Introduction to Model Theory

Lecturer: Will Johnson

2025 Fall

Contents

Basic Information

This class is at Period 3-5 on every Monday. At Period 3, every graduate student should attend, and the lecture is about o-minimality. At Period 4 and 5, it is the main class about Model Theory, for every graduate and undergraduate student.

This note is basically a "faithful" copy of what is on the blackboard. Some proofs are sketches, and some are beyond scope of this class.

Some proofs quite familiar to me are omitted, including some elementary number theory, some group/ring/field theory.

If there are any questions, please contact the email:

22110180006@m.fudan.edu.cn

or:

sxcui22@m.fudan.edu.cn

or:

17300180004@fudan.edu.cn

The tex code is on my overleaf.

1 September 8th

1.1 o-minimality

The **real exponential field** refers to $(\mathbb{R}, +, \cdot, \leqslant, \exp) =: \mathbb{R}_{\exp}$.

Term: an expression of the form like $\exp x + y, x \cdot y$. Built from variables, operations, and parameters from the structure (such as $1, \pi$).

Formula: $x \leqslant y, x = \exp(x+y), \exp z = x \wedge z \cdot z + y \cdot y = w$. Built from variables, operations, parameters, relations, $=, \wedge, \vee, \neg, \forall, \exists$.

In this example, terms take values in $\mathbb{R}$. Formulas take Boolean values (true or false).

Definition 1.1. If M is a structure, $\varphi(x_1, \cdots, x_n)$ is a formula, then we say $D \subseteq M^n$ is **definable** if

$$D = \{(a_1, \cdots, a_n) \in M^n | \varphi(a_1, \cdots, a_n)\}.$$

This set is also denoted by $\varphi(M^n)$.

For example, let $\varphi(x) \equiv (\exists y : y \cdot y = x)$. In $\mathbb{R}$, $\varphi(5)$ is true, $\varphi(-5)$ is false, $\varphi(\mathbb{R}) = [0, +\infty)$.

In $\mathbb{R}_{\exp}$, the set $\{(x, y) | y = \sqrt[3]{x} + e^x\}$ is definable. The formula can be written as $\exists z : z \cdot z \cdot z = x \wedge y = z + e^x$. This set is not definable in $\mathbb{R}$.

Definition 1.2. If X, Y are definable sets, $f : X \to Y$ is a map, then we say f is a **definable** function if its graph $\Gamma(f) = \{(x, y) \in X \times Y | y = f(x)\}$ is definable.

In $\mathbb{R}_{\exp}$, $f(x) = \sqrt[3]{x} + e^x$ is definable. The function $\log : (0, +\infty) \to \mathbb{R}$ is definable.

In $\mathbb{R}_{\exp}$, if f, g are definable, then $f \circ g$ is definable, and $f + g, f - g, f^{-1}, f', fg, f/g$ are definable whenever the operation makes sense.

Theorem 1.3 (Tarski-Seidenberg)**.** In $(\mathbb{R}, +, \cdot, \leqslant)$, a set $D \subseteq \mathbb{R}^n$ is definable iff D is semi-algebraic.

A set is **semi-algebraic** if it is a finite union of sets of the form

$$\{\overline{a} := (a_1, \cdots, a_n) \in \mathbb{R}^n | P_1(\overline{a}) = \cdots = P_m(\overline{a}) = 0, Q_1(\overline{a}) > 0, \cdots, Q_l(\overline{a}) > 0\},$$

where each P_i, Q_j is a polynomial function.

Corollary 1.4. In $(\mathbb{R}, +, \cdot, \leqslant)$, a set $D \subseteq \mathbb{R}^1$ is definable iff D is a finite union of open intervals (with endpoints possibly $\pm\infty$) and points.

Definition 1.5. An ordered structure $(M, \leqslant, \cdots)$ is **o-minimal** if for any definable $D \subseteq M^1$, D is a finite union of points and open intervals.

Theorem 1.6. (1) $(\mathbb{R}, +, \cdot, \leqslant)$ is o-minimal.

(2) $(\mathbb{R}, +, \cdot, \leqslant, \exp)$ is o-minimal.

However, $(\mathbb{R}, +, \cdot, \leqslant, \sin)$ is not o-minimal. The set $\{x \in \mathbb{R} | \sin x = 0\}$ is not a finite union of points. In this structure, every Borel set is definable.

If $(\mathbb{R}, +, \cdot, \leqslant, \cdots)$ is o-minimal, then every definable $D \subseteq \mathbb{R}^n$

1. has finitely many connected components.
2. is path-connected iff D is connected.
3. is not fractal.
4. has a well-defined dimension.
5. has a triangulation.
6. is homeomorphic to a finite simplicial complex if D is also compact.

And definable functions f are piecewise C^k for any k.

So definable objects are not pathological. They are "tame" in o-minimal structures.

$(\mathbb{Q}, \leqslant)$ is o-minimal. $(\mathbb{Q}, \leqslant, +)$ is o-minimal. But $(\mathbb{Q}, \leqslant, \cdot, +)$ is not o-minimal, since $\{x \in \mathbb{Q} | \sqrt{x} \in \mathbb{Q}\}$ is not of the required form.

1.2 main class

Monoid: $(M, \star, e)$.

Group: $(G, \star, e, ')$.

If A is a set, a **binary relation** on A is a subset $R \subseteq A^2$. Often $R(x, y)$ means $(x, y) \in R$. Practically, we write $R : A^2 \to \{\top, \bot\}$. ($\top$ is true, $\bot$ is false)

Definition 1.7. A **k-ary operation** on A is a function $f : A^k \to A$.

A **k-ary relation** on A is a subset $R \subseteq A^k$. (or $R : A^k \to \{\top, \bot\}$)

k is called the **arity** of the operation or relation.

A^0 is a set with one element. 0-ary (nullary) operations are elements in A.

Definition 1.8. A **partially ordered set** (poset) is $(P, \leqslant)$ such that P is a set, $\leqslant$ is a binary relation with

$$\forall x : x \leqslant x,$$
$$\forall x, \forall y, \forall z : x \leqslant y \wedge y \leqslant z \implies x \leqslant z,$$
$$\forall x, \forall y : x \leqslant y \wedge y \leqslant x \implies x = y.$$

Linear order : $\forall x, \forall y : x \leqslant y \vee y \leqslant x$.

Definition 1.9. A **group** is a set G with

1. a binary operation $\star : G^2 \to G$,
2. a nullary operation $e \in G$,
3. a unary operation $' : G \to G$,

satisfying

1. $\forall x, \forall y, \forall z : (x \star y) \star z = x \star (y \star z)$,
2. $\forall x : x = x \star e \wedge x = e \star x$,
3. $\forall x : x \star x' = e \wedge x' \star x = e$.

These two concepts are defined in the pattern:
A WIDGET is a set with
[certain operations, certain relations]
satisfying [certain axioms].
An **elementary class** is a class with such a definition.

Definition 1.10. A **language** is

1. a set of "function symbols",

2. a set of "relation symbols",

3. a map assigning each symbol X a number $n_X \in \mathbb{N} = \{0, 1, 2, \cdots\}$, called arity of X.

Example 1.1. (1) The language of groups $\mathcal{L}_{grp}$.

1. Three function symbols $\star, e, '$,

2. No relation symbols,

3. Arity: 2,0,1.

(1) The language of orders $\mathcal{L}_{\leqslant}$.

1. No function symbols.

2. One relation symbol of arity 2, $\leqslant$.

A **constant symbol** is a 0-ary function symbol.

Definition 1.11. If $\mathcal{L}$ is a language, then an **$\mathcal{L}$-structure** $\mathcal{M}$ is

1. a set M called the underlying set of $\mathcal{M}$.

2. For each k-ary function symbol f, we have a k-ary operation $f^{\mathcal{M}} : M^k \to M$.

3. For each k-ary relation symbol R, we have a k-ary relation $R^{\mathcal{M}} \subseteq M^k$.

$X^{\mathcal{M}}$ is the **interpretation** of X in $\mathcal{M}$.

Example 1.2. (1) An $\mathcal{L}_{\leqslant}$-structure $\mathcal{M}$ is

1. a set M,

2. a binary relation $\leqslant^{\mathcal{M}}$.

$(\mathrm{Pow}(\mathbb{R}), \subseteq)$ is an $\mathcal{L}_{\leqslant}$-structure. $\leqslant^{\mathrm{Pow}(\mathbb{R})}$ is $\subseteq$.
(2) An $\mathcal{L}_{grp}$-structure $\mathcal{M}$ is

1. a set M,

2. $\star^{\mathcal{M}} : M^2 \to M$,

3. $e^{\mathcal{M}} \in M$,

4. $'^{\mathcal{M}} : M \to M$.

$(\mathbb{R}, -, 7, \sqrt[3]{\cdot}$ is an $\mathcal{L}_{grp}$-structure. Here $-$ is the minus operation.

Language is just language. Nothing is born to be true. An $\mathcal{L}$-structure only need to have functions and relations whose arities match up with the corresponding symbols in $\mathcal{L}$. So an $\mathcal{L}_{grp}$-structure may not be a group in the interpretation.

Terms, Formulas, Sentences. $\{Terms\} \cap \{Formulas\} = \varnothing$, and $\{Sentences\} \subseteq \{Formulas\}$.

Definition 1.12. In the formula $\forall y : y \leqslant x$, x is called a **free variable**, y is called a **bound variable**, $\forall$ is called a **quantifier**. So a bound variable is bounded by a quantifier.

Sentences are formulas with no free variables.

If M is an $\mathcal{L}$-structure, φ is an $\mathcal{L}$-sentence, then $M \models \varphi$ (M satisfies φ, φ is true in M) makes sense.

Example 1.3. (1) $\mathcal{L}_{grp}$. We have

$$(\mathbb{R}, +, 0, -) \models \forall x, \forall y : x \star y = y \star x,$$

$$(\mathrm{Perm}(S), 0, \mathrm{id}_S, ^{-1}) \not\models \forall x, \forall y : x \star y = y \star x.$$

(2) $\mathcal{L}_{\leqslant}$. We have

$$(\mathbb{N}, \leqslant) \models \exists x (\forall y : x \leqslant y),$$

$$(\mathbb{Z}, \leqslant) \not\models \exists x (\forall y : x \leqslant y),$$

$$(\mathrm{Pow}(\mathbb{R}), \subseteq) \models \exists x (\forall y : x \leqslant y).$$

(3)

$$(\mathbb{Z}, \leqslant) \models \exists y : x \leqslant y$$

does not make sense. This is because x is a free variable.

Definition 1.13. An **$\mathcal{L}$-theory** T is a set $\mathcal{L}$-sentences.

A **model** of T is an $\mathcal{L}$-structure M s.t. $\forall \varphi \in T, M \models \varphi$. In notation, write $M \models T$.

$\mathrm{Mod}(T) := \{M | M \models T\}$.

$$(\mathbb{Z}, \leqslant) \models \forall x, \forall y : x \star y = y \star x$$

does not make sense, because they are not in the same language.

Example 1.4. T_{grp} is a set containing the following three sentences:

$$\forall x, \forall y, \forall z : x \star (y \star z) = (x \star y) \star z,$$
$$\forall x : x \star e = x \wedge x = e \star x,$$
$$\forall x : x \star x' = e \wedge e = x' \star x.$$

$M \models T_{grp}$ iff M is a group.

Definition 1.14. $\mathcal{K}$ is an **elementary class** if $\mathcal{K} = \mathrm{Mod}(T)$ for some T.

2 September 15th

2.1 o-minimality

Assume $(M, \leqslant, \cdots)$ linear order and densely ordered, i.e. M is totally ordered, and for any $a < b$, we have $(a, b) \neq \varnothing, (a, +\infty) \neq \varnothing, (-\infty, a) \neq \varnothing$. Also assume M is o-minimal.

In this section, intervals always mean non-empty intervals $(x, y) = \{z \in M | x < z < y\}$. For simplicity, some notations are valid only in $\mathbb{R}$ but can be modified in general cases.

The goal of this week and next week is to introduce the Monotonicity Theorem.

Fact 1. If $D \subseteq M$ is definable and D is infinite, then D has non-empty interior, i.e. there exists an interval contained in D.

Note that "densely ordered" implies every interval is infinite.

Fact 2. If $D \subseteq I$ is definable and D is dense in I, an interval in M, then $I \setminus D$ is finite.

D is dense in I means that $I \subseteq \mathrm{cl}(D)$, i.e. for any $a \in I, \varepsilon > 0$, we have $(a - \varepsilon, a + \varepsilon) \cap D \neq \varnothing$. Equivalently, for any interval $J \subseteq I$, we have $J \cap D \neq \varnothing$.

Fact 3. If $D \subseteq M$ is definable, non-empty, bounded, ($D \subseteq [a, b]$ for some $-\infty < a < b < +\infty$), then $\inf(D), \sup(D)$ exist.

Generally, $\inf(D), \sup(D)$ always exist in $M \cup \{\pm\infty\}$ if D is definable.

Fact 4. If $S \subseteq M^n$ is finite, $f : M \to S$ is a definable function, then $\exists a_1, \cdots, a_n \in M$ with $a_1 < \cdots < a_n$ such that f is constant on $(-\infty, a_1), \{a_1\}, (a_1, a_2), \cdots, \{a_n\}, (a_n, +\infty)$.

Definition 2.1. $\forall x \approx a^+ : \varphi(x)$ means $\exists \varepsilon > 0 : (\forall x \in (a, a + \varepsilon) : \varphi(x))$.

$\forall x \approx a^- : \varphi(x)$ means $\exists \varepsilon > 0 : (\forall x \in (a - \varepsilon, a) : \varphi(x))$.

Lemma 2.2. If $f : M \to S$ definable and $S \subseteq M^n$ is finite, $a \in M$, then $\exists i : \forall x \approx a^+, f(x) = i$, and $\exists j : \forall x \approx a^-, f(x) = j$.

Proof. By Fact 4. $\square$

Example 2.1. $f : M \to M$ definable, $a \in M$.

Then $\forall x \approx a^+, f(x) > f(a)$ or $\forall x \approx a^+, f(x) = f(a)$ or $\forall x \approx a^+, f(x) < f(a)$.

Proof. Define $g : M \to \{0, 1, 2\}$ by $g(x) = \begin{cases} 0, & f(x) < f(a), \\ 1, & f(x) = f(a), \\ 2, & f(x) > f(a). \end{cases}$

Then apply Lemma 2.2 to g. □

Lemma 2.3 (Transition Lemma)**.** If $f : M \to \{0, 1\}$ is definable, $I < J$ are intervals, $f = 1$ on I, $f = 0$ on J, then $\exists c : I < c < J$ and $\forall x \approx c^- : f(x) = 1 \wedge \forall x \approx c^+ : f(x) = 0$.

Proof. By Fact 4, $\exists a_0 < \cdots < a_n$ such that f is constant on (a_i, a_{i+1}). If such a c does not exist, then $f = 1$ on (a_{i-1}, a_i) implies $f = 1$ on (a_i, a_{i+1}). So $f = 1$ on I would imply $f = 1$ on J. □

Now assume $f : I \to M$ is definable, I an interval.

Definition 2.4. $b \in I$ is an **ascending point** for f if $\forall x \approx b^- : f(x) < f(b)$ and $\forall y \approx b^+ : f(b) < f(y)$.

Lemma 2.5. If every $a \in I$ is ascending, then f is strictly increasing. ($x < y$ implies $f(x) < f(y)$)

Proof. Suppose $a < b$ but $f(a) \geqslant f(b)$.

Let $g(x) = \begin{cases} 0, & f(x) \leqslant f(x), \\ 1, & f(x) > f(a). \end{cases}$

Then $\forall x \approx a^+ : g(x) = 1$ and $\forall x \approx b^- : g(x) = 0$.

By Lemma 2.3, there exists $c \in (a, b)$ such that $\forall x \approx c^- : g(x) = 1$ and $\forall y \approx c^+ : g(y) = 0$. By definition of g, this is $\forall x \approx c^- : f(x) > f(a)$ and $\forall y \approx c^+ : f(y) \leqslant f(a)$. But c is also an ascending point, so $f(x) < f(y)$. Contradiction. □

Definition 2.6. $a \in I$ is a **valley point** if $\forall x \approx a^- : f(x) > f(a)$ and $\forall y \approx a^- : f(a) < f(y)$.

Lemma 2.7. If f is weakly increasing ($x < y \Longrightarrow f(x) \leqslant f(y)$), then f has no valley points.

Lemma 2.8. Every definable $f : I \to M$ has a non-valley point.

Proof. Suppose every $a \in M$ is a valley point.

Let
$$P = \{a \in I | \forall x \approx a^-, \forall y \approx a^+ : f(x) > f(y)\}$$
and
$$Q = \{a \in I | \forall x \approx a^-, \forall y \approx a^+ : f(x) < f(y)\}.$$

Then one can prove P is non-empty. (hard to prove)

From this, P is dense in I, since one can consider any interval $J \subseteq I$ and $J \cap P$ should be non-empty.

By Fact 2, $I \setminus P$ is finite.

By symmetry, $I \setminus Q$ is finite.

Note that $P \cap Q$ is empty. This leads to a contradiction. □

2.2 main class

Definition 2.9. If M is an $\mathcal{L}$-structure, $A \subseteq M$, we define $\mathcal{L}(A) = \mathcal{L}_A$ to be the language $\mathcal{L}$ plus every element of A added as a new constant symbol.

If $\varphi(x_1, \cdots, x_n)$ is an $\mathcal{L}(A)$-formula, then $\varphi(M) = \{(a_1, \cdots, a_n) \in M^n | M \models \varphi(\overline{a})\}$.

Example 2.2. $\mathcal{L}_{ring}$: $\varphi(x, y) \equiv (\exists z : x^2 + y^2 + z^2 = 1)$.

$\varphi(\mathbb{R})$ is the closed unit disk.

$\varphi(\mathbb{C}) = \mathbb{C}^2$.

$\varphi(\mathbb{Q})$ is something weird.

For $\mathbb{R}$, the formula $\exists z : x^2 + y^2 + z^2 = \pi^2$ is an $\mathcal{L}_{ring}(\mathbb{R})$-formula.

Definition 2.10. M is an $\mathcal{L}$-structure, $A \subseteq M, D \subseteq M^n$. Then D is **definable over** A if $D = \varphi(M)$ for some $\mathcal{L}(A)$-formula φ. Such a D is also called **A-definable**.

"Definable" means M-definable.

0-definable means $\varnothing$-definable.

In $(\mathbb{R}, \leqslant)$, the set $D = [2, +\infty)$ is not $\varnothing$-definable.

Definition 2.11. If φ is an $\mathcal{L}$-formula, and $b_1, \cdots, b_n \in M$, then we define $\varphi(M, b_1, \cdots, b_n) = \{\overline{a} | M \models \varphi(\overline{a}, \overline{b})\}$.

So D is A-definable iff $D = \varphi(M, \overline{b})$ for some $\mathcal{L}$-formula φ and $b_1, \cdots, b_n \in A$.

Theorem 2.12. Suppose $X, Y \subseteq M^n$ are definable. Then $X \cup Y$ is definable.

Proof. Let $X = \varphi(M), Y = \psi(M)$, where φ, ψ are $\mathcal{L}(M)$-formulas. Then we define $\theta(\overline{x}) \equiv \varphi(\overline{x}) \vee \psi(\overline{x})$. $X \cup Y$ is $\theta(M)$. □

Definition 2.13. If X, Y are definable sets, $f : X \to Y$ is definable if $\{(x, y) \in X \times Y | y = f(x)\}$ is definable.

Theorem 2.14. (1) If $f : X \to Y, g : Y \to Z$ are definable, then $g \circ f : X \to Z$ is definable.

(2) $\mathrm{id}_X : X \to X$ is definable.

(3) If $f : X \to Y$ is a definable bijection, then $f^{-1} : Y \to X$ is definable.

Proof. (1)

Suppose $\varphi(x, y)$ defines f, $\psi(y, z)$ defines g. This means (for example for f) $\forall a, \forall b : f(a) = b \leftrightarrow M \models \varphi(a, b)$. Then we have

$$c = (g \circ f)(a) \leftrightarrow M \models \exists y : \varphi(a, y) \wedge \psi(y, c).$$

(2)

Obvious.

(3)

Obvious. □

Definition 2.15. Two $\mathcal{L}$-structures M, N are **elementarily equivalent** if for any $\mathcal{L}$-sentence φ, $M \models \varphi \leftrightarrow N \models \varphi$.

$\mathrm{Th}(M) := \{\varphi | M \models \varphi\}, M \equiv N \Leftrightarrow \mathrm{Th}(M) = \mathrm{Th}(N)$.

Example 2.3. In $\mathcal{L}_{ring}$, four structures $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

We can use three sentences to distinguish them;

$\exists x : x + x = 1$, $\exists x : x \cdot x = 1 + 1$, $\exists x : x \cdot x + 1 = 0$.

Definition 2.16. M, N are two $\mathcal{L}$-structures. An **isomorphism** from M to N is a bijection $j : M \to N$ such that

(1) for every function symbol $f \in \mathcal{L}$,

$$j(f^M(a_1, \cdots, a_n)) = f^N(j(a_1), \cdots, j(a_n)).$$

(2) If $c \in \mathcal{L}$ is a constant symbol, $j(c^M) = c^N$.

(3) For any relation symbol $R \in \mathcal{L}$,

$$R^M(a_1, \cdots, a_n) \Leftrightarrow R^N(j(a_1), \cdots, j(a_n)).$$

Example 2.4. (1) $\mathcal{L}_{graph} = \{E\}$ a binary relation symbol.

$j : (M, E) \to (N, E)$ is a bijection, and $x\ E^M\ y \Leftrightarrow j(x)\ E^N\ j(y)$.

(2) $\mathcal{L}_{grp} = \{\cdot, 1, ^{-1}\}$.

$j(x \cdot y) = j(x) \cdot j(y), j(1) = 1, j(x^{-1}) = j(x)^{-1}$.

For $(\mathbb{R}, +, 0, -)$ and $(\mathbb{R}_{>0}, \cdot, 1, ^{-1})$, we can use $j(x) = e^x, x \in \mathbb{R}$.

Definition 2.17. M is **isomorphic** to N if there is an isomorphism $f : M \to N$, and denoted by $M \cong N$.

Theorem 2.18. (1) $M \cong N$ implies $M \equiv N$.

(2) $M \cong M$.

(3) $M \cong N$ implies $N \cong M$.

(4) $M \cong N, N \cong X$ implies $M \cong X$. We denote $[M]_{\cong} = \{N | N \cong M\}$.

Only the first statement is non-trivial. It is the case $n = 0$ in the following theorem.

Theorem 2.19. If $j : A \to B$ is an isomorphism, then

(1) t is a term implies $j(t^A(a_1, \cdots, a_n)) = t^B(j(a_1), \cdots, j(a_n))$.

(2) $A \models \varphi(a_1, \cdots, a_n) \Leftrightarrow B \models \varphi(b_1, \cdots, b_n)$ where $b_i = j(a_i)$.

Definition 2.20. An **automorphism** of M is an isomorphism $j : M \to M$.

$(\mathrm{Aut}(M), \circ, \mathrm{id}_A, ^{-1})$ is a group.

This can show that some sets are not definable.

Proposition 2.21. In $(\mathbb{R}, \leqslant)$, the set $D = [2, +\infty)$ is not $\varnothing$-definable.

Proof. If $D = \varphi(\mathbb{R})$ for some $\mathcal{L}$-formula φ, then $\mathbb{R} \models \varphi(a) \Leftrightarrow a \in D \Leftrightarrow a \geqslant 2$.

Consider $j(x) = x + 1$. It is an automorphism of $(\mathbb{R}, \leqslant)$. So $\mathbb{R} \models \varphi(x) \Leftrightarrow \mathbb{R} \models \varphi(j(x))$, which is not true. $\square$

A fact on o-minimality of $\mathbb{R}$: if $A \subseteq \mathbb{R}$ is finite, $D \subseteq \mathbb{R}$ is A-definable, then D is a union of some of $(-\infty, a_1), \{a_1\}, \cdots, \{a_n\}, (a_n, +\infty)$, where $A = \{a_1, \cdots, a_n\}, a_1 < \cdots < a_n$.

Definition 2.22. M: an $\mathcal{L}$-structure. $A \subseteq M$ is a **substructure** if for any function symbol $f \in \mathcal{L}$, we have $a_1, \cdots, a_n \in A \Rightarrow f^M(a_1, \cdots, a_n) \in A$, and for any constant symbol $c \in \mathcal{L}$, we have $c^M \in A$.

We also say M is an **extension** of A.

In $(\mathbb{R}, +, \cdot, 0, 1, -)$, $\mathbb{Z}$ is a substructure, $\mathbb{Q}$ is, but $\mathbb{N}$ is not. Also note that a substructure can be non-definable.

In $(\mathbb{R}, \leqslant)$, any $A \subseteq \mathbb{R}$ is a substructure.

If $A \subseteq M$ is a substructure, then A is naturally an $\mathcal{L}$-structure.

Lemma 2.23. If $A \subseteq M$ is a substructure, $\varphi(\overline{x})$ is quantifier-free, and $a_1, \cdots, a_n \in A$, then $A \models \varphi(\overline{a}) \Leftrightarrow M \models \varphi(\overline{a})$.

Definition 2.24. A is an **elementary substructure** of M if for any $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$ and $a_1, \cdots, a_n \in A$, we have $A \models \varphi(\overline{a}) \Leftrightarrow M \models \varphi(\overline{a})$.

We also say M is an **elementary extension** of A, and denote by $A \preceq M$.

This is to say A, M satisfy the same $\mathcal{L}(A)$-sentences.

If $A \preceq M$, then $A \equiv M$. Take $n = 0$ in the definition.

Example 2.5. (1) $(\mathbb{Q}, +, \cdot, 0, 1) \npreceq (\mathbb{R}, +, \cdot, 0, 1)$.

Take $\varphi(x) \equiv (\exists y : y \cdot y = x)$. $\mathbb{Q} \not\models \varphi(2), \mathbb{R} \models \varphi(2)$.

(2) $(2\mathbb{Z}, \leqslant)$ is a substructure of $(\mathbb{Z}, \leqslant)$.

$(2\mathbb{Z}, \leqslant)$ is isomorphic to $(\mathbb{Z}, \leqslant)$. In notation, $(2\mathbb{Z}, \leqslant) \cong (\mathbb{Z}, \leqslant)$.

But $(2\mathbb{Z}, \leqslant) \npreceq (\mathbb{Z}, \leqslant)$. Take $\varphi \equiv (\exists x : 0 < x \wedge x < 2)$.

3 September 22nd

3.1 o-minimality

We continue the proof of the Monotonicity Theorem.

Proof. (Lemma 2.8) Say $I = (b, c)$.

Define $g(x) = \begin{cases} 1, & \exists y \in (x, c), f(y) < f(x), \\ 0, & \text{else.} \end{cases}$

Case 1: $\forall x \approx c^- : g(x) = 0$. $g = 0$ on $(c - \varepsilon, c)$. If $c - \varepsilon < x < y < c$, then $f(x) \leqslant f(y)$ by definition of g. So f is weakly increasing on $(c - \varepsilon, c)$, and there are no valley points on $(c - \varepsilon, c)$.

Case 2: $\forall x \approx c^-, g(x) = 1$. $g = 1$ on $(c - \varepsilon, c)$. We can choose a sequence $c - \varepsilon < x_0 < x_1 < \cdots < c$ such that $f(x_0) > f(x_1) > \cdots$. Define $h : I \to \{0, 1\}$ by $h(x) = \begin{cases} 1, & f(x) \geqslant f(x_0) \\ 0, & \text{else.} \end{cases}$

Then $h(x_1) = h(x_2) = \cdots = 0$, so the set $\{x \in (x_1, c) | h(x) = 0\}$ is infinite, and by Fact 1, it contains an interval J. Meanwhile, x_0 is a valley point, so $\forall x \approx x_0^+ : f(x) > f(x_0)$ or equivalently, $h(x) = 1$.

By Transition Lemma, there exists c such that $\forall x \approx c^-, h(x) = 1 \wedge \forall y \approx c^+, h(y) = 0$. This c is in P, so P is non-empty.

□

Lemma 3.1. Let $f : I \to M$ be definable.

(1) f has only finitely many valley points.

(2) f has only finitely many peak points.

Proof. Otherwise by Fact 1, there exists an interval $J \subseteq \{\text{valley points}\}$, and $f|_J$ has every point in J as valley points, which is a contradiction to Lemma 2.8.

The case for peak points is similar, since peaks are just upside-down valleys. □

Lemma 3.2. If $f : I \to M$ is injective, then $I = \{\text{valleys}\} \cup \{\text{peaks}\} \cup \{\text{ascending}\} \cup \{\text{descending}\}$.

Proof. For $a \in I$, we have $\forall x \approx a^+ : f(x) > f(a)$ or $\forall x \approx a^- : f(x) < f(a)$. The last possibility is eliminated since f is injective. Similarly for $x \approx a^-$. Then we combine these cases. $\square$

Lemma 3.3. If $f : I \to M$ is definable, then $\exists J \subseteq I$ such that $f|_J$ is injective or constant.

Proof. For any $b \in \mathrm{im}(f)$, the fibre $f^{-1}(b)$ is definable. By Fact 1, if $|f^{-1}(b)| = \infty$, then the fibre contains an interval J, so $f = b$ on J.

If for any $b \in \mathrm{im}(f)$, $|f^{-1}(b)| < \infty$, then $|\mathrm{im}(f)| = \infty$. For any $b \in \mathrm{im}(f)$, take $g(b) = \min f^{-1}(b)$, then $g : \mathrm{im}(f) \to \mathrm{dom}(f)$ is definable and injective. Also $|\mathrm{im}(g)| = |\mathrm{im}(f)| = \infty$. So $\exists J \subseteq \mathrm{im}(g)$, and $f|_J$ is injective. $\square$

Lemma 3.4. If every $a \in I$ is

(1) ascending, then f is increasing.

(2) descending, then f is decreasing.

(3) locally constant, then f is constant.

Proof. The first two statements are just Lemma 2.5.

The last statement follows from Transition Lemma. $\square$

Lemma 3.5. If f is a monotone (increasing, decreasing or constant), then $\exists a \in I$ such that f is continuous at a.

Proof. Constant is surely ok.

If f is increasing, take $b \in \mathrm{Int}(\mathrm{im}(f)), b = f(a)$. For any $\varepsilon > 0$, take $\varepsilon' > 0, \varepsilon' \leqslant \varepsilon$ such that $[b-\varepsilon', b+\varepsilon'] \subseteq \mathrm{im}(f)$. Say $b-\varepsilon' = f(c), b+\varepsilon' = f(d)$, then $c < a < d$. So $\forall x \in (c,d) : f(x) \in (b-\varepsilon', b+\varepsilon') \subseteq (b-\varepsilon, b+\varepsilon)$. This exactly shows f is continuous at a. $\square$

Theorem 3.6. If $f : (a,b) \to M$, then $\exists a = c_0 < c_1 < \cdots < c_n = b$ such that $f|_{(c_i, c_{i+1})}$ is continuous and monotone.

Proof. Define $g(x) = \begin{cases} 0, & f \text{ ascending at } x, \\ 1, & f \text{ descending at } x, \\ 2, & f \text{ locally constant at } x, \\ 3, & \text{else.} \end{cases}$

Define $h(x) = \begin{cases} 1, & f \text{ continuous at } x, \\ 0, & \text{else.} \end{cases}$

The map $(g, h) : (a, b) \to \{0, 1, 2, 3\} \times \{0, 1\}$ given by $x \mapsto (g(x), h(x))$. By Fact 4, $\exists a = c_0 < c_1 < \cdots < c_n = b$ such that (g, h) is constant on (c_i, c_{i+1}).

Fix (c_i, c_{i+1}). If $g = 3$ on (c_i, c_{i+1}), take $J \subseteq (c_i, c_{i+1})$. By Lemma 3.3, $f|_J$ is constant or injective. If $f|_J$ is constant, then $g = 2$ on J, a contradiction. If $f|_J$ is injective, by Lemma 3.2, every point is valley or peak, but by Lemma 3.1, there are only finitely many these points, a contradiction. So $g = 3$ is impossible.

Finally, by Lemma 3.4, f is monotone on (c_i, c_{i+1}), then by Lemma 3.5, f is continuous at some point, but h is constant, so f is continuous on (c_i, c_{i+1}). $\square$

3.2 main class

Theorem 3.7. If $A \subseteq M$ is a substructure,

(1) If $\varphi(\overline{x})$ is quantifier-free, $\overline{a} \in A^n$, then $A \models \varphi(\overline{a}) \Leftrightarrow M \models \varphi(\overline{a})$.

(2) If $t(\overline{x})$ is a term, $\overline{a} \in A^n$, then $t(\overline{a})^A = t(\overline{a})^M$.

The theorem is proved by induction on complexity.

Example 3.1. $\mathbb{Z} \subseteq \mathbb{R}, \mathcal{L} = \{+, \cdot, -, 0, 1\}$ the language of ring.

$t(x, y) = x \cdot (x + y), t(2, 5)^{\mathbb{Z}} = t(2, 5)^{\mathbb{R}} = 14.$

$\varphi(x, y) \equiv (x \cdot x = y).\mathbb{Z} \models \varphi(5, 25) \Leftrightarrow \mathbb{R} \models \varphi(5, 25).$

$\psi(x) \equiv (\exists y : y \cdot y = x), \mathbb{R} \models \psi(2), \mathbb{Z} \not\models \psi(2).$

Recall that $A \preceq M$ means $A \models \varphi(\overline{a}) \Leftrightarrow M \models \varphi(\overline{a})$ for any $\varphi(\overline{x})$ and any $\overline{a} \in A^n$.

Theorem 3.8. If $A \subseteq M$ substructure, $M \models \forall \overline{x} : \varphi(\overline{x})$ for some q'free $\varphi(\overline{x})$, then $A \models \forall \overline{x} : \varphi(\overline{x})$.

Proof. If $\overline{a} \in A^n$, then $M \models \varphi(\overline{a})$, so $A \models \varphi(\overline{a})$. $\square$

Example 3.2. (1) All of group/ring axioms can be written in this way. So any substructure of a group/ring is a group/ring.

But we need to note that it depends on the language and axioms we use. We can use different languages to define the same concept.

(2) In our language of fields, substructures may not be fields.

$$\forall x : (x \neq 0 \rightarrow \exists y : x \cdot y = 1).$$

Theorem 3.9. M a structure, $A \subseteq M$. The following are equivalent:

(1) $A \preceq M$.

(2) Tarski-Vaught criterion: If $D \subseteq M$ is A-definable and $D \neq \varnothing$, then $D \cap A \neq \varnothing$.

Proof. (1)$\Rightarrow$(2): Suppose $D = \varphi(M)$, $\varphi(x) = \psi(x, \overline{a})$ an $\mathcal{L}(A)$-formula, $\overline{a} \in A^n$, ψ $\mathcal{L}$-formula.

$D \neq \varnothing$, so $M \models \exists x : \varphi(x)$ an $\mathcal{L}(A)$-sentence. Since $A \preceq M$, we know

$$\begin{aligned} A \preceq M &\Rightarrow A \models \exists x : \varphi(x) \\ &\Rightarrow \exists b \in A : A \models \varphi(b) \qquad \mathcal{L}(A)\text{-sentence} \\ &\Rightarrow \exists b \in A : M \models \varphi(b) \\ &\Rightarrow \exists b \in A : b \in D \\ &\Rightarrow D \cap A \neq \varnothing. \end{aligned}$$

(2)$\Rightarrow$(1): First we prove A is a substructure. Given any function symbol $f \in \mathcal{L}, a_1, \cdots, a_n \in A$, the set $D = \{f(\overline{a})\}$ is A-definable, so $D \cap A \neq \varnothing, f(\overline{a}) \in A$.

Next we prove that for any $\mathcal{L}$-formula $\varphi(\overline{x})$ and $\overline{a} \in A^n$, we have $A \models \varphi(\overline{a}) \Leftrightarrow M \models \varphi(\overline{a})$. We prove by induction on complexity of φ.

$\varphi(\overline{x})$ could be

(i) $t(\overline{x}) = s(\overline{x})$, $R(t_1(\overline{x}), \cdots, t_m(\overline{x}))$,$\psi(\overline{x}) \wedge \theta(\overline{x})$, $\neg\psi(\overline{x})$,$\top$. These can be proved by induction or q'freeness.

(ii) $\exists y : \psi(\overline{x}, y)$.

$$\begin{aligned} A \models \varphi(\overline{a}) &\Leftrightarrow \exists b \in A : A \models \psi(\overline{a}, b) \\ &\Leftrightarrow \exists b \in A : M \models \psi(\overline{a}, b) \quad \text{by induction} \\ &\Leftrightarrow \exists b \in A : b \in D := \psi(\overline{a}, M) \\ &\Leftrightarrow D \neq \varnothing \quad \text{by TV criterion} \\ &\Leftrightarrow \exists b \in M : M \models \psi(\overline{a}, b) \\ &\Leftrightarrow M \models \varphi(\overline{a}). \end{aligned}$$

□

Theorem 3.10 (Downward Löwenheim-Skolem Theorem)**.** If $|\mathcal{L}| \leqslant \aleph_0$, M is an $\mathcal{L}$-structure, then $\exists A \preceq M, |A| \leqslant \aleph_0$.

Proof. Choose $F : \text{Pow}(M) \setminus \{\varnothing\} \to M$ such that $F(X) \in X$. Define $A_0 = \varnothing$, and

$$A_{i+1} = A_i \cup \{F(X) | X \subseteq M, X \neq \varnothing, X \text{ is } A_i\text{-definable}\}.$$

Then $|A_i| \leqslant \aleph_0 \Rightarrow |A_{i+1}| \leqslant \aleph_0$ since $\mathcal{L}$ is countable. So $B = \cup_{i=0}^{\infty} A_i$ is countable.

Claim: $B \preceq M$ by TV criterion.

Let $D \subseteq M, D \neq \varnothing, D$ be B-definable. Suppose $D = \varphi(M, \overline{b}), \overline{b} \in B^n$, so $\exists i >> 0 : \overline{b} \in A_i^n$. So D is A_i-definable, $F(D) \in A_{i+1} \subseteq B$, and $F(D) \in D$, so $D \cap B \neq \varnothing$. □

Example 3.3. $(\mathbb{R}, +, \cdot, 0, 1, -)$ a field. There exists $K \preceq \mathbb{R}, |K| = \aleph_0$. This K satisfies $K \equiv \mathbb{R}, K \ncong \mathbb{R}$.

Definition 3.11. DLO means dense linear order, containing the following $\mathcal{L}_{\leqslant}$ axioms:

(1) Linear orders: $\forall x : x \leqslant x$, $\forall x, y : x \leqslant y \wedge y \leqslant x \to x = y$, $\forall x, y, z : x \leqslant y \wedge y \leqslant z \to x \leqslant z$, $\forall x, y : x \leqslant y \vee y \leqslant x$.

(2) Densely ordered: $x < y \to \exists z : x < z < y$. Here $x < y$ means $x \leqslant y \wedge x \neq y$.

(3) $\forall x \exists y : x < y, \forall x \exists y : y < x$.

(4) $\exists x : \top$.(non-empty)

Definition 3.12. M, N structures. A **partial isomorphism** $f : M \to N$ is an isomorphism $f : A \to B$ where $A \subseteq M, B \subseteq N$ substructures.

$\mathrm{dom}(f) = A, \mathrm{im}(f) = B, |f| := |A| = |B|$. $g \supseteq f$ if g extends f. f is **finite** if $|f| < \infty$. An **FPI** is a finite partial isomorhpism.

Lemma 3.13. If $M, N \models \mathrm{DLO}, f : M \to N$ an FPI, then

(1) $\forall \alpha \in M, \exists$ FPI $g \supseteq f, \alpha \in \mathrm{dom}(g)$.

(2) $\forall \beta \in N, \exists$ FPI $g \supseteq f, \beta \in \mathrm{im}(g)$.

Theorem 3.14 (Cantor)**.** If $M, N \models$ DLO, and $|M| = |N| = \aleph_0$, then $M \cong N$.

Proof. Write $M = \{a_1, a_2, \cdots\}, N = \{b_1, b_2, \cdots\}$. Construct $f_0 \subseteq f_1 \subseteq \cdots$ by $f_0 = \varnothing, a_1 \in \mathrm{dom}(f_1), b_1 \in \mathrm{im}(f_2), a_2 \in \mathrm{dom}(f_3), b_2 \in \mathrm{im}(f_4), \cdots$ by the previous lemma. Then $f = \cup_{i=0}^{\infty} f_i$ is a partial isomorphism, and $\mathrm{dom}(f) = M, \mathrm{im}(f) = N$, so f is an isomorphism. $\square$

Theorem 3.15. If $M, N \models$ DLO, then $M \equiv N$.

Proof. Take $M_0 \preceq M, N_0 \preceq N$ countable. Then we have

$$M \equiv M_0 \equiv N_0 \equiv N.$$

$\square$

For example, $(\mathbb{Q}, \leqslant) \equiv (\mathbb{R}, \leqslant)$.

Definition 3.16. M an $\mathcal{L}$-structure.

A **congruence** on M is an equivalence relation $\approx$ on M such that if $a_1 \approx b_1, \cdots, a_n \approx b_n$, then for any function symbol $f \in \mathcal{L}, f(\overline{a}) = f(\overline{b})$, and for any relation symbol $R \in \mathcal{L}, R(\overline{a}) \Leftrightarrow R(\overline{b})$.

Definition 3.17. Let M be an $\mathcal{L}$-structure, $\approx$ a congruence on M, we define $M/\approx = \{[a] | a \in M\}$, where $[a] = \{b | b \approx a\}$.

The interpretation of symbols is given by

$$R^{M/\approx}([a_1], \cdots, [a_n]) \Leftrightarrow R^M(a_1, \cdots, a_n)$$
$$f^{M/\approx}([a_1], \cdots [a_n]) = [f^M(a_1, \cdots, a_n)].$$

$\mathcal{L}(\approx) = \mathcal{L} \cup \{\approx\}$. If φ is an $\mathcal{L}$-formula, then $\varphi^{\approx}$ is φ with all $=$ replaced by $\approx$.

An **$\mathcal{L}$-prestructure** is $(M, \approx)$, where M is an $\mathcal{L}$-structure, $\approx$ a congruence.

$(M, \approx) \models \varphi^{\approx}$ is called $(M, \approx)$ **presatisfies** $\varphi^{\approx}$.

$(M, \approx)$ is a **premodel** of T if $(M, \approx) \models T^{\approx}$.

Example 3.4. In ring $(\mathbb{Z}, +, \cdot, -, 0, 1)$, we have a congruence $\equiv_n$, meaning modulo n. When n is prime, we know that $(\mathbb{Z}, \equiv_n)$ is a premodel of the field theory.

$$(\mathbb{Z}, \equiv_n) \models (\forall x : x \neq 0 \to \exists y : xy = 1)^{\approx}.$$

Lemma 3.18. Suppose $(M, \approx)$ is an $\mathcal{L}$-prestructure.

(1) If $t(\overline{x})$ is an $\mathcal{L}$-term, then $t([a_1], \cdots, [a_n])^{M/\approx} = [t(a_1, \cdots, a_n)^M]$.

(2) If $\varphi(x_!, \cdots, x_n)$ is an $\mathcal{L}$-formula, then

$$(M/\approx) \models \varphi([a_1], \cdots, [a_n]) \Leftrightarrow M \models \varphi(a_1, \cdots, a_n)^{\approx}.$$

(3) If φ is an $\mathcal{L}$-sentence, then

$$(M/\approx) \models \varphi \Leftrightarrow M \models \varphi^{\approx}.$$

(4) If $(M, \approx)$ is a premodel of T, then $M/\approx$ is a model of T.

Theorem 3.19. If R is a ring and $\approx$ is a congruence, then $R/\approx$ is a ring.

4 September 29th

4.1 o-minimality

Assume M is a dense o-minimal structure.

Fact 1. $D \subseteq_{def} M^n \times [a, b]$ is closed, then $\pi(D)$ is closed. Here $\pi : M^n \times [a, b] \to M^n$ is the projection, $a, b \in M \cup \{\pm\infty\}$.

If $D \subseteq M^n \times M$, then define $D_{\overline{a}} = \{b \in M | (\overline{a}, b) \in D\}$ for $\overline{a} \in M^n$. A **definable family** is a family of the form $\{D_{\overline{a}} | \overline{a} \in Y\}$ for some definable sets D, Y. A non-empty family of sets $\mathcal{F}$ is **directed** if for any $X, Y \in \mathcal{F}$, there is $Z \in \mathcal{F}$ with $Z \subseteq X \cap Y$.

Lemma 4.1. Let $I = [a, b]$ be a closed bounded interval in M. Let $\mathcal{F}$ be a directed definable family of non-empty closed subsets of I. Then $\bigcap \mathcal{F} \neq \varnothing$.

Proof. If $X \in \mathcal{F}$, then X is closed, bounded, non-empty, definable. So $\min(X)$ exists, let $B = \{\min(X) : X \in \mathcal{F}\}$, it is a definable subset of I. Let $c = \sup(B) \in I$. We claim that $c \in \bigcap \mathcal{F}$.

If $c \notin X$ for some $X \in \mathcal{F}$, since X is closed, there exists $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \cap X = \varnothing$. Since $c = \sup(B)$, there is some point in $(c-\varepsilon, c] \cap B$. So there is some $Y \in \mathcal{F}$ with $c - \varepsilon < \min(Y) \leqslant c$. Take $Z \in \mathcal{F}$ with $Z \subseteq X \cap Y$, then $\min(Z) \geqslant \min(Y) > c - \varepsilon$ and $\min(Z) \leqslant c$. But $\min(Z) \in Z \subseteq X$, which contradicts $(c - \varepsilon, c + \varepsilon) \cap X = \varnothing$. $\square$

Lemma 4.2. Suppose $I = [a, b]$ a closed interval and $D \subseteq M^n \times I$ definable and closed. Let $\pi_1 : M^n \times I \to M^n$ be the projection, then $\pi_1(D)$ is closed.

Proof. Take a point $\overline{a} \in \mathrm{cl}(\pi_1(D))$. Let $\mathcal{B}$ be the family of open boxes around $\overline{a}$. For each $B \in \mathcal{B}$, let

$$X_B = \mathrm{cl}(\pi_2((B \times I) \cap D)$$

where $\pi_2 : M^n \times I \to I$ the projection. The family $\{X_B : B \in \mathcal{B}\}$ is directed by the set inclusion. Each X_B is closed.

If some $X_B = \varnothing$, then $(B \times I) \cap D = \varnothing$, so $B \cap \pi_1(D) = \varnothing$, contradicting to $\overline{a} \in \mathrm{cl}(\pi_1(D))$.

We can restrict each X_B in a fixed bounded interval. Then by the previous lemma, there exists $b \in \bigcap_{B\in\mathcal{B}} X_B$.

If $(\overline{a}, b) \in D$, then $\overline{a} = \pi_1(\overline{a}, b) \in \pi_1(D)$, which is as desired. Otherwise, there exists an open box $B \ni \overline{a}$ and an open interval $J \ni b$ such that $(B \times J) \cap D = \varnothing$ since D is closed. Then

$$\pi_2((B \times I) \cap D) \cap J = \varnothing.$$

But $b \in X_B$, so any open neighbourhood of b should intersect $\pi_2((B\times I)\cap D)$, which is a contradiction. □

So we prove Fact 1.

Lemma 4.3. [Prism lemma] Suppose $B \subseteq M^n \times M$ a box (a product of closed intervals), $f : B \to M$, and $f(\overline{x}, b)$ is continuous for all b, $f(\overline{a}, y)$ is continuous and weakly increasing in y for all $\overline{a}$, then f is continuous.

Proof. Fix $(\overline{a}, b) \in B, c = f(\overline{a}, b)$. Choose $U \ni c$ an open interval, $\delta > 0$ such that $f(\overline{a}, b \pm \delta) \in U$. There exists an open neighbourhood $V \ni \overline{a}$ such that

$$\forall \overline{x} \in V : f(x, b + \delta) \in U; \quad \forall \overline{x} \in V : f(\overline{x}, b - \delta) \in U.$$

If $\overline{x} \in V, y \in (b - \delta, b + \delta)$, then $f(\overline{x}, b - \delta) \leqslant f(\overline{x}, y) \leqslant f(\overline{x}, b + \delta)$, both ends in U, so the middle point lies in U. □

Definition 4.4. If $D \subseteq_{def} M^n$, let $\mathcal{C}(D) = \{f : D \to M | f$ is continuous, definable$\}$.

(1) $\{a\} \subseteq M^1$ is a **0-dimensional cell**.

(2) $(a, b) \subseteq M^1$ is a **1-dimensional cell**.

(3) If $D \subseteq M^n$ is a k-dimensional cell, $f \in \mathcal{C}(D)$, then $\Gamma(f)$, the graph of f, is a k-dimensional cell.

(4) If $D \subseteq M^n$ is a k-dimensional cell, $f, g \in \mathcal{C}(D) \cup \{\pm\infty\}$ and $f < g$ on D, then

$$(f, g)_D = \{(\overline{x}, y) | \overline{x} \in D, f(\overline{x}) < y < g(\overline{x})\}$$

is a $(k + 1)$-dimensional cell.

Warning. The definition of cells depend on coordinates. A curve drawn horizontally can be a cell, but usually not when vertically. For example, the figure ξ is not a cell, the figure v is a cell, the figure $>$ is not a cell.

Fact 3. (1) Cells are definable. (by induction)

(2) $C \subseteq M^n$ a k-dimensional cell, C is open iff $k = n$. Otherwise, C is nowhere dense.

(3) If $C \subseteq M^n$ a k-dimensional cell, then $\exists \pi : M^n \to M^k$ a coordinate projection such that $\pi(C) \subseteq M^k$ is an open cell, $C \to \pi(C)$ is a homeomorphism, $D \subseteq C$ is a cell iff $\pi(D)$ is a cell.

(4) If $C \subseteq M^n$ a cell, $f : C \to \{0, 1\}$ definable and continuous, then f is constant.

(5) If $C_1, \cdots, C_k \subseteq M^n$ are non-open cells, then $\operatorname{int}(\bigcup_{i=1}^{k} C_i) = \varnothing$.

An example for (3):

$$C = \{(x, y, z, w)|a < x < b, y = f(x), g(x, y) < z < h(x, y), w = l(x, y, z)\}.$$

Let $\pi(x, y, z, w) = (x, z)$, the "free" part of C.

$$\pi(C) = \{(x, z)|a < x < b, g(x, f(x)) < z < h(x, f(x))\}.$$

Now assume $D \subseteq M^n \times M, \overline{a} \in M^n$, and write $D_{\overline{a}} = \{b \in M|(\overline{a}, b) \in D\}$.

Theorem 4.5. For all $n \geqslant 1$.

(Cell_n) If $D \subseteq_{def} M^n$, then $D = \bigsqcup_{i=1}^{m} C_i$, C_i cells.

(Cont_n) If $D \subseteq_{def} M^n, f : D \to M$ definable, then $D = \bigsqcup_{i=1}^{m} C_i$, C_i cells, $f|_{C_i}$ continuous.

(Bound_n) If $D \subseteq_{def} M^n \times M$, and $\forall \overline{a}, |D_{\overline{a}}| < \infty$, then $\exists N < \infty$ such that $\forall \overline{a}, |D_{\overline{a}}| < N$.

We prove by order:

Cell_1(o-minimality), Cont_1(Monotonicity Theorem), Bound_1,

Cell_2, Cont_2, Bound_2, $\cdots$

Theorem 4.6. (Cell_{n+1}) holds if ($\text{Cell}_{\leqslant n}$), ($\text{Cont}_{\leqslant n}$), ($\text{Bound}_{\leqslant n}$).

Proof. If $D \subseteq_{def} M$, $\mathrm{bd}(D) = \{a_1, \cdots, a_n\}, a_1 < a_2 < \cdots < a_n$, we define $\mathrm{sgn}(D) \in \{0,1\}^{2n+1}$ by whether $(-\infty, a_1), \{a_1\}, (a_1, a_2), \cdots$ are in D.

For example, $D = [0,1) \cup \{2\}$, then $\mathrm{sgn}(D) = 0110010$.

Given $D \subseteq M^{n+1}$, there is some definable $B \subseteq M^{n+1}$ such that $B_{\overline{a}} = \mathrm{bd}(D_{\overline{a}}), \forall \overline{a}$. Note that B itself may not be closed.

By (Bound_n), there exists $N < \infty$ such that $\forall \overline{a}, |B_{\overline{a}}| < N$. Define a function $M^n \to \{0,1\}^{\leqslant 2N+1}, \overline{a} \mapsto \mathrm{sgn}(D_{\overline{a}})$. This function has finite image, and each fibre is definable. So by (Cell_n), $M^n = \bigsqcup_{i=1}^{m} C_i$, C_i cells, $\mathrm{sgn}(D_{\overline{a}}) = s_i$ constant for $\overline{a} \in C_i$. Intuitively, on each C_i, we have a $0-1$ string encoding the information that on each section $D_{\overline{a}}$, which part lies in D.

Let $f_i(\overline{a})$ be the i-th smallest element of $\mathrm{bd}(D_{\overline{a}})$. Each f_i is partially defined on M^n, valued in M, and definable. By (Cont_n), we can further decompose M^n into finitely many cells so that on each cell $\mathrm{sgn}(D_{\overline{a}})$ is constant, each f_j is continuous.

Then we can write D as a disjoint union of finitely many cells according to the above results. For example, $D = [0,1] \times (0,1]$ in $M^2, n = 1$. We have $N = 3$, and $\mathrm{sgn}(D_a)$ has length as most 5. We decompose M into two parts $(-\infty,0) \cup (1,+\infty), C = [0,1]$. On $(-\infty,0) \cup (1,+\infty)$, the section D_a is empty. On $[0,1]$, the sign is 01100. We have only two f_i's, namely $f_1 = 0$ on $[0,1]$ and $f_2 = 1$ on $[0,1]$. From the code 01100 and the functions f_1, f_2, we can completely write D as the disjoint union of the two cells $(f_1, f_2)_C \bigsqcup \Gamma(f_2)_C$. □

Theorem 4.7. (Cont_{n+1}) holds if $(\mathrm{Cell}_{\leqslant n+1})$, $(\mathrm{Cont}_{\leqslant n})$, $(\mathrm{Bound}_{\leqslant n})$.

Proof. Suppose $D \subseteq M^{n+1}, f : D \to M$ definable.

Define

$$g(\overline{a}, b) = \begin{cases} 1, & f(\overline{a}, y) \text{ continuous and locally weakly increasing at } y = b, \\ -1, & f(\overline{a}, y) \text{ continuous and locally strictly decreasing at } y = b, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(x) = \begin{cases} 1, & f(\overline{x}, b) \text{ continuous at } \overline{x} = \overline{a}, \\ 0, & \text{otherwise.} \end{cases}$$

By (Cell_{n+1}), $D = \bigsqcup_{i=1}^{m} C_i$, C_i cells, g, h constant on each cell. If $\dim(C_i) < n+1$, then Fact 3(3) gives $\pi : C_i \to C', C' \subseteq M^k, n < n+1$, and we use ($\text{Cont}_k$) on C'. If C_i is open, $g = 0$ contradicts the Monotonicity Theorem, $h = 0$ contradicts Fact 3(5) and (Cont_n). Then we apply Lemma 4.3. $\square$

4.2 main class

Definition 4.8. R is an **integral domain** if $R \models 0 \neq 1$ and

$$R \models (\forall x, y : x \cdot y = 0 \to x = 0 \vee y = 0).$$

Lemma 4.9. If R is an integral domain, $ax = ay, a \neq 0$, then $x = y$.

Theorem 4.10. If R is a finite integral domain, then R is a field.

Theorem 4.11. $\mathbb{Z}/\equiv_p$ is a field if p is prime.

T is an $\mathcal{L}$-theory.

Definition 4.12. T is **satisfiable** if $\exists M \models T$.

T is **finitely satisfiable** if $\forall T_0 \subseteq_f T : \exists M, M \models T_0$.

Theorem 4.13 (Compactness Theorem)**.** If T is finitely satisfiable, then T is satisfiable.

Example 4.1. Let PA be the Peano Arithmetic. $\mathbb{N} \models PA, \mathcal{L} = \{0, 1, +, \cdot\}$. Let $\mathcal{L}' = \mathcal{L} \cup \{c\}, T = PA \cup \{c \neq 0, c \neq 1, c \neq 1+1, \cdots\}$. Then T is finitely satisfiable, with $\mathbb{N}$ a model for any finite subset of T, c interpreted as a sufficiently large number.

Then T has a model $(M, +, \cdot, 0, 1, c)$. We know $M \models PA$, and $M \ncong \mathbb{N}$.

Example 4.2. $\mathcal{L} = \mathcal{L}_{ring} = \{+, \cdot, -, 0, 1\}$. The theory of finite fields is

$$FF = \{\varphi \ \mathcal{L}\text{-sentences} | \forall \text{ field } K, |K| < \infty \Rightarrow K \models \varphi\}.$$

Then FF contains the field axioms. If K is a finite field, then $K \models FF$.

Theorem 4.14. $\exists K$ a field, $|K| = \infty$, and $K \models FF$. (Such a K is a **pseudofinite field**.)

Proof. $T = FF \cup \{\varphi_n | n < \infty\}$, where $\varphi_n \equiv (\exists^{\geqslant n} x : x = x)$.

If $T_0 \subseteq_f T$, then take a sufficiently large prime p, we have $(\mathbb{Z}/\equiv_p) \models T_0$, so T has a model K. This K has to be infinite. □

$\mathbb{R} \not\models FF$. If K is a finite field, then $f(x) = (x-1)^2 x$ is never injective, so never surjective. This means $K \models \neg(\forall y \exists x : (x-1)^2 x = y)$. However, $\mathbb{R} \models (\forall y \exists x : (x-1)^2 x = y$.

Definition 4.15. T in $\mathcal{L}$-theory is

(1) **strongly complete** if for any $\mathcal{L}$-sentence φ, we have $\varphi \in T$ or $\neg\varphi \in T$.

(2) **Henkinized** if whenever $(\exists x : \varphi(x)) \in T$, then $\exists$ constant symbol c such that $\varphi(c) \in T$.

(2) is also called the witness property.

To explain the witness property, in group theory, we need an identity element. This element may not explicitly appear in the language, since we can use $\exists x \forall y : x \cdot y = y \cdot x = y$. If we desire the witness property, we have to use a constant symbol 1 for this x. Here $\varphi(x) \equiv (\forall y : x \cdot y = y \cdot x = y)$, and the constant symbol c is the identity element.

Now suppose T is finitely satisfiable, strongly complete, Henkinized. $T \models \varphi$ means for every model M of T, we have $M \models \varphi$. For example, $T_{field} \models (\forall x : x \cdot 0 = 0)$.

Lemma 4.16. If $T_0 \subseteq_f T$ and $T_0 \models \varphi$, then $\varphi \in T$.

Proof. Otherwise $\neg\varphi \in T$, and $T_0 \cup \{\neg\varphi\} \subseteq_f T$ but has no models, contradicting T finitely satisfiable. □

Lemma 4.17. (1) $(\varphi \wedge \psi) \in T \Leftrightarrow (\varphi \in T \wedge \psi \in T)$.

(2) $\neg\varphi \in T \Leftrightarrow \varphi \notin T$.

(3) $(\exists x : \varphi(x)) \in T \Leftrightarrow \exists$ closed term $t : \varphi(t) \in T$. A **closed term** is a term with no variables.

(4) t, s closed terms, $(t = s) \in T \Leftrightarrow (s = t) \in T$.

(5) $(t = t) \in T$.

(6) $(t = s) \in T, (s = r) \in T \Rightarrow (t = r) \in T$.

(7) If $(t_1 = s_1), \cdots, (t_n = s_n) \in T$, then $(f(t_1, \cdots, t_k) = f(s_1, \cdots, s_k)) \in T$ for function symbols f, $R(t_1, \cdots, t_l) \in T \Leftrightarrow R(s_1, \cdots, s_l) \in T$ for relation symbols R.

Proof. (1) $\varphi \wedge \psi \models \varphi$ and $\varphi \wedge \psi \models \psi$, so by Lemma 4.16, $\varphi \in T$ and $\psi \in T$. Conversely, $\{\varphi, \psi\} \subseteq_f T$ and $\{\varphi, \psi\} \models \varphi \wedge \psi$.

(2) $\{\varphi, \neg\varphi\}$ has no models, so T contains at most one of them. But T contains at least one because T is strongly complete.

(3) $\varphi(t) \models \exists x : \varphi(x)$, so $\varphi(t) \in T$ implies $(\exists x : \varphi(x)) \in T$ by Lemma 4.16. The reverse direction is by the definition of Henkinized.

(4),(5),(6),(7) Similarly proved from Lemma 4.16. □

Theorem 4.18. If T is finitely satisfiable, strongly complete, Henkinized, then T has a model.

Proof. Let M be the set of closed terms. We make it an $\mathcal{L}$-structure as follows. For a function symbol f, let $f^M(t_1, \cdots, t_k) = f(t_1, \cdots, t_k)$ as a closed term. For a relation symbol R, let $R^M(t_1, \cdots, t_k) \Leftrightarrow R(t_1, \cdots, t_k) \in T$. Then define $t \approx s \Leftrightarrow (t = s) \in T$. The relation $\approx$ is a congruence on M.

We claim that $(M, \approx)$ is a premodel of T, then $M/\approx$ will be a model of T.

If t is a term, then $t^M = t$.

If $\varphi(x_1, \cdots, x_k)$ is a formula, $t_1, \cdots, t_k$ are closed terms, then $M \models \varphi(t_1, \cdots, t_k)^{\approx} \Leftrightarrow \varphi(t_1, \cdots, t_k) \in T$ by induction on complexity of formulas and Lemma 4.17. For example, when $\varphi \equiv R(t_1, \cdots, t_k)$, we have $M \models R(t_1, \cdots, t_k) \Leftrightarrow R(t_1, \cdots, t_k) \in T$. When $\varphi \equiv (t = s)$, then $M \models (t \approx s) \Leftrightarrow (t = s) \in T$. When $\varphi \equiv (\exists x : \psi(x))$, then $M \models \varphi^{\approx} \Leftrightarrow \exists t \in M : M \models \varphi(t)^{\approx} \Leftrightarrow \exists t \in M : \varphi(t) \in T \Leftrightarrow \varphi \in T$. □

Now we extend a finitely satisfiable theory to some desired theory so that we can apply this Theorem.

Lemma 4.19. If T is an $\mathcal{L}$-theory, T is finitely satisfiable, φ is an $\mathcal{L}$-sentence, then $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is finitely satisfiable.

Proof. Otherwise, we have $T_0 \subseteq_f T$ with $T_0 \cup \{\varphi\}$ not satisfiable and $T_1 \subseteq_f T$ with $T_1 \cup \{\neg\varphi\}$ not satisfiable. Then $T_0 \models \neg\varphi, T_1 \models \varphi$, so $T_0 \cup T_1$ has no models. $\square$

Lemma 4.20. If $\{T_i\}_{i \in (I, \leqslant)}$ is a chain of finitely satisfiable theories, then $\bigcup\limits_{i \in I} T_i$ is finitely satisfiable.

So by Zorn's lemma, we can prove

Theorem 4.21. If T is a finitely satisfiable $\mathcal{L}$-theory, then $\exists T' \supseteq T$ an $\mathcal{L}$-theory, T' finitely satisfiable and strongly complete.

Lemma 4.22. If T is finitely satisfiable, $(\exists x : \varphi(x)) \in T$, T an $\mathcal{L}$-theory, let $\mathcal{L}' = \mathcal{L} \cup \{c\}, T' = T \cup \{\varphi(c)\}$, then T' is finitely satisfiable.

Proof. Suppose $T_0 \subseteq_f T'$, take $M \models T_0 \cup \{(\exists x : \varphi(x))\}$, so there exists $b \in M, M \models \varphi(b)$. Then if we interpret c as b, $(M, b) \models T_0 \cup \{\varphi(c)\}$. $\square$

Theorem 4.23. If T is an $\mathcal{L}$-theory, T is finitely satisfiable, then $\exists \mathcal{L}' \supseteq \mathcal{L}$ and $\mathcal{L}'$-theory $T' \supseteq T$ such that T' is finitely satisfiable, strongly complete, Henkinized.

Proof. We recursively build chains:

$$\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots$$
$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$$

such that T_i is an $\mathcal{L}_i$-theory, finitely satisfiable, and when $i > 0$, T_i is strongly complete, whenever $(\exists x : \varphi(x)) \in T_i$, there exists a constant symbol $c \in \mathcal{L}_{i+1}$ such that $\varphi(c) \in T_{i+1}$.

To do this, for step $i > 0$, let $S = \{\varphi(x) \text{ in } \mathcal{L}_{i-1} | (\exists x : \varphi(x)) \in T_{i-1}\}$, and define $\mathcal{L}_i = \mathcal{L}_{i-1} \cup \{c_\varphi | \varphi \in S\}$. We know $T_{i-1} \cup \{\varphi(c_\varphi) | \varphi \in S\}$ is finitely satisfiable, because any finite subset only involves finitely many sentences from T_{i-1} and finitely many $\varphi(c_\varphi)$'s, so this statement is true from the previous lemma. We extend $T_{i-1} \cup \{\varphi(c_\varphi) | \varphi \in S\}$ to a finitely satisfiable, strongly complete theory T_i.

Take $\mathcal{L}' = \bigcup\limits_{i<\omega} \mathcal{L}_i, T' = \bigcup\limits_{i<\omega} T_i$. Then T' is finitely satisfiable, T' is strongly complete because any $\mathcal{L}'$-sentence is an $\mathcal{L}_i$-sentence for some i, so $\varphi \in T_i \subseteq T'$

or $\neg\varphi \in T_i \subseteq T'$. T' is Henkinized because if $(\exists x : \varphi(x)) \in T'$, then $(\exists x : \varphi(x)) \in T_i$ for some $i >> 0$. Then $\exists c \in \mathcal{L}_{i+1}, \varphi(c) \in T_{i+1} \subseteq T'$. $\square$

Combine everything above, we prove the Compactness Theorem.

5 October 13th

5.1 o-minimality

This time we mainly go through the proofs in detail. Here are some revisions to the proofs in the last lecture.

Proof. Theorem 4.6

$B \subseteq M^{n+1}$ such that $B_{\overline{a}} = \text{bd}(D_{\overline{a}}), \overline{a} \in M^n$. This B can taken definable,

$$B = \{(\overline{a}, b) | b \in \text{bd}(D_{\overline{a}})\}.$$

$D = \bigsqcup D \cap (C_i \times M)$. On $D' = D \cap (C_i \times M)$, $\forall \overline{a} \in C_i$, the shape of $D'_{\overline{a}}$ is constant (the shape is the sgn in the last lecture), $|\text{bd}(D'_{\overline{a}})| = k$, say $\text{bd}(D'_{\overline{a}}) = \{f_1(\overline{a}), \cdots, f_k(\overline{a})\}$, $f_1, \cdots, f_k$ continuous. Then D' is uniquely determined with a cell decomposition.

□

Proof. Theorem 4.7

When we define g, h, we need to include $(\overline{a}, b) \in \text{Int}(D)$ in the condition for $h = 1, g = \pm 1$.

□

5.2 main class

We recall the downward Löwenheim-Skolem theorem: If M is an $\mathcal{L}$-structure, $|\mathcal{L}| = \aleph_0$, $\aleph_0 \leqslant \kappa \leqslant |M|$, then $\exists N \preceq M, |N| = \kappa$.

For a language $\mathcal{L}$, $|\mathcal{L}| = \aleph_0 +$ size of $\mathcal{L}$.

Theorem 5.1 (LS)**.** $|\mathcal{L}| = \aleph_0$. If T is an $\mathcal{L}$-theory, and $\forall n < \infty, \exists M \models T, |M| \geqslant n$, then $\forall \kappa \geqslant \aleph_0, \exists M \models T, |M| = \kappa$.

Proof. Take $|I| = \kappa, \mathcal{L}' = \mathcal{L} \cup \{c_i | i \in I\}$ with new constant symbols added. Let $T' = T \cup \{c_i \neq c_j | i, j \in I, i \neq j\}$. T' is finitely satisfiable by assumption, so T' has a model $(M, \overline{c}) \models T'$. This $M \models T, |M| \geqslant |I| = \kappa$.

By DLS, $\exists N \preceq M, |N| = \kappa, N \equiv M \models T$. □

$\mathrm{Th}(M) = \mathrm{Th}(N) \Leftrightarrow M \equiv N \Leftrightarrow M \models \mathrm{Th}(N) \Leftrightarrow \mathrm{Th}(M) \supseteq \mathrm{Th}(N)$.

To prove this, if $\mathrm{Th}(M) \supsetneq \mathrm{Th}(N)$, take $\varphi \in \mathrm{Th}(M) \setminus \mathrm{Th}(N), M \models \varphi$, but $\varphi \notin \mathrm{Th}(N)$ implies $\neg\varphi \in \mathrm{Th}(N)$, so $\neg\varphi \in \mathrm{Th}(M)$, which is a contradiction when $\mathcal{L}$ is complete (or M does exist).

Theorem 5.2 (LS)**.** If $|\mathcal{L}| = \aleph_0, M$ is an $\mathcal{L}$-structure, $|M| = \infty$, then $\forall \kappa \geqslant \aleph_0, \exists N \equiv M, |N| = \kappa$.

Proof. Apply LS to $\mathrm{Th}(M)$. □

Definition 5.3. M, N are $\mathcal{L}$-structures. An **embedding** is an injection $f : M \to N$ such that for every relatino symbol R, $M \models R(a_1, \cdots, a_n) \Leftrightarrow N \models R(f(a_1), \cdots, f(a_n))$, and if g is a function symbol, $g(f(a_1), \cdots, f(a_n)) = f(g(a_1, \cdots, a_n))$ for all $\overline{a} \in M^{<\omega}$.

An isomorphism $f : M \to N$ is exactly a bijective embedding.

If M is a substructure of N, then the inclusion map $M \hookrightarrow N$ is an embedding.

Proposition 5.4. If $f : M \to N$ is an embedding, then $\mathrm{im}(f)$ is a substructure of N and $f : M \to \mathrm{im}(f)$ is an isomorphism.

Proof. Let g be a function symbol, $a_i \in \mathrm{im}(f), a_i = f(b_i), b_i \in M$. Then $g(a_1, \cdots, a_n) = g(f(b_1), \cdots, f(b_n)) = f(g(b_1, \cdots, b_n)) \in \mathrm{im}(f)$. So $\mathrm{im}(f)$ is a substructure. □

Theorem 5.5. If M, N are $\mathcal{L}$-structures, $f : M \to N$ a function, then TFAE:

(1) f is an embedding.

(2) For every quantifier-free formula $\varphi(x_1, \cdots, x_n), M \models \varphi(a_1, \cdots, a_n) \Leftrightarrow N \models \varphi(f(a_1), \cdots, f(a_n))$.

Proof. (1) $\Rightarrow$ (2): When f is an isomorphism or the inclusion map of a substructure, (2) is true.

(2) $\Rightarrow$ (1): The quantifier-free formulas $x = y, R(x_1, \cdots, x_n), g(x_1, \cdots, x_n) = y$ are preserved. □

Definition 5.6. $f : M \to N$ is an **elementary embedding** if for every $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n), M \models \varphi(a_1, \cdots, a_n) \models N \models \varphi(f(a_1), \cdots, f(a_n)), a_1 \cdots, a_n \in M$.

Elementary embeddings are embeddings.

Isomorphisms are elementary embeddings.

If $M \subseteq N$ is a substructure, then $M \hookrightarrow N$ is an elementary embedding iff $M \preceq N$.

Theorem 5.7. If $f : M \to N$ is an elementary embedding, then $\mathrm{im}(f) \preceq N$ and $M \cong \mathrm{im}(f)$.

Proof. The inclusion map $\mathrm{im}(f) \hookrightarrow N$ is the composition of $\mathrm{im}(f) \xrightarrow{\cong} M \xrightarrow{f} N$, so it is an elementary embedding. □

Definition 5.8. If M is an $\mathcal{L}$-structure, the **diagram** of M is

$$\mathrm{diag}(M) = \{\varphi(\overline{a})|\varphi \text{ q'free formula}, \overline{a} \in M^n, M \models \varphi(\overline{a})\} = \{\psi|\psi \text{ q'free } \mathcal{L}(M)\text{-sentence}, M \models \psi\}.$$

The **elementary diagram** of M is

$$\mathrm{eldiag}(M) = \{\varphi(\overline{a})|\varphi\ \mathcal{L}\text{-formula}, \overline{a} \in M^n, M \models \varphi(\overline{a})\}.$$

$(\mathbb{R}, +, \cdot, -, 0, 1)$, the diagram includes $2 + 5 < 23, e < \pi, 2 + 2 = 4$.

The elementary diagram includes $\exists x : x \cdot x + e = \pi$.

In $\mathcal{L}_{graph} = \{E\}$, the graph M has four points a, b, c, d, the edges cd, ab, bc, ca. Then $\mathrm{diag}(M)$ includes $a \neq b, aEb, \neg aEd, \cdots$. A model of $\mathrm{diag}(M)$ is an $\mathcal{L}(M)$-structure, which can be viewed as an extension of M.

An $\mathcal{L}(M)$-structure N can be viewed as an $\mathcal{L}$-structure, and there exists a function $f : M \to N, a \mapsto a^N$. One sees that

$$\{\mathcal{L}(M)\text{-structures}\} = \{(N, f)|N\ \mathcal{L}\text{-structure}, f : M \to N\}.$$

Theorem 5.9. $(N, f) \models \mathrm{diag}(M) \Leftrightarrow f$ is an embedding.

So $\mathrm{Mod}(\mathrm{diag}(M)) = \{(N, f)|N\ \mathcal{L}\text{-structure}, f : M \to N \text{ embedding}\}$.

A model of $\mathrm{eldiag}(M)$ is basically a pair (N, f) where N is an $\mathcal{L}$-structure, $f : M \to N$ is an elementary embedding.

If $M \preceq N$, then $N \models \mathrm{eldiag}(M)$.

Theorem 5.10. If $N \models \mathrm{eldiag}(M)$, then $\exists N' \succeq M, N \cong N'$ as $\mathcal{L}(M)$-structures.

Proof. Fact. If $\mathfrak{M}$ is an $\mathcal{L}$-structure on a set M, N a set, $f : M \to N$ a bijection, then $\exists$ an $\mathcal{L}$-structure $\mathfrak{N}$ on N such that $f : \mathfrak{M} \to \mathfrak{N}$ is an isomorphism.

Fact. If $\mathfrak{M}$ is an $\mathcal{L}$-structure on M, $A \subseteq M, f : A \to B$ bijection, then $\exists$ isomorphism $g : \mathfrak{M} \to \mathfrak{N}$ such that $B \subseteq \mathfrak{N}$ and g extends f.

So given an elementary embedding $f : M \to N$, there exists an isomorphism $g : N \to N'$ such that $(g \circ f)(x) = x, \forall x \in M$. To show this, we let g extend $f^{-1} : \mathrm{im}(f) \to M$. Then $g \circ f : M \to N'$ is an elementary embedding and restricts to the identity map on M, so $M \preceq N'$. □

Theorem 5.11 (Upward LS theorem)**.** If $|\mathcal{L}| = \aleph_0, M$ is an $\mathcal{L}$-structure and $|M| = \infty$, then $\forall \kappa \geqslant |M|, \exists N \succeq M, |N| = \kappa$.

Proof. Recall that if T is an $\mathcal{L}$-theory, T has infinite models, $\kappa \geqslant |\mathcal{L}|$, then T has a model of size κ.

$M \models \mathrm{eldiag}(M)$ which is an $\mathcal{L}(M)$-theory. The size of $\mathcal{L}(M)$ is $\leqslant \kappa$, so $\exists N \models \mathrm{eldiag}(M), |N| = \kappa$. Now apply the previous theorem. □

Definition 5.12. A theory T is **κ-categorical** if $\forall M, N \models T, |M| = |N| = \kappa$, then $M \cong N$.

DLO is $\aleph_0$-categorical.

Theorem 5.13 (Łoś-Vaught criterion)**.** If T is an $\mathcal{L}$-theory, $|\mathcal{L}| = \aleph_0$, all models of T are infinite, T is κ-categorical for some $\kappa \geqslant \aleph_0$, then $M, N \models T$ implies $M \equiv N$.

Proof. Take $M' \equiv M, N \equiv N', |M'| = |N'| = \kappa$ by LS theorem, then $M', N' \models T$, so $M' \cong N'$. □

Example 5.1. (1) $\mathcal{L} = \{\approx\}$, T says: (i) $\approx$ is an equivalence relation. (ii) each $\approx$-class is infinite. (iii) there are infinitely many classes.

T is $\aleph_0$-categorical but not $\aleph_1$-categorical.

(2) $\mathcal{L}_{graph} = \{E\}$, T says: (i) at least one vertex. (ii) every vertex has degree 2. (iii) no cycles.

T is not $\aleph_0$-categorical but κ-categorical if $\kappa > \aleph_0$

(3) $\mathbb{Q}$-vector spaces is κ-categorical for any $\kappa > \aleph_0$.

$\mathbb{Q}$-vector spaces can be axiomized as abelian groups, torsion free, divisible, non-trivial.

(4) Infinite $\mathbb{Z}/2\mathbb{Z}$-vector spaces is κ-categorical for any $\kappa \geqslant \aleph_0$.

This class can be axiomized as infinite abelian groups such that $\forall x : x + x = 0$.

6 October 20th

6.1 o-minimality

Now we prove (Bound_n) in the induction.

Lemma 6.1. If $C \subseteq M^n$ a non-open cell, then $\exists N, \forall \overline{a} \in C, |D_{\overline{a}}| < N$.

Proof. Say C is k-dimensional, $k < n$. Then $\exists \pi : M^n \to M^k$ a coordinate projection such that $C' = \pi(C)$ is an open cell in M^k.

Let $D' = \{(\pi(\overline{a}), b) | \overline{a} \in C, b \in D_{\overline{a}}\} \subseteq M^k \times M$. Then $D'_{\pi(\overline{a})} = D_{\overline{a}}$ for $\overline{a} \in C$. The conclusion follows from (Bound_k). □

Let $M_\infty = M \cup \{\pm\infty\}$. This can be viewed as a subset of M^2, namely $\{(0, m) | m \in M\} \cup \{(1, 0), (2, 0)\}$, so we can talk about definable sets in M_∞ and definable functions between M_∞ and M.

Definition 6.2. Let $(\overline{a}, b) \in M^n \times M_\infty$.

$(\overline{a}, b)$ is a ***D*-good point** if $\exists$ box $B \ni \overline{a}$ and interval $I \ni b$ such that either $(B \times I) \cap D = \varnothing$ or $(B \times I) \cap D = \Gamma(f)$ where $f : B \to I$ is continuous definable and $f(\overline{a}) = b$.

$(\overline{a}, b)$ is **bad** if it is not good.

The set of good points/bad points are definable.

The set of good points is open. The set of bad points is closed.

Lemma 6.3. Suppose $C \subseteq M^n$ open cell, and if $\forall \overline{a} \in C, \exists b \in M_\infty$ such that $(\overline{a}, b)$ bad, then $\perp$.

Proof. Define $f : C \to M_\infty, f(\overline{a}) = \min\{b \in M_\infty | (\overline{a}, b) \text{ bad}\}$. For a closed, non-empty, definable set, min = inf exists in M_∞.

Let $s : C \to \{0, 1\}$ be

$$s(\overline{a}) = \begin{cases} 1, & f(\overline{a}) \in D_{\overline{a}}, \\ 0, & \text{else.} \end{cases}$$

And $g, h : C \to M \cup \{\perp\}$ are defined by

$$g(\overline{a}) = \min\{b \in D_{\overline{a}} | b > f(\overline{a})\}$$
$$h(\overline{a}) = \max\{b \in D_{\overline{a}} | b < f(\overline{a})\}$$
$$\min(\varnothing) = \max(\varnothing) = \perp.$$

By (Cell_n), (Cont_n), we break C into cells where f is continuous, s is constant, g is continuous or $\perp$, h is continuous or $\perp$. Then at least one cell C' is open by considering interior.

We consider one case: on C' $s = 1$, g is defined, h is defined. For $\overline{a} \in C', f(\overline{a}) \in D_{\overline{a}}, f, g, h$ continuous. In $D_{\overline{a}}$, we must have $\cdots < h(\overline{a}) < f(\overline{a}) < g(\overline{a}) < \cdots$ are consecutive elements. Then a figure of D shows that $(a, f(\overline{a})$ cannot be a bad point. A contradiction. $\square$

Lemma 6.4. Suppose C is an open cell in M^n, $\forall \overline{a} \in C, b \in M_\infty, (\overline{a}, b)$ is good. For $k \in \mathbb{N}$, let $C_k = \{\overline{a} \in C || D_{\overline{a}}| = k\}$, so $C = \bigsqcup_{k=0}^{\infty} C_k$. Then

(1) Each C_k is definable and open.
(2) $\exists k : C = C_k$.
(3) $\exists k, \forall \overline{a} \in C : |D_{\overline{a}}| = k$.

Proof. Clearly (2) $\Leftrightarrow$ (3).

(1)$\Rightarrow$(2): Each C_k is open, C is definably connected, so for each C_k, $C_k = \varnothing$ or $C_k = C$. (Consider χ_{C_k} continuous, $\{0, 1\}$-valued, so it is constant.)

(1) Fix k, take $\overline{a} \in C_k, |D_{\overline{a}}| = k$. Let $D_{\overline{a}} = \{b_1, \cdots, b_k\}, b_1 < \cdots < b_k$.

Each $(\overline{a}, b_i)$ is good, so $\exists$ small enough box $B \ni \overline{a}$ and continuous functions $f_1, \cdots, f_k : B \to M$ such that D looks like $\Gamma(f_i)$ near $(\overline{a}, b_i)$. Since $(\overline{a}, b_i) \in D$, we have $f_i(\overline{a}) = b_i$. By continuity, we can assume in B, we have $f_1(\overline{x}) < f_2(\overline{x}) < \cdots < f_k(\overline{x})$. If $\overline{x} \in B$, we have $|D_{\overline{x}}| \geqslant k$. We need to show there are no extra points near $\overline{a}$ with $|D_{\overline{x}}| > k$, then C_k is open.

Let $Y = \{(\overline{c}, d) | \overline{c} \in B, (\overline{c}, d) \in D, d \notin \{f_1(\overline{c}), \cdots, f_k(\overline{c})\}\}$. We claim that if $b \in M_\infty$, then $(\overline{a}, b) \notin \text{cl}(Y)$.

Since $(\overline{a}, b)$ is good, either $(\overline{a}, b) \notin \text{cl}(D)$ or $b = b_i$ for some i. This proves the claim. If $\pi : M^n \times M_\infty \to M^n$ is the projection, then $\overline{a} \notin \pi(\text{cl}(Y))$, so by definable compactness, $\pi(\text{cl}(Y))$ is closed, so there is a small box $B', \overline{a} \in B' \subseteq$

$B, B' \cap \pi(Y) = \varnothing$. If $\bar{c} \in B'$, then $Y_{\bar{c}} = \varnothing$, $D_{\bar{c}} = \{f_1(\bar{c}), \cdots, f_k(\bar{c})\}, |D_{\bar{c}}| = k$. □

Now we can prove (Bound$_n$).

Theorem 6.5. (Bound$_n$) holds if (Cell$_{\leqslant n}$),(Cont$_{\leqslant n}$), (Bound$< n$).

Proof. Let $F : M^n \to \{0, 1\}$,

$$f(\overline{a}) = \begin{cases} 1, & \exists b, (\overline{a}, b) \text{ bad}, \\ 0, & \text{else}. \end{cases}$$

By (Cell$_n$), we can decompose M^n into finitely many cells such that f constant on C_i. For each C_i, we need to bound $|D_{\overline{a}}|$.

(1) C_i non-open, by Lemma 6.1.

(2) C_i open, $f = 0$, by Lemma 6.4.

(3) C_i open, $f = 1$, impossible by Lemma 6.3. □

6.2 main class

We recall Łoś-Vaught criterion: If T a theory, κ a cardinality, $\kappa \geqslant |\mathcal{L}|$, and T has a unique model of size κ under $\cong$, then any $M_1, M_2 \models T$ satisfy $M_1 \equiv M_2$.

Definition 6.6. Let T be an $\mathcal{L}$-theory. Let $\overline{x} = (x_1, \cdots, x_n)$ be a tuple of variables. Let p be a set of $\mathcal{L}$-formulas in $\overline{x}$. If $M \models T$ and $\overline{a} \in M^n$, we say $\overline{a}$ **satisfies** p ($\overline{a} \models p$) iff for every $\varphi(\cdot)$ in p, $M \models \varphi(\overline{a})$.

p is a **partial type** over T if $\exists M \models T, \exists \overline{a} \in M^n, \overline{a} \models p$.

Example 6.1. $T = PA$ or $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1)$ (called true arithmetic).

$p(x) = \{x \neq 1, x \neq 1 + 1 + 1, x \neq 1 + 1 + 1 + 1 + 1, \cdots\}$. Then $2 \models p, 4 \models p, 3 \not\models p$. p is a partial type over PA or $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1)$.

Theorem 6.7. If p is finitely satisfiable in T, $(\forall \varphi_1 \cdots, \varphi_n \in p, \exists M \models T, \exists \overline{a} \in M^n, \overline{a} \models \{\varphi_1, \cdots, \varphi_n\})$ then p is a partial type.

Proof. Add n new constant symbols to $\mathcal{L}$, and add $\varphi(\overline{c})$ for each $\varphi(\overline{x}) \in p$. Then use compactness theorem. □

p is a partial type over T iff $T \cup \{\varphi(\overline{c}) | \varphi(\overline{x}) \in p\}$ is satisfiable.

Example 6.2. $T = PA$. $p(x) = \{x \neq 0, x \neq 1, x \neq 1+1, \cdots\}$.

It is not satisfiable in $\mathbb{N}$, but a partial type over PA because it is finitely satisfiable in $\mathbb{N}$.

Definition 6.8. Suppose M is an $\mathcal{L}$-structure and $A \subseteq M$, $p(\overline{x})$ is a set of $\mathcal{L}(A)$-formulas in $\overline{x}$. Then say p is a **partial type over** A if

(1) p is finitely satisfiable in M.

(2) p is satisfied by some $\overline{a} \in N^n$ where $N \succeq M$.

Actually (1)$\Leftrightarrow$(2).

Proof. (1)$\Rightarrow$(2): If p is finitely satisfiable in M, then p is a partial type over $\mathrm{eldiag}(M)$, so $\exists N \models \mathrm{eldiag}(M)$, p is satisfied in N. Move N by an isomorphism, we can assume $N \succeq M$.

(2)$\Rightarrow$(1): If p is satisfied in $N \succeq M$, then $\forall \varphi_1, \cdots, \varphi_n \in p, N \models (\exists x : \varphi(x) \wedge \cdots \wedge \varphi_n(x))$, so $M \models (\exists x : \varphi_1(x) \wedge \cdots \wedge \varphi_n(x))$, p is finitely satisfiable in M. $\square$

Example 6.3. $(\mathbb{N}, +, \cdot, 0, 1), p(x) = \{x \neq 0, x \neq 1, x \neq 1+1, \cdots\}$ is realized in some $M \succeq \mathbb{N}$.

$q(x) = \{\mathrm{prime}(x), x \neq 2, x \neq 3, x \neq 5, \cdots\}$. Here $\mathrm{prime}(x) = (x \neq 0 \wedge x \neq 1 \wedge \forall y, z : y \cdot z = x \to y = x \vee z = x)$. There is some $M \succeq \mathbb{N}, a \in M$ such that a is "prime" but $a \neq 2, 3, 5, 7, \cdots$.

$r(x) = \{\varphi(x), x \neq 1, x \neq 2, \cdots\}$, where $\varphi(x)$ says x is a power of 2.

$P(x) = \{x \neq 1, 2 \nmid x, 3 \nmid x, 5 \nmid x, \cdots\}$.

We can also consider $r(x) \cup q(y)$.

Definition 6.9. $T \models \varphi$ (T **semantically implies** φ) if $\forall M \models T, M \models \varphi$.

$T \vdash \varphi$ (T **syntactically implies** φ) if T proves φ.

Definition 6.10. If φ, ψ are sentences, $\varphi \vdash \psi$ is generated by a list, including:

$a \vdash a$, $a \vdash b, b \vdash c \Rightarrow a \vdash c$.

$a \vdash b, a \vdash c \Rightarrow a \vdash b \wedge c$.

$\varphi(t) \vdash \exists x : \varphi(x)$.

$a \wedge \neg a \vdash b$.

And so on.

Then $T \vdash \varphi$ means $\exists \psi_1, \cdots, \psi_n \in T$ such that $\psi_1 \wedge \cdots \wedge \psi_n \vdash \varphi$.

Theorem 6.11 (Completeness)**.** $T \models \varphi \Leftrightarrow T \vdash \varphi$.

Proof. (sketch) $\Leftarrow$: If $T \vdash \varphi, M \models T$ then check $\models$ is closed under the rules taht generate $\vdash$, and we get $M \models \varphi$.

$\Rightarrow$: If $T \nvdash \perp$ (T is consistent), then T has a model. This statement is proved as the proof of compactness theorem, with "finitely satisfiable" replaced by "consistent".

Suppose $T \models \varphi$, then $\forall M \models T, M \models \varphi$. $T \cup \{\neg\varphi\}$ has no models, so $T \cup \{\neg\varphi\} \vdash \perp$ by the first paragraph. By proof by contradiction, we have $T \vdash \varphi$. (If $a \wedge \neg b \vdash \perp$, then $a \vdash b$.) □

Theorem 6.12. If T is a theory, TFAE:

(1) $\exists\varphi : T \vdash \varphi$ and $T \vdash \neg\varphi$.

(2) T has no models.

(3) $T \vdash \perp$.

(4) $\forall\varphi : T \vdash \varphi$.

Proof. It is clear that $(4) \Rightarrow (1), (4) \Rightarrow (3), (1) \Rightarrow (2), (3) \Rightarrow (2)$.

That $(2) \Rightarrow (4)$ is followed by vacuous truth. □

Definition 6.13. T is **inconsistent** if (1)-(4) hold. Otherwise, T is **consistent**.

Definition 6.14. A consistent theory T is **complete** if

(1) $\forall\varphi$, $T \vdash \varphi$ or $T \vdash \neg\varphi$.

(2) $\forall M, N \models T : M \equiv N$.

Actually, (1)$\Leftrightarrow$(2).

Proof. $\neg(1) \Leftrightarrow \exists\varphi, T \nvdash \varphi, T \nvdash \neg\varphi \Leftrightarrow \exists\varphi : \exists M \models T, M \models \neg\varphi, \exists N \models T, N \models \varphi \Leftrightarrow \exists M, N \models T : M \not\equiv N \Leftrightarrow \neg(2)$. □

T is complete means for all $\mathcal{L}$-sentences φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Theorem 6.15 (Łoś-Vaught test)**.** If T is κ-categorical (and $\kappa \geqslant |\mathcal{L}|$), consistent, and all $M \models T$ implies $|M| = \infty$, then T is complete.

Example 6.4. (1) DLO is complete.

(2) $(M, \approx)$: $\approx$ is an equivalence relation, all equivalence classes are infinite, infinitely many classes. $\aleph_0$-categorical, so it is complete.

(3) Atomless Boolean algebras $\aleph_0$-categorical.

(4) Algebraically closed fields of char 0. $\aleph_1$-categorical.

(5) RCF (real closed fields) is complete, not κ-categorical for any κ.

Theorem 6.16. If T is complete, $M \models T$, then $\forall \varphi, M \models \varphi \Leftrightarrow T \vdash \varphi$.

Proof. $T \vdash \varphi \Rightarrow M \models \varphi$.

$T \nvdash \varphi \Leftrightarrow T \vdash \neg\varphi \Rightarrow M \models \neg\varphi$. □

Theorem 6.17. If T is complete, T is finite/computably enumerable, $M \models T$, then $\mathrm{Th}(M)$ is decidable/computable. (i.e. $\exists$ an algorithm to decide whether $M \models \varphi$ for every φ)

$\mathrm{Th}(\mathbb{R}, \leqslant)$ is decidable (DLO).

Definition 6.18. Let M, N be $\mathcal{L}$-structures. A **partial isomorphism** between M and N is an isomorphism f where $\mathrm{dom}(f) \subseteq M, \mathrm{im}(f) \subseteq N$ are substructures.

Definition 6.19. A **back-and-forth system** between M and N is a family $\mathcal{F}$ of partial isomorphisms between M and N such that $\mathcal{F} \neq \varnothing$ and

(1) (forth) If $f \in \mathcal{F}, a \in M$, then $\exists g \in \mathcal{F}, g \supseteq f, \mathrm{dom}(g) \ni a$.

(2) (back) If $f \in \mathcal{F}, b \in N$, then $\exists g \in \mathcal{F}, g \supseteq f, \mathrm{im}(g) \ni b$.

Example 6.5. If $M, N \models DLO, \mathcal{F} = \{f | f \text{ a finite partial isomorphism}\}$, then $\mathcal{F}$ is a B&FS.

Theorem 6.20. If M, N countable and $\mathcal{F}$ is a B&FS between M and N, then $M \cong N$. Moreover, for any $f \in \mathcal{F}$, $\exists g : M \to N$ isomorphism and $g \supseteq f$.

Theorem 6.21. Let T be a theory in a countable language. Suppose $\forall M, N \models T$, there is a B&FS between M and N, then T is $\aleph_0$-categorical and complete.

Example 6.6. (1) DLO.

(2) Random graphs. $T_{rand} = T_{graph} \cup \{\varphi_{n,m} | n, m \in \mathbb{N}\}$.

$(V, E) \models \varphi_{n,m}$ iff for any $A \subseteq V, B \subseteq V$ with $|A| = n, |B| = m, A \cap B = \varnothing$, there exists $v \in V \setminus (A \cup B)$ such that $\forall x \in A, xEv$ and $\forall y \in B, yEv$.

If $M \models T_{rand}$, then $|M| = \infty$.

Definition 6.22. $P_N(\varphi)$ is the probability that φ holds in a random graph with N vertices for $\mathcal{L}_{graph}$-sentences. Each edge has probability 1/2 to appear.

Theorem 6.23. T_{rand} is complete, and for any sentence φ,

$$\lim_{N \to \infty} P_N(\varphi) = \begin{cases} 1, & T_{rand} \vdash \varphi, \\ 0, & T_{rand} \nvdash \varphi. \end{cases}$$

Proof. (sketch) Check the expression for $\varphi_{n,m}$ and sentences in T_{graph}.

If $T_{rand} \vdash \varphi$, then $\lim_{N \to \infty} P_N(\varphi) = 1$. This shows that T is consistent.

If $M, N \models T_{rand}$, $\mathcal{F} = \{f | f \text{ a finite partial isomorphism}\}$, then $\mathcal{F}$ is a B&FS. This shows T_{rand} complete, and proves the expression. □

7 October 27th

7.1 o-minimality

Theorem 7.1. If M is o-minimal, $M \equiv N$, then N is o-minimal.

Proof. Recall that for every formula $\varphi(x, y_1, \cdots, y_n)$, we have defined

$$\varphi(M, \bar{b}) = \{a \in M | M \models \varphi(a, \bar{b})\}, \quad \bar{b} \in M^n.$$

For every $\bar{b} \in M^n$, we have $|\mathrm{bd}(\varphi(M, \bar{b}))| < \infty$. By (Bound$_n$), these cardinalities are uniformly bounded by some $k < \infty$. We also note that this can be written in first-order language. So by $M \equiv N$, we have

$$\forall \bar{b} \in N^n, |\mathrm{bd}(\varphi(N, \bar{b}))| < k.$$

For $D \subseteq M$ definable, $\mathrm{bd}(D) = \{a_1, \cdots, a_m\}, a_1 < \cdots < a_m$, the set D is a finite union of points and intervals. This can be characterized as follows:

$$x, y \in M, x < y, [x, y] \cap \mathrm{bd}(D) = \varnothing \Rightarrow x \in D \text{ iff } y \in D.$$

This is also first-order, so for all $\bar{b} \in N^n, x, y \in N$ with $x < y, [x, y] \cap \mathrm{bd}(\varphi(N, \bar{b})) = \varnothing$, we have $x \in \varphi(N, \bar{b}) \Leftrightarrow y \in \varphi(N, \bar{b})$.

These two results together imply $\varphi(N, \bar{b})$ is a finite union of points and intervals. Since φ, b are arbitrary, N is o-minimal. □

Corollary 7.2. If M is o-minimal, then $\mathrm{Th}(M)$ is an o-minimal theory. (Every $N \models \mathrm{Th}(M)$ is o-minimal.)

Let $(M, \leqslant, \cdots)$ be o-minimal.

Definition 7.3. $A \subseteq M^n$ definable. A is **definably connected** iff for any definable continuous $f : A \to \{0, 1\}$, f is constant iff for any definable relatively clopen $B \subseteq A$, we have $B = \varnothing$ or $B = A$.

$(\mathbb{Q}, \leqslant) \equiv (\mathbb{R}, \leqslant)$ because DLO is a complete theory. $(\mathbb{Q}, \leqslant)$ is o-minimal, so it is definably connected. (Warning. The endpoints of an interval must be in $M \cup \{\pm\infty\}$, so $(\sqrt{2}, \infty) \cap \mathbb{Q}$ is not an interval, not definable, but an open subset of $(\mathbb{Q}, \leqslant)$.)

In M, intervals are exactly definably connected sets.

If $C \subseteq M^n$ is a cell, then C is definably connected.

In $(\mathbb{R}, \leqslant)$, cells are connected.

If $f : X \to Y$ definable and continuous, X definably connected, then $\mathrm{im}(f)$ is definably connected.

Corollary 7.4. If $D \subseteq M^n$ definable, and for all $a, b \in D$, there exists a definable continuous function $f : [0,1] \to D$ with $f(0) = a, f(1) = b$, then D is definably connected.

Theorem 7.5. If $D \subseteq M^n$ definable,

(1) $D = \bigsqcup_{i=1}^{m} X_i$ uniquely, X_i definably connected, $X_i \neq \varnothing$.

(2) If $(M, \leqslant) = (\mathbb{R}, \leqslant)$, then these X_i are the connected components.

Proof. $D = \bigsqcup_{i=1}^{l} C_i$ cell decomposition. So there are $\leqslant 2^l$ definable, continuous $f : D \to \{0, 1\}$. Define $x \sim y$ iff $f(x) = f(y), \forall f$. The partition of equivalence classes gives X_i. $\square$

Another explanation: $V = \{C_1, \cdots, C_l\}$, let $E(C_i, C_j)$ iff $C_i \cap \mathrm{cl}(C_j) \neq \varnothing$ or $C_j \cap \mathrm{cl}(C_i) \neq \varnothing$. The connected components of (V, E) give rise to X_i.

Definition 7.6. $D \subseteq M^n$ is **definably compact** if D is definable, bounded and closed.

If A, B definably compact, then $A \cup B, A \times B$ definably compact.

Fact. If $I = [a, b]$, $D \subseteq M^n \times I, \pi : M^n \times I \to M^n$ the projection, then D closed implies $\pi(D)$ closed. Similarly for I^m, since we can compose $M^n \times I \times I \to M^n \times I \to M^n$.

Theorem 7.7. If $A \subseteq M^n$ definably compact, $f : A \to M^m$ definable and continuous, then $\mathrm{im}(f)$ is definably compact.

Proof. $\Gamma = \{(f(\overline{a}), \overline{a}) | \overline{a} \in A\}$. Γ is closed in $M^m \times A$ because f continuous and the order topology of a DLO is Hausdorff. So Γ is closed in $M^m \times M^n$ due to A closed.

For large I, we have $\Gamma \subseteq M^m \times I^n$, so by the fact, $\pi(\Gamma)$ is closed.

The boundedness part will use the definable finite intersection property. See Prof. Johnson's paper, or Lemma 4.1 on September 29th. □

Corollary 7.8. If $D \subseteq M^n$ is definably compact, $D \neq \varnothing$, $f : D \to M$ definable and continuous, then $\max_{a \in D} f(a)$ exists.

Proof. $\operatorname{im}(f) \subseteq M$ closed, bounded, non-empty, definable. □

7.2 main class

Definition 7.9. $\mathcal{L}$ is a **functional** language if it has no relation symbols.

For example, $\mathcal{L}_{grp}, \mathcal{L}_{ring}$.

Now assume $\mathcal{L}$ functional.

Definition 7.10. An **$\mathcal{L}$-algebra** is an $\mathcal{L}$-structure.

Definition 7.11. An **equation** is an $\mathcal{L}$-structure of the form

$$\forall x_1, \cdots, \forall x_n : t(x_1, \cdots, x_n) = s(x_1, \cdots, x_n),$$

where s, t are terms.

Definition 7.12. An $\mathcal{L}$-theory T is **equational** if T is a set of equations and $\mathcal{L}$ is functional.

Definition 7.13. A class $\mathcal{K}$ of $\mathcal{L}$-structures is an **equational class** if $\operatorname{Mod}(T) = \mathcal{K}$ for some equational $\mathcal{L}$-theory T.

Suppose T is an equational theory.

Theorem 7.14. If $M \models T, N \subseteq M$ a subalgebra(substructure), then $N \models T$.

Proof. Say $\varphi \in T$ is of the form $\forall \overline{x} : t(\overline{x}) = s(\overline{x})$.

Suppose $\overline{a} \in N$, then $M \models (t(\overline{a}) = s(\overline{a}))$, which is quantifier-free, so $N \models (t(\overline{a}) = s(\overline{a}))$. Since $\overline{a}$ is arbitrary, $N \models \varphi$. □

Theorem 7.15. If $M \models T, \approx$ a congruence on M, then $(M/\approx) \models T$.

Proof. Say $\varphi \in T$ is of the form $\forall \overline{x} : t(\overline{x}) = s(\overline{x})$. Then $M \models (\forall \overline{x} : t(\overline{x}) = s(\overline{x}))$, $(M, \approx) \models (\forall \overline{x} : t(\overline{x}) \approx s(\overline{x}))$, so $(M/\approx) \models \varphi$. $\square$

Definition 7.16. Let M, N be $\mathcal{L}$-algebras. We define the **product** $M \times N$ by setting for any k-ary $f \in \mathcal{L}$

$$f^{M\times N}((x_1, y_1), \cdots, (x_k, y_k)) = (f^M(x_1, \cdots, x_k), f^N(y_1, \cdots, y_k)).$$

Theorem 7.17. If T is equational, $M, N \models T$, then $M \times N \models T$.

Proof. Induct on complexity of terms to show that

$$t^{M\times N}((x_1, y_1), \cdots, (x_k, y_k)) = (t^M(x_1, \cdots, x_k), t^N(y_1, \cdots, y_k)).$$

$\square$

Corollary 7.18. The class of fields is not equational.

"Equational" depends on language: If $\mathcal{L}_{grp} = \{\cdot\}$, then not all group axioms are equations. But in universal algebra and model theory, we use $\{\cdot, 1, ^{-1}\}$ to make groups an equational class.

A binary relation R on an $\mathcal{L}$-algebra is a congruence iff R is an equivalence relation on A and R is a subalgebra of $A \times A$.

Let R be a ring.

Theorem 7.19. (1) If $I \subseteq R$ an ideal, let $x \equiv_I y \Leftrightarrow x \equiv y \mod I \Leftrightarrow x - y \in I$, then $\equiv_I$ is a congruence on R.

(2) $I \mapsto \equiv_I$ is a bijection from ideals to congruences.

Proof. For (2), let $I = \{x \in R | x \approx 0\}$. $\square$

Let G be a group.

Theorem 7.20. (1) For any normal subgroup $N \subseteq G$, let $x \equiv_N y \Leftrightarrow x \equiv y \mod N \Leftrightarrow xy^{-1} \in N$, then $\equiv_N$ is a congruence on G.

(2) $N \mapsto \equiv_N$ is a bijection from normal subgroups to congruences.

Definition 7.21. A, B are $\mathcal{L}$-algebras. $f : A \to B$ is a **homomorphism** if $\forall g \in \mathcal{L}$,

$$g^B(f(a_1), \cdots, f(a_k)) = f(g^A(a_1, \cdots, a_k)).$$

For $\mathcal{L}$ functional, $f : A \to B$ isomorphism iff f a homomorphism and bijection. $f : A \to B$ embedding iff f a homomorphism and injection.

Theorem 7.22. If $f : A \to B$ a homomorphism, then $\mathrm{im}(f)$ is a subalgebra of B.

Definition 7.23. If $f : A \to B$ a homomorphism, then its **kernel** $\ker(f)$ is an equivalence relation

$$x \approx y \Leftrightarrow f(x) = f(y).$$

Theorem 7.24. $\ker(f)$ is a congruence on A.

If A is an $\mathcal{L}$-algebra, $\approx$ a congruence on A, $A \to A/\approx, x \mapsto [x]_\approx$ is a homomorphism with kernel $\approx$.

So $R \subseteq A^2$ is a congruence iff R is $\ker(f)$ for some homomorphism $f : A \to B$.

Similarly $S \subseteq A$ is a subalgebra iff S is $\mathrm{im}(f)$ for some homomorphism $f : B \to A$. (Take the inclusion map $S \hookrightarrow A$.)

If A, B are $\mathcal{L}$-algebras, then $\pi_1 : A \times B \to A, \pi_2 : A \times B \to B$ are homomorphisms.

$F \subseteq A \times B$ is a homomorphism iff F is a function and F is a subalgebra of $A \times B$.

Theorem 7.25. If $f : A \to B$ is a homomorphism, then $A/\ker(f) \cong \mathrm{im}(f)$.

Let $\beta : \mathbb{N}^3 \to \mathbb{N}$ be given by

$$\beta(a, b, x) = a \mod ((x+1)b + 1),$$

where $a \mod b$ is the unique $0 \leqslant n < b$ such that $a \equiv n \mod b$. This β is definable.

Lemma 7.26. If $y_0, \cdots, y_{n-1} \in \mathbb{N}$, there exists $a, b \in \mathbb{N}$ such that $\beta(a, b, 0) = y_0, \cdots, \beta(a, b, n-1) = y_{n-1}$.

Proof. Take $b = m!, m >> 1$. Then $b + 1, 2b + 1, \cdots, nb + 1$ are pairwise coprime. Now use Chinese Remainder Theorem. $\square$

Example 7.1. $f(0) = 1, f(1) = 1, f(n+2) = f(n+1)+f(n)$ is the Fibonacci sequence.

Theorem 7.27. *f is definable in $(\mathbb{N}, +, \cdot)$.*

Proof. $f(n) = m$ is equivalent to $\exists a, b, n \in \mathbb{N}$ such that $\beta(a, b, 0) = 1, \beta(a, b, 1) = 1, \beta(a, b, i + 2) = \beta(a, b, i + 1) + \beta(a, b, i)$ for $i < n$ and $\beta(a, b, n) = m$. $\square$

This method works for any recursively defined function.

In $(\mathbb{N}, +, \cdot)$, every computable function is definable. (Computable means there is an algorithm to compute the function in finite steps.)

PA is incomplete. Otherwise, $\mathrm{Th}(\mathbb{N}, +, \cdot)$ is decidable, and Halting problem is decidable. A contradiction.

Also, list all $\mathcal{L}$-formulas $\varphi(x)$ as $\psi_0, \psi_1, \cdots$, let $\varphi(n) = (PA \nvdash \psi_n(n))$. If PA complete, then $\varphi(n) \Leftrightarrow \mathbb{N} \not\models \psi_n(n)$, but $\varphi = \psi_k$ for some k, $\psi_k(k) \Leftrightarrow \varphi(k) \Leftrightarrow \neg\psi_k(k)$, a contradiction.

8 November 3rd

8.1 o-minimality

M is an o-minimal DLO.

Definition 8.1. If $X \subseteq_{def} M$, the **dimension** of X is

$$\dim(X) = \begin{cases} -\infty, & X = \varnothing, \\ 0, & 0 < |X| < \infty, \\ 1, & |X| = \infty. \end{cases}$$

If $X \subseteq_{def} M^{n+1}$, for $\overline{a} \in M^n, X_{\overline{a}} = \{b \in M | (\overline{a}, b) \in X\} \subseteq M$, and $S_k = \{\overline{a} \in M^n | \dim(X_{\overline{a}}) = k\}$ for $k \in \{-\infty, 0, 1\}$. One can show S_k is definable, and we define the **dimension** of X to be $\dim(X) = \max\limits_{k \in \{-\infty,0,1\}} k + \dim(S_k)$.

If $C \subseteq M^n$ is a k-dimensional cell, then $\dim(C) = k$.

Fact. (1) X, Y definable sets, $\dim(X \times Y) = \dim(X) + \dim(Y)$.

(2) $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$ if $X, Y \subseteq M^n$.

(3) If $X \subseteq Y$, then $\dim(X) \leqslant \dim(Y)$.

(4) $\dim(X) > 0$ iff $|X| = \infty$. $\dim(X) \geqslant 0$ iff $X \neq \varnothing$.

(5) If $X \subseteq M^n \times M^m$, then $S_k = \{\overline{a} \in M^n | \dim(X_{\overline{a}}) = k\} \subseteq M^n$ is definable.

(6) If $X = \bigsqcup\limits_{i=1}^{m} C_i$ is a cell decomposition, and C_i is a k_i-dimensional cell, then $\dim(X) = \max\limits_{1 \leqslant i \leqslant m} k_i$.

Theorem 8.2. If $X \subseteq_{def} M^n$, then $\dim(X) \leqslant n = \dim(M^n)$. Moreover, $\dim(X) = n$ iff $\operatorname{int}(X) \neq \varnothing$, $\dim(X) < n$ iff X is nowhere dense.

Lemma 8.3. If $X \subseteq_{def} M^2, X^T = \{(y, x) | (x, y) \in X\}$, then $\dim(X) = \dim(X^T)$.

Proof. $\dim(X) = -\infty$ iff $X = \varnothing$.

$\dim(X) = 0$ iff $0 < |X| < \infty$.

$\dim(X) = 2$ iff $\operatorname{int}(X) \neq \varnothing$.

$\dim(X) = 1$ iff X is the remaining case.

These conditions do not depend on the order of coordinates. $\square$

Fact. Let $f : X \to Y$ be definable.

(1) f is a bijection implies $\dim(X) = \dim(Y)$.

(2) f is injective implies $\dim(X) \leqslant \dim(Y)$.

(3) f is surjective implies $\dim(X) \geqslant \dim(Y)$.

(4) $k \in \mathbb{N}, f$ surjective, $\forall b \in Y, \dim(f^{-1}(b)) = k$, then $\dim(X) = k + \dim(Y)$.

Example 8.1. If $f : G \to H$ is a definable group homomorphism, then $\dim(G) = \dim(\mathrm{im}(f)) + \dim(\ker(f))$.

Theorem 8.4. If $X \subseteq_{def} M^n, \mathrm{bd}(X) = \mathrm{cl}(X) \setminus \mathrm{int}(X), \partial X = \mathrm{cl}(X) \setminus X$, then

(1) $\dim(\mathrm{bd}(X)) < n$.

(2) For $0 \leqslant k \leqslant n$, $\dim(X) \geqslant k$ iff $\exists \pi : M^n \to M^k$ such that $\mathrm{int}(\pi(X)) \neq \varnothing$.

(3) If $X \neq \varnothing$, then $\dim(\partial X) < \dim(X)$.

(4) $\dim(\mathrm{cl}(X)) = \dim(X)$.

Proof. (1) $\mathrm{int}(\partial X) = \varnothing$, so $\dim(\partial X) < n$. Let $Y = M^n \setminus X, \dim(\partial Y) < n$, $\mathrm{bd}(X) = \partial X \cup \partial Y$, so $\dim(\mathrm{bd}(X)) < n$.

(2) If $\pi : M^n \to M^k, \mathrm{int}(\pi(X)) \neq \varnothing$, then due to surjectivity, $\dim(X) \geqslant \dim(\pi(X)) = k$.

Conversely, if $\dim(X) \geqslant k$, take a cell $C \subseteq X$ in the cell decomposition with $\dim(C) \geqslant k$. Then there exists a coordinate projection $\pi : M^n \to M^k$ such that $\mathrm{int}(\pi(C)) \neq \varnothing$.

(3) Suppose $k = \dim(\partial X) \geqslant \dim(X)$. Take $\pi : M^n \to M^k, \mathrm{int}(\pi(\partial X)) \neq \varnothing$. Assume $\pi(x_1, \cdots, x_n) = (x_1, \cdots, x_k)$. $X = \bigsqcup_{i=1}^{m} C_i$ cells, $\partial X \subseteq \bigsqcup_{i=1}^{m} \partial C_i$, there exists $C = C_i, \dim(\pi(\partial C)) = k, \mathrm{int}(\pi(\partial C)) \neq \varnothing$.

Claim: If $C \subseteq M^n$ a cell, $\pi(\partial C)$ has non-empty interior, then $\dim(C) > k$.

The proof is by induction?

Then a contradiction.

(4) $\mathrm{cl}(X) = \partial X \cup X$. $\square$

Theorem 8.5. If $f : X \to Y$ definable, $X \neq \varnothing, S = \{a \in X | f \text{ discontinuous at } a\}$, then $S \subsetneq X, \dim(S) < \dim(X)$.

Proof. $X = \bigsqcup_{i=1}^{m} C_i$ and $f|_{C_i}$ continuous. Then $S \subseteq \bigcup_{i=1}^{m} \partial C_i$ (if $p \in X \setminus \bigcup \partial C_i$, then $p \in C_j, p \notin \mathrm{cl}(C_i), \forall i \neq j$). Then the conclusion follows from the previous theorem. $\square$

8.2 main class

There is the midterm exam today, so no main class.

9 November 10th

9.1 o-minimality

Fact.(1) If $f : X \to Y$ definable, then

$$\dim\{a \in X | f \text{ not continuous at } a\} < \dim X$$

or $X = \varnothing$.

(2) If $f : X \to Y$ definable surjection, and $\exists k \in \mathbb{N}, \forall b \in Y, \dim(f^{-1}(b)) = k$, then $\dim X = k + \dim Y$.

Let $(G, \star)$ be a definable group, $G \subseteq M^n$. Let

$$B = \{(x, y) \in G^2 | \star \text{ is not continuous at } (x, y)\}$$

be the set of "bad" points. Then $\dim B \leqslant \dim G^2 - 1 = 2\dim G - 1$. Let $k = \dim G$.

Let

$$E = \{x \in G | \dim\{y \in G | (x, y) \in B\} = k\}$$

be the set of "evil" points. E is definable. **Claim** that $\dim E < k$.

In fact, let $f : \{(x, y) \in B | x \in E\} \to E, f(x, y) = x$. Then $\forall x \in E, \dim f^{-1}(x) = k$, so

$$k + \dim E = \dim\{(x, y) \in G^2 | (x, y) \in B, x \in E\} \leqslant 2k - 1.$$

This proves the claim.

Let $G \setminus E = \bigsqcup\limits_{i=1}^{l} C_i$ be the cell decomposition. **Claim** that if $N \lhd G$ definable normal subgroup, $|G/N| < \infty$, then $|G/N| \leqslant l$.

Let $m = |G/N|$ and suppose $m > l$. We know $G = \bigsqcup\limits_{i=1}^{m} a_i N$, and

$$\dim N = \dim a_i N = \dim G = k.$$

Subclaim: the map $f : G \setminus E \to G/N, x \mapsto xN$ is surjective. Each coset has dimension strictly greater than E, so each coset intersects $G \setminus E$. This proves the subclaim.

So f cannot be constant on each C_i, otherwise $|\mathrm{im}(f)| \leqslant l < m$. Say $C = C_i$ and f is not constant on C. Since C is definably connected, f is not continuous at some $a \in C \subseteq G \setminus E$. By definition of E, $\dim\{y \in G | (a, y) \in B\} < k$.

Take $b \in G$ such that $(a, b) \notin B$ and $a \star b \notin \bigcup_{i=1}^{m} \partial(a_i N)$, which is possible because these finitely many sets all have dimension $< k$. But $a \star b \in a_i N$ for some i and $a \star b \notin \mathrm{cl}(a_j N)$ for $j \neq i$. So $a_i N$ is a nbhd of $a \star b$. Since $\star$ is continuous at $a \star b$, the set $\{x \in G | x \star b \in a_i N\}$ is a nbhd of a. This set is the coset $a_i N b^{-1}$, so f is constant on a nbhd of a, but not continuous at a, which is a contradiction. So $m = |G/N| \leqslant l$.

Fact. If $H \leqslant G$ definable, $|G/H| < \infty$, then $\exists N \lhd G$, N definable, $N \subseteq H$, $|G/N| < \infty$.

Let G act on G/H by $g(a_i H) = (g a_i) H$, this gives a group homomorphism $G \to \mathrm{Perm}(G/H)$, and the kernel N satisfies the condition.

Claim. If $N \leqslant G$ definable, $|G/N| < \infty$, then $|G/N| \leqslant l$.

Let $\mathcal{F} = \{H \leqslant G | H \text{ definable}, |G/H| < \infty\}$. **Claim** that $\mathcal{F}$ has at least one minimal element.

If $G > H_0 > H_1 > \cdots$, then $|G/H_0| < |G/H_1| < \cdots \leqslant l$. The chain must stop somewhere at the minimal element.

Theorem 9.1. There exists $H \leqslant G$ such that $H = \min \mathcal{F}$.

Proof. Take H minimal. If $X \in \mathcal{F}$, then $X \cap H \in \mathcal{F}$. □

Definition 9.2. G^0 is the smallest definable subgroup of finite index in G.

We must have $G^0 \lhd G$ and $f(G^0) = G^0$ for any definable automorphism $f : G \to G$.

$G = O(n)$ in $(\mathbb{R}, +, \cdot, \leqslant)$, then $G^0 = SO(n)$.

Definition 9.3. $D \subseteq M^n$ definable, $D = \bigsqcup_{i=1}^{m} C_i$ the cell decomposition. Then $\chi(D) = \sum_{i=1}^{m} (-1)^{\dim C_i}$ is called the **Euler characteristic** of D.

Fact. (1) This is well-defined. (The cell decomposition is not unique)

(2) If $f : X \to Y$ bijection, then $\chi(X) = \chi(Y)$.

(3) $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.
(4) $\chi(A \times B) = \chi(A) \cdot \chi(B)$.
(5) $f : A \to B$, and if $\forall b \in B, \chi(f^{-1}(B)) = k$, then $\chi(A) = k\chi(B)$.
(6) If $|A| < \infty$, then $\chi(A) = |A|$.

9.2 main class

Definition 9.4. A formula $\varphi(x_1, \cdots, x_n)$ is **quantifier-free** if it has no quantifiers.

Definition 9.5. If T is a theory, φ, ψ are **equivalent modulo** T if $T \vdash (\forall \overline{x} : \varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$, i.e. $\forall M \models T, \varphi(M) = \psi(M)$.

We denote this by $T \vdash \varphi \leftrightarrow \psi$, though $\varphi \leftrightarrow \psi$ does not mean a formula.

Example 9.1. In $\mathcal{L}_{ring}$, $\varphi(x) \equiv (x \neq 0), \psi(x) \equiv (\exists y : xy = 1)$. Then $T_{field} \vdash \varphi \leftrightarrow \psi$, but $T_{ring} \not\vdash \varphi \leftrightarrow \psi$.

Definition 9.6. An $\mathcal{L}$-theory T has **quantifier elimination (QE)** if for every $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, there is a qfree $\mathcal{L}$-formula $\psi(x_1, \cdots, x_n)$ such that $T \vdash \varphi \leftrightarrow \psi$.

An $\mathcal{L}$-structure M has QE if $\mathrm{Th}(M)$ does.

M has QE iff for any $\mathcal{L}$-formula φ, there is a qfree formula ψ such that $\varphi(M) = \psi(M)$.

Fact. DLO has QE. $\varphi(x, y) \equiv (\forall z : z \leqslant x \to z \leqslant y)$ is equivalent to $x \leqslant y$. Similarly $\exists z : x < z \wedge z < y$ is equivalent to $x < y$.

Fact. $(\mathbb{Z}, \leqslant)$ does not have QE. $\exists z : x < z < y$ is not equivalent to a qfree formula.

Lemma 9.7. If $f : M \to N$ is an embedding, then f preserves qfree formulas.

Corollary 9.8. If T has QE, $M, N \models T$, $f : M \to N$ is an embedding, then f is an elementary embedding.

Proof. If $\varphi(x_1, \cdots, x_n)$ is a formula, take a qfree $\psi(x_1, \cdots, x_n)$ such that $T \vdash \varphi \leftrightarrow \psi$. Then for $\overline{a} \in M^n$,

$$M \models \varphi(\overline{a}) \Leftrightarrow M \models \psi(\overline{a}) \Leftrightarrow N \models \psi(f(\overline{a})) \Leftrightarrow N \models \varphi(f(\overline{a})).$$

□

Corollary 9.9. If $M, N \models T$, T has QE, M a substructure of N, then M is an elementary substructure.

Example 9.2. The interval $(0,1) \subseteq \mathbb{R}$ is DLO, so $(0,1) \preceq \mathbb{R}$ in $\mathcal{L}_{\leqslant}$.

Theorem 9.10. If T has QE, $M \models T$, $A \subseteq T$ a substructure, then $A \preceq M \Leftrightarrow A \models T$.

Theorem 9.11. If M has QE, then the boolean algebra

$$\{\varphi(M) | \varphi(x_1, \cdots, x_n) \text{ an } \mathcal{L}(M)\text{-formula}\}$$

is generated by $\{\varphi(M) | \varphi \text{ atomic}\}$.

Atomic formulas are like $R(t_1(\overline{x}), \cdots, t_n(\overline{x}))$ and $t(\overline{x}) = s(\overline{x})$.

Fact. $(\mathbb{C}, +, \cdot, 0, 1, -)$ has QE. If $t(x)$ is a term, then $t^{\mathbb{C}}(x)$ is a polynomial with coefficients in $\mathbb{Z}$. If $t(x)$ is a $\mathcal{L}_{ring}(\mathbb{C})$-term, then $t^{\mathbb{C}}(x)$ is a polynomial in $\mathbb{C}[x]$.

If $\varphi(x)$ is an atomic $\mathcal{L}_{ring}(\mathbb{C})$-formula, then $\varphi(x)$ is $P(x) = Q(x)$ where $P, Q \in \mathbb{C}[x]$, so $\varphi(\mathbb{C})$ is finite or $\mathbb{C}$.

Theorem 9.12. $D \subseteq \mathbb{C}$ is definable in $(\mathbb{C}, +, \cdot, -, 0, 1)$ iff D is finite or cofinite.

Fact. $D \subseteq \mathbb{C}^n$ is definable iff D is constructible in algebraic geometry sense.

Fact. DLO has QE, and atomic formulas are $x = a, x < a, a < b, a = b$. So DLO is o-minimal.

Definition 9.13. If M is a structure and $A \subseteq M$ subset, then $\langle A \rangle$ (or $\langle A \rangle_M$) means

$$\min\{B \subseteq M | B \text{ is a substructure of } M, B \supseteq A\}.$$

Concretely, $\langle A \rangle = \{t^M(\overline{a}) | t(x_1, \cdots, x_n) \text{ an } \mathcal{L}\text{-term}, a_1, \cdots, a_n \in A\}$ is the **structure generated by** A.

Example 9.3. (1) $\mathcal{L}_{grp}$. In $(\mathbb{R}, +, -, 0)$, $\langle 2 \rangle$ is the set of even numbers, $\langle 2, 3 \rangle = \mathbb{Z}, \langle 1, \pi \rangle = \mathbb{Z} + \mathbb{Z}\pi, \langle \varnothing \rangle = \{0\}$.

(2) $\mathcal{L}_{ring}$. In $(\mathbb{R}, +, \cdot, -, 0, 1)$, $\langle 1 \rangle = \mathbb{Z}, \langle \varnothing \rangle = \mathbb{Z}, \langle 1, \pi \rangle = \mathbb{Z}[\pi]$.

(3) $\mathcal{L}_{\leqslant}$. In $(\mathbb{R}, \leqslant)$, $\langle A \rangle = A$.

Theorem 9.14. If T has QE, $M, N \models T$, then $M \equiv N$ iff $\langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$.

Proof. Suppose $A = \langle \varnothing \rangle_M, B = \langle \varnothing \rangle_N, A \cong B$.

Let φ be an $\mathcal{L}$-sentence, $\exists$ qfree $\mathcal{L}$-sentence $\psi, T \vdash \varphi \leftrightarrow \psi$. Then

$$M \models \varphi \Leftrightarrow M \models \psi \Leftrightarrow A \models \psi \Leftrightarrow B \models \psi \Leftrightarrow N \models \psi \Leftrightarrow N \models \varphi.$$

So $M \equiv N$.

Conversely, if $M \equiv N$, then $\langle \varnothing \rangle_M = \{t^M | t \text{ a closed term}\}$, and we just map t^M to t^N. The rest proof proceeds like

$$t^M = s^M \Leftrightarrow M \models t = s \Leftrightarrow N \models t = s \Leftrightarrow t^N = s^N.$$

□

Example 9.4. (1) DLO. $M, N \models$ DLO, then $\langle \varnothing \rangle_M = \varnothing = \langle \varnothing \rangle_N$, so DLO is complete.

This criterion is in fact more useful than categoricity.

(2) RCF (real closed field), $(K, +, \cdot, -, 0, 1, \leqslant)$. The theory includes ordered field and if $P(x) \in K[x], P(0) < 0 < P(1)$, then $\exists c \in [0, 1] : P(c) = 0$.

RCF has QE. $\mathbb{R} \models$ RCF.

If K is an ordered field, $\langle \varnothing \rangle_K \cong (\mathbb{Z}, +, \cdot, -, 0, 1, \leqslant)$, so RCF is complete.

(3) $(\mathbb{Z}, \leqslant)$ does not have QE. $f : (\mathbb{Z}, \leqslant) \to (\mathbb{Z}, \leqslant), x \mapsto 2x$ is an embedding but not an elementary embedding. So $\mathrm{Th}(\mathbb{Z})$ is complete but not QE.

Let the relation symbol $R_n(x, y)$ means $|x - y| = n$, which is $\varnothing$-definable in $(\mathbb{Z}, \leqslant)$. Then $(\mathbb{Z}, \leqslant, R_1, R_2, \cdots)$ has same definable sets as $(\mathbb{Z}, \leqslant)$ and has QE.

This is the method of expanding non-QE to QE. See 9.5 and 11.3 in version 3.5 notes.

M a structure, $\overline{a} = (a_1, \cdots, a_n) \in M^n$.

Definition 9.15. The **type** of $\overline{a}$ is

$$\mathrm{tp}(\overline{a}) = \{\varphi(x_1, \cdots, x_n) | \varphi \text{ an } \mathcal{L}\text{-formula}, M \models \varphi(\overline{a})\}.$$

$\overline{a}$ satisfies a partial type p iff $\mathrm{tp}(\overline{a}) \supseteq p$.

For $\overline{a} \in M^n, \overline{b} \in N^n$, TFAE:

(1) $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$.

(2) $\forall \varphi$ $\mathcal{L}$-formula, $M \models \varphi(\overline{a}) \Leftrightarrow N \models \varphi(\overline{b})$.

(3) $\overline{a}$ satisfies $\mathrm{tp}(\overline{b})$.

(4) $\mathrm{tp}(\overline{a}) \supseteq \mathrm{tp}(\overline{b})$.

Definition 9.16. The **quantifier-free type** of $\overline{a}$ is

$$\mathrm{qftp}(\overline{a}) = \{\varphi \in \mathrm{tp}(\overline{a}) | \varphi \text{ qfree}\}.$$

Theorem 9.17. If $\overline{a} \in M^n, \overline{b} \in N^n$, then $\mathrm{qftp}(\overline{a}) = \mathrm{qftp}(\overline{b})$ iff $\exists$ an isomorphism $f : \langle \overline{a} \rangle_M \to \langle \overline{b} \rangle_N$ such that $f(\overline{a}) = \overline{b}$.

Here $\langle \overline{a} \rangle_M = \langle a_1, \cdots, a_n \rangle_M$ and $f(a_i) = b_i$ for each i.

Theorem 9.18. If T has QE, $M, N \models T, \overline{a} \in M^n, \overline{b} \in N^n$, then $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$ iff $\mathrm{qftp}(\overline{a}) = \mathrm{qftp}(\overline{b})$.

Example 9.5. $(\mathbb{R}, \leqslant)$. DLO has QE. $\mathrm{tp}(2,3) = \mathrm{tp}(1,18)$ because $\mathrm{qftp}(2,3) = \mathrm{qftp}(1,18)$ because $(\{2,3\}, \leqslant) \cong (\{1,18\}, \leqslant)$ by a map such that $2 \mapsto 1, 3 \mapsto 18$.

Definition 9.19. If $\overline{a} \in M^n, B \subseteq M$, then the **type over** B is

$$\mathrm{tp}(\overline{a}/B) = \{\varphi \text{ an } \mathcal{L}(B)\text{-formula} | M \models \varphi(\overline{a})\}.$$

Similarly,

$$\mathrm{qftp}(\overline{a}/B) = \{\varphi \in \mathrm{tp}(\overline{a}/B) | \varphi \text{ qfree}\}.$$

If $\overline{a}, \overline{b} \in M^n, C \subseteq M$, then $\mathrm{tp}(\overline{a}/C) = \mathrm{tp}(\overline{b}/C)$ iff $\forall$ C-definable D, $\overline{a} \in D$ iff $\overline{b} \in D$.

Fact. $\mathrm{qftp}(\overline{a}/C) = \mathrm{qftp}(\overline{b}/C)$ iff $\exists$ an isomorphism $f : \langle \overline{a}, C \rangle \to \langle \overline{b}, C \rangle, f(\overline{a}) = \overline{b}, f \supseteq \mathrm{id}_C$.

Example 9.6. In $(\mathbb{R}, \leqslant)$, $\mathrm{tp}(\pi/\mathbb{Z}) = \mathrm{tp}(3.5/\mathbb{Z})$.

Theorem 9.20. Let T be an $\mathcal{L}$-theory. TFAE:

(1) T has QE.

(2) If $M, N \models T$, A a common $\mathcal{L}$-substructure of M and N, A finitely generated, $b \in M$, then $\exists N' \succeq N, c \in N'$ and c realizes $\mathrm{qftp}(b/A)$, i.e. $\mathrm{qftp}(c/A) = \mathrm{qftp}(b/A)$.

(3) With the same configuration as (2), $\exists N' \succeq N$ and a partial isomorphism $f : M \dashrightarrow N', \mathrm{dom}(f) \ni b, f \supseteq \mathrm{id}_A$.

Proof. (2) implies (3): If $c \models \mathrm{qftp}(b/A), \mathrm{qftp}(c/A) = \mathrm{qftp}(b/A)$, we get $f : \langle A, b\rangle_M \to \langle A, c\rangle_{N'}$ extending id_A.

(3) implies (2): Let $c = f(b)$. $f|_{\langle A,b\rangle}$ is an isomorphism $\langle A, b\rangle \to \langle A, c\rangle$, so $\mathrm{qftp}(b/A) = \mathrm{qftp}(c/A)$.

The rest of the proof is left till the next week. □

Example 9.7. (1) DLO has QE. $M, N \models \mathrm{DLO}$, and $A = (\{a_1, \cdots, a_n\}, \leqslant)$. Given $b \in M$, then $\mathrm{qftp}(b/A)$ is like $a_1 < a_2 < \cdots < a_i < b < a_{i+1} < \cdots < a_n$. So we only need to take $c \in N$ such that $a_i < c < a_{i+1}$. Then $\mathrm{qftp}^M(b/A) = \mathrm{qftp}^N(c/A)$, so DLO has QE.

(2) T_{rand} the theory of random graphs has QE.

(3) $(M, \approx)$, $\approx$ equivalence relation, and there is exactly one class of size n. This theory does not have QE since we can find two models, one being a substructure but not an elementary substructure of another.

$(M, \approx, P_1, P_2, \cdots)$ adding P_n is the unique class of size n. Then for any $M, N \models T, A \subseteq M, N$ and $b \in M$,

Case 1: $b \in A$, take $N' = N, c = b$.

Case 2: $b \in P_n, b \notin A$, take $c \in P_n^N \setminus A$.

Case 3: $b \notin \bigcup P_n, b \notin A, b \approx a \in A$, take $c \in N', c \approx a, c \notin A$.

Then this theory has QE.

10 November 17th

10.1 o-minimality

Suppose $D \subseteq M^n$ definable, M o-minimal, then the cell decomposition $D = \bigsqcup_{i=1}^{m} C_i$, and $\chi(D) = \sum_{i=1}^{m} (-1)^{\dim(C_i)}$.

For the open interval $(0,1)$, the Euler characteristic is -1. For the closed interval $[0,1]$, the Euler characteristic is 1. For most compact cases, the definable Euler characteristic coincides with the algebraic topological one.

Fact. If $D \subseteq M^n \times M^m$, $\overline{a} \in M^n, D_{\overline{a}} = \{\overline{b} \in M^m | (\overline{a}, \overline{b}) \in D\}$, then the map $M^n \to \mathbb{Z}, \overline{a} \mapsto \chi(D_{\overline{a}})$ is "definable", i.e. $M^n = \bigsqcup_{i=1}^{l} E_i$, E_i definable, $\chi(D_{\overline{a}})$ constant on E_i, i.e. $S_k = \{\overline{a} \in M^n | \chi(D_{\overline{a}}) = k\}$ are definable and almost all are empty.

Theorem 10.1. If $D \subseteq_{def} M^n, a \notin D$, then there is no definable bijection $D \cup \{a\} \to D$.

Proof. Note that $\chi(D \cup \{a\}) > \chi(D)$. □

Theorem 10.2. If $f : G \to H$ a definable group homomorphism, then $\chi(G) = \chi(\mathrm{im}(f))\chi(\ker(f))$.

Theorem 10.3. If G is a definable group, p a prime number, then $p | \chi(G)$ iff $\{g \in G | g^p = 1\} \supsetneq \{1\}$.

Proof.

Lemma 10.4. If D a definable set, $\approx$ a definable equivalence relation, every $\approx$=class has size k for a fixed $k \in \mathbb{N}_{>0}$, then $\chi(D) \equiv 0 \mod k$.

To prove this lemma, take $\leqslant$ a definable linear order on D, which can be the lexicographic order on M^n like $(x, y) < (x', y')$ iff $x < x'$ or $x = x', y < y'$.

Let $E = \{\min[a]_{\approx} | a \in D\}$. E is definable, $f : D \to E, f(a) = \min[a]_{\approx}$, then fibres of f are $\approx$-classes, so $\chi(D) = k\chi(E)$.

Back to the theorem. If $g^p = 1, g \neq 1$, then partition G into cosets of $\{1, g, \cdots, g^{p-1}\}$, so $p | \chi(D)$.

Conversely, suppose $\{g \in G | g^p = 1\} = \{1\}$. Let

$$D = \{(a_1, \cdots, a_p) | a_1 a_2 \cdots a_p = 1\},$$

and

$$G^{p-1} \to D, (a_1, \cdots, a_{p-1}) \mapsto (a_1, \cdots, a_{p-1}, a_{p-1}^{-1} \cdots a_1^{-1}).$$

Then $\chi(D) = \chi(G)^{p-1}$. If $(a_1, \cdots, a_p) \in D$, then any cyclic permutation of $(a_1, \cdots, a_p)$ is in D. If $(a_1, \cdots, a_p) \neq (1, \cdots, 1)$ in D, then it is different from any of its cyclic permutation.

Let $D' = D \setminus \{(1, \cdots, 1)\}$. Then $\chi(D') = \chi(D) - 1 = \chi(G)^{p-1} - 1$, and $\chi(D')$ is a multiple of p. So $\chi(G)$ is not a multiple of p. $\square$

Now $(M, \cdot, +, \leqslant, \cdots)$ is an o-minimal ordered field.

Fact. If $f : M \to M, a \in M$, then the one-sided limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to a-} f(x)$ exists in $M \cup \{\pm\infty\}$.

So

$$f^+(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}, f^-(x) = \lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$$

exists in $M \cup \{\pm\infty\}$.

Remark 10.1. If f is increasing on I an open interval, $a \in I$, then $f^+(a) \geqslant 0, f^-(a) \geqslant 0$.

If $f^+(a), g^+(a)$ exist, then $(f + g)^+(a) = f^+(a) + g^+(a)$ if well-defined.

Lemma 10.5. f continuous, $f : I \to M$ definable, then $\{a \in I | f^+(a) = -\infty\}$ is finite.

Proof. Otherwise there is an open interval $J \subseteq \{a \in I | f^+(a) = -\infty\}$. Replace f by $f|_J$, WLOG assume $\forall a \in I, f^+(a) = -\infty$.

Let $g(x) = cx + d, g'(x) = c, h(x) = f(x) + g(x)$, then $h^+(x) = f^+(x) + g^+(x) = -\infty$. Suppose $0, 1 \in I$, by choosing c, d, we can make $h(0) < h(1)$. Since h is continuous, there exists an interval where h is increasing, a contradiction. $\square$

So $\{a \in I | f^+(a) \text{ or } f^-(a) \in \{\pm\infty\}\}$ is finite.

Lemma 10.6. If f is continuous, and $\forall a \in I, f^{\pm}(a) \in M$, then $\{a \in I | f^+(a) \neq f^-(a)\}$ is finite.

Proof. If not, we may assume $\forall a \in I, f^+(a) < f^-(a)$, since one of $\{a \in I | f^+(a) < f^-(a)\}, \{a \in I | f^+(a) > f^-(a)\}$ is infinite. Let $g(x) = cx^2 + dx + e, h = f + g$. Then $h^+(a) < h^-(a)$. We can make $h(0) > h(1) < h(2)$, h has a local minimum at $c \in (0, 2)$, then $h^-(a) \leqslant 0 \leqslant h^+(a)$, a contradiction. $\square$

Repeat this process, we prove that f is almost C^k for every k.

10.2 main class

Fact. If M is an $\mathcal{L}$-structure, $\mathcal{F}$ a family of subsets of M^n, suppose $\mathcal{F}$ is boolean, M has QE, $\varphi(x_1, \cdots, x_n)$ atomic $\mathcal{L}(M)$-formula implies $\varphi(M) \in \mathcal{F}$, then every definable $D \subseteq M^n$ belongs to $\mathcal{F}$.

Theorem 10.7. If M, N are $\mathcal{L}$-structures, $\overline{a} \in M^n, \overline{b} \in N^n$, then TFAE:

(1) $\mathrm{qftp}^M(\overline{a}) = \mathrm{qftp}^N(\overline{b})$.

(2) For any qfree $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, $M \models \varphi(\overline{a})$ iff $N \models \varphi(\overline{b})$.

(3) For any atomic $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, $M \models \varphi(\overline{a})$ iff $N \models \varphi(\overline{b})$.

(4) For any $\mathcal{L}$-terms $t_1(\overline{x}), \cdots, t_k(\overline{x})$ and k-ary relation symbol R,

$$R(t_1(\overline{a}), \cdots, t_k(\overline{a})) \Leftrightarrow R(t_1(\overline{b}), \cdots, t_k(\overline{b})).$$
$$t_1(\overline{a}) = t_2(\overline{a}) \Leftrightarrow t_1(\overline{b}) = t_2(\overline{b}).$$

(5) There exists a well-defined isomorphism

$$\langle a_1, \cdots, a_n \rangle_M \to \langle b_1, \cdots, b_n \rangle_N, \quad t(\overline{a}) \mapsto t(\overline{b}).$$

(6) There is a partial isomorphism $f : M \dashrightarrow N, f(\overline{a}) = \overline{b}$.

$(\mathbb{N}, s)$. We have $\mathrm{qftp}(0) = \mathrm{qftp}(1), \mathrm{tp}(0) \neq \mathrm{tp}(1)$ because $\langle 0 \rangle \cong \langle 1 \rangle$, and one can find a formula to distinguish 0 and 1. So $(\mathbb{N}, s)$ does not have QE.

Theorem 10.8. If $A \subseteq M \cap N, b \in M^n, c \in N^n$, then $\mathrm{qftp}(b/A) = \mathrm{qftp}(c/A)$ iff there exists an isomorphism $\langle A, \overline{b} \rangle_M \to \langle A, \overline{c} \rangle_N$ such that $f(\overline{b}) = \overline{c}, f(a) = a, \forall a \in A$.

Proof. Let $M_A = M$ be the $\mathcal{L}(A)$-structure. Apply the previous theorem to M_A, and note that $\langle \bar{b} \rangle_{M_A} = \langle \bar{b}, A \rangle_M$. $\square$

Theorem 10.9. If T has QE, $M, N \models T$, TFAE:

(1) $M \equiv N$.
(2) $\mathrm{tp}^M() = \mathrm{tp}^N()$.
(3) $\mathrm{qftp}^M() = \mathrm{qftp}^N()$.
(4) $\langle \varnothing \rangle_M \cong \langle \varnothing \rangle_N$.
(5) There is a partial isomorphism $f : M \dashrightarrow N$.

Here $\mathrm{tp}^M(\overline{a}) = \{\varphi(\overline{x}) | M \models \varphi(\overline{a})\}$, so $\mathrm{tp}^M() = \{\varphi | M \models \varphi\} = \mathrm{Th}(M)$.

Theorem 10.10. *T has QE iff whenever $M, N \models T$ extending an $\mathcal{L}$-structure $A = \langle a_1, \cdots, a_n \rangle_A, b \in M$, there is an elementary extension $N' \succeq N, c \in N'$ such that $\mathrm{qftp}^{N'}(c/A) = \mathrm{qftp}^M(b/A)$.*

Example 10.1. (1) $(M, \leqslant), \mathcal{L}_{\leqslant} = \{\leqslant\}$. A finitely generated structure is finite, so $A = \{a_1, \cdots, a_n\}, a_1 < \cdots < a_n, b \in M$.

For DLO, we only need to take $N' = N$.

Case 1. $a_i < b < a_{i+1}$. Take $a_i < c < a_{i+1}$, then the map $A \cup \{b\} \to A \cup \{c\}, a_j \mapsto a_j, b \mapsto c$ is an isomorphism.

Case 2. $b < a_1$, take $c < a_1$.

Case 3. $b > a_n$, take $c > a_n$.

Case 4. $b = a_i$, take $c = a_i$.

Case 5. $A = \varnothing$, take any $c \in N$.

So DLO has QE.

(2) $(M, P), \mathcal{L}_P = \{P\}$ and P is a unary relation, so we can view $P \subseteq M$. T_P says P and $M \setminus P$ are both infinite.

Case 1. $b \in A$. Take $c = b \in A \subseteq N$, then $\mathrm{qftp}^M(b/A) = \mathrm{qftp}^A(c/A) = \mathrm{qftp}^N(c/A)$ because extensions preserve qfree formulas.

Case 2. $b \notin A, P(b)$. Take $c \in N, P(c), c \notin A$. Then $\langle A, b \rangle_M = A \cup \{b\} \cong A \cup \{c\} = \langle A, c \rangle_N$ by $a \mapsto a, b \mapsto c$.

Case 3. $b \notin A, \neg P(b)$. Take $c \in N, c \notin A, \neg P(c)$.

So T_P has QE.

(3) $T_\approx$: $\approx$ an equivalence relation, $|[a]_\approx| = \infty, \forall a \in M$, and $|M/\approx| = \infty$. $\mathcal{L}_\approx = \{\approx\}$. A finite, $b \in M$.

Case 1. $b \in A$. Take $c = b$.

Case 2. $b \notin A, b \approx a \in A$ in M. Take $c \in N, c \notin A, c \approx a$ in N. Then $A \cup \{b\} \to A \cup \{c\}, a \mapsto a, b \mapsto c$ is an isomorphism.

Case 3. $b \notin A, b \not\approx a, \forall a \in A$. Take $c \in N, c \notin A, c \not\approx a, \forall a \in A$.

So $T_\approx$ has QE.

(4) T_{rand} the theory of random graphs. (V, E). The theory says that for any finite $A, B \subseteq V$ with $A \cap B = \varnothing$, there exists $v \in V$ such that $xEv, \forall x \in A$ and $\neg yEv, \forall y \in B$.

Assume $b \notin A$. Let $A^+ = \{x \in A | xEb\}, A^- = \{x \in A | \neg xEb\}$. Take $c \in N$ such that $xEc, \forall x \in A^+$ and $\neg yEc, \forall y \in A^-$. Then $A \cup \{b\} \to A \cup \{c\}, a \mapsto a, b \mapsto c$ is an isomorphism.

So T_{rand} has QE.

(5) $\mathcal{L} = \{s\}$ a unary function symbol. T says $M \neq \varnothing, s^n(x) \neq x, \forall n > 0$ and s is a bijection. T has QE.

Suppose A finitely generated. This time A can be infinite. Assume $b \notin A$.

Case 1. $s^n(b) \notin A, \forall n$. Then $\langle A, b\rangle_M$ looks like $A \bigsqcup \{b, s(b), s^2(b), \cdots\}$. Let $p(x) = \{s^n(x) \neq a | a \in A, n \in \mathbb{N}\}$, then $p(x)$ is finitely satisfiable in N. Take $c \in N', N' \succeq N$ and c realizes $p(x)$. Then $\langle A, c\rangle_{N'} \cong \langle A, b\rangle_M$ mapping c to b and extending id_A.

Case 2. $s^n(b) = a$ for some $a \in A$. Take n minimal, so $b, s(b), \cdots, s^{n-1}(n) \notin A$. Take $c \in N$ such that $s^n(c) = a$. Then $\langle A, b\rangle_M \cong \langle A, c\rangle_N$ in the way we need.

Suppose T satisfies the criterion. Fix some $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n, y)$ qfree, let $\psi(x_1, \cdots, x_n) \equiv \exists y : \varphi(x_1, \cdots, x_n, y)$.

Lemma 10.11. Suppose $M, N \models T, \overline{a} \in M^n, \overline{b} \in N^n, \mathrm{qftp}(\overline{a}) = \mathrm{qftp}(\overline{b})$, then $M \models \psi(\overline{a})$ iff $N \models \psi(\overline{b})$.

Proof. $\langle \overline{a} \rangle_M \cong \langle \overline{b} \rangle_N$. Move by an isomorphism, we can assume $\overline{a} = \overline{b}, \langle \overline{a} \rangle_M = \langle \overline{b} \rangle_N = A$, $A \subseteq M \cap N$.

$M \models \psi(\overline{a})$, so $\exists b \in M, M \models \varphi(\overline{a}, b)$. Take $N' \succeq N, c \in N'$ and $\mathrm{qftp}(c/A) = \mathrm{qftp}(b/A)$ by the criterion. Then

$$M \models \varphi(\overline{a}, b) \Rightarrow N' \models \varphi(\overline{a}, c) \Rightarrow N' \models \psi(\overline{a}) \Rightarrow N \models \psi(\overline{a}).$$

The last implication is from elementary extension. □

11 November 24th

11.1 o-minimality

Assume $(M, +, \cdot, \leqslant, \cdots)$ o-minimal.

Goal: Choose $\gamma(D) \in D$ for definable $\varnothing \neq D \subseteq M^n$ in a "canonical" way.

$\gamma(\{a\}) = a$.

$\gamma((a,b)) = (a+b)/2$. $\gamma((a,+\infty)) = a+1, \gamma((-\infty,a)) = a-1, \gamma((-\infty,\infty)) = 0$.

If $D \subseteq_{def} M, D \neq \varnothing$, then $\mathrm{bd}(D) = \{a_1, \cdots, a_n\}, a_1 < \cdots < a_n$. So

$$M = (-\infty, a_1) \cup \{a_1\} \cup (a_1, a_2) \cup \cdots \cup \{a_n\} \cup (a_n, \infty),$$

written as

$$M = S_1 \cup S_2 \cup \cdots \cup S_{2n+1}.$$

D is a finite union of some of these sets. Take the first S_i contained in D and let $\gamma(D) = \gamma(S_i)$.

If $D \subseteq_{def} M^n \times M, D \neq \varnothing$, let $\pi : M^n \times M \to M^n, \pi(\overline{a}, b) = \overline{a}$. Then $\pi(D) \subseteq M^n$, let $\overline{a} = \gamma(\pi(D)), D_{\overline{a}} = \{b \in M | (\overline{a}, b) \in D\} \subseteq M$ is non-empty. Let $b = \gamma(D_{\overline{a}})$ and define $\gamma(D) = (\overline{a}, b)$.

Example 11.1. The annulus. $D = \{(x,y) \in \mathbb{R}^2 | 2 < \sqrt{x^2+y^2} < 5\}$. $\pi(D) = (-5,5), \gamma(\pi(D)) = 0, D_0 = (-5,-2) \cup (2,5), \gamma(D_0) = -3.5$, so $\gamma(D) = (0, -3.5)$.

For the closed annulus $\overline{D}$, we have $\gamma(\overline{D}) = (-5, 0)$.

Fact. If $D \subseteq_{def} M^n \times M^m, \pi : M^n \times M^m \to M^n$, then $\pi(D) \to M^n, \overline{a} \mapsto \gamma(D_{\overline{a}})$ is definable.

Theorem 11.1 (Definable choice of representatives)**.** If D definable, $\approx$ a definable equivalence relation on D, then $\exists S \subseteq_{def} D$ such that $\forall a \in D, |[a]_{\approx} \cap S| = 1$.

Proof. Let $E = \{(a,b) \in D^2 | a \approx b\}$. Then $E_a = [a]_{\approx}$. The map $a \mapsto \gamma(E_a)$ is definable. Let $S = \{\gamma(E_a) | a \in D\}$. $\square$

Example 11.2. If G is a definable group, $N \triangleleft G$ definable normal subgroup, then there is a definable surjective group homomorphism $f : G \to H$ with $\ker(f) = N$. So H "is" G/N.

Theorem 11.2 (Curve selection)**.** If $D \subseteq_{def} M^n, \overline{a} \in M^n$, then TFAE:

(1) $\overline{a} \in \mathrm{cl}(D)$.

(2) There is a definable continuous $f : [0,1) \to D$ such that $\lim_{x\to 1} f(x) = a$.

Proof. Only need to prove (1) $\Rightarrow$ (2). $\forall \varepsilon > 0, B_\varepsilon(\overline{a}) \cap D \neq \varnothing$, so choose $g(\varepsilon) = \gamma(B_\varepsilon(\overline{a}) \cap D)$. Then $g(\varepsilon) \in D, d(g(\varepsilon), \overline{a}) < \varepsilon$. So $\lim_{\varepsilon\to 0} g(\varepsilon) = \overline{a}$.

By monotonicity theorem, $\exists a$ such that g is continuous on $(0, a]$. Let $f(x) = g((1-x)a) : [0,1) \to D$. □

Recall that $D \subseteq_{def} M^n$ is definable compact if it is closed and bounded.

Theorem 11.3. $D \subseteq M^n$ is definably compact iff for any definable continuous $f : (0,1) \to D, \lim_{x\to 1} f(x) = c \in D$.

Proof. $\Rightarrow$: It can be shown that the limit exists. Then the limit is clearly in D.

$\Leftarrow$: If $\overline{a} \in \mathrm{cl}(D) \setminus D$, use the previous theorem. So D is closed.

If D is unbounded, let $f(x) = \gamma(D \setminus B_x(0))$. By monotonicity theorem, f is continuous on (a, ∞). Take an order-preserving homeomorphism $\theta : (0,1) \to (a, \infty)$ and consider $f \circ \theta : (0,1) \to D$. We cannot have $\lim_{x\to 1} f(\theta(x))$ exists and in D. □

Fix $D \subseteq_{def} M^n$. Define $a \underset{D}{\sim} b$ if there is a definable continuous $f : [0,1] \to D$ such that $f(0) = a, f(1) = b$. Then $\underset{D}{\sim}$ is an equivalence relation.

Fact. If $C \subseteq M^n$ is a k-dim cell, then there is a definable homeomorphism $C \to (0,1)^k$.

Example 11.3. $(f, g)_{(a,b)}$. The map can be $(x, y) \mapsto (\frac{x-a}{b-a}, \frac{y-f(x)}{g(x)-f(x)})$.

For $(f, \infty)_{(a,b)}$, we need an order-preserving definable homeomorphism $\theta : (0,1) \to (0, \infty)$.

Corollary 11.4. If $C \subseteq M^n$ a cell, $a, b \in C$, then $a \underset{C}{\sim} b$.

Lemma 11.5. If $D \subseteq_{def} M^n$, then $\underset{D}{\sim}$ has only finitely many equivalence classes and the classes are definable.

Proof. $D = \bigsqcup_{i=1}^{k} C_i$ the cell decomposition. Each class is a finite union of some cells. □

Theorem 11.6. D is definably connected iff $\forall a, b \in D, a \underset{D}{\sim} b$.

Proof. $\Leftarrow$: Any $\{0,1\}$-valued definable continuous function has to be constant.

$\Rightarrow$: Suppose D is definably connected but $a \underset{D}{\not\sim} b$. Split D into $X \bigsqcup Y, X = \{x \in D | x \underset{D}{\sim} a\}, a \in X, b \in Y$. Define

$$g : D \to \{0,1\}, g(x) = \begin{cases} 0, & x \in X, \\ 1, & x \in Y. \end{cases}$$

Then g cannot be continuous. Say g is not continuous at $c \in D$. Then $c \in X \cap \text{cl}(Y)$ or $Y \cap \text{cl}(X)$. Assume the former, by curve selection, there is a continuous $f : (0,1) \to Y, \lim_{x \to 1} f(x) = c$. This is a contradiction to $\forall x \in X, \forall y \in Y, x \underset{D}{\not\sim} y$. □

11.2 main class

Recall $\text{qftp}(b/A) = \text{qftp}(c/A)$ iff there is an isomorphism $f : \langle A, b \rangle \to \langle A, c \rangle$ such that $f(b) = c, f(a) = a, \forall a \in A$.

Theorem 11.7. T has QE iff:

$(\star)$: If $M, N \models T, A = \langle a_1, \cdots, a_n \rangle \subseteq_{sub} M, N$ and $b \in M$, then $\exists c \in N' \succeq N, \text{qftp}(b/A) = \text{qftp}(c/A)$.

Lemma 11.8. QE implies $(\star)$.

Proof. If $\varphi(\overline{a}, y) \in \text{qftp}(b/A)$, then $M \models \exists y : \varphi(\overline{a}, y)$ because b satisfies this.

This is a formula for $\overline{x}$, $\psi(\overline{x}) = \exists y : \varphi(\overline{x}, y)$. $M \models \psi(\overline{a})$.

Note that $\text{qftp}^M(\overline{a}) = \text{qftp}^N(\overline{a}) = \text{qftp}^A(\overline{a})$,so $\text{tp}^M(\overline{a}) = \text{tp}^N(\overline{a})$ by QE. So $N \models \psi(\overline{a})$. This means $\text{qftp}(b/A)$ is finitely satisfiable in N. □

Now assume $(\star)$. Fix qfree $\varphi(\overline{x}, y)$, let $\psi(\overline{x}) = \exists y : \varphi(\overline{x}, y)$. Recall that last week we proved

Lemma 11.9. If $M, N \models T, \overline{a} \in M^n, \overline{b} \in N^n, \text{qftp}(\overline{a}) = \text{qftp}(\overline{b})$, then $M \models \psi(\overline{a})$ iff $N \models \psi(\overline{b})$.

Lemma 11.10. If $M \models T, M \models \psi(\overline{a})$, then there is a qfree $\theta(\overline{x})$ such that $M \models \theta(\overline{a})$ and $T \vdash \theta \to \psi$.

Proof. Let $p(\overline{x}) = \text{qftp}(\overline{a}) \cup \{\neg\psi(\overline{x})\}$.

If p is fintiely satisfiable in T, then there is $\overline{b} \in N^n, N \models T, \overline{b} \models p$, so $N \not\models \psi(\overline{b})$ and $\text{qftp}(\overline{b}) = \text{qftp}(\overline{a})$, a contradiction to the previous lemma.

If p is not finitely satisfiable in T, then there is $\theta \in \text{qftp}(\overline{a}), \{\theta, \neg\psi\}$ not satisfiable in T, so $T \vdash \theta \to \psi$. □

Lemma 11.11. There is qfree $\theta(\overline{x})$ such that $T \vdash \psi \leftrightarrow \theta$.

Proof. Let $p(\overline{x}) = \{\psi(\overline{x})\} \cup \{\neg\theta |\text{ qfree } \theta, T \vdash \theta \to \psi\}$.

p is not finitely satisfiable in T by the previous lemma. Suppose $\theta_1, \cdots, \theta_n$ qfree and $T \vdash \theta_i \to \psi$, then $\{\psi, \neg\theta_1, \cdots, \neg\theta_n\}$ not satisfiable in T. These two statements imply $T \vdash \bigvee_{i=1}^{b} \theta_i \to \psi$ and $T \vdash \psi \to \bigvee_{i=1}^{n} \theta_i$ respectively. □

So we have actually proved

Lemma 11.12. Assuming $(\star)$, if $\varphi(\overline{x}, y)$ is qfree, then there is a qfree $\theta(\overline{x}), T \vdash \theta(\overline{x}) \leftrightarrow \exists y : \varphi(\overline{x}, y)$.

Theorem 11.13. $(\star)$ implies QE.

Proof. Induct on complexity of a formula. □

Now we come to the theory of algebraically closed fields ACF.

Let K be a field. The theory ACF is adding that every polynomial equation $a_n x^n + \cdots + a_1 x + a_0 = 0$ with $a_n \neq 0, n > 0$ has a solution to the theory of fields. It is well known that $\mathbb{C} \models ACF$.

Fact. If $a, b \in \mathbb{Z}, b \neq 0$, then there is $r \in \mathbb{Z}, a \equiv r \mod b$ and $|r| < |b|$. There is $q \in \mathbb{Z}$ such that $a = q \cdot b + r$.

Lemma 11.14. If $B(x) \in K[x] \setminus \{0\}, A(x) \in K[x]$, then there is $R(x) \in K[x], \deg R < \deg B, A \equiv R \mod B$ and $\exists Q : A = Q \cdot B + R$.

Recall R a ring, $I \lhd R$ an ideal. I is **principal** if $\exists a \in R, I = aR$.

Theorem 11.15. If $I \lhd \mathbb{Z}$, then $I = n\mathbb{Z}$.

Theorem 11.16. If $I \lhd K[x]$, then $I = P \cdot K[x]$, P is monic or 0.

$L \supseteq K$ fields, $a \in L$, a is **algebraic** over K if $P(a) = 0$ for some $P \in K[x] \setminus \{0\}$. a is **transcendental** over K otherwise.

Fact. π, e are transcendental over $\mathbb{Q}$.

Definition 11.17. $I_{a/K} = \{P(x) \in K[x] | P(a) = 0\} \lhd K[x] = \ker(K[x] \to L, P \mapsto P(a))$.

So a is transcendental iff $I_{a/K} = \{0\}$. a is algebraic iff $I_{a/K} = P \cdot K[x], P \neq 0$. This P is the minimal polynomial of a over K, denoted by $\operatorname{minpoly}(a/K)$. Usually we require P monic.

Remark 11.1. If $P(x) \in K[x]$ monic and $P(a) = 0$, then $P = \operatorname{minpoly}(a/K)$ iff P is irreducible over K.

Lemma 11.18. If $M, N \models ACF, M, N \supseteq K, a \in M, b \in N$, then $\operatorname{qftp}(a/K) = \operatorname{qftp}(b/K)$ iff $I_{a/K} = I_{b/K}$ iff a, b are both transcendental over K or both algebraic over K with the same minimal polynomials.

Proof. Every atomic $\mathcal{L}(K)$-formula has the form $P(x) = Q(x)$ for some polynomials $P, Q \in K[x]$, and is equivalent to $P(x) - Q(x) = 0$. $\square$

Lemma 11.19. If $K \models ACF$, then $|K| = \infty$.

Lemma 11.20. If $P(x) \in K[x].P(a) = 0$, then $P(x) = (x - a)Q(x)$.

Corollary 11.21. If $P(x) \in K[x] \setminus \{0\}$, then $|\{a \in K | P(a) = 0\}| \leqslant \deg P$.

Corollary 11.22. If $L \supseteq K$, then $|\{a \in L | a \text{ algebraic over } K\}| \leqslant \aleph_0 + |K|$.

Lemma 11.23. If $N \models ACF$, $K \subseteq N$ subfield, then there exists $c \in N' \succeq N$, c is transcendental over K.

Proof. $|N| = \infty$, so by LS theorem, there is $N' \succeq N, |N'| > \aleph_0 + |K|$, then N' has a transcendental element. $\square$

Lemma 11.24. If $M, N \models ACF$, both extending a common subfield K and $b \in M$, then $\exists c \in N' \succeq N, \operatorname{qftp}(b/K) = \operatorname{qftp}(c/K)$.

Proof. If b is transcendental over K, then use the previous lemma.

If b is algebraic over K, $P(x) = \mathrm{minpoly}(b/K)$, then P is irreducible over K. $N \models ACF$, so $\exists c \in N, P(c) = 0$. Since P is irreducible over K, $P = \mathrm{minpoly}(c/K)$. □

Lemma 11.25. If K_1, K_2 fields, $R_i \subseteq K_i$ subring, $f : R_1 \to R_2$ isomorphism, then $\exists$ F_i field, $R_i \subseteq F_i \subseteq K_i, g : F_1 \to F_2$ isomorphism and $g \supseteq f$.

Proof. R_i subring of a field, so R_i is an integral domain. Then one can use fields of fractions.

Or one just define $F_i = \{x/y | x, y \in R_i, y \neq 0\}$ and $g(x/y) = f(x)/f(y)$. Chech this is well-defined. □

Lemma 11.26. If $M, N \models ACF$, both extending a common subring R and $b \in M$, then there is $c \in N' \succeq N$ such that $\mathrm{qftp}(b/R) = \mathrm{qftp}(c/R)$.

Proof. By the previous lemma, we can take $R \subseteq F_1 \subseteq M, R \subseteq F_2 \subseteq N$, F_i fields, $f : F_1 \to F_2$ a field isomorphism extending id_R. Then use Lemma 11.25. □

Theorem 11.27. ACF has QE.

Fact. If R is a ring, there is a unique ring homomorphism $f : \mathbb{Z} \to R$ such that $f(n) = 1 + 1 + \cdots + 1$, $f(0) = 0^R, f(-n) = -(1 + 1 + \cdots + 1)$. And $\mathrm{im}(f) = \langle \varnothing \rangle_R$.

Definition 11.28. $\mathrm{char}(R) = n$ iff $\ker(\mathbb{Z} \to R) = n\mathbb{Z}, n \in \mathbb{N}$.

Theorem 11.29. If $M, N \models ACF$, then $M \equiv N$ iff $\mathrm{char}(M) = \mathrm{char}(N)$.

12 December 1st

12.1 o-minimality

$(M, +, \cdot, \leqslant, \cdots)$ o-minimal.

Suppose $f : U \to M$ definable, $U \subseteq M^n$ open, $\overline{a} \in U, f$ is C^1 on U, $Df = (\partial f_i / \partial x_j)_{i,j}$. Assume $Df(\overline{a})$ is invertible.

Fact. Inverse Function Theorem. $\exists U' \ni \overline{a}, V \ni f(\overline{a})$ neighbourhoods, $f : U' \to V$ is a homeomorphism.

Corollary 12.1. $\mathrm{im}(f)$ has non-empty interior, so $\dim(\mathrm{im}(f)) = n$.

Proof. Here is a proof without using IFT, otherwise it is quite obvious.

Take a cell decomposition $\mathrm{im}(f) = \bigsqcup_{i=1}^{l} C_i$, if $\dim(C_i) < n$ for all i, then $U \subseteq \bigcup_{i=1}^{n} f^{-1}(C_i)$, so $\exists i, f^{-1}(C_i)$ has dimension n. WLOG $f^{-1}(C_1)$.

There is an open $U' \neq \varnothing, U' \subseteq f^{-1}(C_1), f(U') \subseteq C_1$. Say $\dim(C_1) = k < n$, there is a coordinate projection $\pi : C_1 \to C', C' \subseteq M^k$, π homeomorphism.

Then $f = e \circ \pi \circ f$ where $e = \pi^{-1} : C' \to C_1$. f, π are C^1, and e is generically C^1. Shrink U' so that e is C^1 on $\pi(f(U^1))$. $Df = De \cdot D\pi \cdot Df$ cannot be invertible because $k < n$. $\square$

Fix $D \subseteq M^n, \dim(D) < n$.

Definition 12.2. $\overrightarrow{v} \in M^n \setminus \{0\}$ is a **good direction** for D if $\pi : D \to \overrightarrow{v}^{\perp}$ has finite fibers. Here $\overrightarrow{v}^{\perp} = \{w \in M^n | \langle \overrightarrow{w}, \overrightarrow{v} \rangle = 0\}$, and $\pi(\overrightarrow{x}) = \overrightarrow{x} - \frac{\langle \overrightarrow{x}, \overrightarrow{v} \rangle}{\langle \overrightarrow{v}, \overrightarrow{v} \rangle} \overrightarrow{v}$.

Note that if $\overrightarrow{v}$ is a bad direction, then there exists $l, l \parallel \overrightarrow{v}, |l \cap D| = \infty$, so $\exists$ a line segment $\overline{AB} \subseteq l, \overline{AB} \subseteq D$ by o-minimality.

Lemma 12.3 (Good directions)**.** D has a good direction.

Proof. $(n = 4)$ If every $\overrightarrow{v} \in M^4 \setminus \{0\}$ is bad, for every $(x, y, z) \in M^3$, $(1, x, y, z)$ is bad.

So $\exists \overline{AB} \subseteq D, \overline{AB} \parallel (1, x, y, z)$. We can let $\overline{AB}$ depend definably on x, y, z by definably choosing representatives from

$$D_{x,y,z} = \{(A, B) | \overline{AB} \subseteq D, \overline{AB} \parallel (1, x, y, z)\}.$$

$A = A(x,y,z) = (a,b,c,d), u = u(x,y,z) \in M_{>0}$ such that $B = A + u\overrightarrow{v}$. Then

$$\{(a,b,c,d) + t(1,x,y,z) | 0 \leqslant t \leqslant u\} \subseteq D.$$

By generic differentiability, $\exists V \subseteq M^3$ open and non-empty such that a,b,c,d,u are C^1 on V.

$$S = \{(x,y,z,t) | (x,y,z) \in V, t \in (0, u(x,y,z))\}$$

is open in M^4, then $g(x,y,z,t) = (a,b,c,d) + t(1,x,y,z) : S \to D$. We compute

$$Dg = \begin{pmatrix} \partial a/\partial x & \partial a/\partial y & \partial a/\partial z & 1 \\ \partial b/\partial x + t & \partial b/\partial y & \partial b/\partial z & x \\ \partial c/\partial x & \partial c/\partial y + t & \partial c/\partial z & y \\ \partial d/\partial x & \partial d/\partial y & \partial d/\partial z + t & z \end{pmatrix}$$

has determinant $\det(D_g) = -t^3 + (\cdots)t^2 + (\cdots)t + (\cdots)$ where $(\cdots)$ are terms of x,y,z. So vary t, there exists $(x,y,z,t) \in S$ where Dg is invertible. So $\dim(D) = n$. $\square$

Theorem 12.4 (Inverse Function Theorem)**.** If f is C^1 near $\overline{0}$, $f(\overline{0}) = \overline{0}$, and Df is invertible at $\overline{0}$, then f is a local homeomorphism at $\overline{0}$.

Proof. Change coordinates and assume $Df(\overline{0}) = I_n$ by replacing f with $g \circ f, g^{-1}(\overline{x}) = (Df(\overline{0})) \cdot \overline{x}$.

(1) f is injective near $\overline{0}$. Suppose $\overline{a} \approx \overline{0} \approx \overline{b} \neq \overline{a}$, we have something like

$$0 = f(\overline{b}) - f(\overline{a}) = Df(\overline{0}) \cdot (\overline{b} - \overline{a}) + o(\|\overline{b} - \overline{a}\|),$$

so $\overline{b} - \overline{a} = Df(\overline{0})^{-1} \cdot o(\|\overline{b} - \overline{a}\|)$ which is impossible.

(2) If $\overline{a} \approx \overline{0}$, then $\overline{a} \in \mathrm{im}(f)$. $\overline{x} = \overline{x} - f(\overline{x}) + \overline{a}$, let $g(\overline{x}) = \overline{x} - f(\overline{x}) + \overline{a}$. Then $Dg(\overline{0}) = \overline{0}$.

Use mean value theorem to show g contractive near $\overline{0}$, take $\overline{x}$ minimizing $\|\overline{x} - g(\overline{x})\|$, then $\|g(\overline{x}) - g(g(\overline{x}))\| \leqslant \|\overline{x} - g(\overline{x})\|$, we must have $\overline{x} = g(\overline{x})$. $\square$

12.2 main class

Fact. If R is a ring, $\exists!$ homomorphism $f:\mathbb{Z}\to R$.

Proposition 12.5. $\operatorname{im}(f)=\langle\varnothing\rangle_R=\{t^R|t$ a closed $\mathcal{L}_{ring}$-term$\}$ is the smallest subring of R

Proof. First, $\operatorname{im}(f)$ is a subring.

Second, if S is a subring of R, then $S\supseteq\operatorname{im}(f)$ because $S\supseteq\langle\varnothing\rangle_R\supseteq\{n^R|n\in\mathbb{Z}\}\supseteq\operatorname{im}(f)$. □

Theorem 12.6. If R is a ring, then $\langle\varnothing\rangle_R\cong\mathbb{Z}/n\mathbb{Z}$ where $n=\operatorname{char}(R)$.

If K is a field, then $\operatorname{char}(K)\in\{0,2,3,5,7,\cdots\}$.

Recall that if T has QE, $M,N\models T$, then $M\equiv N$ iff $\langle\varnothing\rangle_M\cong\langle\varnothing\rangle_N$.

Corollary 12.7. If $M,N\models ACF$, then $M\equiv N$ iff $\operatorname{char}(M)=\operatorname{char}(N)$.

Fact. If K is a field, $\exists M\supseteq K, M\models ACF$.

Corollary 12.8. $\forall p\in\{0,2,3,5,7,\cdots\},\exists M\models ACF,\operatorname{char}(M)=p$.

Definition 12.9.

$$ACF_0=ACF\cup\{1+1\neq 0,1+1+1\neq 0,\cdots\}.$$

$$ACF_p=ACF\cup\{\underbrace{1+1+\cdots+1}_{p\text{ times}}=0\}.$$

ACF_0, ACF_p are consistent and complete. $\mathbb{C}\models ACF_0$, so if $\mathbb{C}\models\varphi$ an $\mathcal{L}_{ring}$-sentence, then $ACF_0\vdash\varphi$.

Th$(\mathbb{C})$ is decidable.

If $K\models ACF_0$, then $K\equiv\mathbb{C}$.

Suppose K is a subfield of $M\models ACF$.

$$K^{alg}=\{b\in M|b\text{ is algebraic over }K\}.$$

Recall that if $a,b\in M$, then $\operatorname{tp}(a/K)=\operatorname{tp}(b/K)$ iff $\operatorname{qftp}(a/K)=\operatorname{qftp}(b/K)$ iff $I_{a/K}=I_{b/K}$.

Lemma 12.10. If $D\subseteq M$ is K-definable, then $D=S$ or $D=M\setminus S$ for some $S\subseteq_f K^{alg}$.

Proof. Let $\mathcal{F} = \{S, M \setminus S | S \subseteq_f K^{alg}\}$, $\mathcal{F}$ is closed under Boolean operations.

Then for $\varphi(x)$ an atomic $\mathcal{L}_{ring}(K)$-formula, $\varphi(x) = (P(x) = Q(x)), P, Q \in K[x]$. So $\varphi(M) = M$ or a finite subset of K^{alg}. This shows $\varphi(M) \in \mathcal{F}$.

Another proof. Suppose $D = \varphi(M)$, $\varphi(x)$ an $\mathcal{L}_{ring}(K)$-formula. Take $N \succeq M$ large enough so that $\exists c \in N, c \notin K^{alg}$.

Replace φ with $\neg\varphi$ and D with $M \setminus D$ if necessary so that $N \models \neg\varphi(c)$.

Claim. $D \subseteq_f K^{alg}$.

Otherwise, $\{\varphi(x)\} \cup \{P(x) \neq 0 | P \in K[x] \setminus K\}$ is finitely satisfiable in M because either D has a transcendental element over K or D has infinitely many elements.

So □

Theorem 12.11. K^{alg} is a subfield of M.

Proof. Suppose $a, b \in K^{alg}, P(a) = 0, Q(b) = 0, P, Q \in K[x] \setminus K$. Let

$$D = \{x + y | x, y \in M, P(x) = 0, Q(y) = 0\},$$

so

$$D = \{z \in M | \exists x : P(x) = 0 \wedge Q(z - x) = 0\}$$

is K-definable. By the lemma and D is finite, $D \subseteq_f K^{alg}$, so $a + b \in K^{alg}$. Similarly for $a - b, a \cdot b, a/b$. □

Theorem 12.12. $K^{alg} \models ACF$.

Proof. $P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in K^{alg}[x], n > 0$. There exists $b \in M, P(b) = 0$ because $M \models ACF$.

Take $Q_i(x) \in K[x] \setminus K$ such that $Q_i(c_i) = 0$. Let

$$D = \{x \in M | \exists y_0, \cdots, y_{n-1}, \bigwedge_{i=0}^{n-1} Q_i(y_i) = 0 \wedge (x^n + y_{n-1}x^{n-1} + \cdots + y_0 = 0)\}.$$

So D is K-definable, D is finite. By the lemma, $b \in K^{alg}$. □

Corollary 12.13. $K^{alg} \preceq M$.

Proof. Because in a QE theory, a substructure is an elementary substructure. $\square$

Now M is a structure, $A \subseteq M$.

Definition 12.14. $\operatorname{acl}(A) = \bigcup\{X \subseteq_f M | X \text{ is } A\text{-definable}\}$ is the **algebraic closure** of A.

Example 12.1. If $M \models ACF, K \subseteq M$ a subfield, then $\operatorname{acl}(K) = K^{alg}$.

Proof. If $b \in \operatorname{acl}(K)$, then $b \in D \subseteq_f M$ for some K-definable set D. By the lemma, $D \subseteq_f K^{alg}$, so $b \in K^{alg}$.

Conversely, if $b \in K^{alg}$, then $P(b) = 0$ for some $P \in K[x] \setminus K$. The set $D = \{x \in M | P(x) = 0\}$ is K-definable and finite. So $b \in \operatorname{acl}(K)$. $\square$

Theorem 12.15. (1) $A \subseteq \operatorname{acl}(A)$.

(2) $A \subseteq B$ implies $\operatorname{acl}(A) \subseteq \operatorname{acl}(B)$.

(3) $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$.

So $\operatorname{acl}(-)$ is a closure operation on M.

Proof. (1) If $a \in A$, then $\{a\}$ is A-definable and finite.

(2) A-definable sets are B-definable.

(3) Claim. If $b \in \operatorname{acl}(A)$, then $\exists \mathcal{L}$-formula $\varphi(x_1, \cdots, x_n, y), k \in \mathbb{Z}, \overline{a} \in A^n$ such that $b \in \varphi(\overline{a}, M)$ and $\forall \overline{c} \in M^n, |\varphi(\overline{c}, M)| \leqslant k$.

Proof of Claim. $b \in D, D$ is A-definable, $|D| = k < \infty$. Let $D = \psi(a_1, \cdots, a_n, M)$, and

$$\varphi(\overline{x}, y) = \psi(\overline{x}, y) \wedge \exists^{=k} z : \psi(\overline{x}, z).$$

Then one has

$$\forall \overline{c} \in M^n, \varphi(\overline{c}, M) = \begin{cases} \psi(\overline{c}, M), & \text{if } |\psi(\overline{c}, M)| = k, \\ \varnothing, & \text{if } |\psi(\overline{c}, M)| \neq k. \end{cases}$$

Then $\varphi(\overline{a}, M) = \psi(\overline{a}, M) \ni b$ and $\forall \overline{c} \in M^n, |\varphi(\overline{c}, M)| \leqslant k$. $\square$

[Claim proved.]

Now by (1), $\mathrm{acl}(A) \subseteq \mathrm{acl}(\mathrm{acl}(A))$. Suppose $b \in \mathrm{acl}(\mathrm{acl}(A))$, there is a formula $\varphi(\overline{x}, y), k \in \mathbb{N}, \forall \overline{x} \exists^{\leqslant k} y : \varphi(\overline{x}, y), \exists c_1, \cdots, c_n \in \mathrm{acl}(A), \varphi(c_1, \cdots, c_n, b)$.

Now $c_i \in D_i, D_i$ finite and A-definable. So

$$D = \{y | \exists x_1, \cdots, x_n, x_i \in D_i, \varphi(\overline{x}, y) \text{ true}\}$$

is A-definable and finite. And $b \in D \subseteq \mathrm{acl}(A)$. □

Definition 12.16. A is **algebraically closed** if $A = \mathrm{acl}(A)$.

Theorem 12.17. If $A_i = \mathrm{acl}(A_i), \forall i \in I$, then $B = \bigcap_{i \in I} A_i$ is algebraically closed.

Theorem 12.18. If $A \subseteq M$, then $\mathrm{acl}(A) = \bigcap \{B \subseteq M | B = \mathrm{acl}(B), B \supseteq A\} = \min\{B \subseteq M | B = \mathrm{acl}(B), B \supseteq A\}$.

Theorem 12.19. If $M \models ACF, K \subseteq M$ not necessarily a subring, then TFAE:

(1) K is a subfield and $K = K^{alg}$.
(2) K is a subring and $K \models ACF$.
(3) $K \preceq M$.
(4) $K = \mathrm{acl}(K)$.

Proof. (1) implies (2): $K^{alg} \models ACF$, so $K \models ACF$.

(2) implies (3): ACF has QE, so $K \hookrightarrow M$ is an elementary embedding.

(3) implies (4): If $D \subseteq M$ is K-definable in M and D is finite, $D = \varphi(M)$, $\varphi(x)$ an $\mathcal{L}(K)$-formula, $|D| = n < \infty$, then $M \models \exists^{=n} x : \varphi(x)$. So $K \models \exists^{=n} x : \varphi(x)$. Say $\varphi(K) = \{a_1, \cdots, a_n\}$, then $K \models \varphi(a_i)$, so $M \models \varphi(a_i)$. Then $\varphi(M)$ has to be exactly $\{a_1, \cdots, a_n\}$. So $D \subseteq K, \mathrm{acl}(K) \subseteq K$.

(4) implies (1): Claim K is a field.

If $a, b \in K$, then $\{a/b\}$ is K-definable by $xb = a$, so $a/b \in \mathrm{acl}(K) = K$. This proves the claim.

When K is a field, $\mathrm{acl}(K) = K^{alg}$, so $K = K^{alg}$. □

Definition 12.20. A theory T is **strongly minimal** if $\forall M \models T$, one has $|M| = \infty, \forall D \subseteq_{def} M, D$ is finite or cofinite.

Theorem 12.21. If $M \models T, T$ is strongly minimal, $A \subseteq M$, then $A \preceq M$ iff $|A| = \infty, \operatorname{acl}(A) = A$.

Proof. If $A \preceq M$, then $A \models T$, so $|A| = \infty$. By the same proof as that of the previous theorem, one can prove $A = \operatorname{acl}(A)$.

Conversely, if $D \subseteq M$ is A-definable, non-empty, we want $D \cap A \neq \varnothing$. By strong minimality, D is finite or cofinite. If D is finite, then $D \subseteq \operatorname{acl}(A) = A$, so $D \cap A = D \neq \varnothing$. If D is cofinite, then $D \cap A \neq \varnothing$ because $|A| = \infty$. Then use Tarski-Vaught criterion. □

Definition 12.22. $\operatorname{dcl}(A) = \{b \in A | \{b\}$ is A-definable$\}$ is the **definable closure** of A.

Theorem 12.23. $A \subseteq \langle A \rangle \subseteq \operatorname{dcl}(A) \subseteq \operatorname{acl}(A)$.

Theorem 12.24. $\operatorname{dcl}(-)$ is a closure operation.

Definition 12.25. A is **definably closed** if $A = \operatorname{dcl}(A)$.

$\operatorname{dcl}(A) = \operatorname{dcl}(B)$ iff $\forall n, \forall D \subseteq M^n$, D is A-definable iff D is B-definable.

$$\operatorname{dcl}(A) = \{f(a_1, \cdots, a_n) | f : X \to M \ \varnothing\text{-definable}, \overline{a} \in A^n\}.$$

In $(\mathbb{R}, +, \cdot)$, $f(x) = \sqrt{x}$ is definable, so $\sqrt{\pi} \in \operatorname{dcl}(\{\pi\})$.

Fact. If $M \models ACF, A \subseteq M$, then $A = \operatorname{dcl}(A)$ iff A is a perfect subfield, i.e. $\operatorname{char}(A) = 0$ or $\operatorname{char}(A) = p$, the Frobenius homomorphism is a bijection.

In ACF_p, the map $x \mapsto x^{1/p}$ is definable.

In $ACF_0, f : M \to M$ is K-definable, then f is piecewise given by rational functions with coefficients from K. The proof idea is to show $\{f(x) = y\} \cup \{y \neq P(x)/Q(x) | P, Q \in K[x]\}$ not finitely satisfiable.

13 December 8th

13.1 o-minimality

Definition 13.1. An **abstract simplicial complex** is $(S, \Delta), \Delta \subseteq P(S)$ such that

(1) If $X \in \Delta$, then $0 < |X| < \omega$.

(2) If $\varnothing \neq X \subseteq Y \in \Delta$, then $X \in \Delta$.

(3) $v \in S \Rightarrow \{v\} \in \Delta$. Equivalently, $\bigcup \Delta = S$.

For example, a graph (V, E) can be viewed as an abstract simplicial complex.

Now M o-minimal, and expands the ordered field structure.

Definition 13.2. A **geometric realization** of a finite ASC $(S, \Delta), S = \{v_1, \cdots, v_n\}$ is as follows:

e_i the i-th basis vector in M^n. For any $X \in \Delta$, let $|X|$ = convex hull of $\{e_i | v_i \in X\}$. So $(S, \Delta)| = \bigcup\limits_{X \in \Delta} |X|$ is the **geometric realization** of (S, Δ).

Recall that $D \subseteq_{def} M^n$ is definably compact if D is closed and bounded.

Fact. $\exists$ an embedding $M^n \to [-1, 1]^n$. $\forall$ definable $D \subseteq M^n$, $\exists$ definable homeomorphism $D \to X, X$ is bounded and $\overline{X}$ is definably compact. (The field structure is needed here, it seems not true for $(\mathbb{R}, +, \leqslant)$.)

Theorem 13.3. If D is definably compact, then $\exists$ finite abstract simplicial complex K and definable homeomorphism $D \to |K|$, called a **triangulation** of D.

Moreover, we can make the induced partition from the triangulation of D define any given partition $D = \bigsqcup\limits_{i=1}^{n} Y_i$, Y_i definable.

Any definable D has a triangulation in some sense. Assume D bounded, $\overline{D}$ definably compact, triangulate $\overline{D}$ in a way that respects the partition $\overline{D} = D \bigsqcup \partial D$.

Theorem 13.4. If $\{D_\alpha\}_{\alpha \in Y}$ is a definable family of definably compact sets, then $\exists a_1, \cdots, a_n \in Y$ such that $\forall b \in Y, \exists$ definable homeomorphism $D_b \to D_{a_i}$.

Proof. Recall that o-minimality is preserved under elementary equivalence (see Theorem 7.1). Let $q(x, y)$ be the set of formulas such that $(a, b) \models q$ iff D_a is not definably homeomorphic to D_b. One can use an axiom schema, saying that every definable map does not map D_a homeomorphically onto D_b.

For any finite abstract simplicial complex K, let $p_K(x)$ say D_x is not homeomorphic to $|K|$. Let $q(x) = \bigcup\limits_{K} p_K(x)$, q cannot be finitely satisfiable, otherwise $\exists N \succeq M, b \in N, q(b)$ true, so D_b is not definably homeomorphic to any finite ASC, a contradiction to the triangulation theorem.

So $\exists K_1, \cdots, K_n$ finite ASC such that $\bigcup\limits_{i=1}^{n} P_{K_i}(x)$ inconsistent, then $\forall b \in Y, D_b$ is definably homeomorphic to one of the K_i's. $\square$

Recall the fact that $(\mathbb{R}, +, \cdot, \leqslant, \exp)$ is o-minimal.

Corollary 13.5. $\forall n, m, V(P) = \{\overline{x} \in \mathbb{R}^m | P(\overline{x}) = 0\}$ for $P \in \mathbb{R}[x_1, \cdots, x_m]$. Then $\{V(P) | P \in \mathbb{R}[x_1, \cdots, x_m], P$ a sum of $\leqslant n$ monomials$\}$ has only finitely many homeomorphism types.

This fails over $\mathbb{C}$ because $x^n = 1$ gives infinitely many homeomorphism types.

We explain the corollary by an example. $n = 3, m = 2$, the polynomial

$$P(x, y) = a_1 x^{k_1} y^{j_1} + a_2 x^{k_2} y^{j_2} + a_3 x^{k_3} y^{j_3}$$

is a definable function of x, y, a, k, j in $\mathbb{R}_{\exp}$ because $x^y = \exp(y \log(x))$.

Now we give a proof sketch of the triangulation theorem when $n = 2$. $D \subseteq M^2$ definably compact, $D = \bigsqcup\limits_{i=1}^{l} Y_i$ definable sets. Let $E = \mathrm{bd}(D) \cup \bigcup\limits_{i=1}^{l} \mathrm{bd}(Y_i) \subseteq M^2$, then $\dim(E) \leqslant 1$. Rotate so that the vertical direction is good for E, so every vertical line hits E in only finitely many points, i.e. $|E_a| < \infty, \forall a \in M^1$. Now E can only have points and graphs of functions in its cell decomposition, then we can triangulate D in the way we need.

13.2 main class

Lemma 13.6. If $I \lhd R$, then there is a bijection (actually an isomorphism of lattice)

$$\{J | J \lhd R/I\} \leftrightarrow \{I' \lhd R | I' \supseteq I\}.$$

Proof. Given $J \lhd R/I$, let I' be

$$\ker(R \to R/I \to (R/I)/J).$$

Given $I' \supseteq I, I' \lhd R$, the map $R/I \to R/I', [x]_I \mapsto [x]_{I'}$ is well-defined. □

$\exists! R : R \models 0 = 1$. The number of ideals of R is 1 iff $R \models 0 = 1$, is 2 iff R a field, is $\geqslant 3$ in all other cases.

Definition 13.7. $I \lhd R$ is **proper** iff $I \neq R$ iff $1 \notin I$ iff $R/I \not\models 0 = 1$.

Definition 13.8. $I \lhd R$ is **maximal** if it is a maximal proper ideal.

Theorem 13.9. If $I \lhd R$ is maximal, then R/I is a field.

Proof. $\{I' \lhd R | I' \supseteq I\} = \{I, R\}$, so R/I has only 2 ideals. □

Corollary 13.10. If $R \models 0 \neq 1$, then $\exists$ homomorphism $R \to K, K$ a field.

Remark 13.1. If K, L are fields, $f : K \to L$ a homomorphism, then f is an embedding.

Lemma 13.11. If K is a field, $P(x) \in K[x] \setminus K$, then $\exists$ a field $L \supseteq K$ such that $L \models \exists x : P(x) = 0$.

Proof. $K[x]/P(x)K[x]$ is non-trivial, so $\exists$ homomorphism $K[x]/P(x) \to L$, L a field. Suppose $x \mapsto a$.

$$K \to K[x] \to K[x]/P(x) \to L$$

is an embedding, so $L \models \operatorname{diag}(K)$ and can be viewed as an $\mathcal{L}_{ring}(K)$-structure, so $P(x) \mapsto P(a)$, $P(a) = 0$. Then $L \models T_{field} \cup \operatorname{diag}(K) \cup \{\exists x : P(x) = 0\}$, so move by an isomorphism so that $L \supseteq K$. □

Lemma 13.12. If $P_1, \cdots, P_m \in K[x_1, \cdots, x_n]$ and the ideal they generate is proper, then $\exists L \supseteq K, a_1, \cdots, a_n \in L, P_1(\overline{a}) = \cdots = P_m(\overline{a}) = 0$.

Proof. Same as above. $\square$

Lemma 13.13. If K is a field, then there is $L \supseteq K, L$ a field, $\forall P(x) \in K[x] \setminus K, P$ has a root in L.

Proof. If $P_1, \cdots, P_n \in K[x] \setminus K$, then there exists $L \supseteq K, L \models \bigwedge_{i=1}^{n} \exists x : P_i(x) = 0$. Apply compactness to $\mathcal{L}(K)$-theory $T_{field} \cup \operatorname{diag}(K) \cup \{\exists x : P(x) = 0 | P \in K[x] \setminus K\}$. $\square$

Theorem 13.14. If K is a field, then $\exists M \supseteq K, M \models ACF$.

Proof. Build a chain $K = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ of length ω such that L_{i+1} has a root of every $P \in L_i[x] \setminus L_i$. Take $M = \bigcup_{i=0}^{\infty} L_i$. $\square$

Corollary 13.15. ACF_p is consistent for $p = 2, 3, 5, 7, \cdots$.

Definition 13.16. An **$\forall\exists$-sentence** (reads AE-sentence) is a sentence

$$\forall x_1, \cdots, x_n \exists y_1, \cdots, y_m : \varphi(\overline{x}, \overline{y})$$

where φ is qfree.

An **$\forall\exists$-theory** is a set of $\forall\exists$-sentences.

For example, the theory of fields is an $\forall\exists$-theory.

Remark 13.2. If $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots, M_i \models T$ an $\forall\exists$-theory, then $N = \bigcup_{i=1}^{\infty} M_i \models T$.

Proof. Let $\forall \overline{x} \exists \overline{y} : \varphi(\overline{x}, \overline{y}) \in T$. If $\overline{a} \in N^n$, then $\overline{a} \in M_i^n$ for some $i >> 1$, so $\exists \overline{b} \in M_i^m$ such that $M_i \models \varphi(\overline{a}, \overline{b})$. Since φ is qfree, $N \models \varphi(\overline{a}, \overline{b})$, so $N \models \forall \overline{x} \exists \overline{y} : \varphi(\overline{x}, \overline{y})$. $\square$

$\operatorname{Th}(\mathbb{Z}, \leqslant)$ is not an $\forall\exists$-theory. Because $\mathbb{Z} \subseteq \frac{1}{2}\mathbb{Z} \subseteq \frac{1}{4}\mathbb{Z} \subseteq \cdots, N = \bigcup_i \frac{1}{2^i}\mathbb{Z}, N \models DLO$ so $N \not\models \operatorname{Th}(\mathbb{Z}, \leqslant)$.

Suppose $M \subseteq N$.

Definition 13.17. $M \preceq_\exists N$ (called **existstentially closed**) if $\forall$ qfree $\mathcal{L}(M)$-formula $\varphi(x_1, \cdots, x_n), N \models \exists \overline{x} : \varphi(\overline{x}) \Rightarrow M \models \exists \overline{x} : \varphi(\overline{x})$.

For example, $(\mathbb{R}, +, \cdot, 0, 1, -) \not\preceq_\exists \mathbb{C}$ because $\mathbb{C} \models \exists x : x^2 + 1 = 0$ but $\mathbb{R}$ does not.

If $M \preceq N$, then $M \preceq_\exists N$.

Now fix a theory T.

Definition 13.18. $M \models T$ is **existentially closed** relative to T if $\forall N \supseteq M, N \models T$, we have $M \preceq_\exists N$.

Example 13.1. An e.c. field is a field K such that for any field $L \supseteq K, K \preceq_\exists L$.

Theorem 13.19. If T is an $\forall\exists$-theory, $M \models T$, then $\exists N \supseteq M, N$ is an e.c. model of T.

Proof. Just like how we prove Theorem 13.14 and Remark 13.2. □

Theorem 13.20. K is an e.c. field iff $K \models ACF$.

Proof. $K \not\models ACF$ implies $\exists P \in K[x] \setminus K$ no roots in K implies $\exists L \supseteq K, P$ has a root in L. Then $K \not\preceq_\exists L$ because $L \models \exists x : P(x) = 0, K \not\models \exists x : P(x) = 0$.

Conversely, if $K \models ACF$, suppose field $L \supseteq K$, we need to prove $K \preceq_\exists L$. Take a field $M \supseteq L, M$ e.c. so $M \models ACF$. Since $K \subseteq L \subseteq M$ and ACF has QE, so $K \preceq M$, so $K \preceq_\exists M$. Then

$$L \models \exists \overline{x} : \varphi(\overline{x}) \Rightarrow M \models \exists \overline{x} : \varphi(\overline{x}) \Rightarrow K \models \exists \overline{x} : \varphi(\overline{x})$$

for any qfree $\mathcal{L}(K)$-formula φ. □

Similarly, one can prove e.c. LO is exactly DLO.

Fact. {e.c. groups} is not an elementary class.

Theorem 13.21. If $K \models ACF, L \supseteq K, P_1, \cdots, P_n \in K[x_1, \cdots, x_m], P_1(\overline{x}) = P_2(\overline{x}) = \cdots = P_m(\overline{x}) = 0$ has a solution in L, then it has a solution in K.

Proof. Because $K \preceq_\exists L$. □

Theorem 13.22 (Weak Nullstellensatz)**.** If $K \models ACF, P_1, \cdots, P_m \in K[x_1, \cdots, x_n]$, they generate a proper ideal, then $\exists \overline{a} \in K^n : P_1(\overline{a}) = \cdots = P_m(\overline{a}) = 0$.

Proof.

$$K \hookrightarrow K[\overline{x}] \to K[\overline{x}]/(P_1, \cdots, P_m) \to L.$$

So there exists a field $L \supseteq K, P_1 = \cdots = P_m = 0$ has a root in L. $\square$

Now fix $\kappa \geqslant \aleph_0$, A is **small** if $|A| < \kappa$.

Definition 13.23. M is **κ-saturated** if $\forall$ small $A \subseteq M, \forall$ partial n-type p over A, p is realized in M.

A non-example: $(\mathbb{R}, +, \cdot, 0, 1, -)$ is not $\aleph_0$-saturated because $\{x > 0, x > 1, \cdots\}$ is not realized (with $A = \varnothing$). Similarly, $(\mathbb{R}, \leqslant)$ is not $\aleph_1$ saturated by the same set of formulas, but this time we have to take $A = \mathbb{N}$.

Fact. $(\mathbb{C}, +, \cdot, 0, 1, -)$ is $2^{\aleph_0}$-saturated.

$(\mathbb{Q}, \leqslant)$ and $(\mathbb{R}, \leqslant)$ are $\aleph_0$-saturated.

Fact. $\forall$ structure $M, \kappa \geqslant \aleph_0, \exists N \succeq M, N$ is κ-saturated.

Now suppose $\mathbb{M}$ is κ-saturated.

Theorem 13.24 (κ-compactness)**.** (1) If p is a small partial type over $\mathbb{M}$ ($|p| < \kappa$), then p is realized in $\mathbb{M}$.

Suppose $\mathcal{F} \subseteq \{D \subseteq \mathbb{M}^n | D \text{ definable}\}$ is small.

(2) If $\forall X_1, \cdots, X_m \in \mathcal{F}, \bigcap\limits_{i=1}^{m} X_i \neq \varnothing$, then $\bigcap \mathcal{F} \neq \varnothing$.

(3) If $D \subseteq_{def} \mathbb{M}^n, D \subseteq \bigcup \mathcal{F}$, then $\exists X_1, \cdots, X_m \in \mathcal{F}, D \subseteq \bigcup\limits_{i=1}^{m} X_i$.

So if $\mathcal{F}$ is a small chain of non-empty definable sets, then $\bigcap \mathcal{F} \neq \varnothing$.

Proof. (1) p is a partial type over a small $A \subseteq \mathbb{M}$. Here A can be taken as the set of parameters used in p. So p is realized in $\mathbb{M}$.

(2) $\mathcal{F} = \{\varphi(\mathbb{M}) | \varphi \in p\}$ for some small set of $\mathcal{L}(M)$-formulas p. Now p is finitely satisfiable, so p is a small partial type over $\mathbb{M}$. By (1), p is realized in $\mathbb{M}$, i.e. $\bigcap \mathcal{F} \neq \varnothing$.

(3) Apply (2) to $\{D\} \cup \{\mathbb{M}^n \setminus X | X \in \mathcal{F}\}$. The intersection of this family is empty, so there are $X_1, \cdots, X_m \in \mathcal{F}$ such that $D \cap \bigcap\limits_{i=1}^{m} (\mathbb{M}^n \setminus X_i) = \varnothing$. $\square$

Corollary 13.25. If $D \subseteq_{def} \mathbb{M}^n$ is small, then $|D| < \aleph_0$.

Proof. $D = \bigcup_{b \in D} \{b\}$ is a small cover, so there is a finite subcover.

Another way is to think of $D = \varphi(\mathbb{M}), \{\varphi(x)\} \cup \{x \neq b | b \in D\}$ finitely satisfiable when D is infinite. □

Example 13.2 (Non-standard analysis)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$, and $(\mathbb{R}, +, \cdot, f) \preceq (\mathbb{M}, +, \cdot, f)$ is κ-saturated for some $\kappa > |\mathbb{R}|$.

Let $I = \bigcap_{n=1}^{\infty} [-1/n, 1/n]_{\mathbb{M}}$ be the set of **infinitesimals**. It can be shown that $I = \bigcap_{\varepsilon \in \mathbb{R}_+} [-\varepsilon, \varepsilon]_{\mathbb{M}}$. $a \in \mathbb{M}$ is an **infinitesimal** if $\forall n > 0$, we have $-1/n < a < 1/n$.

$\{0\} = I \cap \mathbb{R}$, but $I \supsetneq \{0\}$ because $\bigcap_{n=1}^{\infty} ([1/n, 1/n] \setminus \{0\}) \neq \varnothing$ by saturation.

Claim. $(I, +)$ is a subgroup of $(\mathbb{M}, +)$.

Proof. $\forall a, b \in I, \forall \varepsilon \in \mathbb{R}_+, \delta = \varepsilon/2$. we have $|a| < \delta, |b| < \delta$, so $|a - b| < \varepsilon$. Since $\varepsilon \in \mathbb{R}_+$ is arbitrarily chosen, $a - b \in I$. □

Definition 13.26. $x \approx y$ if $x - y \in I$. $\approx$ is an equivalence relation on $\mathbb{M}$.

Theorem 13.27. If $a \in \mathbb{R}$, then f is continuous at a iff $\forall x \in \mathbb{M} : x \approx a \Rightarrow f(x) \approx f(a)$.

Proof. Suppose f is continuous at a and $x \approx a$. Now $\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+$, so

$$\begin{aligned} &\mathbb{R} \models \forall x : |x - a| < \delta \to |f(x) - f(a)| < \varepsilon \\ \Rightarrow &\mathbb{M} \models \forall x : |x - a| < \delta \to |f(x) - f(a)| < \varepsilon. \end{aligned}$$

Now $x \approx a \Rightarrow |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, since $\varepsilon \in \mathbb{R}_+$ is arbitrarily chosen, we have $f(x) \approx f(a)$.

Conversely, suppose f is not continuous at a, there exists $\varepsilon \in \mathbb{R}_+$ such that $\forall \delta \in \mathbb{R}_+$,

$$\begin{aligned} &\mathbb{R} \models \exists x : |x - a| < \delta \wedge |f(x) - f(a)| \geqslant \varepsilon \\ \Rightarrow &\mathbb{M} \models \exists x : |x - a| < \delta \wedge |f(x) - f(a)| \geqslant \varepsilon. \end{aligned}$$

Take $x \in \bigcap_{\delta \in \mathbb{R}_+} \{x \in \mathbb{M} | |x - a| < \delta \wedge |f(x) - f(a)| \geqslant \varepsilon\}$, which is possible because of saturation. Then $x \approx a, f(x) \not\approx f(a)$.

Fact. $f'(a) = b (a, b \in \mathbb{R})$ iff $\forall x \in \mathbb{M} : x \approx a, x \neq a \Rightarrow \frac{f(x)-f(a)}{x-a} \approx b$.

Fact. f is uniformly continuous iff $\forall x, y \in \mathbb{M} : x \approx y \Rightarrow f(x) \approx f(y)$.

A similar thing is $D \subseteq \mathbb{R}$, consider $(\mathbb{R}, +, \cdot, D) \preceq (\mathbb{M}, +, \cdot, D^*)$ κ-saturated, $\kappa > |\mathbb{R}|$.

Fact. $(a \in \mathbb{R}) a \in \mathrm{cl}(D)$ iff $\exists x \in D^* : x \approx a$.

Fact. D is compact iff $\forall x \in D^* \exists a \in \mathbb{R} : x \approx a$. $\square$

14 December 15th

14.1 o-minimality

$B = \{\overline{a} \in M^k : |\overline{a}| < 1\}$. All things are definable.

Definition 14.1. $C \subseteq M^n$ is a **nice disk** if $\exists$ definable homeomorphism $f : B \to C$ extending to a homeomorphism $\overline{B} \to C'$. In this case, we necessarily have $C' = \overline{C}, f : \partial B \to \partial C, \dim(C) = k$.

Warning. $X = (0,1) \times (-1,1), f(x,y) = (\frac{x^2-y^2}{x^2+y^2}, \frac{2xy}{x^2+y^2}), X \to \mathbb{R}^2$. Then $\Gamma(f)_X$ is a cell but not a nice disk. The boundary is a circle with two points identified. So a homeomorphism image of a cell may not be a nice disk.

Fact. If C is a nice disk, then any triangulation of ∂C extends to a triangulation of $\overline{C}$.

Definition 14.2. A **pretriangulation** of D is $D = \bigsqcup_{i=1}^{l} C_i$, C_i a nice disk, ∂C_i is a union of some C_j.

Theorem 14.3. If D has a pretriangulation, then it has a triangulation.

Proof sketch. Induction on $k = \dim(D)$.

$D' = \cup\{C_i | \dim(C_i) < k\}$, by induction D' has a triangulation, then use the fact to extend to a triangulation of D. □

Fact. If B is an open ball, $f, g : B \to M$ continuous, $f < g$ on B, suppose f, g extend continuously to $\overline{B}$. Then $f \leqslant g$ on $\overline{B}$, and

$$(f, g)_B = \{(\overline{x}, y) | \overline{x} \in B, y \in (f(\overline{x}), g(\overline{x}))\}$$

is a nice disk, the frontier is $\Gamma(f) \cup \Gamma(g) \cup (f, g)_{\partial B}$.

Lemma 14.4. Suppose $f : B \to M$ definable and continuous, suppose $(0, \cdots, 0, 1)$ is a good direction for $\overline{\Gamma(f)}$. Then f extends continuously to $\overline{B}$.

Proof. It suffices to show f uniformly continuous on B.

Suppose not, $\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in B, |x - y| < \delta$ but $|f(x) - f(y)| \geqslant \varepsilon$.

By definable choice, choose $x = x(\delta), y = y(\delta)$, then suppose $\lim_{\delta \to 0} x(\delta) = \lim_{\delta \to 0} y(\delta) = p \in \overline{B}$ and $a = \lim_{\delta \to 0} f(x(\delta)) \in M_\infty = M \cup \{\pm\infty\}, b = \lim_{\delta \to 0} f(y(\delta)) \in M_\infty$. Then $a \neq b$.

If $p \in B$, then $a = b = f(p)$, a contradiction.

If $p \in \partial B$, then $(p, a), (p, b) \in \overline{\Gamma(f)}$. Then one can show $(a, b) \subseteq E_p$ but E_p is finite by the good direction. A contradiction. □

Theorem 14.5. If $D \subseteq M^n$ definably compact, then D has a pretriangulation.

Moreover, if $D = \bigsqcup_{i=1}^{l} X_i$, we can make the pretriangulation define this partition.

Proof. Let $E = \mathrm{bd}(D) \cup \bigcup \mathrm{bd}(X_i)$, E is closed, bounded, $\dim(E) < n$. So E has a good direction, rotate so that $(0, \cdots, 0, 1)$ is good. $\pi : M^n \to M^{n-1}, E \to \pi(E)$ has finite fibres.

$\pi(D)$ definably compact, by cell decomposition, $\pi(D) = \bigsqcup_{i=1}^{l} Y_i$ such that $\forall i$, E above Y_i is the graph of $f_1 < \cdots < f_m$ continuous functions and $1_D, 1_{X_i}$ constant on each $\Gamma(f_i)$.

Finally refine so that $\bigsqcup Y_i$ is a pretriangulation of $\pi(D)$ (by induction). Let $\mathcal{F}_i$ be the set $f_1, \cdots, f_m$ respect to Y_i as above. Then we have:

(1) $\forall i, \mathcal{F}_i$ a set of continuous definable functions $Y_i \to M$.

(2) $\mathcal{F}_i$ finite.

(3) $f \in \mathcal{F}_i$, $1_D, 1_{X_i}$ constant on $\Gamma(f)$.

(4) $f, g \in \mathcal{F}_i, f < g$ or $g < f$ or $f = g$.

Then $E_{Y_i} = \bigcup_{f \in \mathcal{F}_i} \Gamma(f)$.

If $f \in \mathcal{F}_i, f : Y_i \to M, \Gamma(f) \subseteq E, \overline{\Gamma(f)} \subseteq E$, then f extends continuously to $\overline{Y_i}, \overline{f} : \overline{Y_i} \to M$, $\Gamma(\overline{f}) \subseteq E$, so if $Y_j \subseteq \partial Y_i$, then $\overline{f}|_{Y_j} \in \mathcal{F}_i$.

All the sets $\Gamma(f)_{Y_i}, f \in \mathcal{F}_i$ or $(f, g)_{Y_i}, f, g$ consecutive in $\mathcal{F}_i$ is a pretriangulation. □

14.2 main class

Definition 14.6. $f : M \supseteq A \to B \subseteq N$ bijection, f is a **partial elementary map (PEM)** if $M \models \varphi(\overline{a}) \Leftrightarrow N \models \varphi(f(\overline{a})), \forall \overline{a} \in A^n$.

Example 14.1. (1) $f : M \overset{\preceq}{\to} N$, f is a PEM.

(2) $\overline{a} \in M^n, \overline{b} \in N^n, \mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$, then $\{a_1, \cdots, a_n\} \to \{b_1, \cdots, b_n\}, a_i \mapsto b_i$ is a PEM.

(3) $M \equiv N$, then $\varnothing : \varnothing \to \varnothing$ is a PEM.

Definition 14.7. $\mathrm{Aut}(M) = \{\sigma | \sigma : M \to M \text{ isomorphism}\}$ is called the **automorphism group**.

Definition 14.8. M is **κ-strongly homogeneous** if $\forall$ small PEM $f : A \to B$ extends to $\sigma\mathrm{Aut}(M)$.

Fact. $\forall M, \forall \kappa \geqslant \aleph_0, \exists N \succeq M, N$ is κ-saturated and κ-strongly homogeneous.

Definition 14.9. $\mathrm{Aut}(M/A) = \{\sigma \in \mathrm{Aut}(M) | \sigma \supseteq \mathrm{id}_A\}$.

Theorem 14.10. If M is κ-strongly homogeneous, $A \subseteq M$ is small, $\overline{b}, \overline{c} \in M^n$, then TFAE:

(1) $\mathrm{tp}(\overline{b}/A) = \mathrm{tp}(\overline{c}/A)$.

(2) $\exists \sigma \in \mathrm{Aut}(M/A) : \sigma(\overline{b}) = \overline{c}$.

Proof. Clearly (2) implies (1).

Now prove (1) implies (2). Let $f : A \cup \{b_1, \cdots, b_n\} \to A \cup \{c_1, \cdots, c_n\}, f(b_i) = c_i, f(x) = x, \forall x \in A$. This is a small PEM, so we can extend it to an automorhpism of M. □

Now let $p(x_1, \cdots, x_n)$ be a partial n-type over $A \subseteq M$.

Definition 14.11. p is **complete** iff p is a maximal partial n-type iff $\forall \mathcal{L}(A)$-formula $\varphi(x_1, \cdots, x_n), \varphi \in p$ or $\neg\varphi \in p$ iff $p = \mathrm{tp}(\overline{b}/A)$ for some $\overline{b} \in N^n, N \succeq M$.

Let $S_n(A) = \{\text{complete } n\text{-types over } A\}$.

Theorem 14.12. If M is κ-saturated and κ-strongly homogeneous, $A \subseteq M$ small, then we have a bijection

$$M^n/\mathrm{Aut}(M/A) \leftrightarrow S_n(A)$$
$$\mathrm{orbit}(\overline{b}) \mapsto \mathrm{tp}(\overline{b}/A).$$

The left side means the set of orbits of the group action $\mathrm{Aut}(M/A) \curvearrowright M^n$ coordinate-wise.

Definition 14.13. A **monster model** means $\mathbb{M}$ κ-saturated and κ-strongly homogeneous for $\kappa >>$ (any cardinal we care about).

Fix a monster model $\mathbb{M}$.

Definition 14.14. $\bar{b} \underset{A}{\equiv} \bar{c}$ iff $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$ iff $\exists \sigma \in \mathrm{Aut}(\mathbb{M}/A) : \sigma(\bar{b}) = \bar{c}$ for A small.

Theorem 14.15. If $D \subseteq \mathbb{M}^n$ definable, $A \subseteq \mathbb{M}$ small, then D is A-definable iff $\forall \sigma \in \mathrm{Aut}(\mathbb{M}/A), \sigma(D) = D$. We say D is A-invariant in this case.

Definition 14.16. D is A**-invariant** iff whenever $\bar{b} \underset{A}{\equiv} \bar{c}$, we have $\bar{b} \in D \Leftrightarrow \bar{c} \in D$ iff $\exists X \subseteq S_n(A), D = \{\bar{b} | \mathrm{tp}(\bar{b}/A) \in X\}$.

Proof. Suppose D is definable and A-invariant. Claim that if $\bar{b} \in D$, then $\exists A$-definable set $U, \bar{b} \in U \subseteq D$.

Let $p = \mathrm{tp}(\bar{b}/A), p \cup \{\bar{b} \notin D\}$ has no realizations in $\mathbb{M}$ because D is A-invariant. By saturation, it is not finitely satisfiable, so $\exists \psi \in \mathrm{tp}(\bar{b}/A), \{\psi\} \cup \{\bar{x} \notin D\}$ inconsistent, so $b \in \psi(\mathbb{M}) \subseteq D$, and $\psi(\mathbb{M})$ is A-definable. This proves the claim.

So $D = \bigcup\{U | U \text{ is } A\text{-definable}, U \subseteq D\}$. By κ-compactness, there is a finite subcover, so D is A-definable.

Conversely, write $D = \psi(\mathbb{M}), \psi$ an $\mathcal{L}(A)$-formula. If σ fixes every element in A, then $\sigma(D) = D$. $\square$

Example 14.2. $b \in \mathrm{dcl}(A)$ iff $\{b\}$ is A-definable iff $\{b\}$ is A-invariant iff $\forall \sigma \in \mathrm{Aut}(\mathbb{M}/A), \sigma(b) = b$ iff $\mathrm{Aut}(\mathbb{M}/Ab) = \mathrm{Aut}(\mathbb{M}/A)$.

Here, we follow the convention that $A\bar{b} = A \cup \{b_1, \cdots, b_n\}$.

Another proof that $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$**.**

First we note that if $A \subseteq M \preceq N$, then $\mathrm{acl}_M(A) = \mathrm{acl}_N(A)$. Generally we have $\varphi(M) \subseteq \varphi(N)$ for any $\mathcal{L}(A)$-formula $\varphi(x)$. If $\varphi(M)$ is finite, then $M \models \exists^{=n} x : \varphi(x)$, so $N \models \exists^{=n} x : \varphi(x)$. This forces $\varphi(M) = \varphi(N)$. So $\mathrm{acl}_M(A) = \mathrm{acl}_N(A)$. So we can work in a monster model $\mathbb{M}$.

Theorem 14.17. $b \in \mathrm{acl}(A)$ iff $\mathrm{Aut}(\mathbb{M}/A)b$ is finite iff $\mathrm{Aut}(\mathbb{M}/Ab)$ has finite index in $\mathrm{Aut}(\mathbb{M}/A)$.

Proof. If $b \in \operatorname{acl}(A)$, then there is some A-definable $D, b \in D, |D| < \infty$. Then $\operatorname{Aut}(\mathbb{M}/A)b \subseteq D$ because D is A-invariant.

Conversely, if the orbit is finite, then let D be the orbit set. We have $\sigma(D) = D$ and D is definable, so D is A-definable. Then $b \in D, b \in \operatorname{acl}(A)$. □

Lemma 14.18. If $b \in \operatorname{acl}(A), c \in \operatorname{acl}(Ab)$, then $c \in \operatorname{acl}(A)$.

Proof. Let $G_A = \operatorname{Aut}(\mathbb{M}/A)$. Then consider

$$G_{Abc} \leqslant G_{Ab} \leqslant G_A, \quad G_{Abc} \leqslant G_{Ac} \leqslant G_A.$$

From the first sequence, we know G_{Abc} is of finite index in G_A, so in the second sequence, G_{Ac} is of finite index in G_A. □

Lemma 14.19. If $b \in \operatorname{acl}(A)$, then $\operatorname{acl}(Ab) = \operatorname{acl}(A)$.

Theorem 14.20. $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$.

Proof. If $c \in \operatorname{acl}(\operatorname{acl}(A))$, then $c \in \operatorname{acl}(b_1 \cdots b_n)$ for some $b_i \in \operatorname{acl}(A)$. Then

$$\operatorname{acl}(A) = \operatorname{acl}(Ab_1) = \operatorname{acl}(Ab_1b_2) = \cdots = \operatorname{acl}(Ab_1 \cdots b_n) \ni c.$$

□

Lemma 14.21. If $M \supseteq A \subseteq N$, and if id_A is a PEM, then $S_1^M(A) = S_1^N(A)$.

Proof. If $\overline{a} \in A^n, \varphi(x, y_1, \cdots, y_n)$ an $\mathcal{L}$-formula. By assumption,

$$M \models \exists x : \varphi(x, \overline{a}) \Leftrightarrow N \models \exists x : \varphi(x, \overline{a}).$$

So M, N satisfy the same partial types, and thus the same complete types. □

Lemma 14.22. If $f : N \supseteq A \to B \subseteq M$ is a small PEM, M is κ-saturated, then $\forall c \in N, \exists d \in M$ such that $g : A \cup \{c\} \to B \cup \{d\}, g(x) = f(x), \forall x \in A, g(c) = d, g$ is a PEM.

Proof. Move N, A by an isomorphism, so we can assume $A = B, f = \mathrm{id}_A$. Then $\mathrm{tp}(c/A) \in S_1^N(A) = S_1^M(A)$, take $d \in M, d \models \mathrm{tp}(c/A)$. Then $d \underset{A}{\equiv} c$, so g is a PEM. □

Theorem 14.23. If M is κ-saturated for 1-types $p(x)$, then M is κ-saturated.

Proof. Suppose $A \subseteq M$ small, $p(x_1, \cdots, x_n) \in S_n(A)$. Take $N \succeq M, \bar{b} \in N^n, \bar{b} \models p$, then $\mathrm{id}_A : N \supseteq A \to A \subseteq M$ is a PEM. Extend it to $A \cup \{b_1\} \to A \cup \{c_1\}$, then $Ab_1b_2 \to Ac_1c_2, \cdots$, so we get a PEM $Ab_1 \cdots b_n \to Ac_1 \cdots c_n$. So $\bar{b} \underset{A}{\equiv} \bar{c}, \bar{c} \models p, \bar{c} \in M^n$. □

Example 14.3. (1) If $(M, \leqslant) \models \mathrm{DLO}$, then M is $\aleph_0$-saturated. Suppose $A \subseteq_f M, p \in S_1(A), A = \{a_1, \cdots, a_n\}$, then p is determined by one of $x \in (-\infty, a_1), \{a_1\}, \cdots, \{a_n\}, (a_n, +\infty)$, so p is realized in M.

(2) $(\mathbb{C}, +, \cdot)$ is $2^{\aleph_0}$-saturated.

Proof. $A \subseteq \mathbb{C}, |A| < 2^{\aleph_0}, p \in S_1(A)$. $\mathcal{F} = \{\varphi(\mathbb{C}) | \varphi \in p\}$ closed under intersection, $\varnothing \notin \mathcal{F}$.

If some $D \in \mathcal{F}$ is finite, take $X = \min\{X \subseteq D | X \in \mathcal{F}\}$, then $X = \min \mathcal{F}$ non-empty. Take $b \in X$, then $b \in \bigcap \mathcal{F}$, so $b \models p$.

If every $D \in \mathcal{F}$ is cofinite, then $\bigcap \mathcal{F}$ is co-small, so $\bigcap \mathcal{F}$ is non-empty. □

Theorem 14.24 (κ-universality)**.** If M is κ-saturated, $N \equiv M, N$ small, then $\exists N \overset{\preceq}{\hookrightarrow} M$ $(N \cong N' \preceq M)$.

Proof. $\varnothing : N \supseteq \varnothing \to \varnothing \subseteq M$ is a PEM. By Zorn's lemma, we get $f : N \supseteq A \to B \subseteq M$ a maximal PEM. If $A = N$, then f is already an elementary embedding. If $A \subsetneq N$, take $c \in N \setminus A$, we can get a larger PEM, which contradicts maximality. □

Now suppose T complete, strongly minimal, $\mathbb{M} \models T$ a monster model.

Theorem 14.25. If $A \subseteq \mathbb{M}$, then TFAE:

(1) $A \preceq \mathbb{M}$.

(2) $A = \mathrm{acl}(A)$ and $|A| = \infty$.

Proof. (1) implies (2): $A \equiv \mathbb{M}, |\mathbb{M}| = \infty$, so $|A| = \infty$. Suppose D is a finite A-definable set, $D = \psi(\mathbb{M})$, then use as before $\exists^{=k} x : \psi(x)$, we can prove $D \subseteq A$.

(2) implies (1): Suppose D is A-definable, $D \subseteq \mathbb{M}, D \neq \varnothing$, we want $D \cap A \neq \varnothing$.

If $|D| < \infty$, then $D \subseteq \mathrm{acl}(A) = A$. If $|D| = \infty$, then $|\mathbb{M} \setminus D| < \infty$, so $D \cap A \neq \varnothing$. □

Lemma 14.26. If $A \subseteq_{small} M, b, c \notin \mathrm{acl}(A)$, then $b \underset{A}{\equiv} c$.

Proof. If not, then $b \in D, c \notin D$ for some A-definable set D. But D is finite implies $b \in \mathrm{acl}(A)$, D is cofinite implies $c \in \mathrm{acl}(A)$, always a contradiction. □

So $b, c \notin \mathrm{acl}(A)$ implies $\exists \sigma \in \mathrm{Aut}(\mathbb{M}/A), \sigma(b) = c$.

Lemma 14.27. If $b_1, \cdots, b_n, c_1, \cdots, c_n$ and $b_i \notin \mathrm{acl}(Ab_1 \cdots b_{i-1}), c_i \notin \mathrm{acl}(Ac_1 \cdots c_{i-1})$, then $\bar{b} \underset{A}{\equiv} \bar{c}$.

Proof. Induct on n. $n = 1$ is clear.

For example, $n = 4$. We have an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(b_1 b_2 b_3) = c_1 c_2 c_3$. Suppose $d = \sigma(b_4)$, so $d \notin \mathrm{acl}(Ac_1 c_2 c_3)$, then we have $d \equiv c_4$, so we can compose σ with another automorphism $\tau \in \mathrm{Aut}(\mathbb{M}/Ac_1 c_2 c_3)$. □

Lemma 14.28 (Exchange property)**.** If $b, c \notin \mathrm{acl}(A)$, then $b \in \mathrm{acl}(Ac)$ iff $c \in \mathrm{Aut}(Ab)$.

Proof. We say (b, c) independent over A if $b, c \notin \mathrm{acl}(A), c \notin \mathrm{acl}(Ab)$.

Let $(v_i)_{i \in I}$ be some small set with $|I| > |A| + |\mathcal{L}|$ (We make $\mathbb{M}$ sufficiently large). Let $u \notin \mathrm{acl}(A\bar{v})$. Since $|I| > |A| + |\mathcal{L}|$, there is some $v_i \notin \mathrm{acl}(Au)$, so $(u, v_i), (v_i, u)$ are both independent. So

$$(b, c) \underset{A}{\equiv} (u, v_i) \underset{A}{\equiv} (v_i, u) \underset{A}{\equiv} (c, b),$$

and $b \notin \mathrm{acl}(Ac)$. □

Definition 14.29. $I \subseteq \mathbb{M}$ is **independent** if $\forall b \in I$, we have $b \notin \mathrm{acl}(I \setminus \{b\})$.

Lemma 14.30. If I is independent, $b \notin \operatorname{acl}(I)$, then $I \cup \{b\}$ is independent.

Proof. If $c \in I' = I \cup \{b\}, c \in \operatorname{acl}(I' \setminus \{c\})$,

(1) $c = b$, then $b \in \operatorname{acl}(I)$, impossible.

(2) $c \neq b$, then $c \in \operatorname{acl}((I \cup \{b\}) \setminus \{c\}), c \notin \operatorname{acl}(I \setminus \{c\})$ by I independent, then exchange, so $b \in \operatorname{acl}(I)$, impossible. □

Lemma 14.31. If $I, J \subseteq \mathbb{M}$ independent, $f : I \to J$ bijection, then f is a PEM.

Proof. We may assume I, J finite, since we only need to show $\overline{a} \underset{\varnothing}{\equiv} \overline{b}$ whenever $f(\overline{a}) = \overline{b}$.

Then use the previous lemmas. □

Lemma 14.32. If $N \preceq \mathbb{M}, I$ a maximal independent subset of N, then $N = \operatorname{acl}(I)$.

Proof. $I \subseteq N$ implies $\operatorname{acl}(I) \subseteq \operatorname{acl}(N) = N$.

If $b \in N \setminus \operatorname{acl}(I)$, then $I \cup \{b\}$ is a larger independent set. □

Theorem 14.33. T is κ-categorical for $\kappa > |\mathcal{L}|$.

Proof. Take $M, N \models T, |M| = |N| = \kappa$. Then we can assume they are elementary substructures of $\mathbb{M}$. Let I, J be maximal independent subsets respectively. Then $|I| = |J| = \kappa$. Take $f : I \to J$ a bijection, then f is a PEM, so extend it to $\sigma \in \operatorname{Aut}(\mathbb{M}), \sigma(I) = J$, so $\sigma(\operatorname{acl}(I)) = \operatorname{acl}(J)$, i.e. $\sigma(M) = N$. So $\sigma|_M : M \to N$ is an isomorphism. □

15 December 22nd

15.1 o-minimality

15.2 main class