Introduction to Model Theory

Homework

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1 Information

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The tex code is on my overleaf.

2 Homework 1, September 8

1

Read or skim the Introduction and Chapter 1 of the course notes (version 3.1). Write down at least one question or correction about the notes. You don't need to read the exercises.

There are some possible typos in Section 1.6, subsection *Peano arithmetic* on page 39.

In the list of axioms, there are "strange" variables y in the fourth and sixth lines. These two y's should be deleted.(?)

It seems that by definition, these two y's are not free variables, since they occur within the quantifier $\forall$. But they do not appear in the formulae elsewhere. They are kind of redundant, but intuitively I think it is mathematically legal.

2

Write down a set of axioms in the language of graphs which characterize the class of graphs in which every vertex has degree 3 (Exercise 1.7.3 in version 3.1 of the notes).

$$\forall x : \neg(x \ E \ x),$$
$$\forall x, \forall y : x \ E \ x \to y \ E \ x,$$
$$\forall x, \exists^{=3} y : x \ E \ y.$$

The first two axioms are the axioms of graphs. The last line is the axiom that every vertex has degree 3. It is an abbreviation for

$$.$$

3 Homework 2, September 15

1

Take $A = \mathbb{N}$. For any $x \in \mathbb{N}$, we have $s(x) = x + 1 \in \mathbb{N}$, so $(\mathbb{N}, s)$ is a substructure of $(\mathbb{Z}, s)$.

This (A, s) is not elementarily equivalent to $(\mathbb{Z}, s)$, since for the sentence $\varphi \equiv (\forall x : (\exists y : x = f(y)))$, we have $\mathbb{Z} \models \varphi, A \not\models \varphi$. Here f is the function symbol in $\mathcal{L}_f$. The sentence means "the unary function is surjective".

2

Let $B = 2\mathbb{Z} \subseteq \mathbb{Z}$. If $x \in B(\Leftrightarrow x$ is even), then $g(x) = x + 2 \in B$, so (B, g) is a substructure of $(\mathbb{Z}, g)$.

Define $j : \mathbb{Z} \to 2\mathbb{Z}$ by $j(x) = 2x$. This is a bijection. Also $j(s(x)) = j(x+1) = 2x + 2 = g(j(x))$. So j is an isomorphism from $(\mathbb{Z}, s)$ to (B, g).

3

Any finite subset of $\mathbb{Z}$ is definable.

The set $\{1, 2, 3\}$ is

$$\{x \in \mathbb{Z} | x = 1 \vee x = 2 \vee x = 3\}.$$

4

Suppose $\{1, 2, 3\}$ is $\{1, 3\}$-definable. Then there exists a formula $\varphi(x, \overline{y})$ such that

$$\{1, 2, 3\} = \{x \in \mathbb{Z} | (\mathbb{Z}, g) \models \varphi(x, \overline{b})\},$$

where $\overline{b} = (b_1, \cdots, b_m)$, with each $b_i \in \{1, 3\}$ fixed. So

$$x \in \{1, 2, 3\} \Leftrightarrow (\mathbb{Z}, g) \models \varphi(x, \overline{b}).$$

Consider the function $j : \mathbb{Z} \to \mathbb{Z}$ defined by

$$j(x) = \begin{cases} x, & x \text{ is odd,} \\ x + 2, & x \text{ is even.} \end{cases}$$

One can directly check that j is an automorphism of $(\mathbb{Z}, g)$. Moreover, j fixes 1 and 3. So we must have

$$(\mathbb{Z}, g) \models \varphi(x, \bar{b}) \Leftrightarrow (\mathbb{Z}, g) \models \varphi(j(x), j(\bar{b})).$$

But $j(\bar{b}) = \bar{b}$ since j fixes 1 and 3, so we actually have

$$x \in \{1, 2, 3\} \Leftrightarrow j(x) \in \{1, 2, 3\},$$

which does not hold. This is a contradiction, implying that $\{1, 2, 3\}$ cannot be $\{1, 3\}$-definable.

4 Homework 3, September 22

1

$$D = \{x \in \mathbb{R} | x \cdot x = 1 + 1\} = \{\pm\sqrt{2}\}.$$

This set is $\mathbb{Q}$-definable, non-empty, and does not intersect $\mathbb{Q}$.

2

$$\begin{aligned} 60 &= 5+5+5+5+8+8+8+8+8 \\ &\approx 5+5+8+8+8+8+8+8+8 \\ &= 66. \end{aligned}$$

3

First, we check $\approx$ is an equivalence relation.

$x \approx x$ because $x = x$.

If $x \approx y$, then WLOG we assume $x \neq y$, then $x \geqslant 23, y \geqslant 23, x - y \in 2\mathbb{Z}$. Then $y \geqslant 23, x \geqslant 23, y - x = -(x - y) \in 2\mathbb{Z}$, so $y \approx x$.

If $x \approx y, y \approx z$, WLOG assume $x \neq y \wedge y \neq z \wedge x \neq z$ (otherwise $x \approx z$ obviously holds). Then $x \geqslant 23, y \geqslant 23, x - y \in 2\mathbb{Z}$ and $y \geqslant 23, z \geqslant 23, y - z \in 2\mathbb{Z}$. So $x \geqslant 23, z \geqslant 23, x - z = (x - y) + (y - z) \in 2\mathbb{Z}$, i.e. $x \approx z$.

Next, we show that $\approx$ is a congruence. In this language, we have two binary function symbols and two constant symbols. We only need to check the function symbols.

Suppose $a \approx a', b \in \mathbb{N}$. WLOG $a \neq a'$, so $a \geqslant 23, a' \geqslant 23, a - a' \in 2\mathbb{Z}$. Since $b \geqslant 0$, we have $a + b \geqslant 23, a' + b \geqslant 23$. Also $(a + b) - (a' + b) = a - a' \in 2\mathbb{Z}$, so $a + b \approx a' + b$.

If $b = 0$, then $ab = a'b = 0$, so $ab \approx a'b$. If $b \neq 0$, then $b \geqslant 1$, so $ab \geqslant 23, a'b \geqslant 23$, and $ab - a'b = (a - a')b \in 2\mathbb{Z}$. So $ab \approx a'b$ in both cases.

Since the addition and multiplication is commutative, we can prove in the same way that if $a \in \mathbb{N}, b \approx b'$, then $a + b \approx a + b', ab \approx ab'$. The conclusion then follows from Exercise 4.1.2.

5 Homework 4, October 7

1

Suppose the class of well-founded posets is an elementary class. Assume the language is $\mathcal{L}$. So there is some set of $\mathcal{L}$-sentences T such that $\mathrm{Mod}(T)$ is the class of well-founded posets. All the models of T are exactly well-founded posets.

Consider a new language $\mathcal{L}' = \mathcal{L} \bigsqcup \{c_1, \cdots, c_n, \cdots\}$, where c_i's are new constant symbols. Sentences in T can be viewed as $\mathcal{L}'$-sentences without any change. Let T' be the union of T and the following sentences ($n \in \mathbb{N}^*$):

$$\varphi_n \equiv (c_n > c_{n+1}).$$

Any finite subset F of T' contains finitely many φ_n's, so $F \backslash T \subseteq \{\varphi_1, \cdots, \varphi_{N-1}\}$ for some $N >> 1$.

We claim the subset $\{1, 2, \cdots, N\}$ of $\mathbb{Z}$ with the natural order satisfies F. This set is a finite poset, and hence cannot contain an infinite descending chain, which implies this set is a well-founded poset and satisfies T. Also if we interpret

$$c_j = \begin{cases} N + 1 - j, & j = 1, \cdots, N, \\ N, & j \geqslant N + 1, \end{cases}$$

then $\varphi_1, \cdots, \varphi_{N-1}$ holds in this set. This proves the claim.

So T' is finitely satisfiable, and there is a model P of T' by Compactness Theorem. In P we have $c_1^P > c_2^P > c_3^P > \cdots$, and P is a well-founded poset since P satisfies T. This is a contradiction. So the class of well-founded posets is not an elementary class.

6 Homework 5, October 15

1

Let $\psi_0 \equiv (\forall x : x \approx x), \psi_1 \equiv (\forall x, y : x \approx y \to y \approx x), \psi_2 \equiv (\forall x, y, z : x \approx y \wedge y \approx z \to x \approx z)$. These sentences form the theory of an equivalence relation.

We define a formula $\theta_n(x_1, x_2, \cdots, x_n)$ by

$$(x_1 \approx x_2 \approx \cdots \approx x_n) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant n} x_i \neq x_j \right)$$
$$\wedge (\forall y : (\bigvee_{i=1}^{n} y = x_i) \vee \neg y \approx x_1).$$

This formula says $x_1, \cdots, x_n$ form an equivalence class of exactly these n distinct elements.

Now define a sentence for each positive integer n:

$$\varphi_n \equiv \left(\exists x_1, \cdots, x_n : \theta(x_1, \cdots, x_n) \wedge (\forall z_1, \cdots, z_n : \theta(z_1, \cdots, z_n) \to (\bigvee_{i=1}^{n} z_1 = x_i)) \right).$$

This sentence says that there exists an equivalence class consisting of $\{x_1, \cdots, x_n\}$, and any other equivalence class of exactly n elements has to intersect this equivalence class (so the two equivalence classes coincide).

Then let $T = \{\psi_0, \psi_1, \psi_2, \varphi_n | n \geqslant 1, n \in \mathbb{N}^*\}$. We have $\mathcal{K} = \mathrm{Mod}(T)$.

2

A partition into equivalence classes is exactly a partition into disjoint subsets, since one can define an equivalence relation from a partition into disjoint subsets $A = \bigsqcup_{i \in I} A_i$ by $x \approx y \Leftrightarrow \exists i \in I : x \in A_i \wedge y \in A_i$.

Let $[n]$ stand for a subset of exactly n elements, where n is a cardinal number which can be infinite. Then we can denote a set A with an equivalence relation $\approx$ by $\{[\kappa_1], [\kappa_2], \cdots, [\kappa_i], \cdots | i \in I\}$. $|I|$ is the number of the equivalence classes of $(A, \approx)$. $\sum_{i \in I} \kappa_i = |A|$. Unfortunately we have to allow $\kappa_i = \kappa_j$ for some $i \neq j$, but the meaning is unambiguous, i.e. there may be several equivalence classes of the same cardinal.

For example, $\{[1], [2], [2], [3]\}$ means a set of 8 elements into 4 equivalence classes, one of size 1, one of size 3, two of size 2.

So $(A,\approx)\in\mathcal{K}$ iff $(A,\approx)$ can be written as

$$\{[1],[2],\cdots,[n],\cdots,[\kappa_i]|i\in I\},$$

where $[\kappa_i]$ are infinite equivalence classes which may not appear, and each $[n]\quad(n\in\mathbb{N}^*)$ appears exactly once.

For any infinite κ, let

$$A_1=\{[1],\cdots,[n],\cdots,[\kappa]\},A_2=\{[1],\cdots,[n],\cdots,[\kappa],[\kappa]\}.$$

Then

$$|A_1|=\sum_{n\in\mathbb{N}^*}n+\kappa=\aleph_0+\kappa=\kappa,|A_2|=\sum_{n\in\mathbb{N}^*}n+\kappa+\kappa=\kappa.$$

If A_1 is isomorphic to A_2, then they must have same number of equivalence classes of size κ, which is not satisfied by the construction.

A proof of the red part:

Let $A_1'=\{x\in A_1|\text{the equivalence class containing } x \text{ is size } \kappa\}$ and similarly for A_2'. If $f:A_1\to A_2$ is an isomorphism, then take $x\in A_1'$ and the equivalence class containing x, say (x). For $y\in A_1$, we have $y\in(x)$ iff $f(y)\in(f(x))$. For $z\in A_2$, we have $z\in(f(x))$ iff $f^{-1}(z)\in(x)$, so $(f(x))$ is of the same size as (x), which implies $f(x)\in A_2'$. By considering f^{-1}, we know $f|_{A_1'}:A_1'\to A_2'$ is bijective.

Also note that $x\in A_1',y\approx x\Rightarrow y\in A_1'$. So we can decompose $A_1'=\bigsqcup_{i\in I}(x_i)$ into the disjoint union of equivalence classes, and similarly $A_2'=\bigsqcup_{j\in J}(z_j)$. The representatives are fixed as mentioned. Define $g:I\to J$ by $g(i)=j$ iff $f(x_i)\in(z_j)$. g is injective because if $g(i)=g(i')=j$, then $f(x_i)\approx f(x_{i'})$, so $x_i\approx x_{i'},(x_i)=(x_{i'}),i=i'$. Define $h:J\to I$ by $h(j)=i$ iff $f^{-1}(z_j)\in(x_i)$, so we have the composition $g\circ h:J\to J$. If $h(j)=i$, then $f^{-1}(z_j)\in(x_i)$, so $f^{-1}(z_j)\approx x_i,z_j\approx f(x_i),f(x_i)\in(z_j),g(i)=j$. This shows $(g\circ h)(j)=j$, so g is surjective. This completes the proof that $|I|=|J|$.

3

Let $N=\{[1],\cdots,[n],\cdots,[\aleph_0],[\aleph_0],\cdots\}$ with countably infinitely many $[\aleph_0]$'s. Claim that if $M_0\in\mathcal{K}'$ is countable, then $M_0\cong N$.

By definition of $\mathcal{K}'$, M_0 contains one $[n]$ for each $n \in \mathbb{N}^*$, and the remaining equivalence classes of M_0 are $\{[\kappa_i] | i \in I\}$, where I is infinite, each κ_i is an infinite cardinal number. Since M_0 is countable, $\kappa_i \leqslant |M_0| = \aleph_0$ and κ_i infinite together implies $\kappa_i = \aleph_0$. Now

$$|M_0| = \sum_{n \in \mathbb{N}^*} n + |I| \cdot \aleph_0 = \aleph_0 + |I| \cdot \aleph_0.$$

Since I is infinite, $|I| \geqslant \aleph_0$, so $|I| \cdot \aleph_0 = \max(|I|, \aleph_0) = |I|$. Then $\aleph_0 = \aleph_0 + |I|$ and I infinite together implies $|I| = \aleph_0$. So we have completely determined $M_0 = \{[1], \cdots, [n], \cdots, [\aleph_0], [\aleph_0], \cdots\}$ where $[\aleph_0]$ appears $\aleph_0$ times. The number of equivalence classes of each size matches up, so $M_0 \cong N$. An isomorphism can be constructed by gluing up the bijections between equivalence classes of the same size after choosing bijections between indices of equivalence classes of the same size.

4

$|\mathcal{L}| = \aleph_0$, $M \in \mathcal{K}'$ implies M is an infinite set, and more precisely,

$$M = \{[1]_M, [2]_M, \cdots, [n]_M, \cdots, [\kappa_i]_M | i \in I\},$$

where each κ_i is infinite, I is infinite, the subscripts mean the equivalence classes are taken in M. Then we take a countable infinite subset $I_0 \subseteq I$, and for each $i \in I_0$ take a countable infinite subset $R_i \subseteq [\kappa_i]_M$. Let $A \subseteq M$ be the union of all $[n]_M, n \in \mathbb{N}^*$ and all $R_i, i \in I_0$. Then $|A| = \aleph_0$.

By the downward Löwenheim-Skolem theorem (the stronger version: 1. of Theorem 3.5.3 in version 3.4, though I believe this version is not proved in class) applied to A, M above, we get an elementary substructure $N \preceq M$ with $N \supseteq A$ and $|N| \leqslant |A| + |\mathcal{L}| = \aleph_0$. From $N \preceq M$ we have $N \equiv M$, which together with $M \models T$ implies $N \models T$ and $|N| = \aleph_0$. Now we write

$$N = \{[1]_N, [2]_N, \cdots, [n]_N, \cdots, [\kappa_i']_N | i \in I'\},$$

where each κ_i' is infinite, I' may be empty. Since $|N| = \aleph_0$, each κ_i' must be $\aleph_0$ (if I' is non-empty). By construction $N \supseteq A$, so N contains all $[n]_M$ and all R_i.

Now recall the formulas $\theta(x_1, \cdots, x_n)$ defined at the beginning of this section, which says $x_1, \cdots, x_n$ form an equivalence class of exactly these n

distinct elements. Let $\{x_1, x_2, \cdots, x_n\} = [n]_N$, then $N \models \theta_n(x_1, \cdots, x_n)$, so $M \models \theta_n(x_1, \cdots, x_n)$. This shows that $[n]_N$ has to coincide with $[n]_M$. The rest part of A, those $R_i \quad (i \in I_0)$'s, have to lie in distinct equivalence classes in N because they do so in M and $N \preceq M$ ($x \approx y$ in M iff $x \approx y$ in N), so $|I'| \geqslant |I_0| = \aleph_0$, and a check on cardinality of N shows $|I'| = \aleph_0$. So $N \in \mathcal{K}'$ and $|N| = \aleph_0$. By the previous exercise, or by this partition of N, we have $N \cong M_0$. Since $N \preceq M$, we have $M \equiv N \equiv M_0$.

5

Let $\mathcal{L}' = \mathcal{L} \bigsqcup \{c_{m,n} | m, n \in \mathbb{N}^*\}$ be the language $\mathcal{L}$ with $\aleph_0$-many new constant symbols. Let $\mathrm{Th}(M)$ be the complete theory of M, viewed as $\mathcal{L}'$-sentences without any change. Define $\mathcal{L}'$-sentences for each $p \in \mathbb{N}^*$ by

$$\tau_p \equiv (\bigwedge_{\substack{(i,j)\neq(k,l) \\ 1\leqslant i,j,k,l\leqslant p}} c_{i,j} \neq c_{k,l}) \wedge (\bigwedge_{i=1}^{p-1} \bigwedge_{j=1}^{p} c_{i,j} \approx c_{i+1,j}) \wedge (\bigwedge_{i=1}^{p} \bigwedge_{j=1}^{p-1} \bigwedge_{k=j+1}^{p} \neg c_{i,j} \approx c_{i,k}).$$

One can think of $(c_{i,j})_{p\times p}$ as a $p \times p$ square lattice, and any two elements are $\approx$-equivalent iff they are on the same row.

Let TT be the theory $\mathrm{Th}(M) \cup \{\tau_p | p \in \mathbb{N}^*\}$. We claim that TT is finitely satisfiable, and M is always a model of any finite subset of TT. Suppose T_f is a finite subset of TT, so for some $p >> 1$, we have $T_f \subseteq \mathrm{Th}(M) \cup \{\tau_1, \cdots, \tau_p\}$. These sentences only involve the constant symbols $(c_{i,j})$ with $1 \leqslant i, j \leqslant p$, and it is always true that $\tau_p \to \tau_{p-1} \to \cdots \to \tau_1$. Now we interpret $c_{1,j} (1 \leqslant j \leqslant p)$ as the p elements in $[p]$, $c_{2,j} (1 \leqslant j \leqslant p)$ as any distinct p elements in $[p+1]$, ..., $c_{p,j} (1 \leqslant j \leqslant p)$ as any distinct p elements in $[2p-1]$. The rest $c_{i,j}$'s are arbitrarily interpreted in M. Then $M \models T_f$.

Since TT is finitely satisfiable, we can take a model $N \models TT$. Since $N \models \mathrm{Th}(M)$, we have $\mathrm{Th}(N) \supseteq \mathrm{Th}(M)$. If $\exists \varphi \in \mathrm{Th}(N) \setminus \mathrm{Th}(M)$, then $\neg\varphi \in \mathrm{Th}(M) \subseteq \mathrm{Th}(N)$, so $\{\varphi, \neg\varphi\} \subseteq \mathrm{Th}(N)$, which is impossible. So $\mathrm{Th}(N) = \mathrm{Th}(M)$, implying $M \equiv N$.

Finally we prove $N \in \mathcal{K}'$. From $N \equiv M$ we know $N \models T$, so $N \in \mathcal{K}$. Now note that in N, the elements $c^N_{1,j} \quad (j \geqslant 1)$ all lie in infinite equivalence classes respectively, because (for example) $c^N_{1,j} \approx c^N_{2,j} \approx c^N_{3,j} \approx \cdots$ are distinct elements according to $\tau_p \quad (p \geqslant j)$. And $c^N_{1,j} \quad (j \geqslant 1)$ are pairwise not $\approx$-

equivalent by $\tau_p \quad (p \geqslant 1)$. So there are infinitely many infinite equivalence classes in N, which exactly means $N \in \mathcal{K}'$.

6

Let M_1, M_2 be two models of T. By the previous exercise, we can take $N_1, N_2 \in \mathcal{K}', N_1 \equiv M_1, N_2 \equiv M_2$. By exercise 4, we have $N_1 \equiv M_0 \equiv N_2$, so $M_1 \equiv N_1 \equiv N_2 \equiv M_2$.

7

Suppose $\mathcal{K}'$ is an elementary class, say $\mathcal{K}' = \mathrm{Mod}(T')$ for a set of $\mathcal{L}$-sentences T'. Since $\mathcal{K}' \subseteq \mathcal{K}$, we can replace T' by $T' \cup T$. So WLOG we can assume $T' \supseteq T$.

Now take $M_1 \in \mathcal{K} \setminus \mathcal{K}', M_2 \in \mathcal{K}'$. Such an M_1 exists, for example $\{[1], [2], \cdots, [n], \cdots\}$ with no infinite equivalence classes. Then $M_1 \models T, M_2 \models T'$, so $M_2 \models T$. By the previous exercise, $M_1 \equiv M_2$, so they should satisfy the same sentences. But $M_2 \models T'$, so $M_1 \models T'$, which contradicts $M_1 \notin \mathcal{K}'$. Therefore, $\mathcal{K}'$ cannot be an elementary class.

Some strange ideas

This part is written after the discussion in the Wechat group.

To be precise, we have proved there is no $\mathcal{L}$-theory T' such that $\mathcal{K}' = \mathrm{Mod}(T')$. But according to exercise 5, we do find a language $\mathcal{L}'$ and under this language, we can find a $\mathcal{L}'$-theory TTT such that $\mathrm{Mod}(TTT)$ is the class of sets with exactly one for each $[n]$ and infinitely many infinite equivalence classes.

$\mathcal{L}' = \{\approx, c_{m,n} | m, n \in \mathbb{N}^*\}$. $\approx$ is a binary relation, $c_{m,n}$'s are constant symbols. The theory TTT is made up of:

(1) $\approx$ is an equivalence relation.

(2) $c_{m,n} \approx c_{m',n'}$ iff $m = m'$. $c_{m,n} = c_{m',n'}$ iff $(m, n) = (m', n')$. These can be written down as an infinite list of sentences.

(3) Exactly one equivalence class of size n for each $n \in \mathbb{N}^*$.

So there is a strong similarity in technique between constructing a new model by compactness theorem and modeling a class by a theory.

But this is not the end.

Recall that in Homework 4, the class of well-founded posets is not an elementary class. I believe my proof shows that there is no language and no theory (surely we need to include at least the language and theory of posets to make sense) to make this class an elementary class. This is essentially different from the case of $\mathcal{K}'$. I believe this is due to the formation. The definition of well-founded posets is negating some ”finitely generated” statements, which is against the core of compactness theorem.

7 Homework 6, October 20

1

The theory T says that f is injective, and any f has no periodic points (with a dynamical system perspective).

Let $(\mathbb{R}, h)$ be an $\mathcal{L}$-structure, where $h(x) = x+1, \forall x \in \mathbb{R}$. This is clearly a model of T.

Since T has a model, T is consistent.

2

Let $(\mathbb{N}, g)$ be another model, with $g(x) = x + 1, \forall x \in \mathbb{N}$. Then g is not surjective, but h is surjective. So $(\mathbb{R}, h)$ and $(\mathbb{N}, g)$ can be distinguished by

$$\varphi \equiv (\forall y, \exists x : f(x) = y).$$

Since T has two models which are not elementarily equivalent, T is not complete.

3

We first give the theory in natural languages. The theory T says:

(0) For every x, we have $f(f(x)) = x$. So every point has minimum period 1 or 2.

(1) (M, f) has only one fixed point. Then with (0), all the other points automatically have minimum period 2.

(2) There are infinitely many points in M. Then with (0) and (1), there are infinitely many points of minimum period 2, and thus infinitely many orbits of size 2.

Then T is $\aleph_0$-categorical and all models of T are infinite. The second statement is clear from (2).

To prove T is $\aleph_0$-categorical, given two models $|M_1| = |M_2| = \aleph_0$ with the unary functions f_1, f_2 respectively, note that $|M| = 1+2|\{\text{orbits of size } 2\}|$, so there are exactly $\aleph_0$-many orbits of size 2. List all 2-orbits of $M_i (i = 1, 2)$ as $O_1^i, O_2^i, O_3^i, \cdots$, and the only fixed points $x_1 \in M_1, x_2 \in M_2$. Fix any element $y_j \in O_j^1, z_j \in O_j^2$, so $O_j^1 = \{y_j, f_1(y_j)\}, O_j^2 = \{z_j, f_2(z_j)\}$. Now we define $\rho : M_1 \to M_2$ by $\rho(x_1) = x_2, \rho(y_j) = z_j, \rho(f_1(y_j)) = f_2(z_j), \forall j \geqslant 1.$

Then one directly checks that ρ is an isomorphism by examining every 2-orbit. This shows that T has a unique model of size $\aleph_0$ up to isomorphism.

Now that T is $\aleph_0$-categorical, all models of T are infinite, $|\mathcal{L}| = \aleph_0$, by Łoś-Vaught criterion, T is complete. Clearly $(\mathbb{R}, f) \models T$.

Finally, we give explicit sentences for T.

(0) $\forall x : f(f(x)) = x$. Or $\forall x, \exists y : f(x) = y \land f(y) = x$.

(1) $\exists x, \forall y : f(x) = x \land (f(y) = y \to x = y)$.

(2) For each $n \in \mathbb{N}^*$,

$$\varphi_n \equiv (\exists^{\geqslant n} x : \top).$$

4

Proof without quantifier elimination:

Consider $p = \{x \geqslant 0 \land x \neq 0, x \leqslant 1/n \land x \neq 1/n, \neg(y \leqslant t \land y \neq t), y \leqslant s \land y \neq s | n \in \mathbb{N}^*, t \in \mathbb{Q}, s \in \mathbb{Q}, t < \sqrt{2}, s > \sqrt{2}\}$, an infinite set of $\mathcal{L}(\mathbb{Q})$-formulas.

This set of $\mathcal{L}(\mathbb{Q})$-formulas in (x, y) is finitely satisfiable in $(\mathbb{Q}, \leqslant)$, so p is realizable in some $(M, \leqslant) \succeq (\mathbb{Q}, \leqslant)$. In M, say p is satisfied by (a, b), then $a > 0$ and $a < 1/n$ for every positive integer n (with $0, 1/n \in \mathbb{Q} \subseteq M$), and $b < q \Leftrightarrow \sqrt{2} < q$ for $q \in \mathbb{Q}$.

It remains to prove that p is finitely satisfiable in $(\mathbb{Q}, \leqslant)$. For finitely many formulas in p involving x, there exists a sufficiently small positive rational number satisfying these formulas. For finitely many formulas involving y, say $\neg(y < t_1), \cdots, \neg(y < t_k), y < s_1, \cdots, y < s_l$, we have $\max(t_1, \cdots, t_k) < \sqrt{2} < \min(s_1, \cdots, s_l)$ (assume $\max \varnothing = -\infty, \min \varnothing = +\infty$), so there exists a rational number $> \max(t_1, \cdots, t_k)$ and $< \min(s_1, \cdots, s_l)$.

Proof with quantifier elimination:

A skim on quantifier elimination seems to show that we can construct $\mathbb{Q} \cup \{\sqrt{2}, a\} \cup A$, where $0 < A < a < \mathbb{Q}_+$, A a copy of $(0,1) \cap \mathbb{Q}$. Then by quantifier elimination, this is a DLO containing $\mathbb{Q}$, so it is an elementary extension of $(\mathbb{Q}, \leqslant)$.

5

Consider $p = \{\varphi_n(x) | n \in \mathbb{N}^*\}$ an infinite set of $\mathcal{L}(\mathbb{N})$-formulas, where $\varphi_n(x) \equiv (\exists y : n \cdot y = x + 1)$ with $n = 1 + 1 + \cdots + 1$.

p is finitely satisfiable, because any finite subset pf p is contained in $\{\varphi_1(x), \cdots, \varphi_N(x)\}$ for some $N >> 1$. Then this finite subset is satisfied by $N! - 1 \in \mathbb{N}$. So p is a partial type over $\mathbb{N}$, and there is an elementary extension $M \succeq \mathbb{N}$, which contains some b satisfying every formula in p. This means $b + 1$ is a multiple of every $n \in \mathbb{N}^*$.

Challenge 6

1

2

3

4

8 Homework 7, October 27

1

Only need to check the axioms for being a partial order.

(i) $x \leqslant x$: $x \oplus x = x$ by (1).

(ii) $x \leqslant y \wedge y \leqslant x \to x = y$: When $x \oplus y = y$ and $y \oplus x = x$, by (2) we have $x \oplus y = y \oplus x$, si $x = y$.

(iii) $x \leqslant y \wedge y \leqslant z \to x \leqslant z$: If $x \oplus y = y, y \oplus z = z$, then by (3) we have $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, so $x \oplus z = y \oplus z = z$, which exactly means $x \leqslant z$.

2

T is an equational theory and $(\mathbb{N}, \oplus) \models T$, so $(\mathbb{N} \times \mathbb{N}, \oplus)$ is a model of T. In $\mathbb{N} \times \mathbb{N}$, we have

$$(0,1) \oplus (1,0) = (1,1) \neq (1,0), \quad (1,0) \oplus (0,1) = (1,1) \neq (0,1),$$

so this is what we want.

3

Given $x, y \in S$, we have $x \leqslant y$ or $y \leqslant x$, so $x \oplus y$ is either x or y. In any case, $x \oplus y \in S$, so S is closed under the operation $\oplus$, and hence is a substructure of M.

Since S is a substructure of M, $M \models T$, and T is an equational theory, we have $S \models T$.

4

We use $\mathbb{N}^k$ for the product of k copies of $(\mathbb{N}, \oplus)$. Again since T is equational, $\mathbb{N}^k \models T$. In $\mathbb{N}^k$, let $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ where the only 1 is on the j-th coordinate, and j ranges from 1 to k. It is clear that $\{e_j | j = 1, 2, \cdots, k\}$ is an antichain.

Now define $\mathcal{L}' = \mathcal{L} \cup \{c_n | n \in \mathbb{N}^*\}$ with countably infinitely many new constant symbols. Define $T' = T \cup \{\varphi_n | n \in \mathbb{N}^*\}$, where φ_n is given by

$$\varphi_n \equiv \left((\bigwedge_{i,j=1}^{n} c_i \neq c_j) \wedge (\bigwedge_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} c_i \oplus c_j \neq c_j) \right).$$

Then for any finite subset $T_0 \subseteq T'$, say $T_0 \subseteq T \cup \{\varphi_1, \cdots, \varphi_N\}$, the $\mathcal{L}'$-structure $(\mathbb{N}^N, \oplus)$ is a model of T_0 if we interpret $c_1, \cdots, c_N$ as $e_1, \cdots, e_N$ respectively (and $c_{N+1}, \cdots$ chosen arbitrarily). So T' is finitely satisfiable.

By compactness theorem, T' is satisfiable, say $M \models T'$. Then c_n^M are distinct elements, and $\{c_n^M | n \in \mathbb{N}^*\}$ is an infinite antichain in M.

5

Consider the formula $\varphi_n(\overline{x})$ ($\overline{x} = (x_m)_{m\in\mathbb{Z}}$, I want an infinitary type) for each $n \in \mathbb{N}^*$:

$$\varphi_n(\overline{x}) \equiv ((\bigwedge_{-n\leqslant m_1<m_2\leqslant n} x_{m_1} \neq x_{m_2}) \wedge (x_{-n} < x_{-n+1} < \cdots < x_{n-1} < x_n)).$$

That $y_1 < y_2$ is an abbreviation for $y_1 \oplus y_2 = y_2 \wedge \neg y_1 \oplus y_2 = y_1$.

Let $\Sigma(\overline{x}) = \{\varphi_n(\overline{x}) | n \in \mathbb{N}^*\}$. Then $\Sigma(\overline{x})$ is finitely satisfiable in $(\mathbb{N}, \oplus)$, because to realize $\varphi_1(\overline{x}) \wedge \cdots \wedge \varphi_n(\overline{x})$, we can find the element $(b_m)_{m\in\mathbb{Z}}$ where $b_m = m + n$ when $-n \leqslant m \leqslant n$ and $b_m = 0$ when $|m| > n$. So $\Sigma(\overline{x})$ is a partial type over $(\mathbb{N}, \oplus)$, and hence realized by some element $\overline{c}$ in an elementary extension $(M, \oplus) \succeq (\mathbb{N}, \oplus)$. In M, we have $\cdots < c_{-1} < c_0 < c_1 < \cdots$ are distinct elements, and

$$c_{m_1} \oplus c_{m_2} = \begin{cases} c_{m_2}, & m_1 \leqslant m_2, \\ c_{m_1}, & m_1 > m_2. \end{cases}$$

This can be written as $c_{m_1} \oplus c_{m_2} = c_{\max(m_1, m_2)}$.

Now we can map $(\mathbb{Z}, \oplus)$ to $(M, \oplus)$ by sending $f(m) = c_m$. This is an embedding because $c_{m_1} = c_{m_2}$ implies $m_1 = m_2$, and

$$f(m_1 \oplus m_2) = f(\max(m_1, m_2)) = c_{\max(m_1, m_2)} = c_{m_1} \oplus c_{m_2}.$$

9 Challenge 7

1

Actually there is a wide range of formulas that can be written as equations by adding function symbols.

$$\forall x_1 \forall x_2 \cdots \forall x_n \exists y : t(\overline{x}, y) = s(\overline{x}, y),$$

where s, t are terms. If we write $y = f(\overline{x})$ by adding an n-ary function symbol f, this becomes an equation.

For quasigroups, consider the language $(\cdot, L, R)$, where $\cdot, L, R$ are binary function symbols. Let $x \cdot y := \cdot(x, y)$.

The axioms are

$$\begin{gathered}\forall x, y, z : (x \cdot y) \cdot z = x \cdot (y \cdot z) \\ \forall x, y : L(x, y) \cdot x = y \\ \forall x, y : x \cdot R(x, y) = y \\ \forall x, y : L(y, x \cdot y) = x \\ \forall x, y : R(x, x \cdot y) = y\end{gathered}$$

The only question is to relate the last two axioms to the uniqueness of z, w in $zx = y, xw = y$.

Clearly the uniqueness implies the last two axioms. Conversely, if we add the last two axioms, let $z_1 x = z_2 x = y$, then $z_1 = L(x, z_1 \cdot x) = L(x, y) = z_2$. The other case is similar.

2

Suppose B is a finite Boolean algebra. The definition of a Boolean algebra is $(B, 0, 1, \wedge, \vee, \neg)$ with $\wedge, \vee$ commutative, associative, distributive (like "and, or" with "true, false"), and $x \wedge 1 = x, x \vee 0 = x, x \wedge \neg x = 0, x \vee \neg x = 1, 0 = \neg 1$, etc.

Let R be the underlying set of B. For $x, y \in R$, define

$$x \cdot y = x \wedge y, \quad x + y = (\neg x \wedge y) \vee (x \wedge \neg y).$$

Then $(R, +, \cdot, 0, 1)$ is a commutative unital ring such that $x^2 = x, \forall x \in R$.

Conversely, if $(R, +, \cdot, 0, 1)$ is such a ring, then let B be the underlying set of R, and define

$$x \wedge y = x \cdot y, x \vee y = x + y + x \cdot y, \neg x = x + 1.$$

Then $(B, 0, 1, \wedge, \vee, \neg)$ is a Boolean algebra.

Now B is a finite Boolean algebra, so R is a finite commutative unital ring such that $x^2 = x, \forall x \in R$. This means R is a linear space over $\mathbb{F}_2$ because $2x = 0, \forall x \in R$, and as an $\mathbb{F}_2$-linear space, it is uniquely determined by its dimension (which is necessarily finite in this case). So B has size 2^n for some $n \in \mathbb{N}^*$.

Now we want to prove a category-theory statement: if B_1, B_2 are two Boolean algebras, R_1, R_2 the corresponding rings, then $B_1 \times B_2$ corresponds to $R_1 \times R_2$; If R_1, R_2 are two such rings, B_1, B_2 the corresponding Boolean algebras, then $R_1 \times R_2$ corresponds to $B_1 \times B_2$. The proof is just a careful examination of the product $\mathcal{L}$-algebra ($\mathcal{L}$ is the language of Boolean algebras or rings).

Finally, if R is a finite commutative unital ring such that $x^2 = x, \forall x \in R$, then R is isomorphic to a finite product of $\{0, 1\}$-ring (with $1 \neq 0$). Suppose R as an $\mathbb{F}_2$-linear space is n-dimensional. We first prove by induction on n that R can be decomposed as a direct sum of "orthogonal" one-dimensional subspaces.

If $n = 1$, nothing needs to be proved. For $n \geqslant 2$, let $a \neq 0, 1$ be an element in R. Then $aRa = aR, (1 - a)R(1 - a) = (1 - a)R$ are two non-trivial ideals of R and $R = aR \oplus (1 - a)R$ as linear spaces. Moreover, aR is "orthogonal" to $(1 - a)R$: $\forall x \in aR, \forall y \in (1 - a)R$ we have $xy = yx = 0$. And aR is a finite commutative unital ring (the unit is a) such that $x^2 = x, \forall x \in aR$, so is $(1 - a)R$. They have dimension $\leqslant n - 1$, so we can put the decomposition of aR and $(1 - a)R$ together to prove the case n.

Now we have an "orthogonal" basis $e_1, \cdots, e_n$, so every element of S can be uniquely written as $e_S =: \sum_{i \in S} e_i$ for some $S \subseteq \{1, 2, \cdots, n\}$, and

$$e_S + e_{S'} = e_{S \Delta S'}$$

$$e_1 + \cdots + e_n = 1$$

$$e_S e_{S'} = e_{S \cap S'}$$

The first equality is from $2x = 0, \forall x \in R$. The third equality is from $e_ie_j = 0, i \neq j$. The second equality is from $(e_1 + \cdots + e_n)e_i = e_i$, so $(e_1 + \cdots + e_n)x = x, \forall x \in R$. Each $\{0, e_i\}$ is a two-element ring with $\forall x : x^2 = x$. And the map (with notation $S = \{i_1, \cdots, i_k\}$)

$$R \to \prod_{i=1}^{n}\{0, e_i\}, \quad e_S \mapsto (0, \cdots, 0, e_{i_1}, 0, \cdots, 0, e_{i_k}, 0, \cdots, 0)$$

is a bijective ring homomorphism due to the "orthogonal" condition, so it is an isomorphism.

Combining everything, we prove the problem.

3

This actually works like the orthogonal decompositon of Hilbert spaces. This can be characterized by projections.

(Maybe I have to deal with unital rings, but not necessarily commutative.)

Fix the language $\mathcal{L} = \{+, \cdot, -, 0, 1\}$. Consider the sentence φ:

$$\exists p : (p \cdot p = p) \wedge \Big(\forall x : x = pxp + (1-p)x(1-p)\Big) \wedge (p \neq 0) \wedge (p \neq 1).$$

If $R \cong A \times B$ where A, B non-trivial, then in $A \times B$, the element $(1_A, 0)$ can be p in the above.

Conversely, if $R \models \varphi$, then $pRp, (1-p)R(1-p)$ two subsets of R. The elements p and $1-p$ are projections, and $pRp, (1-p)R(1-p)$ are non-trivial subrings, with units $p, 1-p$ respectively. Define the map $f : R \to pRp \times (1-p)R(1-p)$ by $x \mapsto (pxp, (1-p)x(1-p))$ and the map $g : pRp \times (1-p)R(1-p) \to R$ by $g(x, y) = x + y$. By conditions on p, we have p commutes with all $x \in R$, so $pxyp = ppxyp = pxpyp = (pxp)(pyp)$ and this vertifies that f is a ring homomorphism. g is also a ring homomorphism by $p = p^2$. Moreover, f, g are inverse to each other, so both are ring isomorphisms.

Also note that in the sentence, $\forall x : x = pxp + (1-p)x(1-p)$ can be replaced by $\forall x : xp = px$.

4

10 Midterm practice

1

Let

$$p(x) \equiv ((x \neq 0) \land (x \neq 1) \land (\forall y \forall z : yz = x \to y = x \lor z = x)) \,.$$

Then $p(x)$ says x is a prime.

Let $\varphi \equiv (\forall x \exists y : p(x+y) \land p(x+y+2))$. There are infinitely many twin primes iff for any $x \in \mathbb{N}$, there exists a twin prime both of which larger than x.

2

$\mathbb{Z}/\equiv_3$ is such a ring of size 3. The given sentence is an equation, and the theory of ring is equational, so take the product of k copies of $\mathbb{Z}/\equiv_3$.

3

By the downward Löwenheim-Skolem theorem, we can take an elementary substructure $(R, +, \cdot, \leqslant \mathbb{Q}) \preceq (\mathbb{R}, +, \cdot, \leqslant, \mathbb{Q})$ with $|R| = \aleph_0$.

Note that $(\mathbb{R}, +, \cdot, \leqslant, \mathbb{Q}) \models \exists^{\geqslant n} x : Q(x)$ for every $n \in \mathbb{N}^*$, so does R. So $|Q| = \aleph_0$.

Also $(\mathbb{R}, +, \cdot, \leqslant, \mathbb{Q})$ satisfies both $(\mathbb{R}, \leqslant)$ and $(\mathbb{Q}, \leqslant)$ are DLO. For example, these two sentences:

$$\forall x \forall y : x < y \to \exists z : x < z < y$$
$$\forall x \forall y : (x < y \land Q(x) \land Q(y)) \to \exists z : x < z < y \land Q(z)$$

is a part of DLO for R and Q. We just need to add $\land Q(\cdot)$. So $(R, \leqslant) \cong (Q, \leqslant)$. As an elementary substructure, the two structures are elementarily equivalent.

4

Let $\mathcal{L}' = (\{\approx\} \cup \{c_i | i \in I\}) \bigsqcup M$ where $|I| = \aleph_1$, M viewed as constant symbols. Let $T' = T_{\approx} \cup \{\varphi_{ij} | i, j \in I, i \neq j\} \cup \mathrm{eldiag}(M)$, where $\varphi_{ij} \equiv (c_i \neq c_j \land c_i \approx c_j)$.

T' is finitely satisfiable by M, and $|\mathcal{L}'| = \aleph_1$, so apply the downward Löwenheim-Skolem theorem to a model N_0 of T', we can get $N \preceq N_0$ with $|N| = \aleph_1$. Then N has an equivalence class of size $\aleph_1$, and satisfies $\mathrm{eldiag}(M)$. Move by an isomorphism.

5

Consider the theory T by (each $n \in \mathbb{N}^*$ appears)

$$\exists x : \top.$$
$$\forall x \exists ! y : f(y) = x.$$
$$\forall x : f^n(x) \neq x.$$

All models of T are infinite, T is $\aleph_1$-categorical, $|\mathcal{L}| = \aleph_0$, so all models are elementarily equivalent.

6

s maps $\mathbb{N}$ into $\mathbb{N}$, and so does s^2.

7

$\exists^{=2} x \forall y : f(y) \neq x$. This sentence distinguishes $(\mathbb{N}, s)$ from $(\mathbb{N}, s^2)$.

8

One can draw a graph to show that there exists a model (M, f) in which every element has two preimages.

9

Check it by definition.

10

$\exists ! x : xE0$, and $\forall x : x \neq 0 \to (\exists^{=2} y : yEx)$. Since (M, E) is an elementary extension of $(\mathbb{N}, E)$, M must satisfy these two formulas.

11

We prove by induction.

$\{0\}$ is $\{x \in \mathbb{N} | \exists! y : yEx\}$, so it is 0-definable.

Then $\{1\} = \{x \in \mathbb{N} | xE0\}$ is 0-definable. $\{0, 2\} = \{x \in \mathbb{N} | xE1\}$ is 0-definable, and $A \setminus B$ is 0-definable whenever A and B are 0-definable, so $\{2\}$ is 0-definable. Repeat this process, we can prove any finite subset is 0-definable in this structure.

12

Suppose $\{1, 3, 5, 7, \cdots\} = \varphi(\mathbb{N}, \bar{b})$ for some formula $\varphi(x, y_1, \cdots, y_n)$ and $\bar{b} \in \mathbb{N}^n$. By LS theorem, we can take an elementary extension $M \succeq \mathbb{N}$ with $M \neq \mathbb{N}$. Take $c \in M \setminus \mathbb{N}$, so by problem 10, c has degree 2, take any e adjacent to c.

Note that

$$\mathbb{N} \models (\forall x \forall y : xEy \to ([(\varphi(x, \bar{b}) \vee \varphi(y, \bar{b})] \wedge \neg[(\varphi(x, \bar{b}) \wedge \varphi(y, \bar{b})]).$$

This says that for any xEy in $\mathbb{N}$, only one of them is in $\{1, 3, 5, 7, \cdots\}$. Since M is an elementary extension, M also has this property, so only one of c, e is in $\varphi(M, \bar{b})$.

Now $(\mathbb{N}, E)$ satisfies the sentences: 2-regular except one vertex degree 1; no cycles. So the connected component containing c and e must look like $\mathbb{Z}$, so there exists an automorphism of M which moves c to e and fixes $\mathbb{N} \subseteq M$ (and also fixes all the other components). Then $\varphi(M, \bar{b})$ should remain invariant under this automorphism, which contradicts to the construction of the automorphism (moving c to e) and that only one of c, e is in $\varphi(M, \bar{b})$.

13

Let K_n be the complete graph of n vertices, i.e. with all $(n-1)n/2$ edges. Then form the disjoint union of all $K_n, n \geqslant 1$ with no extra edges. Now this graph V_0 has exactly n vertices of degree $n - 1$.

Consider $V_1 := V_0 \bigsqcup \{x\}$, and add all edges between x and all the other points. Then (V_1, E_1) is a model of T, so T is consistent.

Consider $V_2 := V_0 \bigsqcup \{x, y\}$, and add all edges between x and vertices in $K_{2n}, n \geqslant 1$, all edges between y and vertices in $K_{2n-1}, n \geqslant 1$. This is also a model of T.

However, the sentence $\exists x, \forall y : y \neq x \to yEx$ distinguishes V_1 from V_2, so $V_1 \not\equiv V_2$, and T is not complete.

14

If the class of all connected graphs is elementary, say $\mathrm{Mod}(T)$, then we can assume T contains the theory of graphs. Define formulas $\psi_n(u,v)$ $(n \in \mathbb{N}^*)$ by

$$\exists x_1, \cdots, x_{n-1} : (\bigwedge_{1 \leqslant i < j \leqslant n-1} x_i \neq x_j) \wedge uEx_1 \wedge x_1Ex_2 \wedge \cdots \wedge x_{n-1}Ev.$$

The formula $\psi_n(u,v)$ says that the distance between u and v is $\leqslant n$.

Let $p(u,v) = \{u \neq v, \neg\psi_n(u,v) | n \in \mathbb{N}^*\}$ be a partial type over the theory T. It is finitely realizable, so it can be realized in some model $M \models T$. In the graph (M, E), let S be the connected component containing u, i.e. all the vertices of finite distance from u. Then S is disconnected from $M \setminus S$, so M is not connected.

15

Use axiom schema.

First include the theory of graphs.

Then for each $\mathcal{L}$-formula $\varphi(x_1, \cdots, x_n)$, add a sentence

$$\forall \overline{x} \exists y \exists z \exists w : \Big(\varphi(y, \overline{x}) \wedge \neg\varphi(z, \overline{x}) \wedge yEw \wedge zEw\Big) \vee \Big((\forall s : \neg\varphi(s, \overline{x})) \vee (\forall t : \varphi(t, \overline{x}))\Big).$$

This sentence says if S is definable, by the formula $\varphi(M, \overline{x})$, then either some point in S and some point not in S are connected by some point, or one of S, $M \setminus S$ is empty.

16

Consider the same partial type as in problem 14, but over the theory of problem 15. The partial type is finitely realizable in connected graphs, and every connected graph is definably connected. So the partial type is realized in some definably connected graph (V, E), say realized by (u, v). Then consider S, the connected component containing u. The set S as a substructure is a connected graph, but it cannot be definable, otherwise V cannot be definably connected.

There are no edges between vertices in S and vertices in $V \setminus S$ because S is the largest connected component containing u.

17

By B&FS system, one can show that non-empty, acyclic k-regular connected graphs are isomorphic to each other ($1 < k < \infty$). The graphs in this isomorphism class are of size $\aleph_0$.

So size $\aleph_1$ models of the given theory are isomorphic, and clearly any model is infinite, so this theory is complete.

18

Consider $(\mathbb{Z}, E)$, where $|x - y| = 1$ iff xEy. This is a non-empty, acyclic 2-regular connected graph. Two copies of $\mathbb{Z}$ is also non-empty, acyclic 2-regular, but disconnected. By problem 17, they are elementarily equivalent. So the class of disconnected graphs is not elementary, otherwise it would contain $(\mathbb{Z}, E)$.

19

Again, the only difficulty is to prove the B&FS statement in the proof of problem 17.

20

Claim: if $C_n \models \varphi$ for infinitely many n, then there is a non-empty acyclic 2-regular graph G with $G \models \varphi$.

If the claim holds, then we cannot have $C_n \models \varphi$ for infinitely many n and $C_m \models \neg\varphi$ for infinitely many m at the same time, otherwise there would be two non-empty acyclic 2-regular graphs G_1, G_2 with $G_1 \models \varphi, G_2 \models \neg\varphi$, which contradicts completeness of the theory in problem 17.

Now we prove the claim.

Consider $\mathcal{L}' = \{E\} \cup \{v_n | n \in \mathbb{N}^*\}$, and a new theory which consists of φ, the theory of graphs, 2-regularity, v_n being distinct, no cycles of length

k (for each $k \in \mathbb{N}^*$). This theory is finitely satisfiable, so it has a model G. Then G is a non-empty acyclic 2-regular graph with $G \models \varphi$.

11 Homework 8, November 10

1

First, we note that $\mathbb{N} \models (\exists x : s(x) = 1)$ but $\mathbb{N} \not\models (\exists x : s(x) = 0)$, so $\mathrm{tp}(0) \neq \mathrm{tp}(1)$.

Next, we show that there is an isomorphism $f : \langle 0 \rangle \to \langle 1 \rangle$ with $f(0) = 1$. The structures they generate are $\mathbb{N}$ and $\mathbb{N}^*$ respectively, because the language contains only one function symbol, which is unary, so $\langle x \rangle$, the structure generated by $x \in \mathbb{N}$. is

$$\{x, s(x), s^2(x), \cdots\} = \{x, x+1, x+2, \cdots\}.$$

Then we define $f(x) = x + 1, f : \langle 0 \rangle \to \langle 1 \rangle$. Clearly $f(0) = 1$, f preserves s and $\leqslant$, and f is a bijection. This proves $\mathrm{qftp}(0) = \mathrm{qftp}(1)$.

2

Since T_s has QE, we only need to prove that the structures generated by $\varnothing$ in every model are always isomorphic to each other.

However, there are no constant symbols in the language, so the structure generated by $\varnothing$ is $\varnothing$. So $\langle \varnothing \rangle_{T_s} \cong \langle \varnothing \rangle_{\mathbb{Z}}$.

In fact, any QE theory with no constant symbols in the language is complete, from the same proof.

3

It is clear that $M \models T_s$. Since T_s has QE, a substructure A of M is elementary iff $A \models T_s$.

Let $A = \{a_i | i \in \mathbb{Z}\} \subseteq M, B = \{b_i | i \in \mathbb{Z}\} \subseteq M$. It is clear that A and B are substructures of M and models of T_s, so A, B, M are three elementary substructures of M.

Conversely, suppose $N \subseteq M$ is an elementary substructure, so $N \models T_s$. Then N is non-empty. Since $M = A \bigsqcup B$, we have $N = (N \cap A) \bigsqcup (N \cap B)$, and at least one of $N \cap A, N \cap B$ is non-empty.

By the Tarski-Vaught criterion, N intersects every non-empty N-definable sets. For any $i \in \mathbb{Z}$, the set $\{a_j\}$ is $\{a_i\}$-definable because

$$\{a_j\} = \{x \in M | s^{i-j}(x) = x_i\}$$

or

$$\{a_j\} = \{x \in M | s^{j-i}(x_i) = x\},$$

according to $i \geqslant j$ or $i < j$. Similar for $\{b_i\}$.

So if $N \cap A \neq \varnothing$, then $N \cap \{a_i\} \neq \varnothing$ for every $i \in \mathbb{Z}$, so $a_i \in N$ for every $i \in \mathbb{Z}$, i.e. $A \subseteq N$. Similarly if $N \cap B \neq \varnothing$, then $B \subseteq N$. Since $N = (N \cap A) \bigsqcup (N \cap B)$, the only possible choices for N are $A, B, A \cup B = M$.

4

Let D be a definable set in M. Let

$$\mathcal{F} = \{A \subseteq M | A \text{ is finite or } M \setminus A \text{ is finite}\}.$$

Then $\mathcal{F}$ is a boolean algebra, closed under finite intersection, finite union and complement. Since T has QE, the boolean algebra of definable subsets of M is generated by $\{\varphi(M) | \varphi \text{ atomic } \mathcal{L}(M)\text{-formula}\}$. So we only need to show that if $\varphi(x)$ is an atomic $\mathcal{L}(M)$-formula, then $\varphi(M) \in \mathcal{F}$.

Now we only have one binary relation symbol in the language, so an atomic $\mathcal{L}(M)$-formula is one of the following (x stands for a variable, a, b stand for elements in M) :

(1) $x \approx a$. $\varphi(M)$ is a set of size 2.

(2) $x \approx x$. $\varphi(M) = M$ is cofinite.

(3) $a \approx b$. $\varphi(M)$ is M or $\varnothing$, and is thus cofinite or finite.

(4) $a \approx x$. The same as (1).

(5) $x = a$. $\varphi(M)$ is a set of size 1.

(6) $x = x$. $\varphi(M) = M$ is cofinite.

(7) $a = b$. $\varphi(M)$ is M or $\varnothing$, and is thus cofinite or finite.

(8) $a = x$. The same as (5).

In all cases, $\varphi(M) \in \mathcal{F}$, which completes the proof.

12 Challenge 8

13 Homework 9, November 17

1

Define a set of formulas $\{x > a, x < b | a \in A, b \in B\}$. For any finite subset $\{a_1, \cdots, a_n\}$ of A and finite subset $\{b_1, \cdots, b_m\}$ of B, we have $\max(a_1, \cdots, a_n) < \min(b_1, \cdots, b_m)$ by assumption, so by DLO, there exists an element $x \in M$ such that $x > a_i$ for $1 \leqslant i \leqslant n$ and $x < b_j$ for $1 \leqslant j \leqslant m$. So the set of formulas is a partial type in M, and there is an elementary extension $M' \succeq M$ containing $c \in M'$ which realizes the partial type, i.e. $c > a$ for all $a \in A$ and $c < b$ for all $b \in B$.

2

Since p and s are inverse to each other, any composition of finitely many p's and s's is of the form $p^k, k \in \mathbb{Z}$ (or equivalently s^{-k}), regardless of the order of p and s in the composition.

Suppose $M \models \varphi(b)$ where $\varphi(x)$ is an atomic $\mathcal{L}(A)$-formula. We check all the possibilities of an atomic formula. Note that an $\mathcal{L}(A)$-term must be of the form $p^k(a), a \in A$ or $p^k(x), x$ a variable, according to the previous discussion.

Case 1. $\varphi(x) \equiv p^k(a) \leqslant p^l(x)$. Then $M \models \varphi(b)$ is equivalent to $p^{k-l}(a) \leqslant b$. Note that A is a substructure, so $p^{k-l}(a) \in A$ whenever $a \in A$. Since $\{a \in A | a < b\} = \{a \in A | a < c\}$ and $b \notin A, c \notin A$, we have $p^{k-l}(a) < b$ iff $p^{k-l}(a) < c$. So $M \models \varphi(b)$ iff $N \models \varphi(c)$.

Case 2. $\varphi(x) \equiv p^k(x) \leqslant p^l(x)$. Then $M \models \varphi(b)$ is equivalent to $k \geqslant l$, so $M \models \varphi(b)$ iff $N \models \varphi(c)$.

Case 3. $\varphi(x) \equiv p^k(a) \leqslant p^l(a')$. Clearly $M \models \varphi(b)$ iff $N \models \varphi(c)$ since there are actually no free variables.

Case 4. $\varphi(x) \equiv p^k(x) \leqslant p^l(a)$. Then $M \models \varphi(b)$ iff $b \leqslant p^{l-k}(a)$. For the same reason as in case 1, we have $M \models \varphi(b)$ iff $N \models \varphi(c)$.

Case 5. $\varphi(x) \equiv p^k(x) = p^l(x)$. Then $M \models \varphi(b)$ iff $k = l$, so $M \models \varphi(b)$ iff $N \models \varphi(c)$.

Case 6. $\varphi(x) \equiv p^k(x) = p^l(a)$. Then $M \models \varphi(b)$ iff $b = p^{l-k}(a)$. But $p^{l-k}(a) \in A$ and $b \notin A$, so this is impossible, i.e. $M \not\models \varphi(b)$ and $N \not\models \varphi(c)$.

Case 7. $\varphi(x) \equiv p^k(a) = p^l(x)$. This is same as case 6.

Case 8. $\varphi(x) \equiv p^k(a) = p^l(a)$. This happens iff $k = l$, so $M \models \varphi(b)$ iff $N \models \varphi(c)$.

In all cases $M \models \varphi(b)$ iff $N \models \varphi(c)$ whenever $\varphi(x)$ is an atomic $\mathcal{L}(A)$-formula, so $\mathrm{qftp}(b/A) = \mathrm{qftp}(c/A)$.

3

Assume the same configuration as the previous exercise, but $b \in M$ a general element and A finitely generated by $a_1, \cdots, a_n$. We want to find an elementary extension $N' \succeq N$ and $c \in N'$ such that $\mathrm{qftp}^M(b/A) = \mathrm{qftp}^{N'}(c/A)$

Case 1. $b \in A$. Then we take $N' = N, c = b \in A$. The map id_A gives an isomorphism

$$A = \langle A, b\rangle_M \to \langle A, c\rangle_N = A$$

which maps b to c, so $\mathrm{qftp}(b/A) = \mathrm{qftp}(c/A)$.

Case 2. $b \notin A$. Let $A^+ = \{a \in A | a < b\}, A^- = \{a \in A | a > b\}$. Since $b \notin A$ and M is a linear order, we have $A^+ \bigsqcup A^- = A$, and if $a \in A^+, a' \in A^-$, then $a < a'$. So by exercise 1, there is an elementary extension $N' \succeq N$ and $c \in N'$ such that $a < c < a'$ for every $a \in A^+, a' \in A^-$. Since $A = A^+ \bigsqcup A^-$, this implies $c \notin A$, and $\{a \in A | a < c\} = A^+$. By exercise 2, $\mathrm{qftp}(b/A) = \mathrm{qftp}(c/A)$.

So by the criterion, T has QE.

4

As I pointed out last week, if a theory has QE and the language has no constant symbols, then the theory is complete, because the structure generated by $\varnothing$ is still $\varnothing$. Now the language has a binary relation symbol and two unary function symbols and the theory has QE by the previous exercise.

Warning. The language should also have no 0-ary relation symbols so that a theory with QE is complete.

14 Challenge 9

15 Homework 10, November 24

1

$(a-\sqrt{2})^2 = 3$, so $a^2 - 2\sqrt{2}a - 1 = 0$. Then $(a^2-1)^2 = 8a^2$, so we can take $P(x) = a^4 - 10a^2 + 1$.

2

Claim: Let $a = (1+\sqrt{5})/2, b = (1-\sqrt{5})/2$. We have $\mathrm{qftp}(a) = \mathrm{qftp}(b)$ and $\mathrm{tp}(a) \neq \mathrm{tp}(b)$.

Let $\varphi(x) \equiv (\exists y : y \cdot y = x)$, so in $(\mathbb{R}, +, \cdot, -, 0, 1)$, $\varphi(x)$ says x is non-negative. Since $a > 0, b < 0$, we have $\varphi(x) \in \mathrm{tp}(a), \varphi(x) \notin \mathrm{tp}(b)$.

Let $\psi(x)$ be an atomic $\mathcal{L}_{ring}$-formula. Here $\mathcal{L}_{ring} = \{+, \cdot, -, 0, 1\}$ as in the problem. So $\psi(x)$ is of the form $P(x) = Q(x)$, where P, Q are polynomials with integer coefficients. Then $\psi(x)$ is equivalent to $R(x) = 0$ for some $R \in \mathbb{Z}[x]$. Note that any $\tilde{S}(x) \in \mathbb{Q}[x]$ can be written as $cS(x)$ for some $c \in \mathbb{Z} \setminus \{0\}, S \in \mathbb{Z}[x]$. So $\mathrm{qftp}(a) = \mathrm{qftp}(b)$ iff $I_{a/\mathbb{Q}} = I_{b/\mathbb{Q}}$. (If the character is p, then $\mathrm{qftp}(a) = \mathrm{qftp}(b)$ iff $I_{a/\mathbb{F}_p} = I_{b/\mathbb{F}_p}$.)

And a, b have the same minimal polynomial $x^2 - x - 1$. So we prove the claim. This shows that $(\mathbb{R}, +, \cdot, -, 0, 1)$ does not have QE.

To see $\mathrm{qftp}(a) = \mathrm{qftp}(b)$, we can also consider $\langle a \rangle$ and show that $\langle a \rangle = \mathbb{Q}[a]$.

3

In the language of rings, $\mathbb{C}$ has QE. So we only need to prove the statement for atomic formulas φ, i.e. $\mathrm{qftp}(\tau) = \mathrm{qftp}(-1/\tau)$.

From the discussion in the previous solution, $\mathrm{qftp}(\tau) = \mathrm{qftp}(-1/\tau)$ iff $I_{\tau/\mathbb{Q}} = I_{(-1/\tau)/\mathbb{Q}}$. But it is easy to find that they have the same minimal polynomial $x^2 - x - 1$.

16 Challenge 10

17 Homework 11, December 1

1

Prove by contradiction. Suppose $b \in \operatorname{acl}(A)$. Then $b \in D$ for some A-definable finite set $D \subseteq M$. Write $D = \varphi(M)$ for some $\mathcal{L}(A)$-formula $\varphi(x)$.

A realization of $\operatorname{tp}(b/A)$ in M is an element $c \in M$ such that $\forall \mathcal{L}(A)$-formula $\psi(x)$ with $M \models \psi(b)$, one has $M \models \psi(c)$. Since $b \in D$, one has $M \models \varphi(b)$, so $M \models \varphi(c)$ for any realization c of $\operatorname{tp}(b/A)$ in M, i.e. $c \in \varphi(M) = D$. Then D is infinite, which is a contradiction.

2

Only need to prove $\operatorname{acl}(A) \subseteq A$.

Take any $b \in \operatorname{acl}(A)$. By the previous exercise, $\operatorname{tp}(b/A)$ has finitely many realizations in M. Write $A = \{a_1, \cdots, a_n\}$ with $a_1 < \cdots < a_n$. If A is empty, then $\operatorname{tp}(b/A)$ is completely determined by $\operatorname{qftp}(b)$ since $M \models DLO$ and DLO has QE. But for any $c \in M$, the map $b \mapsto c$ gives an isomorphism $\{b\} = \langle b \rangle_M \to \langle c \rangle_M = \{c\}$, so $\operatorname{qftp}(b) = \operatorname{qftp}(c)$. This means any $c \in M$ is a realization of $\operatorname{tp}(b/A)$, so we cannot have $b \in \operatorname{acl}(\varnothing)$. This proves $\operatorname{acl}(\varnothing) = \varnothing$.

Now suppose $n \geqslant 1$. By Theorem 10.3.2 and the fact that DLO has QE, there are finitely many realizations of $\operatorname{qftp}(b/A)$ in M, and every $\operatorname{qftp}(b/A)$ is p_ξ or q_a, where p_ξ or q_a is characterized as Theorem 10.3.2 in the notes version 3.5.

If $\operatorname{qftp}(b/A)$ is p_ξ, then $\operatorname{qftp}(b/A)$ contains formulas

$$\{x > a : a \in \xi^-\} \cup \{x < a : a \in \xi^+\}$$

for some cut $\xi = (\xi^-, \xi^+)$ of A. Since A is finite, this set of formulas is equivalent to $a_i < x < a_{i+1}$ for some $i \in \{0, 1, \cdots, n\}$, where we regard $a_0 = -\infty, a_{n+1} = +\infty$. But whenever $c \in M$ and $a_i < c < a_{i+1}$, we have an obvious isomorphism $\langle b, A \rangle_M \to \langle c, A \rangle_M$ mapping b to c and extending id_A, so $\operatorname{qftp}(b/A) = \operatorname{qftp}(c/A)$. This implies that there are infinitely many realizations of $\operatorname{qftp}(b/A)$ in M, namely every element $c \in (a_i, a_{i+1})$, since M is a DLO. So it is impossible that $\operatorname{qftp}(b/A)$ is p_ξ for some cut ξ of A.

If $\operatorname{qftp}(b/A)$ is q_a, then $b = a \in A$. Since $b \in \operatorname{acl}(A)$ is chosen arbitrarily, we have $\operatorname{acl}(A) = A$ whenever $1 \leqslant |A| < \infty$.

So $\mathrm{acl}(A) = A$ in any case.

3

Define $p(a, x) = a < x \wedge \neg\exists y : a < y < x$. So $p(a, x)$ says that x is the smallest element which is strictly greater than a. In $(\mathbb{Z}, \leqslant)$, $\mathbb{Z} \models p(a, x)$ iff $x = a + 1$.

We claim that $n \pm 1 \in \mathrm{dcl}(\{n\})$ for any $n \in \mathbb{Z}$.

In fact, $\{n + 1\} = p(n, \mathbb{Z})$ and $\{n - 1\} = p(\mathbb{Z}, n)$ are both $\{n\}$-definable. This proves the claim.

Now by the claim, we have $\{0, \pm 1\} \subseteq \mathrm{dcl}(\{0\})$. Apply $\mathrm{dcl}(-)$ to this inclusion, we get

$$\mathrm{dcl}(\{0, \pm 1\}) \subseteq \mathrm{dcl}(\mathrm{dcl}(\{0\})) = \mathrm{dcl}(0).$$

Use the claim again, we have $\pm 2 \in \mathrm{dcl}(\{0, \pm 1\})$. Repeat this process (or precisely by induction), and we prove that $\pm n \in \mathrm{dcl}(\{0\})$ for any $n \in \mathbb{Z}$, so $\mathrm{dcl}(\{0\}) = \mathbb{Z}$.

4

Let $p(x, y) = xEy$. So $\{n-1, n+1\} = p(\mathbb{Z}, n)$ is $\{n\}$-definable, so $1 \in \{-1, 1\}$ a finite $\{0\}$-definable set, which shows that $1 \in \mathrm{acl}(\{0\})$.

Consider $\varphi(x) = -x, \mathbb{Z} \to \mathbb{Z}$. It is clear that $\varphi(x)E\varphi(y)$ iff xEy, and φ is a bijection, so φ is an automorphism of $(\mathbb{Z}, E)$. Suppose $1 \in \mathrm{dcl}(\{0\})$. Then $\{1\} = \psi(\mathbb{Z}, \overline{0})$ for some $\mathcal{L}_E$-formula $\psi(x, \overline{y})$, i.e. $x = 1$ iff $\mathbb{Z} \models \psi(x, \overline{0})$. However, $\varphi(0) = 0$ and φ is an automorphism, so $\mathbb{Z} \models \psi(x, \overline{0})$ iff $\mathbb{Z} \models \psi(-x, \overline{0})$. This implies that $x = 1$ iff $x = -1$, which is absurd. So $1 \notin \mathrm{dcl}(\{0\})$.

18 Challenge 11

19 Homework 12, December 8

1

Since the language contains only a binary relation symbol, any $N \models T, N \supseteq M$ makes M a substructure of N. Now we take an $N \models T, N \supseteq M$ such that $N/\approx$ is infinite, and every $\approx$-equivalence class of N is infinite. This is possible because we can simply add elements to M. Then $M \preceq_\exists N$.

Consider $\varphi_n = \exists^{\geqslant n} x : \top$ which is qfree. We know for any $n \in \mathbb{N}^*$, $N \models \varphi_n$, so $M \models \varphi_n$. This shows that M is infinite.

In particular, M is non-empty. Fix any element $a \in M$. Consider $\psi_n(y) = \exists^{\geqslant n} x : x \approx y$, which is of the form $\exists \overline{x}$: qfree formula on $(\overline{x}, y)$ in the definition of "existentially closed". Then for any $n \in \mathbb{N}^*$, $N \models \psi_n(a)$, so $M \models \psi_n(a)$. This shows that $[a]_M$ is infinite. Since $a \in M$ is arbitrarily chosen, every $\approx$-equivalence class in M is infinite.

Consider

$$\rho = \exists x_1, \cdots, x_n : \left(\bigwedge_{1 \leqslant i < j \leqslant n} x_i \not\approx x_j \right).$$

Again this is of the required form. Since $N \models \rho$, we know $M \models \rho$. This shwos that $M/\approx$ is infinite.

2

Now $\{M \setminus D_n | n \in \mathbb{N}^*\}$ is a small family of definable subsets. Each of them is non-empty because (for example) $M \setminus D_1 \supseteq D_2 \neq \varnothing$.

For any $D_{i_1}, \cdots, D_{i_n}$ (distinct subscripts), the finite intersection $\bigcap_{j=1}^{n} (M \setminus D_{i_j}) = M \setminus \bigcup_{j=1}^{n} D_{i_j}$ is non-empty because if $i \notin \{i_1, \cdots, i_n\}$, then $\varnothing \neq D_i \subseteq (M \setminus \bigcup_{j=1}^{n} D_{i_j})$. By κ-compactness, the intersection $\bigcap_{n \in \mathbb{N}^*} (M \setminus D_n) = M \setminus \bigcup_{n=1}^{\infty} D_n$ is non-empty.

3

Let $\mathcal{F} = \{[a]_E | a \in X\}$ be the set of E-equivalence classes in X. Since E is not defined on M, we write out explicitly that $[a]_E = \{b \in X | bEa\}$. Since E is a definable equivalence relation on X, the set $Y := \{(a, b) \in X^2 | aEb\}$

is definable, so

$$[a]_E = \{b \in X | (a,b) \in Y\}$$

is definable. In fact, if we write $X = \varphi(M), Y = \psi(M)$ for some $\mathcal{L}(M)$-formulas $\varphi(x), \psi(y,z)$, then $b \in [a]_E$ iff $M \models \varphi(b) \wedge \psi(a,b)$.

Then $X \subseteq_{def} M, X \subseteq \bigcup \mathcal{F}$, $|\mathcal{F}| = |X/E|$ is small, so there are $a_1, \cdots, a_n \in X$ such that $X \subseteq \bigcup_{i=1}^{n} [a_i]_E$. But it can happen iff X/E is finite because $X = \bigsqcup_{\substack{\text{one representative} \\ \text{from each class}}} [a]_E$.

4

Let $A = \{a_n | n \in \mathbb{N}^*\}$ be a small subset of M. Let $p = \{\varphi_n(x) \equiv (x > a_n) | n \in \mathbb{N}^*\}$ be a set of $\mathcal{L}(A)$-formulas. It is finitely satisfiable because for $\varphi_1(x), \cdots, \varphi_N(x)$, the element a_{N+1} realizes them, so p is a partial type over the small set A. Now by definition of saturation, p is realized in M, so there is some $b \in M$ such that $b > a_n$ for all $n \in \mathbb{N}^*$, i.e.

$$a_1 < a_2 < \cdots < b.$$

20 Challenge 12

21 Homework 13, December 15

1

Suppose $f : A \to B$ is a PEM with $|f| < \aleph_0$, so $|A| = |B|$ both are finite. Write $A = \{a_1, \cdots, a_n\}, B = \{b_1, \cdots, b_n\}$ both in the increasing order. Consider $\varphi(x, y) \equiv (x < y)$, we should have $\mathbb{R} \models \varphi(a_i, a_j)$ iff $\mathbb{R} \models \varphi(f(a_i), f(a_j))$, so f must be order-preserving, which forces $f(a_i) = b_i$ for $1 \leqslant i \leqslant n$.

Now define

$$g(x) = \begin{cases} x - a_1 + b_1, & x \leqslant a_1, \\ \frac{b_{i+1}-b_i}{a_{i+1}-a_i} x + \frac{b_i a_{i+1} - a_i b_{i+1}}{a_{i+1}-a_i}, & x \in (a_i, a_{i+1}], \\ x - a_n + b_n, & x > a_n. \end{cases}$$

This extends $f : A \to B$ to a strictly increasing, bijective $g : \mathbb{R} \to \mathbb{R}$. This $g \in \mathrm{Aut}(\mathbb{R})$.

Since f is arbitrarily chosen, $(\mathbb{R}, \leqslant)$ is $\aleph_0$-strongly homogeneous.

2

Suppose $(M, \leqslant)$ is $\aleph_0$-strongly homogeneous. Then there is an automorphism $f : M \to M$ such that $f(-1) = 1$. This is because $M \models \mathrm{DLO}$, $\{-1\} = \langle -1 \rangle \overset{\cong}{\to} \langle 1 \rangle = \{1\}$, so $\mathrm{qftp}(-1) = \mathrm{qftp}(1)$ which further implies $\mathrm{tp}(-1) = \mathrm{tp}(1)$.

As in exercise 1, f is strictly increasing. Now any subset is a substructure because we have only a binary relation symbol, so f maps $\{x \in M | x > -1\}$ isomorphically to $\{x \in M | x > f(-1)\}$, i.e. $f : (-1, 0) \cup (0, +\infty) \overset{\cong}{\to} (1, +\infty)$.

Let $A = (-1, 0), B = (0, +\infty)$, then $f(A) \bigsqcup f(B) = (1, +\infty)$. Since f is strictly increasing, we have:

(1) $\forall x \in f(A), \forall y \in f(B) : x < y$.

(2) $f(A)$ has no maximal elements because A does not, $f(B)$ has no minimal elements because B does not.

However, this is impossible in $(1, +\infty)$. $f(A) \bigsqcup f(B)$ forms a cut of $(1, +\infty)$, and $(1, +\infty)$ is order complete, so either $f(A)$ has a maximum or $f(B)$ has a minimum, which is a contradiction.

(It seems that completeness is second-order, but isomorphisms between structures should preserve second-order things.)

3

Take any two elements a, b in M such that $a \not\approx b$. As in exercise 2, we have only one binary relation symbol, so the structures they generate are isomorphic, so they have the same qfree types. Since T has QE, they have the same complete types, then by $\aleph_0$-strongly homogeneous, there is an automorphism $\sigma : M \to M$ such that $\sigma(a) = b$.

If $x \approx a$, then $\sigma(x) \approx \sigma(a) = b$, and vice versa. This shows that $x \in [a]_\approx$ iff $\sigma(x) \in [b]_\approx$. Since σ is a bijection, it maps $[a]_\approx$ bijectively to $[b]_\approx$, so $|[a]_\approx| = |[b]_\approx|$.

Since a, b are arbitrarily chosen, any two equivalence classes have the same cardinality.

22 Challenge 13

1

Take any elementary substructure $B \preceq \mathbb{M}$ containing A, then $B = \mathrm{acl}(B)$ and $|B| = \infty$. We have $\mathrm{acl}(A) \subseteq \mathrm{acl}(B) = B$, so

$$\mathrm{acl}(A) \subseteq \bigcap \{B | B \preceq \mathbb{M}, B \supseteq A\}.$$

Conversely, we want to prove

$$\mathrm{acl}(A) \supseteq \bigcap \{B | B \preceq \mathbb{M}, B \supseteq A\}.$$

Take any $b \notin \mathrm{acl}(A)$. We want to prove there exists some B with $A \subseteq B \preceq \mathbb{M}$ and $b \notin B$.

We know by κ-strongly homogeneity that for any $c \notin \mathrm{acl}(A)$, there exists some $\sigma \in \mathrm{Aut}(\mathbb{M}/A), \sigma(b) = c$. By Löwenheim-Skolem Theorem, there is an elementary substructure $N \preceq \mathbb{M}$ with $A \subseteq N, |N| = |A| + |\mathcal{L}|$, so N is small. This implies that there is some $c \in \mathbb{M} \setminus N$ because $\mathbb{M}$ is large. Note that $\mathrm{acl}_{\mathbb{M}}(A) = \mathrm{acl}_N(A) \subseteq N$ because $A \subseteq N \preceq \mathbb{M}$, so this c is automatically outside $\mathrm{acl}(A)$, and thus there is some $\sigma \in \mathrm{Aut}(\mathbb{M}/A)$ such that $\sigma(b) = c$. Then $c \notin N$ implies $b \notin \sigma^{-1}(N)$, and $A \subseteq \sigma^{-1}(N) \preceq \mathbb{M}$, so we complete the proof.

2

Take any small set $A \subseteq M$ and a partial n-type p in the language $\mathcal{L}_0$ over A. Then p can certainly be viewed as a partial n-type in the language $\mathcal{L}$, so it is realized in M.

Now take any $\mathcal{L}$-structure M which is not κ-strongly homogeneous for some infinite cardinal κ. Then M is naturally an $\mathcal{L}(M)$-structure. As an $\mathcal{L}(M)$-structure, the map $f : M \supseteq A \to B \subseteq M$ is a PEM iff $A = B$ and $f = \mathrm{id}_A$. So any PEM can be extended to id_M, which shows that M as an $\mathcal{L}(M)$-structure is κ-strongly homogeneous for any κ. Now $M \restriction \mathcal{L}$ can be viewed as a reduction to $\mathcal{L} \subseteq \mathcal{L}(M)$.

3

Inspired by the previous problem, we just need to prove that any PEM must be of the form id_A for some $A \subseteq \mathbb{R}$. It suffices to show that $\mathrm{tp}(a) = \mathrm{tp}(b)$ iff

$a = b$. (In fact, there is only one automorphism of $(\mathbb{R}, +, \cdot)$.)

Suppose $\text{tp}(a) = \text{tp}(b)$ for some $a, b \in \mathbb{R}$. We define several kinds of formulas:

(1) $\varphi_0(x) \equiv (\forall y : x + y = y)$, so $\mathbb{R} \models \varphi_0(x)$ iff $x = 0$.

(2) $\varphi_1(x) \equiv (\forall y : x \cdot y = y)$, so $\mathbb{R} \models \varphi_1(x)$ iff $x = 1$.

(3) $\varphi_+(x) \equiv (\exists y : y \cdot y = x)$, so $\mathbb{R} \models \varphi_+(x)$ iff $x \geqslant 0$.

(4) For $n \in \mathbb{N}^*$, define $\varphi_n(x) \equiv (\exists y : x = \underbrace{y + y + \cdots + y}_{n \text{ times}} \wedge \varphi_1(y)$.

(5) For $k \in \mathbb{Z}_{<0}$, define $\varphi_k(x) \equiv (\exists y, z : \varphi_{-k}(y) \wedge z = x + y \wedge \varphi_0(z)$.

(6) For $p \in \mathbb{Z}, q \in \mathbb{N}^*$, define $\varphi_{p/q}(x) \equiv (\exists y : \varphi_p(y) \wedge y = \underbrace{x + x + \cdots + x}_{q \text{ times}})$.

Then we know

$$\{\exists y, z : \varphi_r(y) \wedge x = y + z \wedge \varphi_+(z) | r \leqslant a, r \in \mathbb{Q}\} \subseteq \text{tp}(a)$$

and

$$\{\neg \exists y, z : \varphi_r(y) \wedge x = y + z \wedge \varphi_+(z) | r > a, r \in \mathbb{Q}\} \subseteq \text{tp}(a).$$

From this we know $\text{tp}(a) = \text{tp}(b)$ iff $a = b$.

4

Suppose a point A in $\mathbb{N}^{\mathbb{N}}$ is in the closure of the image. Then it is the limit of some sequence $T(v_1), \cdots, T(v_m), \cdots$, where

$$T(v_m) = (f_1(v_m), f_2(v_m), f_3(v_m), \cdots).$$

The convergence in $\mathbb{N}^{\mathbb{N}}$ means that for any fixed k, there exists $M = M_k$ such that for all $m > M$, $T(v_m)$ has the same first k numbers as A.

Write $A = (r_1, r_2, r_3, \cdots)$, so for any $n \in \mathbb{N}^*$, when $m > M_n$, $T(v_m)$ looks like $(r_1, \cdots, r_n, \text{then something unknown}, \cdots)$. By definition, for $1 \leqslant k \leqslant n$, there are exactly r_k vertices of distance k from $v_m, m > M_n$.

I guess that every vertex has degree 3 is only used to show that $f_k(v)$ is always finite.

Let $\varphi_{k,r}(x)$ be the formula saying that there are exactly r vertices of distance k from x. Let $p = \{\varphi_{k,r_k}(x) | k \geqslant 1\}$. It is finitely satisfiable because

$$v_m \models \varphi_{1,r_1}(x) \wedge \cdots \wedge \varphi_{n,r_n}(x), \quad \forall m > M_n.$$

So p is a partial type. Moreover, it is a partial type over $\varnothing$, so it is realized in V. The realization v satisfies $T(v) = A$.

5

Take any $a, b \in V$. I want to prove there is an automorphism σ such that $\sigma(a) = b$. So I want to prove $a \mapsto b$ is a PEM, and I have to prove $\mathrm{tp}(a) = \mathrm{tp}(b)$ first, but complete types are difficult to determine. The only way I can imagine is to prove V has QE, but I am not able to show that it does have QE by the criterion.

If $\mathrm{tp}(a) = \mathrm{tp}(b)$ can be proved directly, the only case in question is $\exists x : \varphi$ where φ is a formula. This is too complicated.

Another idea is to consider $(V, E, D_1, \cdots)$ as in challenge 8. This has QE. But I don't know how to relate homogeneity in different language. Maybe consider the connected components?