Power series solutions of differential equations

There are some differential equations which can be solved using power series. This is sometimes a good method of solution, especially when the problem concerns the behavior of a solution near a particular point. In some cases the power series can even be based at a singular point, that is a point which is at the boundary of the domain of validity for the solution.

Let's look at some examples. First, Airy's equation:

$$y'' = ty. \tag{1}$$

We look for a power series solution:

$$y = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{k=0}^{\infty} a_k t^k.$$

This means we need to determine values for the a_k. In fact such a task is not possible unambiguously, since there are infinitely many solutions to equation (1). Since equation (1) is a second-order homogeneous linear equation we instead look for two linearly independent solutions. As always, this will mean specifying initial values for y, at least implicitly. Since $y(0) = a_0$ and $y'(0) = a_1$ this is the same as choosing values for a_0 and a_1.

Our strategy is a familiar one: substitute the expression for y into the differential equation and deduce information on the parameters.

$$\begin{aligned}
y' &= a_1 + 2\cdot a_2 t + 3\cdot a_3 t^2 + \cdots = \sum_{k=0}^{\infty}(k+1)a_{k+1}t^k \\
y'' &= 2\cdot 1\cdot a_2 + 3\cdot 2\cdot a_3 t + 4\cdot 3\cdot a_4 t^2 + \cdots = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k \\
ty &= a_0 t + a_1 t^2 + a_2 t^3 + \cdots = \sum_{k=1}^{\infty} a_{k-1}t^k
\end{aligned}$$

When we compare the expressions for y'' and for ty term by term, we deduce that

$$\begin{aligned}
&2\cdot 1\cdot a_2 = 0 \implies a_2 = 0 \\
&3\cdot 2\cdot a_3 = a_0 \implies a_3 = \frac{1}{3\cdot 2}a_0 \\
&4\cdot 3\cdot a_4 = a_1 \implies a_4 = \frac{1}{4\cdot 3}a_1 \\
&5\cdot 4\cdot a_5 = a_2 \implies a_5 = \frac{1}{5\cdot 4}a_2 = 0 \\
&6\cdot 5\cdot a_6 = a_3 \implies a_6 = \frac{1}{6\cdot 5}a_3 = \frac{1}{6\cdot 5\cdot 3\cdot 2}a_1 \\
&7\cdot 6\cdot a_7 = a_4 \implies a_7 = \frac{1}{7\cdot 6}a_4 = \frac{1}{7\cdot 6\cdot 4\cdot 3}a_0 \\
&8\cdot 7\cdot a_8 = a_5 \implies a_8 = \frac{1}{8\cdot 7}a_5 = 0
\end{aligned}$$

Note the "recursive" nature of the equations: we use a_0, a_1, and a_2 to in turn evaluate a_3, then a_4, then a_5, and so forth. For instance we have already ascertained that $a_2 = a_5 = a_8 = \cdots = 0$. To finish we need to choose values for a_0 and a_1 and use these to determine all the other coefficients.

Let y_1 be the solution with initial values $y_1(0) = 1$ and $y_1'(0) = 0$. We obtain that

$$\begin{aligned}
y_1 &= 1 + \frac{t^3}{3\cdot 2} + \frac{t^6}{6\cdot 5\cdot 3\cdot 2} + \frac{t^9}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} + \cdots \\
&= 1 + \sum_{k=1}^{\infty}\frac{t^{3k}}{3k(3k-1)(3k-3)(3k-4)\cdots 3\cdot 2}.
\end{aligned}$$

On the other hand if y_2 is the solution with initial values $y_1(0) = 0$ and $y_1'(0) = 1$ then we obtain that

$$\begin{aligned} y_2 &= t + \frac{t^4}{4 \cdot 3} + \frac{t^7}{7 \cdot 6 \cdot 4 \cdot 3} + \frac{t^{10}}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} + \cdots \\ &= t + \sum_{k=1}^{\infty} \frac{t^{3k+1}}{(3k+1)3k(3k-2)(3k-3)\cdots 4 \cdot 3}. \end{aligned}$$

For the second example we look at Hermite's equation:

$$y'' - 2ty' + \lambda y = 0 \tag{2}$$

where λ is a constant. We proceed as above:

$$\begin{aligned} y' &= a_1 + 2 \cdot a_2 t + 3 \cdot a_3 t^2 + \cdots = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k \\ ty' &= a_1 t + 2 \cdot a_2 t^2 + 3 \cdot a_3 t^3 + \cdots = \sum_{k=1}^{\infty} k a_k t^k \\ y'' &= 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 t + 4 \cdot 3 \cdot a_4 t^2 + \cdots = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k \end{aligned}$$

If we substitute these expressions into equation (2) and look at the result term by term we find that

$$\begin{aligned} 2 \cdot 1 \cdot a_2 + \lambda a_0 = 0 &\implies a_2 = \frac{\lambda}{2 \cdot 1} a_0 \\ 3 \cdot 2 \cdot a_3 - 2a_1 + \lambda a_1 = 0 &\implies a_3 = \frac{(\lambda - 2)}{3 \cdot 2} a_1 \\ 4 \cdot 3 \cdot a_4 - 4a_2 + \lambda a_2 = 0 &\implies a_4 = \frac{(\lambda - 4)}{4 \cdot 3} a_2 = \frac{\lambda(\lambda - 4)}{4 \cdot 3 \cdot 2 \cdot 1} a_0 \\ 5 \cdot 4 \cdot a_5 - 6a_3 + \lambda a_3 = 0 &\implies a_5 = \frac{(\lambda - 6)}{5 \cdot 4} a_3 = \frac{(\lambda - 2)(\lambda - 6)}{5 \cdot 4 \cdot 3 \cdot 2} a_1 \end{aligned}$$

As above, if we take $a_0 = 1$ and $a_1 = 0$ we obtain the solution

$$\begin{aligned} y_1 &= 1 + \lambda \frac{t^2}{2!} + \lambda(\lambda - 4) \frac{t^4}{4!} + \lambda(\lambda - 4)(\lambda - 8) \frac{t^6}{6!} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \lambda(\lambda - 4) \cdots (\lambda - 4k + 4) \frac{t^{2k}}{(2k)!}. \end{aligned}$$

If instead we take $a_0 = 0$ and $a_1 = 1$ we obtain the linearly independent solution

$$\begin{aligned} y_2 &= t + (\lambda - 2) \frac{t^3}{3!} + (\lambda - 2)(\lambda - 6) \frac{t^5}{5!} + (\lambda - 2)(\lambda - 6)(\lambda - 10) \frac{t^7}{7!} + \cdots \\ &= t + \sum_{k=1}^{\infty} (\lambda - 2)(\lambda - 6) \cdots (\lambda - 4k + 2) \frac{t^{2k+1}}{(2k+1)!}. \end{aligned}$$

For the last example we consider

$$y'' + (1 + t)y' + ty = 0, \tag{3}$$

In this case we have that

$$ty = a_0 t + a_1 t^2 + a_2 t^3 + \cdots = \sum_{k=1}^{\infty} a_{k-1} t^k$$

$$y' = a_1 + 2 \cdot a_2 t + 3 \cdot a_3 t^2 + \cdots = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k$$

$$ty' = a_1 t + 2 \cdot a_2 t^2 + 3 \cdot a_3 t^3 + \cdots = \sum_{k=1}^{\infty} k a_k t^k$$

$$y'' = 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 t + 4 \cdot 3 \cdot a_4 t^2 + \cdots = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k$$

Again we substitute these expressions into our differential equation and examine the result term by term:

$$\begin{aligned}
2 \cdot 1 \cdot a_2 + a_1 = 0 &\implies a_2 = -\frac{a_1}{2!} \\
3 \cdot 2 \cdot a_3 + 2 \cdot a_2 + a_1 + a_0 = 0 &\implies a_3 = -\frac{2 \cdot a_2 + a_1 + a_0}{3 \cdot 2} = -\frac{a_0}{3!} \\
4 \cdot 3 \cdot a_4 + 3 \cdot a_3 + 2 \cdot a_2 + a_1 = 0 &\implies a_4 = -\frac{3 \cdot a_3 + 2 \cdot a_2 + a_1}{4 \cdot 3} = \frac{a_0}{4!} \\
5 \cdot 4 \cdot a_5 + 4 \cdot a_4 + 3 \cdot a_3 + a_2 = 0 &\implies a_5 = -\frac{4 \cdot a_4 + 3 \cdot a_3 + a_2}{5 \cdot 4} \\
& = \frac{2a_0}{5!} + \frac{3a_1}{5!} \\
6 \cdot 5 \cdot a_6 + 5 \cdot a_5 + 4 \cdot a_4 + a_3 = 0 &\implies a_6 = -\frac{5 \cdot a_5 + 4 \cdot a_4 + a_3}{6 \cdot 5} \\
& = -\frac{2a_0}{6!} - \frac{3a_1}{6!} \\
7 \cdot 6 \cdot a_7 + 6 \cdot a_6 + 5 \cdot a_5 + a_4 = 0 &\implies a_7 = -\frac{6 \cdot a_6 + 5 \cdot a_5 + a_4}{7 \cdot 6} \\
& = -\frac{13a_0}{7!} - \frac{12a_1}{7!} \\
8 \cdot 7 \cdot a_8 + 7 \cdot a_7 + 6 \cdot a_6 + a_5 = 0 &\implies a_8 = -\frac{7 \cdot a_7 + 6 \cdot a_6 + a_5}{8 \cdot 7} \\
& = \frac{13a_0}{8!} + \frac{12a_1}{8!}
\end{aligned}$$

Summary and further reading

The examples above start with a second-order homogeneous linear differential equation and determine two linearly independent solution in the form of power series. The series are based at an *ordinary point* — that is, a point where the coefficients p and q of the standard form of the equation $y'' + py' + qy = 0$ are continuous. This material is covered in more detail in sections 5.2–5.3. In particular theorem 5.3.1 in section 5.3 gives a precise statement about the existence of series solutions in this situation, including a lower bound for the radius of convergence. Good practice problems are 1, 2, 4–7, 9–13, page 259; and 5–10, page 265.

Assignment 14: due Friday, 24 March

For each of the following equations, find two linearly independent power series solutions. Determine the radius of convergence (or at least a lower bound for it).

1. $y'' + k^2 y = 0$, where k is a constant.
2. $(3 - t^2)y'' + ty' - 2y = 0$.
3. $(1 - t^2)y'' - ty' + \alpha^2 y = 0$, where α is a constant. (Chebyshev's equation)