VI. 1 and VI. 6

VI. 1 Topologies on bounded operators

For Banach spaces X, Y, we denote the set of all bounded linear operators from X to Y by $B(X,Y)$ (or $\mathcal{L}(X,Y)$). When $X = Y = \mathcal{H}$ for some separable Hilbert space $\mathcal{H}$, we also denote $B(\mathcal{H}) = B(\mathcal{H},\mathcal{H})$ (or $\mathcal{L}(\mathcal{H})$).

$B(X,Y)$ is a Banach space under the operator norm

$$\|T\| = \sup_{\|x\|=1} \frac{\|Tx\|_Y}{\|x\|_X}.$$

The corresponding norm topology is also called the **uniform operator topology**.

There are two more topologies on $B(X,Y)$: the weak operator topology and the strong operator topology.

(1) The strong operator topology (SOT) is the weakest topology on $B(X,Y)$ such that the maps $E_x \colon B(X,Y) \to Y, T \mapsto Tx$ are continuous for all $x \in X$. Equivalently, it is determined by a separating family of seminorm $\{T \mapsto \|Tx\|\}_{x\in X}$. A neighbourhood basis at 0 is given by the collection of sets of the form

$$U(x_1,\ldots,x_n;\varepsilon) := \{T \in B(X,Y) : \|Tx_i\|_Y < \varepsilon, i = 1,\ldots,n\}$$

where $\{x_i\}_{i=1}^n$ is a finite collection of vectors in X, and $\varepsilon > 0$.

When considering convergence in this topology, we have to consider net convergence. A net in $B(X,Y)$ is a map from a directed set Λ to $B(X,Y)$, usually denoted by $(T_\lambda)_{\lambda\in\Lambda}$. We say the net (T_λ) converges to T in the given topology if for any neighbourhood U of T, there exists some $\mu \in \Lambda$ such that for any $\lambda \in \Lambda$ with $\lambda \geqslant \mu$, we have $T_\lambda \in U$. In the case of the strong operator topology, $T_\lambda \xrightarrow{s} T$ if and only if for any $x \in X$, we have $T_\lambda x \to Tx$.

Remark 1. Norm convergent net may not be bounded.

The multiplication map $B(X,Y) \times B(Y,Z) \to B(X,Z), (S,T) \mapsto TS$ is obviously separately continuous, but not jointly continuous.

(2) The weak operator topology (WOT) is the weakest topology on $B(X,Y)$ such that the maps $E_{x,l} \colon B(X,Y) \to \mathbb{C}, T \mapsto l(Tx)$ are continuous for all $x \in X, l \in Y^*$.

The multiplication map is also separately continuous, but not jointly continuous. (Both examples are provided in the last of this file.)

Obviously the norm topology is stronger than the strong operator topology, which is stronger than the weak operator topology.

Example 2. Suppose $\mathcal{H} = l^2(\mathbb{N})$ and consider $B(\mathcal{H})$.

(1) Define T_n by

$$T_n(c_1, c_2, \ldots) = (\frac{1}{n}c_1, \frac{1}{n}c_2, \ldots).$$

Then $T_n \to 0$ in the norm topology.

(2) Define S_n by

$$S_n(c_1, c_2, \ldots) = (0, \ldots, 0, c_{n+1}, c_{n+2}, \ldots),$$

where the first n coordinates of the image are zero. Then $S_n \to 0$ in the strong operator topology but not in the norm topology.

(3) Define W_n by

$$W_n(c_1, c_2, \ldots) = (0, \ldots, 0, c_1, c_2, \ldots),$$

where the first n coordinates of the image are zero. Then $W_n \to 0$ in the weak operator topology but not in the strong operator topology.

To see this, we first observe that $W_n^*(c_1, c_2, \ldots) = (c_{n+1}, c_{n+2}, \ldots)$, so $W_n^* \to 0$ in the strong operator topology. Then it is clear that for all $x, y \in \mathcal{H}$, we have

$$\langle W_n x, y\rangle = \langle x, W_n^* y\rangle \to 0.$$

On the other hand, W_n is an isometry.

Remark 3. This example also shows that the involution is not continuous with respect to the strong operator topology.

Theorem 4. (1) Let $B(\mathcal{H})$ denote the bounded operators on a Hilbert space $\mathcal{H}$. Let T_n be a sequence of bounded operators and suppose that $\langle T_n x, y\rangle$ converges as $n \to \infty$ for each $x, y \in \mathcal{H}$. Then there exists $T \in B(\mathcal{H})$ such that $T_n \xrightarrow{w} T$.

(2) If a sequence of bounded operators T_n has the property that $T_n x$ converges for each $x \in \mathcal{H}$, then there exists $T \in B(\mathcal{H})$ such that $T_n \xrightarrow{s} T$.

Proof. (1) Let $Q(x, y) = \lim_{n\to\infty}\langle T_n x, y\rangle \quad (x, y \in \mathcal{H})$ be a sesquilinear form. For any fixed $x \in \mathcal{H}$ and $n \geqslant 1$, the maps $y \mapsto \langle T_n x, y\rangle$ are bounded linear functionals on $\mathcal{H}$ with norm $\|T_n x\|$. By the convergence of $\langle T_n x, y\rangle$, the set $\{\langle T_n x, y\rangle : n \geqslant 1\}$ is bounded for every $y \in \mathcal{H}$. By the uniform boundedness principle, $\{\|T_n x\|\}$ is a bounded set of numbers. Then by the uniform boundedness principle again, $\{\|T_n\|\}$ is a bounded set. Therefore, $Q(x, y)$ is a bounded sesquilinear form, and thus there exists a bounded linear operator T such that $Q(x, y) = \langle Tx, y\rangle$.

(2) Quite similar. Let T be the linear operator $Tx = \lim_{n\to\infty} T_n x$. By the uniform boundedness principle, T is bounded. $\square$

Example 5. The range space of a bounded operator may not be closed. Consider $\mathcal{H} = l^2(\mathbb{N})$. Define a bounded operator T by

$$T(c_1, c_2, \dots) = (c_1, \frac{c_2}{2}, \dots, \frac{c_n}{n}, \dots).$$

Then $\|T\| = 1$ and T is injective. Also note that the range space of T contains all compactly supported sequences, so ran T is dense in $\mathcal{H}$. But T cannot be surjective, otherwise T is a bijective bounded linear operator, so by the open mapping theorem (or the inverse operator theorem, anything equivalent), T^{-1} has to be bounded and T has to be bounded below, which contradicts the construction of T.

VI. 6 The trace class and Hilbert-Schmidt ideals

In this section, all Hilbert spaces are separable.

Theorem 6. Let $\mathcal{H}$ be a separable Hilbert space, $(e_n)_{n=1}^{\infty}$ an orthonormal basis. Then for any positive operator $A \in B(\mathcal{H})$ we define

$$\text{tr } A = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle.$$

The number tr $A \in [0, \infty]$ is called the **trace** of A and is independent of the orthonormal basis chosen. The trace has the following properties:

(1) $\text{tr } (A + B) = \text{tr } A + \text{tr } B$.

(2) $\text{tr}(\lambda A) = \lambda \text{tr } A$ for all $\lambda \geqslant 0$. Here $0 \cdot \infty = 0$.

(3) $\text{tr}(UAU^{-1}) = \text{tr } A$ for any unitary operator U.

(4) If $0 \leqslant A \leqslant B$, then $\text{tr } A \leqslant \text{tr } B$.

Proof. If $(f_m)_{m=1}^{\infty}$ is another orthonormal basis, then

$$\begin{aligned}
\text{tr } A = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle &= \sum_{n=1}^{\infty} \|A^{1/2} e_n\|^2 \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle A^{1/2} e_n, f_m \rangle|^2 \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle e_n, A^{1/2} f_m \rangle|^2 \\
&= \sum_{m=1}^{\infty} \|A^{1/2} f_m\|^2 \\
&= \sum_{m=1}^{\infty} \langle Af_m, f_m \rangle.
\end{aligned}$$

Here, we have used the existence of (positive) square roots of a positive operator, the Parseval identity with respect to an orthonormal basis and interchanged the order of the positive series. So the value of the trace of a positive operator is independent of the choice of the orthonormal basis.

(1) It is obvious.

(2) It is also obvious.

(3) Note that for unitary operators U, we have

$$\langle UAU^{-1} e_n, e_n \rangle = \langle AU^{-1} e_n, U^{-1} e_n \rangle.$$

The set $(U^{-1} e_n)_{n=1}^{\infty}$ forms an orthonormal basis of $\mathcal{H}$.

(4) For each n, we have $\langle Ae_n, e_n \rangle \leqslant \langle Be_n, e_n \rangle$. $\square$

Definition 7. An operator $A \in B(\mathcal{H})$ is called **trace-class** if and only if tr $|A| < \infty$. The family of all trace class operators is denoted by $B_1(\mathcal{H})$.

Theorem 8. $B_1(\mathcal{H})$ is a *-ideal in $B(\mathcal{H})$. This consists of three non-trivial parts:

(1) If $A, B \in B_1(\mathcal{H})$, then $A + B \in B_1(\mathcal{H})$.
(2) If $A \in B_1(\mathcal{H}), B \in B(\mathcal{H})$, then $AB, BA \in B_1(\mathcal{H})$.
(3) If $A \in B_1(\mathcal{H})$, then $A^* \in B_1(\mathcal{H})$.

Proof. Clearly $B_1(\mathcal{H})$ is closed under scalar multiplication.

(1) For $A, B \in B_1(\mathcal{H})$, let U, V, W be the partial isometries given by the polar decompositions

$$A + B = U|A + B|, A = V|A|, B = W|B|.$$

For any positive integer N,

$$\begin{aligned}
\sum_{n=1}^{N}\langle|A+B|e_n, e_n\rangle &= \sum_{n=1}^{N}\langle U^*(A+B)e_n, e_n\rangle \\
&\leqslant \sum_{n=1}^{N}\langle U^*V|A|e_n, e_n\rangle + \sum_{n=1}^{N}\langle U^*W|B|e_n, e_n\rangle \\
&\leqslant \sum_{n=1}^{N}\langle|A|^{1/2}e_n, |A|^{1/2}V^*Ue_n\rangle + \sum_{n=1}^{N}\langle|B|^{1/2}e_n, |B|^{1/2}W^*Ue_n\rangle \\
&\leqslant (\sum_{n=1}^{N}\||A|^{1/2}e_n\|^2)^{1/2}(\sum_{n=1}^{N}\||A|^{1/2}V^*Ue_n\|^2)^{1/2} + \\
&\quad (\sum_{n=1}^{N}\||B|^{1/2}e_n\|^2)^{1/2}(\sum_{n=1}^{N}\||B|^{1/2}W^*Ue_n\|^2)^{1/2} \\
&\leqslant (\text{tr }|A|)^{1/2}\cdot(\sum_{n=1}^{N}\||A|^{1/2}V^*Ue_n\|^2)^{1/2} + \\
&\quad (\text{tr }|B|)^{1/2}\cdot(\sum_{n=1}^{N}\||B|^{1/2}W^*Ue_n\|^2)^{1/2}.
\end{aligned}$$

Here we have used $U^*A = |A|$ (see 25) and Cauchy-Schwarz inequality. Note that

$$\sum_{n=1}^{N}\||A|^{1/2}V^*Ue_n\|^2 = \text{tr }(U^*V|A|V^*U).$$

Use an orthonormal basis from $\ker U \oplus (\ker U)^\perp$ and by the independence of the chosen orthonormal basis, we have $\text{tr}(U^*V|A|V^*U) \leqslant \text{tr }(V|A|V^*)$. Again from an orthonormal basis from $\ker V^* \oplus (\ker V^*)^\perp$, we have $\text{tr }(V|A|V^*) \leqslant \text{tr }|A|$. Therefore, $\text{tr }|A+B| \leqslant \text{tr }|A| + \text{tr }|B|$.

(2) Write B as a linear combination of four unitaries, then we only need to prove for the case when $B = U$ is unitary.

Note that $|UA| = (A^*U^*UA)^{1/2} = |A|$, and $|AU| = (U^*A^*AU)^{1/2} = U^*|A|U$, so $UA, AU \in B_1(\mathcal{H})$ by part (3) of the preceding theorem.

(3) We prove a more general result. If I is an ideal of $B(\mathcal{H})$, then I is closed under involution (adjoint).

Suppose $A \in I$, then write $A = U|A|$ where U is a partial isometry, by the polar decomposition of A. Note that the theorem of polar decomposition (see Theorem VI.10) actually tells more.

Here we use the fact that $U^*A = |A|$ (see 25). So $|A| \in I$, since I is an ideal. Therefore, $A^* = |A|U^*$ lies in I. □

Remark 9. (1) One may be confused about the expression of $|AU|$. To verify this, recall that the square root $(A^*A)^{1/2}$ can be expressed as a series (see Theorem VI. 9), which is the norm limit of a sequence of polynomials in A^*A. Since one can easily verify that $U^*(A^*A)^nU = (U^*A^*AU)^n$ for all positive integers n, one can then pass to the limit.

By using such an argument, one can prove the following result with the help of the Stone-Weierstrass theorem:

If A is a self-adjoint operator, U is a unitary operator, then for any continuous function f on the spectrum of A, we have $U^*f(A)U = f(U^*AU)$.

(2) Generally, in a C*-algebra, an ideal may not be a *-ideal. The proof here is valid for von Neumann algebras.

Theorem 10. Let $\|\cdot\|_1$ be defined on $B_1(\mathcal{H})$ by $\|A\|_1 := \mathrm{tr}\ |A|$. Then $B_1(\mathcal{H})$ is a Banach space with norm $\|\cdot\|_1$ and $\|A\| \leqslant \|A\|_1$.

Proof. Clearly it is a normed vector space. For any unit vector $x \in \mathcal{H}$, extend $\{x\}$ to an orthonormal basis $(e_n)_{n=1}^{\infty}$ of $\mathcal{H}$ where $e_1 = x$. Then

$$\mathrm{tr}\ |A| = \sum_{n=1}^{\infty} \||A|^{1/2}e_n\|^2.$$

So $\||A|^{1/2}x\|^2 \leqslant \|A\|_1$. Take supremum over $\|x\| \leqslant 1$, then $\||A|^{1/2}\|^2 \leqslant \|A\|_1$. Note that $\||A|^{1/2}\|^2 = \||A|\| = \|(A^*A)^{1/2}\| = \|A^*A\|^{1/2} = \|A\|$.

To prove that $B_1(\mathcal{H})$ is complete under the trace class norm, first suppose that $(A_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\|\cdot\|_1$. Then this sequence is also Cauchy in the operator norm, and has a limit A in $B(\mathcal{H})$. Then $|A_n|$ also converges to $|A|$ in the operator norm (see 26). Then we have

$$\sum_{n=1}^{N} \langle |A|e_n, e_n\rangle \leqslant \sup\{\|A_n\|_1 : n \geqslant 1\} < \infty$$

for all N, so $A \in B_1(\mathcal{H})$.

The last thing is to prove that A is actually the limit of (A_n) in $\|\cdot\|_1$. For any $\varepsilon > 0$, there exists some N such that whenever $m, n > N$, we have

$$\operatorname{tr} |A_n - A_m| = \sum_{k=1}^{\infty} \langle |A_n - A_m| e_k, e_k \rangle < \varepsilon.$$

Then by Fatou's lemma (or take partial sums and then pass to the limit), we have $\operatorname{tr} |A_n - A| \leqslant \varepsilon$ whenever $n > N$. This completes the proof. □

Remark 11. This is almost the same as the proof of the completeness of $l^1(\mathbb{N})$.

Theorem 12. Every $A \in B_1(\mathcal{H})$ is compact. A compact operator A is in $B_1(\mathcal{H})$ if and only if $\sum_{n=1}^{\infty} \lambda_n < \infty$ where $(\lambda_n)_{n=1}^{\infty}$ are the singular values of A.

Proof. Suppose $A \in B_1(\mathcal{H})$, then $|A|^2 = A^*A \in B_1(\mathcal{H})$ is positive. For any orthonormal basis $(e_n)_{n=1}^{\infty}$, we have

$$\operatorname{tr} |A|^2 = \sum_{n=1}^{\infty} \langle |A|^2 e_n, e_n \rangle = \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

Let $A_N = \sum\limits_{n=1}^{N} \langle \cdot, e_n \rangle A e_n$ be finite-rank operators, then $A_N - A$ is zero on $\operatorname{span}(e_1, \cdots, e_N)$. For $x \in \mathcal{H}$, write $x = x_1 + x_2$ where $x_1 \in \operatorname{span}(e_1, \cdots, e_N)$, $x_2 \perp \operatorname{span}(e_1, \cdots, e_N)$. Then $\|x_2\| \leqslant \|x\|$ and $(A - A_N)x = Ax_2$. So $\|A - A_N\| = \sup\{\|Ax_2\| : \|x_2\| = 1, x_2 \perp \operatorname{span}(e_1, \cdots, e_N)\}$. Now suppose x_2 is a unit vector with these conditions.

Extend $e_1, \cdots, e_N, x_2$ to an orthonormal basis of $\mathcal{H}$, then we have

$$\|Ax_2\|^2 \leqslant \operatorname{tr} |A|^2 - \sum_{n=1}^{N} \|Ae_n\|^2 = \sum_{n=N+1}^{\infty} \|Ae_n\|^2 \to 0, \quad N \to \infty.$$

Therefore, A is the operator norm limit of A_N, and is thus compact.

Now suppose A is compact. By Theorem VI.17, there exist two orthonormal sets $(e_n)_{n=1}^{N}, (f_n)_{n=1}^{N}$ and positive real numbers $(\lambda_n)_{n=1}^{N}$ with $\lambda_n \to 0$ such that

$$A = \sum_{n=1}^{N} \lambda_n \langle \cdot, e_n \rangle f_n$$

is the operator norm limit (N can be ∞). In the proof of Theorem VI.17, it is proved that $(\lambda_n)_{n=1}^{N}$ are precisely the non-zero eigenvalues of $|A|$, and the corresponding eigenvectors are $(e_n)_{n=1}^{N}$. Extend $(e_n)_{n=1}^{N}$ to an orthonormal basis of $\mathcal{H}$, then we find that

$$\operatorname{tr} |A| = \sum_{n=1}^{N} \lambda_n.$$

□

Corollary 13. The finite-rank operators are $\|\cdot\|_1$-dense in $B_1(\mathcal{H})$.

Proof. If F is a finite-rank operator, then $|F|$ is a finite-rank operator (also by the polar decomposition, see 25). Extend an orthonormal basis of $\operatorname{ran}|F|$ to an orthonormal basis of $\mathcal{H}$, and we see that $F \in B_1(\mathcal{H})$.

For a trace class operator $A \in B_1(\mathcal{H})$, it has to be compact. By Theorem VI.17, we have (if we allow some λ_n's to be zero)

$$A = \sum_{n=1}^{\infty} \lambda_n \langle \cdot, e_n \rangle f_n.$$

As in the proof of Theorem VI.17, we have $\lambda_n f_n = Ae_n$. Construct A_N as in the first paragraph of the proof of the preceding theorem, we have

$$A_N = \sum_{n=1}^{N} \lambda_n \langle \cdot, e_n \rangle f_n.$$

Since $A \in B_1(\mathcal{H})$, we have $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$. By the proof of the preceding theorem,

$$\|A_N - A\|_1 = \sum_{n=N+1}^{\infty} \lambda_n \to 0, \quad N \to \infty.$$

□

Definition 14. An operator $T \in B(\mathcal{H})$ is called **Hilbert-Schmidt** if and only if $\mathrm{tr}\ T^*T < \infty$. The family of all Hilbert-Schmidt operators is denoted by $B_2(\mathcal{H})$.

Clearly $A \in B_1(\mathcal{H})$ if and only if $|A|^{1/2} \in B_2(\mathcal{H})$. One can immediately compute that

$$\mathrm{tr}\ T^*T = \sum_{n=1}^{\infty} \|Te_n\|^2$$

for any orthonormal basis $(e_n)_{n=1}^{\infty}$. Moreover, if it is finite for some orthonormal basis, then it is finite for any orthonormal basis.

Theorem 15. (1) $B_2(\mathcal{H})$ is a *-ideal.

(2) If $A, B \in B_2(\mathcal{H})$, then for any orthonormal basis (e_n),

$$\sum_{n=1}^{\infty} \langle e_n, A^*Be_n\rangle$$

is absolutely summable, and its limit, denoted by $\langle A, B\rangle_2$, is independent of the orthonormal basis chosen.

(3) $B_2(\mathcal{H})$ with inner product $\langle \cdot, \cdot\rangle_2$ is a Hilbert space.

(4) If $\|\cdot\|_2$ denote the induced norm by the given inner product, then

$$\|A\| \leqslant \|A\|_2 \leqslant \|A\|_1, \qquad \text{and } \|A\|_2 = \|A^*\|_2.$$

(5) Every $A \in B_2(\mathcal{H})$ is compact and a compact operator A is in $B_2(\mathcal{H})$ if and only if $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ where λ_n are the singular values of A.

(6) The finite rank operators are $\|\cdot\|_2$-dense in $B_2(\mathcal{H})$.

(7) $A \in B_2(\mathcal{H})$ if and only if $(\|Ae_n\|)_n \in l^2(\mathbb{N})$ for some orthonormal basis (e_n).

(8) $A \in B_1(\mathcal{H})$ if and only if $A = BC$ with $B, C \in B_2(\mathcal{H})$.

Proof. (1) Clearly it is closed under scalar multiplication. If $A, B \in B_2(\mathcal{H})$, then $A^*A, B^*B \in B_1(\mathcal{H})$. Note that

$$(A+B)^*(A+B) \leqslant 2A^*A + 2B^*B.$$

So $A + B \in B_2(\mathcal{H})$.

If $A \in B_2(\mathcal{H}), B \in B(\mathcal{H})$, then $(BA)^*BA \leqslant \|B\|^2A^*A$, so $BA \in B_2(\mathcal{H})$.

We also note that $\mathrm{tr}\ A^*A = \mathrm{tr}\ AA^*$, whether it is finite or not. This follows from the Parseval Identity:

$$\begin{aligned}\sum_{n=1}^{\infty}\|Ae_n\|^2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|\langle Ae_n, f_m\rangle|^2\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\langle e_n, A^*f_m\rangle|^2\\ &= \sum_{m=1}^{\infty}\|A^*f_m\|^2,\end{aligned}$$

where $(e_n),(f_m)$ are arbitrarily chosen orthonormal basis for $\mathcal{H}$. This shows that $A\in B_2(\mathcal{H})$ if and only if $A^*\in B_2(\mathcal{H})$, and moreover, in the notation of (4), we have $\|A\|_2=\|A^*\|_2$ whether it is finite or not.

(2) By Cauchy-Schwarz Inequality, the series is absolutely summable. Note that $\langle A,B\rangle_2$ is a sesquilinear form on $B_2(\mathcal{H})$, so we may form the polarization identity with notations in (4):

$$\langle A,B\rangle_2=\frac{1}{4}(\|A+B\|_2^2-\|A-B\|_2^2+i\|A+iB\|_2^2-i\|A-iB\|_2^2).$$

So the inner product is independent of the orthonormal basis chosen.

(8) If $A\in B_1(\mathcal{H})$, then $|A|^{1/2}\in B_2(\mathcal{H})$. By the polar decomposition, $A=U|A|^{1/2}|A|^{1/2}$. Since $B_2(\mathcal{H})$ is a *-ideal, we have $U|A|^{1/2}\in B_2(\mathcal{H})$.

Conversely, if $B,C\in B_2(\mathcal{H})$ and $A=BC$, then by the polar decomposition, $|A|=U^*A=U^*BC$. Then we have

$$\operatorname{tr}|A|=\langle C,B^*U\rangle_2<\infty.$$

(4) The second part has been proved in (1). For any unit vector $x\in\mathcal{H}$, extend it to an orthonormal basis for $\mathcal{H}$, then we immediately have $\|Ax\|\leqslant\|A\|_2$, so $\|A\|\leqslant\|A\|_2$.

The inequality $\|A\|_2\leqslant\|A\|_1$ will be proved later.

(3) Almost the same as the case in $B_1(\mathcal{H})$. For a Cauchy sequence in $\|\cdot\|_2$, it has a limit in $B(\mathcal{H})$ with respect to the operator norm. Then we can prove that it is actually the $\|\cdot\|_2$ limit and lies in $B_2(\mathcal{H})$.

(5)&(6) Also the same as the case in $B_2(\mathcal{H})$. In fact, in this case, the computation is much easier, since

$$\|A\|_2^2=\sum_{n=1}^{\infty}\|Ae_n\|^2$$

for any orthonormal basis (e_n) and we do not need to rely on $|A|$.

If A is compact, we write as before

$$A=\sum_{n=1}^{N}\lambda_n\langle\cdot,e_n\rangle f_n.$$

Then extend $(e_n)_{n=1}^N$ to an orthonormal basis, we have $\|A\|_2 = (\sum\limits_{n=1}^{\infty} |\lambda_n|^2)^{1/2}$ if we add 0's when necessary.

(7) Obvious.

(4) (Continued) The trace class norm and the Hilbert-Schmidt norm is totally determined by the singular values. It is obvious that for non-negative numbers $(\lambda_n)_n$, we have

$$\sum_{n=1}^{\infty} \lambda_n^2 \leqslant (\sum_{n=1}^{\infty} \lambda_n)^2.$$

It then follows that $\|A\|_2 \leqslant \|A\|_1$ for all $A \in B(\mathcal{H})$. □

Theorem 16. Let (M, μ) be a σ-finite measure space and $\mathcal{H} = L^2(M, \mathrm{d}\mu)$. Then $\mathcal{H}$ is separable. Then $A \in B(\mathcal{H})$ is Hilbert-Schmidt if and only if there is a function

$$K \in L^2(M \times M, \mathrm{d}\mu \otimes \mathrm{d}\mu)$$

with

$$Af(x) = \int_M K(x, y) f(y) \mathrm{d}\mu(y).$$

Moreover,

$$\|A\|_2^2 = \int_M |K(x, y)|^2 \mathrm{d}\mu(x) \mathrm{d}\mu(y).$$

Proof. Let $K \in L^2(M \times M, \mathrm{d}\mu \otimes \mathrm{d}\mu)$ and A_K be the associated integral operator. By Fubini theorem, $K(x, \cdot)$ is square-integrable for almost every x, and for such x, $A_K f(x)$ is well-defined. Then for $f \in L^2(M, \mathrm{d}\mu)$, we have

$$\begin{aligned} \|A_K f\|_{L^2} &= \sup\{\left|\int_M A_K f(x) \overline{g(x)} \mathrm{d}\mu(x)\right| : g \in L^2(M, \mathrm{d}\mu), \|g\|_{L^2} = 1\} \\ &\leqslant \sup\{\int_{M \times M} |K(x, y)||f(y)||g(x)| : g \in L^2(M, \mathrm{d}\mu), \|g\|_{L^2} = 1\} \\ &\leqslant \|K\|_{L^2} \|f\|_{L^2} \end{aligned}$$

where we have used Fubini theorem and Cauchy-Schwarz inequality. So $\|A_K\| \leqslant \|K\|_{L^2}$.

Choose an orthonormal basis (e_n) for $L^2(M, \mathrm{d}\mu)$. Then $(e_n(x)\overline{e_m(y)})_{m,n}$ is an orthonormal basis for $L^2(M \times M, \mathrm{d}\mu \otimes \mathrm{d}\mu)$. (To see this, first one can easily verify that it is an orthonormal set. Then prove that its orthogonal complement is zero. See this post on MSE.)

Expand K as

$$K = \sum_{n,m=1}^{\infty} \alpha_{nm} e_n(x) \overline{e_m(y)}, \quad \|K\|_{L^2} = \sum_{n,m=1}^{\infty} |\alpha_{nm}|^2 < \infty.$$

So we find that $A_K e_m = \sum\limits_{n=1}^{\infty} \alpha_{nm} e_n$, and $\|A_K\|_2^2 = \sum\limits_{m,n=1}^{\infty} |\alpha_{nm}|^2 = \|K\|_{L^2}^2$. Therefore, $A_K \in B_2(\mathcal{H})$ and $K \mapsto A_K$ is an isometry from $L^2(M \times M, \mathrm{d}\mu \otimes \mathrm{d}\mu)$ to $B_2(\mathcal{H})$. The range has to be closed. Recall that finite-rank operators are $\|\cdot\|_2$-dense in $B_2(\mathcal{H})$. For a rank-one operator, it can be written as $\langle \cdot, e\rangle f$. By approximation and linearity, we only need to prove that operators of the form $\langle \cdot, e_n\rangle e_m$ lie in the range. This is given by $K(x,y) = e_m(x)\overline{e_n(y)}$. □

Theorem 17. If $A \in B_1(\mathcal{H})$ and (e_n) is an orthonormal basis, then

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$$

converges absolutely and the limit is independent of the choice of basis.

Proof. By (8) of 15, we can write $A = BC$ where $B, C \in B_2(\mathcal{H})$. Then

$$\sum_{n=1}^{\infty} |\langle Ae_n, e_n\rangle| \leqslant \|B\|_2^2 \|C\|_2^2 < \infty.$$

Also note that

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle = \langle C, B^*\rangle_2,$$

so by (2) of 15, it is independent of the choice of basis. □

Definition 18. The map $\mathrm{tr}\colon B_1(\mathcal{H}) \to \mathbb{C}$ given by $\mathrm{tr}\ A = \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$ where (e_n) is any orthonormal basis is called the **trace**.

Remark 19. It is not true that $\sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$ converges absolutely for some orthonormal basis (e_n) is sufficient to imply $A \in B_1(\mathcal{H})$.

For an example, see 27.

Theorem 20. (1) $\mathrm{tr}(\cdot)$ is linear.

(2) $\mathrm{tr}\ A^* = \overline{\mathrm{tr}\ A}$.

(3) $\mathrm{tr}\ AB = \mathrm{tr}\ BA$ if one of the following holds: $A \in B_1(\mathcal{H})$ and $B \in B(\mathcal{H})$; or $A, B \in B_2(\mathcal{H})$.

Proof. (1) Obvious.

(2) Obvious.

(3) Recall that $(A, B) \mapsto \mathrm{tr}\ (AB^*) = \langle B^*, A^*\rangle_2$ is a sesquilinear form on $B_2(\mathcal{H})$. Also recall that

$$\langle A, B\rangle_2 = \frac{1}{4}(\|A+B\|_2^2 - \|A-B\|_2^2 + i\|A+iB\|_2^2 - i\|A-iB\|_2^2),$$

and

$$\|A\|_2 = \|A^*\|_2.$$

So

$$\begin{aligned}\text{tr }(AB) &= \langle B, A^* \rangle_2 \\ &= \frac{1}{4}(\|B+A^*\|_2^2 - \|B-A^*\|_2^2 + i\|B+iA^*\|_2^2 - i\|B-iA^*\|_2^2) \\ &= \frac{1}{4}(\|A+B^*\|_2^2 - \|A-B^*\|_2^2 + i\|A+iB^*\|_2^2 - i\|A-iB^*\|_2^2) \\ &= \langle A, B^* \rangle_2 \\ &= \text{tr }(BA).\end{aligned}$$

If $A \in B_1(\mathcal{H}), B \in B(\mathcal{H})$, we write $A = A_1 A_2$ where $A_1, A_2 \in B_2(\mathcal{H})$. Then

$$\text{tr }(AB) = \text{tr }(A_1 A_2 B) = \text{tr }(A_2 B A_1) = \text{tr }(B A_1 A_2) = \text{tr }(BA).$$

$\square$

Remark 21. We can now prove that $\|A\|_1 = \|A^*\|_1$ if $A \in B_1(\mathcal{H})$. Suppose $A = U|A|$ is the polar decomposition, then $|A^*|^2 = AA^* = U|A|^2U^*$. By uniqueness of the square root, $|A^*| = U|A|U^*$. Then tr $|A^*| =$ tr $(U|A|U^*) =$ tr $(U^*U|A|) =$ tr $|A|$.

Some examples, results and exercises

Example 22. Multiplication is not jointly SOT-continuous, not jointly WOT-continuous. For the "WOT" part, see the next example. For other proofs of the "SOT" part, see [Halmos] Problem 111. The proof given here comes from Exercise 6.

Given an infinite-dimensional Hilbert space $\mathcal{H}$. If multiplication is jointly SOT-continuous, the pre-image of an open set is open, so $U := \{(S,T) : \|STz\| < 1\}$ is an open neighbourhood of $(0,0) \in B(\mathcal{H}) \times B(\mathcal{H})$ for every vector $z \in \mathcal{H}$. By definition of the product topology, U contains a set of the form $V(x_1, \dots, x_n; \varepsilon) \times V(y_1, \dots, y_m; \delta)$. Recall that

$$V(x_1, \dots, x_n; \varepsilon) := \{T \in B(\mathcal{H}) : \|Tx_i\| < \varepsilon, i = 1, \dots, n\}.$$

In other words, whenever $\|Sx_i\| < \varepsilon, \|Ty_j\| < \delta$, we have $\|STz\| < 1$. But it leads to a contradiction. Since $\dim \mathcal{H} = \infty$, we may find some $T \in B(\mathcal{H})$ such that $\|Ty_j\| < \delta$ and $Tz \neq 0$ is orthogonal to the linear span of $\{x_i\}_{i=1}^n$ at the same time. Then we may find some $S \in B(\mathcal{H})$ such that $\|Sx_i\| < \varepsilon$ and $\|STz\| > 1$. This is the contradiction.

There is a weak but positive result about joint continuity. In $B(\mathcal{H})$, if $S_\lambda^* \xrightarrow{s} S^*$ and $T_\lambda \xrightarrow{s} T$, then $S_\lambda T_\lambda \xrightarrow{w} ST$. This follows easily from this fact: For any $x, y \in \mathcal{H}$,

$$\langle S_\lambda T_\lambda x, y\rangle = \langle T_\lambda x, S_\lambda^* y\rangle.$$

Example 23. When considering sequences $T_n \to T, S_n \to S$ in WOT or SOT, it turns out that $T_n S_n \to TS$ holds for SOT, but not WOT.

In fact, as in the proof of Theorem 4, WOT- or SOT-convergent **sequences** are norm bounded, which does not hold in the case of net (Recall Remark 1.). For $x \in \mathcal{H}$,

$$\begin{aligned}\|T_n S_n x - TSx\| &\leqslant \|(T_n - T)Sx\| + \|T_n(S_n x - Sx)\| \\ &\leqslant \|(T_n - T)Sx\| + \|T_n\| \cdot \|S_n x - Sx\| \\ &\to 0.\end{aligned}$$

But consider $\mathcal{H} = l^2(\mathbb{N})$ and the unilateral shift S, then $(S^*)^n S^n = 1$ for all positive integer n. Note that $S^n \xrightarrow{s} 0$, so $S^n \xrightarrow{w} 0$ and $(S^*)^n \xrightarrow{w} 0$.

Proposition 24. Every $B \in B(\mathcal{H})$ can be written as a linear combination of four unitaries.

Proof. First we write $B = B_1 + iB_2$ where B_1, B_2 are self-adjoint. Then there are real numbers $c_1, c_2 \in \mathbb{R}$ such that $c_j B_j$ is contractive for $j = 1, 2$. If A is self-adjoint and contractive, then

$$A = \frac{A + i\sqrt{I - A^2}}{2} + \frac{A - i\sqrt{I - A^2}}{2}.$$

By an easy computation, $A \pm i\sqrt{I - A^2}$ are unitary operators. $\square$

Theorem 25. Recall that for $A \in B(\mathcal{H})$, the partial isometry $U \in B(\mathcal{H})$ in the polar decomposition is uniquely determined by the conditions

$$A = U|A|, \quad \ker U = \ker A.$$

It is proved in Theorem VI.10 that such U automatically satisfies ran $U = \overline{\text{ran } A}$.

We claim that it automatically satisfies $U^*A = |A|$.

Proof. In fact, it is directly seen from the proof. Such U is defined as an isometry (extension of $|A|x \mapsto Ax$) from $\overline{\text{ran } |A|}$ onto $\overline{\text{ran } A}$ and zero on $\overline{\text{ran } |A|}^{\perp}$. The adjoint of U is thus an isometry (extension of $Ax \mapsto |A|x$) from $\overline{\text{ran } A}$ onto $\overline{\text{ran } |A|}$ and zero on $\overline{\text{ran } A}^{\perp} = \ker A^*$. So $U^*A = |A|$. $\square$

Theorem 26. Given a sequence $(A_n)_{n=1}^{\infty}$ in $B(\mathcal{H})$ which converges to A in the operator norm.

(1) $A_n^*A_n$ converges to A^*A in the operator norm.

(2) $|A_n|$ converges to $|A|$ in the operator norm.

Proof. (1) The multiplication is jointly continuous with respect to the operator norm by an easy estimation.

(2) To simplify, suppose A_n are positive operators for all n and converges to $A \geqslant 0$ in the operator norm. Then there is some $R > 0$ such that $\|A_n\| \leqslant R$ for all n. By Stone-Weierstrass theorem, there is a sequence of (real-)polynomials $P_m(t)$ defined on $t \in [0, R]$ which uniformly converges to $f(t) = t^{1/2}$ on $[0, R]$. For any $\varepsilon > 0$, there exists some M such that $\|P_M(A_n) - f(A_n)\| < \varepsilon$, $\|P_M(A) - f(A)\| < \varepsilon$ for all n. For such a fixed M, there exists some N such that whenever $n > N$, we have $\|P_M(A_n) - P_M(A)\| < \varepsilon$. Therefore, $\|f(A_n) - f(A)\| < 3\varepsilon$ whenever $n > N$. This completes the proof.

(If one is not familiar with the continuous functional calculus, the series used in Theorem VI.8 and Theorem VI.9 also serves well.) $\square$

Example 27. (1) Given $A \in B(\mathcal{H})$. If $\sum\limits_{n=1}^{\infty} |\langle Ae_n, e_n\rangle| < \infty$ for some orthonormal basis, A may not lie in $B_1(\mathcal{H})$.

The unilateral shift is an easy example. Let $\mathcal{H} = l^2(\mathbb{N})$ and $(e_n)_{n=1}^{\infty}$ be the standard basis. The operator $S : e_k \mapsto e_{k+1} \quad (k \geqslant 1)$ satisfies $\langle Se_n, e_n\rangle = 0$. But S is not even compact.

(2) If $\sum\limits_{n=1}^{\infty} \|Ae_n\| < \infty$ for some orthonormal basis, then $A \in B_1(\mathcal{H})$.
In fact,

$$\sum_{n=1}^{\infty} \langle |A| e_n, e_n \rangle \leqslant \sum_{n=1}^{\infty} \| |A| e_n \| = \sum_{n=1}^{\infty} \|Ae_n\| < \infty.$$

So tr $|A| < \infty$.

Conversely, if $A \in B_1(\mathcal{H})$, then we take the orthonormal basis (e_n) which consists of unit eigenvectors of $|A|$. Then $\sum\limits_{n=1}^{\infty} \|Ae_n\| < \infty$ is the sum of singular values of A, or equivalently, eigenvalues of $|A|$.

(3) If $\sum\limits_{n=1}^{\infty} |\langle Ae_n, e_n \rangle| < \infty$ holds for every orthonormal basis, then $A \in B_1(\mathcal{H})$.

Write $A = A_1 + iA_2$ where A_1, A_2 are self-adjoint. The condition also holds for A_1, A_2. So if we prove $A_1, A_2 \in B_1(\mathcal{H})$, then we will have $A \in B_1(\mathcal{H})$. Therefore, we only need to consider self-adjoint operators A.

If A is not compact, (I have not found a proof without using spectral projections) then there exists a spectral projection $E(B)$ related to A such that E is infinite-dimensional and $B \subseteq \sigma(A)$ is Borel and $\overline{B}$ does not contain 0, otherwise A would be the norm limit of finite linear combinations of finite-rank spectral projections. An orthonormal basis for the range of $E(B)$ then leads to a contradiction to the condition.

Since A is compact, there are an orthonormal basis (e_n) for $\mathcal{H}$ and a sequence of real numbers $\lambda_n \to 0$ such that $Ae_n = \lambda_n e_n$, by Theorem VI.16. The condition then implies that $\sum |\lambda_n| < \infty$. Note that $(|\lambda_n|)$ are precisely the singular values of A, so $A \in B_1(\mathcal{H})$.

To summary, one need not rely on the polar decomposition to determine whether an operator is Hilbert-Schmidt, and one only needs an arbitrarily chosen orthonormal basis. In contrast, one needs to consider either $|A|$ with an arbitrarily chosen orthonormal basis, or A itself with any orthonormal basis in the case of trace class operators. If this fails (e.g. in (2)), the proof may hide there.

Theorem 28. Equip $K(\mathcal{H}), B(\mathcal{H})$ with the operator norm and $B_1(\mathcal{H})$ with the trace class norm.

(1) The Banach space dual of $K(\mathcal{H})$ is $B_1(\mathcal{H})$. The isometric isomorphism is given by $A \mapsto \text{tr}\ (A\cdot)$.

(2) The Banach space dual of $B_1(\mathcal{H})$ is $B(\mathcal{H})$. The isometric isomorphism is given by $B \mapsto \text{tr}\ (\cdot B)$.

Proof. Proof omitted, since I am exausted now. One can refer to Theorem 4.2.1 and Theorem 4.2.3 in [Murphy]. *C*-algebras and Operator Theory* written by

Gerard J. Murphy. Or also *A Course in Operator Theory* written by John B. Conway. □

Remark 29. The weak*-topology on $B(\mathcal{H})$ as a dual space of $B_1(\mathcal{H})$ is called the **ultraweak** topology. This is valid for von Neumann algebras. Von Neumann algebras can be characterized as C*-algebras possessing a pre-dual (This result is due to Sakai).