H = (W,F) are isomorphic.



G

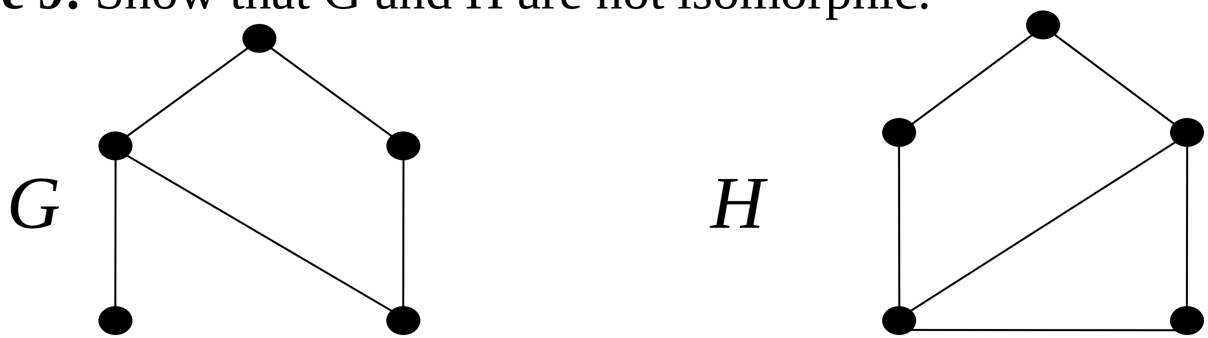


H

Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ is a one-to-one correspondence between V and W. $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_3) = v_3$, and $f(u_1) = v_1$ and $f(u_4) = v_2$ are adjacent in H.

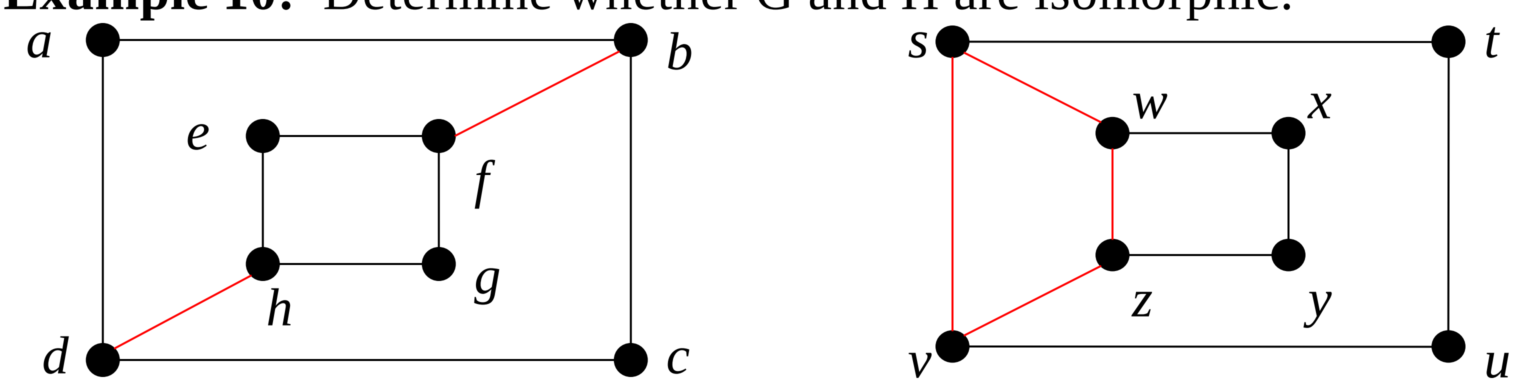
8.3 Representing Graphs and Graph Isomorphism

- **Example 9:** Show that G and H are not isomorphic.



Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e , whereas G has no vertices of degree one. It follows that G and H are not isomorphic .

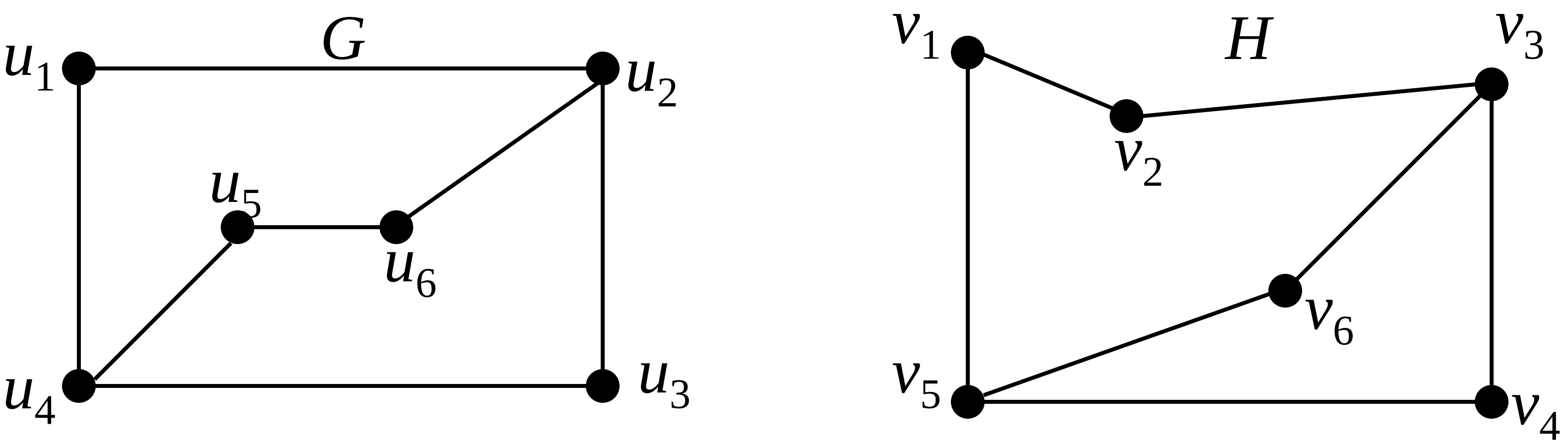
- **Example 10:** Determine whether G and H are isomorphic.



Solution: G and H are not isomorphic. Because deg(a) = 2 in G, a must correspond to either t, u , x , or y in H, because these are the vertices of degree

two in H . However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.
Another way to see that G and H are not isomorphic is to note that the subgraphs of G and H made up of vertices of degree three and the edges connecting them must be isomorphic if these two graphs are isomorphic.

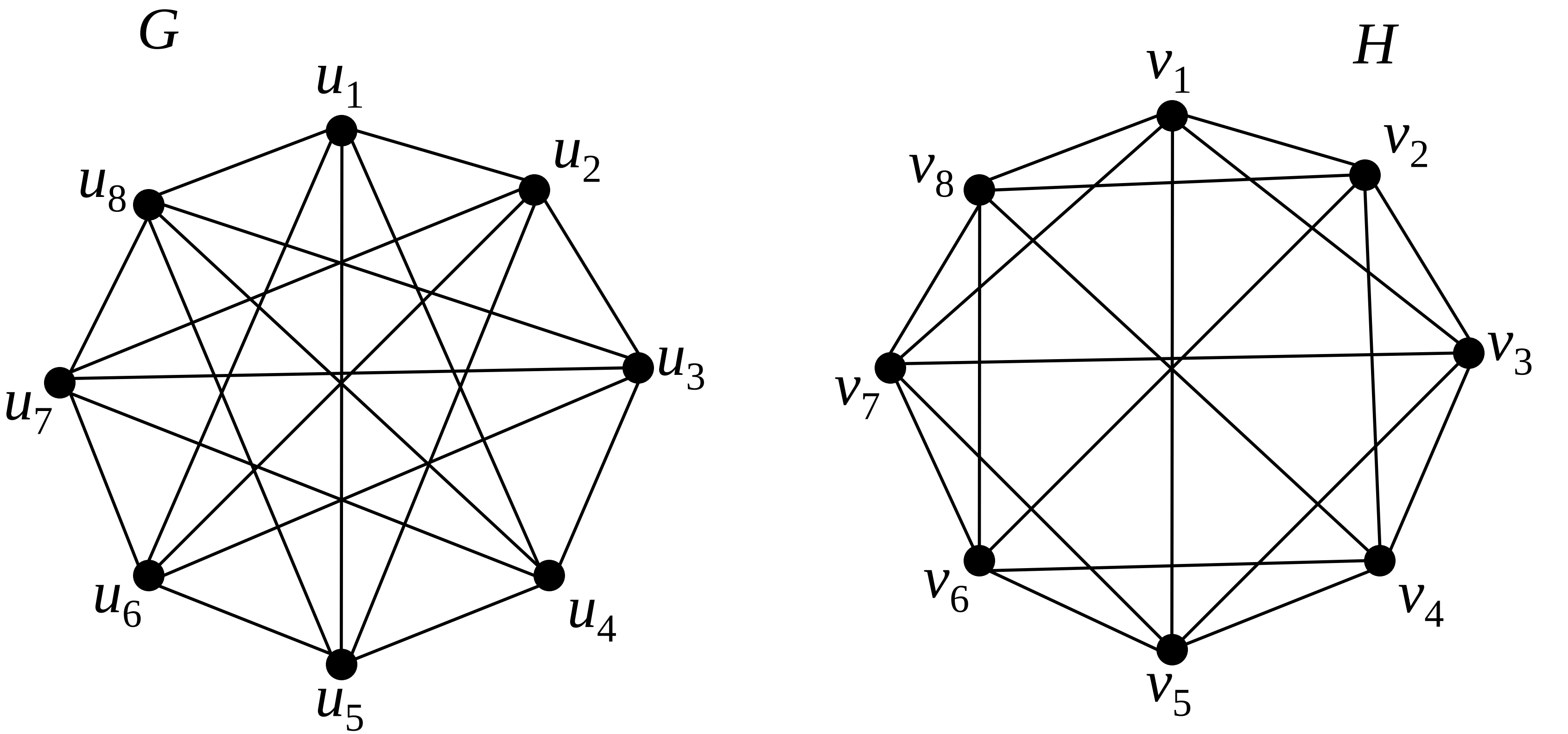
- **Example 11:** Determine whether the graphs G and H are isomorphic.



Solution: $f(u_1)=v_6$, $f(u_2)=v_3$, $f(u_3)=v_4$, $f(u_4)=v_5$, $f(u_5)=v_1$, $f(u_6)=v_2$
$\Rightarrow$Yes

8.3 Representing Graphs and Graph Isomorphism

- **Example:** Determine whether the graphs *G* and *H* are isomorphic.

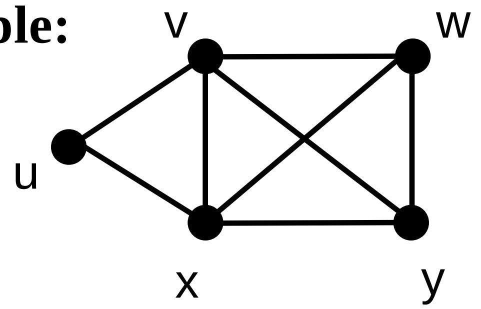


Solution: Yes

8.4 Connectivity

8.4 Connectivity

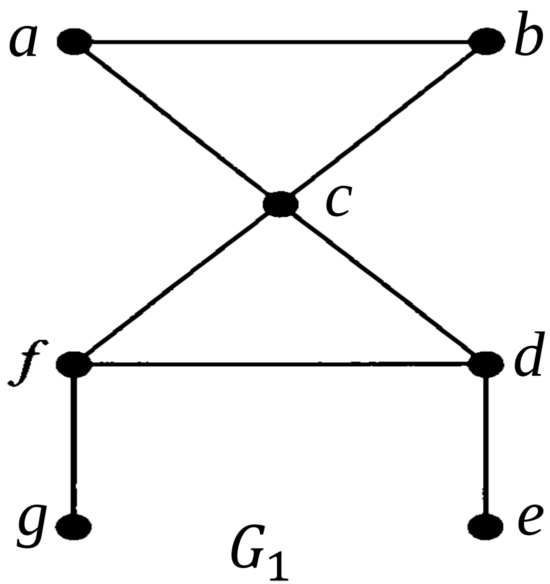
- **Path:**
  - A ***path*** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.
  - Let n be a nonnegative integer and G an undirected graph.
  - A ***path*** of ***length*** n from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G such that $f(e_1) = \{x_0, x_1\}$, $f(e_2) = \{x_1, x_2\}$, …, $f(e_n) = \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.
  - When graph is simple, path is denoted by its vertex sequence $x_0, x_1, \ldots, x_n$.
  - The path is a ***circuit*** if it begins and ends at the same vertex, that is, if u = v, and has length greater than zero.
  - The path or circuit is said to ***pass through*** the vertices $x_1, x_2, \ldots, x_{n-1}$ or ***traverse*** the edges $e_1, e_2, \ldots, e_n$.
  - A path or circuit is ***simple*** if it does not contain the same edge more than once.
- **Example:**

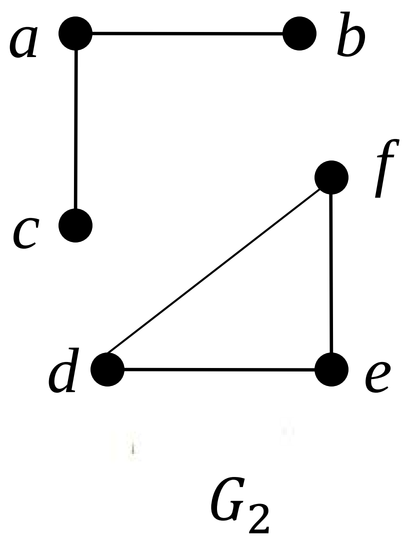


path: u, v, y
cycle: u, v, y, x, u

8.4 Connectivity

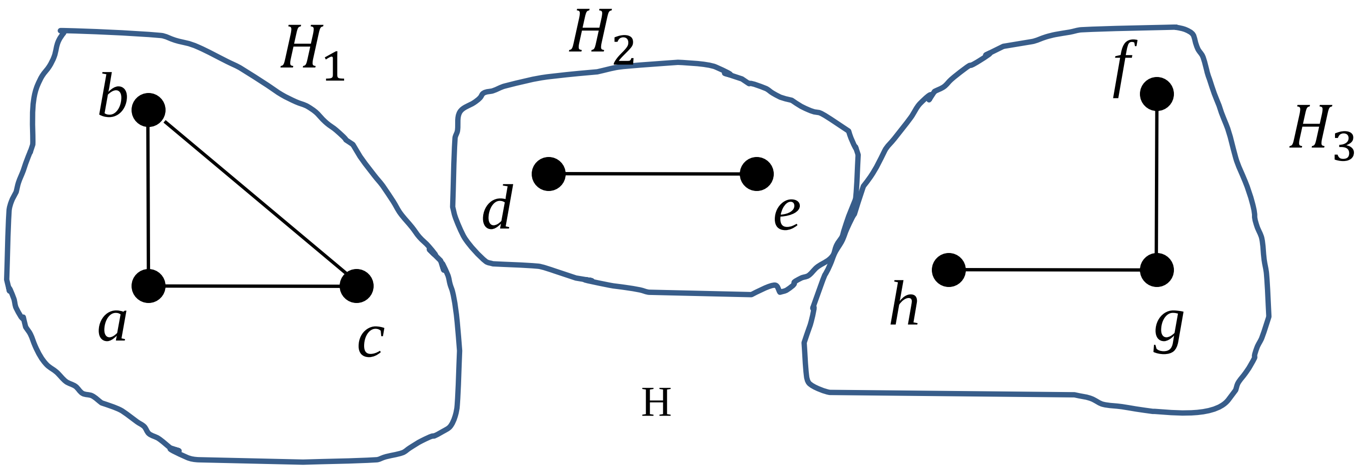
- **Connectedness:**
  - ✓ An undirected graph is ***connected*** if and only if there is a path between every pair of distinct vertices in the graph.
  - ✓ Two computers in the network can communicate if and only if this network is connected.
- **Example 5:** The graph G_1 in Figure is connected, because for every pair of distinct vertices there is a path between them (the reader should verify this). However, the graph G_2 in Figure is not connected. For instance, there is no path in G_2 between vertices a and d.





8.4 Connectivity

- **Theorem 1:**
  There is a simple path between every pair of distinct vertices of a connected undirected graph.
- ***Connected component:*** connected subgraph
- **Example 6:** What are the connected components of the graph H?

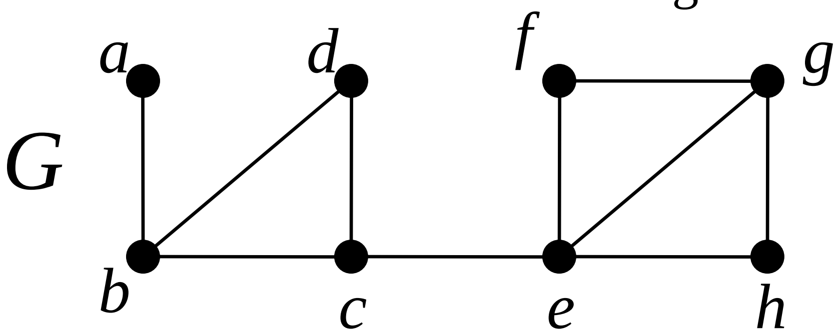


H

Solution: The graph H is the union of three disjoint connected subgraphs H_1 , H_2 , and H_3. These three subgraphs are the connected components of H .

8.4 Connectivity

- **Definition:**
  - ✓ A ***cut vertex*** separates one connected component into several components if it is removed.
  - ✓ A ***cut edge*** separates one connected component into two components if it is removed.
- **Example 8:** Find the cut vertices and cut edges in the graph G.



Solution: cut vertices: b, c, e & cut edges: {a, b}, {c, e}

- **Definition 4:**

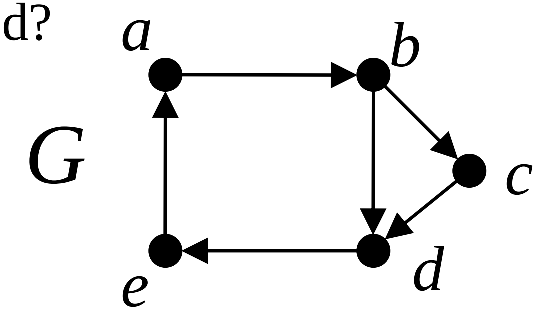
A directed graph is *strongly connected* if there is a path from *a* to *b* for any two vertices *a*, *b*.

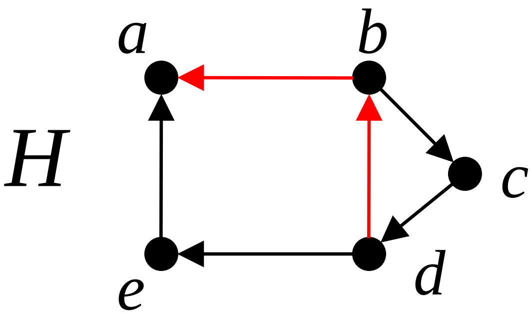
- **Definition 5:**

A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graphs.

8.4 Connectivity

- **Example 9:** Are the directed graphs G and H strongly connected or weakly connected?





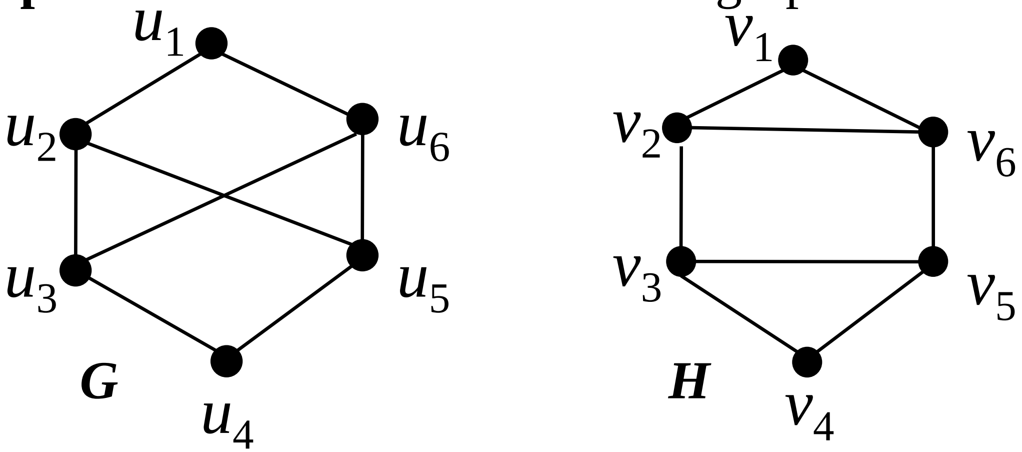
Solution: G is strongly connected because there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.
The graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H.

- **Example 6:** What are the connected components of the graph H?

Solution: G and H are not isomorphic. Because deg(a) = 2 in G, a must correspond to either t, u , x , or y in H, because these are the vertices of degree

8.4 Connectivity

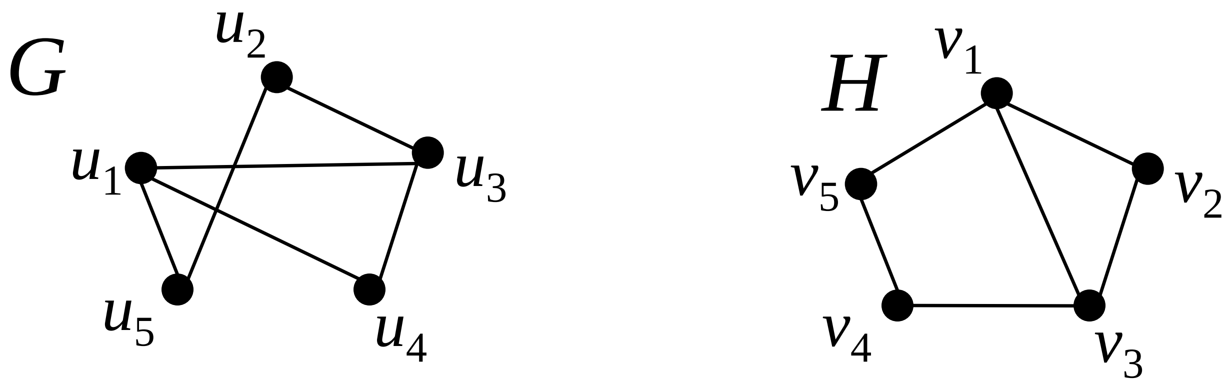
- **Paths and Isomorphism:**
  - ✓ Note that connectedness, and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism.
  - ✓ Paths and circuits can help determine whether 2 graphs are isometric
  - ✓ The existence of a simple circuit (or cycle) of a particular length is a useful invariant to show that 2 graphs are not isomorphic
- **Example 12:** Determine whether the graph G and H are isomorphic.



Solution: Both G and H have 6 vertices and 8 edges. Each has 4 vertices of degree 3, and 2 vertices of degree 2. However, H has a simple circuit of length 3, namely, v_1, v_2, v_6, v_1 whereas G has no simple circuit of length 3, as can be determined by inspection (all simple circuits in G have length at least four). G and H are not isomorphic.

8.4 Connectivity

- **Example 13.** Determine whether the graphs *G* and *H* are isomorphic.



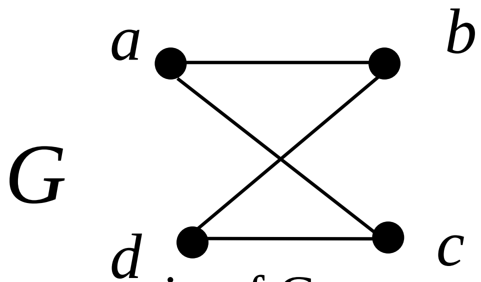
Solution: Both *G* and *H* have 5 vertices, 6 edges, two vertices of deg 3, three vertices of deg 2, a 3-cycle, a 4-cycle, and a 5-cycle. $\Rightarrow$ *G* and *H* may be isomorphic. The function *f* with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ and $f(u_5) = v_5$ is a one-to-one correspondence between $V(G)$ and $V(H)$. $\Rightarrow$ *G* and *H* are isomorphic.

Counting Paths between Vertices

- **Theorem 2:**
  Let G be a graph with adjacency matrix A with respect to the ordering v_1, v_2, …, v_n (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer is equals to the (i, j)th entry of A^r.

8.4 Connectivity

- **Example 14:** How many paths of length 4 are there from a to d in the simple graph G?



Solution: The adjacency matrix of G

$a \quad b \quad c \quad d$

a

b

c

d

$\Rightarrow$ $\Rightarrow 8$

there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a , b, a , b, d; a , b, a , c, d; a , b, d, b, d; a , b, d, c, d; a , c, a , b, d; a , c, a , c, d; a , c, d, b, d; and a , c, d, c, d are the eight paths from a to d. (ordering as a, b, c, d) is

8.5 Euler & Hamilton Paths

8.5 Euler & Hamilton Paths

- **The Seven Bridges of Konigsberg:**
  - ✓ In Konigsberg, Germany, a river ran through the city such that in its center was an island, and after passing the island, the river broke into two parts. Seven bridges were built so that the people of the city could get from one part to another.
  - ✓ Is it possible to walk through the city that would cross each bridge once and only once?

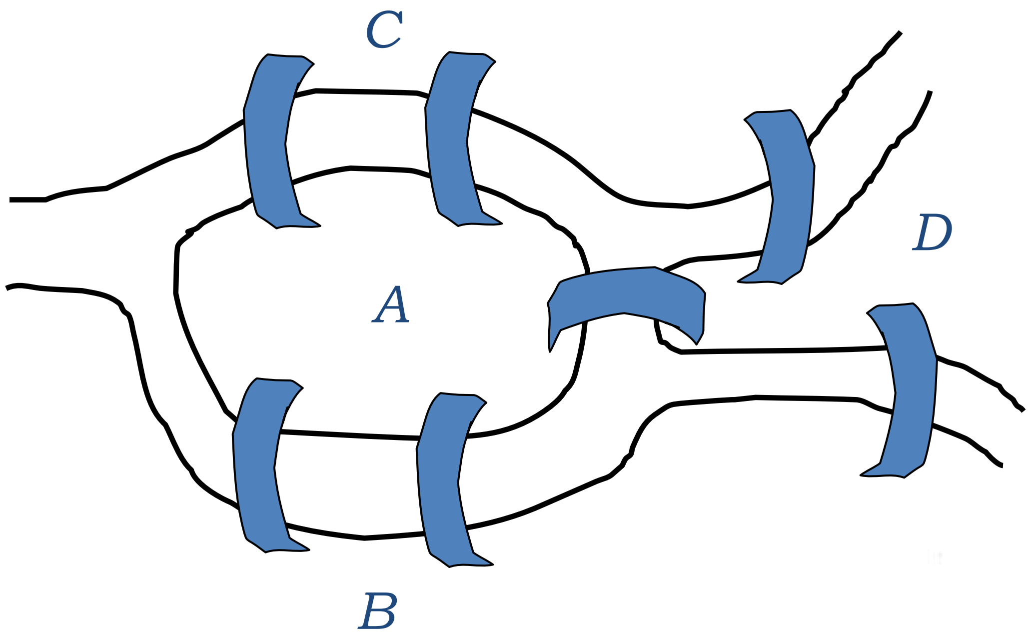


Fig-a: the seven bridges of konigsberg

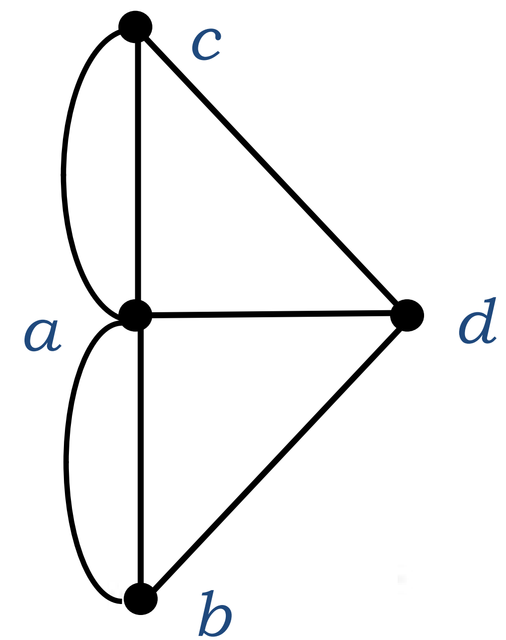
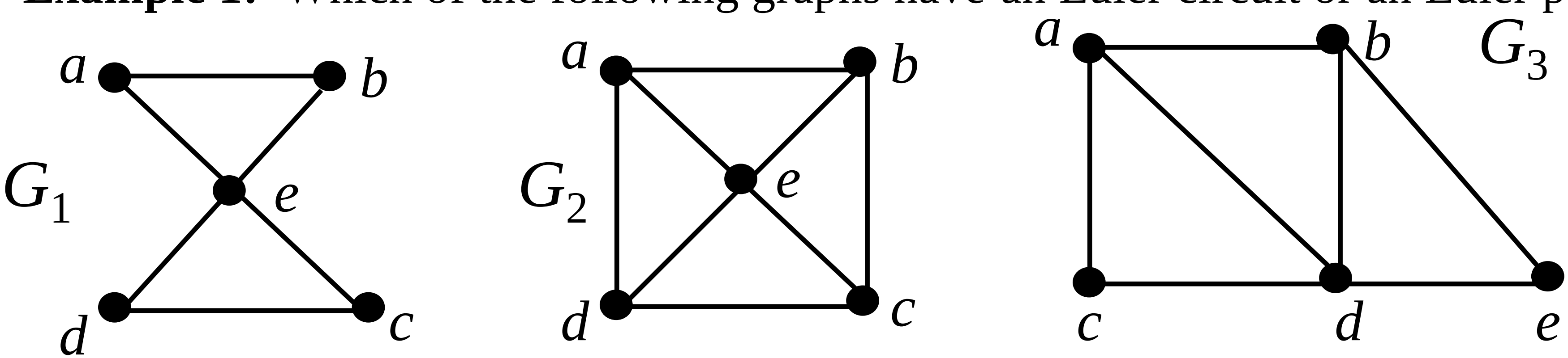


Fig-2: multigraph model of the town of konigsberg

8.5 Euler & Hamilton Paths

- **Definition 1:**
  - ✓ An ***Euler circuit*** in a graph G is a simple circuit containing every edge of G.
  - ✓ An ***Euler path*** in G is a simple path containing every edge of G.
- **Example 1:** Which of the following graphs have an Euler circuit or an Euler path?



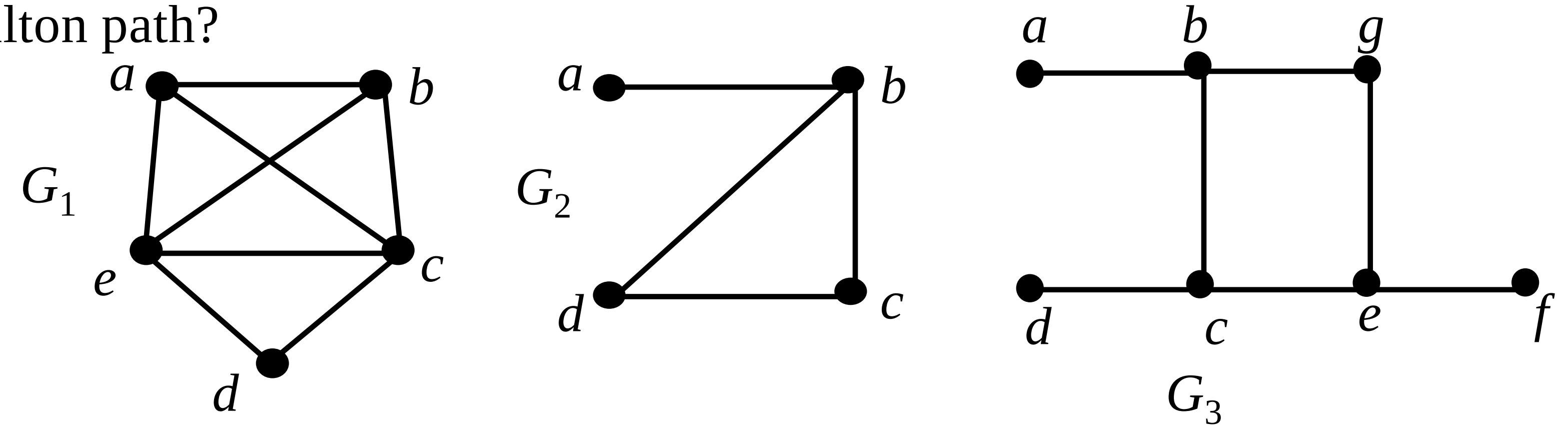
Solution: G_1 is Euler circuit, for example, a , e, c, d, e, b, a. G_2 and G_3 are not an Euler circuit. However, G_3 has an Euler path, namely, a , c, d, e, b, d, a , b. G_2 does not have an Euler path.

<u>Necessary and sufficient conditions for Euler path and Euler circuit</u>

- **Theorem 1:** A connected multigraph has a Euler circuit if and only if each of its vertices has an even degree.
- **Theorem 2:** A connected multigraph has a Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

8.5 Euler & Hamilton Paths

- **Definition 2:**
  - ✓ A ***Hamilton path*** is a path that traverses each vertex in a graph G exactly once.
  - ✓ A ***Hamilton circuit*** is a circuit that traverses each vertex in G exactly once.
- **Example 5:** Which of the following graphs have a Hamilton circuit or a Hamilton path?



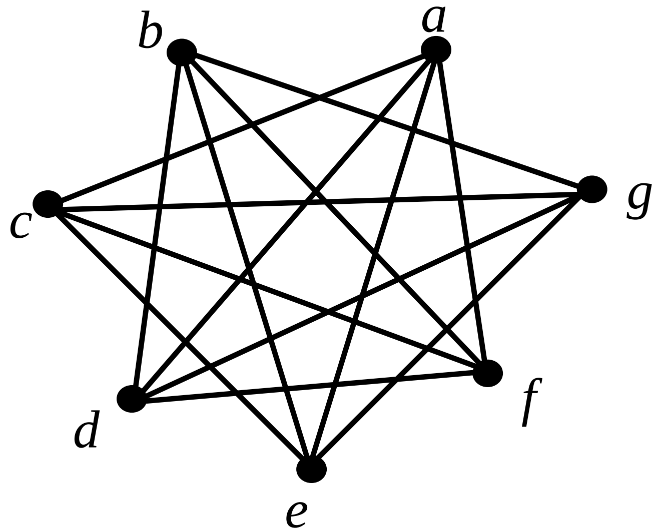
Solution: G_1 has a Hamilton circuit: a , b, c, d, e, a . There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge {a , b} twice), but G_2 does have a Hamilton path, namely, a , b, c, d. G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges {a, b}, {e, f}, and {c, d} more than once.

8.5 Euler & Hamilton Paths

- **Theorem 3:**

  If (but not only if) G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

- **Example:**



each vertex has deg $\geq n/2 = 3.5$

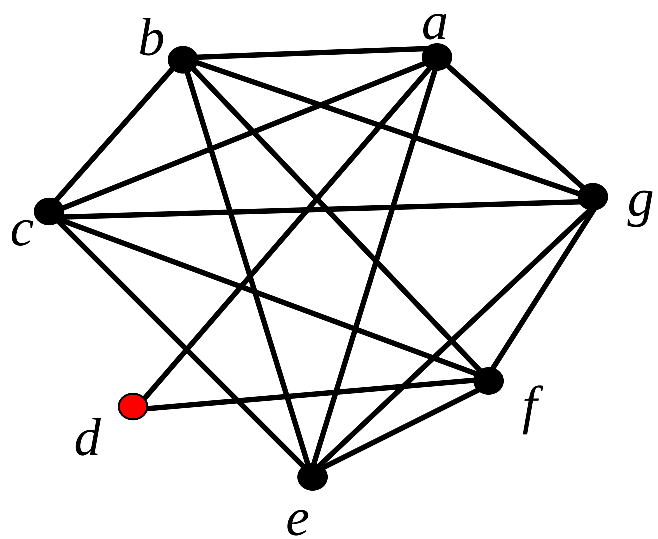
$\Rightarrow$Hamilton circuit exists: a, c, e, g, b, d, f, a

8.5 Euler & Hamilton Paths

- **Theorem 4:**

  If G is a simple graph with $n \geq 3$ vertices such that $\deg(u)+\deg(v) \geq n$ for every pair of nonadjacent vertices u and v, then G has a Hamilton circuit.

- **Example:**



each nonadjacent vertex pair has deg sum $\geq n = 7$

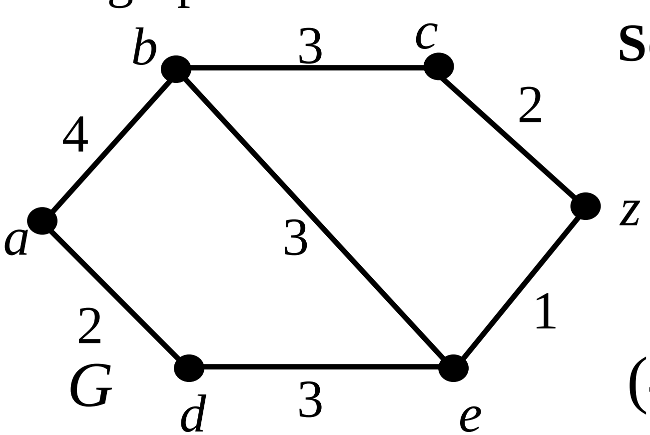
$\Rightarrow$Hamilton circuit exists: a, d, f, e, c, b, g, a

8.6 Shortest-Path Problems

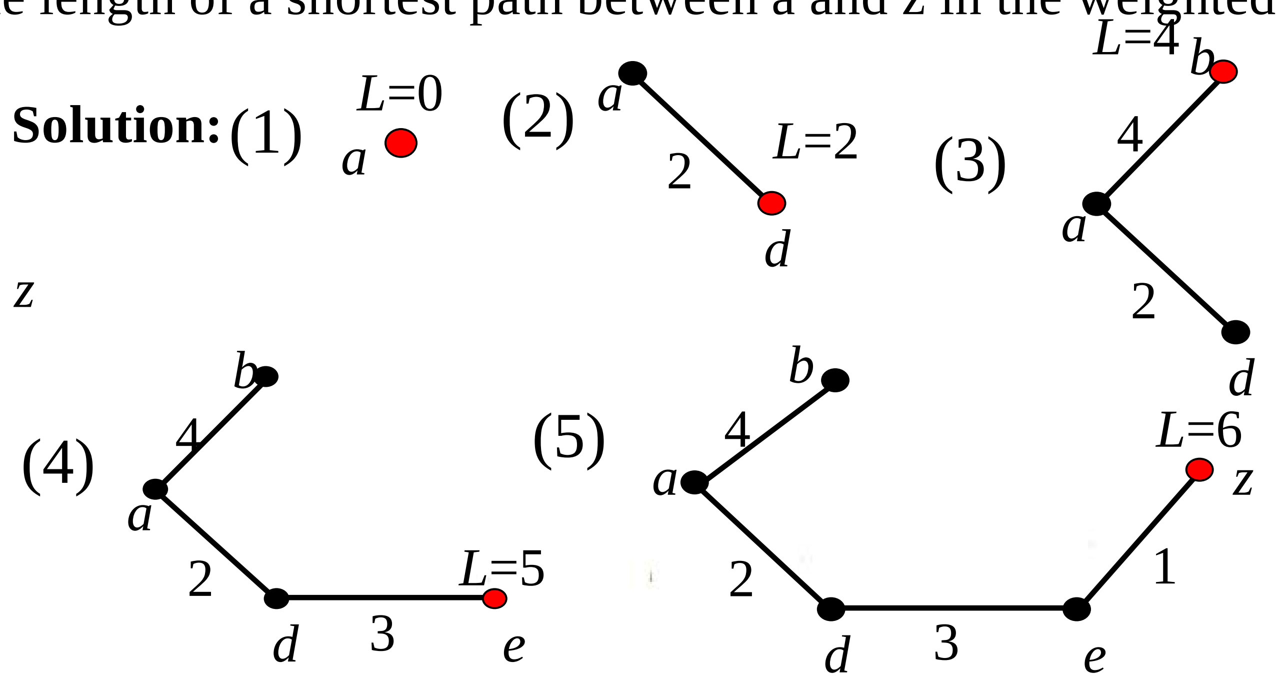
8.6 Shortest-Path Problems

- **Definition:**
  - ✓ Graphs that have a number assigned to each edge are called ***weighted graphs***.
  - ✓ The ***length*** of a path in a weighted graph is the sum of the weights of the edges of this path.
- **Shortest path Problem:**

  Determining the path of least sum of the weights between two vertices in a weighted graph.
- **Example 1:** What is the length of a shortest path between a and z in the weighted graph G?



Solution:



length=6

8.6 Shortest-Path Problems

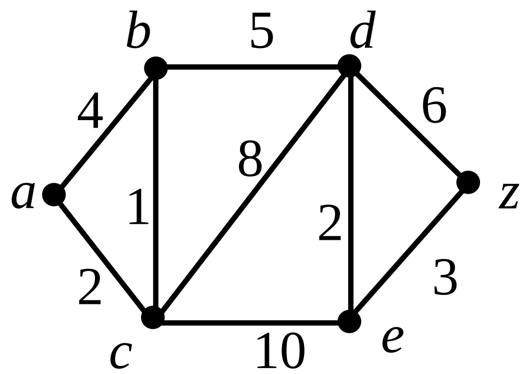
- **Dijkstra's Algorithm:** (find the length of a shortest path from a to z)

Procedure *Dijkstra*(G: weighted connected simple graph, with all weights positive)

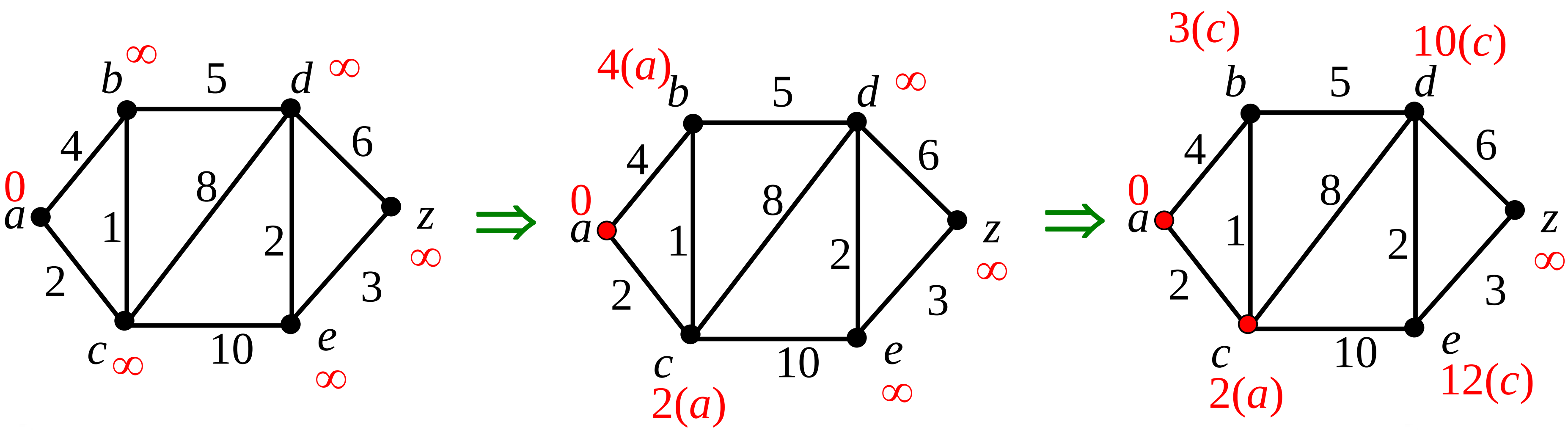
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
while $z \notin S$
begin
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(\mathrm{u}) + w(u, v) < L(v)$ **then** $L(v) := L(\mathrm{u}) + w(u, v)$
end {$L(z)$ = length of a shortest path from a to z}

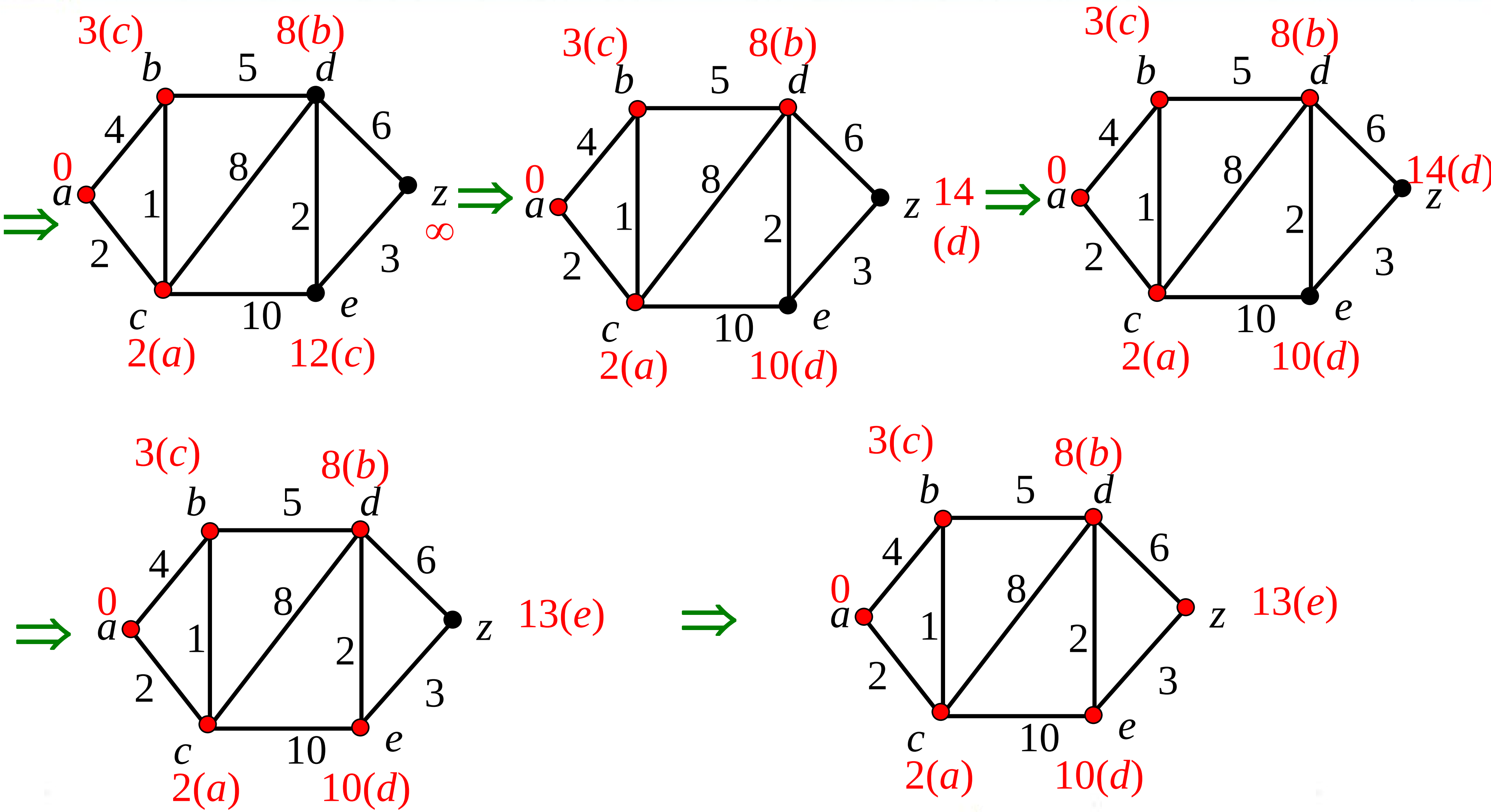
- **Example 2:** Use Dijkstra’s algorithm to find the length of a shortest path between a and z in the weighted graph.



Solution:



8.6 Shortest-Path Problems

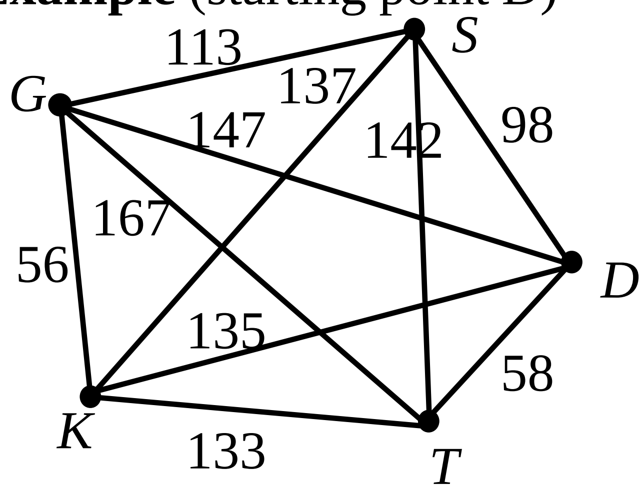


⇒ path: *a*, *c*, *b*, *d*, *e*, *z*

length: 13

8.6 Shortest-Path Problems

- **Theorem 1:**
  Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.
- **Theorem 2:**
  Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.
- **The Traveling Salesman Problem:**
  A traveling salesman wants to visit each of n cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?
- **Example** (starting point D)



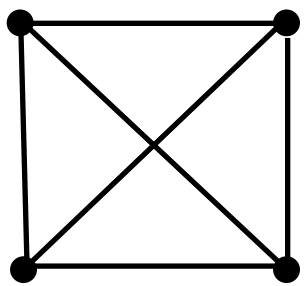
$D \to T \to K \to G \to S \to D$: 458
$D \to T \to S \to G \to K \to D$: 504
$D \to T \to S \to K \to G \to D$: 540

8.7 Planar Graphs

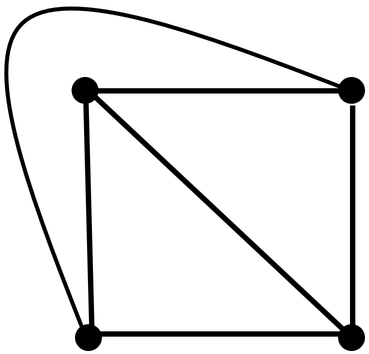
8.7 Planar Graphs

- **Definition 1:**
  - ✓ A graph is called ***planar*** if it can be drawn in the plane without any edge crossing.
  - ✓ Such a drawing is called a ***planar representation*** of the graph.
- **Example 1:** Is K_4 planar?

  Solution: K_4 is planar because it can be drawn without crossings, as shown in Figure

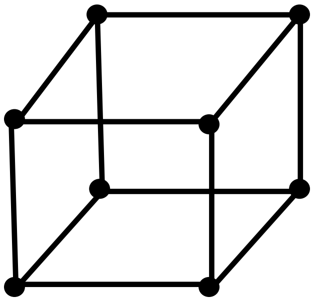


K_4

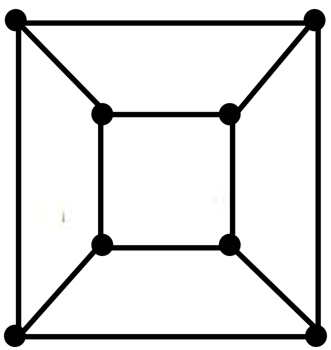


- **Example 2:** Is Q_3 planar?

  Solution: Q_3 is planar, because it can be drawn without any edges crossing.

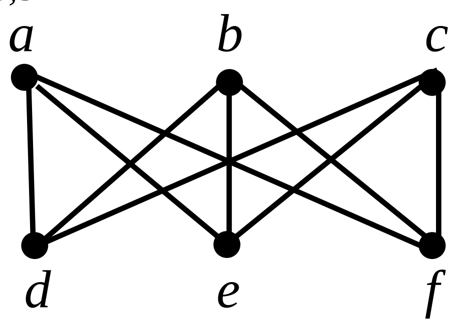


Q_3

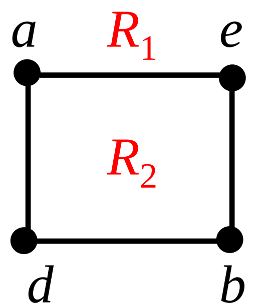


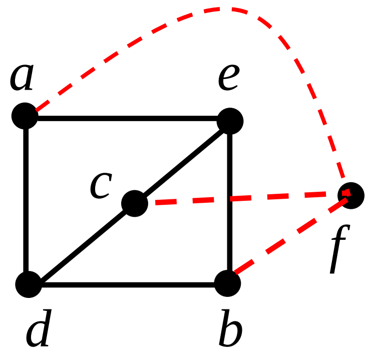
8.7 Planar Graphs

- **Example 3:** Show that $K_{3,3}$ is nonplanar.



Solution: Not possible to drawing without crossing.



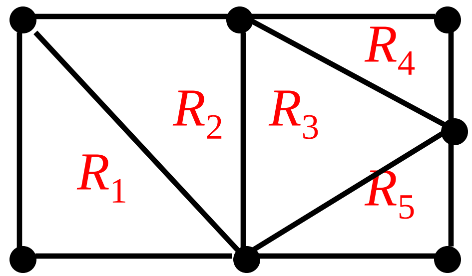


8.7 Planar Graphs

- **Euler's Formula:**

  A planar representation of a graph splits the plane into ***regions***, including an unbounded region.

- **Example:** How many regions are there in the following graph?



  Solution: 6

- **Theorem 1: (Euler's Formula)**

  Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then r = e-v +2.

- **Example 4:** Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

  Solution: $v = 20$, $2e = 3\times 20 = 60$, $e = 30$

  $r = e\text{-}v\text{+}2 = 30\text{-}20\text{+}2 = 12$

8.7 Planar Graphs

- **Corollary 1:**
  If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.
- **Example 5:** Show that K_5 is nonplanar
  Solution: $v = 5$, $e = 10$, but $3v - 6 = 9$.
- **Corollary 2:**
  If G is a connected planar simple graph, then G has a vertex of degree ≤ 5.
  Proof: Let G be a planar graph of v vertices and e edges.
  If $\deg(v) \geq 6$ for every $v \in V(G)$

$$\Rightarrow \sum_{v \in V(G)} \deg(v) \geq 6v$$

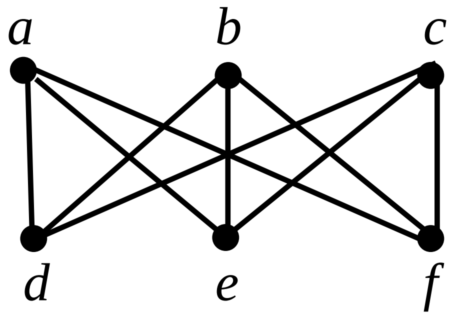
$$\Rightarrow \quad 2e \geq 6v \qquad \rightarrow\leftarrow \ (e \leq 3v - 6)$$

- **Corollary 3:**
  If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

8.7 Planar Graphs

- **Example 6:** Show that $K_{3,3}$ is nonplanar by Cor. 3.

  Solution: Because $K_{3,3}$ has no circuits of length three, and $v = 6$, $e = 9$, but $e = 9 > 2v - 4$.



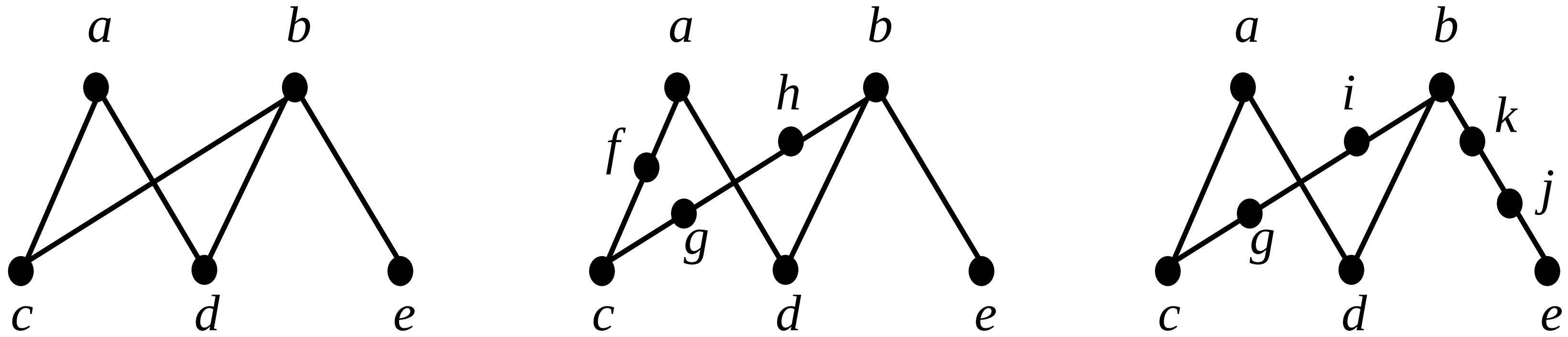
- **Kuratowski's Theorem:**

  If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{v, w\}$.

  Such an operation is called an elementary subdivision.
  Two graphs $G_1 = (V_1, E_1)$, $G_2=(V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisons.

8.7 Planar Graphs

- **Example 7:** Show that the graphs G_1, G_2, and G_3 are all homeomorphic.

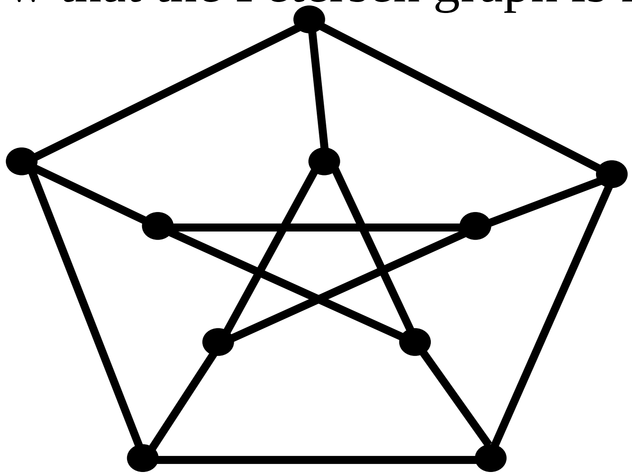


Solution: all three can be obtained from G_1

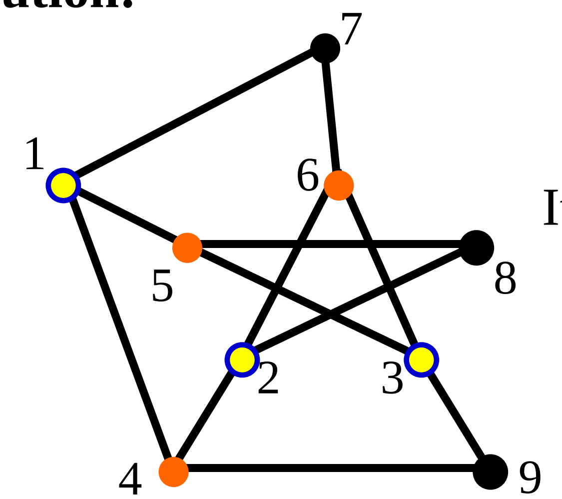
- **Theorem 2: (Kuratowski Theorem)**
  A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

8.7 Planar Graphs

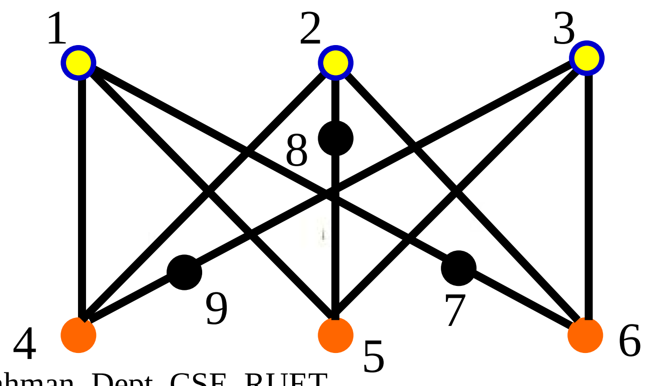
Example 9: Show that the Petersen graph is not planar.



Solution:



It is homeomorphic to $K_{3,3}$.

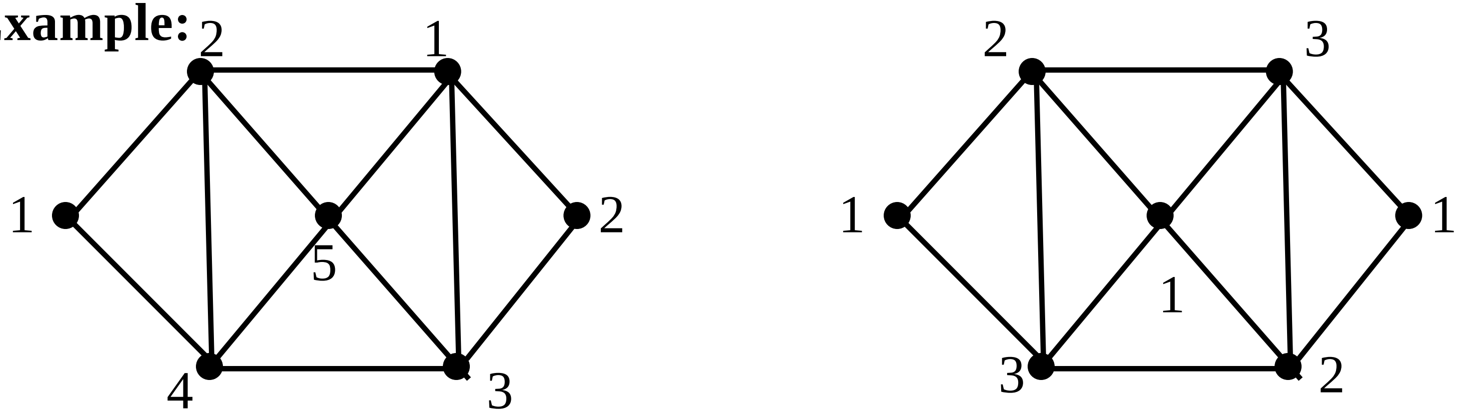


8.8 Graph Coloring

8.8 Graph Coloring

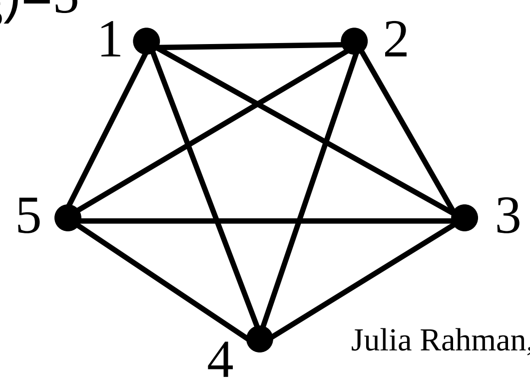
- **Definition 1:**

  A ***coloring*** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- **Example:**



- **Definition 2:**

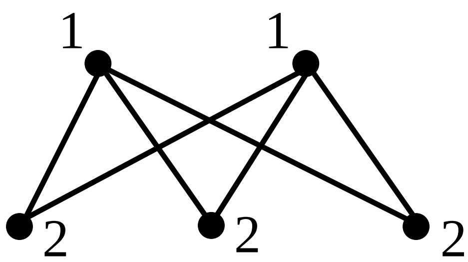
  The ***chromatic number*** of a graph is the least number of colors needed for a coloring of this graph. (denoted by $\chi(G)$)
- **Example 2:** $\chi(K_5)=5$



Note: $\chi(K_n)=n$

8.8 Graph Coloring

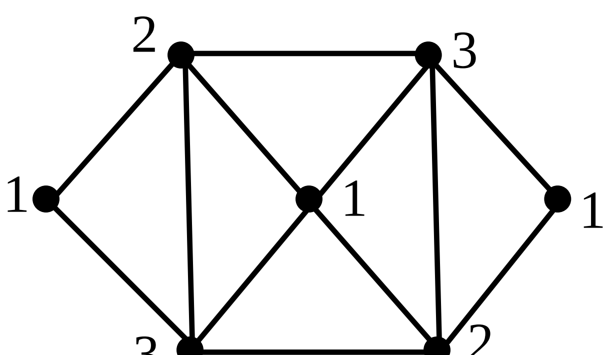
- **Example:** $\chi(K_{2,3}) = 2$.

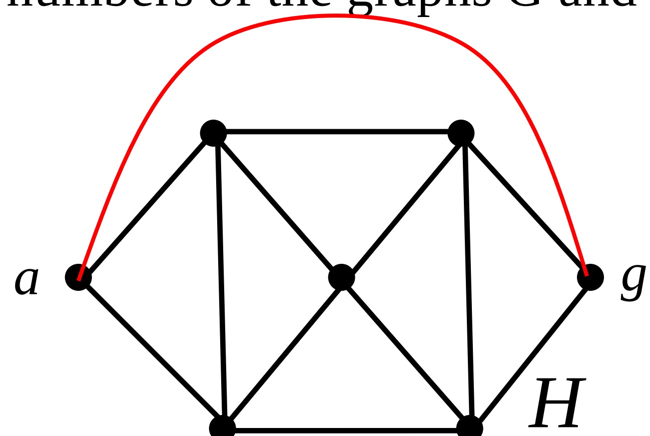


Note: $\chi(K_{m,n}) = 2$

Note: If G is a bipartite graph, $\chi(G) = 2$.

- **Example 1:** What are the chromatic numbers of the graphs G and H?





Solution: G has a 3-cycle

$\Rightarrow \chi(G) \geq 3$

G has a 3-coloring

$\Rightarrow \chi(G) \leq 3$

$\Rightarrow \chi(G) = 3$

Solution: any 3-coloring for

H-$\{(a,g)\}$ gives the same color to a and g

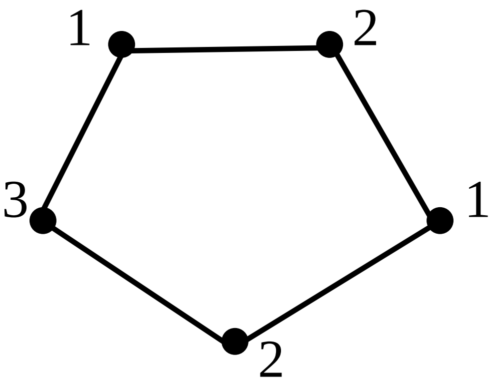
$\Rightarrow c(H) > 3$

4-coloring exists $\Rightarrow \chi(H) = 4$

8.8 Graph Coloring

- **Example 4:** $\chi(C_n) = \begin{cases} 2 \text{ if } n \text{ is even,} \\ 3 \text{ if } n \text{ is odd.} \end{cases}$

  C_n is bipartite when n is even.



- **Theorem 1:** (The Four Color Theorem)
  The chromatic number of a planar graph is no greater than four.
- **Corollary:**
  Any graph with chromatic number > 4 is nonplanar.
- **Applications of graph coloring:**
  1) Scheduling final exam
  2) Frequency assignment
  3) Index registering