Chapter 1

Elementary Spectral Theory

Problem 1. Let $(A_\lambda)_{\lambda\in\Lambda}$ denote a family of Banach algebras. The *direct sum* $A = \oplus_\lambda A_\lambda$ is the set of all $(a_\lambda) \in \prod_\lambda A_\lambda$ such that $\|(a_\lambda)\| = \sup_\lambda \|a_\lambda\|$ is finite. Show that this is a Banach algebra under the pointwise-defined operations

$$(a_\lambda) + (b_\lambda) = (a_\lambda + b_\lambda)$$
$$\mu(a_\lambda) = (\mu a_\lambda)$$
$$(a_\lambda)(b_\lambda) = (a_\lambda b_\lambda),$$

and norm given by $(a_\lambda) \mapsto \|(a_\lambda)\|$. Show that A is unital or abelian if this is the case for all of the algebras A_λ.

The *restricted sum* $B = \oplus_\lambda^{c_0} A_\lambda$ is the set of all elements $(a_\lambda) \in A$ such that for each $\varepsilon > 0$ there exists a finite subset F of Λ for which $\|a_\lambda\| < \varepsilon$ if $\lambda \in \Lambda \setminus F$. Show that B is a closed ideal in A.

Solution. It is easily seen that A is a $\mathbb{C}$-algebra with the given norm being submultiplicative. To prove that it is complete, suppose that $\{a^m = (a_\lambda^m)\}$ is a Cauchy sequence. Then for any fixed λ, the pointwise sequence $\{a_\lambda^m\}$ is Cauchy in A_λ, and thus converges to some c_λ. Denote $c = (c_\lambda)$ and it is easy to see that $\sup_\lambda \|c_\lambda\|$ is finite, so $c \in A$. For any $\varepsilon > 0$, there exists some N such that $m, n > N$ implies $\|a^n - a^m\| \leqslant \varepsilon$, i.e. $\|a_\lambda^n - a_\lambda^m\| \leqslant \varepsilon$ for any $\lambda \in \Lambda$. Let $m \to \infty$, we get $\|a_\lambda^n - c_\lambda\| \leqslant \varepsilon$ for any $\lambda \in \Lambda$. This is exactly $\|a^n - c\| \leqslant \varepsilon$ when $n > N$, so c is the limit of $\{a^m\}$ in A.

When all A_λ are unital with the identity e_λ, then $\|e_\lambda\| = 1$ and $e = (e_\lambda) \in A$ must be the identity. When all A_λ are abelian, it is obvious that A is abelian.

Suppose $b = (b_\lambda) \in B$ and $a = (a_\lambda) \in A$ with $\|a\| \leqslant M < \infty$, then for each $\varepsilon > 0$, there exists a finite subset F of Λ such that $\|b_\lambda\| < \varepsilon/M$ if $\lambda \in \Lambda \setminus F$. This implies that $\|a_\lambda b_\lambda\| < \varepsilon$ when $\lambda \in \Lambda \setminus F$, so $ab \in B$. Similarly, $ba \in B$. For a sequence $\{b^n = (b^n_\lambda)\} \subseteq B$ converging to $a = (a_\lambda) \in A$. For any $\varepsilon > 0$, there exists by definition some n and a finite subset F of Λ such that $\|b^n - a\| < \varepsilon/3$ and $\|b^n_\lambda\| < \varepsilon/3$ when $\lambda \in \Lambda \setminus F$. So $\|a_\lambda\| < \varepsilon$ when $\lambda \in \Lambda \setminus F$.This means $a \in B$, i.e. B is closed in A.

Problem 2. Let A be a Banach algebra and Ω a non-empty set. Denote by $l^\infty(\Omega, A)$ the set of all bounded maps f from Ω to A. Show that $l^\infty(\Omega, A)$ is a Banach algebra with the pointwise-defined operations and the sup-norm $\|f\| = \sup\{\|f(\omega\| : \omega \in \Omega)\}$. If Ω is a compact Hausdorff space, show that the set $C(\Omega, A)$ of all continuous functions from Ω to A is a closed subalgebra of $l^\infty(\Omega, A)$.

Solution. For the first statement, it is just the case in Problem 1 where A_λ are all equal to A and the index set $\Lambda = \Omega$.

If Ω is compact Hausdorff, first see why $C(\Omega, A)$ is contained in $l^\infty(\Omega, A)$. It follows from considering the composition of two continuous maps $\Omega \xrightarrow{f} A \xrightarrow{\|\cdot\|} \mathbb{R}$. For algebraic operations, consider $\Omega \xrightarrow{f\times g} A \times A \xrightarrow{+,\cdot} A$. Every map is continuous (the first one follows from the universal property of product topology), so $C(\Omega, A)$ is closed under pointwise addition and multiplication. It is certainly closed under scalar multiplication.

The last thing is its completeness. Suppose $\{f_n\} \subseteq C(\Omega, A)$ converges to $f \in l^\infty(\Omega, A)$. For $\omega \in \Omega$ and $\varepsilon > 0$, there is some n and an open neighbourhood U of ω such that $\|f_n - f\| < \varepsilon/3$ and $\|f_n(\omega) - f_n(\omega')\| < \varepsilon/3$ when $\omega' \in U$. Then $\|f(\omega') - f(\omega)\| < \varepsilon$ when $\omega' \in U$. So f is continuous.

(Actually, the proof is also similar to the last part of Problem 1, where we consider the continuity of the point at infinity with Λ equipped with discrete topology being a locally compact Hausdorff space.)

Problem 3. Give an example of a unital non-abelian Banach algebra A in which 0 and A are the only close ideals.

Solution. The matrix algebra $A = M_n(\mathbb{C})$ is an example when $n > 1$. The norm is given by the operator norm on the Hilbert space $\mathbb{C}^n$.

Suppose an ideal $I \leqslant A$ contains a non-zero element M, then there exist two matrices P, Q such that $PMQ = E_{1,1}$ where $E_{1,1}$ is the elementary matrix, i.e. its only non-zero entry lies at $(1, 1)$ and equals 1.

It is easy to see that $E_{1,1}$ generates the whole A as an ideal, so $I = A$.

Problem 4. Give an example of a non-modular maximal ideal in an abelian Banach algebra. (If A is the disc algebra, let $A_0 = \{f \in A : f(0) = 0\}$. Then A_0 is a closed subalgebra of A and admits an ideal of the type required.)

Solution. An ideal I of A is modular if there exist an element $u \in A$ such that $a - au, a - ua \in I$ for all $a \in A$.

Let $A_0 = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}), f(0) = 0\}$ and $J = \{f \in A_0 : f'(0) = 0\}$.

Consider the evaluation of derivative $A_0 \to \mathbb{C}, f \mapsto f'(0)$. This is a surjective linear map with kernel J. In other words, J is a linear subspace of A_0 with codimension 1, so J is maximal.

If J is modular, suppose $u \in A_0$ satisfy the definition. For any $a \in A_0$, we can calculate the derivative of au at 0: $(au)'(0) = a'(0)u(0) + a(0)u'(0)$. By construction, $a(0) = u(0) = 0$, so $(au)'(0) = 0$, i.e. $au \in J$. By definition, $a - au \in J$, so $a \in J$ for all $a \in A_0$, which is absurd.

Problem 5. Let A be a unital abelian Banach algebra.

(a) Show that $\sigma(a+b) \subseteq \sigma(a)+\sigma(b)$ and $\sigma(ab) \subseteq \sigma(a)\sigma(b)$ for all $a, b \in A$. Show that this is not true for *all* Banach algebras.

(b) Show that if A contains an *idempotent* e (that is, $e = e^2$) other than 0 and 1, then $\Omega(A)$ is disconnected.

(c) Let $a_1, \ldots, a_n$ generate A as a Banach algebra. Show that $\Omega(A)$ is homeomorphic to a compact subset of $\mathbb{C}^n$. More precisely, set $\sigma(a_1, \ldots, a_n) = \{(\tau(a_1), \ldots, \tau(a_n)) : \tau \in \Omega(A)\}$. Show that the canonical map from $\Omega(A)$ to $\sigma(a_1, \ldots, a_n)$ is a homeomorphism.

Solution. (a) Since A is unital abelian, we can use characters to represent the spectrum and the statement is certainly true.

In detail,

$$\begin{aligned}
\sigma(a+b) &= \{\tau(a+b) : \tau \in \Omega(A)\} \\
&= \{\tau(a)+\tau(b) : \tau \in \Omega(A)\} \\
&\subseteq \{\tau(a) : \tau \in \Omega(A)\} + \{\tau(b) : \tau \in \Omega(A)\} \\
&= \sigma(a)+\sigma(b).
\end{aligned}$$

Similar for $\sigma(ab) \subseteq \sigma(a)\sigma(b)$.

In fact, the statement only requires $ab = ba$, since we can consider the subalgebra generated by $1, a, b, (\lambda - a)^{-1}, (\mu - b)^{-1}$ when taking inverse makes sense.

In general, for example $A = M_2(\mathbb{C})$, take $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The spectrum is exactly the set of eigenvalues, so $\sigma(a+b) = \{1, -1\}, \sigma(a) = \sigma(b) = \{0\}$.

(b) Denote $A_1 = eA, A_2 = (1-e)A$, then A_1, A_2 are closed ideals. It can be seen from elementary linear algebra: consider two linear maps from A to A, $T_1 : a \mapsto ea$ and $T_2 : a \mapsto (1-e)a$. Then $T_1^2 = T_1, T_2^2 = T_2, T_1 + T_2 = \mathrm{id}$. So $A_1 = \mathrm{ran}(T_1) = \ker(T_2), A_2 = \mathrm{ran}(T_2) = \ker(T_1)$ are closed. Since e is non-trivial, A_1, A_2 are non-trivial.

Note that A_1, A_2 are unital abelian with identities e and $1-e$ respectively, so we can form $\Omega_1 := \Omega(A_1), \Omega_2 := \Omega(A_2)$. For any character τ_1 on A_1, we can define τ to be 0 on A_2, and linearly extend it to A, still denoted by τ_1. It is easy to check that the extension is a character on A. We can think of it as an inclusion map $\Omega_1 \subseteq \Omega(A)$, and the inclusion is obviously continuous. Similarly Ω_2 can be regarded as a subset of $\Omega(A)$.

Now observe that $\Omega(A) = \Omega_1 \cup \Omega_2$ is a decomposition into the union of two disjoint closed subsets. If $\tau \in \Omega_1 \cap \Omega_2$, then τ is zero on the whole A, which is not allowed as a character. Each $\Omega_i, i = 1, 2$ is the continuous image of a compact space, so each is a closed subset. And every $\tau \in \Omega(A)$ must take values 0 or 1 at $e, 1-e, 1$, so the only possibilities are that $\tau(e) = 0, \tau(1-e) = 1$ and $\tau(1-e) = 0, \tau(1-e) = 1$. These two cases are symmetric by interchanging e with $1-e$. WLOG, $\tau(e) = 0, \tau(1-e) = 1$, so τ is zero on A_1, i.e. $\tau \in \Omega_1$. The other case implies $\tau \in \Omega_2$.

This disjoint union decomposition shows that $\Omega(A)$ is disconnected.

(c) The given map $\Omega(A) \to \mathbb{C}^n$ is certainly continuous and injective. So it is a homeomorphism onto its image.

In detail, continuity follows from the definition of weak*-topology, and injectivity can be argued as follows: given $\tau_1, \tau_2 \in \Omega(A)$, the set $\{a \in A : \tau_1(a) = \tau_2(a)\}$ must form a Banach subalgebra which contains $a_1, \ldots, a_n$. But these elements generate the whole A, so $\tau_1 = \tau_2$.

Problem 6. Let A be a unital Banach algebra.

(a) If a is invertible in A, show that $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$.

(b) For any element $a \in A$, show that $r(a^n) = (r(a))^n$.

(c) If A is abelian, show that the Gelfand representation is isometric if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.

Solution. (a) If $\lambda \notin \sigma(a)$ and $\lambda \neq 0$, then $\lambda(\lambda^{-1} - a^{-1})a = a - \lambda$ is invertible, so $\lambda^{-1} \notin \sigma(a^{-1})$. By definition, $0 \notin \sigma(a^{-1})$, so $\sigma(a^{-1}) \subseteq \sigma(a)^{-1}$.

Interchange a and a^{-1}, and we get $\sigma(a) \subseteq \sigma(a^{-1})^{-1}$. Combine these two inclusions together, and we reach the conclusion.

(b) We invoke the formula of spectral radius $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. Therefore,

$$r(a^n) = \lim_{m\to\infty} \|a^{mn}\|^{\frac{1}{m}} = (\lim_{m\to\infty} \|a^{mn}\|^{\frac{1}{mn}})^n = (r(a))^n.$$

(c) By definition, the Gelfand representation is isometric if and only if $\|a\| = r(a), \forall a \in A$.

If $\|a^2\| = \|a\|^2, \forall a \in A$, then $\|a^{2^n}\| = \|a\|^{2^n}$ and hence

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|.$$

Conversely, $\|a^2\| = r(a^2) = (r(a))^2 = \|a\|^2, \forall a \in A$.

Problem 7. Let A be a Banach algebra. Show that the spectral radius function $r : A \to \mathbb{R}$ is upper semi-continuous.

Solution. Since $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n}$ is the infimum of a family of continuous functions, it is upper semi-continuous.

Another proof is to use the definition. If $\lambda \in \mathbb{C}, |\lambda| > r(a)$, then $\lambda - a$ is invertible. Since the set of all invertible elements is open in A, so for any sequence $\{a_n\}$ converging to a, $\lambda - a_n$ is invertible for sufficiently large n. Therefore, $\lambda \notin \sigma(a_n)$ for n large enough, and $r(a_n) \leqslant r(a)$ for large n. This is the definition of upper semi-continuous function.

Problem 8. Show that if B is a maximal abelian subalgebra of a unital Banach algebra A, then B is closed and contains the unit. Show that $\sigma_A(b) = \sigma_B(b)$ for all $b \in B$.

Solution. It is easy to prove that $\overline{B}$ is an abelian subalgebra which contains B, and the unitalization B^+ is too. Since B is maximal, it must be true that $B = \overline{B}$ and $B = B^+$, i.e. B is closed and contains the unit.

For $b \in B$, it is obvious by definition that $\sigma_A(b) \subseteq \sigma_B(b)$. Conversely, if $\lambda \notin \sigma_A(b)$, there exists in A an inverse of $\lambda - b$, denoted by a. Then a commutes with B.

To see this, we start from the assumption that B is abelian. For any $c \in B$, we know $bc = cb$ and $(\lambda - b)c = c(\lambda - b)$. Then multiply this by a both on the left and the right, we get $ca = ac$. Now that a commutes with B and B is maximal abelian, we must have $a \in B$, i.e. the inverse of $\lambda - b$ lies in B. This proves the required statement.

Problem 9. Let (Ω, μ) be a measure space. Show that the linear span of the idempotents is dense in $L^\infty(\Omega, \mu)$. Show that the spectrum of the Banach algebra $L^\infty(\Omega, \mu)$ is totally disconnected, by showing that if A is and arbitrary abelian Banach algebra in which the idempotents have dense linear span, its spectrum $\Omega(A)$ is totally disconnected.

Solution. The first statement is routine in real analysis. An idempotent in $L^\infty(\Omega, \mu)$ is exactly a characteristic function of a measurable set, so for a bounded measurable function on Ω, we divide its range into finitely many parts such that the diameter of each part is sufficiently small. Then take a linear combination of characteristic functions like $\{\omega \in \Omega : f(\omega) \in D\}$ where D is a "small part".

A topological space is called disconnected if the only connected subsets are singletons. If a subset $X \subseteq \Omega(A)$ contains two different elements $\varphi \neq \psi$, then there exists an idempotent e such that $\varphi(e) \neq \psi(e)$. Obviously, such an idempotent cannot be trivial. WLOG, $\varphi(e) = 0, \psi(e) = 1$. In Problem 5(b), we have proved that φ is zero on eA and ψ is zero on $(1-e)A$. Moreover, ψ and φ are separated by two disjoint closed subsets $\Omega(eA)$ and $\Omega((1-e)A)$ whose union is $\Omega(A)$. Therefore, a subset containing more than one element cannot be connected.

Problem 10. Let $A = C^1[0,1]$, as in Example 1.2.6. Let $x : [0,1] \to \mathbb{C}$ be the inclusion. Show that x generates A as a Banach algebra. If $t \in [0,1]$, show that τ_t belongs to $\Omega(A)$, where τ_t is defined by $\tau_t(f) = f(t)$, and show that the map $[0,1] \to \Omega(A), t \mapsto \tau_t$ is a homeomorphism. Deduce that $r(f) = \|f\|_\infty$ $(f \in A)$. Show that the Gelfand representation is not surjective for this example.

Solution. In Example 1.2.6, the norm is given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Obviously, A is a unital abelian Banach algebra.

By Weierstrass Theorem, for any complex-valued continuous function on $[0,1]$, we can use polynomials with complex coefficients to approximate it uniformly. So if $f \in A$ and $\varepsilon > 0$, we find a polynomial p such that $\|p - f'\|_\infty < \varepsilon$. Then $q := f(0) + \int_0^t p(s)\mathrm{d}s$ is a polynomial, and $\|q - f\|_\infty \leqslant \varepsilon, q' = p$. This means that we can approximate f in the norm of A with polynomials, and x generates A as a Banach algebra.

τ_t is surely a character, and $t \mapsto \tau_t$ is certainly a continuous injective map. If $\tau \in \Omega(A)$, then let $t = \tau(x)$. Since $\sigma(x) = \{\tau(x) : x \in \Omega(A)\}$, and we can directly calculate the spectrum of x– $\sigma(x) = [0,1]$–we know that $t \in [0,1]$. Note that $\tau(x) = \tau_t(x)$ and x generates A as a Banach algebra, we have $\tau = \tau_t$. So the map $t \mapsto \tau_t$ is surjective. A continuous bijection between compact Hausdorff spaces must be a homeomorphism.

$r(f)$ is the maximum modulus of the spectrum of f, and

$$\sigma(f) = \{\tau(f) : \tau \in \Omega(A)\} = \{\tau_t(f) : t \in [0,1]\} = \{f(t) : t \in [0,1]\}.$$

Therefore, $r(f) = \|f\|_\infty$.

For the last statement, surely $C^1[0,1] \neq C[0,1]$.

Problem 11. Let A be a unital Banach algebra and set

$$\zeta(a) = \inf_{\|b\|=1} \|ab\| \qquad (a \in A).$$

We say that an element of a of A is a *left topological zero divisor* if there is a sequence of unit vectors (a_n) of A such that $\lim_{n\to\infty} aa_n = 0$. Equivalently, $\zeta(a) = 0$.

(a) Show that left topological zero divisor are not invertible.

(b) Show that $|\zeta(a) - \zeta(b)| \leqslant \|a-b\|$ for all $a, b \in A$. Hence, ζ is a continuous function.

(c) If a is a boundary point of the set $\mathrm{Inv}(A)$ in A, show that there is a sequence of invertible elements (v_n) converging to a such that $\lim_{n\to\infty} \|v_n^{-1}\|^{-1} = 0$. Using the continuity of ζ, deduce that $\zeta(a) = 0$. Thus, boundary points of $\mathrm{Inv}(A)$ are left topological zero divisors. In particular, if λ is a boundary point of the spectrum of an element a of A, then $\lambda - a$ is a left topological zero divisor.

(d) Let Ω be a compact Hausdorff space and let $A = C(\Omega)$. Show that in this case the topological zero divisors are precisely the non-invertible elements (if f is non-invertible, then 0 is a boundary point of the spectrum of $\overline{f}f$).

(e) Give an example of a unital Banach algebra and a non-invertible element that is not a left topological zero divisor.

Solution. (a) If a is invertible, then given $\|b\| = 1$, we know that $\|ab\| \cdot \|a^{-1}\| \geqslant \|b\| = 1$, so $\|ab\| \geqslant \|a^{-1}\|^{-1} > 0$ and $\zeta(a) \geqslant \|a^{-1}\|^{-1}$.

(b) By definition,

$$\begin{aligned} \zeta(a) &= \inf_{\|c\|=1} \|ac\| \\ &\leqslant \inf_{\|c\|=1} (\|bc\| + \|(a-b)c\|) \\ &\leqslant \inf_{\|c\|=1} (\|bc\| + \|a-b\|). \end{aligned}$$

Taking infimum over c on the last term, we get $\zeta(a) \leqslant \zeta(b) + \|a-b\|$. Interchange a and b, and we get the required inequality.

(c) If not, then there exists $\delta > 0, \varepsilon \in (0, \delta)$ such that whenever $\|x - a\| < \varepsilon$ and x is invertible, $\|x^{-1}\|^{-1} > \delta$, i.e. $\|x^{-1}\| < \delta^{-1}$. But it is well-known that if x is invertible and $\|y\| < \|x^{-1}\|^{-1}$, the element $x - y$ is invertible. So if $\|y\| < \delta$, then $x - y$ is invertible. Now put $y = x - a$, then a is invertible. But $\mathrm{Inv}(A)$ is an open subset of A, which is a contradiction to a being a boundary point.

(d) If f is not invertible, then f has a zero point ω. Using Urysohn's lemma, we can construct a continuous function g on Ω such that g is supported in an open neighbourhood U of ω and $\|g\| = 1$. Since $f(\omega) = 0$, we let U be such that whenever $\omega' \in U$, $|f(\omega')| < \varepsilon$ where $\varepsilon > 0$. Then $\|fg\| \leqslant \varepsilon$. It then follows that $\zeta(f) = 0$.

(e) If some element is left invertible but not invertible, it will be an example. In $B(H)$, there are plenty of examples.

Problem 12. A derivation on an algebra A is a linear map $d : A \to A$ such that $d(ab) = adb + d(a)b$. Show that the Leibnitz formula,

$$d^n(ab) = \sum_{r=0}^{n} \binom{n}{r} d^r(a)d^{n-r}(b) \qquad (n = 1, 2, \ldots),$$

holds.

Solution. We prove by induction. If the formula holds for $n-1$ where $n > 1$, then

$$\begin{aligned}
d^n(ab) &= dd^{n-1}(ab) \\
&= d\left(\sum_{r=0}^{n-1} \binom{n-1}{r} d^r(a)d^{n-1-r}(b)\right) \\
&= \sum_{r=0}^{n-1} \binom{n-1}{r} (d^{r+1}(a)d^{n-1-r}(b) + d^r(a)d^{n-r}(b)) \\
&= \sum_{r=0}^{n} \binom{n}{r} d^r(a)d^{n-r}(b).
\end{aligned}$$

In the last equality, we use $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$.

Problem 13. Suppose that d is a bounded derivation on a unital Banach algebra A and $\lambda \in \mathbb{C} \setminus \{0\}$ such that $da = \lambda a$. Show that a is nilpotent, that is, that $a^n = 0$ for some positive integer n.

Solution. By induction, we find that $d(a^n) = n\lambda a^n$. Since $\lambda \neq 0$, when n is sufficiently large, $d - n\lambda$ is invertible. But $(d - n\lambda)(a^n) = 0$, so $a^n = 0$.

Problem 14. Suppose that d is a bounded derivation on a unital Banach algebra A, and that $a \in A$ and $d^2 a = 0$. Show that da is quasinilpotent. For $a \in A$, the map $b \mapsto [a, b] = ab - ba$ is a bounded derivation on A. Therefore, the Kleinecke-Shirokov theorem holds: If $[a, [a, b]] = 0$, then $[a, b]$ is nilpotent.

Solution. By Leibnitz formula, we have

$$d^n(a^m) = \sum_{r=0}^{n} \binom{n}{r} d^r(a) d^{n-r}(a^m - 1).$$

Since $d^r(a) = 0$ when $r \geqslant 2$, the above equality is actually very simple:

$$d^n(a^m) = a d^n(a^{m-1}) + n d(a) d^{n-1}(a^{m-1}).$$

So by induction, whenever $n > m \geqslant 1$, we have $d^n(a^m) = 0$ and $d^n(a^n) = n!(da)^n$. Then the spectral radius of da is

$$\lim_n \|(da)^n\|^{1/n} = \lim_n \|d^n(a^n)/n!\|^{1/n} \leqslant \lim_n \|1/n!\|^{1/n} \|d\| \|a\| = 0,$$

i.e. da is quasinilpotent.

The Kleinecke-Shirikov theorem follows from the first statement where d is given by $d(b) := ab - ba$.

Problem 15. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=1}^{\infty}$, and let u be an operator in $B(H)$ diagonal with respect to (e_n) with diagonal the sequence (λ_n). Show that u is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.

Solution. It is easy to calculate that if u is of the given form, then $\|u\| = \sup\{\lambda_n : n \geqslant 1\}$. Let p_n denote the orthogonal projection onto the space spanned by $e_1, \dots, e_n$.

If $\lim_{n\to\infty} \lambda_n = 0$, then $p_n u p_n$ are finite-rank operators and by the observation in the beginning, $\|p_n u p_n - u\| = \sup\{\lambda_k : k \leqslant n+1\}$ tends to zero when $n \to \infty$. So u is the norm limit of a sequence of finite-rank operators, and is hence compact.

Conversely, if u is compact, then for any subsequence (λ_{n_k}) of (λ_n), the set $\{e_{n_k}\}$ consists of vectors of norm 1, so $\{ue_{n_k}\}$ contains a convergent (sub-)subsequence indexed by n_{k_l}. By assumption, $ue_{n_k} = \lambda_{n_k} e_{n_k}$. The e_{n_k} are orthogonal to each other, so by Cauchy criterion, we have $\lambda_{n_{k_l}} \to 0$. By a routine argument in mathematical analysis (proof by contradiction and Bolzano-Weierstrass theorem), $\lambda_n \to 0$.

Problem 16. Let X be a Banach space. If $p \in B(X)$ is a compact idempotent, show that its rank is finite.

Solution. By assumption, X can be decomposed into direct sum of pX and $(1-p)X$, both of which are closed subspace of X and are hence Banach space. Composing the maps $pX \xrightarrow{\text{inclusion}} X \xrightarrow{p} pX$ gives the identity map on pX, which is compact since p is. However, the identity map on a Banach space is compact if and only if the space is finite-dimensional. So pX is finite-dimensional, i.e. p is of finite rank.

Problem 17. Let $u : X \to Y$ be a compact operator between Banach spaces. Show that if the range of u is closed, then it is finite-dimensional.

Solution. If the range of u is closed, the map $X \xrightarrow{u} uX$ is a surjective, compact linear operator between Banach spaces. By the open mapping theorem, u is an open map. So the image under u of the closed unit ball of X must contain an open neighbourhood of $0 \in uX$, and since u is compact, the image of the closed unit ball is relatively compact. This implies that the chosen open neighbourhood of 0 is relatively compact, so uX is finite-dimensional.

Problem 18. Let X, Y be Banach spaces and suppose that $u \in B(X, Y)$ has compact transpose u^*. Show that u is compact using that fact that u^{**} is compact.

Solution. By Theorem 1.4.4 which says $u^* \in B(Y^*, X^*)$ is compact when $u \in B(X, Y)$ is compact, we know that in this exercise, $u^{**} \in B(X^{**}, Y^{**})$ is compact. Consider the natural inclusion $X \to X^{**}, Y \to Y^{**}$, denoted by ι_1, ι_2 respectively. A simple check by definition gives the following commuting diagram:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\iota_1"'] & Y \arrow[d, "\iota_2"] \\ X^{**} \arrow[r, "u^{**}"] & Y^{**} \end{tikzcd}$$

Now suppose (x_n) is a bounded sequence in X. Since ι_1 is an isometry and u^{**} is compact, we know that $(u^{**}\iota_1(x_n))$ admits a subsequence that converges in Y^{**}. So $(\iota_2 u(x_n))$ admits a subsequence that converges in Y^{**}. Now use that ι_2 is an isometry, and we know that the convergent subsequence in Y^{**} gives rise to a convergent subsequence of $(u(x_n))$ in Y with the same subscripts, so u is compact.

Problem 19. Let $u : X \to Y$ and $u' : X' \to Y'$ be bounded operators between Banach spaces. Show that the linear map

$$u \oplus u' : X \oplus X' \to Y \oplus Y', (x, x') \mapsto (u(x), u'(x')),$$

is bounded with norm $\max\{\|u\|, \|u'\|\}$. Show that if u and u' are Fredholm operators, so is $u \oplus u'$, and $\operatorname{ind}(u \oplus u') = \operatorname{ind}(u) + \operatorname{ind}(u')$.

Solution. There are several choices for the norm of direct sum of Banach spaces, and I do not know whether the statement is true for all choices of norms. Now fix $\|(x, x')\| := \max(\|x\|, \|x'\|)$.

Then

$$\|(u \oplus u')(x, x')\| = \max(\|ux\|, \|u'x'\|) \leqslant \max(\|u\|, \|u'\|) \max(\|x\|, \|x'\|),$$

so $\|u \oplus u'\| \leqslant \max(\|u\|, \|u'\|)$. Conversely, $(u \oplus u')(x, 0) = (ux, 0)$ and $\|(x, 0)\| = \|x\|, \|(ux, 0)\| = \|ux\|$, so take supremum over $x \in X, \|x\| \leqslant 1$, and we get $\|u \oplus u'\| \geqslant \|u\|$. Do the same for u' and we prove the first statement.

The kernel of $u \oplus u'$ is $\ker u \oplus \ker u'$, and the cokernel is $(Y \oplus Y')/(uX \oplus u'X') \cong (Y/uX) \oplus (Y'/u'X')$. So the index of $u \oplus u'$ is the sum of those of u and u'.

Problem 20. If X is an infinite-dimensional Banach space and $u \in B(X)$, show that

$$\bigcap_{v \in K(X)} \sigma(u+v) = \sigma(u) \setminus \{\lambda \in \mathbb{C} : u - \lambda \quad \text{is Fredholm of index zero}\}.$$

Solution. If a is invertible and v is compact, then $\operatorname{ind}(a+v) = \operatorname{ind}(a) = 0$. This holds when a is Fredholm of index zero. We denote the set on the left by L and the one on the right by R.

If $\lambda \notin R$, then $u - \lambda$ is invertible or $u - \lambda$ is Fredholm of index zero. In both cases, $u - \lambda + v$ must be Fredholm of index zero for any compact v, by the observation in the beginning. Then by Remark 1.4.3, any Fredholm operator of index zero is a compact perturbation of an invertible element, so there exists $v \in K(X)$ such that $u + v - \lambda$ is invertible, i.e. $\lambda \notin L$. This shows that $L \subseteq R$.

Conversely, if $\lambda \notin L$, then there exists a $v \in K(X)$ such that $u + v - \lambda$ is invertible. Then the Fredholm index of $u + v - \lambda$ is zero, and thus the index of $u - \lambda$ is zero. This shows that $\lambda \notin R$ and $R \subseteq L$.