Ellipses of minimal area and of minimal eccentricity circumscribed about a convex quadrilateral

Alan Horwitz
Penn State University
25 Yearsley Mill Rd.
Media, PA 19063
alh4@psu.edu

7/24/08

Abstract

First, we fill in key gaps in Steiner's nice characterization of the most nearly circular ellipse which passes through the vertices of a convex quadrilateral, Ð. Steiner proved that there is only one pair of conjugate directions, M_1 and M_2, that belong to all ellipses of circumscription. Then he proves that **if** there is an ellipse, E, whose **equal** conjugate diameters possess the directional constants M_1 and M_2, then E must be an ellipse of circumscription which has minimal eccentricity. Howvere, Steiner does not show the existence or uniqueness of such an ellipse. We prove that there is a unique ellipse of minimal eccentricity which passes through the vertices of Ð. We also show that there exists an ellipse which passes through the vertices of Ð and whose *equal* conjugate diameters possess the directional constants M_1 and M_2. We also show that there exists a unique ellipse of minimal area which passes through the vertices of Ð. Finally, we call a convex quadrilateral, Ð, bielliptic if the unique inscribed and circumscribed ellipses of minimal eccentricity have the same eccentricity. This generalizes the notion of bicentric quadrilaterals. In particular, we show the existence of a bielliptic convex quadrilateral which is not bicentric.

1 Introduction

Let Đ be a convex quadrilateral in the xy plane. An ellipse which passes through the vertices of Đ is called a circumscribed ellipse or ellipse of circumscription. In the book [1], Dörrie presents Steiner's nice characterization of the ellipse of circumscription which has minimal eccentricity, which he calls the most nearly circular ellipse. A pair of **conjugate diameters** are two diameters of an ellipse such that each bisects all chords drawn parallel to the other. Every non circular ellipse has a unique pair of **equal** conjugate diameters. Let θ_1 and θ_2 be the angles which a pair of conjugate diameters make with the positive x axis. Then $\tan\theta_1$ and $\tan\theta_2$ are called a pair of **conjugate directions**. First, Steiner proves that there is only one pair of conjugate directions, M_1 and M_2, that belong to all ellipses of circumscription. Then he proves in essence that **if** there is an ellipse, E, whose **equal** conjugate diameters possess the directional constants M_1 and M_2, then E must be an ellipse of circumscription which has minimal eccentricity. There are several gaps and missing pieces in Steiner's result. Steiner does **not** show that there **exists** an ellipse of circumscription, E, whose equal conjugate diameters possess the directional constants M_1 and M_2, or that such an ellipse is **unique**. He also does **not** prove in general the **uniqueness** of an ellipse of circumscription which has minimal eccentricity. That leaves open the possibility that there exists a circumscribed ellipse of minimal eccentricity that might **not** have **equal** conjugate diameters which possess the directional constants M_1 and M_2. Steiner's proof does show that if there exists an ellipse of circumscription, E, whose equal conjugate diameters possess the directional constants M_1 and M_2, then any other ellipse of circumscription of minimal eccentricity must also have equal conjugate diameters which possess the directional constants M_1 and M_2.

In Propositions 1 and 2 we fill in these gaps in Steiner's proof. We prove(Proposition 1), without using the directional constants M_1 and M_2, that there is a unique ellipse, E_O, of minimal eccentricity which passes through the vertices of Đ. Then we show(Proposition 2) that there exists an ellipse which passes through the vertices of Đ and whose *equal* conjugate diameters possess the directional constants M_1 and M_2. In addition, our methods enable us to prove(Theorem 2) that there is a unique ellipse of **minimal area** which passes through the vertices of Đ. Our proof applies to the case when Đ is not a trapezoid, though the results can be proven in that case by using a limiting argument or by directly deriving the corresponding formulas as done for the non–trapezoid case.

In [2] the author proved numerous results about ellipses **inscribed** in convex quadrilaterals, where we filled in similar gaps in a classical solution to Newton's problem, which was to determine the locus of centers of ellipses inscribed in Đ. In addition, in [2] the author proved that there exists a unique ellipse of minimal eccentricity, E_I, inscribed in Đ. This leads to the last section of this paper, where we discuss a special class of convex quadrilaterals which we call bielliptic and which generalize the bicentric quadrilaterals. A convex quadrilateral, Đ, is called bicentric if there exists a circle inscribed in Đ and a circle circumscribed about Đ. Đ is called **bielliptic** if E_I and E_O have the **same** eccentricity. We prove(Theorem 4), that there exists a bielliptic convex quadrilateral which is not bicentric. We also prove(Theorem 5), that there exists a bielliptic trapezoid which is not bicentric.

Finally we prove the perhaps not so obvious result(Theorem 3), that if Đ is not a parallelogram, and E_1 and E_2 are each ellipses, with E_1 inscribed in Đ and E_2 circumscribed about Đ, then E_1 and E_2 cannot have the same center.

In a forthcoming paper, we shall focus on ellipses inscribed in, and circumscribed about, parallelograms. In particular, there is a nice characterization of the ellipse of minimal eccentricity inscribed in a parallelogram.

2 Minimal Eccentricity

We state the following lemma without proof(see [6]).

Lemma 1 *: The equation* $Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0$, *with* $A, B > 0$, *is the equation of an ellipse,* E_0, *if and only if* $AB - C^2 > 0$ *and* $AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF > 0$. *Let* a *and* b *denote the lengths of the semi–major and semi–minor axes, respectively, of* E_0. *Let* ϕ *denote the acute rotation angle of the axes of* E_0 *going counterclockwise from the positive* x *axis and let* (x_0, y_0) *denote the center of* E_0. *Then*

$$a^2 = \frac{AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF}{2(AB - C^2)\left(A + B - \sqrt{(B-A)^2 + 4C^2}\right)}, \tag{2.1}$$

$$b^2 = \frac{AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF}{2(AB - C^2)\left(A + B + \sqrt{(B-A)^2 + 4C^2}\right)}, \tag{2.2}$$

$$\phi = \frac{1}{2}\cot^{-1}\left(\frac{A-B}{2C}\right), C \neq 0 \textit{ and } \phi = 0 \textit{ if } C = 0, \tag{2.3}$$

and

$$x_0 = -\frac{1}{2}\frac{BD-CE}{AB-C^2}, y_0 = \frac{1}{2}\frac{CD-AE}{AB-C^2} \tag{2.4}$$

Throughout this section, we let Đ be a given convex quadrilateral and we assume throughout that Đ is not a trapezoid. We use the notation and terminology of Steiner in [1]. Let $OPRQ$ denote the vertices of Đ, in counterclockwise order. Use the oblique coordinate system with $\overrightarrow{OP}$ as the x axis and $\overrightarrow{OQ}$ as the y axis, with those sides given by $y = 0$ and $x = 0$. By using an isometry of the plane, we can assume that $O = (0,0)$, P lies on the positive x axis, and that R and Q are in the first quadrant. Let $H = \overleftrightarrow{QR} \cap \overleftrightarrow{OP}, K = \overleftrightarrow{PR} \cap \overleftrightarrow{OQ}, p = \left|\overline{OP}\right|, q = \left|\overline{OQ}\right|, h = \left|\overline{OH}\right|$, and $k = \left|\overline{OK}\right|$. The sides $\overleftrightarrow{PR}$ and $\overleftrightarrow{QR}$ are given by $z = 0$ and $w = 0$, respectively, where $z = kx+py-kp$ and $w = qx+hy-hq$. As in the diagram shown in [1], we assume that R is to the right of, and below, Q, and the slope of $\overleftrightarrow{PR}$ is less than the slope of $\overleftrightarrow{OQ}$. Other shapes for a convex quadrilateral are possible, of course, but we do not consider those cases in the proofs below, the details being similar. It follows that

$$0 < p < h, 0 < q < k. \tag{2.5}$$

Any ellipse passing through the vertices of Đ has equation $\lambda xz+\mu yw = 0$, where λ and μ are *nonzero* real numbers. Letting $v = \frac{\lambda}{\mu}$, the equation becomes $vxz + yw = 0$, or

$$kvx^2 + hy^2 + (vp + q)xy - vkpx - hqy = 0. \tag{2.6}$$

Let $A = kv$, $B = h$, $C = \frac{1}{2}(vp + q)$, $D = -vkp$, $E = -hq$, and $F = 0$. Then $AB - C^2 = kvh - \frac{1}{4}(vp + q)^2 =$
$\frac{1}{4}\left[-p^2v^2 + (4kh - 2pq)\,v - q^2\right]$. Let

$$g(v) = 4khv - (vp + q)^2 = 4\left(AB - C^2\right)$$

Note that $g(v) = 0 \iff v = \frac{1}{p^2}\left(2kh - pq \pm 2\sqrt{kh\,(kh - pq)}\right)$. Hence $g(v) > 0$, and thus $AB - C^2 > 0$, if and only if $v \in I$, where

$$I = \left(\frac{1}{p^2}\left(2kh - pq - 2\sqrt{kh\,(kh - pq)}\right), \frac{1}{p^2}\left(2kh - pq + 2\sqrt{kh\,(kh - pq)}\right)\right).$$

Also, $(2kh-pq)^2 - 4(kh(kh-pq)) = q^2p^2 > 0$. Since $kh > pq$ by (2.5), $2kh - pq > 2\sqrt{kh(kh-pq)}$. Hence $I \subset (0, \infty)$, which implies that $v > 0$ whenever $v \in I$. Now $AE^2 + BD^2 + 4FC^2 - 2CDE - 4ABF = khv\left[vp^2(k-q) + q^2(h-p)\right] > 0$ if $v \in I$ by (2.5). By Lemma 1, (2.6) is the equation of a nontrivial ellipse if and only if $v \in I$. Our first main result is the following.

Proposition 1 : *There is a unique ellipse, E_O, of minimal eccentricity which passes through the vertices of Đ.*

Proof. By Lemma 1,

$$a^2 = \frac{2khv\left[vp^2(k-q) + q^2(h-p)\right]}{\left(4khv - (vp+q)^2\right)\left(kv + h - \sqrt{(kv-h)^2 + (vp+q)^2}\right)} \tag{2.7}$$

and

$$b^2 = \frac{2khv\left[vp^2(k-q) + q^2(h-p)\right]}{\left(4khv - (vp+q)^2\right)\left(kv + h + \sqrt{(kv-h)^2 + (vp+q)^2}\right)}, \tag{2.8}$$

which implies that $\frac{b^2}{a^2} = \frac{kv + h - \sqrt{(kv-h)^2 + (vp+q)^2}}{kv + h + \sqrt{(kv-h)^2 + (vp+q)^2}}$. Some simplification yields

$$\frac{b^2}{a^2} = f(v) = \frac{g(v)}{\left(kv + h + \sqrt{(kv-h)^2 + (vp+q)^2}\right)^2}. \tag{2.9}$$

We shall now minimize the eccentricity by maximizing $\frac{b^2}{a^2}$. Differentiating f with respect to v yields $f'(v) = \frac{-2(2hk-pq)(vk-h) + p^2hv - q^2k}{\sqrt{(kv-h)^2 + (vp+q)^2}\left(kv + h + \sqrt{(kv-h)^2 + (vp+q)^2}\right)^2}$. Thus

$f'(v) = 0 \iff (2hk-pq)(vk-h) + p^2hv - q^2k = 0 \iff v = v_0$, where

$$v_0 = \frac{q^2k + 2kh^2 - hpq}{2k^2h - kpq + hp^2}. \tag{2.10}$$

Some more simplification yields $(kv_0 - h)^2 + (v_0 p + q)^2 = \frac{(ph+qk)^2 W}{(2k^2 h - kpq + hp^2)^2}$, where

$$W = 4k^2 h^2 + (hp - qk)^2 . \tag{2.11}$$

It follows that

$$g(v_0) = \frac{4kh\,(kh - pq)\,W}{(2k^2 h - kpq + hp^2)^2}. \tag{2.12}$$

Thus $g(v_0) > 0$ by by (2.5) and (2.12), which implies that $v_0 \in I$. Note that $kv + h + \sqrt{(kv-h)^2 + (vp+q)^2} > 0$ for all $v > 0$, and $g(v) > 0, v \in I$. Thus f is differentiable on I and has a unique real critical point in I. Since g vanishes at the endpoints of I, f also vanishes at the endpoints of I by (2.9). Since $f(v) > 0$ on I, $f(v_0)$ must give the unique maximum of f on I. ■

Note that the quadrilateral Đ above, with vertices $OPRQ$, is **not cyclic** since $\frac{b^2}{a^2} = 1 \iff (kv-h)^2 + (vp+q)^2 = 0$, which cannot hold if $v \in I$. Thus any ellipse of circumscription is not a circle. In [1], Steiner shows that the unique pair of conjugate directions that belong to all ellipses which pass through the vertices of Đ is given by

$$M_1 = -\frac{k}{p} + \frac{r}{hp}, M_2 = -\frac{k}{p} - \frac{r}{hp}, \text{ where } r = \sqrt{hk}\sqrt{hk - pq}. \tag{2.13}$$

Proposition 2 *There exists an ellipse which passes through the vertices of Đ and whose equal conjugate diameters possess the directional constants M_1 and M_2.*

Proof. Let E_O denote the the unique ellipse from Proposition 1 of minimal eccentricity which passes through the vertices of Đ. As noted above, the quadrilateral Đ, with vertices $OPRQ$, is not cyclic, which implies that E_O is not a circle. Let L and L' denote *equal* conjugate diameters of E_O with directional constants M and M', respectively. Let ϕ denote the acute angle of counterclockwise rotation of the axes of E_O and let a and b denote the lengths of the semi–major and semi–minor axes, respectively, of E_O. It is known(see, for example, [5]) that L and L' make equal acute angles, on opposite sides, with the semi–major axis of E_O. Let θ denote the acute angle going counterclockwise from the major axis of E_O to one of the equal conjugate diameters, which implies that $\tan\theta = \dfrac{b}{a}$. By Lemma

1, with $A = kv$, $B = h$, $C = \dfrac{1}{2}(vp + q)$, $D = -kpv$, $E = -hq$, and $F = 0$, $\cot(2\phi) = \dfrac{kv - h}{vp + q}$. Note that $C \neq 0$, which implies that $\phi \neq 0$. As one would expect from the results in [1], *if* there is some ellipse whose equal conjugate diameters possess the directional constants M_1 and M_2, then that ellipse minimizes the eccentricity among all ellipses of circumscription. By the proof of Proposition 1, the point v_0 given in (2.10) yields the ellipse which minimizes the eccentricity. Thus, to prove Proposition 2, we let $v = v_0$. Then $\cot(2\phi) = \frac{kq-hp}{2kh} \Rightarrow \frac{\cot^2\phi - 1}{2\cot\phi} = \frac{kq-hp}{2kh} \Rightarrow \cot\phi = \frac{1}{2kh}\left(kq - hp \pm \sqrt{4k^2h^2 + (kq-hp)^2}\right) = \frac{kq-hp\pm\sqrt{W}}{2kh}$. We first need to determine whether to choose the positive or the negative root. If $kq - hp \geq 0$, then $\cot(2\phi) = \frac{kq-hp}{2kh} \geq 0 \Rightarrow 0 < 2\phi \leq \frac{\pi}{2} \Rightarrow 0 < \phi \leq \frac{\pi}{4} \Rightarrow 1 \leq \cot\phi < \infty$. Let $x = 2kh$, $y = kq - hp$, $0 < x < \infty$, $0 \leq y < \infty$. If $\cot\phi = \frac{kq-hp-\sqrt{W}}{2kh}$, then $\cot\phi = \frac{y-\sqrt{x^2+y^2}}{x} = \frac{y}{x} - \sqrt{1 + \left(\frac{y}{x}\right)^2} = u - \sqrt{1+u^2}$, where $u = \frac{y}{x}$, $0 \leq u < \infty$. Let $z(u) = u - \sqrt{1+u^2}$. Then $z'(u) = \frac{\sqrt{1+u^2}-u}{\sqrt{1+u^2}} > 0$, $z(0) = -1$, and $\lim\limits_{u\to\infty} z(u) = 0$. Thus $-1 \leq z(u) < 0 \Rightarrow -1 \leq \cot\phi < 0$, which contradicts $1 \leq \cot\phi < \infty$. If $kq - hp < 0$, then $\cot(2\phi) = \frac{kq-hp}{2kh} < 0 \Rightarrow \frac{\pi}{2} < 2\phi < \pi \Rightarrow \frac{\pi}{4} < \phi < \frac{\pi}{2} \Rightarrow 0 < \cot\phi < 1$. Again, if $\cot\phi = \frac{kq-hp-\sqrt{W}}{2kh}$, then $\cot\phi = z(u), -\infty < u < 0$. Since $z(0) = -1$ and $\lim\limits_{u\to-\infty} z(u) = -\infty$, $-\infty < z(u) < -1 \Rightarrow \cot\phi < -1$, which contradicts $0 < \cot\phi < 1$. That proves

$$\cot\phi = \frac{kq - hp + \sqrt{W}}{2kh} \tag{2.14}$$

To finish the proof of Proposition 2, note that $M_1 = \frac{-kh+\sqrt{kh}\sqrt{kh-pq}}{hp} = \sqrt{kh}\frac{-\sqrt{kh}+\sqrt{kh-pq}}{hp} < 0$ and $M_2 < 0$. Thus the only way that L and L' can form angles of θ and $-\theta$, respectively, with the semi–major axis of E_O is if the major axis of E_O has a negative slope. In that case the **minor** axis of E_O is rotated by ϕ counterclockwise from the positive x axis. It follows that the two directional constants, M and M', are given by $\tan\left(\phi + \theta - \frac{\pi}{2}\right)$ and $\tan\left(\phi - \theta - \frac{\pi}{2}\right)$. We shall prove that $\tan\left(\phi + \theta - \frac{\pi}{2}\right) = M_1$. We find it convenient to introduce the following variables:

$$s = hp + kq, t = hp - kq.$$

Note that $2k^2h - kpq + hp^2 = k(kh - pq) + k^2h + hp^2 > 0$ by (2.5). Hence $(kv_0 + h) + \sqrt{(kv_0 - h)^2 + (v_0p + q)^2} = kv_0 + h + \frac{(ph+qk)\sqrt{W}}{2k^2h-kpq+hp^2}$, which implies that $\frac{(kv_0+h)\left(2k^2h-kpq+hp^2\right)+(ph+qk)\sqrt{W}}{2k^2h-kpq+hp^2} = \frac{W+(ph+qk)\sqrt{W}}{2k^2h-kpq+hp^2} = \sqrt{W}\frac{\sqrt{W}+(ph+qk)}{2k^2h-kpq+hp^2}$. By (2.9) and (2.12), $f(v_0) = \frac{4kh(kh-pq)W}{(2k^2h-kpq+hp^2)^2}\frac{\left(2k^2h-kpq+hp^2\right)^2}{W\left(\sqrt{W}+(ph+qk)\right)^2} = \frac{4kh(kh-pq)}{\left(\sqrt{W}+(ph+qk)\right)^2} = \frac{4r}{\left(\sqrt{W}+s\right)^2}$. By (2.9) again,

$$\frac{b}{a} = \frac{2r}{\sqrt{W}+s}. \tag{2.15}$$

By (2.14) and (2.15), $\tan\left(\phi + \theta - \frac{\pi}{2}\right) = \frac{\tan\theta\tan\phi - 1}{\tan\theta + \tan\phi} = \frac{\frac{b}{a}\frac{2kh}{kq-hp+\sqrt{W}}-1}{\frac{b}{a}+\frac{2kh}{kq-hp+\sqrt{W}}} = \frac{\frac{2r}{\sqrt{W}+s}\frac{2kh}{\sqrt{W}-t}-1}{\frac{2r}{\sqrt{W}+s}+\frac{2kh}{\sqrt{W}-t}} = \frac{4khr-\left(\sqrt{W}+s\right)\left(\sqrt{W}-t\right)}{2r\left(\sqrt{W}-t\right)+2kh\left(\sqrt{W}+s\right)}$. Hence

$$\tan\left(\phi + \theta - \tfrac{\pi}{2}\right) - M_1 = \frac{1}{2}\frac{4khr-\left(\sqrt{W}+s\right)\left(\sqrt{W}-t\right)}{r\left(\sqrt{W}-t\right)+kh\left(\sqrt{W}+s\right)} - \frac{r-hk}{hp} = 0 \iff$$

$$\begin{gathered}4kh^2rp - hp\left(\sqrt{W}+s\right)\left(\sqrt{W}-t\right) - 2r(r-hk)\left(\sqrt{W}-t\right) \\ -2(r-hk)kh\left(\sqrt{W}+s\right) = 0 \iff\end{gathered}$$

$$\begin{gathered}4kh^2rp + hpst + 2r(r-hk)t - 2s(r-hk)kh + \\ \left(-hp\,(s-t) - 2r(r-hk) - 2(r-hk)kh\right)\sqrt{W} - hpW = 0.\end{gathered}$$

Now $4kh^2rp + hpst + 2r(r-hk)t - 2s(r-hk)kh = hpW$ and $-hp\,(s-t) - 2r(r-hk) - 2(r-hk)kh = 0$. Hence

$\tan\left(\phi + \theta - \frac{\pi}{2}\right) = M_1$. Similarly, one can show that $\tan\left(\phi - \theta - \frac{\pi}{2}\right) = M_2$. ∎

By Propositions 1 and 2 and the main result in ([1]), we have

Theorem 1 *There exists a unique ellipse, E_O, which passes through the vertices of Đ and whose equal conjugate diameters possess the directional constants M_1 and M_2. Furthermore, E_O is the unique ellipse of minimal eccentricity among all ellipses which pass through the vertices of Đ.*

3 Minimal Area

We now prove a result similar to Proposition 1, but which instead minimizes the **area** among all ellipses which pass through the vertices of Đ. This was not discussed by Steiner in [1] and there does not appear to be a nice characterization of the minimal area ellipse. Again we shall prove the case when Đ is not a trapezoid. Since ratios of areas of ellipses and four–sided convex polygons are preserved under one–one affine transformations, we may assume throughout this section, unless stated otherwise, that the vertices of Đ are $(0,0),(1,0),(0,1)$, and (s,t) for some positive real numbers s and t. Furthermore, since Đ is convex and is not a trapezoid, it follows easily that

$$s+t>1 \text{ and } s\neq 1\neq t. \tag{3.1}$$

Lemma 2 *Suppose that the vertices of Đ are* $(0,0),(1,0),(0,1)$*, and* (s,t) *for some positive real numbers* s *and* t *satisfying (3.1). Let*

$$\begin{aligned} m_{s,t} &= \frac{t}{s(s-1)^2}\left(s+t-1+st-2\sqrt{st\,(s+t-1)}\right) \\ M_{s,t} &= \frac{t}{s(s-1)^2}\left(s+t-1+st+2\sqrt{st\,(s+t-1)}\right). \end{aligned}$$

An ellipse, E_0*, passes through the vertices of Đ if and only if* E_0 *has the form*

$$stux^2+sty^2-[s(s-1)u+t(t-1)]\,xy-stux-sty=0, u\in I_{s,t}=(m_{s,t},M_{s,t}). \tag{3.2}$$

If a *and* b *denote the lengths of the semi–major and semi–minor axes, respectively, of* E_0*, then*

$$a^2=\frac{2s^2t^2(s+t-1)u(su+t)}{\left[-s^2(s-1)^2u^2+2st(s+st+t-1)u-t^2(t-1)^2\right]}\times \tag{3.3}$$

$$\frac{1}{st(u+1)-\sqrt{t^2(s^2+(t-1)^2)-2st(s+t-1)u+s^2(t^2+(s-1)^2)u^2}} \tag{3.4}$$

and

$$b^2=\frac{2s^2t^2(s+t-1)u(su+t)}{\left[-s^2(s-1)^2u^2+2st(s+st+t-1)u-t^2(t-1)^2\right]}\times \tag{3.5}$$

$$\frac{1}{st(u+1)+\sqrt{t^2(s^2+(t-1)^2)-2st(s+t-1)u+s^2(t^2+(s-1)^2)u^2}}. \tag{3.6}$$

Finally, the center of $E_0, (x_0, y_0)$, *is given by*

$$x_0 = \frac{st\left[\left(2st + s^2 - s\right)u + \left(t^2 - t\right)\right]}{2st\left(st + s + t - 1\right)u - s^2\left(s-1\right)^2 u^2 - t^2\left(t-1\right)^2} \tag{3.7}$$

and

$$y_0 = \frac{st\left[\left(2st + t^2 - t\right)u + (s^2 - s)u^2\right]}{2st\left(st + s + t - 1\right)u - s^2\left(s-1\right)^2 u^2 - t^2\left(t-1\right)^2}. \tag{3.8}$$

Proof. Substituting the vertices of Đ into the general equation of a conic, $Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0,\ A, B > 0$, yields the equations $F = 0,\ A + D = 0, B + E = 0$, and $As^2 + Bt^2 + 2Cst - As - Bt = 0$, which implies that $As(s-1) + Bt(t-1) + 2Cst = 0$ or $C = -\frac{As(s-1)+Bt(t-1)}{2st}$. Multiplying thru by st and dividing thru by B yields the equation in (3.2), with $u = \dfrac{A}{B}$. Conversely, any conic satisfying (3.2) must pass through the vertices of Đ. By Lemma 1, the curve defined by (3.2) is an ellipse if and only if $s^2t^2u\left(s + t - 1\right)\left(su + t\right) > 0$ and $-s^2\left(s-1\right)^2 u^2 + 2st\left(st + s + t - 1\right)u - t^2\left(t-1\right)^2 > 0$. The first inequality clearly holds since $s + t > 1$ and $u > 0$. We write the second condition as $\alpha(u) < 0$, where

$$\alpha(u) = s^2(s-1)^2u^2 - 2st\left(st + s + t - 1\right)u + t^2(t-1)^2.$$

Now it is easy to show that $\alpha(u) < 0 \iff m_{s,t} < u < M_{s,t}$. That proves (3.2). If E_0 satisfies (3.2), then (3.3) and (3.5) follow immediately from Lemma 1–(2.7) and (2.8), and (3.7) and (3.8) follow immediately from Lemma 1–2.4. ■

Theorem 2 *There exists a unique ellipse,* E_A, *of minimal area which passes through the vertices of Đ.*

Proof. By Lemma 2–(3.3) and (3.5),

$$a^2b^2 = \left(\frac{2s^2t^2(s+t-1)u(su+t)}{-s^2(s-1)^2u^2+2st(s+st+t-1)u-t^2(t-1)^2}\right)^2 \times \frac{1}{[st(u+1)]^2-[t^2(s^2+(t-1)^2)-2st(s+t-1)u+s^2(t^2+(s-1)^2)u^2]}$$

$$= \frac{4u^2(su+t)^2s^2t^2[st(s+t-1)]^2}{\left[-t^2(t-1)^2+(4s^2t^2-2s(s-1)t(t-1))u-s^2(s-1)^2u^2\right]^3} = \beta(u), \text{ where}$$

$$\beta(u) = -\frac{4u^2(su+t)^2s^2t^2\left(st\left(s+t-1\right)\right)^2}{\left(\alpha(u)\right)^3}.$$

Note that β is differentiable on $I_{s,t}$ since $\alpha(u) < 0$ there. Also, $m_{s,t} > 0 \iff s+t-1+st > 2\sqrt{st\,(s+t-1)} \iff (s+t-1+st)^2 > 4st\,(s+t-1)$(since $s+t>1$) $\iff (t-1)^2\,(s-1)^2 > 0$, which holds since $s,t \neq 1$. Thus $m_{s,t} > 0$ and $M_{s,t} > 0$, which implies that $I_{s,t} \subset (0,\infty)$. Now $\lim\limits_{u\to m_{s,t}^+} \alpha(u) = \lim\limits_{u\to M_{s,t}^-} \alpha(u) = 0$, so that $\alpha(u)$ approaches 0 thru negative numbers as u approaches the endpoints of $I_{s,t}$. In addition, the numerator of $\beta(u)$, for given s and t, satisfies $4u^2(su+t)^2 s^2 t^2\,(st\,(s+t-1))^2 > \delta > 0$ for $u \in I_{s,t}$. Thus $\lim\limits_{u\to m_{s,t}^+} \beta(u) = \lim\limits_{u\to M_{s,t}^-} \beta(u) = \infty$, which implies that β must attain its global minimum on $I_{s,t}$. Differentiating with respect to u yields $\beta'(u) = 8u\,(su+t)\,s^2t^2\,(st\,(s+t-1))^2\,\frac{\gamma(u)}{(\alpha(u))^4}$, where

$$\begin{aligned}\gamma(u) &= s^3\,(s-1)^2\,u^3 + s^2 t\left(2s^2 - 3s + st + 1 + t\right)u^2 \\ &\quad - st^2\left(2t^2 + st - 3t + s + 1\right)u - t^3\,(t-1)^2\,.\end{aligned}$$

Now $2s^2-3s+st+1+t = 2(s-1)^2+st+s+t-1 > 0$ and $2t^2+st-3t+s+1 = 2(t-1)^2+st+s+t-1 > 0$ by (3.1). Hence γ has precisely one sign change, which implies that γ has exactly one real root in $(0,\infty)$ by Descartes' Rule of Signs. That in turn implies that β has a **unique** global minimum on $I_{s,t}$, which yields a unique ellipse of minimal area which passes through the vertices of Đ. ■

Remark 1 *In [3] and [4], the authors investigate the problem of constructing and characterizing an ellipse of minimal area containing a finite set of points. The results and methods in § 3 of this paper are different than in those papers, but it is worth pointing out some of the small intersection. In particular, for a convex quadrilateral, Đ, the authors in [3] and [4] construct an algorithm for finding the minimal area ellipse containing Đ and they also prove a uniqueness result. For the case when this ellipse passes thru all four vertices of Đ, this ellipse is then the minimal area ellipse discussed in this paper. However, there is a convex quadrilateral, Đ, for which the minimal area ellipse containing Đ does not pass thru all four vertices of Đ. In that case, the minimal area ellipse discussed in this paper is not the same.*

4 Inscribed versus Circumscribed

In this section and the next, we allow Đ to be a **trapezoid**, so we shall need a version of Lemma 2 for trapezoids. The proof of Lemma 3 below follows immediately from Lemma 1 or from Lemma 2 by allowing s to approach 1. We omit the details here.

Lemma 3 *Suppose that Đ is a **trapezoid** with vertices* $(0,0),(1,0),(0,1)$, *and* $(1,t), 0<t\neq 1$. *An ellipse,* E_0, *passes through the vertices of Đ if and only if* E_0 *has the form*

$$ux^2+y^2-(t-1)xy-ux-y=0, u\in I_t=\left(\frac{1}{4}(t-1)^2,\infty\right). \tag{4.1}$$

If a *and* b *denote the lengths of the semi–major and semi–minor axes, respectively, of* E_0, *then*

$$a^2=\frac{-2u(u+t)}{\left((t-1)^2-4u\right)\left(u+1-\sqrt{(t-1)^2+(u-1)^2}\right)} \tag{4.2}$$

and

$$b^2=\frac{-2u(u+t)}{\left((t-1)^2-4u\right)\left(u+1+\sqrt{(t-1)^2+(u-1)^2}\right)}. \tag{4.3}$$

Finally, the center of $E_0, (x_0,y_0)$, *is given by*

$$x_0=\frac{2u+t-1}{4u-(t-1)^2}, y_0=\frac{(1+t)u}{4u-(t-1)^2}. \tag{4.4}$$

Remark 2 *Lemma 3 actually holds when* $t=1$ *as well, which of course yields the unit square.*

Theorem 3 *Let Đ be a convex quadrilateral in the xy plane which is **not** a parallelogram. Suppose that* E_1 *and* E_2 *are each ellipses, with* E_1 *inscribed in Đ and* E_2 *circumscribed about Đ. Then* E_1 *and* E_2 *cannot have the same center.*

Proof. Assume first that Đ is **not** a **trapezoid**. Since the center of an ellipse is affine invariant, we may assume that the vertices of Đ are $(0,0),(1,0),(0,1)$, and (s,t) as above, where s and t satisfy (3.1). By ([2], Theorem 2.3), if M_1 and M_2 are the midpoints of the diagonals of Đ, then each point on the open line segment, Z, connecting M_1 and M_2 is the center of some ellipse inscribed in Đ. Thus the locus of centers of E_1 is precisely Z. For

Đ above, the equation of Z is $y = L(x) = \frac{1}{2}\frac{s-t+2x(t-1)}{s-1}$, where x lies in the open interval connnecting $\frac{1}{2}$ and $\frac{1}{2}s$. If E_1 and E_2 have the same center, then the center of $E_2, (x_0, y_0)$, must lie on Z. Hence $L(x_0) = y_0$, which implies that $L(x_0) - y_0 = \frac{-(s+t)\left[\left(s-s^2\right)u+t^2-t\right]\left[\left(s^2-s\right)u+t^2-t\right]}{2\left[s^2(s-1)^2u^2-2st(s+st+t-1)u+t^2(t-1)^2\right](s-1)} = 0$. Thus $(s-s^2)u+t^2-t=0$ or $(s^2-s)u+t^2-t=0$, which implies that $u = \pm\frac{t^2-t}{s^2-s}$. If $u = \frac{t^2-t}{s^2-s}$, then some simplification yields, by (3.7) in Lemma 2, $x_0 = \frac{1}{2}s$. Similarly, if $u = -\frac{t^2-t}{s^2-s}$, then $x_0 = \frac{1}{2}$. But $\frac{1}{2}s$ and $\frac{1}{2}$ do not lie on Z, and thus E_1 and E_2 cannot have the same center. Now suppose that Đ is a trapezoid, but not a parallelogram. Then we may assume, again by affine invariance, that the vertices of Đ are $(0,0), (1,0), (0,1)$, and $(1,t), t \neq 1$. The equation of Z is now $x = \frac{1}{2}$, where y lies in the open interval connnecting $\frac{1}{2}$ and $\frac{1}{2}t$. If E_1 and E_2 have the same center, then $x_0 = \frac{1}{2}$. By (4.4) of Lemma 3, $\frac{2u+t-1}{4u-(t-1)^2} = \frac{1}{2} \Rightarrow 4u+2t-2 = 4u-(t-1)^2 \Rightarrow t = \pm 1$, which contradicts the assumption that $t > 0, t \neq 1$. Again E_1 and E_2 cannot have the same center. ■

It is easy to find examples where the center of an ellipse circumscribed about Đ may lie inside Đ, on the boundary of Đ, or outside the closure of Đ. We make the following conjectures.

Conjecture 1 *The center of the ellipse of minimal eccentricity circumscribed about Đ lies inside Đ.*

Conjecture 2 *The center of the ellipse of minimal area circumscribed about Đ lies inside Đ.*

5 Bielliptic Quadrilaterals

The following definition is well–known.

Definition 1 *Let Đ be a convex quadrilateral in the xy plane.*

(A) Đ is called cyclic if there is a circle which passes through the vertices of Đ.

(B) Đ is called tangential if a circle can be inscribed in Đ.

(C) Đ is called bicentric if Đ is both cyclic and tangential.

We generalize the notion of bicentric quadrilaterals as follows. In ([2], Theorem 4.4) the author proved that there is a unique ellipse, E_I, of minimal eccentricity inscribed in a convex quadrilateral, Đ. Using Proposition 1 from this paper, we let E_O be the unique ellipse of minimal eccentricity circumscribed about Đ.

Definition 2 *A convex quadrilateral is called* ***bielliptic*** *if E_I and E_O have the* ***same eccentricity****.*

If Đ is bielliptic, we say that Đ is of class τ, $0 \leq \tau < 1$, if E_I and E_O each have eccentricity τ.

It is natural to ask the following:

Question: Does there exist a bielliptic quadrilateral of class τ for *some* $\tau, \tau > 0$?

We answer this in the affirmative with the following results.

Theorem 4 *There exists a convex quadrilateral, Đ, which is not a parallelogram and which is bielliptic of class τ for some $\tau > 0$. That is, there exists a bielliptic convex quadrilateral which is not a parallelogram and which is not bicentric.*

Proof. Consider the convex quadrilateral, Đ, with vertices $(0,0),(1,0),(0,1)$, and (s,t). We shall show that for some s and t satisfying (3.1), Đ is bielliptic of class τ for some $\tau > 0$. It is easy to show that Đ is cyclic if and only if $(2s-1)^2+(2t-1)^2 = 2$. In general, a convex quadrilateral is tangential if and only if the lengths of opposite sides add up to the same sum. It follows that Đ is tangential if and only if $s = t$. Consider the family of quadrilaterals $Đ_r$ given by

$$s = -\frac{3}{2}r + 2, t = r\left(\frac{1}{2}+\frac{1}{2}\sqrt{2}\right) + 2 - 2r, 0 \leq r \leq 1. \tag{5.1}$$

$r = 0$ gives $s = 2$ and $t = 2$, which yields a tangential quadrilateral which is not cyclic, and $r = 1$ gives $s = \frac{1}{2}$ and $t = \frac{1}{2}\left(1+\sqrt{2}\right)$, which yields a cyclic quadrilateral which is not tangential. Since the eccentricity of the inscribed and circumscribed ellipses of minimal eccentricity, $E_I(r)$ and $E_O(r)$, each vary continuously with r, $Đ_r$ must be bielliptic for some $r, 0 < r < 1$. More precisely, let $\epsilon_I(r)$ and $\epsilon_O(r)$ denote the eccentricities of E_I and E_O, respectively. Then $\epsilon_I(0) = 0$ and $\epsilon_O(0) > 0$ since $E_I(0)$ is a circle and $E_O(0)$ is not a circle. Similarly, $\epsilon_I(1) > 0$ and $\epsilon_O(1) = 0$ since $E_I(1)$ is not a circle and $E_O(1)$ is a circle. Since $\epsilon_I(r)$ and $\epsilon_O(r)$ are each continuous functions of r, by the Intermediate Value Theorem, $\epsilon_I(r_0) = \epsilon_O(r_0)$ for some $0 < r_0 < 1$. Now if s and t satisfy (5.1), then $s = t \iff -\frac{3}{2}r+2 = -\frac{3}{2}r+\frac{1}{2}r\sqrt{2}+2 \iff r = 0$. So for $0 < r < 1$, $Đ_r$ cannot be tangential. One can also easily show that or $0 < r < 1$, $Đ_r$ cannot be cyclic, but we don't need that here. It follows that $\epsilon_I(r_0) = \epsilon_O(r_0) = \tau > 0$, which means that $Đ_{r_0}$ is bielliptic of class τ. ∎

Theorem 5 *There exists a bielliptic trapezoid which is not a parallelogram, and which is of class τ for some $\tau > 0$.*

Proof. Consider the trapezoid, Đ, with vertices $(0,0),(1,0),(0,1)$, and $(1,t), t \neq 1$. We shall show that for some $t \neq 1$, Đ is bielliptic of class $\tau > 0$. By Lemma 3–(4.2) and (4.3), $\frac{b^2}{a^2} = \frac{\left[(t-1)^2-4u\right]\left[u+1-\sqrt{(t-1)^2+(u-1)^2}\right]}{\left[(t-1)^2-4u\right]\left[u+1+\sqrt{(t-1)^2+(u-1)^2}\right]}$. Hence the square of the eccentricity of an ellipse circumscribed about Đ is given by $\epsilon(u) = 1 - \frac{b^2}{a^2} = \frac{2\sqrt{(t-1)^2+(u-1)^2}}{u+1+\sqrt{(t-1)^2+(u-1)^2}}, u \in I_t = \left(\frac{1}{4}(t-1)^2, \infty\right)$. Differentiating with respect to u yields $\epsilon'(u) = \frac{-2\left(3+t^2-2t-2u\right)}{\left(u+1+\sqrt{t^2-2t+2+u^2-2u}\right)^2\sqrt{(t-1)^2+(u-1)^2}} = 0 \iff u = \frac{1}{2}(t^2-2t+3)$. We shall show that this value of u gives the minimal eccentricity. First, $\epsilon\left(\frac{1}{2}(t^2-2t+3)\right) = \frac{2\sqrt{(t^2-2t+5)(t-1)^2}}{t^2-2t+5+\sqrt{(t^2-2t+5)(t-1)^2}} = \frac{2|t-1|\sqrt{t^2-2t+5}}{t^2-2t+5+|t-1|\sqrt{t^2-2t+5}} =$
$\frac{2|t-1|}{\sqrt{t^2-2t+5}+|t-1|} < \frac{2|t-1|}{|t-1|+|t-1|} = 1$. Also, $\lim\limits_{u \to (t-1)^2/4^+} \epsilon(u) = 1$ and $\lim\limits_{u\to\infty} \epsilon(u) = 1$. Thus the square of the minimal eccentricity of an ellipse circumscribed about Đ is given by

$$\epsilon_O = \frac{2|t-1|}{\sqrt{t^2-2t+5}+|t-1|} \tag{5.2}$$

In [2] the author derived formulas for the eccentricity of the unique ellipse of minimal eccentricity inscribed in a convex quadrilateral, Đ. Those formulas apply when Đ is **not** a **trapezoid**. The methods used in [2] can easily be adapted to the case when Đ is a trapezoid. The ellipse of minimal eccentricity inscribed in a trapezoid is also unique, and one can derive the following formulas. Let I_t denote the open interval with $\frac{1}{2}$ and $\frac{1}{2}t$ as endpoints. For fixed t, we define the following functions of $k, k \in I_t$.

$$E(k) = \frac{(2k-1)(2k-t)}{16(t-1)^2 k^4 + 8(t^2+6t+1)k^2 - 32t(t+1)k + 17t^2 - 2t + 1},$$

$$\epsilon(k) = \frac{2}{1 + \sqrt{1 - 16t\,(1-t)^2\,E(k)}}, \tag{5.3}$$

and

$$c(k) = 16k^3 - 12(t+1)k^2 + 4(2t-1)k + t + 1.$$

Then $c(k)$ has a unique root, k_0, in I_t, and $\epsilon\,(k_0)$ equals the square of the minimal eccentricity of an ellipse inscribed in Đ. By (5.2) and (5.3), we want to show that there is a value of $t \neq 1$ and $k \in I_t$ such that $c(k) = 0$ and $\frac{2|t-1|}{\sqrt{(t-1)^2+4}+|t-1|} = \frac{1}{1+\sqrt{1-16t(1-t)^2E(k)}}$. This is equivalent, after some algebraic simplification, to $4t\,(t-1)^4\,E(k)+1 = 0$. Some more algebraic simplification yields the equation

$$\begin{gathered} 16\,(t-1)^2\,k^4 + \left(16t^5 - 64t^4 + 96t^3 - 56t^2 + 64t + 8\right)k^2 \\ -8t\,(1+t)\left(t^2 - 4t + 5\right)\left(t^2+1\right)k + \\ 4t^6 - 16t^5 + 24t^4 - 16t^3 + 21t^2 - 2t + 1 = 0 \end{gathered} \tag{5.4}$$

Thus we want a solution to the simultaneous equations (5.4) and $c(k) = 0$, with $t \neq 1$ and $k \in I_t$. Maple gives the following solutions: $t = 1, k = \frac{1}{2}, t = \frac{1}{2}i, k = \pm\frac{1}{2}i$, and $t = \frac{2\rho_2^3-3\rho_2^2+1-2\rho_2}{3\rho_2^2-4\rho_2-1}, k = \frac{1}{2}\rho_2$ where ρ_2 is a root of

$$\begin{gathered} p(x) = 32x^{11} - 287x^{10} + 1006x^9 - 1487x^8 + 160x^7 + \\ 1762x^6 - 884x^5 - 822x^4 + 80x^3 + 333x^2 + 150x + 21 \end{gathered}$$

$t = 1$ or $t = \frac{1}{2}i$ do not satisfy t real, $t \neq 1$. Since $p(1) = 64 > 0$ and $p(1.5) = -23.07715 < 0, p$ must have a root, x_0, between 1 and 2. Numerically $x_0 \approx 1.2323$. It appears that the real roots of p are approximately -0.8296,

$1.2323, 1.7787$, though we don't need that here. Now $\rho_2 = 1.2323 \Rightarrow t = \frac{2\rho_2^3-3\rho_2^2+1-2\rho_2}{3\rho_2^2-4\rho_2-1} \approx 1.6581$. Then $k = \frac{1}{2}\rho_2 = 0.6161 \in I_t$. The corresponding common value of the eccentricity is approximately 0.6901. ■

Remark 3 *It is interersting to note here that the bielliptic quadrilateral in Theorem 4 is not a trapezoid. The family of quadrilaterals Đ$_r$ given in the proof of Theorem 4 yields a trapezoid if and only if $s = 1$ or $t = 1$. Now $s = 1 \iff -\frac{3}{2}r + 2 = 1 \iff r = \frac{2}{3}$ and $t = 1 \iff r\left(\frac{1}{2} + \frac{1}{2}\sqrt{2}\right) + 2 - 2r = 1 \iff r = \frac{2}{3-\sqrt{2}} > 1$. Thus Đ$_r$ is a trapezoid $\iff r = \frac{2}{3}$. Now $r = \frac{2}{3} \Rightarrow t = 1 + \frac{1}{3}\sqrt{2}$. By (5.2) in the proof of Theorem 5, the square of the minimal* *eccentricity of an ellipse circumscribed about $Đ_{2/3}$ is $\frac{2}{\sqrt{19}+1} \approx 0.373$. Also, $I_t \approx (0.5, 0.736)$ and $c(k) = 16k^3 + \left(-24 - 4\sqrt{2}\right)k^2 + \left(\frac{8}{3}\sqrt{2} + 4\right)k + 2 + \frac{1}{3}\sqrt{2} = 0$ has the root $k \approx 0.5918$ in I_t. That yields $E(k) \approx -1.429\,5$. By (5.3) in the proof of Theorem 5, the square of the minimal eccentricity of an ellipse inscribed in $Đ_{2/3}$ is $\epsilon(k) \approx 0.5113$. Thus the bielliptic convex quadrilateral from Theorem 4 is not a trapezoid.*

Theorems 4 and 5 show the existence of a bielliptic quadrilateral of class τ for some $0 < \tau < 1$. We cannot yet answer the following:

Question: Does there exist a bielliptic quadrilateral of class τ for *each* $\tau, 0 < \tau < 1$?

Remark 4 *In a future paper we prove that a square is the only bielliptic parallelogram.*

References

[1] HEINRICH DÖRRIE, The most nearly circular ellipse circumscribing a quadrilateral, 100 Great problems of Mathematics, Their History and Solution, Dover Publications, Inc. 1965, 231–236.

[2] ALAN HORWITZ, Ellipses of maximal area and of minimal eccentricity inscribed in a convex quadrilateral, Australian Journal of Mathematical Analysis and Applications, 2(2005), 1-12.

[3] B. V. RUBLEV and YU. I. PETUNIN, Minimum–Area Ellipse Containing a Finite Set of Points. 1, Ukrainian Mathematical Journal, Vol. 50, No. 7, 1998, 1115–1124.

[4] B. V. RUBLEV and YU. I. PETUNIN, Minimum–Area Ellipse Containing a Finite Set of Points. 2, Ukrainian Mathematical Journal, Vol. 50, No. 8, 1998, 1253–1261.

[5] GEORGE SALMON, A treatise on conic sections, 6th edition, Chelsea Publishing Company, New York.

[6] `Online http://mathworld.wolfram.com/Ellipse.html`