The Existence and Uniqueness Theorem

Theorem on Existence and Uniqueness for First-order Equations: If f and $\partial f/\partial y$ are continuous on a rectangle $[a,b] \times [c,d]$ containing the point (t_0, y_0) then the initial value problem (1) has a unique solution $y = \phi(t)$ defined in some interval $(t_0 - \epsilon, t_0 + \epsilon)$ around t_0.

In the text this theorem is proved by constructing a sequence of approximations and then showing that these approximations converge. Today we first review the theorem itself. Next lecture we will look at the approximation methods that can be used to prove the theorem.

The largest interval on which the solution is defined is called the *interval of validity*. The interval of validity may be very small, and may depend on the initial values. Let's return to a familiar example: $y' = y^2 + 1$. (See the notes on separable equations.) In this case $f(t,y) = y^2 + 1$, and $\partial f/\partial y = 2y$, which are both continuous everywhere. However the solution with initial values $y(0) = 0$ is $\phi(t) = \tan(t)$, which is valid only in the interval $(-\pi/2, \pi/2)$. If we were to change the initial values to $y(0) = 1$ then the solution is now $\phi(t) = \tan(t + \pi/4)$, which is valid on the interval $(-3\pi/4, \pi/4)$.

In contrast let's consider the class of linear differential equations. In this case $f(t,y) = g(t) - p(t)y$, and $\partial f/\partial y = -p(t)$, hence the hypotheses are satisfied whenever both $p(t)$ and $g(t)$ are continuous at t_0. However, the conclusion of the theorem is fairly weak in this case, since all it guarantees is a unique solution valid on *some* interval around t_0. On the other hand we have an explicit formula for the solution of these equations, namely

$$\phi(t) = \frac{1}{\mu}(C + \int \mu g), \text{ where } \mu = \exp \int p.$$

This formula shows us that the solution is valid on any interval around t_0 over which both p and g are continuous. In other words the interval of validity for the solution of a linear differential equation can be determined easily by looking at the coefficients and the driving function. More precisely we have the following theorem, which is valid in all orders, even tho there is not formula for the solution of an arbitrary higher-order linear equation.

Theorem on Existence and Uniqueness for Linear Equations: If the coefficients p_1, p_2, $\ldots$, p_n, and the driving function g are all continuous on an interval I then for any point t_0 in I and any initial values y_0, y_0', $\ldots$, $y_0^{(n-1)}$, there exists a unique solution to the initial value problem

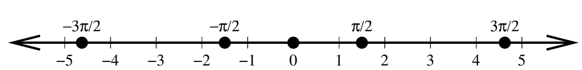
$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = g,$$
$$y(t_0) = y_0, y'(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

This unique solution remains valid throughout the interval I.

For example, consider the initial value problem

$$t^2 y'' + \tan(t) y' + (t-2)y = e^t,$$
$$y(-1) = 2, y'(-1) = -3.$$

If we put the equation in standard form we see that the coefficients are $\tan(t)/t^2$, $(t-2)/t^2$, and the driving function is e^t/t^2. There are discontinuities for these functions at 0 and at at odd multiples of $\pi/2$. The initial point -1 lies between the discontinuities $-\pi/2$ and 0:



Hence the interval of validity of $(-\pi/2, 0)$. Note that the location of t_0 but *not* the initial values determine the interval of validity. If we were to change the initial values to, say $y(-1) = 2.5$, $y'(-1) = 0$, then the new solution would still be valid on $(-\pi/2, 0)$. However if we were to move the initial point t_0 to, say, 3 then the solution to any initial value problem based there would be valid on the interval $(\pi/2, 3\pi/2)$.

Further reading

The theorems on existence and uniqueness for first-order equations are presented in sections 2.4 and 2.8 of the text. The text stresses the difference in behavior between linear and nonlinear equations in section 2.4, and presents a proof of the general theorem using Picard's approximation method in section 2.8. Good practice problems are 1–20, pages 75–76.

Assignment 8: due Monday, 27 February

1. Determine the interval of validity for the following initial value problems *without* integrating. Explain your answers!

   (a) $(t-2)y' + \ln|t|y = 2-t$, $y(-1) = 2$.

   (b) $(t-2)y' + \ln|t|y = 2-t$, $y(3) = 2$.

   (c) $(t-2)y' + \ln|t|y = 2-t$, $y(1) = 2$.

   (d) $y' + t^2 y = \sin(t)$, $y(\pi) = -2.6$.

   (e) $y' + t^2 y = \sin(t)$, $y(-\pi/8) = 1.9$.

2. For each of the following differential equations, draw a slope field, as neatly and accurately as possible. For each of the given initial points (t_0, y_0) sketch the integral curve that passes thru this point. (Sketch the integral curves on your slope field.) Try to estimate the interval of validity using the slope field. When possible, solve the initial value problems and determine the interval of validity exactly by determining the domain.

   (a) $y' = ty(3+y)$; $(t_0, y_0) = (0, -2)$; $(t_0, y_0) = (0, 1)$; $(t_0, y_0) = (0, -4)$.

   (b) $ty' = 4/y$; $(t_0, y_0) = (\pm 1, \pm 1)$ (four separate points).

   (c) $y' = t - 1 - y^2$; $(t_0, y_0) = (0, 2)$; $(t_0, y_0) = (0, -1)$.