Chapter 2

C*-Algebras and Hilbert Space Operators

Problem 1. Let A be a Banach algebra such that for all $a \in A$ the implication

$$Aa = 0 \quad \text{or} \quad aA = 0 \Rightarrow a = 0$$

holds. Let L, R be linear mappings from A to itself such that for all $a, b \in A$,

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad \text{and} \quad R(a)b = aL(b).$$

Show that L and R are necessarily continuous.

Solution. We used the closed graph theorem to prove the statement. If $a_n \to a, L(a_n) \to b$, then for any $c, d \in A$, consider the limit of $R(d)a_nc = dL(a_nc) = dL(a_n)c$, and we get $R(d)ac = dbc$. Note that $R(d)ac = dL(a)c$, so the identity $dL(a)c = dbc$ holds for all $c, d \in A$. By assumption, $L(a) = b$. By the closed graph theorem, L is continuous. Similarly, R is continuous.

Problem 2. Let A be a unital C*-algebra.

(a) If a, b are positive elements of A, show that $\sigma(ab) \subseteq \mathbb{R}^+$.

(b) If a is an invertible element of A, show that $a = u|a|$ for a unique unitary u of A. Give an example of an element of $B(H)$ for some Hilbert space H that cannot be written as a product of a unitary times a positive operator.

(c) Show that if $a \in \mathrm{Inv}(A)$, then $\|a\| = \|a^{-1}\| = 1$ if and only if a is a unitary.

Solution. (a) Recall that in a unital Banach algebra, we always have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

So we write $ab = ab^{1/2}b^{1/2}$, and we only need to prove that $\sigma(b^{1/2}ab^{1/2}) \subseteq \mathbb{R}^+$, which is obviously true by properties of positive elements.

(b) $|a| = (a^*a)^{1/2}$ is invertible in this case, so u is unique and actually equals $a|a|^{-1}$. Compute $u^*u = |a|^{-1}a^*a|a|^{-1} = 1$ and $uu^* = a|a|^{-2}a^*$. Here is a trick (but quite common) on commutativity.

Note that $a(a^*a)^n = (aa^*)^n a$ for any positive integer n, so using approximation by polynomials, we get $af(a^*a) = f(aa^*)a$ for continuous functions f defined on $\mathbb{R}^+$. Here we take $f(t) = t^{-1}$ since the elements involved are invertible. This shows that $a|a|^{-2} = (aa^*)^{-1}a = (a^*)^{-1}$, so $uu^* = 1$ and u is a unitary.

In general, we can only write an operator as a product of a partial isometry times a positive operator. Let s denote the unilateral shift, and suppose $s = ua$ where u is a unitary and a is positive. Then $1 = s^*s = a^*a = a^2$. Since a is positive, the spectrum of a is the singleton $\{1\}$ and thus $a = 1$. However, s is not a unitary, so $s = ua$ can never hold for u unitary and a positive.

(c) If a is a unitary, certainly $\|a\| = \|a^{-1}\| = 1$. Conversely, we write the polar decomposition $a = u|a|$ as in (b). Since u is a unitary, $|a|$ and $|a|^{-1}$ are norm-one. Both the elements are positive, so by the spectral radius formula, the spectrum of $|a|$ is contained in the intersection of the unit circle and the positive real line, i.e. the singleton $\{1\}$. So $|a| = 1$, and $a = u$ is a unitary.

Problem 3. Let Ω be a locally compact Hausdorff space, and suppose that the C*-algebra $C_0(\Omega)$ is generated by a sequence of projections $(p_n)_{n=1}^{\infty}$. Show that the hermitian element $h = \sum_{n=1}^{\infty} p_n/3^n$ generates $C_0(\Omega)$.

Solution. By Stone-Weierstrass theorem, we only need to prove that h separates points of Ω and h vanishes nowhere.

If $h(x) = h(y)$, then since p_n takes values in $\{0, 1\}$, we get that $p_n(x) = p_n(y)$ for every $n \geqslant 1$. Since points of Ω can be realized as characters on $C_0(\Omega)$, which is generated by $(p_n)_{n=1}^{\infty}$, we must have $x = y$.

Meanwhile, if $h(x) = 0$, then $p_n(x) = 0$ for every $n \geqslant 1$, and the C*-algebra generated by $(p_n)_{n=1}^{\infty}$ must be contained in $C_0(\Omega \setminus \{x\})$, which leads to a contradiction.

Problem 4. Give an example of an ideal in the C*-algebra $C(\mathbb{D})$ that is not self-adjoint.

Solution. $\mathbb{D}$ refers to the closed unit disc in the complex plane.

We consider the principal ideal generated by z. An interesting thing is that we actually rarely consider principal ideal in operator algebra, because it really does not make much sense when the ring is not commutative.

Now the chosen principal ideal is $I = \{f(z)z : f \in C(\mathbb{D})\}$, then $z \in I, \overline{z} \notin I$.

Problem 5. Let $\varphi : A \to B$ be an isometric linear map between unital C*-algebras A and B such that $\varphi(a^*) = \varphi(a)^*, (a \in A)$ and $\varphi(1) = 1$. Show that $\varphi(A^+) \subseteq B^+$.

Solution. If $a = a^*, a \in A$, then $\varphi(a^*) = \varphi(a)^*$ implies that $\varphi(a)$ is self-adjoint. So $\varphi(a) \geqslant 0$ is equivalent to $\sigma_B(\varphi(a)) \subseteq \mathbb{R}^+$.

By assumption, φ is unital and injective, so when considering the spectrum, A can be viewed as a C*-subalgebra of B, and thus $\sigma_A(a) = \sigma_B(\varphi(a))$. This proves the statement.

Problem 6. Let A be a unital C*-algebra.

(a) If $r(a) < 1$ and $b = (\sum_{n=0}^{\infty} a^{*n}a^n)^{1/2}$, show that $b \geqslant 1$ and $\|bab^{-1}\| < 1$.

(b) For all $a \in A$, show that

$$r(a) = \inf_{b \in \mathrm{Inv}(A)} \|bab^{-1}\| = \inf_{b \in A_{sa}} \|e^b a e^{-b}\|.$$

Solution. (a) That $r(a) < 1$ implies the convergence of the series defining b. As a limit of positive elements, b is positive. The summand when $n = 0$ is 1, so $\sum_{n=0}^{\infty} a^{*n}a^n \geqslant 1$, and since the square root function is operator monotone, $b \geqslant 1$.

The formula defining b shows that $a^*b^2a = b^2 - 1$, so

$$\|bab^{-1}\|^2 = \|b^{-1}a^*bbab^{-1}\| = \|1 - b^{-2}\|.$$

Since $b \geqslant 1$, we have $\|1 - b^{-2}\| < 1$.

(b) If b is invertible, then bab^{-1} has the same spectrum as a. So $r(a) \leqslant \inf_{b \in \mathrm{Inv}(A)} \|bab^{-1}\|$.

Obviously, $\inf_{b \in \mathrm{Inv}(A)} \|bab^{-1}\| \leqslant \inf_{b \in A_{sa}} \|e^b a e^{-b}\|$ since e^b is always invertible.

Now we only need to prove that $\inf_{b \in A_{sa}} \|e^b a e^{-b}\| \leqslant r(a)$. Suppose $|\lambda| > r(a)$. Let $b = (\sum_{n=0}^{\infty} a^{*n}a^n/|\lambda|^{2n})^{1/2}$. As in (a), b is well-defined, $b \geqslant 1$ and $\|bab^{-1}\| < |\lambda|$. On $[1, \infty)$, the logarithm function is continuous and takes positive values. So $\log b$ is well-defined, self-adjoint and $b = e^{\log b}$. Therefore, $\inf_{b \in A_{sa}} \|e^b a e^{-b}\| < |\lambda|$. The inequality holds for every $|\lambda| > r(a)$, so we prove the required statement.

Problem 7. Let A be a unital C*-algebra.

(a) If $a, b \in A$, show that the map

$$f : \mathbb{C} \to A, \quad \lambda \mapsto e^{i\lambda b} a e^{-i\lambda b},$$

is differentiable and that $f'(0) = i(ba - ab)$.

(b) Let X be a closed vector subspace of A which is unitarily invariant in the sense that $uXu^* \subseteq X$ for all unitaries u of A. Show that $ba - ab \in X$ if $a \in X$ and $b \in A$.

(c) Deduce that the closed linear span X of the projections in A has the property that $a \in X$ and $b \in A$ implies that $ba - ab \in X$.

Solution. (a) First, $\lambda \mapsto e^{i\lambda b}$ is differentiable everywhere, since

$$e^{i\lambda b} - e^{i\mu b} = \frac{e^{i\mu b}(e^{i(\lambda-\mu)b} - 1)}{\lambda - \mu},$$

and we use the power series expansion of $e^{i\lambda b}$ at $\lambda = 0$ to calculate $f'(\lambda) = ie^{i\lambda b}b$.

Second, to calculate the derivative of the product of two differentiable functions, it is the same as how we do in the case of real-valued functions and the result is $(fg)' = f'g + fg'$. Note that here the order of f, g matters.

From the discussions above, we know that f is differentiable, and $f'(\lambda) = i(e^{i\lambda b} bae^{-i\lambda b} - e^{i\lambda b} ae^{-i\lambda b} b)$, so $f'(0) = i(ba - ab)$.

(b) Define f as in (a) when $a \in X, b \in A$. Then by assumption, f always takes values in X whenever b is self-adjoint and $\lambda \in \mathbb{R}$. In this case,

$$\frac{f(\lambda) - f(0)}{\lambda} \in X, \quad \lambda \in \mathbb{R},$$

so we let $\lambda \to 0$, the limit $f'(0) \in X$ since X is closed. This means that $ba - ab \in X$ if $a \in X, b \in A_{sa}$. Since A_{sa} linealy spans A, we prove the statement.

(c) Since a projection conjugated by a unitary is still a projection, such an X satisfies the condition of (b).

Problem 8. Let a be a normal element of a C*-algebra A, and b an element commuting with a. Show that b^* also commutes with a.

Solution. Define $f(\lambda) = e^{i\lambda a^*} b e^{-i\lambda a^*}$ from $\mathbb{C}$ to A^+, the unitalization of A. By Problem 2.7, $f'(0) = i(a^*b - ba^*)$.

Since b commutes with a, b also commutes with $e^{i\overline{\lambda}a}$, so

$$f(\lambda) = e^{i\lambda a^*} e^{i\overline{\lambda}a} b e^{-i\overline{\lambda}a} e^{-i\lambda a^*}.$$

Since a is normal, $e^{i\lambda a^*} e^{i\overline{\lambda}a} = e^{i\lambda a^* + i\overline{\lambda}a}$ is a unitary, so $\|f(\lambda)\| = \|b\|$ for all $\lambda \in \mathbb{C}$. By Liouville's theorem, f must be constant. Therefore, $f'(0) = 0$, i.e. b commutes with a^* and b^* commutes with a.

Problem 9. If I is an ideal of $B(H)$, show that it is self-adjoint.

Solution. If $a \in I$, then by the polar decomposition, $a = u|a|$ where u is a partial isometry and $\ker(u) = \ker(a)$. Such a decomposition satisfies $u^*a = |a|$, by Theorem 2.3.4. So $|a| \in I$.

Note that the polar decomposition also implies that $a^* = |a|u^*$, so $a^* \in I$, which proves the statement.

Problem 10. Let $u \in B(H)$.

(a) Show that u is a left topological zero divisor in $B(H)$ if and only if it is not bounded below.

(b) Define

$$\sigma_{ap}(u) = \{\lambda \in \mathbb{C} : u - \lambda \text{ is not bounded below}\}.$$

This set is called the *approximate point spectrum* of u because $\lambda \in \sigma_{ap}(u)$ if and only if there is a sequence (x_n) of unit vectors of H such that $\lim_{n\to\infty} \|(u - \lambda)(x_n)\| = 0$. Show that $\sigma_{ap}(u)$ is a closed subset of $\sigma(u)$ containing $\partial\sigma(u)$.

(c) Show that u is bounded below if and only if it is left-invertible in $B(H)$.

(d) Show that $\sigma(u) = \sigma_{ap}(u)$ if u is normal.

Solution. (a) If u is bounded below, then there exists $c > 0$ such that $\|ux\| \geqslant c\|x\|, \quad (x \in H)$. So the following inequalities hold:

$$\begin{aligned}\inf_{\|b\|=1} \|ub\| &= \inf_{\|b\|=1} \sup\{\|ubx\| : x \in H, \|x\| = 1\} \\ &\geqslant \inf_{\|b\|=1} \sup\{c\|bx\| : x \in H, \|x\| = 1\} \\ &= \inf_{\|b\|=1} c\|b\| = c.\end{aligned}$$

Hence u is not a left topological zero divisor.

Conversely, if u is not bounded below, then there exists a sequence of unit vectors $(x_n)_{n\geqslant 1}$ such that $u(x_n) \to 0$. Now consider the projection $p_n := x_n \otimes x_n$, or we can write

$$p_n(x) = \langle x, x_n \rangle x_n, \quad x \in H.$$

Then $\|p_n\| = \|x_n\|^2 = 1$, and $\|up_n\| \leqslant \|u(x_n)\|$, so u is a left topological zero divisor.

(b) Since "not bounded below" implies "not invertible", it is clear that $\sigma_{ap}(u) \subseteq \sigma(u)$.

In Problem 11, Chapter 1, we have proved several statements about left topological zero divisors. We can now describe the approximate point spectrum in a new way.

$\sigma_{ap}(u) = \{\lambda \in \mathbb{C} : \zeta(\lambda - u) = 0\}$. Since ζ is a continuous function from $B(H)$ to $\mathbb{R}$, $\lambda \mapsto \zeta(\lambda - u)$ is also continuous, so $\sigma_{ap}(u)$ is closed.

Moreover, for boundary points λ of $\sigma(u)$, we have proved in Problem 11, Chapter 1 that $\zeta(u - \lambda) = 0$, so $\sigma_{ap}(u)$ contains $\partial\sigma(u)$.

(c) If u is left-invertible, say $vu = 1$, then $\zeta(u) \geqslant \|v\|^{-1} > 0$, so u is not a left topological zero divisor.

Conversely, if u is bounded below, then u is a bijection from H to uH. Define a linear operator $v : uH \to H$ to be its inverse. Now uH is a normed vector space and v is bounded, since u is bounded below. Using completeness, we can naturally continuously extend v to $\overline{uH} \to H$ and then define v to be zero on $\overline{uH}^{\perp}$. Such a v is a left inverse of u.

(d) It is well-known that a normal operator which is bounded below is invertible, so $\sigma(u) = \sigma_{ap}(u)$.

In detail, suppose $\lambda \notin \sigma_{ap}(u)$, i.e. $u - \lambda$ is bounded below. Note that $u - \lambda$ is normal, so $\|(u-\lambda)x\| = \|(u^* - \overline{\lambda})x\|$ for every $x \in H$. Hence, $u-\lambda$ and $u^* - \overline{\lambda}$ are both injective. Also recall $\operatorname{ran}(u - \lambda)^{\perp} = \ker(u^* - \overline{\lambda})$, so $u - \lambda$ has dense range. However, "bounded below" implies "closed range", also by an argument of completeness. Combining all these results, $u-\lambda$ is bijective, so by the Inverse Operator theorem, $u - \lambda$ is invertible in $B(H)$.

Problem 11. Let $u \in B(H)$ be a normal operator with spectral resolution of the identity E.

(a) Show that u admits an invariant closed vector subspace other than 0 and H if $\dim(H) > 1$.

(b) If λ is an isolated point of $\sigma(u)$, show that $E(\lambda) = \ker(u - \lambda)$ and that λ is an eigenvalue of u.

Solution. (a) For any Borel subset S of $\sigma(u) \subseteq \mathbb{C}$, consider the characteristic function χ_S. It is a bounded Borel measurable function, so we can define $\chi_S(u)$. Clearly, it is a projection in $B(H)$, and the range of $\chi_S(u)$ is an invariant closed subspace of u.

If $\sigma(u)$ is a singleton, then u is a scalar operator and the statement certainly holds true. If $\sigma(u)$ contains at least two points, then we can find two disjoint open subsets S_1, S_2 of $\sigma(u)$ separating these two points. Then $E(S_1), E(S_2)$ are non-trivial projections, and their images are invariant closed vector subspace of u other than 0 and H.

(b) Actually when λ is an isolated point, we lie in the scope of continuous functional calculus, since $\chi_{\{\lambda\}}$ is then a continuous function on $\sigma(u)$. Let $p = \chi_{\{\lambda\}}(u)$, then p is a non-zero projection. As continuous functions, we know that $z\chi_{\{\lambda\}} = \lambda\chi_{\{\lambda\}}$, so $pu = up = \lambda p$, so $pH \subseteq \ker(u - \lambda)$. Meanwhile, if $ux = \lambda x \quad (x \in H)$, then

$$px = \chi_{\{\lambda\}}(u)x = x - \chi_{\{\lambda\}^c}(u)x = x - \left(\chi_{\{\lambda\}^c}(z - \lambda)^{-1}(z - \lambda)\right)(u)x.$$

But what should be noted is that $(z - \lambda)(u) = u - \lambda$ and $\chi_{\{\lambda\}^c}(z - \lambda)^{-1}$ is a continuous function on $\sigma(u)$ because λ is an isolated point. Therefore, $px = x$. This proves that $\operatorname{ran}(p) = \ker(u - \lambda)$.

Since p is non-trivial, every non-zero vector in the range of p serves as an eigenvector of u corresponding to λ.

Problem 12. An operator u on H is *subnormal* if there is a Hilbert space K containing H as a closed vector subspace and there exists a normal operator v on K such that H is invariant for v, and u is the restriction of v. We call v a *normal extension* of u.

(a) Show that the unilateral shift is a non-normal subnormal operator.

(b) Show that if u is subnormal, then $u^*u \geqslant uu^*$.

(c) A normal extension $v \in B(K)$ of a subnormal operator $u \in B(H)$ is a *minimal* normal extension if the only closed vector subspace of K reducing v and containing H is K itself. Show that u admits a minimal normal extension. In the case that v is a minimal normal extension, show that K is the closed linear span of all $v^{*n}(x)(n \in \mathbb{N}, x \in H)$.

(d) Show that if $v \in B(K)$ and $v' \in B(K')$ are minimal normal extensions of u, then there exists a unitary operator $w : K \to K'$ such that $v' = wvw^*$.

Solution. The definition of subnormal operators has a more intuitive explanation. We can decompose K into $H \oplus H^\perp$, and operators in $B(K)$ can be written as 2×2 matrices. So a subnormal operator u gives rise to a normal operator on K which looks like

$$\begin{pmatrix} u & * \\ 0 & * \end{pmatrix}.$$

(a) Let s be the unilateral shift on H. Then $u^*u = 1$ and uu^* is the projection onto $\{e_1\}^\perp$. Let $K = H \oplus H$ and $a = e_1 \otimes e_1$, i.e. $a(x) = \langle x, e_1 \rangle e_1$. Then consider the operator

$$v = \begin{pmatrix} s & a \\ 0 & s^* \end{pmatrix} \in B(K).$$

It is easy to check that v is normal. In fact, v is a unitary in $B(K)$.

(b) Take a corresponding $v = \begin{pmatrix} u & a \\ 0 & b \end{pmatrix}$. Since v is normal, it gives the following identity:

$$\begin{pmatrix} uu^* + aa^* & ab^* \\ ba^* & bb^* \end{pmatrix} = \begin{pmatrix} u^*u & u^*a \\ a^*u & a^*a + b^*b \end{pmatrix}.$$

This shows that $u^*u = uu^* + aa^* \geqslant uu^*$.

(c) Denote by P the set of all closed vector subspaces of K which reduce v and contain H, then P is non-empty since $K \in P$. The set P is partially ordered by inclusion. Note that H does not necessarily belong to P, since H is invariant but not necessarily reducing. If T is a totally ordered subset of P, then we form the intersection $M = \bigcap_{L \in T} L$. Then M is a closed vector subspace of K containing H. It is also easy to check that M is a reducing subspace of v. By Zorn lemma, there exists a minimal normal extension.

Denote the closed linear span of all $v^{*n}(x)(n \in \mathbb{N}, x \in H)$ by L. Any reducing subspace of v containing H must contain $v^n H, v^{*n} H$ for all $n \geqslant 0$. Since H is invariant for v, $v^n H \subseteq H$ for all $n \geqslant 0$. This means that any reducing subspace contains L. On the other hand, L is definitely reducing since v is normal. By minimality, $K = L$.

(d) Define an operator w as follows:

$$w(\sum_{k=0}^{n} v^{*k} x_k) = \sum_{k=0}^{n} v'^{*k} x_k, \quad \forall x_k \in H.$$

Then w is an isometry since v, v' are normal and have the same restriction on H:

$$\begin{aligned}
\langle \sum_{k=0}^{n} v^{*k} x_k, \sum_{j=0}^{n} v^{*j} x_j \rangle &= \sum_{k=0}^{n} \sum_{j=0}^{n} \langle v^{*k} x_k, v^{*j} x_j \rangle \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \langle x_k, v^k v^{*j} x_j \rangle \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \langle x_k, v^{*j} v^k x_j \rangle \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \langle v^j x_k, v^k x_j \rangle \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \langle u^j x_k, u^k x_j \rangle.
\end{aligned}$$

and similarly for v'. This also shows that w is well-defined. Since w is defined on a dense subspace of K and its range contains a dense subspace of K', we can naturally extend w to a unitary between K and K'.

The identity $v'(x) = wvw^*(x)$ holds for x in a dense subspace, so it holds everywhere.