Chapter 4

Von Neumann Algebras

Problem 1. Let H be a separable Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. Prove that the relative weak topology on the closed unit ball S of $B(H)$ is metrizable by showing that the equation

$$d(u,v) = \sum_{n,m=1}^{\infty} \frac{|\langle (u-v)e_n, e_m\rangle|}{2^{n+m}}$$

defines a metric on S inducing the weak topology (WOT).

Solution. What we have defined is clearly a metric on S, and this metric is translation-invariant, i.e. $d(u,v) = d(u+w, v+w)$ if these operators lie in S. (Actually we can consider the metric on the open ball centered at 0 with radius 2 to get rid of some bothering details on "relative topology".)

We only need to prove that the open sets of two topologies coincide. Moreover, we only need to consider neighbourhoods of 0 by proceeding as follows:

If $V = \{u \in S : d(u,0) < \varepsilon\}$ is an open neighbourhood of 0 in the metric topology, then whenever $n+m > N$, given a fixed N such that $2^N > 1/\varepsilon$, V contains $\{u \in S : |\langle ue_n, e_m\rangle| < 1\}$, which is an open neighbourhood of 0 in the weak topology.

Conversely, suppose V is of the form $\{u \in S : |\langle ux_k, y_k\rangle| < c_k, \text{ for } k = 1, 2, \ldots, K\}$ where x_k, y_k are vectors in H (this is a neighbourhood basis of the weak topology at 0). First handle the case $K = 1$. We may expand $x_1 = \sum_{n=1}^{\infty} a_n e_n, y_1 = \sum_{m=1}^{\infty} b_m e_m$ and set $x = \sum_{n=1}^{N} a_n e_n, y = \sum_{m=1}^{N} b_m e_m$. We have $\|x\| \leqslant \|x_1\|, \|y\| \leqslant \|y_1\|$. Moreover, there is some N such that $\|x -$

$x_1\|, \|y - y_1\|$ are small enough that $|\langle ux_1, y_1\rangle - \langle ux, y\rangle| < c_1/2$, where we have used $u \in S$. Now,

$$\begin{aligned}\langle ux, y\rangle &= \langle\sum_{n=1}^{N} a_n u e_n, \sum_{m=1}^{N} b_m e_m\rangle \\ &= \sum_{n,m=1}^{N} a_n \overline{b_m} \langle ue_n, e_m\rangle.\end{aligned}$$

And let

$$R_1 = \min\{\frac{c_1}{2^{m+n+1} N^2 |a_n b_m|} : n, m = 1, \ldots, N\} > 0.$$

Here, if $a_n b_m = 0$, the term is viewed as ∞ and larger than any finite number. If R_1 is still ∞, we may take R_1 to be any positive number.

We claim that V contains $\{u \in S : d(u, 0) < R_1\}$. If $d(u, 0) < R_1$, the expression of R_1 ensures the following inequality: $|\langle ue_n, e_m\rangle| \leqslant 2^{n+m} d(u, 0)$, and thus

$$|a_n \overline{b_m} \langle ue_n, e_m\rangle| \leqslant \frac{c_1}{2N^2},$$

so $|\langle ux, y\rangle| \leqslant c_1/2$, and $|\langle ux_1, y_1\rangle| < c_1$. This is exactly what we have claimed.

Now for general K, we may find such R_k for each $k = 1, \ldots, K$, then let $R = \min\{R_k : k = 1, \ldots, K\}$. Then V contains $\{u \in S : d(u, 0) < R\}$.

Since we have proved that the each of neighbourhood bases of the two topologies is a refinement of the other one, the two topologies are the same.

There is something needed to point out. We can actually take a countable dense set in the unit ball of H and replace (e_n) with it. The proof of metrizability of WOT will be much easier.

Problem 2. Let H be a Hilbert space.

(a) Show that a weakly convergent sequence of operators on H is necessarily norm-bounded.

(b) Show that if (u_n) and (v_n) are sequences of operators on H converging strongly to the operators u and v, respectively, then $(u_n v_n)$ converges strongly to uv.

(c) Show that if (u_n) is a sequence of operators on H converging strongly to u, and if $v \in K(H)$, then $(u_n v)$ converges in norm to uv. Show that (vu_n) may not converge to vu in norm.

Solution. We first put a remark on (c). It is quite interesting that there is a misbelief that (vu_n) converges to vu in norm, even among great mathematicians. For example, in John B. Conway's book *A Course in Operator Theory* exercise 16.5 on page 81, he falsely states that (vu_n) converges to vu in norm (in our notation).

(a) Suppose (u_n) converges to u in WOT. Then for any $x \in H$, $u_n x$ converges to ux weakly. Viewing $u_n x$ as a bounded linear functional on H, for any $y \in H$, the sequence $\langle y, u_n x\rangle$ is convergent and is thus bounded. (Note that a convergent net may not be bounded.) So by Principle of Uniform Boundedness, $\{\|u_n x\| : n \geqslant 1\}$ is bounded. Then by Principle of Uniform Boundedness again, the sequence (u_n) is norm-bounded.

(b) By (a), the two sequences $(u_n), (v_n)$ are norm-bounded. So for any $x \in H$,

$$\begin{aligned}\|u_n v_n x - uvx\| &= \|(u_n v_n x - u_n vx) + (u_n vx - uvx)\| \\ &\leqslant \|u_n(v_n x - vx)\| + \|(u_n - u)vx\| \\ &\leqslant \|u_n\| \cdot \|(v_n - v)x\| + \|(u_n - u)vx\|\end{aligned}$$

which tends to zero when $n \to \infty$, since (u_n) is norm-bounded, and by SOT convergence, $v_n x \to vx, u_n vx \to uvx$ in norm. So $(u_n v_n)$ converges to uv in SOT.

(c) First consider the case when $v \in F(H)$ is a finite-rank operator. Then $\ker(v)^\perp$ is a finite-dimensional subspace of H, with a chosen orthonormal basis $(e_k)_{k=1}^m$. Then for any $x \in H$ with $\|x\| \leqslant 1$, write $x = \sum_{k=1}^m c_k e_k + x_0$, where

$x_0 \in \ker(v)$, and we have $\sum_{k=1}^{m} |c_k|^2 \leqslant \|x\|^2 \leqslant 1$. Then

$$\begin{aligned}\|u_n vx - uvx\| &= \|(u_n - u) \sum_{k=1}^{m} c_k v e_k\| \\ &= \| \sum_{k=1}^{m} c_k (u_n - u) v e_k\| \\ &\leqslant \sum_{k=1}^{m} \|(u_n - u) v e_k\|.\end{aligned}$$

By SOT convergence of (u_n) to u, $\|u_n v - uv\| = \sup\{\|u_n vx - uvx\| : x \in H, \|x\| \leqslant 1\}$ tends to zero as $n \to \infty$.

Now consider the general case when v is compact. By (a), (u_n) is norm-bounded, say $\|u_n\| < M$. For any $\varepsilon > 0$, take a finite-rank operator v' such that $\|v' - v\| < \varepsilon$, then

$$\|u_n v - uv\| \leqslant \|u_n(v' - v)\| + \|(u_n - u)v'\| + \|u(v' - v)\| \leqslant \varepsilon(M + \|u\|) + \|u_n v' - uv'\|.$$

This proves the general case.

For the last part about the famous misbelief, just consider the unilateral shift s and let $u_n = s^{*n}$. Then u_n converges to 0 in SOT. Let v be the projection onto $\mathbb{C}e_1$, i.e. the kernel of s^*. Then a simple computation shows that $\|vu_n\| = 1$, so (vu_n) cannot converge to 0 in norm.

Problem 3. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$.

(a) Denote by Λ the set of all pairs (n, U) where n is a positive integer, and U is a neighbourhood of 0 in the strong topology of $B(H)$. For (n, U) and (n', U') in Λ, write $(n, U) \leqslant (n', U')$ if $n \leqslant n'$ and $U' \subseteq U$. Show that Λ is a poset under the relation $\leqslant$, and that it is upwards-directed.

(b) Let u denote the unilateral shift on (e_n), and note that (u^{*n}) is strongly convergent to zero. If $\lambda = (n_\lambda, U_\lambda) \in \Lambda$, then $\lim_{n\to\infty}(n_\lambda u^{*n}) = 0$ in the strong topology, so for some n we have $n_\lambda u^{*n} \in U_\lambda$. Set $u_\lambda = n_\lambda u^{*n}$ and $v_\lambda = \frac{1}{n_\lambda} u^n$. Show that $\lim_\lambda u_\lambda = 0$ in the strong topology and $\lim_\lambda v_\lambda = 0$ in the norm topology. Since $u_\lambda v_\lambda = 1$, this shows that the operation of multiplication

$$B(H) \times B(H) \to B(H), \quad (u, v) \mapsto uv,$$

is not jointly continuous in either the weak or the strong topologies.

(c) Show that neither the weak nor the strong topologies on $B(H)$ are metrizable, using Problem 2 and the nets (u_λ) and (v_λ) from part (b) of this problem.

Solution. (a) “Poset” is an abbreviation for partially ordered set. This part is obvious.

(b) For any $x \in H$, write $x = \sum_{m=1}^{\infty} c_m e_m$. Then $\sum_{m=1}^{\infty} |c_m|^2 = \|x\|^2 < \infty$. A computation by definition shows that $u^{*n}x = \sum_{m=n+1}^{\infty} c_m e_m$, so $\|u^{*n}x\| = \sum_{m=n+1}^{\infty} |c_m|^2$ tends to zero as $n \to \infty$. This shows that (u^{*n}) converges to zero in SOT.

Since multiplying by a fixed scalar is SOT continuous, clearly $n_\lambda u^{*n}$ converges to zero in SOT, and there is some n such that $n_\lambda u^{*n} \in U_\lambda$.

For any neighbourhood U of 0 in SOT topology, if $\lambda \geqslant (1, U)$, $u_\lambda \in U_\lambda \subseteq U$, so (u_λ) converges to zero in SOT. Note that $\|v_\lambda\| = \frac{1}{n_\lambda}$, so given any $\varepsilon > 0$, there is some positive integer $N > 1/\varepsilon$, and if $\lambda \geqslant (N, H)$, $\|v_\lambda\| < \varepsilon$, so (v_λ) converges to zero in the norm topology.

Note that part (a) and (b) works well if Λ is replaced by the set of (n, U) where U is a WOT neighbourhood of 0. We will abuse the notation using Λ for this net, and use this substitution in part (c).

(c) If either of the topologies, denoted by $\mathcal{T}$, is metrizable by a metric d, then the net convergence may be substituted with the sequential convergence. To be more precise, we can choose open neighbourhoods of 0 of the form $U_n :=$

$\{u \in B(H) : d(u, 0) < 1/n\}$ with $n \to \infty$. Let $\lambda_n = (n, U_n) \in \Lambda$, and $u_n := u_{\lambda_n}, v_n := v_{\lambda_n}$. Then (u_n) converges to 0 in the topology $\mathcal{T}$ and (v_n) converges to 0 in the norm topology. Recall that $\mathcal{T}$ is either WOT or SOT, so by Problem 2(a), (u_n) is norm-bounded, so $(u_n v_n)$ converges to zero in the norm topology, which is absurd. Therefore, neither WOT or SOT on $B(H)$ is metrizable.

Problem 4. Let A be a von Neumann algebra on a Hilbert space H, and suppose that τ is a bounded linear functional on A. We say that τ is *normal* if, whenever an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ in A_{sa} converges strongly to an operator $u \in A_{sa}$, we have $\lim_\lambda \tau(u_\lambda) = \tau(u)$. Show that every σ-weakly continuous functional $\tau \in A^*$ is normal.

Solution. Suppose an increasing net $(u_\lambda) \subseteq A_{sa}$ converges to $u \in A_{sa}$ in SOT, then fix any $\mu \in \Lambda$, whenever $\lambda \geqslant \mu$, we have $u_\lambda \geqslant u_\mu$. On the other hand, $u_\lambda x$ converges to ux for any $x \in H$, and $(\langle u_\lambda x, x\rangle)$ is an increasing net of real numbers for every $x \in H$, so $\langle u_\lambda x, x\rangle \leqslant \langle ux, x\rangle$. This shows that whenever $\lambda \geqslant \mu$, $u_\mu \leqslant u_\lambda \leqslant u$, so $\|u_\lambda\| \leqslant \max(\|u_\mu\|, \|u\|)$.

So when considering such a net (u_λ), we can assume here that (u_λ) is norm-bounded. Recall that on norm-bounded sets of $B(H)$, WOT coincides with σ-weak topology, and since SOT is stronger than WOT, we have $\lim_\lambda \tau(u_\lambda) = \tau(u)$.

The hint given by the author is strange. He said to write $\tau(v) = \mathrm{tr}(uv)$ for some trace-class operator u, and try to prove that $\lim_\lambda \|v_\lambda u - vu\|_1 = 0$. I cannot prove this, but I also find another way, which is quite different, not using the explicit formula for τ.

Problem 5. Let H be a non-zero Hilbert space.

(a) Show that the extreme points of the closed unit ball of H are precisely the unit vectors.

(b) Deduce that the isometries and co-isometries of $B(H)$ are extreme points of the closed unit ball of $B(H)$. (It can be shown that these are all of the extreme points. This follows from [Tak, Theorem I.10.2].)

Solution. The reference the author mentioned is M. Takesaki, *Theory of Operator Algebras*, Vol.1.

(a) If x is an extreme point of the closed unit ball of H, then $x \neq 0$, and $x = (1 - \|x\|)0 + \|x\| \cdot \frac{x}{\|x\|}$, which shows that $\|x\| = 1$.

Conversely, given $\|x\| = 1$, suppose $x = ty + (1-t)z$ for some y, z in the closed unit ball of H and $t \in (0, 1)$. Then

$$1 = \langle x, x\rangle = t\langle x, y\rangle + (1-t)\langle x, z\rangle.$$

However, $\langle x, y\rangle, \langle x, z\rangle$ are complex numbers with modulus $\leqslant 1$, so $\langle x, y\rangle = \langle x, z\rangle = 1$. Then

$$\langle x - y, x - y\rangle = \langle x, x\rangle + \langle y, y\rangle - \langle x, y\rangle - \langle y, x\rangle = 0.$$

This shows that $x = y$, so x is an extreme point of the closed unit ball of H.

(b) If u is an isometry and $u = tv + (1-t)w$, where $t \in (0, 1), \|v\| \leqslant 1, \|w\| \leqslant 1$. Then for any $x \in H$ with $\|x\| = 1$, $ux = tvx + (1-t)wx$. Note that $\|ux\| = 1, \|vx\| \leqslant 1, \|wx\| \leqslant 1$, so by (a), we have $vx = ux = wx$. This shows that $u = v = w$, so u is an extreme point of the closed unit ball of $B(H)$.

If u is a co-isometry, then u^* is an isometry, and we proceed as above.

Problem 6. Let A be a C*-algebra.

(a) Show that if A is unital, then its unit is an extreme point of its closed unit ball.

(b) If p is a projection of A, show that it is an extreme point of the closed unit ball of A^+. The converse of this result if also true, but more difficult. If follows from [Tak, Lemma I.10.1].

(c) Show that if H is an infinite-dimensional Hilbert space, then the closed unit ball of $B(H)^+$ is not the convex hull of the projections of $B(H)$.

Solution. (a) Suppose $1 = ta + (1-t)b$ for $t \in (0,1)$ and $\|a\| \leqslant 1, \|b\| \leqslant 1$. Set $a' = (a+a^*)/2, b' = (b+b^*)/2$, then $1 = ta' + (1-t)b'$, $\|a'\| \leqslant 1, \|b'\| \leqslant 1$ and a', b' are self-adjoint. So $\sigma(ta') \subseteq [-t, t]$, and $(1-t)b' = 1 - ta'$ gives that $\sigma((1-t)b') \subseteq [1-t, 1+t]$. But $\sigma((1-t)b') \subseteq [t-1, 1-t]$, so $\sigma((1-t)b') = \{1-t\}$. Being self-adjoint, $(1-t)b' = 1-t$. This shows that $a' = b' = 1$ and a is of the form $a = 1 + ix$ where x is self-adjoint. Then $a^*a = 1 + x^2$ has norm $\leqslant 1$. By a similar argument on the spectrum of $1 + x^2$, we have $x = 0$. This proves that 1 is an extreme point of the closed unit ball of A.

(b) Suppose $p = ta + (1-t)b$ where $t \in (0,1)$ and a, b are in the closed unit ball of A^+. Then p is the unit of pAp, and $p = tpap + (1-t)pbp$. By (a), we have $pap = pbp = p$.

The problem now involves a tricky but quite common skill. Now $p(1-a)p = 0$ and $1 - a \geqslant 0$, so let $c = (1-a)^{1/2}p$, then $c^*c = 0$. Hence, $(1-a)^{1/2}p = 0$, and $(1-a)p = (1-a)^{1/2}(1-a)^{1/2}p = 0$. So $p = ap$. Now replace p, a, b with $1-p, 1-a, 1-b$, since clearly adding a unit to A does not change anything. Then by a similar argument on $(1-p)\tilde{A}(1-p)$, we have $1 - p = (1-a)(1-p)$. This shows that $a = ap$, so $a = p$, and certainly $b = p$. Therefore, p is an extreme point of the closed unit ball of A^+, and actually, we have also proved that p is an extreme point of the closed unit ball of $\tilde{A}^+$.

Moreover, the converse of this result is also quite easy. If a is an extreme point of the closed unit ball of A^+, then $\sigma(a) \subseteq [0,1]$. Consider the following:

$$a = \frac{1}{2}a^2 + \frac{1}{2}(2a - a^2).$$

Note that a^2 and $2a - a^2$ both lie in the closed unit ball of A^+ (one can prove this using continuous functional calculus), so by the definition of extreme points, $a = a^2$, which shows that a is a projection.

(c) First some remarks on this result. By Krein-Milman theorem, a compact convex subset of a Hausdorff locally convex topological vector space is equal to the *closed* convex hull of its extreme points. Note that every von Neumann algebra, certainly including $B(H)$, has a natural structure as a dual space, i.e. the dual space of its pre-dual with the ultraweak topology coinciding with the weak* topology. So, the closed unit ball of $B(H)^+$ is the closed (under ultraweak topology) convex hull of the projections of $B(H)$.

Let $(e_n)_{n=1}^{\infty}$ be an orthonormal family of vectors in H, and $x = \sum_{n\geqslant 1} \frac{1}{n} e_n \otimes e_n$. If $x = t_1 p_1 + \ldots + t_m p_m$ is a convex combination of projections with t_i positive, then all p_i are dominated by some compact positive operators ($p_i \leqslant t_i^{-1} x$), so they are compact themselves. Since compact projections must be of finite rank, x is of finite rank, as a finite sum of finite-rank operators, which is a contradiction. Therefore, x cannot be written as a convex combination of projections in $B(H)$ and clearly x is in the closed unit ball of $B(H)^+$.

Problem 7. Let A be a C*-algebra. Show that if p, q are (Murray-von Neumann) equivalent projections in A, and r is a projection orthogonal to both (that is, $rp = rq = 0$), then the projections $r + p$ and $r + q$ are equivalent.

If H is a separable Hilbert space and p is a projection not of finite rank, set $\mathrm{rank}(p) = \infty$. If p has finite rank, set $\mathrm{rank}(p) = \dim p(H)$. Show that $p \sim q$ in $B(H)$ if and only if $\mathrm{rank}(p) = \mathrm{rank}(q)$.

Thus, the equivalence class of a projection in a C*-algebra can be thought of as its "generalized rank".

We say a projection p in a C*-algebra A is *finite* if for any projection q such that $q \sim p$ and $q \leqslant p$ we necessarily have $q = p$. Otherwise, the projection is said to be *infinite*. Show that if p, q are projections such that $q \leqslant p$ and p is finite, then q is finite.

A projection p in a von Neumann algebra A is *abelian* if the algebra pAp is abelian. Show that abelian projections are finite.

A von Neumann algebra is said to be *finite* or *infinite* according as its unit is a finite of infinite projection. If H is a Hilbert space, show that the von Neumann algebra $B(H)$ is finite or infinite according as H is finite- or infinite-dimensional.

Solution. Suppose $p = u^*u, q = uu^*$. We claim that $r + p = (r + u)^*(r + u)$. In fact,

$$(r + u)^*(r + u) = r^*r + u^*r + r^*u + u^*u = r + p + u^*r + ru,$$

and $(u^*r)^*(u^*r) = rqr = 0$, so $u^*r = 0$. Similarly, $r + q = (r + u)(r + u)^*$, so $r + p$ and $r + q$ are equivalent.

Actually, we can replace r with projections $p' \sim q'$ which satisfy $p' \perp p, q' \perp q$, then $(p + p') \sim (q + q')$.

If $p = u^*u, q = uu^*$, then u is a partial isometry, and p is the projection onto $\ker(u)^\perp$, q is the projection onto $\ker(u^*)^\perp$. Since u is a partial isometry, u gives a unitary equivalence from $\ker(u)^\perp$ to $\mathrm{ran}(u) = \ker(u^*)^\perp$, so p, q have the same rank. Conversely, since all separable infinite-dimensional Hilbert spaces are unitarily equivalent, there is a unitary operator u from pH onto qH. Then we extend u by zero on $(pH)^\perp$ and include qH into H, and we get a partial isometry, still denoted by u. Then $u^*u = p, uu^* = q$.

If a projection r satisfies $r \sim q, r \leqslant q$, then $r \leqslant q \leqslant p$, so $(p-q)$ is a projection orthogonal to r and q. By the first paragraph, $(r + p - q) \sim (q + p - q) = p$. Since p is finite, and $r + p - q \leqslant p$, so $r + p - q = p$, i.e. $q = r$. This shows that q is finite.

If a projection p is abelian, and a projection $q \leqslant p, q \sim p$, then pAp is a hereditary C*-subalgebra of A containing p, so $q \in pAp$. Moreover, if we write $p = u^*u, q = uu^*$, then $up = u, qu = u$. Since $q \leqslant p$, we have $pq = q$, so $pu = pqu = qu = u$. Therefore, $pup = up = u \in pAp$. This shows that u commutes with u^*, so $p = q$, i.e. p is finite.

If H is finite-dimensional, suppose $p \leqslant 1, p \sim 1$. Then $\text{rank}(p) = \text{rank}(1) = \dim H$, so p is surjective. By considering the dimension of the kernel and the range of p, we know that p must be injective, thus isometric. This shows that $p = p^2 = p^*p = 1$, i.e. 1 is a finite projection.

If H is infinite-dimensional, then by a well-known theorem in set theory, there is a closed proper subspace $K \subseteq H$ such that $\dim K = \dim H$ (even if H is not separable). Let p be the projection onto K, then $q \leqslant 1, q \sim 1$ but $q \neq 1$, so 1 is infinite.