Complex exponentials: Euler's formula

What happens when we try to solve an equation such as

$$y' - 2y' + 5y = 0. \tag{1}$$

The characteristic equation is $r^2 - 2r + 5 = 0$, whose roots are complex:

$$\frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \tag{2}$$

Since the roots are not real, does this mean there is no solution to the differential equation? No, because the theorem on existence and uniqueness guarantees a solution. Does it mean that our favorite method for solving constant coefficient linear equations is useless here? Not at all. We simply have to understand what expressions like $e^{(1+2i)t}$ mean.

Algebra with complex numbers is essentially the same as with real numbers:

$$(2+3i) + (-5+4i) = (2-5) + (3+4i) = -3+7i$$
$$(2+3i) \times (-5+4i) = -10 + 12i^2 - 15i + 8i = -22 - 7i$$

One important exception is that the relations "greater than" and "less than" makes no sense for complex numbers. Geometrically a complex number $a+bi$ is represented by a point in the plane with coordinates (a, b). Hence the magnitude $|a+bi|$ is simply $\sqrt{a^2+b^2}$, as usual for points in the plane. One important formula is that

$$(a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 + b^2 = |a+bi|^2,$$

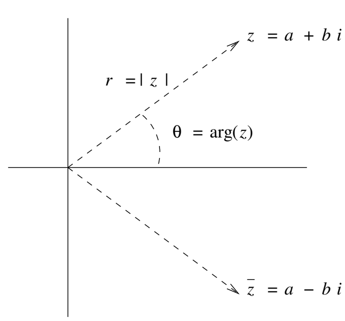
from which we find a simple method to find the reciprocal of a complex number. or to divide one complex number by another:

$$\frac{1}{-5+4i} = \frac{-5-4i}{25+16} = -\frac{5}{41} - \frac{4}{41} i$$
$$\frac{2+3i}{-5+4i} = (2+3i) \times \frac{1}{-5+4i} = (2+3i)(-\frac{5}{41} - \frac{4}{41} i) = \frac{2}{41} - \frac{23}{41} i$$

The *real part* of a complex number $a+bi$ is a and the *imaginary part* is b. The magnitude $|a+bi|$ is also called the *modulus*, for some arcane historical reason. A point in the plane has both cartesian coordinates and polar coordinates. If the polar coordinates of $a+bi$ are r and θ then r is simply the modulus, and θ has the equally arcane name of *argument*, when it refers to a complex number. The modulus and argument behave well with respect to multiplication:

$$|wz| = |w|\,|z|$$
$$\arg(wz) = \arg(w) + \arg(z)$$

The second formula is not quite precise. The argument of a number is not unique, since we can add to it a multiple of 2π and not change the position of the point. Hence we have to interpret the second formula a bit loosely. For example, $\arg(-i) = 3\pi/2$ but $\arg((-i)(-i)) = \arg(-1) = \pi$, altho $3\pi/2 + 3\pi/2 = 3\pi$, not π. However, an argument of either π or 3π determines the same point, and so this is the sense in which we understand the second formula.



The *conjugate* of $a+bi$ is $a-bi$, and is indicated by an overline: $\overline{a+bi}=a-bi$. Usually when a real equation leads to complex roots, even at an intermediate step in the calculations, then the roots come in complex conjugate pairs. So, our equation above led to the roots $1+2i$ and $1-2i$. This often cuts our work almost in half, because we can take one solution and apply complex conjugation to obtain a second solution. The picture illustrates the modulus, argument, and conjugate of a complex number.

There are many useful algebraic properties of complex conjugation.

$$\begin{aligned}
\overline{w+z} &= \overline{w}+\overline{z}\\
\overline{wz} &= \overline{w}\,\overline{z}\\
|z|^2 &= z\,\overline{z}\\
\frac{1}{z} &= \frac{\overline{z}}{|z|^2}\\
\frac{1}{2}(z+\overline{z}) &= \text{ real part of } z\\
\frac{1}{2i}(z-\overline{z}) &= \text{ imaginary part of } z
\end{aligned}$$

In particular, the last two formulas tells us that the real and imaginary parts of z can be expressed as linear combinations of z and $\overline{z}$. This will be important when we look for real-valued solutions to a differential equation.

Since algebra is essentially the same for real or complex numbers, power series are essentially the same, and this is the key to understanding the exponential of a complex number:

$$e^z = 1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5+\frac{1}{720}z^6+\cdots$$

This converges nicely whether we use a real or complex value for z. The resulting function has most of the familiar properties. The most important of these comes from differentiating the series term by term:

$$\frac{d}{dz}e^z = e^z$$

From this formula (and the fact that $e^0=1$) we can derive the algebraic properties:

$$e^{w+z}=e^w e^z \qquad e^{-z}=\frac{1}{e^z}$$

Let's apply these rules to determine what e^{a+bi} is:

$$\begin{aligned}
e^{a+bi} &= e^a e^{bi}\\
&= e^a\left(1+(bi)+\frac{1}{2}(bi)^2+\frac{1}{6}(bi)^3+\frac{1}{24}(bi)^4+\frac{1}{120}(bi)^5+\frac{1}{720}(bi)^6+\cdots\right)\\
&= e^a\left(1+bi-\frac{1}{2}b^2-\frac{1}{6}b^3i+\frac{1}{24}b^4+\frac{1}{120}b^5i-\frac{1}{720}b^6+\cdots\right)\\
&= e^a\left(1-\frac{1}{2}b^2+\frac{1}{24}b^4-\frac{1}{720}b^6+\cdots\right)+e^a\left(bi-\frac{1}{6}b^3i+\frac{1}{120}b^5i-\cdots\right)\\
&= e^a\cos(b)+e^a\sin(b)i.
\end{aligned}$$

What we have derived is called Euler's formula:

$$e^{a+bi}=e^a\cos(b)+e^a\sin(b)i. \tag{3}$$

It gives a geometric interpretation to the complex exponential, since it says that the polar coordinates of e^{a+bi} are e^a and b. In other words,

$$\begin{aligned}
|e^{a+bi}| &= e^a\\
\arg(e^{a+bi}) &= b
\end{aligned}$$

One odd consequence of Euler's formula is that the complex exponential is not one-to-one. In fact, it is periodic: $e^{z+2\pi i} = e^z$. For each complex number z there are infinitely many different values for $\log(z)$. However, they all differ by a multiple of the period $2\pi i$. More precisely $\log(z) = \ln|z| + \arg(z)i$. As we noted above, there are really infinitely different values for $\arg(z)$, altho in this formula each different value produces a different value for $\log(z)$.

Another way of stating Euler's formula is that

$$\text{real part of } e^{a+bi} = e^a \cos(b) \tag{4}$$

$$\text{imaginary part of } e^{a+bi} = e^a \sin(b) \tag{5}$$

Since we have seen that the real and imaginary parts of z are linear combinations of z and $\overline{z}$, Euler's formula (4) and (5) tells us how to use the complex eigenvalues (2) to construct real solutions to equation (1).

$$y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) = e^t(c_1 \cos(2t) + c_2 \sin(2t)). \tag{6}$$

Further reading

The use of Euler's Forumula to solve equations with complex eigenvalues is covered in section 3.4. Good practice problems are 1–16, pages 164.

Reading quiz

1. What is Euler's forumula?
2. What is the real part of a complex number?
3. What is the imaginary part of a complex number?
4. What is the modulus of a complex number?
5. What is the argument of a complex number?
6. What is the complex conjugate of a complex number?
7. How can the real part of a complex number be expressed in terms of the number and its complex conjugate?
8. How can the imaginary part of a complex number be expressed in terms of the number and its complex conjugate?

Extra credit 5: due Monday, 24 April

Solve the initial value problem

$$y'' + \alpha^2 y = \sin(\beta t), y(0) = y'(0) = 0,$$

where α and β are constants. Your solution should have the form

$$y = A(\cos(\alpha t) - \cos(\beta t)),$$

for an appropriate constant A, which will depend on α and β. Use trig identities to rewrite your solution in the form

$$y = B \sin(\tfrac{1}{2}(\alpha - \beta)t) \sin(\tfrac{1}{2}(\alpha + \beta)t),$$

where B is an appropriate constant A, which will depend on α and β. Choose a value of α and a sequence of (say) a half-dozen values of β which are increasingly close to α. Use your second form to graph these solutions. Describe what happens. Speculate on what happens in the limit as $\beta \to \alpha$.