Some collections of solutions to Analysis and Differential Equations

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Contents

1 2010

1

(a) Apply the Jensen inequality to $\log(\sin x/x), x \in (0, \pi)$.

(b) Use a sector of angle $\pi/4$ and the function $f(z) = e^{iz^2}$.

2

Also see 2016, p3.

Let A be the set of points of continuity. Let

$$\omega(x) = \lim_{\varepsilon \to 0} \min\Big(1, \sup\{f(y) : y \in (x-\varepsilon, x+\varepsilon)\} - \inf\{f(z) : z \in (x-\varepsilon, x+\varepsilon)\}\Big).$$

So $A = \{x \in \mathbb{R} : \omega(x) = 0\}$. Consider $\{x \in \mathbb{R} : \omega(x) < 1/n\}$. If $\omega(x) < 1/n$, then there exists $\varepsilon_0 > 0$ such that whenever $0 < \varepsilon < \varepsilon_0$, we have

$$\sup\{f(y) : y \in (x-\varepsilon, x+\varepsilon)\} - \inf\{f(z) : z \in (x-\varepsilon, x+\varepsilon)\} \leqslant c < 1/n.$$

Then for $x' \in (x - \varepsilon_0, x + \varepsilon_0)$, $0 < \varepsilon < \varepsilon_0/2$, we have

$$\sup\{f(y) : y \in (x'-\varepsilon, x'+\varepsilon)\} - \inf\{f(z) : z \in (x'-\varepsilon, x'+\varepsilon)\} \leqslant c.$$

So $\omega(x') \leqslant x < 1/n$. This proves that $\{x \in \mathbb{R} : \omega(x) < 1/n\}$ is open for each $n \in \mathbb{N}^*$. So A is a countable intersection of open sets.

3

Consider the finite Blaschke product

$$g(z) = \Big(\frac{z - z_0}{1 - \overline{z_0}z} \cdot \frac{z + z_0}{1 + \overline{z_0}z}\Big)^m = (\frac{z^2 - z_0^2}{1 - |z_0|^2 z^2})^m.$$

Then $h = f/g$ satisfies: (1) h is holomorphic in D. (2) When $|z| < 1$, we have $|h(z)| \leqslant 1$. We explain (2) in details. The function g is holomorphic on a neighbourhood of $\overline{D}$ and $|g(z)| = 1$ whenever $|z| = 1$, so by uniform continuity, we know there is a function $a(r) \geqslant 0$ such that $\lim_{r \to 1} a(r) = 0$

and $\Big||g(z)|-1\Big| \leqslant a(|z|)$. Then on $\{|z|=r\}$ for any $r<1$, we have $|h(z)| \leqslant 1/(1-a(r))$, so by the maximum principle,

$$|h(z)| \leqslant \frac{1}{1-a(r)}, \quad \forall |z| \leqslant r.$$

Let $r \to 1-$, so (2) holds.

Now we know

$$|f(0)| = |g(0)||h(0)| \leqslant |g(0)| = |z_0|^{2m} \leqslant (1-1/m)^{2m} < e^{-2}.$$

4

Also see 2014, p4. Just an application of the formula there. The answer is

$$\frac{2}{\pi}\arctan(\frac{y}{x}), \quad x>0.$$

Also one can think of the argument function. Since the argument function is formally the imaginary part of the logarithm, it is harmonic on suitable domains.

5

This is a false statement. We first prove for $K(x,y) \in C^0([0,1]\times[0,1])$ and then give a counterexample when $K \in L^1$.

For $\delta \in \mathbb{R}$ such that $x, x+\delta \in [0,1]$, we have

$$\begin{aligned}|Tf(x)-Tf(x+\delta)| &= \Big|\int_0^1 \Big(K(x,y)-K(x+\delta,y)\Big)f(y)dy\Big| \\ &\leqslant \int_0^1 |K(x,y)-K(x+\delta,y)||f(y)|dy.\end{aligned}$$

Since K is continuous on a compact set, it is uniformly continuous, so if we let

$$\omega(r) = \sup\{|K(x_1,y)-K(x_2,y)| : |x_1-x_2| \leqslant r\},$$

we have $\omega(r) \to 0$ as $r \to 0$, and

$$|Tf(x)-Tf(x+\delta)| \leqslant \omega(\delta)\|f\|_\infty.$$

This proves two things. The first is that Tf is continuous. The second is that the family Ω is equi-continuous because ω depends on K only. Note that the family Ω is also uniformly bounded by $\|K\|_\infty$, so by the Arzelà-Ascoli theorem, the set Ω is precompact in $C^0([0,1])$.

If $K \in L^1$ only, it can be directly proved that Tf is only a.e. defined. To be precise, if $K \equiv 0$, then $Tf \equiv 0$, but if $K(x,y) = \chi_{\{1\}\times[0,1]}$, then $Tf(1) = \int_0^1 f(y)dy$. But these two K are identified in L^1, so we cannot have a well-defined value everywhere for Tf.

2 2011

1

(a) The integrand is absolutely integrable because it is $O(|x|^{-3})$ at infinity. Then note that the integrand is an odd function.

(b) Consider $f(x) = x^a$. Choose a suitable a.

3

This is the domain $\{re^{i\theta} : r > 1, -\pi < \theta < \pi\}$. So take the reciprocal $z \mapsto 1/z$, the domain becomes

$$\{re^{i\theta} : 0 < r < 1, -\pi < \theta < \pi\}.$$

Take $z \mapsto \sqrt{z}$ to obtain the right half of the unit disk. By

$$z \mapsto (\frac{z+i}{z-i})^2,$$

mapped onto the lower half plane. Then to the unit disk.

4

I suggest doing this problem after 2012, p3.

For any $x \in [a, b]$, we have

$$f'(x) = \frac{f(x+2h) - f(x)}{2h} - f''(\xi)h,$$

provided $x + 2h \in [a, b], h \neq 0, \xi = \xi_{x,h}$ between x and $x + 2h$. So

$$|f'(x)| \leqslant \frac{A}{h} + Bh.$$

If $h = \sqrt{A/B}$ is possible, then $|f'(x)| \leqslant 2\sqrt{AB}$. If $h = \sqrt{A/B}$ is impossible, then $|x - x_0| < \sqrt{A/B}$. Then

$$f'(x) - f'(x_0) = f''(\eta)(x - x_0) \Longrightarrow |f'(x)| \leqslant D + \sqrt{AB}.$$

So we always have $|f'(x)| \leqslant 2\sqrt{AB} + D$.

5

The statement is similar to the cone version of the Hahn-Banach theorem (also called the Krein-Riesz extension theorem). But the conditions here are much stronger, so the proof is much easier.

Let $\mathbf{1}$ refer to the constant function on $[0,1]$. By density of X, there exists an element e such that $\|e-\mathbf{1}\|_\infty < 1/2$. So the function e is everywhere larger than $1/2$. For any element $x \in X$, we know $x \leqslant \|x\|\mathbf{1}$ pointwise, so $x \leqslant 2\|x\|e$ pointwise. By the assumption, we have

$$l(x) \leqslant 2\|x\|l(e),$$

so l is bounded on X. Then we can use the Hahn-Banach extension to obtain a bounded extension L of l. This $L : C([0,1]) \to \mathbb{R}$ corresponds to a unique Borel measure μ on $[0,1]$ such that

$$L(f) = \int f d\mu, \quad \forall f \in C[0,1]$$

because of the Riesz representation theorem and the fact that any finite Borel measure on a compact metric space is automatically regular and $\mu([0,1]) = L(\mathbf{1}) < \infty$. The uniqueness when considering $f \in X$ only comes from the density of X.

Theorem 2.1 (Krein-Riesz)**.** Let X be a real linear space. Say $P \subseteq X$ is a cone if $P + P \subseteq P$ and $tP \subseteq P, \forall t \geqslant 0$. Let $Y \subseteq X$ be a linear subspace, $\varphi : Y \to \mathbb{C}$ be a real linear functional. Suppose φ is P-positive, i.e. $\varphi(x) \geqslant 0, \forall x \in P \cap Y$. If for any $x \in X$, there exists $y \in Y$ such that $x - y \in P$, then φ can be extended to a P-positive linear functional $\psi : X \to \mathbb{R}$.

One can prove this from the Hahn-Banach theorem. Let

$$\rho(x) = \inf_{y \in Y, y-x \in P} \varphi(y).$$

Then check the conditions for the usual Hahn-Banach theorem.

For discussions on regularity, see Folland Chapter 7.

3 2012

1

One can consider the contour integral, with γ the keyhole contour, the integrand
$$f(z) = \frac{e^{p\log z}}{z^2+1},$$
where the branch of log is chosen so that when z approaches the positive real axis, the function $\log z$ approaches the usual logarithm.

The residues of f at $\pm i$ are
$$\operatorname{Res}(f;i) = \frac{e^{\pi pi/2}}{2i}, \quad \operatorname{Res}(f;-i) = \frac{e^{-\pi pi/2}}{-2i}.$$
Another way is to use the change of variables $x = \tan\theta$ and this is the form of the integral of a Beta function.

2

Under the map $z \mapsto 1/z$, the region becomes $\{x+iy : -2 < y < -1\}$. Then use the map
$$z \mapsto \exp(\pi z)$$
to obtain the upper half plane. Then the rest is clear.

3

Consider the Taylor expansion
$$f(x+2h) = f(x) + f'(x)\cdot 2h + \frac{f''(\xi)}{2}\cdot(2h)^2, \quad \exists\xi = \xi_{x,h} \in (x, x+2h),$$
where h is an arbitrary positive number. Then we have
$$f'(x) = \frac{f(x+2h)-f(x)}{2h} - hf''(\xi).$$
If we denote $A_0 = \sup|f(x)|$, $A_2 = \sup|f''(x)|$, then this implies that
$$|f'(x)| \leqslant \frac{A_0}{h} + A_2 h.$$
Since this holds for any $h > 0$, we let $h = \sqrt{A_0/A_2}$ to obtain $|f'(x)| \leqslant 2\sqrt{A_0 A_2}$.

4

This is a false statement. The function $n(y)$ is called the **Banach indicatrix function** of f. It is not always measurable. A theorem of Banach says that if f is continuous on $[a, b]$, then $n(y)$ is measurable. Moreover, if $m \leqslant f \leqslant M$ on $[a, b]$, then the integral $\int_m^M n(y)dy$ is equal to the total variation of f on $[a, b]$.

A more general theorem proves that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is measurable and $f(B)$ is measurable for any Borel measurable B, then n is measurable. See this article.

Here is a counterexample. If n is measurable on y, then $n^{-1}(\mathbb{N}^* \cup \{\infty\}) = \mathrm{im}(f)$ is Borel measurable in $\mathbb{R}$. We have to emphasize **Borel** for the space y lies in because we are asking that n is defined on the codomain of f. As a measurable function (no matter Borel or Lebesgue), the key is that we are always requiring Borel-Borel measurable or Lebesgue-Borel measurable. Even continuous functions can fail to be Lebesgue-Lebesgue measurable. So now, we only need to give an example to show that a measurable function f on $[a, b]$ may not have a Borel image. A non-trivial fact is that f does have a Lebesgue measurable image if f is Borel measurable.

Assume $[a, b] = [0, 1]$. Take a bijection between the Cantor ternary set $C \subset [0, 1]$ and the Vitali set V which is not Lebesgue measurable. Extend by zero. This gives a Lebesgue measurable example for our need.

An example of Borel measurable function does exist but is beyond our scope.

5

The Hölder inequality shows that $\|F\| \leqslant \|g\|_q$. To prove the equality, we consider (WLOG $g \neq 0$)

$$f = \frac{|g|^{q-1}\mathrm{sign}(g)}{\|g\|_q^{q-1}},$$

where $\mathrm{sign}(g)g = |g|$ and $\mathrm{sign}(g)$ is measurable. So $\|f\|_p = 1$. And we compute that

$$\int fg = \|g\|_q.$$

So $\|F\| \geqslant \|g\|_q$.

This proposition holds when $p = \infty, q = 1$. If we desire the result for $p = 1, q = \infty$, we need to assume that the measure space is semi-finite.

4 2013

1

Consider the non-negative function on $(\mathbb{R}^d \times [0, \infty), \mathrm{Leb} \otimes \mathrm{Leb})$:

$$g(x,t) = \chi_{|f(x)|>t}(x,t).$$

By the Tonelli's theorem, we have

$$\int_{\mathbb{R}^d} \int_0^\infty g(x,t)dtdx = \int_0^\infty \int_{\mathbb{R}^d} g(x,t)dxdt.$$

This is exactly what we need to prove.

2

A technical way is to consider the residues of $1/p(z)$. It has residues

$$\mathrm{Res}(1/p; a_j) = \frac{1}{p'(a_j)}.$$

The contour integral

$$\int_{|z|=R} \frac{1}{p(z)} dz \to 0, \quad R \to \infty.$$

So the sum of the residues is zero. Or we can view $1/p$ as a meromorphic function on $\overline{\mathbb{C}}$ and use the residue theorem for the obviously-related meromorphic 1-form Pdz on the compact Riemann surface $\overline{\mathbb{C}}$. The 1-form has residue zero at ∞ because $d \geqslant 2$ (use local coordinate $z \leftrightarrow 1/w$ on $\overline{\mathbb{C}} \setminus \{0\}$.

Another way is to consider the partial fraction

$$\frac{1}{p(z)} = \sum_{j=1}^{d} \frac{c_j}{z - a_j}.$$

Multiply by $z - a_j$, and let $z = a_j$, we see that $c_j = 1/p'(a_j)$. So

$$1 = \sum_{j=1}^{d} \frac{r(z-a_1)\cdots(z-a_d)}{p'(a_j)(z-a_j)}.$$

Here $r \neq 0$ is the leading coefficient of $p(z)$. The coefficient of z^{d-1} of the right side is

$$r \cdot \sum_{j=1}^{d} \frac{1}{p'(a_j)}.$$

It has to be zero by comparison to the left. In some sense these two proofs are the same.

The third proof is credited to Lin Zeshuan.

Since $a_1, \cdots, a_d$ are distinct, they are all simple zeros pf $p(z)$. So $p'(a_j) \neq 0$ for $j = 1, \cdots, d$. By the inverse function theorem for holomorphic functions, for each $j = 1, \cdots, d$ there exist an open neighbourhood $U_j \ni a_j$, an open neighbourhood $V_j \ni 0$, a holomorphic function $a_j(w) : V_j \to U_j$ satisfying $(p \circ a_j)(w) = w \quad (w \in V_i), a_j(0) = a_j$. The derivative of $a_j(w)$ is

$$\frac{d}{dw} a_j(w) = \frac{1}{p'(a_j(w))}.$$

Let $V = V_1 \cap \cdots V_d$. Then for $w \in V$, the functions $a_1(w), \cdots, a_d(w)$ are the d roots of the polynomial equation $p(z) = w$. Since $d \geqslant 2$, by Vieta's theorem we know $a_1(w) + \cdots + a_d(w)$ is constant on V. Take the derivative with respect to w at $w = 0$, and we obtain

$$\sum_{j=1}^{d} \frac{1}{p'(a_j)} = 0.$$

3

The theorem is called the **Weyl's equi-distribution law**. The problem is taken from Stein, Fourier Analysis.

(a) There is a missing $1/N$ on the left side. And we can only prove for $k \in \mathbb{Z}$. (In fact whenever $k\alpha$ is not an integral multiple of 2π, but we want generality.)

We can directly computed both sides:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{ik(x+n\alpha)} = \lim_{N\to\infty} \frac{1}{N} e^{ikx+ik\alpha} \frac{e^{ikN\alpha} - 1}{e^{ik\alpha} - 1} = 0, N \to \infty \quad \text{if } k \neq 0.$$

While $k = 0$ the conclusion is clear.

(b) For any continuous periodic function $f : \mathbb{R} \to \mathbb{C}$ of period 2π, for any $\varepsilon > 0$, there is a trigonometric polynomial P such that $|P - f| < \varepsilon$ uniformly on $[-\pi, \pi]$ by the Weierstrass approximation theorem. Now we know from (a) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)dt,$$

so

$$\varlimsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt \right| \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows.

4

There are several quite different proofs. Also see 2018, p6 for general Euclidean spaces.

(1) Claim that if u is a harmonic real-valued function on a domain $D \subseteq \mathbb{C}$, f is holomorphic on a domain $\Omega \subseteq \mathbb{C}$ and $f(\Omega) \subseteq D$, then $u \circ f$ is harmonic on Ω.

The function u is locally the real part of a holomorphic function g, so on this open set, $u \circ f$ is the real part of the holomorphic function $g \circ f$. This proves the claim, since being harmonic is a local property.

Now consider $f(z) = e^z, f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, so $u \circ f$ is a positive harmonic function on $\mathbb{C}$. Since $\mathbb{C}$ is simply connected, the function $u \circ f$ is the real part of some entire function g. The entire function has positive real part, so the entire function $\exp(-g)$ is bounded. By Liouville's theorem, g is constant, and thus u is constant.

(2) Consider $D_1 = \mathbb{C} \setminus (-\infty, 0], D_2 = \mathbb{C} \setminus [0, \infty)$. So there are two holomorphic functions f_1 on D_1 and f_2 on D_2 such that their real parts are u. The difference $f_1 - f_2$ is holomorphic on $D_1 \cap D_2$ with vanishing real part, but it is defined on a disconnected open set. So write $U_1 = \{y > 0\}, U_2 =$

$\{y < 0\}$, then $D_1 \cap D_2 = U_1 \cup U_2$, and $f_1 - f_2 \equiv c_1$ on U_1 and $f_1 - f_2 \equiv c_2$ on U_2, where c_1, c_2 are purely imaginary numbers.

If $c_1 = c_2$, then $f_1 = f_2$ is a globally well-defined holomorphic function on $\mathbb{C} \setminus \{0\}$ with positive real part, so $f_1 = f_2$ is constant and thus u is constant.

If $c_1 \neq c_2$, then the function

$$g(z) = \begin{cases} \exp\Big(-\frac{2\pi i}{c_2 - c_1} f_1(z)\Big), & z \in D_1, \\ \exp\Big(-\frac{2\pi i}{c_2 - c_1}(f_2(z) + c_1)\Big), & z \in D_2 \end{cases}$$

is a globally defined function on $\mathbb{C} \setminus \{0\}$. Its modulus is $\exp(-2\pi ui/(c_2 - c_1))$. By choosing one of g or $1/g$, we obtain a bounded holomorphic function on $\mathbb{C} \setminus \{0\}$, so it is constant.

5

This problem is taken from Stein, Real Analysis, Chapter 4 Exercise 28.

This is a little strange because of the contents. In Stein, he discussed Hilbert-Schmidt operators in some sense. He defined Hilbert-Schmidt operators by integral operators with L^2 kernels, and proved Hilbert-Schmidt operators are compact, but he did not talk about the general theory of Hilbert-Schmidt operators (on general Hilbert spaces). If we do not assume any knowledge of Hilbert-Schmidt operators, the proof will be quite long. So we will give a proof from the very beginning, but omit some details.

Lemma 4.1. If $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(B)$, then $\{\varphi_n(x)\overline{\varphi_m(y)}\}_{n,m=1}^{\infty}$ is an orthonormal basis of $L^2(B \times B)$.

Proof. Prove orthogonality by Fubini's theorem. Prove completeness by showing that if f is orthogonal to $\{\varphi_n(x)\overline{\varphi_m(y)}\}$, then $f = 0$. □

Lemma 4.2. If $K(x, y) \in L^2(B \times B)$, then the associated integral operator $T : L^2(B) \to L^2(B)$ defined by

$$Tf(x) = \int_B K(x, y) f(y) dy$$

is compact of norm $\leqslant \|K\|_{L^2(B\times B)}$.

A fact we will not prove is that $\|T\| \leqslant \|T\|_{HS} = \|K\|_{L^2(B\times B)}$.

Proof. The Fourier expansion of K is

$$K(x,y) = \sum_{n,m=1}^{\infty} c_{n,m}\varphi_n(x)\overline{\varphi_m(y)}.$$

Here $\{c_{n,m}\} \in l^2$.

First, we prove that T is bounded.

$$\begin{aligned}\Big|\int K(x,y)f(y)dy\Big| &\leqslant \int |K(x,y)||f(y)|dy \\ &\leqslant \Big(\int |K(x,y)|^2 dy\Big)^{1/2}\Big(\int |f(y)|^2 dy\Big)^{1/2}.\end{aligned}$$

So

$$\begin{aligned}\|Tf\|_{L^2}^2 &= \int |Tf(x)|^2 dx \\ &\leqslant \int \Big(\int |K(x,y)|^2 dy\Big)\Big(\int |f(y)|^2 dy\Big)dx \\ &= \|K\|_{L^2(B\times B)}^2 \|f\|_{L^2}^2.\end{aligned}$$

This proves that T is bounded of norm $\leqslant \|K\|_{L^2(B\times B)}$.

To prove compactness of T, note that in the Fourier expansion of K, if we define the partial sum

$$K_N(x,y) = \sum_{n,m=1}^{N} c_{n,m}\varphi_n(x)\overline{\varphi_m(y)},$$

then $\|K_N - K\|_{L^2(B\times B)} \to 0$ as $N \to \infty$. Each associated T_N is a finite-rank operator, so each T_N is compact, Since $T_N \to T$ in the operator norm, T is also compact. □

Now back to the original problem.

Let

$$K_n(x,y) = \begin{cases} K(x,y), & |x-y| \geqslant 1/n, \\ 0, & \text{else.}\end{cases}$$

So $K_n(x,y)$ are bounded measurable functions on $B \times B$, and thus clearly L^2. So the associated integral operators T_n are compact. To go further, we need another important lemma, called the Schur test.

Lemma 4.3 (The Schur test, general version)**.** Suppose $K(x,y)$ defined on a measure space (Ω,μ) is measurable, T is the associated integral operator

$$Tf(x)=\int_X K(x,y)f(y)d\mu(y).$$

If $1<p,q<\infty, 1/p+1/q=1$ and there exists a measurable, strictly positive function h and a constant $M>0$ such that

$$\int_\Omega |K(x,y)|h(y)^q d\mu(y)\leqslant Mh(x)^q,$$
$$\int_\Omega |K(x,y)|h(x)^p d\mu(x)\leqslant Mh(y)^p,$$

then $\|T\|\leqslant M$ as a linear operator $L^p(\Omega,\mu)\to L^p(\Omega,\mu)$.

Proof. The proof relies on the trick

$$\begin{aligned}|Tf(x)| &\leqslant \int_\Omega h(y)h(y)^{-1}|f(y)||K(x,y)|d\mu(y)\\ &\leqslant \Big(\int_\Omega |K(x,y)|h(y)^q d\mu(y)\Big)^{1/q}\Big(\int_\Omega |K(x,y)|h(y)^{-p}|f(y)|^p d\mu(y)\Big)^{1/p}\\ &\leqslant M^{1/q}h(x)\Big(\int_\Omega |K(x,y)|h(y)^{-p}|f(y)|^p d\mu(y)\Big)^{1/p}.\end{aligned}$$

So

$$\begin{aligned}\|Tf\|_p^p &\leqslant \int_\Omega M^{p/q}h(x)^p\Big(\int_\Omega |K(x,y)|h(y)^{-p}|f(y)|^p d\mu(y)\Big)d\mu(x)\\ &= \int_\Omega\int_\Omega M^{p/q}|K(x,y)|h(x)^p h(y)^{-p}|f(y)|^p d\mu(x)d\mu(y)\\ &\leqslant \int_\Omega M^{p/q}Mh(y)^p h(y)^{-p}|f(y)|^p d\mu(y)\\ &= M^p\|f\|_p^p.\end{aligned}$$

This proves the Schur test. □

When $p=q=2$ and $h=1$, the Schur test can prove that T is bounded in this problem. This is because

$$\int_B |K(x,y)|dy\leqslant M,\quad \int_B |K(x,y)|dx\leqslant M$$

for some constant $M > 0$, for example take

$$M = \int_{B_2(0)} \frac{A}{|x|^{d-\alpha}} dx$$

which is finite because $\alpha > 0$. The norm of the difference $\|T_n - T\|$ can be done similarly, like

$$\int_B |K(x,y) - K_n(x,y)| dx \leqslant \int_{B \cap B_{1/n}(y)} \frac{A}{|x-y|^{d-\alpha}} dx \leqslant \int_{B_{1/n}(0)} \frac{A}{|t|^{d-\alpha}} dt \to 0$$

uniformly on $x \in B$ as $n \to \infty$. This proves that $T_n \to T$ in the operator norm. Since each T_n is compact, we know T is compact.

Note that T itself is Hilbert-Schmidt if and only if $K \in L^2$, so in this problem, only when $\alpha > d/2$ can one prove $K \in L^2$.

6

This is just a scary-looking problem. I will not present a detailed proof here.

The integrand is certainly not integrable in the Lebesgue sense, but is better-behaved in the Riemann sense. Just consider

$$\int_{\mathbb{R}} e^{ix^2} dx = \int_{\mathbb{R}} \cos x^2 + i \sin x^2 dx.$$

This makes sense as the improper Riemann integral. The given definition is an example of the oscillatory integrals, with a single stationary point. We do not need the assumption $\lambda > 0$. A general version is for $\beta \neq 0, \mathrm{Re}\beta \geqslant 0$, we have

$$\int_{\mathbb{R}} \exp(-\beta x^2) dx = \sqrt{\pi} \beta^{-1/2},$$

with $\sqrt{\beta}$ defined on $\mathbb{C} \setminus (-\infty, 0]$.

The problem first asks you to prove this formula by the contour integral. Then a simple generalization.

To compute

$$\int_{\mathbb{R}^n} \exp\left(i\frac{\lambda}{2}\langle Ax, x\rangle - i\langle x, \xi\rangle\right) dx,$$

first diagonalize A by an orthogonal transformation, say O an orthogonal matrix and $O^{-1}AO = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. Translate x by a vector so that there is no linear term of x. Then compute.

5 2014

1

This is a really good problem. I suggest you compare this problem with the following famous lemma:

Lemma 5.1 (Fekete's subadditive lemma)**.** Suppose (a_n) is a sequence of real numbers with the subadditive property: $a_{n+m} \leqslant a_n + a_m$ for all $n, m \in \mathbb{N}^*$. Then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geqslant 1} \frac{a_n}{n} \in [-\infty, \infty) \quad \text{always exists in the extended real line.}$$

Let $g(x) = f(x) - 2014x$. Then we know

$$\sup_{x,y\in\mathbb{R}} |g(x+y) - g(x) - g(y)| < \infty,$$

and

$$\lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

Suppose $-M \leqslant g(x+y) - g(x) - g(y) \leqslant M$ for some constant $M > 0$. Let $h_1(x) = g(x) - M, h_2(x) = g(x) + M$. Then

$$h_1(x+y) \geqslant h_1(x) + h_1(y), \quad h_2(x+y) \leqslant h_2(x) + h_2(y).$$

By Fekete's lemma, we know

$$\lim_{n\to\infty} \frac{h_1(n)}{n} = \sup_{n\in\mathbb{N}^*} \frac{h_1(n)}{n}, \quad \lim_{n\to\infty} \frac{h_2(n)}{n} = \inf_{n\in\mathbb{N}^*} \frac{h_2(n)}{n}.$$

But by the construction of h_1 and h_2, the limit $\lim_{n\to\infty} h_j(n)/n = 0$ for $j = 1, 2$. So we get

$$\frac{h_1(n)}{n} \leqslant 0, \quad \frac{h_2(n)}{n} \geqslant 0, \quad \forall n \in \mathbb{N}^*.$$

This shows that $-M \leqslant g(n) \leqslant M$ for $n \in \mathbb{N}^*$.

Let $x = y = 0$ and we know $-M \leqslant g(0) \leqslant M$. Let $x = n, y = -n$ and we know $|g(k)| \leqslant 3M$ for all $k \in \mathbb{Z}$. Let $x = [r], y = \{r\}$ for a real number

$r \in \mathbb{R}$ and $A := \sup_{t\in[0,1]} |g(t)|$, we know $|g(r)| \leqslant 4M + A$ for all $r \in \mathbb{R}$. So g is a bounded function.

It is certainly possible to avoid using Fekete's lemma, but the idea is similar.

2

This is a property of subharmonic functions. Clearly $|f_k|$ is subharmonic if f_k is holomorphic, and a finite sum of subharmonic functions is subharmonic, so we prove the conclusion.

We can easily solve this problem from this idea.

Since each $f_k(z)$ is holomorphic, we have

$$f_k(z) = \frac{1}{2\pi}\int_0^{2\pi} f_k(z + re^{i\theta})d\theta$$

whenever $\overline{B_r(z)} \subseteq \overline{D}$. Take the moduli of both sides, so we get

$$|f_k(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f_k(z + re^{i\theta})|d\theta.$$

Sum over $k = 1, \cdots, n$, so we get

$$|\phi(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |\phi(z + re^{i\theta)}|d\theta.$$

Then we do as in the case of harmonic functions. Suppose ϕ achieves its maximum M in D. Then the set $\{z \in D | \phi(z) = M\}$ is a non-empty (relatively) closed subset of D. It is also open because of the inequality above. Since D is connected, we must have $\phi(z) = M$ on all of D. So by continuity of ϕ, the maximum is also achieved on ∂D.

3

There are many proofs of this problem.

1. (The proof from Rudin) WLOG assume $r_1 = \rho_1 = 1$. Let $K_R = \{z \in \mathbb{C} : |z| = R\}$ a compact set. If $1 < R < \rho_2$, then $f^{-1}(K_R)$ is compact.

Since K_R divides the second annulus into two disjoint parts, so does $f^{-1}(K_R)$. The set $\{1 < |z| < 1+\varepsilon\}$ does not intersect $f^{-1}(K_R)$ when ε is small enough because $f^{-1}(K_R)$ is compact. So $f(\{1 < |z| < 1+\varepsilon\})$ is contained in one of $\{|z| < R\}$ or $\{|z| > R\}$. WLOG we may assume $f^{-1}(\{1 < |z| < R\})$ contains all the points near $|z| = 1$. By considering limit points, we know $|z| \to 1 \Rightarrow |f(z)| \to 1, |z| \to r_2 \Rightarrow |f(z)| \to \rho_2$.

Define a function

$$u(z) = 2\log|f(z)| - 2c\log|z|, \quad 1 < |z| < r_1.$$

Here c is a constant to be determined later. By a direct computation, u is harmonic in $\{1 < |z| < r_2\}$. And we know

$$u(z) \to 0 \text{ as } |z| \to 1, \quad u(z) \to 2\log\rho_2 - 2c\log r_2 \text{ as } |z| \to r_2.$$

Take $c = \log\rho_2/\log r_2$. So u has boundary value 0 on $\{1 \leqslant |z| \leqslant r_2\}$. By the maximum principle, the function u is constant. So

$$\frac{\partial}{\partial \overline{z}}u = \frac{f'}{f} - \frac{\alpha}{z} = 0.$$

Since for $\gamma = Re^{i\theta}, \theta \in [0, 2\pi], 1 < R < r_2$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \in \mathbb{Z},$$

so $\alpha \in \mathbb{Z}$. Then $z^{-\alpha}f(z)$ has zero derivative, and is thus constant. So $f(z) = Cz^\alpha$, and this $\alpha = 1$ clearly. So $r_2 = \rho_2$.

2. Schwarz reflection First we prove the boundary behaviour as above. So we still assume

$$|f(z)| \to 1, \text{ as } |z| \to 1, \quad |f(z)| \to \rho_2, \text{ as } |z| \to r_2.$$

Then we may apply the Schwarz reflection principle (the strong version, see the notes).

Lemma 5.2. If f is a biholomorphic map from $\{1 < |z| < r_2\}$ onto $\{1 < |z| < \rho_2\}$ and $|f(z)| \to 1$ as $|z| \to 1$, $|f(z)| \to \rho_2$ as $|z| \to r_2$, then f can be extended to a holomorphic function on $\{1/r_2 < |z| < r_2\}$ by defining

$$f(z) = 1/\overline{f(\frac{1}{\overline{z}})}.$$

The proof can be found in the notes.

After the first extension, when $|z| \to 1/r_2$, the function $|f(z)| \to 1/\rho_2$. Also f maps $\{1/r_2 < |z| < 1\}$ biholomorphically onto $\{1/\rho_2 < |z| < 1\}$. So we can repeatedly extend f to a holomorphic map from $\{0 < |z| < r_2\}$ to $\{0 < |z| < \rho_2\}$. So $z = 0$ is a removable singularity. Since $f(z) \to 0$ as $z \to 0$, we must have $f(0) = 0$. Now $|f(z)| = 1$ as $|z| = 1$, and $f(z) = 0$ only when $z = 0$. So f is a finite Blaschke product, and has to be $f(z) = cz^n$ for some $|c| = 1$. Since f is injective on each annulus, we know $n = 1$, so $f(z) = cz$, which proves $r_2 = \rho_2$.

4

Also see 2010, p4. For some related discussion, see Stein, Real Analysis, Chapter 5 Section 2.

This can be written as a convolution with $K_y(x) = y/(x^2 + y^2)$ for $y > 0$. So K_y is non-negative, and we can easily show that

$$\int_{\mathbb{R}} K_y(x)dx = \pi, \quad \int_{|x|>\delta} K_y(x)dx \to 0, \text{ as } y \to 0$$

for any fixed $\delta > 0$. Now see the discussion in 2016, p2. The function U is bounded, so the easier version there is sufficient. Try it yourself to prove that $K_y * U \to U$ at the points of continuity of U.

To show that $(K_y * U)(x)$ is harmonic as a function of $(x, y) \in \{(x, y) \in \mathbb{R}^2 | y > 0\}$, just compute the Laplacian directly.

This is the Poisson kernel for the upper half plane. Compare with the one for the unit disk (see 2017, p1. Since we can find a biholomorphic map from the unit disk to the upper half plane, namely

$$z \mapsto \frac{1}{i}\frac{z-1}{z+1},$$

this map certainly carries the Poisson kernel of the unit disk to the Poisson kernel of the upper half plane. Say $z = u + iv \in \mathbb{D}$ is mapped to $x + iy$, then $P_{\mathbb{D}}(u, v) = P_{\mathbb{H}}(x, y)$.

5

This is taken from Stein, Fourier Analysis.

Suppose $f \in \mathcal{S}(\mathbb{R})$ first.

$$\begin{aligned}\|f\|_{L^2}^2 &= \int_{-\infty}^{\infty} |f(x)|^2 dx = -\int_{-\infty}^{\infty} x\Big(f(x)\overline{f(x)}\Big)' dx \\ &= -\int_{-\infty}^{\infty} xf'(x)\overline{f(x)} + xf(x)\overline{f'(x)} dx \\ &\leqslant \int_{-\infty}^{\infty} |2xf'(x)f(x)| dx \\ &\leqslant \Big(\int_{-\infty}^{\infty} |f'(x)|^2 dx\Big) \cdot \Big(\int_{-\infty}^{\infty} 4|x|^2|f(x)|^2 dx\Big).\end{aligned}$$

By the Plancherel theorem and the fact that $\widehat{f'} = 2\pi i\xi\widehat{f}$, we have

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx = 4\pi^2 \int_{-\infty}^{\infty} \xi^2|\hat{f}(\xi)|^2 d\xi.$$

So we prove the inequality for $f \in \mathcal{S}(\mathbb{R})$.

For general $f \in L^2(\mathbb{R})$, if the integrals are finite, then $xf, \xi\hat{f} \in L^2$. The latter shows that f has a distributional derivative which is in L^2, so $f \in H^1(\mathbb{R})$. Consider a sequence of cut-off functions (χ_n) with the following conditions: $\chi_n \in C_c^\infty(\mathbb{R}), 0 \leqslant \chi_n \leqslant 1, |\chi_n'| \leqslant 3, \chi_n = 0$ when $|x| \geqslant n+1$, $\chi_n = 1$ when $|x| \leqslant n$. Then $f\chi_n$ has compact support. So first check the inequality for compactly supported L^2 functions.

If $f \in H^1(\mathbb{R})$ is compactly supported (say $\{|x| < R\}$), choose a sequence of functions $(g_n) \subset C_c^\infty(\mathbb{R})$ approaching f in the H^1-norm. We can assume the g_n's are all supported in $\{|x| < R+1\}$, so $xg_n \to xf$ in L^2. Then we let $n \to \infty$, and we prove the inequality for f from the inequalities for g_n.

If $f \in H^1(\mathbb{R})$ and $xf \in L^2(\mathbb{R})$, then $\chi_n f$ is in L^2, compactly supported, and

$$|(\chi_n f)'| = |\chi_n' f + \chi_n f'| \leqslant 3|f| + |f'|,$$

so $\chi_n f \in H^1$ is compactly supported. Now we can prove the inequality for f from the inequalities for $\chi_n f$ and MCT/DCT.

6

This problem is taken from Stein, Real Analysis, Chapter 5, Exercise 6 and 7. The space here is an example of Bergman spaces. It is also related to reproducing kernel Hilbert space. There is a mistake in the statement of (b). The correct statement is

$$\sum_{n=0}^{\infty} |\phi_n(z)|^2 \leqslant \frac{C^2}{d(z,\Omega^c)^2}, \quad z \in \Omega.$$

(a) Only one thing needs verification. Suppose $\{f_n\} \subseteq \mathbb{H}$ converges to f in $L^2(\Omega)$, then $f \in \mathbb{H}$. For any closed disk $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$ contained in Ω, we prove that $\{f_n\}$ converges uniformly on $D_r(z_0)$.

To see this, first note that by compactness of $D_r(z_0)$, we can find $\delta > 0$ such that for any $z \in D_r(z_0)$, the closed disk $D_\delta(z)$ is contained in Ω. Then note that holomorphic functions are complex-valued harmonic functions, so we have the mean-value property

$$f_n(z) = \frac{1}{\pi\delta^2} \int_{D_\delta(z)} f_n(x+iy)dxdy.$$

So

$$|f_m(z)-f_n(z)| \leqslant \frac{1}{\sqrt{\pi}\delta}\Big(\int_{D_\delta(z)} |f_n(x+iy)-f_m(x+iy)|^2 dxdy\Big)^{1/2} \leqslant \frac{1}{\sqrt{\pi}\delta}\|f_m-f_n\|_{L^2(\Omega)}.$$

This shows that $\{f_m\}$ is a Cauchy sequence under the supremum norm on $D_r(z_0)$, so it uniformly converges to some function F on $D_r(z_0)$. Since the closed disk $D_r(z_0)$ is arbitrarily chosen and the union of all such disks is clearly Ω, we can glue up the limit function and get a pointwise defined function F in Ω. Since $\{f_m\}$ is a sequence of holomorphic functions and converges pointwise to F uniformly on any compact subset of Ω, we know F is holomorphic in Ω.

Finally we compare f and F. We know $f_n \to f$ in $L^2(\Omega)$-norm and $f_n \to F$ pointwise and uniformly on any compact subset. So $f_n \to f$ in measure, and there is a subsequence $\{f_{n_k}\}$ convergent to f almost everywhere, so $F = f$ almost everywhere. This shows that $f \in \mathbb{H}$.

(b) Now we examine the inequality in (a) more carefully. The inequality we have used says

$$|f(z)| \leqslant \frac{1}{\sqrt{\pi}\delta}\|f\|_{L^2(\Omega)}, \quad \forall \delta < d(z, \Omega^c).$$

Let $\delta \to d(z, \Omega^c)$, we know

$$|f(z)| \leqslant \frac{1}{\sqrt{\pi} d(z, \Omega^c)}\|f\|_{L^2(\Omega)}.$$

Consider $\mathrm{ev}_z : \mathbb{H} \to \mathbb{C}, f \mapsto f(z)$. By the above inequality, $\mathrm{ev}(z)$ is a continuous linear functional for every $z \in \Omega$. By the Riesz representation, there is some $f_z \in \mathbb{H}$ such that $\langle -, f_z\rangle = \mathrm{ev}_z(-)$

For the orthonormal basis $\{\phi_n\}$, we have

$$\Big(\sum_{n=0}^{\infty} |\phi_n(z)|^2\Big)^{1/2} = \sup\{\Big|\sum_{n=0}^{\infty} c_n\phi_n(z)\Big| : (c_n) \in l^2(\mathbb{N}), \|(c_n)\|_{l^2} = 1\}.$$

But the series $\sum_{n=0}^{\infty} c_n\phi_n$ converges in $\mathbb{H}$ when $\|(c_n)\|_{l^2} = 1$. Call this element f, and we have $\|f\|_{L^2(\Omega)} = 1$ because of the Parseval identity. Recall that in (a) we have proved that norm convergence in $\mathbb{H}$ implies pointwise convergence, so the right side is exactly

$$\sup\{|f(z)| : f \in \mathbb{H}, \|f\|_{L^2} = 1\} = \sup\{|\mathrm{ev}_z(f)| : f \in \mathbb{H}, \|f\|_{L^2} = 1\},$$

i.e. the norm of the linear functional ev_z. Combine everything, and we get

$$\Big(\sum_{n=0}^{\infty} |\phi_n(z)|^2\Big)^{1/2} \leqslant \frac{1}{\sqrt{\pi} d(z, \Omega^c)}.$$

(c) The absolute convergence is directly from (b) and the Cauchy inequality. To see that $B(z, w)$ is independent of the choice of $\{\phi_n\}$, we prove that $B(z, w) = \overline{f_z(w)}$, where f_z is the function associated to ev_z. Since f_z is independent of $\{\phi_n\}$, this will prove that $B(z, w)$ is independent of $\{\phi_n\}$.

Fix any $z \in \Omega$. We use $dm(w)$ for the 2D Lebesgue measure $dxdy$ for convenience. Since f_z is uniquely determined by the formula

$$\int_\Omega g(w)\overline{f_z(w)}dm(w) = \langle g, f_z\rangle = \mathrm{ev}_z(g) = g(z).$$

Now if $g(w) = \sum\limits_{n=0}^{\infty} c_n\phi_n(w)$, where the sum here and below is in both the L^2 sense and pointwise sense, we have

$$\begin{aligned}\int_\Omega g(w)B(z,w)dm(w) &= \int_\Omega g(w)\sum_{n=0}^{\infty}\phi_n(z)\overline{\phi_n(w)}dm(w)\\ &= \sum_{n=0}^{\infty}\phi_n(z)\int_\Omega g(w)\overline{\phi_n(w)}dm(w)\\ &= \sum_{n=0}^{\infty}c_n\phi_n(z)\\ &= g(z).\end{aligned}$$

We can exchange the integral and the sum because the inner product is norm-continuous. This completes the proof.

A remark on the Bergman kernel. Compare this with the method of Green's function when the differential operator admits a complete set of eigenvectors.

Proposition 5.3. Let $B(z,w) = \sum_{n=0}^{\infty}\phi_n(z)\overline{\phi_n(w)}$.

$$f(z) = \int_\Omega B(z,w)f(w)dm(w), \quad \forall f \in \mathbb{H}.$$

And $B(z,w) = \overline{B(w,z)}$. If for any fixed $w \in \Omega$, the function $B'(-,w) \in \mathbb{H}$, $B'(z,w) = \overline{B'(w,z)}$, and $B'(z,w)$ satisfies the same reproducing property

$$f(z) = \int_\Omega B'(z,w)f(w)dm(w), \forall f \in \mathbb{H},$$

then $B' = B$. This unique function $B(z,w) = \overline{f_z(w)}$

Proof. The first formula has been proved. The conjugate symmetry in z and w is obvious. Fix $w \in \Omega$, the function $B(-,w)$ is in $\mathbb{H}$ because the coefficients $\{\overline{\phi_n(w)}\} \in l^2$.

Let $B'(z,w)$ be another such function.

$$\begin{aligned} B(z,w) = \overline{B(w,z)} &= \int_\Omega B'(z,t)\overline{B(w,t)}dm(t) \\ &= \overline{\int_\Omega B(w,t)\overline{B'(z,t)}dm(t)} \\ &= B'(z,w). \end{aligned}$$

□

6 2015

1

We need to assume $f \in L^2(\mathbb{R})$. Otherwise, the conclusion is false. This problem is taken from Rudin, Chapter 3 exercise 17.

If not, take a subsequence (f_{n_k}) of (f_n) such that $\|f_{n_k} - f\|_{L^2} > \varepsilon > 0$ for some constant ε. By Fatou's lemma for $g_n = 2|f_n|^2 + 2|f|^2 - |f_n - f|^2 \geqslant 0$, we have

$$4\|f\|_{L^2} = \int_{\mathbb{R}} \underline{\lim}\, g_n dx \leqslant \underline{\lim} \int_{\mathbb{R}} g_n dx = 4\|f\|_{L^2} - \overline{\lim} \int_{\mathbb{R}} |f_n - f|^2 dx.$$

So $\|f_n - f\|_{L^2} \to 0$.

There is another proof in Rudin.

2

A direct application of the Weierstrass approximation theorem. For any $\varepsilon > 0$, take a polynomial P on $[a, b]$ such that $\|P - f\|_\infty < \varepsilon$. By the assumption, we have

$$\int_a^b f(x)\overline{P(x)}dx = 0.$$

So

$$\int_a^b |f(x)|^2 dx \leqslant \varepsilon(b-a)\|f\|_\infty.$$

Since $\varepsilon > 0$ can be arbitrarily chosen, we prove that $f = 0$.

Pay attention to how we use continuity and the mode of convergence. In this problem, we can also use L^2-density of polynomials, but in some other problems, care must be taken.

3

This is a strange question. When $z \in \mathbb{R}$ and $|z|$ is sufficiently large, we have $f(z) = 0$. So $f \equiv 0$.

4

In the notes, we have proved that they are exactly finite Blaschke products.

5

This is the famous Atkinson's theorem on Fredholm operators. The theorem holds for general Banach spaces, but the proof is much easier in Hilbert spaces.

By the assumption, we have

$$\ker T \subseteq \{v \in H_1 : S_1 v = v\}$$

is contained in the eigenspace of S_1 corresponding to the eigenvalue 1. So it has to be finite-dimensional.

To see that $\operatorname{coker}(T)$ is finite-dimensional, we can take adjoint and use the same argument as above. If we do not assume the knowledge of A compact iff A^* compact, we can prove as follows.

Similarly, we have [1]

$$\operatorname{coker} T \subseteq H_2/\overline{\operatorname{Im}(I - S_2)}.$$

For Hilbert spaces, the cokernel can be naturally viewed as the orthogonal complement. Let $X = \overline{\operatorname{Im}(I - S_2)}^{\perp}$, so we need to prove X is finite-dimensional. If not, we can take a sequence $\{v_n\} \subseteq X$ such that $\|v_n\| = 1, \langle v_n, v_m \rangle = 0$ whenever $n \neq m$. Then $\{Sv_n\}$ has a convergent subsequence. WLOG we assume the sequence $\{Sv_n\}$ itself converges. Now note that by the construction of X, we have

$$\langle v_n, v_m - S_2 v_m \rangle = 0, \quad \forall m, n \in \mathbb{N}^*.$$

So whenever $n \neq m$, we have

$$0 = \langle v_n - v_m, (I - S_2)(v_n - v_m) \rangle = \|v_n - v_m\|^2 - \langle v_n - v_m, S_2(v_n - v_m) \rangle.$$

[1] Strictly speaking, this is a naturally-defined injective map, not an inclusion.

Note that $\|v_n - v_m\|^2 = 2, S_2(v_n - v_m) \to 0$, so the right side tends to 2 as $n, m \to \infty$ while keeping $n \neq m$. This is a contradiction, so X is finite-dimensional, and thus coker T is finite-dimensional.

Finally, we prove that $\mathrm{Im}(T)$ is closed. Suppose the sequence $\{Tv_n\}$ converges to $w \in H_2$. We can assume $v_n \in (\ker T)^\perp$ for each $n \in \mathbb{N}^*$. Then

$$(I - S_1)v_n = QTv_n \to Qw.$$

The difficulty is to prove that the sequence $\{v_n\}$ is bounded. If we have proved this, then we know $\{S_1 v_n\}$ has a convergent subsequence $\{S_1 v_{n_k}\}$, so by the above identity, we know $\{v_{n_k}\}$ converges to some $v \in H_1$, and $w = Tv \in \mathrm{Im}(T)$, so $\mathrm{Im}(T)$ is closed.

To show that $\{v_n\}$ is bounded, we prove that there exists $\varepsilon > 0$ such that $\|Tx\| \geqslant \varepsilon\|x\|$ for all $x \in (\ker T)^\perp$. If not, then there exists a sequence $\{x_n\} \subseteq (\ker T)^\perp$ such that $\|x_n\| = 1, Tx_n \to 0$. Then $QTx_n = x_n - S_1 x_n \to 0$. Since $\{S_1 x_n\}$ has a convergent subsequence, we can take a convergent subsequent $\{x_{n_k}\}$ whose limit is $x \in (\ker T)^\perp$. So $Tx = 0$ by continuity of T. But $\ker(T) \cap (\ker T)^\perp = \{0\}$, so $x = 0$. However, $\|x_n\| = 1$, so $\|x\| = 1$, a contradiction. So there exists $\varepsilon > 0$ such that $\|Tx\| \geqslant \varepsilon\|x\|$ for all $x \in (\ker T)^\perp$, and $\{v_n\}$ has to be bounded.

7 2016

1

This is a trap. The problem can be found in Stein, Real Analysis, but from the Problem part, so you know it is very difficult. This also appears in Rudin, Real and Complex Analysis, Theorem 7.21. There is a complete proof of this problem. [2]

A generalization of this problem is the following theorem whose proof is not even harder. Also see J.J. Koliha, A Fundamental Theorem of Calculus for Lebesgue Integration. *The American Mathematical Monthly, 113(6)*, 551-555. https://doi.org/10.1080/00029890.2006.11920335.

Theorem 7.1 (Koliha)**.** If $F(x) \in C[a,b]$ is continuous, $f(x) \in L^1[a,b]$, outside a countable set we have $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

The original proof by Koliha. We first prove that

$$|F(b) - F(a)| \leqslant \int_a^b |f(t)|dt.$$

Let $F'(t) = f(t)$ for all $t \in A := [a,b] \setminus D$, D a countable set. Assume that the one-sided derivatives exist at the endpoints of $[a,b]$; otherwise take $[a_n, b_n]$ with this property and $[a_n, b_n] \nearrow [a,b]$.

Let $\varepsilon > 0$ be given. Set $c_i = i\varepsilon/(b-a)$ for $i = 0, 1, 2, \cdots$, and define

$$E_i = \{t \in A : c_{i-1} \leqslant |f(t)| \leqslant c_i\} \quad (i \in \mathbb{N}).$$

Since f and $|f|$ are Lebesgue integrable, the sets E_i are Lebesgue measurable and A is the disjoint union of the E_i. So $m(A) = b - a = \sum_{i=1}^{\infty} m(E_i)$, and

$$c_{i-1}m(E_i) \leqslant \int_{E_i} |f(t)|dt \leqslant c_i m(E_i),$$

which gives

$$0 \leqslant c_i m(E_i) - \int_{E_i} |f(t)|dt \leqslant \frac{\varepsilon}{b-a} m(E_i).$$

[2] Thanks to Zhao Weida who reminded me of this proof in Rudin.

Sum over $i \in \mathbb{N}$, and we conclude that

$$\sum_{i=1}^{\infty} c_i m(E_i) \leqslant \int_a^b |f(t)|dt + \varepsilon.$$

For each i choose a bounded open subset $G_i \subseteq \mathbb{R}$ such that $G_i \supseteq E_i$ and

$$m(G_i) \leqslant m(E_i) + c_i^{-1}(\frac{1}{2})^i \varepsilon.$$

Define two functions $H, M : [a, b] \to \mathbb{R}$ by $H(a) = M(a) = 0$ and

$$H(t) = \sum_{i=1}^{\infty} c_i m(G_i \cap [a, t]), \quad M(t) = \sum_{u_j \in [a,t)} (\frac{1}{2})^j \varepsilon, \quad (a \leqslant t \leqslant b)$$

where $\{u_j : j \in \mathbb{N}\}$ is a n enumeration of D. Both H and M are nondecreasing.

Let x be the supremum of all $t \in [a, b]$ such that $|F(t) - F(a)| \leqslant H(t) + M(t)$. Assume that $x < b$. If $x \in A$, then $x \in E_k$ for some k. On E_k we know $|f(x)| < c_k$, so there exists $x_1 \in (x, b)$ such that $[x, x_1] \subseteq G_k$ and

$$|F(x_1) - F(x)| < c_k(x_1 - x)$$

because $F'(x) = f(x)$ and by the definition of the derivative. Since F is continuous, we have $|F(x) - F(a)| \leqslant H(x) + M(x)$. Then

$$|F(x_1)-F(a)| \leqslant |F(x_1)-F(x)|+|F(x)-F(a)| \leqslant c_k(x_1-x)+H(x)+M(x),$$

and note that $H(x) + c_k(x_1 - x) \leqslant H(x_1)$. This contradicts the maximality of x.

If $x \in D$. Then $x = u_m$ for some m. Since F is continuous, there exists $x_2 \in (x, b)$ such that $|F(x_2) - F(x)| < \varepsilon/2^m$, so

$$|F(x_2) - F(a)| \leqslant |F(x_2) - F(x)| + |F(x) - F(a)| \leqslant \varepsilon/2^m + H(x) + M(x).$$

This time $\varepsilon/2^m + M(x) \leqslant M(x_2)$. So this contradicts the maximality of x again.

So we must have $x = b$. Then put all the inequalities together, we have

$$|F(b) - F(a)| \leqslant H(b) + M(b) \leqslant \int_a^b |f(t)|dt + 3\varepsilon.$$

Since ε was arbitrary, we have proved that

$$|F(b) - F(a)| \leqslant \int_a^b |f(t)|dt.$$

Now for any subinterval $[u, v] \subseteq [a, b]$, we have $|F(v)-F(u)| \leqslant \int_u^v |f(t)|dt$, so F is absolutely continuous on $[a, b]$. After proving absolute continuity of F, we know the equality holds by the classical fundamental theorem of calculus for Lebesgue integration. $\square$

2

This problem is a standard example of the approximate identity of $L^1(\mathbb{R}^n)$ for convolution, which can be found in any textbooks on real analysis. Also see Stein, Fourier Analysis. But we emphasize the relation between the properties of convolution kernels and the mode of convergence here.

Proposition 7.2. Suppose a family of $L^1(\mathbb{R}^{\ltimes})$ functions $(k_\lambda)_{\lambda>0}$ satisfies:

(1) $\int_{\mathbb{R}^{\ltimes}} k_\lambda(x)dx = 1$.

(2) $\int_{\mathbb{R}^{\ltimes}} |k_\lambda(x)|dx$ is bounded for $\lambda > 0$.

(3) For every $\delta > 0$,

$$\lim_{\lambda\to 0} \int_{|x|>\delta} |k_\lambda(x)|dx = 0.$$

Then for any $f \in L^1(\mathbb{R}^n)$, we have $f * k_\lambda \to f$ in L^1 norm.

Generally for $f \in L^p(\mathbb{R}^n)$ where $1 \leqslant p < \infty$, we have $f * k_\lambda \to f$ in L^p norm.

For its proof, see Grafakos Theorem 1.2.19. One should master this proof. It is a standard 3ε-argument. We first give a lemma which is also used in 2022, p3.

Lemma 7.3 (Strong continuity of the translation)**.** Suppose $f \in L^p(\mathbb{R}^n)$ and the translation $(\tau_d f)(x) = f(x + d)$ for $d \in \mathbb{R}^n$, so $\tau_d : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Then $\tau_d f \to f$ in the L^p-norm.

Proof. First prove for $f \in C_c^\infty(\mathbb{R}^n)$ by boundedness and uniform continuity or using DCT. Then approximate. $\square$

And we will use the following Minkowski's integral inequality:

Theorem 7.4. Let $(X, \mu), (T, \nu)$ be two σ-finite measure spaces and let $1 \leqslant p < \infty$. Show that for every non-negative measurable function F on the product space $(X, \mu) \times (T, \nu)$ we have

$$\left(\int_T \Big(\int_X F(x,t) d\mu(x) \Big)^p d\nu(t) \right)^{1/p} \leqslant \int_X \Big(\int_T F(x,t)^p d\nu(t) \Big)^{1/p} d\mu(x).$$

To memorize this inequality, just remember that the L^p-norm of the integral is smaller than the integral of L^p-norms.

Proof. Say $\int_{\mathbb{R}^n} |k_\lambda(x)| dx \leqslant M$ for all $\lambda > 0$.

For any $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|d| < \delta \Longrightarrow \|\tau_d f - f\|_{L^p} < \varepsilon.$$

Then split into two parts:

$$\begin{aligned} |(k_\lambda * f)(x) - f(x)| &= \Big| \int_{\mathbb{R}^n} \Big(f(x-y) - f(x) \Big) k_\lambda(y) dy \Big| \\ &\leqslant \int_{|y|<\delta} |f(x-y) - f(x)| |k_\lambda(y)| dy \\ &\quad + \int_{|y| \geqslant \delta} |f(x-y) - f(x)| |k_\lambda(y)| dy. \end{aligned}$$

And take L^p norms in x:

$$
\begin{aligned}
\|k_\lambda * f - f\|_{L^p} &\leqslant \left(\int_{\mathbb{R}^n}\Big(\int_{|y|<\delta}|f(x-y)-f(x)||k_\lambda(y)|dy\Big)^p dx\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}^n}\Big(\int_{|y|\geqslant\delta}|f(x-y)-f(x)||k_\lambda(y)|dy\Big)^p dx\right)^{1/p} \\
\text{(Minkowski)} &\leqslant \int_{|y|<\delta}\Big(\int_{\mathbb{R}^n}|f(x-y)-f(x)|^p dx\Big)^{1/p}|k_\lambda(y)|dy \\
&\quad + \int_{|y|\geqslant\delta}\Big(\int_{\mathbb{R}^n}|f(x-y)-f(x)|^p dx\Big)^{1/p}|k_\lambda(y)|dy \\
&\leqslant \int_{|y|<\delta}\|\tau_{-y}f-f\|_{L^p}|k_\lambda(y)|dy \\
&\quad + \int_{|y|\geqslant\delta}2\|f\|_{L^p}|k_\lambda(y)|dy \\
&\leqslant \varepsilon M + 2\|f\|_{L^p}\int_{|y|\geqslant\delta}|k_\lambda(y)|dy.
\end{aligned}
$$

This proves what we want. The case $p=1$ is easier because we can use the Fubini's theorem instead to the Minkowski's inequality. □

Proposition 7.5. Suppose a family of $L^1(\mathbb{R}^\ltimes)$ functions $(k_\lambda)_{\lambda>0}$ satisfies:

(1) $\int_{\mathbb{R}^\ltimes} k_\lambda(x)dx = 1$.

(2) $|k_\lambda(x)| \leqslant A\lambda^{-n}$ for all $\lambda>0$.

(3) $|k_\lambda(x)| \leqslant A\lambda^\varepsilon/|x|^{n+\varepsilon}$ for some fixed $\varepsilon>0$ and all $\lambda>0, x\in\mathbb{R}^n$.

If $k_\lambda(x)$ are induced by a single $k(x)$ by $k_\lambda(x) = \lambda^{-n}k(x/\lambda)$, then the conditions (2) and (3) become:

(2') k is bounded.

(3') $|k(x)| \leqslant A/|x|^{n+\varepsilon}$.

Or (2')+(3'): $|k(x)| \leqslant A(1+|x|)^{-n-\varepsilon}$.

Then k_λ satisfies the conditions of the previous proposition. Moreover, for any $f\in L^p(\mathbb{R}^n)$ where $1\leqslant p\leqslant\infty$, we have $f*k_\lambda\to f$ at any Lebesgue point for f when $\lambda\to 0$, so necessarily almost everywhere.

For its proof, see Folland Theorem 8.15 or Stein, Real Analysis, p109. As an example, see 2017, p1.

So now we are happy to check the conditions. Note that here the family (K_δ) is also induced by a single function. This family is called the **heat kernel**.

3

This is a standard result which can be found in Xie Huimin and many textbooks.

For (a) and (b), see 2010, p2.

For (c), we need to find a partition such that the difference of the upper Darboux sum and the lower Darboux sum is smaller than any given positive number. Since the set of discontinuities has measure zero, we can cover it with open sets of arbitrarily small measure. Since $\{c \in I : \operatorname{osc}(f, c) \geqslant \varepsilon\}$ is compact, we can cover it with a finite union of (relatively) open intervals whose total length is $< \delta$. Then we can bound the difference of the upper Darboux sum and the lower Darboux sum by

$$2\delta\|f\|_{\sup} + (b-a)\varepsilon.$$

This proves that f is Riemann integrable.

4

(a) If $z = re^{i\theta}$, then

$$w = \frac{1}{2}\Big((r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta\Big).$$

Solve this equation for $\overline{\mathbb{C}} \setminus [-1, 1]$ to prove that this is bijective.

(b) To compute the integral, the first way is to use the contour integral along the keyhole contour.

The second way is to show

$$\int_1^\infty \frac{\log x}{x^2-1}dx = \int_1^0 \frac{\log 1/y}{1/y^2-1}d(1/y) = \int_0^1 \frac{\log y}{y^2-1}dy,$$

and for $x \in [0, 1]$, the integral can be computed by MCT and the expansion

$$\frac{\log x}{x^2-1} = -\log x(1 + x^2 + x^4 + \cdots).$$

5

We should assume that f is not constant. And what the conclusion really means is that the numbers are equal in a fundamental domain.

For $z \in \mathbb{C}$, let γ_z be the square with four vertices $z, z+1, z+i, z+1+i$. Let Ω_z be the interior of this square. Since f is not constant, we can choose z such that there are no zeros or poles of f on γ_z. So we only need to prove that the number of zeros in Ω_z and the number of poles in Ω_z are equal.

But the difference of these two numbers is equal to the integral

$$\frac{1}{2\pi i}\int_{\gamma_z}\frac{f'(z)}{f(z)}dz.$$

But the integral is zero, because f'/f is also doubly periodic with periods $1, i$, so the integrals over the top edge and the bottom edge cancel out and the integrals over the left and right edges cancel out.

6

This can be found in any textbooks on functional analysis. We include only a sketch of the proof here. We need to prove three things.

(1) $\sigma(A)$ is bounded.

Lemma 7.6. If $\lambda \in \mathbb{C}, |\lambda| > \|A\|$, then $\lambda - A$ is invertible.

If B_1 is invertible, B_2 is another bounded linear operator and $\|B_2\| < \|B_1^{-1}\|^{-1}$, then $B_2 - B_1$ is invertible.

This proves that $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| \leqslant \|A\|\}$.

(2) $\sigma(A)$ is closed.

If $\lambda \notin \sigma(A)$ and $|\mu - \lambda| < \|(\lambda I - A)^{-1}\|^{-1}$, then $\mu \notin \sigma(A)$.

(3) $\sigma(A) \subseteq \mathbb{R}$.

If $x, y \in \mathbb{R}$, $y \neq 0$ and $\lambda = x + iy$, then $\lambda I - A$ is invertible. To see this, take any $v \in H$. We have

$$\langle(\lambda I - A)v, (\lambda I - A)v\rangle = |\lambda|^2\|v\|^2 + \|Av\|^2 - 2\mathrm{Re}\langle Av, \lambda v\rangle.$$

Note that $\mathrm{Re}\langle Av, \lambda v\rangle = x\langle Av, v\rangle$, so $\|(\lambda I - A)v\|^2 \geqslant y^2\|v\|^2$. This proves that $\lambda I - A$ is injective.

Recall that $\ker(\overline{\lambda}I - A) = \overline{\operatorname{ran}(\lambda I - A)}^{\perp}$, so $\lambda I - A$ has dense range. Suppose $\{(\lambda I - A)v_n\}$ converges to w in H as $n \to \infty$. Then

$$|y|^2\|v_n - v_m\|^2 \leqslant \|(\lambda I - A)v_n - (\lambda I - A)v_m\|^2 \to 0,$$

so $\{v_n\}$ is a Cauchy sequence, and it converges to some $v \in H$. Then $w = (\lambda I - A)v \in \operatorname{ran}(\lambda I - A)$, so $\lambda I - A$ has closed range. This proves that $\lambda I - A$ is surjective. By the open mapping theorem, the inverse map $(\lambda I - A)^{-1}$ is automatically bounded, so $\lambda \notin \sigma(A)$.

Pay attention to part (3). The argument on dense range and closed range also appears in the theory of unbounded self-adjoint operators.

8 2017

1

Pay attention to the conclusion which asks for a proof of a.e. convergence. Compare with 2016, p2. They are different at least on using the Lebesgue points non-trivially. See also 2020, p1.

The difference is that we now work on a finite interval $[-\pi, \pi]$. And one has to familiar with the given series.

The left side is formally

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} \int_{-\pi}^{\pi} \frac{1}{2\pi} f(y) e^{-iny} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(x-y)} dy,$$

so it looks like the convolution of f with

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

For $0 < r < 1$, the series is dominated by the convergent series $\sum r^{|n|}$, so $P_r(x)$ is continuous on $[-\pi, \pi]$. Similarly, we know P_r is smooth. The convolution $P_r * f$ is

$$\begin{aligned} (P_r * f)(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) P_r(x-y) dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx - iny} dy \\ (\text{Fubini on } dy, \sum_n) &= \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}. \end{aligned}$$

The Fubini's theorem is valid because f is integrable. The family $(P_r)_{0<r<1}$ satisfies:

(1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x) dx = 1$, and $|P_r(x)| \leqslant (1+r)/(1-r)$.

(2) $P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$ is non-negative.

(3) If $|x| \geqslant \delta > 0$, then $|P_r(x)| \leqslant (1-r^2)/\sin^2 \delta$.

This justifies the conditions we need. Then we have to do the recitation. Let

$$I = \int_{|y| \leqslant \delta} \cdots, \quad I_k = \int_{2^k \delta < |y| \leqslant 2^{k+1} \delta} \cdots$$

and

$$A(r) = \frac{1}{2r} \int_{|y| \leqslant r} |f(x-y) - f(x)| dy.$$

Then

$$I \leqslant \frac{1+r}{1-r} 2\delta A(\delta),$$

and

$$I_k \leqslant \frac{1-r^2}{\sin^2 2^k \delta} 2^{k+2} \delta A(2^{k+1}\delta).$$

So the difference is not greater than

$$\frac{2+2r}{1-r} A(\delta)\delta + \sum_{k \geqslant 0}^{[\log_2 \pi/\delta]} \frac{1-r^2}{\sin^2 2^k \delta} 2^{k+2} \delta A(2^{k+1}\delta).$$

Take $\delta = 1 - r$ and prove this converges to zero.

I suggest you compare this with the theory of holomorphic functions on $\mathbb{D}$. For the terms with $n < 0$ in the power series, they are anti-holomorphic, and the terms with $n \geqslant 0$ are holomorphic. The power series of the left side (one a power series of $z = re^{ix}$ and another of $\overline{z} = re^{-ix}$) represents a harmonic function on $\mathbb{D}$ with its boundary like f. To be precise, we construct a harmonic function whose a.e. radial limit is f. This actually solves the Dirichlet problem for L^1 boundary values.

If we do not use the dyadic decomposition, we would only get the estimate

$$\frac{2+2r}{1-r} A(\delta)\delta + \frac{1-r^2}{\sin^2 \delta} (\|f\|_{L^1} + 2\pi |f(x)|).$$

I think it is impossible to prove what we want just from this upper bound. At least I failed.

2

Also see Stein, Real Analysis, Chapter 4 Exercise 20.

(a) See part (b) of 2018, p1.

(b) Fix an orthonormal basis $(e_1, \cdots, e_n)$ for H. Then we have a unique expansion

$$f_k = c_{1,k}e_1 + c_{2,k}e_2 + \cdots + c_{n,k}e_n, \quad f = \sum_{j=1}^{n} c_j e_j.$$

From $(f_k, e_j) \to (f, e_j)$ as $k \to \infty$ we know $c_{j,k} \to c_j$ as $k \to \infty$. Then from the Parseval identity,

$$\|f_k - f\|^2 = \sum_{j=1}^{n} |c_{j,k} - c_j|^2 \to 0.$$

So $\{f_k\}$ converges to f.

For an infinite-dimensional H, choose an orthonormal family $\{e_n\}_{n=1}^{\infty}$. Then the sequence $\{e_n\}$ weakly converges to 0, because for any $f \in H$, by the Parseval identity, the sequence of numbers $\{(f, e_n)\}$ is in $l^2(\mathbb{N})$, so $(f, e_n) \to 0$ as $n \to \infty$. This proves the weak convergence of $\{e_n\}$ to 0.

3

All such functions are exactly entire functions.

Clearly entire functions satisfy the condition. If f is such a function, then consider $g(z) = zf(z)$. We have

$$|g(z)| \leqslant |z|^3 + |z|^{1/2}$$

near $z = 0$. So $z = 0$ is a removable singularity of g. We extend g to make it an entire function, and automatically $g(0) = 0$. So $g(z)/z = f(z)$ is holomorphic at $z = 0$, i.e. f is entire.

This basically says one cannot have zeros/poles of fractional orders for holomorphic/meromorphic functions.

4

This region is surrounded by two arcs with angles $\pi/2$. Under the map

$$z \mapsto \frac{z+1}{z-1},$$

the region is mapped onto $\{re^{i\theta} : r > 0, 3\pi/4 < \theta < 5\pi/4\}$. Now take $z \mapsto z^2$, it becomes the right half plane, and the rest is clear.

9 2018

1

Also see Stein, Real Analysis, Chapter 4 Exercise 20.

(a)

$L^2(\mathbb{R}^d)$ is separable, so we can choose a subsequence (f_{n_j}) by the diagonal argument so that for some countable dense set D, we have

$$(f_{n_j}, g) \to c_g, \quad \forall g \in D.$$

Here c_g is a number depending only on g. To do this, just note that $\{(f_n, g) | n \in \mathbb{N}^*\}$ is a bounded set of numbers, so we take a convergent subsequence. Then use the diagonal argument to do this repeatedly for all the elements in D. Finally note that $|c_g| \leqslant \|g\|_{L^2}$ because $\|f_n\|_{L^2} = 1$.

Then for any $h \in L^2(\mathbb{R}^d)$ and any $\varepsilon > 0$, choose $g \in D$ such that $\|g - h\|_{L^2} < \varepsilon$. So

$$|(f_{n_j}, h) - (f_{n_k}, h)| \leqslant |(f_{n_j}, g) - (f_{n_k}, g)| + |(f_{n_j}, g) - (f_{n_j}, h)| + |(f_{n_k}, g) - (f_{n_k}, h)|,$$

from which we can prove the sequence $\{(f_{n_j}, h)\}_{j \geqslant 1}$ is Cauchy. Say $(f_{n_j}, h) \to c_h$. Clearly $|c_h| \leqslant \|h\|_{L^2}$, and the map $h \mapsto c_h$ is linear (for the complex case, assume the inner product is linear on the second component). By the Riesz representation theorem, there exists $f \in L^2(\mathbb{R})$ such that $(f, h) = c_h, \forall h \in L^2(\mathbb{R}^d)$. This proves that $\{f_{n_j}\}$ weakly converges to f.

(b)

Consider

$$(f_n - f, f_n - f) = \|f_n\|^2 + \|f\|^2 - (f, f_n) - (f_n, f) \to 2\|f\|^2 - \|f\|^2 - \|f\|^2 = 0.$$

A remark on (a). Note that $L^2(\mathbb{R}^d)$ is reflexive because it is a Hilbert space. So the closed unit ball is weakly compact. Then note that $L^2(\mathbb{R}^d)$ is separable, so the closed unit ball is weakly metrizable, so we can use sequences to describe convergence.

2

Again we know f is a finite Blaschke product

$$f(z) = c\prod_{j=1}^{n}\frac{z-a_j}{1-\overline{a_j}z}, \quad \exists|c| = 1, |a_j| < 1, \forall z \in U.$$

For the equation $f(z) = w, w \in D$, we only need to solve the polynomial equation

$$\prod_{j=1}^{n}(z-a_j) = \frac{w}{c}\prod_{j=1}^{n}(1-\overline{a_j}z).$$

Since f is non-constant, we have $n \geqslant 1$. The leading coefficients are $1, c^{-1}w(-1)^n\prod\overline{a_j}$ respectively. They are not equal, so there exists $z \in \mathbb{C}$ such that $f(z) = w$. By computing the modulus, we know $|z| < 1$. So $D \subset f(D)$.

4

(a) This is a false statement. WLOG let $\rho = 1$ and d the Euclidean metric. Consider the set $S = \{1 + 1/k | k \in \mathbb{N}^*\}$ and the measure μ defined by

$$\mu(A) = \begin{cases} \mathrm{Leb}(A), & A \cap S = \varnothing, \\ \infty, & A \cap S \neq \varnothing. \end{cases}$$

Then

$$\theta(x) = \begin{cases} c_1, & \neg\exists y : y \in S \wedge d(x,y) \leqslant 1, \\ \infty, & \exists y : y \in S \wedge d(x,y) \leqslant 1. \end{cases}$$

Here c_1 is the volume of the closed unit ball. One immediately finds that $\theta(0) = c_1$ but $\overline{\lim}_{y\to 0}\theta(y) = \infty$.

(a) Clearly we only need to prove the inequality when $\theta(x) < \infty$. We have to assume that μ has some regularity.

The unambiguous interpretation of $\overline{B_\rho(x)}$ being the limit of $\overline{B_\rho(y_n)}$ as $y_n \to x$ is

$$(\overline{B_\rho(x)})^c \subseteq \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\overline{B_\rho(y_k)}^c.$$

This is proved by the following argument: if $d(x,y) > \rho$, then there exists n such that whenever $k > n$, we have $d(y, y_k) > \rho$. So take the complement, and we obtain

$$\overline{B_\rho(x)} \supseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \overline{B_\rho(y_k)}.$$

The inclusion cannot be replaced by an equality by the simple example: $\rho = 1, x = 0, y_k = 1/k$ in $\mathbb{R}^1$.

We cannot go further. We have to use some "regularity" of μ to obtain continuity on decreasing sequences of sets. Note that local finiteness is not enough because a closed ball may not be compact.

If we assume $\mu(\overline{B_{\rho+1}(x)}) < \infty$, then we can say

$$\lim_{n\to\infty} \mu\Big(\bigcup_{k=n}^{\infty} \overline{B_\rho(y_k)} \Big) = \mu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \overline{B_\rho(y_k)} \Big)$$

when all y_k are in $B_1(x)$ because $\overline{B_\rho(y_k)} \subseteq \overline{B_{\rho+1}(x)}$ now.

(b) Consider the Euclidean metric and $\mu = \delta_0$.

6

This is called the Bôcher theorem.

10 2019

1

Let $m = -\|u\|_\infty, M = \|u\|_\infty$. We have the convex combination

$$u(x) = \frac{M - u(x)}{M - m} m + \frac{u(x) - m}{M - m} M.$$

So by convexity, we have

$$F(u(x)) \leqslant \frac{M - u(x)}{M - m} F(m) + \frac{u(x) - m}{M - m} F(M).$$

Then integrate on $[0, 1]$, we obtain

$$\int_0^1 F(u(x))dx \leqslant \frac{M}{M - m} F(m) - \frac{m}{M - m} F(M) = \frac{F(m) + F(M)}{2}.$$

The equality holds if and only if F is linear on $[m, M] = [-\|u\|_\infty, \|u\|_\infty]$, but F is strictly convex by the assumption, so the equality holds if and only if $u \equiv 0$.

3

Recall that for a planar harmonic function $f : \mathbb{R}^2 \to \mathbb{R}$ is the real part of an entire function g. As an entire function, we can expand $\exp(g)$ as

$$e^{g(z)} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

where we can compute each coefficient by

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz.$$

So

$$|a_n| \leqslant r^{-n} \sup\{|e^{g(z)}| : |z| = r\}, \quad \forall r > 0.$$

We know $|e^{g(z)}| = \exp \operatorname{Re}(g(z)) = \exp f(z)$. As $|z| \to \infty$, we have $|f(z)| = o(\log |z|)$, so $|f(z)| \leqslant \log |z|/2$ whenever $|z| >> 1$. This gives the estimate

$$|a_n| \leqslant r^{-n} \exp(\log r/2) = r^{-n+1/2}, \quad \forall r >> 1.$$

So $a_n = 0$ for $n \geqslant 1$, i.e. $\exp(g)$ is constant. This proves that f is constant.

4

(a) If there is such a holomorphic function f, then the contour integral [3]

$$\frac{1}{2\pi i}\int_{|z+1|=1} f'(z)dz = 0.$$

But this is not the case for the given f.

(b) A cheating example is to choose $L = \mathbb{R}$. Another cheating example is to choose $L = (-\infty, 1]$. The second example is to make $\{\pm 1\} \subseteq L$ and $\mathbb{C} \setminus L$ simply connected.

We prove here that $L = [-1, 1]$ is sufficient. For this L, the homology group $H_1(\mathbb{C} \setminus L) \cong \mathbb{Z}$ whose generator can be chosen as $z = 2e^{i\theta}$ with $\theta \in [0, 2\pi]$. Since

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{1}{(z^2-1)^{2019}} = \text{Res}(-1) + \text{Res}(1) = 0,$$

the anti-derivative F exists.

A direct explanation without using homology groups is that first construct F in $\mathbb{C} \setminus [-1, \infty)$. The anti-derivative exists because $\mathbb{C} \setminus [-1, \infty)$ is simply connected. Then examine the limit of $F(z)$ when z approaches the points on $(1, \infty)$. For $x \in (1, \infty)$, define

$$I_1(x) = \lim_{z \to x, \text{Im}(z) > 0} F(z), \quad I_2(x) = \lim_{z \to x, \text{Im}(z) < 0} F(z).$$

The difference $I_1(x) - I_2(x)$ can be represented by a contour integral

$$\int_\gamma \frac{1}{(z^2-1)^{2019}} dz$$

where γ starts from x, travel around 0 counter-clockwise, and goes back to x. Now the same application of the residue theorem shows that $I_1(x) = I_2(x)$, so we may extend F to $\mathbb{C} \setminus [-1, 1]$.

If one still thinks this is not unambiguous, then it can be done on $\mathbb{C} \setminus (-\infty, 1]$ and $\mathbb{C} \setminus [-1, \infty)$ separately. On these two domains, we get two

[3] The trace of a contour is not well-defining the contour, but for convenience, I use a circle to mean a round contour around the center once.

anti-derivatives F_1, F_2 respectively. We can assume $F_1 = F_2$ in the upper half plane. The same application of the residue theorem shows that $F_1 = F_2$ automatically in the lower half plane. So they can be glued up to a holomorphic function on $\mathbb{C} \setminus [-1, 1]$.

11 2020

1

Note that $\chi \in C_c^\infty(\mathbb{R})$, so it satisfies the conditions in Proposition 7.5. The rest is just a recitation.

There is also a much easier proof because the conditions for χ is much stronger than what we need.

Suppose x_0 is a Lebesgue point for f, so

$$\frac{1}{2t}\int_{|x-x_0|\leqslant t} |f(x)-f(x_0)|dx \to 0, \quad t \to 0.$$

Suppose $\operatorname{supp}(\chi) \subseteq [-M, M]$. By definition

$$\begin{aligned}(\chi_\varepsilon * f)(x_0) - f(x_0) &= \int_{\mathbb{R}} (f(x_0-t)-f(x_0))\chi_\varepsilon(t)dt \\ &= \varepsilon^{-1}\int_{\mathbb{R}} (f(x_0-t)-f(x_0))\chi(\varepsilon^{-1}t)dt \\ &= \int_{\mathbb{R}} (f(x_0-\varepsilon t)-f(x_0))\chi(t)dt \\ &= \int_{-M}^{M} (f(x_0-\varepsilon t)-f(x_0))\chi(t)dt.\end{aligned}$$

For any $\delta > 0$, there exists $t_0 > 0$ such that whenever $0 < t < t_0$, we have

$$\int_{|x-x_0|\leqslant t} |f(x)-f(x_0)|dx < 2\delta t.$$

Now for $\varepsilon < t_0/M$, we have

$$\begin{aligned}\Big|\int_{-M}^{M} (f(x_0-\varepsilon t)-f(x_0))\chi(t)dt\Big| &\leqslant \|\chi\|_\infty \int_{-M}^{M} \Big|f(x_0-\varepsilon t)-f(x_0)\Big|dt \\ &\leqslant \|\chi\|_\infty \cdot 2\delta\varepsilon M.\end{aligned}$$

This shows that $\chi_\varepsilon * f \to f$ at x_0.

4

Let $h = f/g$ be a meromorphic function on $\mathbb{C}$. Then

$$h(z)^{2020} + 1 = \frac{1}{g(z)^{2020}}, \quad \forall z \in \mathbb{C}.$$

Then $h(z)$ omits 2020 values, namely all the 2020 roots of $w^{2020} = -1$. So h is constant by the Picard little theorem.

In the official solution, it is solved by considering the map

$$z \mapsto (f(z) : g(z) : 1) \in \mathbb{CP}^2$$

and the projective variety (here it is a curve) defined by the homogeneous equation

$$z_1^{2020} + z_2^{2020} = z_3^{2020}.$$

The genus can be computed by the formula $g = (d-1)(d-2)/2$ where $d = 2020$ here. And the universal cover of a compact Riemann surface has only 3 possibilities: S^2 (the Riemann sphere for $g = 0$); $\mathbb{C}$ ($g = 1$ is the torus); $\mathbb{D}$ ($g \geqslant 2$, the Poincaré disk). Then by the lifting property of the universal cover, we would get a bounded holomorphic map $\mathbb{C} \to \mathbb{D}$.

6

Note that the given f is a finite Blaschke product with the prescribed zeros at $\{0, a_1, \cdots, a_n\}$. We have encountered several times that f maps S^1 to S^1. Note that f maps $\mathbb{D}$ into $\mathbb{D}$.

To prove that f preserves the surface measure, we only need to prove that

$$\int_{S^1} \varphi d\theta = \int_{S^1} \varphi \circ f d\theta, \quad \forall \varphi \in C(S^1).$$

By solving the Dirichlet problem or explicit using the Poisson kernel (see 2017, p1), there exists a continuous function Φ defined on $\overline{\mathbb{D}}$, which is harmonic in $\mathbb{D}$ and has the boundary value φ. Now by the mean-value property of harmonic functions, we have

$$\int_{S^1} \varphi d\theta = 2\pi\varphi(0), \quad \int_{S^1} \varphi \circ f d\theta = 2\pi\varphi(f(0)) = 2\pi\varphi(0).$$

Note that $\varphi \circ f$ is harmonic (see the first proof of 2013, p4). This completes the proof.

12 2021

2

The official proof basically captures everything.

Suppose X has a basis $\varphi_1, \cdots, \varphi_n$ so $\dim X = n$. Write

$$f_k = \sum_{j=1}^{n} c_{k,j}\varphi_j.$$

We have to prove that $c_{k,j}$ converges for any j as $k \to \infty$.

Let us explain how to do this. The most common method to extract the coefficients is to find the value under some linear functionals. Here the functionals we can use are δ_x for $x \in [0,1]$, defined by $\delta_x(f) := f(x), \forall f \in X$.

1. Official version.

Observe that $\cap_{x\in[0,1]} \ker \delta_x = \{0\}$, so there exist $x_1, \cdots, x_n \in [0,1]$ such that $\cap_{j=1}^{n} \ker \delta_{x_j} = \{0\}$. [4] This means that $\det(A) := \det(\varphi_i(x_j)) \neq 0$. So

$$\begin{pmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_n) \end{pmatrix} = A^T \begin{pmatrix} c_{k,1} \\ c_{k,2} \\ \vdots \\ c_{k,n} \end{pmatrix}.$$

Move A^T to the left as $(A^T)^{-1}$ because it is invertible. This is a constant matrix independent of f_k. Let $k \to \infty$, the column vector $(f_k(x_1) \cdots f_k(x_n))^T$ converges, so $(c_{k,1} \cdots c_{k,n})^T$ also converges.

2. Let $\{\varphi_j^*\}$ be the dual basis in X^* with respect to $\{\varphi_j\}$, i.e. $\varphi_j^*(\varphi_k) = \delta_{j,k}$ the Kronecker delta. Claim that there exist $l \in \mathbb{N}^*, r_1, \cdots, r_l \in \mathbb{F}, x_1, \cdots, x_l \in [0,1]$ such that $r_1\delta_{x_1} + \cdots + r_l\delta_{x_l} = \varphi_1^*$.

[4] This certainly needs a proof, but the official answer does not include any explanations on it. The simplest way is to consider the time when the dimension of the intersection strictly decreases (by 1 because $\operatorname{codim} \ker \delta \leqslant 1$). This needs some efforts. See the remark at the end for another proof for this statement, which involves an important lemma.

If not, then for any $x_1, \cdots, x_l \in [0,1]$, we have

$$\begin{cases} r_1\varphi_2(x_1) + \cdots + r_l\varphi_2(x_l) = 0, \\ \quad\vdots \\ r_1\varphi_n(x_1) + \cdots + r_l\varphi_n(x_l) = 0. \end{cases} \implies r_1\varphi_1(x_1) + \cdots + r_l\varphi_1(x_l) = 0.$$

So the following two matrices have the same rank:

$$\mathrm{rank}\begin{pmatrix} \varphi_2(x_1) & \cdots & \varphi_2(x_l) \\ \vdots & & \vdots \\ \varphi_n(x_1) & \cdots & \varphi_n(x_l) \end{pmatrix} = \mathrm{rank}\begin{pmatrix} \varphi_2(x_1) & \cdots & \varphi_2(x_l) \\ \vdots & & \vdots \\ \varphi_n(x_1) & \cdots & \varphi_n(x_l) \\ \varphi_1(x_1) & \cdots & \varphi_1(x_l) \end{pmatrix}.$$

So the new row is a linear combination of the first $(n-1)$ rows. The matrix on the left side (denoted by $B(x_1, \cdots, x_l)$) is of size $(n-1)\times l$, with l arbitrarily chosen. The rank is $\leqslant n-1$, so the set

$$\{\mathrm{rank}(B(x_1, \cdots, x_l)) | x_1, \cdots, x_l \text{ distinct from } [0,1]\}$$

is finite. Then we can choose $x_1, \cdots, x_l$ so that $B(x_1, \cdots, x_l)$ has the maximal possible rank $r \leqslant n-1$. [5] Now we can choose r linearly independent rows from $B(x_1, \cdots, x_l)$, WLOG say the rows of $\varphi_2, \cdots, \varphi_{r+1}$. The advantage of doing so is that now the new row of φ_1 is written uniquely as a linear combination of the rows of $\varphi_2, \cdots, \varphi_{r+1}$. To be precise,

$$r = \mathrm{rank}\begin{pmatrix} \varphi_2(x_1) & \cdots & \varphi_2(x_l) \\ \vdots & & \vdots \\ \varphi_{r+1}(x_1) & \cdots & \varphi_{r+1}(x_l) \end{pmatrix} = \mathrm{rank}\begin{pmatrix} \varphi_2(x_1) & \cdots & \varphi_2(x_l) \\ \vdots & & \vdots \\ \varphi_n(x_1) & \cdots & \varphi_n(x_l) \end{pmatrix},$$

and for any new elements $x_{l+1}, \cdots, x_k \in [0,1]$, the rank does not increase:

$$r = \mathrm{rank}\begin{pmatrix} \varphi_2(x_1) & \cdots & \varphi_2(x_l) & \cdots & \varphi_2(x_k) \\ \vdots & & \vdots & & \vdots \\ \varphi_n(x_1) & \cdots & \varphi_n(x_l) & \cdots & \varphi_n(x_k) \end{pmatrix}.$$

[5] We can prove $r = n-1$ but we do not need this.

Now $(\varphi_1(x_1)\ \cdots\ \varphi_1(x_k))$ can be written as a linear combination of the rows above. But the first r rows are already linearly independent and the rank of the matrix is r, so the row of φ_1 can be uniquely written as a linear combination of the first r rows. The coefficients are uniquely determined by the first l columns. This proves that φ_1 is a linear combination of $\varphi_2, \cdots, \varphi_{r+1}$, a contradiction. So φ_1^* is a linear combination of $\delta_{x_1}, \cdots, \delta_{x_l}$ for some $x_1, \cdots, x_l \in [0,1]$.

Therefore, X^* is generated by the set $\{\delta_x | x \in [0,1]\}$. Since $c_{k,j} = \varphi_j^*(f_k)$, we know $\lim_{k\to\infty} c_{k,j}$ exists.

A lemma one may use.

Lemma 12.1. Let $L, L_j (j = 1, \cdots, n)$ be linear functionals on a linear space X. Then $\cap_{j=1}^n \ker(L_i) \subseteq \ker(L)$ if and only if L is a linear combination of $L_1, \cdots, L_n$.

Now each $\delta_x \in X^*, \forall x \in [0,1]$. Since X is finite-dimensional, its dual space X^* is of the same dimension. So we can take a maximal linearly independent subset of $\{\delta_x | x \in [0,1]\}$. This subset must be finite, and we denote it by $\{\delta_{x_1}, \cdots, \delta_{x_m}\}$. By the lemma, we must have

$$\bigcap_{j=1}^m \ker \delta_{x_j} = \bigcap_{x\in[0,1]} \ker \delta_x = \{0\}.$$

By counting dimensions, we know $m = \dim X = n$. The lemma can be proved by induction.

4

We have to assume $P(z) \not\equiv 0$.

The easiest way is to use the Picard great theorem. Consider $f(z) = \exp(1/z)/P(1/z)$. Then $z = 0$ is an essential singularity for f. By the Picard great theorem, except for at most one value, f assumes every value infinitely many times in every neighbourhood of $z = 0$. But the exceptional value is zero, so 1 is not the exceptional value.

Also one can use the Hadamard factorization. Let $f(z) = e^z - P(z)$. This function has growth order 1, so if it has finitely many zeros, then it must be of the form

$$e^z - P(z) = e^{az+b}Q(z), \quad \exists a, b \in \mathbb{C}, Q(z) \text{ a polynomial.}$$

Divide both sides by e^z, so

$$1 - e^{-z}P(z) = e^{(a-1)z+b}Q(z), \quad \forall z \in \mathbb{C}.$$

Let $z \in \mathbb{R}, z \to +\infty$, the left side tends to 1, then for the right side to be so, we must have $\mathrm{Re}(a) = 1, Q$ constant. This forces

$$P(z) = e^z - e^{az+b'},$$

which is impossible.

5

The Hadamard three-line theorem. See the notes.

13 2022

1

This has probability meaning.

2

The statement is false. We have to assume U is connected, otherwise there are counterexamples.

The set of fixed points $\{z \in U | f(z) = z\}$ is closed by continuity of f. But by the condition, this is exactly $\mathrm{Im}(f)$. By the open mapping theorem, $\mathrm{Im}(f)$ is open since f is non-constant. By connectedness of U, we know

$$\mathrm{Im}(f) = U = \{z \in U | f(z) = z\}.$$

3

Define an operator $(\tau_d f)(x) = f(x+d)$ for $f \in L^p(\mathbb{R}), d \in \mathbb{R}$. If $1 \leqslant p < \infty$, we know $\tau_d : L^p \to L^p$ is **strongly continuous**: $\tau_d f \to f$ in L^p-norm as $d \to 0$. Define a function

$$f(d) = \int_{\mathbb{R}} \chi_E(x)\chi_E(x+d)\cdots\chi_E(x+2021d)dx.$$

Here we replace X by E because x is sometimes confused with X. The integral is well-defined because $\tau_{kd}\chi_E$ is in $L^{2022}(\mathbb{R})$, and by Hölder's inequality and strong continuity of τ_{kd}, the function f is well-defined and continuous. Now note that $f(0) = m(E)^{2022} > 0$, so f is positive on a neighbourhood of 0. Take $d \neq 0, f(d) > 0$. There must be some x such that

$$\chi_E(x)\chi_E(x+d)\cdots\chi_E(x+2021d) > 0,$$

i.e. $x, x+d, \cdots, x+2021d \in E$.

4

There are at least three different proofs.

1. (The official one) Let $I = \{(x, y) \in [0, 1]^2 | x \neq y\}$. For each $(x, y) \in I$, define a map

$$T_{(x,y)} : \mathbf{P} \to \mathbb{C}, \quad u \mapsto \frac{u(x) - u(y)}{|x - y|}.$$

Clearly

$$\sup_{(x,y)\in I} |T_{(x,y)} u| \leqslant \|u'\|_\infty, \quad \|T_{(x,y)}\| \leqslant \frac{2}{|x - y|}$$

so we can apply the Banach-Steinhaus theorem to the family $(T_{(x,y)})_{(x,y)\in I}$ on the Banach space $\mathbf{P}$. This implies that there exists $C > 0$ such that

$$\sup\{\|T_{(x,y)}\| : (x, y) \in I\} \leqslant C.$$

On the unit ball B of $\mathbf{P}$, we have

$$|u(x) - u(y)| \leqslant C|x - y|, \quad \forall u \in B.$$

By the Arzelà-Ascoli theorem, B is compact. So $\mathbf{P}$ is finite-dimensional.

2. (Use properties of polynomials) Suppose $\mathbf{P}$ is infinite-dimensional. Note that $\mathbf{P}$ has algebraic dimension $\aleph_0$, but a Banach space cannot have algebraic dimension $\aleph_0$ due to the Baire's Category theorem. (An $\aleph_0$-dimensional linear space can be written as the countable union of finite-dimensional subspaces.)

3. (Credited to Zhao Weida) Suppose $\dim \mathbf{P} = \infty$. Then there would be elements in $\mathbf{P}$ whose degree can be arbitrarily large.

Now we build recursively several sequences as follows.

(1) Take $0 \neq x_1 \in \mathbf{P}$, let $d_1 = \deg(x_1), n_1 = \|x_1\|, c_1 > 0$ be the coefficient of t^{d_1} in x_1, and choose $r_1 > 0$ so that $r_1 n_1 < 1$.

(2) Take $x_2 \in \mathbf{P}$ such that $d_2 := \deg(x_2) > d_1, n_2 = \|x_2\|, c_2 > 0$ the coefficient of t^{d_2} in x_2, and choose $r_2 > 0$ so that $r_2 n_2 < 2^{-1}$ and the coefficient of t^{d_1} in $r_1 x_1 + r_2 x_2$ is $> r_1 c_1/2$.

(3) If $x_j, r_j (j = 1, \cdots, k)$ have been chosen so that $r_j n_j < 2^{1-j}, d_1 < \cdots < d_k, c_j > 0$ the coefficient of t^{d_j} in x_j, and the coefficient of t^{d_j} in $r_1 x_2 + \cdots + r_k x_k$ is $> r_j c_j/2$ for each $j = 1, \cdots, k - 1$, then we can choose $x_{k+1} \in \mathbf{P}, r_{k+1} > 0$ so that $d_{k+1} = \deg(x_{k+1}) > d_k, n_{k+1} = \|x_{k+1}\|, c_{k+1} > 0$ the coefficient of $t^{d_{k+1}}$ in x_{k+1}, $r_{k+1} n_{k+1} < 2^{-k}$ and the coefficient of t^{d_j} in

$r_1x_1 + \cdots + r_{k+1}x_{k+1}$ is $> r_jc_j/2$ for each $j = 1, \cdots, k$. This is possible because these requirements are simultaneously satisfied whenever $r_{k+1} > 0$ is sufficiently small.

So $x = \sum_{j=1}^{\infty} r_k x_k$ is an element in $\mathbf{P}$. If a sequence of polynomials converges uniformly to a polynomial on $[0, 1]$, then the sequence of the coefficients converges for each degree. But now by the construction, the polynomial x has infinitely many non-zero coefficients. This is a contradiction.

Two remarks.

(1) The first proof certainly proves more. We only use polynomials are C^1, so we can prove the statement where "polynomials" is replaced by $C^1([0, 1])$.

(2) An infinite dimensional Banach space has algebraic dimension $\geqslant 2^{\aleph_0}$. I do not remember who gives the first proof, but the proof relies on the following construction: given an infinite countable set A, we can pick $2^{\aleph_0}$ many subsets $(B_r)_{r\in\mathbb{R}}$ such that each B_r is infinite, $B_r \cap B_s$ is finite if $r \neq s$. (Maybe more conditions)

A way is to consider $A = \mathbb{Q}$, and pick for any irrational number r a sequence of rational numbers convergent to r. The sequence set is B_r.

Another way is to consider strips centered at 0 in the plane. The strips are of fixed width a for some suitable a, and r refers to the slope of each strip, B_r refers to the lattices in each strip.

However, the second proof also proves more in another sense. We can assume $\mathbf{P}$ is a Banach space without knowing any information. This is quite essential. So we can replace the space $C([0, 1])$ by any other Banach space B and assume $\mathbf{P} \subset B$ closed subspace.

14 2023

1

Pay attention. There is an abuse of notation in the size of matrices and the subscript of the sequence. They are certainly different.

(a)

Near $x_0 = 0$, by definition we have

$$\lim_{x \to x_0} \frac{\gamma(x) - \gamma(x_0)}{x - x_0} = A,$$

so we can write $\gamma(x) = I + Ax + G(x)$, where $\|G(x)\| = o(|x|)$ as $x \to 0$. Then

$$\gamma(t/n) = I + At/n + G(t/n), \quad \forall n >> 1.$$

Now the logarithm function $\log(I + X)$ has the power series expansion near $X = 0$ as

$$\log(I + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots$$

So when n is sufficiently large (compared to t), we have $\|G(t/n)\| \leqslant |t|/n$

$$n \log(\gamma(t/n)) = n(At/n + G(t/n)) - n(At/n + G(t/n))^2/2 + \cdots$$

Now $nG(t/n) \to 0$, $\|n(At/n + G(t/n))^k\| \leqslant (\|A\| + 1)^k |t|^k n^{1-k}$ when $k \geqslant 2$, so

$$n \log(\gamma(t/n)) \to At, \quad n \to \infty.$$

So $\gamma^n(t/n) \to \exp(At), \quad n \to \infty.$

Let $\gamma(x) = e^{xA}e^{xB}$. Then

$$\gamma'(0) = \frac{d}{dx}e^{xA}|_{x=0} \cdot e^{0B} + e^{0A}\frac{d}{dx}e^{xB}|_{x=0} = A + B.$$

So

$$e^{t(A+B)} = \lim_{n \to \infty} (e^{\frac{t}{n}A}e^{\frac{t}{n}B})^n.$$

(b)

Taking derivatives of both sides with respect to t gives

$$(A + B)e^{t(A+B)} = Ae^{tA}e^{tB} + e^{tA}Be^{tB}.$$

Taking $\frac{d}{dt}|_{t=0}$ again gives

$$(A+B)^2 = A^2 + AB + AB + B^2,$$

so $AB = BA$.

3

This is related to the Bochner theorem, which says that a continuous function is positive definite if and only if it is the Fourier transform of some finite Borel measure.

5

The condition says that

$$|f(z)|^2 \leqslant C^2|z||\cos^2(z)|, \quad \forall z \in \mathbb{C}.$$

First a useful fact. If f, g are two entire functions and $|f| \leqslant C|g|$, then $f = ag$ for some constant a. To prove this, note that f/g is a bounded meromorphic function, so all its singularities are removable, and thus f/g has to be a bounded entire function, i.e. a constant function.

So now we compare $f^2(z)$ and $z\cos^2(z)$, so we know $f^2(z) = az\cos^2(z)$ for some constant a. Now $f(0) = 0$, so $f(z) = zg(z)$ for some entire function g. Then $zg^2(z) = a\cos^2(z)$. Let $z = 0$, so $a = 0$.

We can explain the second paragraph like this: the condition says that f has a zero at $z = 0$ of order $\leqslant 1/2$, but the order has to be an integer, so f has a zero at $z = 0$.

15 2024

1

This is an application of the implicit function theorem. Consider the function $(F_1, F_2) = F : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$F(\lambda, \alpha, c) = (\langle Q_{\lambda,\alpha} - Q - cT_3, T_1\rangle, \langle Q_{\lambda,\alpha} - Q - cT_3, T_2\rangle).$$

We have $F(1, 0, 0) = 0$. We need to prove F is C^1 and the Jacobian of F with respect to (λ, α) is invertible at $(1, 0, 0)$.

The function is linear with respect to c, so there is nothing to prove. Note that Q, T_3 are even and T_1 is odd, so

$$\langle Q_{\lambda,0}, T_1\rangle = 0, \quad \forall \lambda \in \mathbb{R}.$$

This proves that $\partial F_1/\partial\lambda|_{(1,0,0)} = 0$. Note that

$$F_1(1, \alpha, 0) = \int Q(x - \alpha)xQ(x)dx,$$

so

$$\frac{\partial F_1}{\partial \alpha}\Big|_{(1,0,0)} = -\int Q'(x)xQ(x)dx = \frac{1}{2}\int Q^2(x)dx > 0$$

by the integration by parts. The partial derivative $\partial F_2/\partial\alpha$ has nothing to do with the singularity of the Jacobian. The last thing is $\partial F_2/\partial\lambda$:

$$\frac{\partial F_2}{\partial \lambda}\Big|_{(1,0,0)} = -\frac{1}{2}\int x^2Q(x)^2dx - \int x^3Q'(x)Q(x)dx = \int x^2Q(x)^2dx > 0,$$

so the determinant is non-zero.

2

There is a beautiful proof using very easy functional analysis only.

Let $U : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the Fourier transform, so this is a unitary operator. For $h \in L^\infty(\mathbb{R}^3)$, let $M_h : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the multiplication by h, i.e. $M_h(f) = fh, \forall f \in L^2(\mathbb{R}^3)$. This is a bounded linear operator, and it is self-adjoint if and only if h is (a.e.) real-valued, it is positive if and only if h is (a.e.) non-negative. First a lemma.

Lemma 15.1. If A, B are two bounded positive operators, then $\sigma(AB) \subseteq [0, \infty)$ even if they do not commute.

Proof. Since $AB = A^{1/2}A^{1/2}B$, and $A^{1/2}BA^{1/2}$ is positive, by the property

$$\sigma(XY) \cup \{0\} = \sigma(YX) \cup \{0\}$$

for any bounded linear operator X, Y, we know (with $X = A^{1/2}, Y = A^{1/2}B$)

$$\sigma(AB) \cup \{0\} = \sigma(A^{1/2}BA^{1/2}) \cup \{0\} \subseteq [0, \infty).$$

□

Now we will try to write T as this form. The operator V is a multiplication operator. Since $e^{-|x|^2}$ is bounded by 1 and has positive value everywhere, the operator V is a bounded positive operator. The operator $(-\Delta + 1)^{-1}$ is also easy to handle. Using the Fourier transform, we have

$$(-\Delta + 1)g = f \Longleftrightarrow (4\pi^2\xi_1^2 + 4\pi^2\xi_2^2 + 4\pi^2\xi_3^2 + 1)\widehat{g} = \widehat{f}.$$

If we use operators, this is to say

$$(-\Delta + 1)^{-1} = U^{-1}M_hU,$$

where $h(\xi) = (1 + 4\pi^2|\xi|^2)^{-1}$ is bounded. This also proves the uniqueness of the solution $(-\Delta + 1)g = f$ since $U^{-1}M_hU$ is a bounded linear operator. Note that h is also positive everywhere, so $U^{-1}M_hU$ is a bounded positive operator. Now apply the lemma to $U^{-1}M_hU, V$, we know $(-\Delta + 1)^{-1}V$ has spectrum in $[0, \infty)$. So T is invertible, because -1 is not in the spectrum of $(-\Delta + 1)^{-1}V$.

Warning. One may try to solve this problem from some classical PDE techniques. For example, if we can prove T has the form $I + K$ where K is compact, then we only need to prove this is injective since we have the Fredholm alternative in this case. And this is indeed injective, by an easy energy functional argument.

But this does not make sense at a key point. The operator $(-\Delta+1)^{-1}$ is not compact on $L^2(\mathbb{R}^3)$. It is compact on compact manifolds or bounded domains with some mild regularities. The usual way to prove $(-\Delta+1)^{-1}$ compact requires the inclusion $H^1 \hookrightarrow L^2$ being compact, but this is not the case for $\mathbb{R}^3$.

However, the Fredholm alternative method does work. One can express $(-\Delta+1)^{-1}$ by an integral operator (use the Green's function), and $(-\Delta+1)^{-1}V$ is also an integral operator, just do something with the kernel of $(-\Delta+1)^{-1}$ and the function V. Then one can prove that the integral kernel of $(-\Delta+1)^{-1}$ is L^2, so this is a compact operator.

3

It is said that this is the form of the solution to some Schrödinger equation and the conclusion is the Strichartz estimates.

The function u_1u_2 is

$$u_1u_2(x_1,x_2) = \int_{\mathbb{R}^2} \psi(\xi)\psi(\eta-10)f_1(\xi)f_2(\eta)e^{i(\xi+\eta)x_1}e^{i(\xi^2+\eta^2)x_2}d\xi d\eta.$$

With $F_1(\xi)=\psi(\xi)f_1(\xi), F_2(\eta)=\psi(\eta-10)f_2(\eta), F=F_1F_2$ we get

$$u_1u_2(x_1,x_2) = \int_{\mathbb{R}^2} F(\xi,\eta)e^{ix_1(\xi+\eta)}e^{ix_2(\xi^2+\eta^2)}d\xi d\eta.$$

Now use the change of variables $\Phi : (\xi,\eta)\mapsto(\xi',\eta')=(\xi+\eta,\xi^2+\eta^2)$, whose Jacobian

$$\frac{\partial(\xi',\eta')}{\partial(\xi,\eta)} = 2\eta-2\xi.$$

So for suitable functions f, we have

$$\int_{\mathbb{R}^2} f(\xi+\eta,\xi^2+\eta^2)d\xi d\eta = 2\int_{\Phi(\mathbb{R}^2)} f(\xi',\eta')\cdot|2\sqrt{2\eta'-\xi'^2}|^{-1}d\xi'd\eta'$$

because $\Phi:\mathbb{R}^2\to\Phi(\mathbb{R}^2)$ is a 2-sheet C^1 covering map outside a null set of $\mathbb{R}^2$. We restrict the domain of Φ to $\{\xi<\eta\}$ so that Φ^{-1} is well-defined. This gives

$$u_1u_2(x_1,x_2) = \int_{\Phi(\mathbb{R}^2)} F(\Phi^{-1}(\xi',\eta'))e^{ix_1\xi'}e^{ix_2\eta'}\cdot|\sqrt{2\eta'-\xi'^2}|^{-1}d\xi'd\eta'.$$

Note that F is supported in $[-1,1]\times[9,11]$, so the integrand is non-zero only if $(\xi',\eta')\in\Phi([-1,1]\times[9,11])=:E$. On this set, we have $|\sqrt{2\eta'-\xi'^2}|^{-1}\in[1/12,1/8]$. So u_1u_2 is the inverse Fourier transform of the function

$$F(\Phi^{-1}(\xi',\eta'))|\sqrt{2\eta'-\xi'^2}|^{-1}\chi_E(\xi',\eta')$$

whose L^2 norm is clearly bounded as required.

16 2025

4

We have to assume that γ does not pass through 0 and do all these things for piecewise smooth closed curve γ.

(a) For any smooth segment $\gamma_0(s) = (x(s), y(s)), s \in [t_0, t_1]$, this can be parametrized as $z(s) = x(s) + iy(s)$, so we know

$$\begin{aligned}
\frac{1}{2\pi}\int_{\gamma_0} \frac{xdy - ydx}{x^2+y^2} &= \frac{1}{2\pi}\int_{t_0}^{t_1} \frac{x(s)y'(s) - y(s)x'(s)}{x(s)^2+y(s)^2} ds \\
&= \frac{1}{2\pi}\int_{t_0}^{t_1} \frac{-iz'(s)\overline{z}(s) + ix'(s)x(s) + iy(s)y'(s)}{|z(s)|^2} ds \\
&= \frac{1}{2\pi i}\int_{t_0}^{t_1} \frac{z'(s)}{z(s)} ds + \frac{i}{2\pi}\int_{t_0}^{t_1} \frac{(x(s)^2+y(s)^2)'}{2(x(s)^2+y(s)^2)} ds \\
&= \frac{1}{2\pi i}\int_{t_0}^{t_1} \frac{z'(s)}{z(s)} ds + \frac{i}{2\pi} \log|z(s)|\Big|_{t_0}^{t_1}.
\end{aligned}$$

Now sum over the smooth segments. This number is an integer because if we let

$$k(s) = \frac{1}{2\pi i}\int_0^s \frac{z'(t)}{z(t)} dt,$$

then $k(0) = 0$ and $k'(s) = z'(s)/2\pi i z(s)$. So we know

$$\Big(z(s)\exp(-2\pi i k(s))\Big)' = 0.$$

Taking $s = 0, 1$, we know $z(0) = z(1) \neq 0$, so $\exp(-2\pi i k(0)) = \exp(-2\pi i k(1))$, so $k(0) - k(1) \in \mathbb{Z}$. Since $k(0) = 0$, we know $k(1) \in \mathbb{Z}$.

(b) Surely $r > 0$. By definition $d(\gamma) = n$.

(c) If γ_0, γ_1 are regularly homotopic, then $d(\gamma_t)$ is a continuous function of $t \in [0, 1]$. Since $d(\gamma)$ is integer-valued, it has to be constant. So $d(\gamma_0) = d(\gamma_1)$.

Conversely, we only need to prove that if $d(\gamma) = n$, then γ is regularly homotopic to the curve in (b) since regular homotopy is clearly an equivalence relation.

Write $\gamma(s) = r(s)e^{i\theta(s)}$ with r, θ piecewise smooth. The radius function r is uniquely determined by $r = |\gamma|$, and the argument function θ is defined by gluing up the local argument functions which always exist. The radius function r is non-vanishing. Let $\gamma_t(s) = (tr(s) + 1 - t)e^{i\theta(s)}$ be a closed curve (check it) for each $t \in [0, 1]$. This is a regular homotopy, so $\gamma(s)$ is regularly homotopic to the curve $\gamma_0(s) = e^{i\theta(s)}$. Since $d(\gamma) = n$, we have $\theta(1) - \theta(0) = 2\pi n$. Then use

$$\rho_t(s) = \exp(it\theta(s) + 2\pi in(1 - t)s + i(1 - t)\theta(0))$$

which is also a closed curve for each $t \in [0, 1]$. This completes the proof.

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WLOG $a = 0, b = 1$. Consider

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad x \in X.$$

So when $x \in A$, we have $f(x) = 0$. When $x \in B$, we have $x \notin A$, so $f(x) = 1$. We never have $d(x, A) + d(x, B) = 0$ because $A \cap B = \varnothing$.

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