The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 2, Issue 1, Article 4, pp. 1-12, 2005

ELLIPSES OF MAXIMAL AREA AND OF MINIMAL ECCENTRICITY INSCRIBED IN A CONVEX QUADRILATERAL

ALAN HORWITZ

Received 10 September, 2004; accepted 21 December, 2004; published 15 February, 2005.

PENN STATE UNIVERSITY, 25 YEARSLEY MILL RD., MEDIA, PA 19063
alh4@psu.edu
URL: www.math.psu.edu/horwitz

ABSTRACT. Let Đ be a convex quadrilateral in the plane and let M1 and M2 be the midpoints of the diagonals of Đ. It is well–known that if E is an ellipse inscribed in Đ, then the center of E must lie on Z, the open line segment connecting M1 and M2 . We use a theorem of Marden relating the foci of an ellipse tangent to the lines thru the sides of a triangle and the zeros of a partial fraction expansion to prove the converse: If P lies on Z, then there is a unique ellipse with center P inscribed in Đ. This completely characterizes the locus of centers of ellipses inscribed in Đ. We also show that there is a unique ellipse of maximal area inscribed in Đ. Finally, we prove our most signifigant results: There is a unique ellipse of minimal eccentricity inscribed in Đ.

Key words and phrases: Ellipse, Tangent, Quadrilateral, Eccentricity.

2000 *Mathematics Subject Classification.* 26A99, 51N99, 65D99.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Let Đ be a **convex quadrilateral** in the xy plane. A problem, often referred to in the literature as Newton's problem, was to determine the locus of centers of ellipses inscribed in Đ. Chakerian ([1]) gives a partial solution of Newton's problem using orthogonal projection, which is the solution actually given by Newton, which we state as

Theorem 1.1. *Let M_1 and M_2 be the midpoints of the diagonals of Đ. If E is an ellipse inscribed in Đ, then the center of E must lie on the open line segment, Z, connecting M_1 and M_2.*

However, Theorem 1.1 does not really give the precise locus of centers of ellipses inscribed in Đ. Newton only proved that the center of E must lie on Z, as is noted in ([1]). In [3] we proved that it is indeed the case that **every point** of Z is the center of an ellipse inscribed in Đ. In this paper we give a much shorter and more succint proof (Theorem 2.3) that if $(h, k) \in Z$, then there is a unique ellipse, with center (h, k), inscribed in Đ. In addition, we prove two other important results not proved in [3]. First, we show that there is a unique ellipse of **maximal area** inscribed in Đ (Theorem 3.3). Our most signifigant result is Theorem 4.4: There is a unique ellipse, E, of **minimal eccentricity** inscribed in Đ. Theorem 4.4 is somewhat more difficult to prove, and our proof gives a constructive method for finding such an ellipse by finding the roots of a polynomial of degree four. Only one of those roots lies in a known interval containing the x coordinate of the center of E. Of course, if Đ is a **tangential quadrilateral**, meaning that a **circle** can be inscribed in Đ, then that circle would be the unique ellipse of minimal eccentricity inscribed in Đ.

The approach given here is based on the following theorem of Marden ([4], Theorem 1) relating the foci of an ellipse tangent to the lines thru the sides of a triangle and the zeros of a partial fraction expansion.

Theorem 1.2. *Let z_1, z_2, z_3 be three noncollinear points in the complex plane, and let $F(z) = \dfrac{t_1}{z-z_1} + \dfrac{t_2}{z-z_2} + \dfrac{t_3}{z-z_3}$, $\sum_{k=1}^{3} t_k = 1$, and let Z_1 and Z_2 denote the zeros of $F(z)$. Let L_1, L_2, L_3 be the line segments connecting z_2, z_3, z_1, z_3, and z_1, z_2, respectively. If $t_1t_2t_3 > 0$, then Z_1 and Z_2 are the foci of an ellipse, E, which is tangent to L_1, L_2, and L_3 in the points $\zeta_1, \zeta_2, \zeta_3$, where $\zeta_1 = \dfrac{t_2z_3 + t_3z_2}{t_2+t_3}$, $\zeta_2 = \dfrac{t_1z_3 + t_3z_1}{t_1+t_3}$, $\zeta_3 = \dfrac{t_1z_2 + t_2z_1}{t_1+t_2}$, respectively.*

2. LOCUS OF CENTERS

We shall prove Theorem 2.3 below for the case when no two sides of Đ are parallel. Our methods extend easily to the case when exactly two sides of Đ are parallel, that is, when Đ is a trapezoid. Of course, if Đ is a parallelogram, then the midpoints of the diagonals coincide, and the line segment Z is just a point. Since ellipses, tangent lines to ellipses, and convex quadrilaterals are preserved under affine transformations, we may assume that the vertices of Đ are $(0, 0)$, $(1, 0)$, $(0, 1)$, and (s, t) for some real numbers s and t. Then the midpoints of the diagonals of Đ are $M_1 = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$, $M_2 = \left(\dfrac{1}{2}s, \dfrac{1}{2}t\right)$, and the equation of the line thru M_1 and M_2 is

$$y = L(x) = \frac{1}{2}\frac{s - t + 2x(t-1)}{s-1}.$$

Since Đ is convex, it follows easily that $s > 0, t > 0$ and $s + t \geq 1$. Since Đ is four–sided and no two sides of Đ are parallel, $s + t > 1$ and $s \neq 1 \neq t$.

Let I denote the open interval between $\dfrac{1}{2}$ and $\dfrac{1}{2}s$. We shall need the following lemmas.

Lemma 2.1. *If $h \in I$ and $s + t > 1$, then $s + 2h(t-1) > 0$.*

Proof. If $s \geq 1$, then $I = \left(\frac{1}{2}, \frac{1}{2}s\right) \Rightarrow h < \frac{1}{2}s \Rightarrow s - 2h > 0 \Rightarrow s + 2h(t-1) = s - 2h + 2ht > 0$. If $s \leq 1$, then $I = \left(\frac{1}{2}s, \frac{1}{2}\right) \Rightarrow h < \frac{1}{2} \Rightarrow 1 - 2h > 0 \Rightarrow$
$s + 2h(t-1) = 2h(s+t-1) + (1-2h)s > 0$. ∎

Lemma 2.2. *Let E_1 and E_2 be ellipses with the same foci. Suppose also that E_1 and E_2 pass through a common point, z_0. Then $E_1 = E_2$.*

Proof. Denote the foci by Z_1 and Z_2. Then E_j has equation $|z - Z_1| + |z - Z_2| = k_j, j = 1, 2$, and $|z_0 - Z_1| + |z_0 - Z_2| = k_j, j = 1, 2 \Rightarrow k_1 = k_2 \Rightarrow E_1 = E_2$. ∎

Theorem 2.3. *Let Ð be a convex quadrilateral in the xy plane and let M_1 and M_2 be the midpoints of the diagonals of Ð. Let Z be the open line segment connecting M_1 and M_2. If $(h, k) \in Z$ then there is a unique ellipse with center (h, k) inscribed in Ð.*

Proof. Denote the lines which make up ∂ (Ð) by L_1: $y = 0$, L_2: $x = 0$, L_3: $y = \frac{t}{s-1}(x-1)$, L_4: $y = 1 + \frac{t-1}{s}x$. The three intersection points of the lines L_1, L_2, and L_3 are the complex points $z_1 = 0$, $z_2 = 1$, and $z_3 = -\frac{t}{s-1}i$. Using Theorem 1.2, if t_1 and t_2 are real numbers with $t_1 t_2 (1 - t_1 - t_2) > 0$, there is an ellipse, E_1, tangent to L_1, L_2, and L_3 with foci Z_1 and Z_2 which are the zeros of $F(z) = \frac{t_1}{z} + \frac{t_2}{z-1} + \frac{1 - t_1 - t_2}{z + \frac{t}{s-1}i}$. Z_1 and Z_2 are the zeros of the numerator of $F(z)$, which is the polynomial

$$\begin{aligned} p_1(z) &= (s-1)\, z^2 + (it(t_1 + t_2) + (s-1)(t_2 - 1))\, z - it_1 t \\ &= (s-1)\,(z - Z_1)(z - Z_2). \end{aligned}$$

Thus the center, C_1, of E_1 is

$$\frac{1}{2}(Z_1 + Z_2) = -\frac{1}{2(s-1)}\left(it(t_1 + t_2) + (s-1)(t_2 - 1)\right).$$

Taking real and imaginary parts yields $C_1 = \left(\frac{1}{2} - \frac{1}{2}t_2, -\frac{1}{2}t\frac{t_1 + t_2}{s-1}\right)$. The three intersection points of the lines L_1, L_2, and L_4 are the complex points $w_1 = 0$, $w_2 = i$, and $w_3 = -\frac{s}{t-1}$. Again, using Theorem 1.2, if s_1 and s_2 are real numbers with $s_1 s_2 (1 - s_1 - s_2) > 0$, there is an ellipse, E_2, tangent to L_1, L_2, and L_4 with foci, W_1 and W_2, which are the zeros of $G(z) = \frac{s_1}{z} + \frac{s_2}{z-i} + \frac{1 - s_1 - s_2}{z + \frac{s}{t-1}}$. W_1 and W_2 are the zeros of the numerator of $G(z)$, which is the polynomial

$$\begin{aligned} p_2(z) &= (t-1)\, z^2 + (s(s_1 + s_2) + i\,(s_2 - 1)\,(t-1))\, z - is_1 s \\ &= (t-1)\,(z - W_1)(z - W_2). \end{aligned}$$

A simple computation shows that the center of E_2 is $C_2 = \left(-\frac{1}{2}s\frac{s_1 + s_2}{t-1}, -\frac{1}{2}(s_2 - 1)\right)$. One can solve for t_1 and t_2 to show that the center of E_1 equals $C_1 = (h, k)$ if and only if

$$t_1 = 2h - 1 - 2k\left(\frac{s-1}{t}\right), t_2 = 1 - 2h. \tag{2.1}$$

Similarly, the center of E_2 equals $C_2 = (h, k)$ if and only if

$$s_1 = 2k - 1 - 2h\frac{t-1}{s}, s_2 = 1 - 2k. \tag{2.2}$$

Our objective now is to show that if $(h, k) \in Z$ and if s_1, s_2, t_1, t_2 are defined by (2.1) and (2.2), then $t_1 t_2 (1 - t_1 - t_2) > 0$ and $s_1 s_2 (1 - s_1 - s_2) > 0$, so that the ellipses E_1 and E_2 exist. Then we shall show that $k = L(h)$ forces E_1 and E_2 to be the **same** ellipse! Letting $E = E_1 = E_2$ then gives an ellipse which is inscribed in Đ since $(h, k) \in$ Đ. So given $(h, k) \in Z$, let s_1, s_2, t_1, t_2 be defined by (2.1) and (2.2). Now $(h, k) \in Z \Rightarrow k = L(h) = \frac{1}{2}\frac{s - t + 2h(t-1)}{s-1}$. Substituting $k = L(h)$ into (2.1) and (2.2) gives $t_1 t_2 (1 - t_1 - t_2) = (s - 2h)(2h - 1)\frac{s + 2h(t-1)}{t^2} > 0$ since $h \in I$ and by Lemma 2.1. Similarly,

$$s_1 s_2 (1 - s_1 - s_2) = (s + 2h(t-1))(2h - 1)(s - 2h)\frac{(t-1)^2}{s^2 (s-1)^2} > 0,$$

again since $h \in I$ and by Lemma 2.1. The centers of E_1 and E_2 are now both equal to (h, k), with E_1 tangent to L_1, L_2, and L_3, and E_2 tangent to L_1, L_2, and L_4. By (2.1) and (2.2),

$$\begin{aligned} p_1(z) &= (s-1) z^2 - 2(s-1)(h + ki)z \\ &\quad + i(t(1-2h) + 2k(s-1)) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} p_2(z) &= (t-1) z^2 - 2(t-1)(h + ik)z \\ &\quad - i(2h(1-t) + s(2k-1)). \end{aligned} \tag{2.4}$$

Substituting $k = L(h)$ into (2.3) and (2.4) gives $\frac{p_1(z)}{s-1} = \frac{p_2(z)}{t-1} = z^2 - 2(h + iL(h))z + i\frac{s-2h}{s-1}$. Thus $\frac{p_1(z)}{s-1}$ and $\frac{p_2(z)}{t-1}$ have the **same** coefficients. Recalling that the zeros of p_1 and p_2 are the foci of E_1 and E_2, respectively, we have shown that E_1 and E_2 have the *same foci*. Also, by Theorem 1.2, E_1 is tangent to L_1 at $\zeta_3 = \frac{t_1 z_2 + t_2 z_1}{t_1 + t_2} = \frac{t_1}{t_1 + t_2} = \frac{s - 2h}{s - t + 2ht - 2h} \equiv z_0$. Similarly, E_2 is tangent to L_1 at $\zeta_2 = \frac{s_1 w_3 + s_3 w_1}{s_1 + s_3}$, which, upon simplifying, also equals z_0. Thus E_1 and E_2 are ellipses with the same foci and which pass through the common point, z_0. By Lemma 2.2, $E_1 = E_2$. Hence $E = E_1 = E_2$ is an ellipse, with center (h, k), which is tangent to **all four lines** L_1, L_2, L_3, and L_4. Of course E is **inscribed** in Đ since $(h, k) \in Z \subset$ Đ.

To prove *uniqueness*, if E_1 and E_2 are distinct concentric ellipses, then, as noted in ([1]), their four common tangents would have to form a parallelogram. If Đ is not a parallelogram, then this is a contradiction. We leave the proof of Theorem 2.3 when exactly two sides of Đ are parallel to the reader. ∎

3. MAXIMAL AREA

The following lemma is a generalization of a result which appears in ([1]) on the area of an ellipse inscribed in a triangle. Chakerian's result assumes that the point P lies **inside** ABC, the triangle with vertices A, B, and C, while our result assumes that P lies **outside** ABC. In that case, area$(ABC) =$ area$(CPA) +$ area$(APB) -$ area(BPC). The details of the proof are similar and we omit them.

Lemma 3.1. *Given a triangle ABC and a point $P \notin \partial(ABC)$, let $\alpha = area(BPC), \beta = area(CPA)$, and $\gamma = area(APB)$. Let L_1, L_2, and L_3 be the three lines thru the sides of ABC, and let E be an ellipse with center P which is tangent to L_1, L_2, and L_3. If $\sigma = \frac{1}{2}(\alpha+\beta+\gamma)$, then $area(E) = \frac{4\pi}{area(ABC)}\sqrt{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)}$.*

Lemma 3.2. *Let E be the ellipse in Theorem 1.2 and let U be the triangle formed by z_1, z_2, and z_3. Then $area(E) = \pi\times$ $area(U)\sqrt{t_1t_2t_3}$.*

Proof. If T is the composition of a rotation, a magnification, and/or a translation of the plane, then it is easy to show that the foci of $T(E)$ are $T(Z_1)$ and $T(Z_2)$. Thus we may assume that U has vertices $A=(0,0), B=(s,t)$, and $C=(0,1)$, where $s>0$. Then Z_1 and Z_2 are the zeros of $F(z) = \frac{t_1}{z} + \frac{t_2}{z-i} + \frac{1-t_1-t_2}{z-s-ti}$ and the center of E is $P = \frac{1}{2}(Z_1+Z_2) = (s(t_1 + t_2)/2, (t(t_1+t_2)+1-t_2)/2)$. A simple computation shows that area$(APB) = \frac{1}{4}s\left|1-t_2\right|$, area$(CPA) = \frac{1}{4}s\left|t_1+t_2\right|$, and area$(BPC) = \frac{1}{4}s\left|1-t_1\right|$. Considering the cases $t_1>0, t_2>0$, $t_1<0, t_2<0$, $t_1>1, t_2<0$, or $t_1<0, t_2>1$, it follows that $\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma) = \frac{1}{256}s^4t_1t_2t_3$. By Lemma 3.1, area$(E) = \frac{4\pi}{\text{area}(U)}(\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma))^{1/2} = \frac{1}{2}\pi s\sqrt{t_1t_2t_3} \Rightarrow \frac{\text{area}(E)}{\text{area}(U)} = \frac{\pi(s/2)\sqrt{t_1t_2t_3}}{(s/2)} = \pi\sqrt{t_1t_2t_3}$. ∎

Theorem 3.3. *Let Đ be a convex quadrilateral in the xy plane. Then there is a unique ellipse of maximal area inscribed in Đ.*

Proof. Again, we may assume that the vertices of Đ are $(0,0), (1,0), (0,1)$, and (s,t) where the positive real numbers s and t satisfy the hypotheses in §2. Let A_E = area of an ellipse E inscribed in Đ. We want to maximize A_E as a function of h, where $(h, L(h))$ denotes the center of E. Assume first that no two sides of Đ are parallel. From the proof of Theorem 2.3, $t_1t_2(1-t_1-t_2) = (s-2h)(2h-1)\frac{s+2h(t-1)}{t^2}$. Since E is tangent to L_1, L_2, and L_3 from the proof of Theorem 2.3, by Lemma 3.2, it suffices to maximize $S(h) = (s-2h)(2h-1)(s+2h(t-1)), h \in I$ = the open interval between $\frac{1}{2}$ and $\frac{1}{2}s$. Now $S(1/2) = S(s/2) = 0$, and $S(h) \geq 0$ for $h \in I$ by Lemma 2.1. Hence $S'(h_0) = 0$ for some $h_0 \in I$ with $S(h_0)$ a local maximum. Also, $S(h_0)$ must be the **only** local maximum of $S(h)$ on I, else $S'(h)$ would have **three** zeros in I. Thus $S(h_0)$ is the unique global maximum of $S(h)$ on I. If exactly two sides of Đ are parallel, so that Đ is the trapezoid with vertices $(0,0), (1,0), (0,1)$, and $(1,t)$, $t \neq 1$, then one can show that the area of the ellipse inscribed in Đ is $S(k) = (2k-1)\frac{t-2k}{t^3}$, $k \in I$, where I is the open interval between $\frac{1}{2}$ and $\frac{1}{2}t$. Setting $S'(k) = 0$ yields $k = \frac{1}{4}t + \frac{1}{4}$, which is the *midpoint* of I. ∎

4. Minimal Eccentricity

Unfortunately, since the ratio of the eccentricity of two ellipses is **not** preserved in general under nonsingular affine transformations of the plane, we cannot assume, as earlier, that that the vertices of Đ are $(0,0), (1,0)$, $(0,1)$, and (s,t). However, by using an **isometry** of the plane, we can assume that Đ has vertices $(0,0), (0,C)$, (A,B), and (s,t), where

$$s > 0, A > 0, C > 0, t > B \tag{4.1}$$

Let L_1: $y = \dfrac{B}{A}x, L_2$: $x = 0, L_3$: $y = B + \dfrac{t-B}{s-A}(x-A)$, and L_4: $y = C + \dfrac{t-C}{s}x$ denote the lines which make up the boundary of Đ. As earlier, we shall provide the details for the proof of Theorem 4.4 below with the assumption that no two sides of Đ are parallel.

- Since Đ is convex, (s,t) must lie above $\overleftrightarrow{(0,C)\ (A,B)}$ and (A,B) must lie below $\overleftrightarrow{(0,0)\ (s,t)}$, which implies

$$A(t-C) + (C-B)s > 0, At - Bs > 0. \tag{4.2}$$

- Since no two sides of Đ are parallel, $L_1 \not\parallel L_4$ and $L_2 \not\parallel L_3$, which implies

$$Bs - A(t-C) \neq 0, s \neq A. \tag{4.3}$$

Let

$$I = \begin{cases} (A/2, s/2) & \text{if } A < s \\ (s/2, A/2) & \text{if } s < A. \end{cases}$$

$M_1 = \left(\frac{1}{2}A, \frac{1}{2}(B+C)\right)$ and $M_2 = \left(\frac{1}{2}s, \frac{1}{2}t\right)$ are the midpoints of the diagonals of Đ and the equation of the line thru M_1 and M_2 is

$$y = L(x) = \frac{1}{2}t + \frac{B+C-t}{A-s}\left(x - \frac{1}{2}s\right), x \in I. \tag{4.4}$$

Remark 4.1. It is useful to note that reflection of Đ thru the x axis followed by translation upward by C units is equivalent to permuting s and A, then replacing t by $C-B$, and finally replacing B by $C-t$. That transformation leaves $q(h)$ and D invariant.

We first prove some key lemmas about the following quadratic polynomial in h:

$$\begin{aligned} q(h) &= 4\left((s-A)^2 + (t-B-C)^2\right)(h-A/2)^2 \\ &+ 4(s-A)\left(A(s-A) + B(t-B) + C(t-C)\right)(h-A/2) \\ &+ \left(A^2 + (C-B)^2\right)(s-A)^2. \end{aligned} \tag{4.5}$$

Let D denote (the discriminant of q)/16 $(s-A)^2$. A simple computation yields

$$D = 4BC((t-B)(t-C) + s(s-A)) - (At - s(B+C))^2. \tag{4.6}$$

We shall prove in general that q has no zeros in I. First we show that if $t-C$ and B have **opposite signs**, then q has no real zeros whatsoever.

Lemma 4.1. *If* $(t-C)B < 0$*, then* $D < 0$.

Proof. If (1) $s > A, t > C$ and $B < 0$, or (2) $s < A, t < C$ and $B > 0$, then $D < 0$ by (4.1) and (4.6). If $s < A, t > C$ and $B < 0$, or $s > A, t < C$ and $B > 0$, then permute s and A, replace t by $C - B$, and finally replace B by $C - t$ (that is equivalent to reflection of Đ thru the x axis followed by translation upward by C units). It is easy to show that that transformation leaves $q(h)$ and D invariant and the new parameters $A, B, C, s,$ and t then satisfy (1) or (2). ∎

Now we show that if $t - C$ and B have the **same sign** and $D \geq 0$, then q cannot vanish in I.

Lemma 4.2. *If $D \geq 0$ and $(t - C)B \geq 0$, then $q'(A/2)q'(s/2) > 0$.*

Proof. A simple computation gives

$$q'(A/2)q'(s/2) =$$
$$16\,(s-A)^2\left(D + (As + B(t-C) + C(t-B))\left((B+C-t)^2 + (s-A)^2\right)\right)$$

and the lemma follows immediately from (4.1). ∎

Some simplification yields $q(A/2) = \left(A^2 + (C-B)^2\right)(s-A)^2$ and $q(s/2) = \left(s^2 + t^2\right)(s-A)^2$, which are both positive by (4.1). Thus q has an **even** number of roots in I, which implies that if $q'(A/2)$ and $q'(s/2)$ have the same sign, then q cannot vanish in I . Thus lemmas 4.1 and 4.2 imply

Proposition 4.3. *q has no zeros in I.*

We can now prove

Theorem 4.4. *Let Đ be a **convex quadrilateral** in the xy plane. Then there is a unique ellipse of minimal eccentricity inscribed in Đ.*

Proof. As in the proof of Theorem 2.3, $L_1, L_2,$ and L_3 form a triangle, T_1, whose vertices are the complex points $z_1 = 0$, $z_2 = A + Bi$, and $z_3 = -\dfrac{At - Bs}{s - A}i$. If E is any ellipse inscribed in Đ, then E must be tangent to the three sides of T_1 (though not necessarily inscribed in T_1). By Theorem 1.2, the foci, Z_1 and Z_2, of E are the zeros of $F(z) = \dfrac{t_1}{z} + \dfrac{t_2}{z - (A + Bi)} + \dfrac{1 - t_1 - t_2}{z + \frac{At - Bs}{s - A}i}$. Now $F(z) = 0 \iff p(z) = 0$, where

$$\begin{aligned} p(z) &= (s-A)\,z^2 - \\ &\quad (A((s-A)(1-t_2) - it(t_1+t_2)) + iB((s-A)(1+t_1) + A(t_1+t_2)))\,z + \\ &\quad i\,(Bs - At)\,(A + iB)\,t_1. \end{aligned}$$

The center, Ĉ, of E is

$$\frac{1}{2}\left(Z_1 + Z_2\right) = -p'(0)/p''(0) =$$
$$\frac{1}{2\,(s-A)}\left((A\,(1-t_2)\,(s-A) + (-At\,(t_1+t_2) + B\,(s - A + At_2 + t_1 s))i\right).$$

Taking real and imaginary parts yields

$$\hat{C} = \frac{1}{2\,(s-A)}(A\,(1-t_2)\,(s-A)\,, -At\,(t_1+t_2) + B\,(s - A + At_2 + t_1 s)).$$

If Ĉ $= (h, k) \in$ Đ, then solving for t_1 and t_2 yields

$$t_1 = \frac{2(t-B)h + 2k(A-s) - (At - Bs)}{At - Bs}, t_2 = \frac{A - 2h}{A}. \tag{4.7}$$

Substitute for t_1 and t_2 in the formula above for $p(z)$, let $k = L(h)$ (see (4.4), and denote the resulting polynomial by $p_h(z)$. Some simplification yields

$$p_h(z) = (s-A)z^2 - 2(s-A)(h + iL(h))z - (B - iA)(s - 2h)\,C. \tag{4.8}$$

By Theorems 1.1 and 2.3, the locus of centers of ellipses inscribed in Đ is precisely (h,k) with $k = L(h), h \in I$. We now view the foci, Z_1 and Z_2, as functions of $h \in I$, and we will minimize the eccentricity, $\tau = \tau(h)$, as a function of h. Let $b = b(h)$ and $a = a(h)$ denote the lengths of the semi–minor and semi–major axes of any ellipse, E, inscribed in Đ. Let

$$R = a^2 - b^2 = \frac{1}{4}\left|Z_2 - Z_1\right|^2$$

and let

$$W = a^2 b^2.$$

Solving $a^2 - b^2 = R, a^2b^2 = W$ for a^2 and b^2 in terms of R and W yields $a^2 = \rho_1 + R, b^2 = \rho_1$, where ρ_1 is a root of $\hat{Z}^2 + \hat{Z}R - W$. Thus $\rho_1 = \frac{1}{2}\left(-R + \sqrt{R^2 + 4W}\right)$ since $a^2 > 0$, which implies that $a^2 = \frac{1}{2}\left(R + \sqrt{R^2 + 4W}\right)$, $b^2 = \frac{1}{2}\left(-R + \sqrt{R^2 + 4W}\right) \Rightarrow \tau^2 = 1 - \frac{b^2}{a^2} = \frac{2}{1 + \sqrt{1 + \frac{4W}{R^2}}}$, $R \neq 0$ (If $R = 0$, then Đ is tangential and the ellipse of minimal eccentricity in that case would be a circle). We shall minimize the eccentricity by maximizing $\frac{W}{R^2}$. To derive a formula for R^2, we proceed as follows. First, let $r(h)$ denote the discriminant of $p_h(z)$: Some simplification yields $r(h) = r_1(h) + ir_2(h)$, where

$$\begin{aligned} r_1(h) &= 4\left((s-A)^2 - (t-B-C)^2\right)(h - A/2)^2 + \\ &\quad 4(s-A)\left(A(s-A) + B(B-t) + C(C-t)\right)(h - A/2) + \\ &\quad (s-A)^2\left(A^2 - (C-B)^2\right) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} r_2(h) &= 8\,(t-B-C)\,(s-A)\,(h - A/2)^2 \\ &+ 4\,(s-A)\,(At + sC + Bs - 2AB)\,(h - A/2) \\ &+ 2A\,(s-A)^2\,(B - C)\,. \end{aligned} \tag{4.10}$$

Now $(s-A)(Z_2 - Z_1) = \pm\sqrt{r(h)} \Rightarrow (s-A)^2\left|Z_2 - Z_1\right|^2 = \left|\sqrt{r(h)}\right|^2 = |r(h)| \Rightarrow (s-A)^4\left|Z_2 - Z_1\right|^4 = |r(h)|^2$. $R^2 = \frac{1}{16}\left|Z_2 - Z_1\right|^4 = \frac{1}{16\,(s-A)^4}\,|r(h)|^2$.

Let

$$u(h) = |r(h)|^2 = (r_1(h))^2 + (r_2(h))^2\,,$$

so that u is a polynomial of degree 4 in h. Then

$$R^2 = \frac{1}{16\,(s-A)^4}u(h). \tag{4.11}$$

To obtain W in terms of h, using $k = L(h)$ and (4.7),

$$t_1t_2t_3 = t_1t_2\,(1 - t_1 - t_2) = \left(2\,(Bs - A(t-C))\,h - sAC\right)(2h - A)\,(2h - s)\,\frac{C}{A^2\,(At - Bs)^2}.$$

Thus $t_1t_2t_3$ is a constant multiple of

$$S(h) = (2(Bs - A(t-C))h - sAC)(2h - A)(2h - s) \tag{4.12}$$

S vanishes at $h_1 = \frac{1}{2}A, h_2 = \frac{1}{2}s$, and

$$h_3 = \frac{1}{2}\frac{ACs}{Bs - A(t-C)}.$$

Using (4.1), (4.2), and (4.3), we show now that $h_3 \notin I$. First, if $Bs - A(t-C) < 0$, then $h_3 < 0 \Rightarrow h_3 \notin I$. If $Bs - A(t-C) > 0$ and $s > A$, then $h_3 - \frac{1}{2}s = \frac{1}{2}s\frac{At - Bs}{Bs - A(t-C)} > 0$ by (4.2) $\Rightarrow h_3 \notin I$. Finally, if $Bs - A(t-C) > 0$ and $s < A$, then $h_3 - \frac{1}{2}A = \frac{1}{2}A\frac{A(t-C) + (C-B)s}{Bs - A(t-C)} > 0$ by (4.2) $\Rightarrow h_3 \notin I$. In addition we have shown

$$\begin{aligned} Bs - A(t-C) &< 0 \Rightarrow h_3 < 0 \\ Bs - A(t-C) &> 0 \Rightarrow h_3 > \max(s/2, A/2). \end{aligned} \tag{4.13}$$

Note that $S'(A/2) = 2A(s-A)(A(t-C) + (C-B)s)$ and $S'(s/2) = -2s(s-A)(At - Bs)$. Hence, by (4.1) and (4.2),

$$\begin{cases} S'(A/2) > 0, S'(s/2) < 0 & \text{if } s > A \\ S'(A/2) < 0, S'(s/2) > 0 & \text{if } s < A \end{cases} \tag{4.14}$$

Since $S(h_3) = 0$ and $h_3 \notin I$, (4.14) implies that $S(h) > 0$ on I. Also,

$$S'(h_3) = 2As(At - Bs)\frac{A(t-C) + (C-B)s}{Bs - A(t-C)}.$$

so that, by (4.1) and (4.2),

$$Bs - A(t-C)S'(h_3) > 0. \tag{4.15}$$

Since the area of E equals πab, by Lemma 3.2, $W = a^2b^2$ is also a constant multiple of $S(h)$. Thus, by (4.11), to maximize $\frac{W}{R^2}$ it suffices to maximize

$$E(h) = \frac{S(h)}{u(h)}, h \in I.$$

Write $E'(h) = \frac{N(h)}{u^2(h)}$, where

$$N(h) = u(h)S'(h) - S(h)u'(h)$$

is a polynomial of degree ≤ 6. We shall show that N, and hence E', has precisely one zero in I. Using a computer algebra system (we used Maple within Scientific Workplace 4.1),

$$N(h) = M(h)q(h)$$

where q is the polynomial defined earlier in (4.5) and M is a polynomial of degree ≤ 4. While the expression for M is rather long, we shall use the fact that

$$M(h) = -32(Bs - A(t-C))\left((s-A)^2 + (t-B-C)^2\right)h^4 + \cdots, \tag{4.16}$$

which is again easy to verify using a computer algebra system. Now some algebraic simplification shows that $q(h_3) =$

$\dfrac{\left(A(2Bs-At)(t-C)+B(C-B)s^2\right)^2+A^2s^2C^2\left(s-A\right)^2}{\left(Bs-A(t-C)\right)^2}$, which implies, by (4.1), that

$$q(h_3)>0. \tag{4.17}$$

Also, we showed earlier that

$$q(A/2)>0, q(s/2)>0. \tag{4.18}$$

It follows easily from (4.9), (4.10), and a similar expansion about $h=s/2$ that

$$u(A/2)>0, u(s/2)>0. \tag{4.19}$$

Now $r_1(h_3)=0\Rightarrow A\left(At-2Bs\right)\left(C-t\right)+s^2B(C-B)=\pm ACs\left(s-A\right)$ and $r_2(h_3)=0\Rightarrow A\left(At-2Bs\right)\left(C-t\right)+s^2B\left(C-B\right)=0$. Thus $r_1(h_3)=r_2(h_3)=0\Rightarrow ACs\left(s-A\right)=0$, which has no solution. Thus $u(h_3)=r_1^2(h_3)+r_2^2(h_3)\neq 0$, which implies that

$$u(h_3)>0. \tag{4.20}$$

There are now four cases to consider, depending on the sign of $s-A$ and the sign of $Bs-A(t-C)$. We provide the details for Case 1: $Bs-A(t-C)>0$ and $s>A$. Then $N(A/2)=u(A/2)S'(A/2)>0, N(s/2)=u(s/2)S'(s/2)<0$, and $N(h_3)=u(h_3)S'(h_3)>0$ by (4.14), (4.15), (4.19), and (4.20). Since $M(h)=N(h)/q(h)$, (4.17) and (4.18) imply

$$M(A/2)>0, M(s/2)<0, M(h_3)>0 \tag{4.21}$$

By (4.13), $h_3>s/2$. Consider the four open intervals $I_1=(-\infty,A/2), I_2=I=(A/2,s/2)$, $I_3=(s/2,h_3)$, and $I_4=(h_3,\infty)$. By (4.16), $\lim\limits_{h\to\infty}M(h)=-\infty$. Thus by (4.21) and Rolle's Theorem, M has precisely one zero in each of I_1 thru I_4. The other cases follow in a similar fashion. Since $\deg M=4$, M has precisely one root in I. By Proposition 4.3, $N=Mq$ has precisely one root in I. Assume first that u does not vanish in I. Then $E=S/u$ and $E'=N/u^2$ are continuous on I. Since $E(A/2)=E(s/2)=0$, and E' has precisely one zero in I, E must have a unique global maximum on $\bar{I}$. The existence and uniqueness of the ellipse of minimal eccentricity then follows immediately. Now suppose that $u(h_0)=0$ for some $h_0\in I$. Then $r(h_0)=0$, which implies that $Z_1=Z_2$. $h=h_0$ would yield the ellipse of minimal eccentricity in this case, which would be a circle. In addition, since $u(h)\geq 0$ for all $h, u'(h_0)=0$ as well, which implies that $N(h_0)=0$. Since N cannot have more than one zero in I, u also cannot have more than one zero in I. That proves the uniqueness of an inscribed circle when Ð is a tangential quadrilateral, which is, of course, well known. Again, we have proven the existence and uniqueness of the ellipse of minimal eccentricity. ∎

Remark 4.2. The proof above of Theorem 4.4 yields a precise formula for the eccentricity of an ellipse inscribed in Ð in terms of h: $W=a^2b^2=\dfrac{1}{\pi^2}\left(\text{area}(E)\right)^2=\left(\text{area}(T_1)\right)^2\left(t_1t_2t_3\right)$ by Lemma 3.2. A simple computation yields $\left(\text{area}(T_1)\right)^2=\dfrac{1}{4}A^2\dfrac{(Bs-At)^2}{\left(s-A\right)^2}$, which, by (4.7) gives $W=\dfrac{1}{4}\dfrac{C}{\left(s-A\right)^2}S(h)$. Using $R^2=\dfrac{1}{16\left(s-A\right)^4}u(h)$, $\tau^2=\dfrac{2}{1+\sqrt{1+\frac{4W}{R^2}}}=$ $\dfrac{2}{1+\sqrt{1+16\left(s-A\right)^2CE(h)}}$.

4.1. **Algorithm.** To find the ellipse of minimal eccentricity, E, inscribed in a convex quadrilateral Ð with no parallel sides, one does the following:

- Use an isometry of the plane so that Ð has vertices $(0,0)$, $(0,C)$, (A,B), and (s,t), where $s>0, A>0, C>0$ and $t>B$.
- Use (4.9) and (4.10) to find the quartic polynomial $u(h)=(r_1(h))^2+(r_2(h))^2$
- Use (4.12) to find the sixth degree polynomial $N(h)=u(h)S'(h)-S(h)u'(h)$
- Factor $N(h)=M(h)q(h)$
- The x coordinate of the center of E is the unique root, h_0, in I of the quartic polynomial M. The y coordinate of the center of E is $\frac{1}{2}t+\frac{B+C-t}{A-s}\left(h_0-\frac{1}{2}s\right)$. One could also skip the previous step and take h_0 to be the unique root in I of the sixth degree polynomial N.
- The foci of E are the roots of the polynomial $p_{h_0}(z)$ given in (4.8)
- The length of the major axis of E is $2a$, where $a^2=\frac{1}{2}\left(R+\sqrt{R^2+4W}\right)$,

$R^2=\frac{1}{16(s-A)^4}u(h_0)$, and $W=\frac{1}{4}\frac{C}{(s-A)^2}S(h_0)$.

Example: Suppose that $s=3$, $t=4$, $A=2$, $B=-1$, and $C=3$. Then $M(h)=800h^4+480h^3-12\,000h^2+15\,680h-3840$ and the unique root of M in $I=(1,1.5)$ is $h_0\approx 1.\,232\,8$. The corresponding ellipse, E, of minimal eccentricity has foci $Z_1\approx 1.\,097\,2-0.034\,4i$ and $Z_2\approx 1.\,368\,4+2.\,965\,5i$. The length of the major axis of E is $\approx 3.\,883\,1$ and the equation of E is $60.\,019\,0x^2+24.\,316\,1y^2-6.\,509\,8xy-138.\,440\,2x-63.\,248\,6y+41.\,128\,9=0$. Finally, the minimal eccentricity is $\approx .775\,7$. See Figure 1 below.

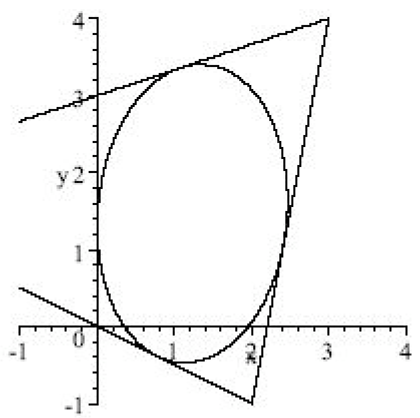

Figure 1: Ellipse of minimal eccentricity inscribed in Ð

4.2. **Trapezoids.** We did not give the details of the proof of Theorem 4.4 when Ð is a trapezoid. We provide here the specifics for finding the x coordinate of the center of the ellipse of minimal eccentricity inscribed in Ð. Assume, without loss of generality, that the lines L_1 and L_3 of Ð are parallel. Then $Bs-A(t-C)=0$, and one can show that

$$\begin{aligned} M(h) &= 16\left(A^2+B^2\right)h^3-12\left(B^2+A^2\right)(A+s)\,h^2+ \\ &\quad 4A\left(2sA^2+ABC-C^2A-CBs+2B^2s\right)h+A^2C^2(A+s)\,. \end{aligned}$$

The x coordinate of the center of the ellipse of minimal eccentricity inscribed in Đ is the unique root of M in I. For example, suppose that $s = 4$, $t = 11$, $A = 1$, $B = 2$, and $C = 3$. Then $M(h) = 80h^3 - 300h^2 + 52h + 45$ and the unique root of M in $I = (.5, 2)$ is $h \approx .531\,0$

5. Future Research and Open Questions

• Theorems 3.3 and 4.4 yield two new points inside a convex quadrilateral, Đ: The centers of the ellipses of maximal area and of minimal eccentricity inscribed in Đ. Is there a nice relationship between these points ?

• In [2], Dorrie characterizes the unique ellipse, E, of minimal eccentricity passing thru the vertices of a convex quadrilateral, Đ. He shows that E is the ellipse whose equal conjugate diameters possess the conjugate directions common to all ellipses passing thru the vertices of Đ. Is there a similar characterization for the unique ellipse of minimal eccentricity *inscribed* in Đ?

Related to this:

• Is there a nice relationship between the ellipse of minimal eccentricity inscribed in Đ and the ellipse of minimal eccentricity passing thru the vertices of Đ ? This would generalize the known relationship between the inscribed and circumscribed circles of bicentric quadrilaterals.

• Show that there is a unique ellipse of maximal *arc length* inscribed in Đ, and provide an algorithm for finding such an ellipse.

References

[1] G. D. CHAKERIAN, *A Distorted View of Geometry*, MAA, Mathematical Plums, Washington, DC, 1979, 130-150.

[2] H. DÖRRIE, *100 Great Problems of Elementary Mathematics*, Dover, New York, 1965.

[3] A. HORWITZ, Finding ellipses and hyperbolas tangent to two, three, or four given lines, *Southwest Journal of Pure and Applied Mathematics*, **1** (2002), 6-32.

[4] M. MARDEN, A note on the zeros of the sections of a partial fraction, *Bulletin of the AMS*, **51** (1945), 935–940.