Chapter 5

Representations of C*-Algebras

Problem 1. Let τ be a pure state on a C*-algebra A, and y a unit vector in H_τ such that $\tau(a) = \langle \varphi_\tau(a)(y), y \rangle$ for all $a \in A$. Show that there is a scalar λ of modulus one such that $y = \lambda x_\tau$.

Solution. Since τ is pure, (H_τ, φ_τ) is an irreducible representation, and hence every non-zero vector is cyclic. Define an operator $u : H_\tau \to H_\tau$, which is given by $\varphi_\tau(a)x_\tau \mapsto \varphi_\tau(a)y$ and extended by density. Then one can directly shows that u is a unitary, $ux_\tau = y$, and u commutes with $\varphi_\tau(a)$ for all $a \in A$, or one can apply Theorem 5.1.4 to prove this. So $u \in \varphi_\tau(A)' = \mathbb{C}1$, and has to be $\lambda 1$ for some complex number λ with modulus one. Since $ux_\tau = y$, we have thus shown that $y = \lambda x_\tau$.

Problem 2. Let H be a Hilbert space and x a unit vector of H. Show that the functional

$$\omega_x : B(H) \to \mathbb{C}, \quad u \mapsto \langle u(x), x \rangle,$$

is a pure state of $B(H)$. Show that not all pure states of $B(H)$ are of this form if H is separable and infinite-dimensional.

Solution. Consider the representation $\mathrm{id} : B(H) \to B(H)$. It is irreducible, so by Theorem 5.1.7, ω_x is pure. The second statement can by proved in two different ways.

Thanks to this post on MathStackExchange!

The first proof uses the existence of irreducible representations of any non-zero C*-algebras. We form the *-homomorphism $B(H) \to B(H)/K(H) =: Q(H)$, and find an irreducible representation $Q(H) \to B(K)$. Since $B(H) \to Q(H)$ is surjective, the compostion of the two maps gives an irreducible representation $B(H) \to B(K)$. Now take any unit vector in K and proceed as in the problem, we get a pure state τ on $B(H)$. Such a pure state is zero on $K(H)$, which is impossible for the pure states of the form ω_x, so we find what we want.

The second proof is by a construction. Consider a diagonal operator T whose diagonal sequence is $(\frac{n-1}{n})_{n\geqslant 1}$, then $\|T\| = 1$ and $\|Tx\| < \|x\|$ for any non-zero $x \in H$. By Theorem 5.1.11, there is a pure state ρ of A such that $\|T^2\| = \rho(T^2)$. If ρ is of the form ω_x, then $\|Tx\|^2 = \langle T^2 x, x\rangle = \|T^2\| = \|T\|^2$, which is impossible. So ρ is what we want.

Note that both proofs rely on the existence of pure states, or irreducible representations, which is a consequence of Krein-Milman Theorem. So to some extent, the second proof is not totally based on constructions.

Problem 3. Give an example to show that a quotient C*-algebra of a primitive C*-algebra need not be primitive.

Solution. The Toeplitz algebra $\mathbb{A}$ is primitive, since it can be irreducibly faithfully represented on the Hardy space $H^2(\mathbb{T})$. One can recall this result in Theorem 3.5.5. Also recall that $C(\mathbb{T})$ is a quotient C*-algebra of $\mathbb{A}$, which is not primitive, so this is the example what we want.

Problem 4. If I is a primitive ideal of a C*-algebra A, show that $M_n(I)$ is a primitive ideal of $M_n(A)$. (Thus, if A is primitive, so is $M_n(A)$.)

Solution. Since I is a primitive ideal of A, there exists a non-zero irreducible representation (H, φ) of A such that $I = \ker(\varphi)$. Intuitively, we can form the tensor product $M_n(\mathbb{C}) \otimes B(H)$, which will be introduced in Chapter 6. The process can be stated as follows:

Form the direct sum of n copies of H, denoted by $H^{(n)}$. Then one can regard operators in $B(H^{(n)})$ as a matrix of size $n \times n$ with entries in $B(H)$. So there is a representation of $M_n(A)$ on $H^{(n)}$, namely

$$\varphi \otimes \mathrm{id} : M_n(A) \to B(H^{(n)}), \quad (a_{ij}) \mapsto (\varphi(a_{ij})).$$

To prove that $\varphi \otimes \mathrm{id}$ is irreducible, one can use Kadison's transitivity theorem. For any non-zero vector $0 \neq x = (x_1, x_2, \ldots, x_n) \in H^{(n)}$ and $y = (y_1, \ldots, y_n) \in H^{(n)}$, there is at least one of $x_j, j = 1, \ldots, n$ which is non-zero. WLOG, suppose $x_1 \neq 0$. By Kadison's transitivity theorem, there exists $a_{i1} \in A$ such that $\varphi(a_{i1})(x_1) = y_i$. When $j > 1$, set $a_{ij} = 0$. Then the $n \times n$ matrix $(a_{ij}) \in M_n(A)$ satisfies that $(\varphi \otimes \mathrm{id})(a_{ij})(x) = y$. This shows the irreducibility of $\varphi \otimes \mathrm{id}$. Obviously, its kernel is $M_n(I)$, so $M_n(I)$ is a primitive ideal of $M_n(A)$.

Problem 5. Let A be a C*-algebra. Show the following conditions are equivalent:

(a) A is prime.

(b) If $aAb = 0$, then a or $b = 0$ $(a, b \in A)$.

Solution. If (a) holds, then by Remark 5.4.2, (b) holds.

If (b) holds, suppose I, J are two closed ideals of A and $I \cap J = IJ = 0$. If I and J are both non-zero, then there exist $0 \neq a \in I, 0 \neq b \in J$. For any $x \in A$, we have $ax \in I$, so $axb \in IJ$, and thus $axb = 0$. This shows that $aAb = 0$, but $a \neq 0, b \neq 0$, which contradicts (b).

Problem 6. Let S be a set of C*-subalgebras of a C*-algebra A that is *upwards-directed*, that is, if $B, C \in S$, then there exists $D \in S$ such that $B, C \subseteq D$. Show that $\overline{\cup S}$ is a C*-subalgebra of A.

Suppose that all the algebras in S are prime and that $A = \overline{\cup S}$. Show that A is prime.

Solution. Note that the norm closure of a *-subalgebra in a C*-algebra is necessarily a C*-subalgebra. Clearly $\cup S$ is a *-subalgebra, so the first statement is obvious.

For the second statement, suppose I, J are closed ideals of A and $IJ = 0$. Then for any $B \in S$, the intersections $I \cap B$ and $J \cap B$ are closed ideals in B, and $(I \cap B)(J \cap B) = 0$. Since B is prime, we have $I \cap B = 0$ or $J \cap B = 0$. Now assume that $\overline{\cup_{B \in S} B \cap I} = I$ (and similarly for J), which will be proved later. Then if $I \neq 0, J \neq 0$, there will exist $B_1, B_2 \in S$ such that $B_1 \cap I \neq 0, B_2 \cap J \neq 0$. Since S is upwards-directed, there exists $B \in S$ such that $B_1 \subseteq B, B_2 \subseteq B$. This implies that $B \cap I \neq 0, B \cap J \neq 0$, which is a contradiction.

Now prove the assumption that $\overline{\cup_{B \in S} B \cap I} = I$. This can be found in Theorem 6.2.6. Denote $\overline{\cup_{B \in S} B \cap I}$ by K, and clearly $K \subseteq I$ is a closed ideal. From this inclusion, one can consider the well-defined *-homomorphism $\varphi : A/K \to A/I, \quad a + K \mapsto a + I$. Note that $A/K = \overline{\cup_{B \in S}(B + K)/K}$ and we want to prove that φ is isometric on each $(B + K)/K$.

Consider the maps $\psi : B/(B \cap K) \to (B+K)/K$ given by $b+(B\cap K) \mapsto b+K$, and $\theta : B/(B \cap I) \to (B+I)/I$ defined similarly. These two maps are well-defined *-isomorphisms. Note that $B \cap K = B \cap I$, so $\theta \circ \psi^{-1} : (B+K)/K \to (B+I)/I$ is a *-isomorphism, whose explicit formula is $b + K \mapsto b + I$, which coincides with the restriction of φ. Therefore, φ is isometric and injective. This shows that $K = I$ and completes the proof.

Problem 7. If A is a C*-algebra, its *center* C is the set of elements of A commuting with every element of A. Show that C is a C*-subalgebra of A. Show that if A is simple, then $C = 0$ if A is non-unital and $C = \mathbb{C}1$ if A is unital.

Solution. This result holds for any prime C*-algebra, see this post.

Since a simple C*-algebra is necessarily primitive, and thus prime, the above link gives a proof of this result. For another generalization, see this post, which states that for a primitive C*-algebra A, the multiplier algebra $M(A)$ has trivial center. One can easily use the argument I will give below to prove this (which is actually similar to the answer.)

Here, I will prove for primitive C*-algebras. In this case, A admits a faithful irreducible representation (H, φ). Since C commutes with A, we have $\varphi(C) \subseteq \varphi(A)' = \mathbb{C}\mathrm{id}_H$, which completes the proof.

Problem 8. Let S be an upwards-directed set of closed ideals in a C*-algebra A. Suppose that $A = \overline{\cup S}$, and that all of the C*-algebras in S are postliminal. Show that A is postliminal.

Solution. Suppose (H, φ) is a non-zero irreducible representation of A. We want to prove that there is a non-zero compact operator in $\varphi(A)$.

By Theorem 5.5.2 and the fact that every closed ideal is a hereditary C*-subalgebra, the restriction $(H_I, \varphi_I) = (H, \varphi)_I$ is an irreducible representation of I for every $I \in S$. Here the representation space H_I is $[\varphi(I)H]$, the closure of $\varphi(I)H$, so we know that $\varphi(b) = \varphi_I(b) \oplus 0$ for every $b \in I$, where 0 is the zero operator on $H_I^{\perp}$.

So we know from this decomposition that $\varphi(b)$ is compact if and only if $\varphi_I(b)$ is compact, and $\varphi(b) = 0$ if and only if $\varphi_I(b) = 0$. Since φ is a non-zero representation of A and $A = \overline{\cup_{I \in S} I}$, there exists some $I \in S$ such that I is not contained in $\ker(\varphi)$. Then (H_I, φ_I) is non-zero, and since I is postliminal, there exists some $b \in I$ such that $\varphi_I(b)$ is non-zero and compact. According to the previous observation, $\varphi(b)$ is non-zero and compact, so we find a non-zero compact operator in $\varphi(A)$. This implies that $K(H) \subseteq \varphi(A)$ by Theorem 2.4.9.

It should be pointed out that "upwards-directed" is not needed to prove "postliminal", but to ensure that $\overline{\cup S}$ is a C*-subalgebra (see Problem 6) when the condition $A = \overline{\cup S}$ does not hold.

Problem 9. Let A be a C*-algebra. If I, J are postliminal ideals in A (that is, closed ideals that are postliminal C*-algebras), show that $I + J$ is postliminal also. Deduce from this and Problem 8 that there is a largest postliminal ideal I in A (which may, of course, be the zero ideal). Show that A/I has no non-zero postliminal ideals.

Solution. We may form a short exact sequence

$$0 \to I \to I + J \to (I + J)/I \to 0,$$

and recall that $(I + J)/I \cong I/(I \cap J)$ by Remark 3.1.3. So by Theorem 5.6.2, $(I+J)/I$ is postliminal, as a quotient of the postliminal C*-algebra I. Then by Theorem 5.6.2 again, applied to the mentioned short exact sequence, we know that $I + J$ is postliminal.

Let S be the set of all postliminal ideals in A. Then S is non-empty since $0 \in S$, and S is upwards-directed by the previous paragraph. So $I := \overline{\cup S}$ is a closed ideal of A, and B is postliminal by Problem 8. Then I is the largest postliminal ideal in A.

Suppose A/I has a postliminal ideal J_0. Let π be the quotient map $A \to A/I$. Then $J := \pi^{-1}(J_0)$ is a closed ideal in A containing I. Again we have a short exact sequence

$$0 \to I \to J \to J_0 \to 0,$$

so J is postliminal by Theorem 5.6.2. However, I is the largest postliminal ideal in A, so $I = J$, which means that $J_0 = 0$. Now we complete the proof.