communications

Graphical Construction of Rays in an Ideal Luneburg Lens*

Introduction

It has been shown by Eaton[1] and others that the paths followed by the rays in an ideal Luneburg lens lie in planes and have elliptical shapes. For the standard Luneburg lens having one focus on its surface and the other at infinity, the equation of a ray is given by

$$\rho^2 = \frac{\sin^2 \delta}{1 - \cos \delta \cos (2\theta - \delta)}, \tag{1}$$

where ρ and θ are the polar coordinates in the plane containing the ray as defined in Fig. 1. The ray makes the initial angle δ with respect to the axis of the lens, and leaves the lens at E in a direction parallel to the lens axis.

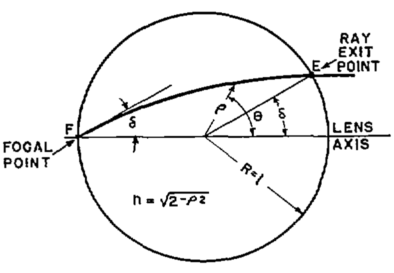

Fig. 1—Typical ray in an ideal Luneberg lens.

A simple graphical construction of these rays is presented, followed by the mathematical derivation.

Construction Method

As derived in the following section, the curved ray from the focus F to the exit point E is a portion of an ellipse having its major axis along a line through the lens center O parallel to the straight line connecting E with F. This line makes an angle of $\delta/2$ with the lens axis FO, as shown in Fig. 2. The length of the semi-major axis of the ellipse is equal to $\sqrt{2} \cos (\delta/2)$, and the semi-minor axis has a length $\sqrt{2} \sin (\delta/2)$. These principal dimensions of the ray ellipse can be laid out by the following graphical construction, after which the ellipse may be constructed by any of the well-known methods, such as the trammel method.[2]

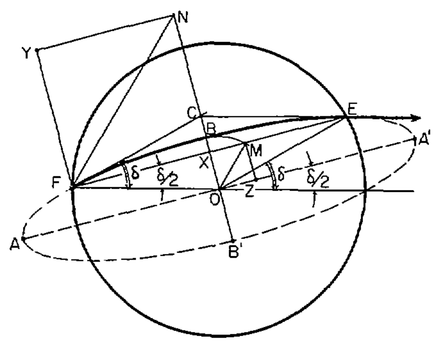

Fig. 2—Construction of ray ellipse in an ideal Luneberg lens.

1) Let the lens radius be normalized, *i.e.*, $FO = 1.00$.
2) Starting at the focal point F, draw the line $\overline{FC}$ in the initial ray direction given by angle δ.
3) Construct $\overline{OE} \parallel \overline{FC}$ through the lens center; the point E where this line intersects the rim of the lens is the exit point of the ray.
4) Complete the rhombus $FCEO$ by constructing $\overline{EC} \parallel \overline{OF}$, the lens axis.
5) Draw the perpendicular bisectors of the rhombus, $\overline{FE}$ and $\overline{CO}$, which intersect at point X.
6) Note that $\overline{OX} = \sin (\delta/2)$, and lay off $\overline{XM} = \overline{OX}$ along $\overline{XE}$. The diagonal $\overline{OM}$ of the small square $OXMZ$ has a length $\overline{OM} = \sqrt{2} \sin (\delta/2)$, which equals the semiminor axis of the ellipse.
7) Using a radius $\overline{OM}$ and the center O, swing an arc to intersect $\overline{OC}$ at points B and B'. The line $\overline{BB'}$ is the minor axis of the desired ellipse.
8) Note that $\overline{FX} = \cos (\delta/2)$, and lay off an equal distance $\overline{XN} = \overline{FX}$ along the direction $XC \perp FX$. The diagonal $\overline{FN}$ of the larger square $FXNY$ has the length of the semimajor axis, *i.e.*, $\overline{FN} = \sqrt{2} \cos (\delta/2)$.
9) Construct a line through O parallel to $\overline{FE}$; lay off the distance $\overline{FN}$ along this line on both sides of point O to produce points A and A', which determine the major axis of the ellipse.
10) With the major and minor axes of the desired ellipse given by $\overline{AA'}$ and $\overline{BB'}$, respectively, the ellipse can now be constructed by means of standard drafting techniques, as shown dotted in Fig. 2; the segment FBE of this ellipse represents the actual ray path through the Luneburg lens.

Mathematical Derivation

The polar expression for the ray, given by (1), can be readily identified as the equation of a tilted ellipse by starting with the more familiar equation of an ellipse in Cartesian coordinates that is

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \tag{2}$$

where a and b are the lengths of the semimajor and semiminor axes, respectively.

* Received by the PGAP, February 9, 1961.

[1] J. E. Eaton, "An Extension of the Luneburg-type Lenses," Naval Res. Lab., Washington, D. C., Rept. No. 4110, p. 13; February 16, 1953.

[2] T. E. French, "Engineering Drawing," McGraw-Hill Book Co., Inc., pp. 63–66; 1935.

Using the well-known relations between polar and rectangular coordinates (Fig. 3), namely

$$X = \rho \cos \theta$$
$$Y = \rho \sin \theta, \tag{3}$$

(2) becomes

$$\frac{\rho^2 \cos^2 \theta}{a^2} + \frac{\rho^2 \sin^2 \theta}{b^2} = 1,$$

from which

$$\rho^2 = \frac{a^2 b^2}{b^2 \cos 4\theta + a^2 \sin^2 \theta}. \tag{4}$$

From trigonometry

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \text{ and } \sin {}^2\theta = \frac{1 - \cos 2\theta}{2},$$

$$\therefore \rho^2 = \frac{a^2 b^2}{\frac{b^2(1 + \cos 2\theta)}{2} + \frac{a^2(1 - \cos 2\theta)}{2}}$$

or

$$\rho^2 = \frac{a^2 b^2}{\frac{(a^2 + b^2)}{2} - \frac{(a^2 - b^2)}{2} \cos 2\theta}. \tag{5}$$

The polar equation of an ellipse whose major axis is inclined at an angle α from the horizontal lens axis can be derived from (5) by using the following relation between angles (see Fig. 4):

$$\theta' = \theta + \alpha$$
$$\therefore \theta = \theta' - \alpha. \tag{6}$$

Substituting (6) into (5) results in the polar equation of the inclined ellipse:

$$\rho^2 = \frac{a^2 b^2}{\left(\frac{a^2+b^2}{2}\right) - \left(\frac{a^2-b^2}{2}\right) \cos (2\theta' - 2\alpha)}. \tag{7}$$

Comparing (7) with Eaton's (1) for the ray path, it can be noted that these equations are of the same form and that the following relations between terms are obtained:

$$a^2 b^2 = \sin^2 \delta \tag{8}$$

$$\left(\frac{a^2 + b^2}{2}\right) = 1 \tag{9}$$

$$\left(\frac{a^2 - b^2}{2}\right) = \cos \delta \tag{10}$$

and

$$2\alpha = \delta. \tag{11}$$

From (11) it can be noted that the ray ellipse is tilted at an angle

$$\alpha = \delta/2. \tag{11a}$$

Adding (9) and (10) results in

$$a^2 = 1 + \cos \delta = 2 \cos^2 (\delta/2) \tag{12}$$

or

$$a = \sqrt{2} \cos (\delta/2). \tag{12a}$$

Subtracting (10) from (9) results in

$$b^2 = 1 - \cos \delta = 2 \sin^2 (\delta/2) \tag{13}$$

or

$$b = \sqrt{2} \sin (\delta/2). \tag{13a}$$

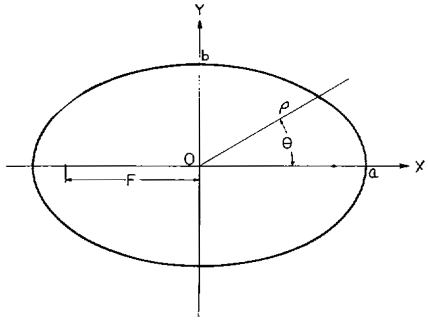

Fig. 3—Ellipse in rectangular coordinates.

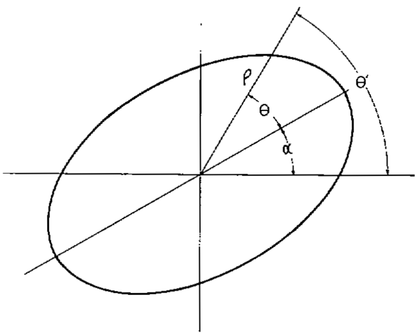

Fig. 4—Inclined ellipse.

Eqs. (12a) and (13a) give the semimajor and semiminor axes of the ray ellipse. Although (8) was not needed in the derivation, it is satisfied by (12) and (13), *i.e.*,

$$a^2 b^2 = (1 + \cos \delta)(1 - \cos \delta)$$
$$= 1 - \cos^2 \delta = \sin^2 \delta.$$

Conclusion

It has been shown that Eaton's equation for a ray in an ideal Luneburg lens represents an ellipse with its major axis inclined at an angle equal to half the initial ray angle δ. The semimajor axis of this ellipse has a length of $a = \sqrt{2} \cos (\delta/2)$, and the semiminor axis is given by $b = \sqrt{2} \sin (\delta/2)$. These expressions lead to a simple graphical method for constructing the principal dimensions of the ray ellipse.

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Backscattering From a Finite Cone—Comparison of Theory and Experiment*

The electromagnetic backscattering cross section σ of a perfectly conducting finite cone was determined by the author[1] by means of the geometrical theory of diffraction. Experimental measurements of σ have been made by Keys and Primich.[2] Fig. 1(a)–(f) shows both the experimental and theoretical results for cones of various half-angles γ for the case of axial incidence. In the figure the ordinate is $\sigma/\pi a^2$ and the abscissa is ka, where a is the radius of the base of the cone and $k = 2\pi/\lambda$ with λ the incident wavelenth. The theoretical curves are based upon (30)[1] which takes account of single and double diffraction by the rear edge of the cone.

Moffatt pointed out the agreement between the curve in the author's[1] article for a cone of half-angle $\gamma = 11.5°$ and the data in Keys and Primich[2] for a cone with $\gamma = 12°$.

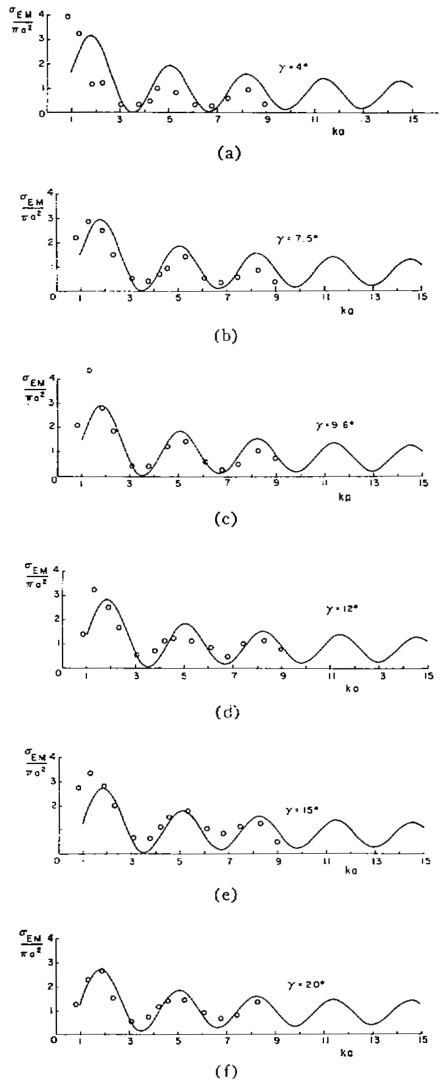

Fig. 1—Experimental and theoretical values of the backscattering cross section σ of a finite cone for axial incidence. The ordinate is $\sigma/\pi a^2$ and the abcissa is ka where a is the radius of the base of the cone and $k = 2\pi/\lambda$. The circles are experimental values and the curves are computed. The half-angle of the cone γ is indicated on each curve.

* Received by the PGAP, February 17, 1961. The research in this paper has been sponsored by the AF Cambridge Res. Labs. under Contract No. AF19(604) 5238.

[1] J. B. Keller, "Backscattering from a finite cone," IRE Trans. on Antennas and Propagation, vol. AP-8, pp. 175–182; March, 1960.

[2] J. E. Keys and R. I. Primich, "The Radar Cross-Section of Right Circular Metal Cones," Defense Res. Telecommun. Estab., Ottawa, Can., Rept. No. 1010, ASTIA Doc. No. AD 217–921; May, 1959.