First-Order Linear Equations

A simple solution: Leibniz' method

A first order linear differential equation is one that can be written in the form

$$y' + py = g. \tag{1}$$

Here both p and g are functions of the independent variable t. (Please be careful: the notation gets messy, so we write p and g instead of $p(t)$ and $g(t)$, but you must not assume that p and g are constants unless we state this explicitly.) In many mathematical models the left-hand side of this equation reflects the internal dynamics of a system, and the term g on the right-hand side is an external driving force.

There is a simple method, due to Leibniz, for solving any first-order linear equation, at least in principle. As we will see, the method requires the computation of two integrals, which may or may not be possible in practice. Briefly, here is how Leibniz' method works. The left-hand side is reminiscent of the result of product rule: $(\mu y)' = \mu y' + \mu' y$. To make this explicit we multiply both sides of equation (1) by a factor μ, as yet unknown:

$$\mu y' + p\mu y = \mu g. \tag{2}$$

If we can find μ so that $\mu' = p\mu$ then we do indeed have $(\mu y)'$ on the left-hand side, whose antiderivative is plainly μy. This leads to the formula

$$y = \frac{1}{\mu}\left(\int \mu g + A\right), \text{ where } \mu = e^{\int p} \text{ and } A \text{ is an arbitrary constant.} \tag{3}$$

For example, to solve the equation

$$y' = t - 2ty \tag{4}$$

we first put the equation in standard form

$$y' + 2ty = t$$

then we compute the integrating factor

$$\mu = \exp \int 2t\, dt = e^{t^2},$$

and then we deduce that

$$y = e^{-t^2}\left(\int te^{t^2}\, dt + A\right) = \frac{1}{2} + Ae^{-t^2}. \tag{5}$$

Note that we do not need a constant of integration for the computation of μ, since we only need one integrating factor, not an entire family. The constant of integration arises in the last integral, which is sufficient to describe the entire family of solutions.

A deeper analysis: variation of parameters

Let's look at the solutions of linear equations in more depth. Note that the solution in equation (5) is a sum of two terms, one of which ($\frac{1}{2}$) is the same for all solutions while the other (Ae^{-t^2}) is a constant multiple of a some function. This is the characteristic form of the solution to any linear equation.

The first term is a particular response to the external driver g, while the second term solves the homogeneous equation $y' + py = 0$. Again, when a system is governed by a first order differential equation the term g typically represents an external force.

Let us go back to the beginning, and study the equation by first considering the special case where $g = 0$. We are now led to the simpler equation

$$y' + py = 0. \tag{6}$$

This is simpler because we can *separate variables*:

$$\frac{dy}{dt} = -py, \text{ or } \frac{dy}{y} = -p\, dt \tag{7}$$

Now we can integrate both sides — on the left with respect to y and on the right with respect to t — then solve the resulting equation for y.

Rather than do the calculations for a general differential equation, let us look at the homogeneous part of our specific example:

$$y' + 2ty = 0. \tag{8}$$

If we separate variables we obtain

$$\frac{dy}{y} = -2t\,dt. \tag{9}$$

If we integrate both sides of equation (9) we obtain

$$\ln|y| = -t^2 + C, \tag{10}$$

where C is some constant. (Remember that the constant can be determined only by some initial conditions.) We can solve equation (10) for y:

$$y = \pm e^C e^{-t^2} = Ae^{-t^2}, \tag{11}$$

where $A = \pm e^C$ — again a constant. The integral curves all depend on the parameter A. Note that all of these curves decay to 0 as $t \to +\infty$, no matter what A is.

What happens if we now re-introduce the (possibly nonconstant) driver g? So, we try to find a factor v so that ve^{-t^2} solves the original (nonhomogeneous) equation. Again, rather than look at the most general situation let us take a specific example:

$$y' + 2ty = t. \tag{12}$$

If we take $y = ve^{-t^2}$ then we find that

$$\begin{aligned} y' &= v'e^{-t^2} - 2tve^{-t^2} \\ y' + 2ty &= v'e^{-t^2} - 2tve^{-t^2} + 2tve^{-t^2} = v'e^{-t^2} \end{aligned}$$

If this y is to solve equation (12) then we must have that

$$v'e^{-t^2} = t, \qquad v = \int te^{t^2}\,dt = \frac{1}{2}e^{t^2} + A,$$

where A is a constant. This means that

$$y = ve^{-t^2} = \frac{1}{2} + Ae^{-t^2}. \tag{13}$$

Yet another perspective

There is a third way to approach linear equations that works well in many cases, especially when the driver is fairly elementary: look at the slope field and make a good guess! Draw the slope field for our equation. On it you should see that all of the solutions approach some constant value as $t \to +\infty$. Based on this we guess that one possible solution is $y_1 = C$, where C is a constant. If y_1 is a solution then we must have that $0 + 2Ct = t$ (why?) and hence $C = \frac{1}{2}$. This gives one solution. What about the others? If there were a second solution y_2 then we would have

$$\begin{aligned} y_1' + 2ty_1 = t &= y_2' + 2ty_2 \\ (y_2 - y_1)' + 2t(y_2 - y_1) &= 0 \end{aligned}$$

In other words, $y_2 - y_1$ is a solution to the homogeneous equation — let's call it y_h. We have already found that $y_h = Ae^{-t^2}$, for some constant A, and therefore

$$y_2 = y_1 + y_h = \frac{1}{2} + Ae^{-t^2}.$$

We have derived this answer three times now, but the answer itself is not the moral of the story. What we have seen is that the solution of the homogeneous equation plays a role in the solution of the nonhomogeneous equation. If we can find *one* solution to the nonhomogeneous equation — perhaps by integrating, perhaps with an inspired guess — then we can find *all* solutions merely by adding the solutions to the homogeneous equation. We will employ this strategy over and over again this semester.

Further reading

Please read examples 1–4 from section 2.1 for more details. Pay attention not only to the formal manipulations, but also to the graphs of the integral curves. Systems governed by first order linear equations have their own distinctive properties, which we will explore in more detail in section 4. Good practice problems are 1–12, page 39. For each of these, draw a direction field, solve the differential equation, and graph several solutions on your direction field. Summarize the behavior of the solutions, paying particular attention to the domain of validity and the asymptotic behavior. How does this behavior depend on the initial values?

Reading quiz

1. What is the standard form for a first-order linear equation? Give some examples.
2. True or false: Every first-order linear equation is separable. Explain!
3. True or false: Every first-order autonomous equation is linear. Explain!
4. True or false: Every first-order linear equation is autonomous. Explain!
5. Outline the steps to solve a first-order linear equation.
6. True or false: Any two solutions to a nonhomogeneous linear equation differ by a solution to the homogeneous equation. Explain!

Assignment 4: due Monday, 6 February

1. Draw the slope field for the example above, $y' + 2ty = t$. Sketch several integral curves. Describe the behavior of the integral curves as $t \to +\infty$.
2. For each of the following equations, use Leibniz' method to find the complete solution. Show all your work!
   (a) $y' + 2y = e^t$.
   (b) $ty' = t^2 - y$.
   (c) $\cos(t)y' + \sin(t)y = \cos^2(t)$.
   (d) $ty' + 2y = \sin(t)$.
   (e) $y' = t(5y + e^{-t})$.