Chapter 3

Ideals and Positive Functionals

Problem 1. Let a, b be normal elements of a C*-algebra A, and c an element of A such that $ac = cb$. Show that $a^*c = cb^*$.

Solution. Consider the C*-algebra $M_2(A)$ and two elements in $M_2(A)$:

$$d = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, d' = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

Then d is normal and commutes with d'. By Fuglede's theorem (Problem 8, Chapter 2), d^* commutes with d'. This is exactly what we need to prove.

Problem 2. Let τ be a positive linear functional on A.

(a) If I is a closed ideal in A, show that $I \subseteq \ker(\tau)$ if and only if $I \subseteq \ker(\varphi_\tau)$.

(b) We say τ is *faithful* if $\tau(a) = 0 \Rightarrow a = 0$ for all $a \in A^+$. Show that if τ is faithful, then the GNS representation (H_τ, φ_τ) is faithful.

(c) Suppose that α is an automorphism of A such that $\tau(\alpha(a)) = \tau(a)$ for all $a \in A$. Define a unitary on H_τ by setting $u(a + N_\tau) = \alpha(a) + N_\tau,\ (a \in A)$. Show that $\varphi_\tau(\alpha(a)) = u\varphi_\tau(a)u^*,\ (a \in A)$.

Solution. (a) For $a \in A$, $\varphi_\tau(a) = 0$ if and only if $ab \in N_\tau = \{x \in A : \tau(x^*x) = 0\}$ for all $b \in A$, or equivalently, $\tau(b^*a^*ab) = 0$ for all $b \in A$.

If $I \subseteq \ker(\tau)$, then for $a \in I, b \in A$, we have $b^*a^*ab \in I$, so $\tau(b^*a^*ab) = 0$. Conversely, if $I \subseteq \ker(\varphi_\tau)$, then for $a \in I_+$, we have $\tau(u_\lambda a^{1/2}a^{1/2}u_\lambda) = 0$ for an approximate unit (u_λ) for A. Since τ is continuous, $\tau(a) = 0$ for all $a \in I_+$, and thus for all $a \in I$.

(b) $I := \ker(\varphi_\tau)$ is a closed ideal in A, so by (a), $I \subseteq \ker(\tau)$. Suppose $a \in I$, then $a^*a \in I$, so $\tau(a^*a) = 0$, which by faithfulness implies that $a^*a = 0$. So $I = 0$.

(c) We only need to check the identity on a dense subspace of H_τ. Note that u is definitely a unitary, since u is a bijective isometry by the assumption.

For any $b \in A$, $u\varphi_\tau(a)u^*(b + N_\tau) = u\varphi_\tau(a)(\alpha^{-1}(b) + N_\tau) = u(a\alpha^{-1}(b) + N_\tau) = \alpha(a)b + N_\tau = \varphi_\tau(\alpha(a))(b + N_\tau)$. Therefore, $\varphi_\tau(\alpha(a)) = u\varphi_\tau(a)u^*$ for all $a \in A$.

Problem 3. If $\varphi : A \to B$ is a positive linear map between C*-algebras, show that φ is necessarily bounded.

Solution. If φ is not bounded, then

$$\sup\{\|\varphi(a)\| : a \in A_+, \|a\| \leqslant 1\} = \infty.$$

Suppose $a_n \in A_+, \|a_n\| \leqslant 1$ and $\|\varphi(a_n)\| \geqslant 4^n$. Consider $a = \sum_{n\geqslant} a_n/2^n$, then a is a positive element in A and $\|a\| \leqslant 1$. Since $a \geqslant a_n/2^n$, we have $\varphi(a) \geqslant \varphi(a_n)/2^n \geqslant 0$, and thus $\|\varphi(a)\| \geqslant \|\varphi(a_n)\|/2^n = 2^n$ for all $n \geqslant 1$, which is impossible.

Another proof comes from a functorial argument and fundamental theorems in functional analysis. Such a φ gives rise to a map $S(B) \to S(A), \tau \mapsto \tau \circ \varphi$. Since we know a positive linear functional is necessarily bounded, and every bounded linear functional on a C*-algebra can be written as a linear combination of 4 states, we actually get a map $B^* \to A^*, \tau \mapsto \tau \circ \varphi$.

Now it is an easy exercise to prove that φ is bounded. One can use the closed graph theorem, or principle of uniform boundedness. This completes the proof.

Problem 4. Suppose that A is unital. Let α be an automorphism of A such that $\alpha^2 = \mathrm{id}_A$. Define B to be the set of all matrices

$$c = \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix},$$

where $a, b \in A$. Show that B is a C*-subalgebra of $M_2(A)$. Define a map $\varphi : A \to B$ by setting

$$\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & \alpha(a) \end{pmatrix}.$$

Show that φ is an injective *-homomorphism. We can thus identify A as a C*-subalgebra of B. If we set $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then u is a self-adjoint unitary and $B = A + Au$. if C is any unital C*-algebra with a self-adjoint unitary element v, and $\psi : A \to C$ is a *-homomorphism such that

$$\psi(\alpha(a)) = v\psi(a)v^* \quad (a \in A),$$

show that there is a unique *-homomorphism $\psi' : B \to C$ extending ψ such that $\psi'(u) = v$.

Solution. It is easy to check that B is a C*-subalgebra of $M_2(A)$ and φ is an injective *-homomorphism.

It is also easy to check that u is a self-adjoint unitary and $B = A + Au$.

Now prove the universal property of B. For $a \in A \subseteq B$, define $\psi'(a) = \psi(a)$, or more precisely, it should be:

$$\psi' \begin{pmatrix} a & 0 \\ 0 & \alpha(a) \end{pmatrix} = \psi(a), \quad \forall a \in A.$$

It extends to B as:

$$\psi'(\varphi(a) + \varphi(b)u) = \psi(a) + \psi(b)v, \quad \forall a, b \in A.$$

The uniqueness comes from the fact that $B = A + Au$.

Problem 5. An element a of A^+ is *strictly positive* if the hereditary C*-subalgebra of A generated by a is A itself, that is, if $\overline{aAa} = A$.

(a) Show that if A is unital, then $a \in A^+$ is strictly positive if and only if a is invertible.

(b) If H is a Hilbert space, show that a positive compact operator on H is strictly positive in $K(H)$ if and only if it has dense range.

(c) Show that if a is strictly positive in A, then $\tau(a) > 0$ for all non-zero positive linear functionals τ on A.

Solution. (a) If a is invertible, then $1 \in \overline{aAa}$, and $\overline{aAa}$ must contain the hereditary C*-subalgebra generated by 1 which is $\overline{1A1} = A$, so a is strictly positive.

Conversely, if a is strictly positive, then $1 \in \overline{aAa}$, so there exists some $b \in A$ such that $\|1 - aba\| < 1$. This implies that aba is invertible, so a is invertible.

(b) Recall that a positive compact operator always has the form

$$a = 0 \oplus \sum_{n\geqslant 1}^{\infty} \lambda_n (x_n \otimes x_n),$$

where $(x_n)_{n\geqslant 1}$ is an orthonormal basis for $\ker(a)^\perp$, x_n is a unit eigenvector of a corresponding to λ_n, $\lambda_n > 0$, and for any given $r > 0$, there is finitely many λ_n greater than r, counting with multiplicities. Then it is clear that a has dense range if and only if $\ker(a) = 0$. More precisely, the closure of the range of a is the closed linear span of $(x_n)_{n\geqslant 1}$, i.e. $\ker(a)^\perp$.

If a has dense range, then $(x_n)_{n\geqslant 1}$ is an orthonormal basis of H. For any rank-one operator b, it can be written as $e \otimes f$, i.e. $b(x) = \langle x, f\rangle e$. Approximate e, f by finite linear combinations of (x_n), b can be approximated in operator norm by finite linear combinations of operators of the form $x_n \otimes x_m$. However,

$$x_n \otimes x_m = \frac{1}{\lambda_m \lambda_n} a(x_n \otimes x_m)a,$$

so $x_n \otimes x_m \in \overline{aAa}$, where $A = K(H)$. So $\overline{aAa}$ contains every rank-one operator and thus equals A.

Conversely, if $x \in \ker(a), x \neq 0$, then for every operator of the form $aba, b \in A$, it must map x to 0. So does every operator in $\overline{aAa}$. Clearly $K(H)$ is not such an algebra, so $\overline{aAa} \neq A$.

(c) If $\tau(a) = 0$ for some positive linear functional τ on A, then by the fact that $0 \leqslant a^{1/2}ba^{1/2} \leqslant \|b\|a$ when $b \geqslant 0$, we know that $\tau(a^{1/2}Aa^{1/2}) = 0$. But $aAa \subseteq a^{1/2}Aa^{1/2}$, so $\tau = 0$.

Problem 6. We say that A is *σ-unital* if it admits a sequence $(u_n)_{n=1}^{\infty}$ which is an approximate unit for A.

(a) Let a be strictly positive element of A, and set $u_n = a(a+1/n)^{-1}$ for each positive integer n. Show that (u_n) is an approximate unit for A.

(b) If $(u_n)_{n=1}^{\infty}$ is an approximate unit for A, show that $a = \sum_{n=1}^{\infty} u_n/2^n$ is a strictly positive element of A.

Thus, A is σ-unital if and only if it admits a strictly positive element.

Solution. (a) Set $g_n(t) = t^2/(t+1/n)^{-1}, t \geqslant 0$. By Dini's theorem, g_n converges to the identity function uniformly on every compact subset of $[0,\infty)$, so $a = \lim_{n\to\infty} au_n = \lim_{n\to\infty} u_n a$. Then for every element of the form $aba, b \in A$, we know that

$$aba = \lim_{n\to\infty} abau_n = \lim_{n\to\infty} u_n aba,$$

and then by approximation, $(u_n)_{n\geqslant 1}$ is an approxiamte unit.

(b) Denote the hereditary C*-subalgebra which a generates by B. Since $0 \leqslant u_n \leqslant 2^n a$, we have $u_n \in B$. Since B is hereditary, $u_n c u_n \in B$ for all $c \in A$. Let $n \to \infty$, then $c \in B$, so $B = A$ and a is strictly positive.

Problem 7. Let Ω be a locally compact Hausdorff space. Show that $C_0(\Omega)$ admits an approximate unit $(p_n)_{n=1}^{\infty}$, where all the p_n are projections, if and only if Ω is the union of a sequence of compact open sets. Deduce that if a C*-algebra A admits a strictly positive element a such that $\sigma(a) \setminus \{0\}$ is discrete, then A admits an approximate unit $(p_n)_{n=1}^{\infty}$ consisting of projections.

Solution. If Ω is the union of a sequence of compact open sets $\Omega = \cup_{n\geqslant 1} A_n$, then $p_n = \chi_{B_n}$ belongs to $C_0(\Omega)$, where $B_n = \cup_{k=1}^{n} A_k$. Such a sequence (p_n) is an approximate unit, because if $f \in C_0(\Omega)$, for any $\varepsilon > 0$, there exists a compact $K \subseteq \Omega$ such that $|f| < \varepsilon$ on K^c. By compactness of K, it is covered by finitely many A_n, so it is contained in B_n for all sufficiently large n. By choice of K, $\|p_n f - f\| \leqslant \varepsilon$.

Conversely, if $C_0(\Omega)$ admits an approximate unit consisting of projections $(p_n)_{n=1}^{\infty}$, then each p_n can be written as χ_{B_n} where B_n are compact open sets. If $\Omega \neq \cup_{n\geqslant 1} B_n$, say $x \notin \cup_{n\geqslant 1} B_n$, then by Urysohn's lemma, there is a function $f \in C_0(\Omega)$ such that $f(x) = 1$. Clearly $\|p_n f - f\| \geqslant 1$, which is a contradiction. Hence $\Omega = \cup_{n\geqslant 1} B_n$ is the union of a sequence of compact open sets.

The (not necessarily unital) C*-subalgebra $C^*(a)$ generated by a is isomorphic to $C_0(\sigma(a) \setminus \{0\})$, so $C^*(a)$ admits an approximate unit consisting of projections. However, a is strictly positive, so by the same argument in Problem 6(a), this approximate unit is also one for A.

Problem 8. Let $z : \mathbb{T} \to \mathbb{C}$ be the inclusion map. Let $\theta \in [0,1]$. Show that there is a unique automorphism α of $C(\mathbb{T})$ such that $\alpha(z) = e^{i2\pi\theta}z$. Define the faithful positive linear functional $\tau : C(\mathbb{T}) \to \mathbb{C}$ by setting $\tau(f) = \int f\mathrm{d}m$ where m is normalized arc length on $\mathbb{T}$. Show that $\tau(\alpha(f)) = \tau(f)$ for all $f \in C(\mathbb{T})$. Deduce from Problem 2 in Chapter 3 that there is a unitary v on the Hilbert space H_τ such that $\varphi_\tau(\alpha(f)) = v\varphi_\tau(f)v^*$ for all $f \in C(\mathbb{T})$. Let u be the unitary $\varphi_\tau(z)$. Show that $vu = e^{i2\pi\theta}uv$. If θ is irrational, the C*-algebra A_θ generated by u and v is called an *irrational rotation* algebra, and A_θ can be shown to be simple.

Solution. Since z generates the whole $C(\mathbb{T})$, so the uniqueness of α is obvious. For any $f \in C(\mathbb{T})$, the automorphism is given explicitly by $\alpha(f)(\zeta) = f(e^{i2\pi\theta}\zeta)$.

Since the measure m is invariant under the transformation $z \mapsto e^{i2\pi\theta}z$, $\tau(\alpha(f)) = \tau(f)$ for all $f \in C(\mathbb{T})$. Actually, if we view $\mathbb{T}$ as the unit interval (with its end points pinned together), m is the usual Borel measure, and α is translating the variable by θ.

Problem 2(c) asserts that for the GNS representation (H_τ, φ_τ) corresponding to τ, there is a unitary v on H_τ such that $\varphi_\tau(\alpha(f)) = v\varphi_\tau(f)v^*$ for all $f \in C(\mathbb{T})$.

Since z is a unitary in $C(\mathbb{T})$, $u := \varphi_\tau(z)$ must be a unitary on H_τ. For $f \in C(\mathbb{T})$,

$$vu(f + N_\tau) = v(zf + N_\tau) = \alpha(zf) + N_\tau,$$

and

$$uv(f + N_\tau) = z\alpha(f) + N_\tau,$$

so $vu = e^{i2\pi\theta}uv$.

Actually more can be said. The GNS representation in this problem is simple. H_τ is $L^2(\mathbb{T})$, and φ_τ is the multiplication operator. So u is multiplication by z, and v is translating by θ (or rotating by $2\pi\theta$). This is a representation of the irrational rotation algebra.

Problem 9. Let m be normalized Haar measure on $\mathbb{T}$. If $\lambda \in \mathbb{C}, |\lambda| < 1$, define $\tau_\lambda : H^1 \to \mathbb{C}$ by setting

$$\tau_\lambda(f) = \int \frac{f(w)}{1-\lambda\overline{w}} \mathrm{d}m(w) \quad (f \in H^1).$$

Show that $\tau_\lambda \in (H^1)^*$. By expanding $(1-\lambda\overline{w})^{-1}$ in a power series, show that $\tau_\lambda(f) = \sum_{n=0}^{\infty} \hat{f}(n)\lambda^n$. Deduce that the function

$$\tilde{f} : \mathrm{int}\ \mathbb{D} \to \mathbb{C},\ \lambda \mapsto \tau_\lambda(f),$$

is holomorphic, where $\mathrm{int}\ \mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. If $f, g \in H^2$, show that $fg \in H^1$ and $\tau_\lambda(fg) = \tau_\lambda(f)\tau_\lambda(g)$.

Solution. Whenever $w \in \mathbb{T}$, $|(1-\lambda\overline{w})^{-1}| \leqslant (1-|\lambda|)^{-1}$, so $|\tau_\lambda(f)| \leqslant (1-|\lambda|)^{-1}\|f\|$ and $\tau_\lambda \in (H^1)^*$.

Whenever $|\lambda| < 1$,

$$(1-\lambda\overline{w})^{-1} = \sum_{n=0}^{\infty} \overline{w}^n\lambda^n \quad (w \in \mathbb{T}).$$

Therefore,

$$\begin{aligned}\tau_\lambda(f) &= \int f(w)\sum_{n=0}^{\infty}\overline{w}^n\lambda^n \mathrm{d}m(w) \\ &= \sum_{n=0}^{\infty}\int f(w)\overline{w}^n \mathrm{d}m(w)\lambda^n \\ &= \sum_{n=0}^{\infty}\hat{f}(n)\lambda^n.\end{aligned}$$

Here the interchange of the summation and the integral is justified by the DCT and the estimation in the beginning.

Since $f \in H^1$, all $\hat{f}(n)$ are bounded, so the convergence radius of $\tilde{f}$ at $\lambda = 0$ is at least 1, which proves that $\tilde{f}$ is holomorphic in $\mathrm{int}\ \mathbb{D}$.

If $f, g \in H^2$, then there exist two sequences of analytic trigonometric polynomials $(\varphi_n), (\psi_n)$ converging to f, g in L^2 norm, respectively. Then $\varphi_n\psi_n$ converges to fg in L^1 norm, and the Fourier coefficients of $\varphi_n\psi_n$ also converge to those of fg. Since $\widehat{\varphi_n\psi_n} = \hat{\varphi}_n * \hat{\psi}_n$, where $*$ is the convolution, it is clear that $\varphi_n\psi_n \in H^1$, so its limit $fg \in H^1$.

The identity $\tau_\lambda(fg) = \tau_\lambda(f)\tau_\lambda(g)$ also follows from a similar argument. The coefficients of the product of two power series behave exactly the same way as convolution, so we only need to prove that $\widehat{fg} = \hat{f} * \hat{g}$, which also follows by letting $n \to \infty$ in the case of φ_n, ψ_n.

Problem 10. If $f : \operatorname{int} \mathbb{D} \to \mathbb{C}$ is an analytic function and $0 < r < 1$, define $f_r \in C(\mathbb{T})$ by setting $f_r(\lambda) = f(r\lambda)$. Set $\|f\|_2 = \sup_{0<r<1} \|f_r\|_2$, and let $H^2(\mathbb{D})$ denote the set of all analytic functions $f : \operatorname{int} \mathbb{D} \to \mathbb{C}$ such that $\|f\|_2 < \infty$. If $f \in H^2(\mathbb{D})$, show that $\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |\lambda_n|^2}$, where $f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n$ is the Taylor series expansion of f. Show that $H^2(\mathbb{D})$ is a Hilbert space with inner product $\langle f, g\rangle = \sum_{n=0}^{\infty} \lambda_n \overline{\mu_n}$, where $\lambda_n = f^{(n)}(0)/n!$ and $\mu_n = g^{(n)}(0)/n!$ (the operations are pointwise-defined), and show also that the map

$$H^2 \to H^2(\mathbb{D}), \quad f \mapsto \tilde{f},$$

is a unitary operator. (Thus, the elements of H^2 can be interpreted as analytic functions on int $\mathbb{D}$ satisfying a growth condition approaching the boundary. A similar interpretation can be given for the other H^p-spaces.)

Solution. $f_r(\lambda) = \sum_{n=0}^{\infty} \lambda_n r^n \lambda^n$, so

$$\begin{aligned}\|f_r\|_2^2 &= \int_{\mathbb{T}} \sum_{n=0}^{\infty} \lambda_n r^n \lambda^n \cdot \sum_{m=0}^{\infty} \overline{\lambda}_m r^m \overline{\lambda}^m \mathrm{d}\lambda \\ &= \sum_{n=0}^{\infty} |\lambda_n|^2 r^{2n}.\end{aligned}$$

The norm is monotonely increasing with r, so $\|f\|_2 = \sum\limits_{n=0}^{\infty} |\lambda_n|^2$.

We only need to prove that $H^2(\mathbb{D})$ is complete with the given inner product. If (f_n) is a Cauchy sequence in $H^2(\mathbb{D})$, we write $f_n(\lambda) = \sum_{k=0}^{\infty} \lambda_{nk} \lambda^k$, corresponding to an element in $l^2(\mathbb{N})$, i.e. $a_n = (\lambda_{nk})_{k\geqslant 0}$. Since $l^2(\mathbb{N})$ is complete, a_n converges to $a = (c_k)_{k\geqslant 0}$. This element in l^2 corresponds to a power series $f(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k$, which is holomorphic in int $\mathbb{D}$, since c_k must be bounded, leading to the convergence radius $\geqslant 1$. The formula for the norm in $H^2(\mathbb{D})$ now shows that f_n converges to f in this norm.

The last statement follows from what we have proved in this problem and in Problem 9.

Problem 11. Show that if φ is a function in $L^\infty(\mathbb{T})$ not almost everywhere zero, then either T_φ or T_φ^* is injective. Deduce that T_φ is invertible if and only if it is a Fredholm operator of index zero.

Solution. T_φ is the Toeplitz operator with symbol φ. If $f \in \ker T_\varphi, g \in \ker T_\varphi^*$, then $\varphi f \perp H^2, \overline{\varphi} g \perp H^2$. Note that $\varphi f \perp H^2 \Rightarrow \overline{\varphi f} \in H^2$, since $\widehat{\overline{\varphi f}}(n) = \overline{\widehat{\varphi f}(-n)}$ and thus $(H^2)^\perp = \{f \in L^2 : \hat{f}(n) = 0, \forall n \geqslant 0\}$. We have proved in Problem 9 that the product of two H^2 functions lie in H^1, so $\varphi f \overline{g}, \overline{\varphi f} g \in H^1$. Any function in H^1 whose conjugate also lies in H^1 must be constant (Lemma 3.5.1), so $\varphi f \overline{g}$ is constant. Moreover, by examing the zeroth Fourier coefficient, $\varphi f \overline{g} = 0$. (One can see this from the expression of the zeroth Fourier coefficient, which is the inner product of φf and g.)

Now by the assumption on φ and Theorem 3.5.4, the set $\{f = 0\} \cup \{g = 0\}$ has positive measure, so $f = 0$ or $g = 0$. This proves the first statement.

For the second statement, if T_φ is a Fredholm operator of index zero, then $\dim \ker T_\varphi = \dim \ker T_\varphi^*$, so both of them are 0. So T_φ is bijective, and thus invertible. The reverse direction is obvious.