Using fences to determine long-term behavior from initial values

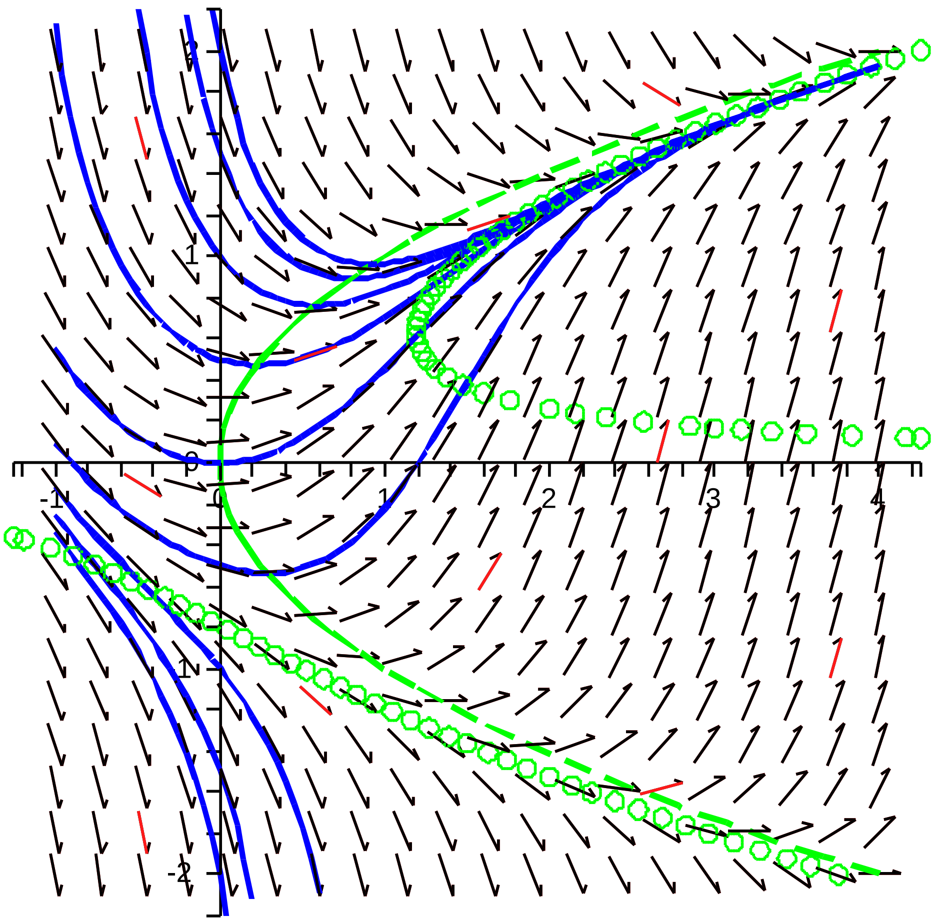
Often the long-term (asymptotic) behavior of an integral curve depends on the initial value. For example, the integral curves of the equation

$$y' = t - y^2 \tag{1}$$

fall into one of two types:

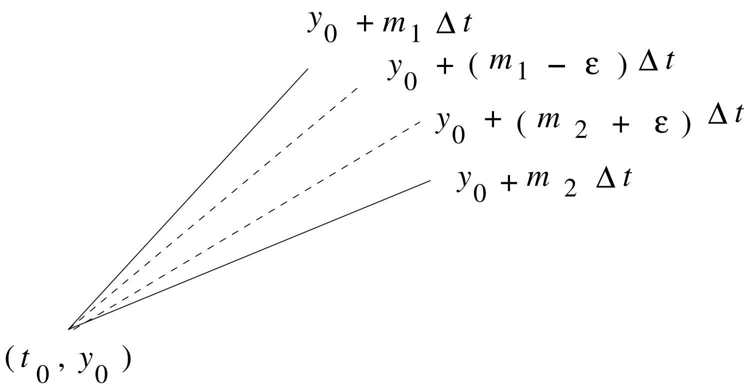
- Those which decrease for a time, then hit the critical curve $t = y^2$, turn up and follow the critical curve asymptotically.
- Those which forever decrease, tending to $-\infty$.

Let's recall the picture:



In this picture, the dashed line is the critical curve and the dotted line is the inflection curve. It appears that the critical curve and the lower portion of the inflection curve are *fences* — boundary lines which contain any critical curves that cross into the territory the mark out. The aim of these notes is to verify these claims.

The main idea is that if two smooth curves pass thru the same point (t_0, y_0), and if the slope of the first curve exceeds the slope of the second at this point, then for (at least) a small time interval the first curve stays above the second curve.



The picture above shows two curves $y_1(t)$ and $y_2(t)$, both emanating from a point (t_0, y_0), with respective slopes m_1 and m_2 at the point (t_0, y_0). We assume that $m_1 > m_2$. We have drawn these curves as if they are straight lines, because when we focus our attention very close to this point, smooth graphs are approximately linear. More precisely, for each positive ϵ then we can find a short t-interval $(t_0, t_0 + \delta)$ such that

$$y_0 + (m_i - \epsilon)\,\Delta t < y_i(t) < y_0 + (m_i + \epsilon)\,\Delta t, \tag{2}$$

for $i = 1, 2$ and $\Delta t = t - t - t_0 < \delta$. If we choose ϵ to "split the difference" between the m_i — that is, $m_1 + \epsilon < m_2 - \epsilon$ — then we have

$$y_2(t) < y_0 + (m_1 + \epsilon)\,\Delta t < y_0 + (m_2 - \epsilon)\,\Delta t < y_1(t). \tag{3}$$

This was exactly the claim we made above.

To apply this to our example, note that if an integral curve $y_2(t)$ crosses the upper half of the critical curve $y_1(t)$ then at the point of crossing (t_0, y_0) we have $m_2 = 0$ and $m_1 = \frac{1}{2}y_0^{-1} > 0$. Hence it must be crossing into the region where $y' > 0$, it can never cross out of this region. A similar argument applies to the bottom half of the critical curve. Thus, the critical curve is indeed a "fence" which corrals all integral curves which pass into its territory.

So too is the lower half of the inflection curve. This curve has equation $t = y^2 + \frac{1}{2}y^{-1}$, where $y < 0$. The slopes of the integral curves $y_2(t)$ as they touch the inflection curve at (t_0, y_0) are therefore

$$m_2 = t_0 - y_0^2 = y_0^2 + \frac{1}{2}y_0^{-1} - y_0^2 = \frac{1}{2}y_0^{-1}. \tag{4}$$

On the other hand using implicit differentiation we find that the slope of the inflection curve itself at the same point (t_0, y_0) is

$$m_1 = (2y_0 - \frac{1}{2}y_0^{-2})^{-1} > (2y_0)^{-1} = m_2. \tag{5}$$

Again we can apply our "fence" argument above to conclude that integral curves can cross into the territory marked out by the curve, but not back out again.

Extra credit 3: due Friday, 3 March

These questions refer to the same example above: $y' = t - y^2$.

1. Is the upper part of the inflection curve also a fence?
2. It looks as if the integral curves which decrease to $-\infty$ might in fact have vertical asymptotes. One way to prove this is to find fences that have vertical asymptotes. Here are two candidate fences.
   - Are there any fences of the form $y = C/(t - a)$, where a and C are constants?
   - Are there any fences of the form $y = C\,\tan(t - a)$, where a and C are constants?