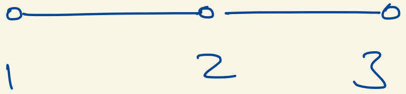
8th March: Finite element algorithms.

$$u \frac{d\phi}{dx} \underline{-} k \frac{d^2\phi}{dx^2} = 0.$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\boxed{\frac{u}{2}}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix} + \boxed{\frac{k}{l}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$



assembled

proved

$$\int_{\Omega} \underset{\uparrow}{W} R \, d\Omega = 0$$

$$\phi = \underline{N_1}\phi_1 + \underline{N_2}\phi_2$$

$$W = N$$

$$\underline{\boxed{\frac{ul}{2k} = Pe}}$$

element Peclet number

Finite element method - Examples.

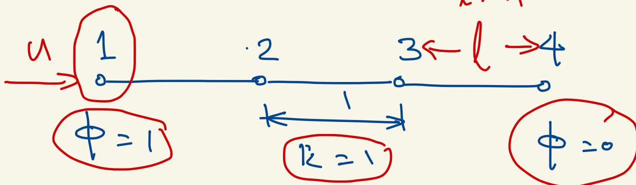
$$u\frac{d\phi}{dx} - k\frac{d^2\phi}{dx^2} = 0$$

$l = h$

Convection-diffusion equation.

Element matrices

$$\frac{u}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix} \quad ; \quad \frac{k}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$



Assembled matrix.

$$\frac{u}{2}\begin{bmatrix}-1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix} + \frac{k}{l}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_1 = 1\\ \phi_2\\ \phi_3\\ \phi_4 = 0\end{Bmatrix} = \begin{Bmatrix}Q_1\\ 0\\ 0\\ Q_4\end{Bmatrix}.$$

Peclet number ; $P_e = \frac{ul}{2k}$ ✓

Example 1: ✓

$u = 0$ ∴ $P_e = 0$

$$\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_1 = 1\\ \phi_2\\ \phi_3\\ \phi_4 = 0\end{Bmatrix} = \begin{Bmatrix}Q_1\\ 0\, + 1\\ 0\\ Q_4\end{Bmatrix}$$

$\phi_1 = 1$

$\phi_4 = 0$

✓

$$\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}\phi_2\\ \phi_3\end{Bmatrix} = \begin{Bmatrix}1\\ 0\end{Bmatrix}.$$

$$\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}\begin{Bmatrix}\phi_2\\ \phi_3\end{Bmatrix} = \begin{Bmatrix}1\\ 1\end{Bmatrix} \Longrightarrow$$

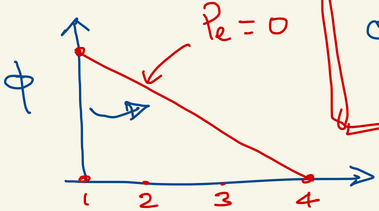
$\phi_1 = 1$ ✓

$\phi_3 = \frac{1}{3}$ ✓

$\phi_2 = \frac{4}{6} = 2/3$ ✓

$\phi_4 = 0$ ✓

Answers



✓

Example 2:

$u = 2$; $P_e = 1$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} + \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ Q_4 \end{Bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ Q_4 \end{Bmatrix}.$$

$\phi_1 = 1$

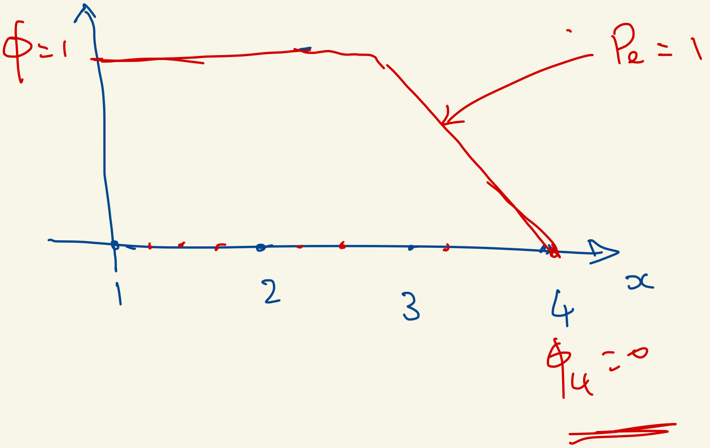
$+ 2$

$\phi_4 = 0$

$$\begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}. \Rightarrow \boxed{\begin{aligned} \phi_2 &= 1 \\ \phi_3 &= 1 \end{aligned}}$$

Answers.

✓



Example 3 :

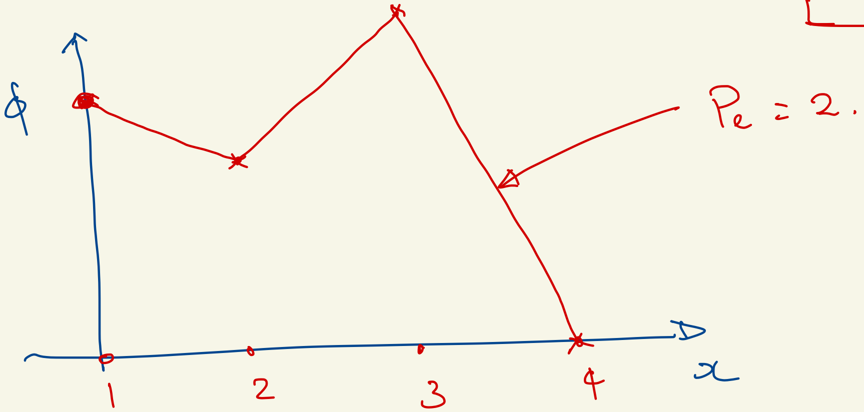
$u = 4$; $P_e = 2$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ Q_4 \end{Bmatrix} \quad \begin{matrix} \phi_1 = 1 \\ +3 \\ \\ \phi_4 = 0 . \end{matrix}$$

$$\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3.5 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.5 \end{Bmatrix} \Rightarrow$$

$\phi_1 = 1$ ✓
$\phi_2 = 0.85$ ✓
$\phi_3 = 1.28$ ✓
$\phi_4 = 0$
Answers.



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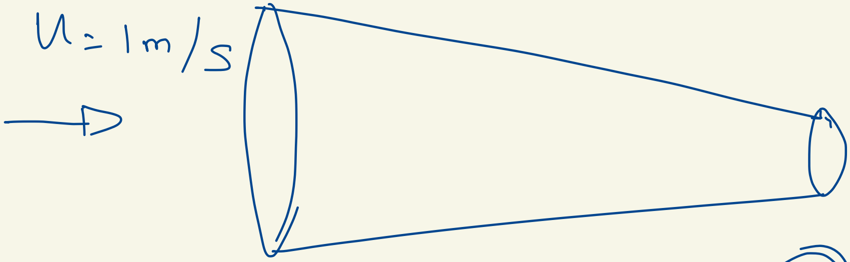


$$P_e = \frac{u\,\boxed{\ell}\longleftarrow}{2k}$$

Assignment 2 :

$$u\frac{d\phi}{dx} - k\frac{d^2\phi}{dx^2} = 0$$

Element convection matrix $= \frac{u}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}$

Element diffusion matrix $= \frac{k}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$.

Implementation in Matlab

pre-processing. — input number of points / NP, velocity, u and diffusion coefficient, k and total length, L

```
% No elements.
NE = NP - 1
% Create connectivity ✓
For ie = 1, NE
    intma (ie, 1) = ie
    intma (ie, 2) = ie+1
End
```

ie = 1
intma (1, 1) = 1
intma (1, 2) = 2
ie = 2
intma(2,1) = 2
intma (2,2) = 3

```
% Element length ✓
el = L/NE
% Element matrix
emat (1, 1) = -0.5 *u + (k /el)
emat (1, 2) = 0.5 *u - (k /el)
emat (2, 1) = -0.5 * u - (k /el)
emat (2,2) = 0.5 * u + (k /el)
```

1, 2

gmat(1, 1)
gmat(1, 2)
gmat(2, 1)
gmat(2, 2)

```
% matrix assembly
% initialize matrix
gmat (np, np) = 0.0

For ie = 1, NE
    ip1 = intma(ie, 1)
    ip2 = intma(ie, 2)
    gmat (ip1, ip1) = gmat (ip1, ip1) + emat(1, 1)
    gmat (ip1, ip2) = gmat (ip1, ip2) + emat (1, 2)
    gmat (ip2, ip1) = gmat (ip2, ip1) + emat (2, 1)
    gmat (ip2, ip2) = gmat (ip2, ip2) + emat (2, 2)

end.
% gmat is the assembled matrix.
```

$$\left[\ gmat\ \right]\{\phi\} = \{0\}$$

$$\left[\quad\right]\{\phi\} = \{f\}$$

$$u \int_{\Omega^{(e)}} \alpha \underline{W_a^*} \frac{d\hat{\phi}}{dx} \, d\Omega$$

Petrov-Galerkin Weighting

$$\begin{bmatrix} \left(\frac{dN_1}{dx}\right)^2 & \frac{dN_1}{dx}\frac{dN_2}{dx} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & \left(\frac{dN_2}{dx}\right)^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{l\cancel{x}} & -\frac{1}{l\cancel{x}} \\ -\frac{1}{l\cancel{x}} & \frac{1}{l\cancel{x}} \end{bmatrix} l$$

$$\underline{\underline{W_a^* = \frac{h}{2}\frac{dN^T}{dx}}} \checkmark$$

$$\frac{h u \alpha}{2} \int \frac{dN^T}{dx}\frac{dN}{dx}\{\phi\} \, d\Omega$$

$$\boxed{= \frac{u\alpha}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}}$$

$$\underline{\frac{k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}}$$

Petrov-Galerkin method.

$$u\frac{d\phi}{dx} - k\frac{d^2\phi}{dx^2} = 0.$$

$$\frac{u}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix}$$ — convection matrix.

$$\frac{k}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix}$$ — diffusion matrix.

$$\boxed{\frac{\alpha u}{2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix}$$ + additional diffusion.

Exercise: (*) Assemble these matrices for 2 elements and write the nodal equation for node 2

Substitute $\alpha = 1$ $\boxed{\Rightarrow u\frac{\phi_i - \phi_{i-1}}{\Delta x}}$

First order upwind.

Transient problems - FEM.

Lax-Windroff method.

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0$$

$$\phi^{n+1} = \phi^n + \Delta t \underline{\frac{\partial \phi^n}{\partial t}} + \frac{\Delta t^2}{2} \underline{\frac{\partial^2 \phi^n}{\partial t^2}}$$

$$\frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial t^2} = -u \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) = -u \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)$$

$$= u^2 \frac{\partial^2 \phi}{\partial x^2}.$$

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = -u \frac{\partial \phi^n}{\partial x} + \frac{\Delta t u^2}{2} \frac{\partial^2 \phi^n}{\partial x^2}$$

Central FDM — Lax-Windroff

Standard Galerkin — Tayor Galerkin

Navier - Stokes equation :

Incompressible flow equations;

$$\rho \left(\cancel{\frac{\partial u_i}{\partial t}} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

— momentum.

$$\frac{\partial u_i}{\partial x_i} = 0$$ — continuity.

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

$$[C]\{u\} = -[G]\{p\} + [K]\{u\} + \{f\}_1$$

$$[D]\{u\} = \{f\}_2$$

$$\begin{bmatrix} [C]-[K] & [G] \\ [D] & \overset{\frac{1}{\gamma}[M]}{\boxed{0}} \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{p\} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Pressure instability.

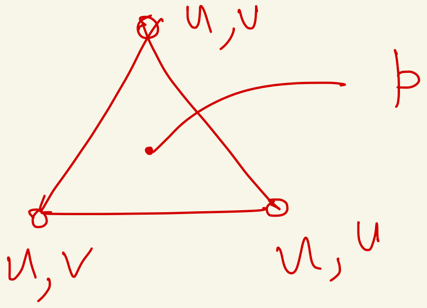
1. Penalty method.

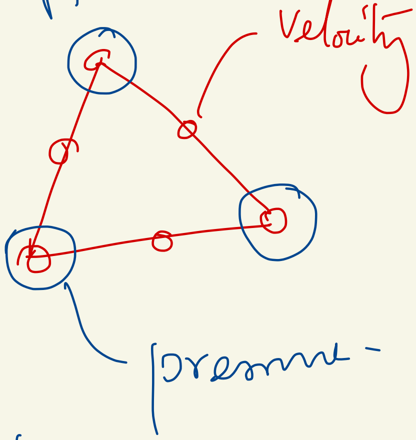
$$\frac{p}{\gamma} + \frac{\partial u_i}{\partial x_i} = 0 \qquad \gamma \sim 10^8$$

$$\underline{\frac{1}{\gamma}[M]\{p\}} + [D]\{u\} = f_2$$

2. Mixed interpolation

Staggered approach.





3. fractional step method.