Introduction to Differential Equations:
Terminology and Basic Concepts

A differential equation is simply an equation involving an unknown function and its derivatives. A famous example described the motion of a pendulum:

$$\theta'' = -\frac{g}{L}\sin(\theta). \tag{1}$$

Here θ is an unknown function of the independent variable t, and g and L are constant parameters. (In this equation, g represents the acceleration due to gravity and L represents the length of the pendulum). The goal is to find an explicit solution for the equation, that is an explicit formula for $\theta(t)$ which makes the equation a true statement. Finding the solution is sometimes referred to as *integrating* the equation. In fact, most of the methods for explicit solution require the evaluation of several integrals, in the sense of calculus.

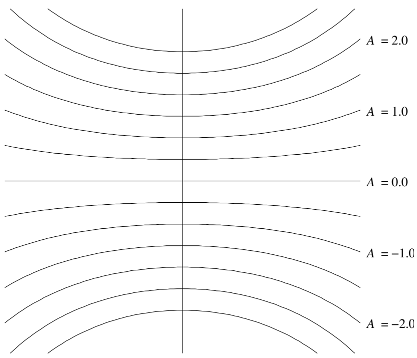
The graph of a solution is called an *integral curve*. Because the solution to a differential equation involves integration, the solutions come in families, parametrized by the constants of integration. Let's look at a simple example:

$$y' = ty. \tag{2}$$

We can check that $\phi(t) = A\exp(\frac{1}{2}t^2)$ is a solution, no matter what the constant A is. To do this we substitute the formula for $\phi(t)$ everywhere y appears in the equation:

$$\phi'(t) = (A\exp(\tfrac{1}{2}t^2))' = A \cdot \tfrac{1}{2} \cdot 2t\exp(\tfrac{1}{2}t^2) = t(A\exp(\tfrac{1}{2}t^2)) = t\phi(t).$$

Here is the graph, for a variety of values of A:



For most differential equations there is no explicit formula for the solution (even theoretically). Nevertheless there are two important tools which are always available: slope fields and numerical approximation. In fact, even when we can find an explicit solution the slope field often gives us better insight into the system we are studying. We look at slope fields in the next lecture.

There are two basic types of differential equations: *ordinary* and *partial*. Partial differential equations involve partial derivatives, and their solutions are functions of 2 or more variables. Ordinary differential equations involve only ordinary derivatives, and their solutions are functions of a single variable. In this class we only look at ordinary differential equations.

Often (but not always!) the independent variable of a differential equation represents time, and is denoted t. This is because in many applications of differential equations, we are examining a time-dependent system. Usually in such applications we are interested the *asymptotic behavior* of the solutions — that is, the behavior of the solution as $t \to +\infty$. This is a broader concept than that of asymptotes for a solution. Horizontal asymptotes are one possible example of asymptotic behavior, but their are many other possible behaviors: the solution might be monotonically increasing, or it might be oscillatory; it might be bounded or it might be unbounded; it might decay to 0 or it might blow up to ∞.

An ordinary differential equation can be put in the form

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}) \tag{3}$$

where $y^{(n)}$ denotes the n-th derivative of y and f is some function of the quantities t, y, y', ... For example, in equation (1) we have $f(t, \theta, \theta') = -\frac{g}{L}\sin(\theta)$ and in equation (2) we have $f(t, y) = ty$.

An important class of differential equations are the *autonomous* equations. These are equations where the right-hand side f is a function of y and its derivatives, but not of t. For example, the pendulum equation (1) is autonomous, but equation (2) is not.

In equation (3) the order of differential equation is n. For example a differential equation of order 3 has derivatives of order 3 in it, but no higher-order derivatives. Equation (1) has order 2 while equation (2) is of first order.

As we mentioned above, most differential equations cannot be solved explicitly, just as most functions cannot be integrated explicitly. Fortunately, many applications lead to *linear* differential equations, and there are methods for solving linear equations. What is a linear differential equation? A linear differential equation of order n can be put in the form

$$\boxed{}\, y^{(n)} + \boxed{}\, y^{(n-1)} + \cdots + \boxed{}\, y' + \boxed{}\, y = \boxed{}, \tag{4}$$

where we can put in the empty boxes any function of t alone. So equation (2) above is linear but equation (1) is nonlinear — the nonlinear term is the term $-\frac{g}{L}\sin(\theta)$. Do you see why? Some more examples: the equation

$$t^3 y'' + \sin(t) y' + \ln(t) y = t^3 + t^2 + 5$$

is a linear equation of order 2 (what are the coefficients?) but the equation

$$y' y'' + y = 0$$

is nonlinear of order 2 (what is the nonlinear term?).

The boxes on the left side of equation (4) hold the coefficients, and the box on the right-hand side holds what is called the *driving function*. Again, this terminology comes from applications: typically the left-hand side of a linear equation describes the internal laws of a system (perhaps an electrical circuit) while the right-hand side is an external force (perhaps the impressed voltage). For example, the equation

$$3y'''' - 5y''' + 3y'' - 2y' + 4y = t - \sin(t)$$

is a fourth order linear equation with constant coefficients, where the driving function equals $t - \sin(t)$. Note that a constant-coefficient linear equation might have a nonconstant driving function.

In the special case of a linear equation where the driving function is 0 we say that the equation is *homogeneous*. Our strategy to analyze a linear differential equation is to examine the homogeneous equation obtained by removing the external driver, and then, only after we understand the internal dynamics, examine the behavior under the influence of the driving function.

Further reading

The basic terminology of differential equations is introduced in section 1.3 of the text. Good practice problems are 1–20, pages 24–25.

Reading quiz

1. What is a differential equation?
2. What is a solution to a differential equation?
3. What is an integral curve?
4. What is a linear differential equation?
5. What is the order of a differential equation?
6. What is an autonomous differential equation?

Assignment 2: due Monday, 30 January

1. Give examples of differential equations satisfying the following conditions. Explain why your answers meet the conditions.

    (a) A second-order linear differential equation with constant coefficients and a nonconstant driving function.

    (b) A homogeneous third-order linear differential equation with nonconstant coefficients.

    (c) A fourth-order nonlinear differential equation.

    (d) A nonlinear, first-order, autonomous equation.

2. For each of the following equations, determine whether or not it is linear, and give its order. Underline the term of highest order. If the equation is nonlinear then put a box around all nonlinear terms. If the equation is linear then put it in standard form (as above) and draw a box around each of the coefficients and draw a circle around the driving function.

    (a) $ty'' = \sin(t)y$.

    (b) $\sin(y)y' = 2t + 5$.

    (c) $(y'')^2 + y''' = y^4$.

    (d) $y' + (t^2 - t)y'' + \dfrac{\sqrt{t}}{\ln|t|}y + t^3 \exp(t) = 0$.

3. For each of the following determine whether the function is a solution of the differential equation. Show all your work.

    (a) $y = \sin(2t)$, $y'' + 4y = 0$.

    (b) $y = 2t + 1$, $yy' = y''$.

    (c) $y = e^{-2t}$, $y'' - 3y' + 2y = 8e^{2t}$.

    (d) $y = \ln(t)$, $t^2y'' + ty' = 0$.

    (e) $y = e^{-t^2} \int t^3\, e^{t^2}\, dt$, $y' + 2ty = t^3$.

    (f) $y = Ae^t$, $y''' - 2y'' + y' - 2y = 2e^t$, for some constant A.