Chapter 6

Direct Limits and Tensor Products

Problem 1. Let $(A_n, \varphi_n)_{n=1}^{\infty}$ and $(B_n, \psi_n)_{n=1}^{\infty}$ be direct sequences of C*-algebras with direct limits A and B, respectively. Let $\varphi^n : A_n \to A$ and $\psi^n : B_n \to B$ be the natural maps. Suppose there are *-homomorphisms $\pi_n : A_n \to B_n$ such that for each n the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ \big\downarrow{\scriptstyle \pi_n} & & \big\downarrow{\scriptstyle \pi_{n+1}} \\ B_n & \xrightarrow{\psi_n} & B_{n+1} \end{array}$$

Show that there exists a unique *-homomorphism $\pi : A \to B$ such that for each n the following diagram commutes:

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi^n} & A \\ \big\downarrow{\scriptstyle \pi_n} & & \big\downarrow{\scriptstyle \pi} \\ B_n & \xrightarrow{\psi^n} & B \end{array}$$

Show that if all the π_n are *-isomorphisms, then π is a *-isomorphism.

Solution. This is an easy application of the universal property of the direct limit, i.e. part (2) of Theorem 6.1.2.

Now we have the following commuting diagram:

$$
\begin{tikzcd}
 & & A \arrow[ddd, dashed, "\pi"] \\
A_n \arrow[r, "\varphi_n"'] \arrow[urr, "\varphi^n"] \arrow[d, "\pi_n"] & A_{n+1} \arrow[ur, "\varphi^{n+1}"'] \arrow[d, "\pi_{n+1}"] & \\
B_n \arrow[r, "\psi_n"] \arrow[drr, "\psi^n"'] & B_{n+1} \arrow[dr, "\psi_{n+1}"] & \\
 & & B
\end{tikzcd}
$$

The *-homomorphism π comes from applying the theorem to the compositions $\psi^n \circ \pi_n$ for all n.

If all the π_n are *-isomorphisms, we interchange A_n with B_n, repeat the whole process, and obtain a map $\rho : B \to A$. Then we can consider again a commuting diagram:

$$
\begin{tikzcd}
 & & A \arrow[ddd, dashed, "\pi"] \\
A_n \arrow[r, "\varphi_n"'] \arrow[urr, "\varphi^n"] \arrow[d, "\pi_n"] \arrow[dd, bend right, "\mathrm{id}"'] & A_{n+1} \arrow[ur, "\varphi^{n+1}"'] \arrow[d, "\pi_{n+1}"] \arrow[dd, bend left, "\mathrm{id}"] & \\
B_n \arrow[r, "\psi_n"] \arrow[d, "\pi_n^{-1}"] & B_{n+1} \arrow[d, "\pi_{n+1}^{-1}"] & \\
A_n \arrow[r, "\varphi_n"] \arrow[drr, "\varphi^n"'] & A_{n+1} \arrow[dr, "\varphi^{n+1}"] & B \arrow[d, "\rho"] \\
 & & A
\end{tikzcd}
$$

By uniqueness of the right vertical arrow, $\rho \circ \pi = \mathrm{id}_A$. Repeat the process for $\pi \circ \rho$, and ρ is the inverse of π, so we complete the proof.

Problem 2. Show that every non-zero finite-dimensional C*-algebra admits a faithful tracial state. Give an example of a unital simple C*-algebra not having a tracial state.

Solution. By Theorem 6.3.8, every non-zero finite-dimensional C*-algebra is *-isomorphic to $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \ldots M_{n_k}(\mathbb{C})$. There is a unique faithful tracial state $\tau_j : M_{n_j}(\mathbb{C}) \to \mathbb{C}$ for each $j = 1, 2, \ldots, k$. Then $(\tau_1 + \ldots + \tau_k)/k$ is a faithful tracial state.

The Calkin algebra $Q(H)$ for an infinite-dimensional separable Hilbert space H is a unital simple C*-algebra. If it admits a tracial state τ, then $\tau \circ \pi$ is a tracial state on $B(H)$, where π is the quotient map from $B(H)$ onto $Q(H)$. However, $B(H)$ does not admit a tracial state, which leads to a contradiction.

Problem 3. Let A be a C*-algebra. A *trace* on A is a function $\tau : A^+ \to [0, +\infty]$ such that

$$\tau(a+b) = \tau(a) + \tau(b)$$

$$\tau(ta) = t\tau(a)$$

$$\tau(c^*c) = \tau(cc^*)$$

for all $a, b \in A^+, c \in A$, and all $t \in \mathbb{R}^+$. We use the convention that $0 \cdot (+\infty) = 0$.

The motivating example is the usual trace function on $B(H)$. Another example is got on $C_0(\mathbb{R})$ by setting $\tau(f) = \int f \mathrm{d}m$ where $f \in C_0(\mathbb{R})^+$ and m is ordinary Lebesgue measure on $\mathbb{R}$.

Traces (and their generalization, *weights*) play a fundamental role, especially in von Neumann algebra theory.

Let

$$A_\tau^2 = \{a \in A : \tau(a^*a) < \infty\}.$$

Show that

$$(a+b)^*(a+b) \leqslant 2a^*a + 2b^*b$$

and

$$(ab)^*ab \leqslant \|a\|^2 b^*b,$$

and deduce that A_τ^2 is a self-adjoint ideal of A.

Let A_τ be the linear span of all products ab, where $a, b \in A_\tau^2$. Show that A_τ is a self-adjoint ideal of A.

Show that for arbitrary $a, b \in A$,

$$a^*b = \frac{1}{4}\sum_{k=0}^{3} i^k (b + i^k a)^*(b + i^k a),$$

and if a^*b is self-adjoint,

$$a^*b = \frac{1}{4}[(b+a)^*(b+a) - (b-a)^*(b-a)].$$

Let

$$A_\tau^+ = \{a \in A^+ : \tau(a) < \infty\}.$$

Show that A_τ is the linear span of A_τ^+ and $A_\tau^+ = A_\tau \cap A^+$.

Show that there is a unique positive linear extension (also denoted τ) of τ to A_τ. Show that

$$\tau(ab) = \tau(ba)$$

for all $a, b \in A_\tau^2$, and deduce that this equation also holds for all $a \in A$ and $b \in A_\tau$.

Solution. Note that $(a+b)^*(a+b) + (a-b)^*(a-b) = 2a^*a + 2b^*b$, so $(a+b)^*(a+b) \leqslant 2a^*a + 2b^*b$. Also note that $a^*a \leqslant \|a\|^2$ (if A is not unital, we can surely add a unit), so $(ab)^*ab \leqslant \|a\|^2 b^*b$.

Therefore, the first inequality says that $a, b \in A_\tau^2 \Rightarrow a + b \in A_\tau^2$, and the second inequality says that $b \in A_\tau^2, a \in A \Rightarrow ab \in A_\tau^2$. By definition of a trace, A_τ^2 is self-adjoint. Combining all these results, A_τ^2 is a self-adjoint ideal of A.

Clearly A_τ is a self-adjoint ideal of A since A_τ^2 is.

It follows from a direct computation that $a^*b = \frac{1}{4}\sum\limits_{k=0}^{3} i^k(b + i^k a)^*(b + i^k a)$. One may compute the term $k = 0$ and $k = 2$ first to make the computation easier.

If a^*b is self-adjoint, then $\frac{1}{4}[(b+a)^*(b+a) - (b-a)^*(b-a)] = \frac{1}{2}(b^*a + a^*b) = a^*b$.

If $a \in A_\tau^+$, then $a^{1/2} \in A_\tau^2$, so $a \in A_\tau$. Conversely, if $a, b \in A_\tau^2$, then

$$ab = \frac{1}{4}\sum_{k=0}^{3} i^k(b + i^k a^*)^*(b + i^k a^*).$$

Note that $b + i^k a^* \in A_\tau^2$, since A_τ^2 is a self-adjoint ideal. Therefore, $(b+i^k a^*)^*(b+i^k a^*) \in A_\tau^+$. This shows that ab lies in the linear span of A_τ^+, and also A_τ is contained in the linear span of A_τ^+.

Clearly $A_\tau^+ \subseteq A_\tau \cap A^+$. Conversely, if $a \in A_\tau \cap A^+$, then a can be written as a finite sum $a = \sum_{j=0}^{k} \lambda_j a_j$ where $a_j \in A_\tau^+, \lambda_j \in \mathbb{C}$. Now we observe that:

(1) $0 \leqslant a \leqslant b, b \in A_\tau^+ \Rightarrow a \in A_\tau^+$;

(2) a and a_j are self-adjoint, so consider $a = \frac{a+a^*}{2}$, we can assume $\lambda_j \in \mathbb{R}$ WLOG.

So we can futhermore assume WLOG that $\lambda_j \geqslant 0$ for all j. But then it is obvious to see $a \in A_\tau^+$, since A_τ^+ is certainly a positive cone.

Finally, we first restrict τ on A_τ^+. The restriction takes values in $[0, +\infty)$. Then linearly extend the restriction to A_τ. The extension is well-defined: if $\sum_{j=0}^{k} \lambda_j a_j = \sum_{n=0}^{m} \mu_n b_n$, where $\lambda_j, \mu_n \in \mathbb{C}, a_j, b_n \in A_\tau^+$, we take the real part of each side and assume that $\lambda_j, \mu_n \in \mathbb{R}$. Then we put all the terms with negative coefficients to the other side, and both sides will be a linear combination of elements in A_τ^+ with positive coefficients, so the two sums lie in A_τ^+. Now apply τ which is originally defined on A_τ^+. All the numbers are finite, so we can freely move back the terms with negative coefficients.

Since $A_\tau \cap A^+ = A_\tau^+$, by construction τ is a positive linear functional. The uniqueness also follows from that A_τ is the linear span of A_τ^+.

If $a, b \in A_\tau^2$ and self-adjoint, then $a + ib \in A_\tau^2$ and $\tau((a+ib)^*(a+ib)) = \tau((a+ib)(a+ib)^*)$. So

$$\tau(a^2 - iba + iab + b^2) = \tau(a^2 + iba - iab + b^2) < \infty.$$

Since $a, b \in A_\tau^2$ and $a^* = a, b^* = b$, we have $\tau(a^2) < \infty, \tau(b^2) < \infty$, so we can safely eliminate these common terms on both sides to get $\tau(ab) = \tau(ba)$. Now we use can linearity to prove the general case where a, b may not be self-adjoint.

If $a \in A, b \in A_\tau$, then b can be written as a finite sum $b = \sum_{j=0}^k c_j d_j$ where $c_j, d_j \in A_\tau^2$. Since A_τ^2 is an ideal, $ac_j, d_j a \in A_\tau^2$. By the previous paragraph,

$$\tau(ac_j d_j) = \tau(d_j ac_j) = \tau(c_j d_j a).$$

So $\tau(ab) = \tau(ba)$. Now we complete the proof.

Problem 4. Show that an AF-algebra admits a sequential approximate unit consisting of projections.

Solution. Suppose A is an AF-algebra with $A = \overline{\cup A_n}$, where (A_n) is an increasing sequence of finite-dimensional C*-subalgebras of A. By Theorem 6.3.8, every finite-dimensional C*-algebra is unital, say p_n is the unit of $A_n \subseteq A$. Then p_n are projections in A. We claim that (p_n) is an approximate unit for A.

For any $a \in \cup A_n$, there exists some n such that $x \in A_n$. Then whenever $m \geqslant n$, $p_m a = a$, so $\lim_{n\to\infty} p_n a = a$. For any $b \in A$ and $\varepsilon > 0$, there exists some $b \in \cup A_n$ such that $\|b-a\| < \varepsilon$, so $\|p_n b - b\| \leqslant \|p_n b - p_n a\| + \|p_n a - a\| + \|a - b\| < 2\varepsilon + \|p_n a - a\|$. This shows that $\lim_{n\to\infty} p_n b = b$ and thus we complete the proof.

Problem 5. Let u be a normal operator on a Hilbert space H. Show that there is a commuting sequence of projections on H such that the C*-algebra that they generate contains u.

Use this to construct an example of a C*-subalgebra of an AF-algebra which is not an AF-algebra.

Solution. Suppose $K \subseteq \mathbb{C}$ is the spectrum of u, then K is a non-empty compact set. Now we use the Borel functional calculus. For any non-empty Borel subset A of K, denote $p_A := \chi_A(u)$. Then (p_A) are a family of commuting projections on H. Recall that $\mathcal{B}_K$, the Borel σ-algebra on K, is countably generated, so we may take a countable set of generators $\{A_n\}_{n\geqslant 0}$ and denote $p_n := p_{A_n}$, and the unital C*-algebra they generate by $B \ni \mathrm{id}_H$.

Now check what B contains. Denote $\mathcal{C} = \{C \in \mathcal{B}_K : \chi_C(u) \in B\}$. It is easy to check that $A_n \in \mathcal{C}$, $\mathcal{C}$ contains K, and is closed under finite unions and complements. By taking norm limits, it can be shown that $\mathcal{C}$ is closed under countable unions. Therefore, $\mathcal{C} = \mathcal{B}_K$. Since the coordinate function z on K can be approximated by simple functions in the supremum norm, B contains u.

Note that for any positive integer m, the C*-subalgebra of B generated by $p_1, \ldots, p_m$ is finite-dimensional, so B is an AF-algebra. But if $K = [0, 1]$ (such a normal operator exists when H is infintie-dimensional), then the non-unital C*-subalgebra generated by u does not contain any non-trivial projections, so it cannot be an AF-algebra, since every AF-algebra contains a sequential approximate unit.

Problem 6. If A is an AF-algebra, show that $M_n(A)$ is one also.

Solution. Suppose $A = \overline{\cup A_m}$, where (A_m) is an increasing sequence of finite-dimensional C*-subalgebras of A. Then it is easy to check that $\cup M_n(A_m)$ is dense in $M_n(A)$, since convergence in the matrix algebra is equivalent to convergence of each entry. Obviously $M_n(A_m)$ is finite-dimensional, so $M_n(A)$ is an AF-algebra.

Problem 7. Show that if A and B are AF-algebras, then $A \otimes_* B$ is an AF-algebra.

Solution. Suppose $A = \overline{\cup A_m}$ and $B = \overline{\cup B_m}$, where (A_m) and (B_m) are two increasing sequences of finite-dimensional C*-subalgebras of A, B, respectively.

Recall that in Lemma 6.3.4, it is proved that $A \times B \to A \otimes_\gamma B, \quad (a, b) \mapsto a \otimes b$ is separately continuous. An application of uniform boundedness principle shows that this map is also jointly continuous. Therefore, $\cup A_m \otimes B_m$ is dense in $A \otimes_* B$, and $A \otimes_* B$ is thus an AF-algebra.

Problem 8. Show that if A, B, and C are *-algebras, then the unique linear map $\varphi : (A \otimes B) \otimes C \to B \otimes (A \otimes C)$, such that $\varphi((a \otimes b) \otimes c) = b \otimes (a \otimes c)$ for all $a \in A, b \in B, c \in C$, is a *-isomorphism.

Deduce that if A is a nuclear C*-algebra, so is $M_n(A)$.

Solution. In fact, the only thing to check is that φ is well-defined. If it has been done, then we can similarly define a well-defined map $\psi : B \otimes (A \otimes C) \to (A \otimes B) \otimes C, \psi(b \otimes (a \otimes c)) = (a \otimes b) \otimes c$. Then it is easy to see that φ, ψ are inverse to each other, and they clearly preserve involutions, so φ is a *-isomorphism as we want.

(If the readers feel unsafe about this, they can identify "φ is well-defined" with "ψ is injective" and "ψ is well-defined" with "φ is injective". The surjectiveness is obvious.)

Now suppose $\sum_{n=0}^{m}(\sum_{p=0}^{q_n} a_p \otimes b_p) \otimes c_n = 0$, and we need to prove that $\sum_{n=0}^{m}(\sum_{p=0}^{q_n} b_p \otimes (a_p \otimes c_n)) = 0$. We may find a basis for the linear span of $(c_n)_{n=0}^{m}$, so WLOG suppose $(c_n)_{n=0}^{m}$ is linearly independent. Similarly, $(b_p)_{p=0}^{q_n}$ is linearly independent for each n. Then the assumption implies that $a_p = 0$ for all $p = 0, \dots, q_n$ and all $n = 0, \dots, m$. The conclusion then follows.

If A is nuclear, then for any C*-algebra B, there is only one C*-norm on $A \otimes B$. By the first part, $M_n(A) \otimes B \cong A \otimes M_n(B)$, so there is only one C*-norm on $M_n(A) \otimes B$, which completes the proof.

Problem 9. If H_1, H_2, and H_3 are Hilbert spaces, show that there exists a unique unitary $u : (H_1 \hat{\otimes} H_2) \hat{\otimes} H_3 \to H_1 \hat{\otimes} (H_2 \hat{\otimes} H_3)$ such that

$$u((x_1 \otimes x_2) \otimes x_3) = x_1 \otimes (x_2 \otimes x_3) \quad (x_j \in H_j, j = 1, 2, 3).$$

Show that

$$u((v_1 \hat{\otimes} v_2) \hat{\otimes} v_3)u^* = v_1 \hat{\otimes} (v_2 \hat{\otimes} v_3) \quad (v_j \in B(H_j), j = 1, 2, 3).$$

Deduce that if A_1, A_2, and A_3 are C*-algebras, then there exists a unique *-isomorphism $\theta : (A_1 \otimes_* A_2) \otimes_* A_3 \to A_1 \otimes_* (A_2 \otimes_* A_3)$ such that

$$\theta((a_1 \otimes a_2) \otimes a_3) = a_1 \otimes (a_2 \otimes a_3) \quad (a_j \in A_j, j = 1, 2, 3).$$

Solution. It is routine to check that $u_0 : (H_1 \otimes H_2) \otimes H_3 \to H_1 \otimes (H_2 \otimes H_3), (x_1 \otimes x_2) \otimes x_3 \mapsto x_1 \otimes (x_2 \otimes x_3)$ defines a surjective isometry, so by density, u_0 uniquely extends to a unitary u as required.

The second statements follows from a similar argument by checking the identity on the dense subspace $H_1 \otimes (H_2 \otimes H_3)$.

For the third statement, we can first define $\theta : (A_1 \otimes A_2) \otimes A_3 \to A_1 \otimes (A_2 \otimes A_3)$, which is a *-isomorphism between *-algebras. Then we take faithful non-degenerate representations (H_i, φ_i) of A_i $(i = 1, 2, 3)$, respectively. By Theorem 6.4.19 and Theorem 6.3.3, the representations $((H_1 \hat{\otimes} H_2) \hat{\otimes} H_3, (\varphi_1 \hat{\otimes} \varphi_2) \hat{\otimes} \varphi_3)$ and $(H_1 \hat{\otimes} (H_2 \hat{\otimes} H_3), \varphi_1 \hat{\otimes} (\varphi_2 \hat{\otimes} \varphi_3))$ of $(A_1 \otimes_* A_2) \otimes_* A_3$ and $A_1 \otimes_* (A_2 \otimes_* A_3)$, respectively, both give rise to the spatial C*-norms. From the unitary intertwining operator u, we have that θ is an isometry, and thus θ extends uniquely to a *-isomorphism between C*-algebras as required.

Problem 10. If A, B are C*-algebras, show that there exists a unique *-isomorphism $\theta : A \otimes_* B \to B \otimes_* A$ such that $\theta(a \otimes b) = b \otimes a$ $(a \in A, b \in B)$.

Solution. Obviously we can define a *-homomorphism on the algebraic tensor products $\rho : A \otimes B \to B \otimes A$ such that $\rho(a \otimes b) = b \otimes a$ for all $a \in A, b \in B$, and clearly it is a *-isomorphism. If ρ is isometric with respect to the spatial norms, then ρ can be extended to θ as required, which proves the existence of θ. The uniqueness follows easily from that $\theta(a \otimes b) = b \otimes a$ for all $a \in A, b \in B$ and that $A \otimes B$ is dense in $A \otimes_* B$.

Now we will prove that ρ is isometric. Recall that the spatial norm is determined by the universal representation. (In fact, any faithful representation gives the spatial norm.) Suppose $\varphi : A \to B(H), \psi : B \to B(K)$ are the corresponding universal representations for A and B. Then there is a unique unitary between Hilbert spaces $u : H \hat{\otimes} K \to K \hat{\otimes} H$ such that $u(x \otimes y) = y \otimes x$ for all $x \in H, y \in K$.

The spatial norms are given by two injective *-homomomrphisms $\pi_1 : A \otimes B \to B(H \hat{\otimes} K)$ and $\pi_2 : B \otimes A \to B(K \hat{\otimes} H)$, which clearly satisfies that $\pi_1(c) = u^* \pi_2(\rho(c)) u$ for all $c \in A \otimes B$. One can check this property by a direct computation and use linearity:

$$\begin{aligned} u^* \pi_2(\rho(a \otimes b)) u(x \otimes y) &= u^* \pi_2(b \otimes a)(y \otimes x) \\ &= \varphi(a) x \otimes \psi(b) y \\ &= \pi_1(a \otimes b)(x \otimes y), \quad \forall a \in A, b \in B, x \in H, y \in K. \end{aligned}$$

Since the operator norm is fixed under conjugation by unitaries, $\|\pi_1(c)\| = \|\pi_2(\rho(c))\|$ for all $c \in A \otimes B$. This shows that ρ is an isometry from $A \otimes B$ onto $B \otimes A$, and completes the proof.