Finding an Ellipse Tangent to finitely many given Lines

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Abstract

Given a finite set Σ of lines in the plane, we discuss necessary and sufficient conditions on when there is an ellipse E, with specified center (h,k) and angle of rotation α, which is tangent to each line in Σ. In all cases we assume that no three of the lines are parallel or have a common intersection point, else no such ellipse could exist. If such an ellipse exists, we say that (h,k) is α admissible, or just admissible if $\alpha = 0$. For *two* given lines $T_1 : y-k = m_1(x-h)+b_1$ and

$T_2 : y-k = m_2(x-h)+b_2$, with $|m_1| \neq |m_2|$, (h,k) is admissible if and only if $\frac{b_2^2-b_1^2}{m_2^2-m_1^2}$ and $\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}$ are both positive. Further, the ellipse is unique. We prove similar results when $|m_1| = |m_2|$, in which case the ellipse may not be unique. In certain cases we allow α to vary. We then show that every $(h,k) \notin T_1 \cup T_2$ is α admissible for some α(if $m_1 \neq m_2$). We prove various results for *three* given lines. In particular, if none of the slopes of the lines are equal in absolute value, and none of the lines are horizontal, then there are cubic polynomials $q(k)$ and $r(h)$ such that the set of admissible centers is precisely the set of points on a hyperbola where q and r are positive. q and r are obtained using the intersection points of the given lines. Finally, for *four* given lines, we show that there is always *some* ellipse, rotated of course, tangent to the given lines.

1 Introduction

Many curve fitting problems involve finding a curve C of a certain type(such as a polynomial of degree n or conic section)which passes through specified points(interpolation), perhaps with specified slopes at those points as well(Hermite interpolation). In this paper we consider a different type of curve fitting: Given a set Σ of finitely many lines and a family $\mathcal{F}$ of curves in the plane, find, if possible, a curve $\sigma \in \mathcal{F}$ such that each line in Σ is tangent to σ. We emphasize here that the *points of tangency are unknown.* This problem is similar to finding the envelope of a family of lines, except that here we only have *finitely* many lines.[1] The tradeoff is that you then must have some information about σ, the more information the less tangents are needed to reconstruct σ. In this paper we assume that $\mathcal{F}$ is the family, or some subfamily, of **ellipses** in the plane. Most of our results, however, extend easily to hyperbolas as well. The subfamilies we discuss are ellipses with a specified center or angle of rotation. Many of our results are proven for non-rotated ellipses $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, but extend to any fixed angle of rotation.

It is useful to make the following definition.

Definition 1 *Given a finite set of distinct lines $T_1, T_2, ..., T_n$ in the plane, and an angle α, $-\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$, suppose that there is an ellipse with center (h, k) and rotation angle α*[2] *which is tangent to each of the T_j. Then we say that (h, k) is α* ***admissible***. *If $\alpha = 0$ we just call (h, k) admissible.*

Of course α admissibility depends on the given lines T_j. In all cases we assume that no three of the lines are parallel or have a common intersection point, else no such ellipse could exist for any α or (h, k).

We discuss the following questions:

(1) Given two distinct lines in the plane, T_1 and T_2, and a point $C = (h, k) \notin T_1 \cup T_2$, under what conditions is C admissible. Is the ellipse always unique ?

(2) Is *any* point $(h, k) \notin T_1 \cup T_2$ admissible ? If not, is *any* point
$(h, k) \notin T_1 \cup T_2$ α admissible for some α ?

(3) For three given lines, what does the set of admissible centers look like for fixed α ? What about specifying the eccentricity in advance ?

(4) Suppose that we are given four lines, such that no three of the lines are parallel or have a common intersection point. Is there always some ellipse(rotated) tangent to the four lines ?

[1] To reconstruct a general curve given *all* of its tangents, see [4].

[2] By rotation angle α, we mean rotated *clockwise* about (h, k) through the angle α.

We summarize our results. In Section 2, we give necessary and sufficient conditions for when two lines T_1 and T_2 are tangents to some non-rotated ellipse with center (h,k)(see Theorems 4 and 5). More precisely, in Theorem 4 we assume that T_1 and T_2 are distinct, non–vertical lines with equations $y-k=m_j(x-h)+b_j,\ m_1^2\neq m_2^2$. Then then there is an ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, which has T_1 and T_2 as tangents, if and only if $\frac{b_2^2-b_1^2}{m_2^2-m_1^2}$ and $\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}$ are both positive. Further, the ellipse is unique with $a^2=\frac{b_2^2-b_1^2}{m_2^2-m_1^2}$ and $b^2=\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}$. In certain cases such an ellipse may *not exist* or be *unique*. For example, if T_1 and T_2 have equations $y=x+2$ and $y=2x-3$, then there is *no* ellipse E(nonrotated) with center $(2,0)$, such that T_1 and T_2 are tangents to E. On the other hand, if T_1 and T_2 have equations $y=2x+3$ and $y=-2x+3$, and $h=k=0$, then T_1 and T_2 are tangent to any ellipse in the family $F=\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1:4a^2+b^2=9\}$.

If one allows the angle of rotation to vary, then for any $(h,k)\notin T_1\cup T_2$(with $m_1\neq m_2$) is α admissible for some α.(see Theorem 7). Also, for fixed α, there is always either an ellipse or hyperbola with angle of rotation α tangent to T_1 and T_2(see Theorem 9).

For three given lines, the set of admissible centers either lies on a hyperbola or a straight line. See Theorems 11, 12, 14, and 15. In addition, one can choose the center so that the ellipse has any specified eccentricity. In particular, in Theorem 11, we assume that none of the slopes of the lines are equal in absolute value, and none of the lines are horizontal. We apply Theorem 4 to T_1 and T_2, and to T_1 and T_3. To get one ellipse tangent to all three lines, the a^2 and b^2 obtained in both cases must be equal. This leads to the equation of a hyperbola γ on which the set of admissible centers S must lie. More precisely, $S=\{(h,k)\in\gamma: q_3(k)>0$ and $r_3(h)>0\}$, where $q(k)$ and $r(h)$ are cubic polynomials obtained using the intersection points of the T_js.

For *four* given lines, there is always *some* ellipse, rotated of course, tangent to the given lines(see Theorem 18). By using an affine map, one only need prove this result for the special case when the four lines have interesection points $(0,0),(1,0),(0,1)$, and (s,t), with $s>0,t>0,\ s\neq 1\neq t,\ s+t\neq 1$.

2 Two Given Tangents

The theorems in this section are stated for ellipses with major and minor axes parallel to the x and y *axes.* Later we indicate how to extend the results easily using a rotation map. Indeed, the proofs we give all assume that the center C of E is $(0,0)$. If $C=(h,k)$, let $u=x-h, v=y-k$, and let $\bar{E}$ be the ellipse in the $uv-plane$ which is the image of E under the translation map $T:(x,y)\to(u,v)$. Then $\bar{E}$ has center $(0,0)$, and T maps tangents to E onto tangents to $\bar{E}$.

Lemma 2 *Let E be the ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, and suppose that T is a nonvertical tangent to E at (c,d). Write the equation of T in the form*

$$y-k=m(x-h)+B$$

Then $B\neq 0$ and $d\neq 0$.

Proof. The equation of T can be written in the form $\frac{c(x-h)}{a^2}+\frac{d(y-k)}{b^2}=1$. If $B=0$, then T passes thru (h,k), which is impossible. If $d=0$, then T is vertical. ■

Suppose that T_1 and T_2 are given distinct lines in the $xy-plane$, which are known to be tangent to an ellipse E with center (h,k). If h and k are known, how do you find the equation of E–i.e. what are a and b in $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$? Also, is E unique ? The next two theorems answer this question.

Proposition 3 *Suppose that we are given distinct, non-vertical lines T_1 and T_2, which are known to be tangent to some ellipse E with center $C=(h,k)$. Suppose that T_j has equation*

$$y-k=m_j(x-h)+b_j \tag{1}$$

$j=1,2$, with $m_1^2\neq m_2^2$. Then the equation of E is $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $a^2=\dfrac{b_2^2-b_1^2}{m_2^2-m_1^2}$ and $b^2=\dfrac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}$.

Proof. We assume first that $C=(0,0)$. Then E has equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. We shall derive formulas for a^2 and b^2. Suppose that T_j is tangent to E

at $(x_j, y_j), j = 1, 2$. Since $dy/dx = (-b^2/a^2)(x/y)$ at any point of $E(y \neq 0)$, we have

$$m_j = -\frac{b^2}{a^2}\frac{x_j}{y_j} \tag{2}$$

Note that T_j not vertical implies that $y_j \neq 0$, by Lemma 2. Also

$$y_j = m_j x_j + b_j \tag{3}$$

Using (2) and substituting (3) gives
$a^2 m_j(m_j x_j + b_j) + b^2 x_j = 0$, which implies

$$x_j = -\frac{a^2 m_j b_j}{a^2 m_j^2 + b^2} \tag{4}$$

By (3) we also have

$$y_j = \frac{b^2 b_j}{a^2 m_j^2 + b^2} \tag{5}$$

Using (4), (5), and $\dfrac{x_j^2}{a^2} + \dfrac{y_j^2}{b^2} = 1$ implies that
$\dfrac{a^2 m_j^2 b_j^2 + b^2 b_j^2}{(a^2 m_j^2 + b^2)^2} = 1 \Rightarrow b_j^2 = a^2 m_j^2 + b^2$. This gives the following linear system in the unknowns a^2 and b^2, which is nonsingular since $m_1^2 \neq m_2^2$:

$$\begin{aligned} m_1^2 a^2 + b^2 &= b_1^2 \\ m_2^2 a^2 + b^2 &= b_2^2 \end{aligned} \tag{6}$$

with unique solution

$$a^2 = \frac{b_2^2 - b_1^2}{m_2^2 - m_1^2},\ b^2 = \frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2} \tag{7}$$

Now suppose that $C = (h, k)$, and suppose that the equation of E is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Let $u = x - h, v = y - k$, and let $\bar{E}$ be the ellipse in the $uv-plane$ which is the image of E under the translation map $(x, y) \to (u, v)$. Then $\bar{E}$ has center $(0, 0)$, the equation of $\bar{E}$ is $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$, and the lines $\bar{T}_j$ with equation $v = m_j u + b_j$ are tangent to $\bar{E}$ at $(u_j, v_j) = (x_j - h, v_j - k)$. By the case just proven, a^2 and b^2 are given by (7). ■

Remark 1 *If $m_1^2 = m_2^2$, then the ellipse E in Theorem 3 is not unique. This will follow from Theorem 5.*

Now suppose that two distinct lines T_1 and T_2 are given which are *not* known to be tangent lines to some ellipse, as in Theorem 3. Given $C = (h,k)$, are T_1 and T_2 tangent to some ellipse with a *specified* center C ? The conditions on T_1 and T_2 are more restrictive, as the following theorem shows.

Theorem 4 *Let h and k be given real numbers, and let T_1 and T_2 be distinct, non–vertical lines with equations $y-k = m_j(x-h)+b_j,\ j=1,2$. Suppose that $m_1^2 \neq m_2^2$.*

Part 1: If

(A)

$$\frac{b_2^2-b_1^2}{m_2^2-m_1^2} > 0$$

and

(B)

$$\frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2-m_1^2} > 0$$

then there is a <u>unique</u> ellipse E, with center (h,k), which has T_1 and T_2 as tangents. Furthermore, E has equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where a^2 and b^2 are given by (7).

Part 2: (A) and (B) are also necessary conditions for T_1 and T_2 to be tangent to some ellipse with center (h,k).

Proof. Part 1: It is not hard to show that (A) and (B) imply

$$b_1 \neq 0 \neq b_2$$

We prove the case $h=k=0$, the general case following easily by a translation map, as in the proof of Proposition 3. Then T_j has equation $y = m_j x + b_j,\ j = 1,2$. Let a^2 and b^2 be defined as in (7), and let x_j and y_j be defined by (4) and (5). By (A) and (B), a^2 and b^2 are well–defined and positive, and x_j and y_j are well–defined. Now $m_j x_j + b_j = \frac{-a^2m_j^2b_j^2+a^2m_j^2b_j+b^2b_j}{a^2m_j^2+b^2} = \frac{b^2b_j}{a^2m_j^2+b^2} = y_j$. Note that by Lemma2, $y_j \neq 0$. Also, since the b_j are nonzero, $\frac{x_j}{y_j} = \frac{-a^2m_jb_j}{b_jb^2} = -\frac{a^2m_j}{b^2} \Rightarrow dy/dx\ |_{x=x_j,y=y_j} = \frac{-b^2x_j}{a^2y_j} = m_j$. Letting E be the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have proven that the T_j are tangent to E(at (x_j,y_j). Now if $\tilde{E}$ is any ellipse with center $(0,0)$, with the T_j tangent to $\tilde{E}$, then the equation of $\tilde{E}$ is $\frac{x^2}{\tilde{a}^2}+\frac{y^2}{\tilde{b}^2}=1$. By Proposition 3, $\tilde{a}$ and $\tilde{b}$ are given by (7). But then $\tilde{a}=a$ and $\tilde{b}=b$, which implies that $\tilde{E}=E$, and that proves uniqueness.

Part 2: This follows immediately from Proposition 3. ■

2.0.1 Examples

(1) Suppose that the lines $T_1: \ y=x+3$ and $T_2: \ y=2x-3$ are given, and we want an ellipse E with center $(1,2)$, such that T_1 and T_2 are tangents to E. First we must expand about $(1,2)$. $T_1: \ y-2=(x-1)+2$ and $T_2: \ y-2=2(x-1)-3$. Then $m_1=1, b_1=2, m_2=2, b_2=-3$. Since the conditions of Theorem 4 are satisfied, E exists and is unique. From $a^2=\frac{b_2^2-b_1^2}{m_2^2-m_1^2}$ and $b^2=\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}$ we obtain the equation of $E: \frac{(x-1)^2}{\frac{5}{3}}+\frac{(y-2)^2}{\frac{7}{3}}=1$. See Figure 2.1 for a plot of T_1, T_2, and E.

Note that by using (4) and (5), along with the translation $(x,y)\rightarrow(x+1,y+2)$, one obtains the *points of tangency*: $(\frac{1}{6},\frac{19}{6})$ and $(\frac{19}{9},\frac{11}{9})$.

(2) The lines $T_1: \ y=x+2$ and $T_2: \ y=2x-3$ are given, and we want an ellipse E(nonrotated) with center $(2,0)$, such that T_1 and T_2 are tangents to E. Then $m_1=1, b_1=4, m_2=2, b_2=1$, which implies that $\frac{b_2^2-b_1^2}{m_2^2-m_1^2}<0$. By Theorem 4, Part 2, there is **no** such ellipse E.

Remark 2 *One can also obtain results similar to Proposition 3 or Theorem 4 if one of the T_j(but not **both**) is vertical. If T_1 has equation $x-h=b_1$,then $b_1=a$ or $-a$, which implies that $a^2=b_1^2$. Since $m_2^2a^2+b^2=b_2^2$ still holds, one then solves to get b^2.*

Theorem 5 *Let h and k be given real numbers, and let T_1 and T_2 be distinct, non–vertical lines with equations $y-k=m_j(x-h)+b_j, \ j=1,2$.*

Suppose that $m_1^2=m_2^2$.

(i) If $b_1^2=b_2^2\neq 0$, then T_1 and T_2 are tangent to any ellipse $E\in \digamma$ = the family of ellipses $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where a^2 and b^2 are any positive real numbers satisfying the equation

$$a^2m_1^2+b^2=b_1^2 \tag{8}$$

(ii) If $b_1^2\neq b_2^2$ and/or $b_1=0=b_2$, then T_1 and T_2 are not tangent to any ellipse with center (h,k).

Proof. Again, we prove the case $h = k = 0$, the general case following from a translation. Note first that the equation

$m_1^2 u + v = b_1^2$ always has infinitely many positive solutions (u, v) since the line $m_1^2 u + v = b_1^2$ has a positive v intercept and a nonpositive slope. Now let a^2 and b^2 be any positive real numbers satisfying (8). To prove (i), let $x_j = -\frac{a^2 b_j m_j}{a^2 m_j^2 + b^2}$ and $y_j = \frac{b^2 b_j}{a^2 m_j^2 + b^2}$. Note that $b_1 \neq 0 \neq b_2$. Consider first the case $m_1 = m_2 \equiv m$. Then $b_2 = -b_1$ since $T_1 \neq T_2$. Also, $x_j = -\frac{a^2 b_j m}{b_1^2}$ and $y_j = \frac{b^2 b_j}{b_1^2}$, which implies that $dy/dx \mid_{x=x_j, y=y_j} = \frac{-b^2 x_j}{a^2 y_j} = \frac{-b^2}{a^2} \frac{-a^2 b_j m}{b^2 b_j} = m$. Finally, $m x_1 + b_1 = \frac{-a^2 m^2 b_1 + b_1^3}{b_1^2} = \frac{b^2}{b_1} = y_1$ and $m x_2 + b_2 = \frac{-a^2 m^2 b_2 + b_1^2 b_2}{b_1^2} = \frac{b^2 b_2}{b_1^2} = y_2$. Hence T_1 and T_2 are tangents to E(at (x_j, y_j)).

Now suppose that $m_1 = -m_2$. Then $b_2 = b_1$ since $T_1 \neq T_2$, and $x_j = -\frac{a^2 b_j m_j}{b_1^2}$ and $y_j = \frac{b^2 b_j}{b_1^2}$, which implies that $dy/dx \mid_{x=x_j, y=y_j} = \frac{-b^2 x_j}{a^2 y_j} = \frac{-b^2}{a^2} \frac{-a^2 b_j m_j}{b^2 b_j} = m_j$. Also, $m x_1 + b_1 = \frac{-a^2 m_1^2 b_1 + b_1^3}{b_1^2} = \frac{b^2}{b_1} = y_1$ and $m x_2 + b_2 = \frac{-a^2 m_2^2 b_2 + b_1^2 b_2}{b_1^2} = \frac{b^2 b_2}{b_1^2} = y_2$. Hence T_1 and T_2 are tangents to E(at (x_j, y_j)).

To prove (ii), suppose that T_1 and T_2 are tangent to an ellipse E with center (h, k). Proving the case $h = k = 0$ and proceeding as in the proof of Theorem 3, $b_j^2 = a^2 m_j^2 + b^2$, $j = 1, 2$. Since $m_1^2 = m_2^2$, $b_1^2 = b_2^2$. Also, by Lemma2, $b_1^2 \neq 0 \neq b_2^2$. ■

Remark 3 *A result similar toTheorem 5 can be proved if both T_1 and T_2 are vertical, say with equations $x - h = b_j$, $j = 1, 2$. It is easy to show that $b_2 = -b_1$(so that $b_1^2 = b_2^2$) and $b_1 \neq 0 \neq b_2$. Also, $a^2 = 2b_1$, while b can be any positive real number.*

Remark 4 *One can obtain the expression for b^2 in (7) from the expression for a^2 in a natural way as follows: Write the T_j in the form $x - h = \frac{1}{m_j}(y - k) - \frac{b_j}{m_j}$(assuming $m_1 \neq 0 \neq m_2$). This suggests replacing m_j by $\frac{1}{m_j}$ and b_j by $-\frac{b_j}{m_j}$. Then $\frac{b_2^2 - b_1^2}{m_2^2 - m_1^2}$ becomes $\frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2}$.*

2.0.2 Examples

(1) Let $h = k = 0$, $T_1 : \ y = 2x + 3$, $T_2 : \ y = -2x + 3$. Then by Theorem 5, T_1 and T_2 are tangent to any ellipse $E \in F = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 : 4a^2 + b^2 = 9\}$.

See Figure 2.2 for a plot of $T_1,\ T_2$, and the two ellipses in F, $x^2+\dfrac{y^2}{5}=1$ and $y^2+\dfrac{x^2}{2}=1$

(2) Let $h=k=0,\ T_1:\ y=x+2,\ T_2:\ y=-x+1$. Then by Theorem 5, T_1 and T_2 are not tangent to any ellipse with center $(0,0)$.

2.1 Rotated Ellipses

For simplicity of exposition, we have only considered non-rotated ellipses. However, the previous results extend with little effort to ellipses whose axes are *rotated* thru a specified angle α. Let L represent rotation clockwise[3] about (h,k) thru an angle $\alpha, -\dfrac{\pi}{2}<\alpha<\dfrac{\pi}{2}$. Then the line $y-k=m(x-h)+b$ is transformed into the line

$$v-k=\left(\frac{m+\tan\alpha}{1-m\tan\alpha}\right)(u-h)+\frac{b\sec\alpha}{1-m\tan\alpha},\ 1-m\tan\alpha\neq 0$$

Now let T_1 and T_2 be distinct, non–vertical lines with equations $y-k=m_j(x-h)+b_j,\ j=1,2,\ m_1^2\neq m_2^2$.

We want to apply conditions (A) and (B) of Theorem 4 in the *new* coordinates u and v. Thus we replace m_j by $m_{j,\alpha}$ and b_j by $b_{j,\alpha}$, where

$$m_{j,\alpha}=\frac{m_j+\tan\alpha}{1-m_j\tan\alpha}$$

and

$$b_{j,\alpha}=\frac{b_j\sec\alpha}{1-m_j\tan\alpha}$$

This yields the expressions

$$A=\frac{b_{2,\alpha}^2-b_{1,\alpha}^2}{m_{2,\alpha}^2-m_{1,\alpha}^2}$$

$$B=\frac{b_{1,\alpha}^2m_{2,\alpha}^2-b_{2,\alpha}^2m_{1,\alpha}^2}{m_{2,\alpha}^2-m_{1,\alpha}^2}$$

We now state the following generalization of Theorems 4 without proof.

[3] One usually rotates *counterclockwise*, but the computations are slightly easier with a clockwise rotation.

Theorem 6 *Let h and k be given real numbers, and let T_1 and T_2 be distinct lines with equations $y - k = m_j(x - h) + b_j$, $j = 1, 2$. Suppose that $-\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$ and $1 - m_1 \tan\alpha \neq 0 \neq 1 - m_2 \tan\alpha$(thus T_1 and T_2 are not vertical when rotated by α)*

Part 1: If $A > 0$ and $B > 0$, then there is a unique ellipse E, with center (h, k) and rotation angle α, which has T_1 and T_2 as tangents. Furthermore, in the new rotated coordinates u and v, E has equation $\dfrac{(u-h)^2}{a^2} + \dfrac{(v-k)^2}{b^2} = 1$, where

$$a^2 = \frac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}, \; b^2 = \frac{b_{1,\alpha}^2 m_{2,\alpha}^2 - b_{2,\alpha}^2 m_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$$

Part 2: If $m_{2,\alpha}^2 \neq m_{1,\alpha}^2$, then $A > 0$ and $B > 0$ are also necessary conditions for T_1 and T_2 to be tangent to some ellipse with center (h, k) and rotation angle α.

It is more interesting to allow the rotation angle to *vary*. This leads to the following

Question: Given two distinct lines T_1 and T_2, is any point $(h, k) \notin T_1 \cup T_2$ α admissible for some α, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$? The following theorem answers this question in the affirmative.

Theorem 7 *Let h and k be given real numbers, and let T_1 and T_2 be distinct, non–vertical lines with equations $y-k = m_j(x-h)+b_j$, $j = 1, 2$. Assume that $m_1 \neq m_2$, and that $(h, k) \notin T_1 \cup T_2$. Then there is an angle α, $-\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$, such that the ellipse E with certain (h, k) and rotation angle α has T_1 and T_2 as tangents.*

Proof. Note that $(h, k) \notin T_1 \cup T_2 \Rightarrow b_1 \neq 0 \neq b_2$

Case 1: $m_1^2 \neq m_2^2$

We want to choose α so that $A > 0$ and $B > 0$. Note that $\frac{m+\tan\alpha}{1-m\tan\alpha}$ is a strictly increasing function of m. Since $m_1^2 \neq m_2^2$, we can assume without loss of generality that $m_2 \neq 0$, and that $m_2^2 > m_1^2$. Let $\alpha_0 = \arctan(1/m_2)$. Then $\lim_{\alpha\to\alpha_0}\left(\frac{m_2+\tan\alpha}{1-m_2\tan\alpha}\right)^2 = \infty$ and $\lim_{\alpha\to\alpha_0}\left(\frac{b_2\sec\alpha}{1-m_2\tan\alpha}\right)^2 = \infty$ since $b_2 \neq 0$.

(Note that $\tan(\alpha_0) = 1/m_2 \Rightarrow m_2 + \tan\alpha_0 \neq 0$).

Hence $\lim_{\alpha\to\alpha_0} A = \infty$. Now $B = \left(\left(\frac{m_2+\tan\alpha}{1-m_2\tan\alpha}\right)^2 - \left(\frac{m_1+\tan\alpha}{1-m_1\tan\alpha}\right)^2\right)^2 \left(\frac{b_1\sec\alpha}{1-m_1\tan\alpha}\right)^2 - \left(\frac{m_1+\tan\alpha}{1-m_1\tan\alpha}\right)^2 A =$
$\left(\left(\frac{m_2+\tan\alpha}{1-m_2\tan\alpha}\right)^2 - \left(\frac{m_1+\tan\alpha}{1-m_1\tan\alpha}\right)^2\right) C$, where $C = \left((\frac{m_2+\tan\alpha}{1-m_2\tan\alpha})^2 - (\frac{m_1+\tan\alpha}{1-m_1\tan\alpha})^2\right)(\frac{b_1\sec\alpha}{1-m_1\tan\alpha})^2 - (\frac{m_1+\tan\alpha}{1-m_1\tan\alpha})^2\left(\left(\frac{b_2\sec\alpha}{1-m_2\tan\alpha}\right)^2 - \left(\frac{b_1\sec\alpha}{1-m_1\tan\alpha}\right)^2\right)$.

Clearly $\frac{m_{2,\alpha}}{b_{2,\alpha}} \to \infty$ as $\alpha \to \alpha_0$ since $-\frac{\pi}{2} < \alpha_0 < \frac{\pi}{2}$ implies that $\sec\alpha_0 \neq 0$. Hence $C \to \infty \Rightarrow B \to \infty$, as $\alpha \to \alpha_0$. Thus for α sufficiently close to α_0, $A > 0$ and $B > 0$.

Case 2: $0 \neq m_2 = -m_1$. Since $b_1 \neq 0 \neq b_2$, by Theorem 5, we need to choose α so that $b_{1,\alpha}^2 = b_{2,\alpha}^2$, which holds if and only if $(\frac{b_1\sec\alpha}{1-m_1\tan\alpha})^2 = (\frac{b_1\sec\alpha}{1+m_1\tan\alpha})^2 \Leftrightarrow \frac{b_1^2}{b_2^2} = \frac{1-m_1\tan\alpha}{1+m_1\tan\alpha} \Leftrightarrow \tan\alpha = \frac{1}{m_1}\frac{b_2^2-b_1^2}{b_2^2+b_1^2}$. Solving gives the required α.

■

2.1.1 Example

$T_1 : y = x + 2,\ T_2 : y = 2x + 1$ Then $(0, 0)$ is not admissible if one does not allow rotation. However, letting $\alpha = \arctan(1/m_2) = \arctan(.5) \approx .46365$ implies that $A > 0$ and $B > 0$. Hence the ellipse E with center $(0, 0)$ and rotation angle α has T_1 and T_2 as tangents.

Remark 5 *Clearly rotation is no help when $m_1 = m_2$. Given (h, k), by Theorem 5, we need to choose α so that $b_{1,\alpha}^2 = b_{2,\alpha}^2$. However, since $m_{1,\alpha} = m_{2,\alpha}$ for any α, $b_{1,\alpha}^2 = b_{2,\alpha}^2$ if and only if $b_1^2 = b_2^2$. In that case a proof similar to that of Theorem 5 shows that any angle α will do. If $b_1^2 \neq b_2^2$, no α will work.*

2.2 An Aside on Hyperbolas

Using similar techniques, it is not hard to prove a version of Theorem 6 for hyperbolas.

Theorem 8 *Suppose that we are given distinct lines T_1 and T_2, which are known to be tangent to some hyperbola H with center $C = (h, k)$. Suppose that T_j has equation*

$$y - k = m_j(x - h) + b_j$$

$j = 1, 2$. Finally, assume that $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $1 - m_1 \tan\alpha \neq 0 \neq 1 - m_2 \tan\alpha$.

Part 1:

(a) If

$$\frac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2} > 0 \text{ and } \frac{b_{2,\alpha}^2 m_{1,\alpha}^2 - b_{1,\alpha}^2 m_{2,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2} > 0 \tag{9}$$

, then the equation of H in the rotated coordinates (u, v) is $\dfrac{(u-h)^2}{a^2} - \dfrac{(v-k)^2}{b^2} = 1$, where $a^2 = \dfrac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$ and $b^2 = \dfrac{b_{2,\alpha}^2 m_{1,\alpha}^2 - b_{1,\alpha}^2 m_{2,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$[4]

(b) If

$$\frac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2} < 0 \text{ and } \frac{b_{2,\alpha}^2 m_{1,\alpha}^2 - b_{1,\alpha}^2 m_{2,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2} < 0 \tag{10}$$

, then the equation of H in the rotated coordinates (u, v) is $\dfrac{(v-k)^2}{a^2} - \dfrac{(u-h)^2}{b^2} = 1$, where $a^2 = -\dfrac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$ and $b^2 = -\dfrac{b_{2,\alpha}^2 m_{1,\alpha}^2 - b_{1,\alpha}^2 m_{2,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$

Part 2: If $m_{2,\alpha}^2 \neq m_{1,\alpha}^2$, then it follows that a necessary condition for a nonrotated hyperbola with T_1 and T_2 as tangents, and center (h, k), to exist is that $\dfrac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$ and $\dfrac{b_{2,\alpha}^2 m_{1,\alpha}^2 - b_{1,\alpha}^2 m_{2,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2}$ are either both positive or both negative.

2.2.1 Example

The lines $T_1 : \ y = x + 2$ and $T_2 : \ y = 2x - 3$ are given, and we want a hyperbola(nonrotated) H with center $(2, 0)$, such that T_1 and T_2 are tangents to H. Then $m_1 = 1$, $b_1 = 4$, $m_2 = 2$, $b_2 = 1$, which implies that $\frac{b_{2,\alpha}^2 - b_{1,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2} < 0$ and $\frac{b_{2,\alpha}^2 m_{1,\alpha}^2 - b_{1,\alpha}^2 m_{2,\alpha}^2}{m_{2,\alpha}^2 - m_{1,\alpha}^2} < 0$. By Theorem 8, Part 2, the hyperbola $\dfrac{y^2}{21} - \dfrac{(x-2)^2}{5} = 1$ is tangent to T_1 and T_2.

[4] Note that a^2 is the same as for the ellipse, but b^2 has the opposite sign.

Recall that for the example above there was no *ellipse*$(\alpha = 0)$ with center $(2, 0)$ with T_1 and T_2 as tangents. This leads to the question of whether, for fixed center $(h, k) \notin T_1 \cup T_2$ and fixed rotation angle α, there is always either a hyperbola or ellipse which is tangent to T_1 and T_2. The following theorem answers this in the affirmative. For simplicity of exposition, we first state the result for nonrotated conics.

Theorem 9 *(Nonrotated Version)Let h and k be given real numbers, and let T_1 and T_2 be distinct, non–vertical lines with equations $y - k = m_j(x - h) + b_j,\ j = 1, 2$. Assume that $(h, k) \notin T_1 \cup T_2$, and that $m_1^2 \neq m_2^2$. Then there is either a non-rotated hyperbola or a non-rotated ellipse, with center (h, k), which is tangent to T_1 and T_2.*

Proof. Assume, w.l.o.g., that $m_2^2 - m_1^2 > 0$. It is clear that *both* (A) and (B) of Theorem 4 and eq(9) cannot hold. Similarly, *both* (A) and (B) of Theorem 4 and eq(10) cannot hold. Hence there cannot be *both* an ellipse and a hyperbola, with center (h, k), which is tangent to T_1 and T_2. Now suppose that there is no ellipse, with center (h, k), which is tangent to T_1 and T_2. By Theorem 4, Part 2, either $b_2^2 - b_1^2 < 0$ or $b_1^2 m_2^2 - b_2^2 m_1^2 < 0$. In the first case, $b_2^2 m_1^2 - b_1^2 m_2^2 < b_2^2 m_2^2 - b_1^2 m_2^2 = m_2^2(b_2^2 - b_1^2) < 0$. By Theorem 8, Part 1, there is a hyperbola, with center (h, k), which is tangent to T_1 and T_2. In the second case, $b_2^2 m_1^2 - b_1^2 m_2^2 > 0$ and $m_2^2(b_2^2 - b_1^2) > b_2^2 m_1^2 - b_1^2 m_2^2 > 0 \Rightarrow b_2^2 - b_1^2 > 0$. Again, by Theorem 8, Part 1, there is a hyperbola, with center (h, k), which is tangent to T_1 and T_2. Finally, suppose that there is no hyperbola, with center (h, k), which is tangent to T_1 and T_2. By Theorem 8, Part 2, either $b_2^2 - b_1^2 > 0$ and $b_2^2 m_1^2 - b_1^2 m_2^2 < 0$, or $b_2^2 - b_1^2 < 0$ and $b_2^2 m_1^2 - b_1^2 m_2^2 > 0$. In the first case, $b_1^2 m_2^2 - b_2^2 m_1^2 > 0$. By Theorem 4, Part 1, there is an ellipse, with center (h, k), which is tangent to T_1 and T_2. Since the second case cannot occur, that completes the proof of the theorem. $\blacksquare$.

The proof of the rotated version of Theorem 9 follows in a similar fashion by replacing m_j by $m_{j,\alpha}$ and b_j by $b_{j,\alpha}$. We leave the details to the reader.

Theorem 10 *(Rotated Version)Let h and k be given real numbers, and let T_1 and T_2 be distinct lines with equations $y - k = m_j(x - h) + b_j,\ j = 1, 2$. Suppose that $-\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$ and that $1 - m_1 \tan\alpha \neq 0 \neq 1 - m_2 \tan\alpha$. Finally, assume that $(h, k) \notin T_1 \cup T_2$, and that $m_{1,\alpha}^2 \neq m_{2,\alpha}^2$. Then there is either a hyperbola or an ellipse, with center (h, k) and rotation angle α, which is tangent to T_1 and T_2.*

3 Three Given Tangents

To simplify things, at first we only consider nonrotated ellipses. Later we indicate how the results can be modified for a fixed angle of rotation. Most interesting is what happens when one allows the angle of rotation to vary.

Given three lines $T_j : y = m_j x + c_j$, $j = 1, 2, 3$, we want to find an ellipse $E : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ which is tangent to each of the T_j. If we consider each pair of tangents $\{T_i, T_j\}$, we can find an ellipse $E_{i,j}$, with center $C = (h, k)$, which is tangent to T_i and T_j, using Theorem 4. The formula for a^2 and b^2 is given by (7), provided the a^2 and b^2 obtained from each pair of tangents are equal. This yields the curve γ below, which is a hyperbola, on which the admissible centers must lie. Several cases must be considered depending upon whether $m_i^2 = m_j^2$ and/or $m_i = m_j$ for some $i \neq j$. Of course it is geometrically obvious that one must also assume that the T_j do not have a common intersection point.

3.1 No two of the T_j have slopes equal in absolute value

We first state our results when none of the T_j is horizontal or vertical–that is, when none of the T_j are parallel to the major or minor axis of the ellipse. In that case we shall show that the set of admissible centers is given precisely by

$\{(h, k) \in \gamma : q_3(k) > 0 \text{ and } r_3(h) > 0\}$, where γ is a hyperbola and q_3 and r_3 are polynomials of degree three with all real zeroes. Further, the zeroes of q_3 and r_3 can be obtained from the intersection points of the T_j using γ. Similar, but less concise results, can be proved if one of the T_j is horizontal or vertical.

Theorem 11 *Let T_j, $j = 1, 2, 3$ be three distinct non-horizontal and non–vertical lines with equations $y = m_j x + c_j$, and assume that $i \neq j \Rightarrow m_i^2 \neq m_j^2$. Assume also that the T_j do not have a common intersection point. Let (x_1, y_1) equal the intersection point of T_2 and T_3, (x_2, y_2) equal the intersection point of T_1 and T_3, and (x_3, y_3) equal the intersection point of T_1 and T_2.*[5] *Let $D = (m_2 - m_1)(m_3 - m_2)(m_3 - m_1)$*

Let γ be the curve with equation

[5] In this section (x_j, y_j) denotes the intersection point of two tangents, and *not* the point of tangency, as in the previous section.

$$Dhk + a_1 h + a_2 k + a_3 = 0 \tag{11}$$

where
$a_1 = m_1 c_1\,(m_3^2 - m_2^2) - m_2 c_2\,(m_3^2 - m_1^2) + m_3 c_3\,(m_2^2 - m_1^2)$,
$a_2 = -c_1\,(m_3^2 - m_2^2) + c_2\,(m_3^2 - m_1^2) - c_3\,(m_2^2 - m_1^2)$,
$a_3 = (c_1^2(m_3^2 - m_2^2) + c_2^2(m_1^2 - m_3^2) + (m_2^2 - m_1^2)c_3^2)/2$

Write equation (11) in the form $k = f(h)$ *or* $h = g(k)$. *Let* $v_j = f(x_j)$, $w_j = g(y_j)$, $j = 1, 2, 3$, *and let* $r_3(h) = \tau(h - w_1)(h - w_2)(h - w_3)$, $q_3(k) = \beta(k - v_1)(k - v_2)(k - v_3)$, *where* $\tau = -m_1 m_2 m_3 \dfrac{x_2 - x_3}{m_3 - m_2}$ *and* $\beta = \dfrac{x_2 - x_3}{m_3 - m_2}$. *Let* S *denote the set of admissible centers* (h, k). *Then*

$$S = \{(h, k) \in \gamma : q_3(k) > 0 \text{ and } r_3(h) > 0\} \tag{12}$$

For each $(h, k) \in S$, *if* E *is the ellipse* $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$ *which is tangent to each of the* T_j, *then* E *is unique, and* a^2 *and* b^2 *are given by (18) and (19) below, with* $b_j = m_j h + c_j - k$.

Finally, given any $0 \le e_0 < 1$, *there is an* $(h, k) \in S$ *such that* E *has eccentricity* e_0.

Proof.

Note that $x_i = x_j$ for $i \neq j$ if and only if the T_j have a common intersection point. Since none of the T_j are horizontal, it also follows that $y_i = y_j$ for $i \neq j$ if and only if the T_j have a common intersection point. Since we assumed that the T_j do **not** have a common intersection point,

$$x_i \neq x_j \text{ and } y_i \neq y_j \text{ for } i \neq j \tag{13}$$

Solving (11) for k yields

$$k = f(h) = -\frac{a_1 h + a_3}{Dh + a_2} \tag{14}$$

and solving (11) for h yields

$$h = g(k) = -\frac{a_2 k + a_3}{Dk + a_1} \tag{15}$$

For any distinct $\{i, j, k\}$ from $\{1, 2, 3\}$, T_i and T_j intersect at

$$(x_l, y_l) = (\frac{c_j - c_i}{m_i - m_j}, \frac{m_i c_j - m_j c_i}{m_i - m_j}) \tag{16}$$

We shall use the following easily proven identity many times.

For any $i, j, l \in \{1, 2, 3\}$

$$(m_l - m_i)(m_j - m_i)(x_j - x_l) = -c_2 m_3 + c_2 m_1 + c_1 m_3 + c_3 m_2 - c_3 m_1 - c_1 m_2 \tag{17}$$

Using (17), it is not hard to show that

$Dx_l + a_2 = (m_i^2 - m_j^2)(m_l - m_i)(x_j - x_l)$,

$Dy_l + a_1 = m_l(m_j^2 - m_i^2)(m_l - m_i)(x_j - x_l)$

Since $x_j \neq x_l$ for $j \neq l$ by (13), $Dx_l + a_2 \neq 0$ and $Dy_l + a_1 \neq 0$ for any l. Hence by (14) and (15), $v_j = f(x_j)$ and $w_j = g(y_j)$ are each *finite*.

Now given $C = (h, k)$, write T_j in the form $y - k = m_j(x - h) + b_j$, $j = 1, 2, 3$. By Theorem 4, for each pair of tangents $\{T_i, T_j\}$, we can find an ellipse $E_{i,j}$, with center C which is tangent to T_i and T_j, if and only if

$$\frac{b_j^2 - b_i^2}{m_j^2 - m_i^2} > 0 \text{ and } \frac{b_i^2 m_j^2 - b_j^2 m_i^2}{m_j^2 - m_i^2} > 0$$

Since we want E to be tangent to all three lines, we need the a^2 and b^2 obtained from each pair of tangents to be **equal**. First, using (7) for a^2, we want

$$\frac{b_2^2 - b_1^2}{m_2^2 - m_1^2} = \frac{b_3^2 - b_2^2}{m_3^2 - m_2^2} = \frac{b_3^2 - b_1^2}{m_3^2 - m_1^2} \tag{18}$$

Second, using (7) for b^2, we want

$$\frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2} = \frac{b_1^2 m_3^2 - b_3^2 m_1^2}{m_3^2 - m_1^2} = \frac{b_2^2 m_3^2 - b_3^2 m_2^2}{m_3^2 - m_2^2} \tag{19}$$

For simplicity, let $e_j = b_j^2$ and $f_j = m_j^2$, $j = 1, 2, 3$. Cross multiplying in (18) and (19) yields a system of four polynomial equations in the variables e_j and f_j :

$s_1 \equiv (e_2 - e_1)(f_3 - f_1) - (e_3 - e_1)(f_2 - f_1) = 0$, $s_2 \equiv (e_2 - e_1)(f_3 - f_2) - (e_3 - e_2)(f_2 - f_1) = 0$, $s_3 \equiv (e_1 f_2 - e_2 f_1)(f_3 - f_1) - (e_1 f_3 - e_3 f_1)(f_2 - f_1) = 0$, $s_4 \equiv (e_1 f_2 - e_2 f_1)(f_3 - f_2) - (e_2 f_3 - e_3 f_2)(f_2 - f_1) = 0$. We found a

Grobner Basis(see [5]) GB for $\{s_1, s_2, s_3, s_4\}$ using Maple. GB contains the one polynomial

$(e_2-e_3)f_1+(e_3-e_1)f_2+(e_1-e_2)f_3$. Hence the solution set of the system of equations in (18) and (19) is equivalent to the one equation

$$(b_2^2-b_3^2)m_1^2+(b_3^2-b_1^2)m_2^2+(b_1^2-b_2^2)m_3^2=0 \tag{20}$$

Expanding the LHS of (20) and using $b_j = m_jh + c_j - k$ yields

$(b_2^2-b_3^2)m_1^2+(b_3^2-b_1^2)m_2^2+(b_1^2-b_2^2)m_3^2 = 2m_2^2c_1k-2m_1^2c_2k-m_2^2c_1^2+m_1^2c_2^2-2m_2^2m_1hc_1+2m_2^2m_1hk+2m_1^2m_2hc_2-2m_1^2m_2hk-m_1^2c_3^2+m_2^2c_3^2+m_3^2c_1^2-m_3^2c_2^2+2m_1^2c_3k-2m_2^2c_3k-2m_3^2c_1k+2m_3^2c_2k-2m_1^2m_3hc_3+2m_1^2m_3hk+2m_2^2m_3hc_3-2m_2^2m_3hk+2m_3^2m_1hc_1-2m_3^2m_1hk-2m_3^2m_2hc_2+2m_3^2m_2hk$

Dividing thru by 2 and simplifying gives eq.(11).

Claim: (14) never reduces to k = constant and (15) never reduces to h = constant.

Proof. of claim: To prove the claim, we must show that $a_1a_2 - a_3D \neq 0$. Now $a_1a_2 - a_3D =$

$-\frac{1}{2}(m_3+m_2)(m_2+m_1)(m_3+m_1)(m_1c_2-m_3c_2-m_1c_3+m_2c_3+m_3c_1-m_2c_1)^2 =$

$-\frac{1}{2}(m_3+m_2)(m_2+m_1)(m_3+m_1)\left(\frac{x_3-x_2}{(m_1-m_3)(m_1-m_2)}\right)^2 \neq 0$ by (17). Since $m_i \neq -m_j$ for $i \neq j$ and $x_2 \neq x_3$, $a_1a_2 - a_3D \neq 0$.

Note that γ is a nontrivial curve since $D \neq 0$, and that part of γ must lie inside the triangle T formed by the T_j since the incenter of T must lie on γ.

If $b_1 = m_1h + c_1 - k$, $b_2 = m_2h + c_2 - k$, then by (14),

$\frac{b_2^2-b_1^2}{m_2^2-m_1^2} = \left(\frac{1}{m_2^2-m_1^2}\right)((m_2-m_1)h+c_2-c_1)((m_1+m_2)h+c_1-2k+c_2)$

$=\frac{p_3(h)}{p_1(h)}$, where p_3 is a monic polynomial of degree ≤ 3 and $p_1(h) = h + \frac{a_2}{D}$, and by (15),

$\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2} =$

$-\left(\frac{1}{m_2^2-m_1^2}\right)((m_1-m_2)k+m_2c_1-m_1c_2)((m_1+m_2)k-2m_1m_2h-m_1c_2-m_2c_1)$

$=\frac{s_3(k)}{s_1(k)}$, where s_3 is a monic polynomial of degree ≤ 3 and $s_1(k) = k + \frac{a_1}{D}$

We can determine the zeroes of p_3 and s_3 as follows:

Using the fact that $\frac{c_j-c_i}{m_i-m_j} = \frac{b_j-b_i}{m_i-m_j} + h$ and $\frac{m_ic_j-m_jc_i}{m_i-m_j} = \frac{m_ib_j-m_jb_i}{m_i-m_j} + k$, we have

$$x_l = \frac{b_j-b_i}{m_i-m_j} + h,\ y_l = \frac{m_ib_j-m_jb_i}{m_i-m_j} + k$$

Hence $b_j - b_i = 0$ if and only if $h = x_l$, and $m_ib_j - m_jb_i = 0$ if and only if $k = y_l$.

This implies that the polynomials $p_3(h)$ and $s_3(k)$ above have all real roots.

p_3 vanishes at the x coordinates of the intersection points of the T_j, and s_3 vanishes at the y coordinates of the intersection points of the T_j. Hence $p_3(h) = (h - x_1)(h - x_2)(h - x_3)$ and $s_3(k) = (k - y_1)(k - y_2)(k - y_3)$.

Now we want to know when both $\frac{b_2^2 - b_1^2}{m_2^2 - m_1^2}$ and $\frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2}$ are positive for $(h, k) \in \gamma$.

First, substituting $h = g(k)$, one can write $\frac{p_3(h)}{p_1(h)} = \frac{(g(k)-x_1)(g(k)-x_2)(g(k)-x_3)}{g(k)+\frac{a_2}{D}}$,

which, using the fact that $v_j = f(x_j)$, vanishes when $k = v_j,\ j = 1, 2, 3$. Simplifying yields

$\frac{p_3(h)}{p_1(h)} = \frac{2((m_2-m_1)a_2-(c_2-c_1)D)}{(m_2^2-m_1^2)D} \frac{(k-v_1)(k-v_2)(k-v_3)}{\left(k+\frac{a_1}{D}\right)^2} = 2\frac{x_2-x_3}{m_3-m_2} \frac{(k-v_1)(k-v_2)(k-v_3)}{\left(k+\frac{a_1}{D}\right)^2}$,

which is positive precisely when $q_3(k) > 0$. Thus $\frac{b_2^2 - b_1^2}{m_2^2 - m_1^2} > 0$ precisely when $q_3(k) > 0$. Similarly, substituting $k = f(h)$, one can write

$\frac{s_3(k)}{s_1(k)} = -2m_1 m_2 m_3 \frac{x_2-x_3}{m_3-m_2} \frac{(f(h)-y_1)(f(h)-y_2)(f(h)-y_3)}{\left(h+\frac{a_2}{D}\right)^2}$,

which, using the fact that $w_j = g(y_j)$, vanishes when $h = w_j,\ j = 1, 2, 3$. Hence $\frac{s_3(k)}{s_1(k)} = -2m_1 m_2 m_3 \frac{x_2-x_3}{m_3-m_2} \frac{(h-w_1)(h-w_2)(h-w_3)}{\left(h+\frac{a_2}{D}\right)^2}$, which is positive precisely when $r_3(h) > 0$. Thus $\frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2} > 0$ precisely when $r_3(h) > 0$.

Since $h + \frac{a_2}{D} = 0$ or $k + \frac{a_1}{D} = 0$ cannot yield a point on γ, the set of admissible centers is given by (12). Note that $S \neq \emptyset$ since the incenter of T is admissible. One could also prove directly that S is nonempty. The *uniqueness* follows from Theorem 3.

Finally, to prove that there is $(h, k) \in S$ such that E has eccentricity e_0: Using (16), it follows easily that for any set of distinct $i, j, l \in \{1, 2, 3\}$,

$r_3(x_i) = \frac{1}{8}(m_j + m_l)^2 \frac{(m_j-m_i)^2(m_l-m_i)^2(x_j-x_l)^4}{(m_j-m_l)^4}$, which implies that

$$r_3(x_i) \neq 0 \text{ for any } i \tag{21}$$

and $r_3(-\frac{a_2}{D}) = \frac{1}{8}\left(m_1 + m_2\right)^2 \left(x_3 - x_2\right)^4 \left(m_1 + m_3\right)^2 \frac{(m_3+m_2)^2}{(m_3-m_2)^4}$, which implies that

$$r_3(-\tfrac{a_2}{D}) \neq 0 \tag{22}$$

Also, $p_3(g(y_i)) = \frac{1}{8} \frac{(m_i+m_j)(m_i+m_l)(x_j-x_l)^3(m_i-m_j)^2(m_i-m_l)^2}{m_i^3(m_l-m_j)^3}$ and $p_1(g(y_i)) = \frac{1}{2} \frac{(m_i+m_j)(m_i+m_l)(x_j-x_l)}{m_i(m_l-m_j)}$, which implies that

$$p_3(w_i)p_1(w_i) \neq 0 \tag{23}$$

Note that from the proof of Theorem 11, since $\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2} = \frac{p_3(h)}{p_1(h)}$,

$$S = \{(h, f(h)) : \frac{p_3(h)}{p_1(h)} > 0 \text{ and } r_3(h) > 0\} \tag{24}$$

It follows from(7) that $\frac{a^2}{b^2} = \left(\frac{b_2^2-b_1^2}{m_2^2-m_1^2}\right) / \left(\frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}\right) = \frac{p_1(h)p_3(h)}{r_3(h)} \equiv R(h)$. Let T be the triangle enclosed by the T_j, and let (h_0, k_0) equal the incenter of T. Then $R(h_0) = 1, p_1(h_0)p_3(h_0) > 0$, and $r_3(h_0) > 0$. Let $s = \inf\{h < h_0 : p_1(h)p_3(h) > 0 \text{ and } r_3(h) > 0\}$. Since $x_j < h_0$ for some j, $-\infty < s$. First, if $r_3(s) = 0$, then $s = w_j \Rightarrow p_1(s)p_3(s) \neq 0$ by (23), and thus $\lim_{h\to s^+} R(h) = \infty$. In that case $\frac{b^2}{a^2}$ varies from 0 to 1 on (s, h_0), and the equation $1 - \frac{b^2}{a^2} = e_0^2$ has a solution (h, k) which is an admissible center by (24). Second, if $p_1(s)p_3(s) = 0$, then $s = x_j$ or $s = -\frac{a_2}{D} \Rightarrow r_3(s) \neq 0$ by (21) or (22), and thus $\lim_{h\to s^+} R(h) = 0$. In that case $\frac{a^2}{b^2}$ varies from 0 to 1 on (s, h_0), and again the equation $1 - \frac{b^2}{a^2} = e_0^2$ has a solution (h, k) which is an admissible center by (24).

■

3.1.1 Example

$T_1 : y = x + 1,\ T_2 : y = 2x - 2,\ T_3 : y = -3x$. The intersection points of the $T_j's$ are $(\frac{2}{5}, -\frac{6}{5}), (-\frac{1}{4}, \frac{3}{4}), (3, 4)$, and γ has equation $40hk + 74h - 27 - 42k = 0$. Hence $f(h) = -\frac{74h-27}{40h-42}$ and $g(k) = \frac{42k+27}{40k+74}$. It follows that

$p_3(h) = (h - 3)\left(h - \frac{2}{5}\right)\left(h + \frac{1}{4}\right)$, $s_3(k) = (k + \frac{6}{5})(k - 4)(k - \frac{3}{4})$, $r_3(h) = \frac{39}{10}\left(h - \frac{9}{16}\right)\left(h + \frac{9}{10}\right)(h - \frac{5}{6})$,

$q_3(k) = \frac{13}{10}(k + \frac{5}{2})(k + \frac{7}{8})(k - \frac{1}{10})$, $p_1(h) = h - \frac{21}{20}$, and $s_1(k) = k + \frac{37}{20}$. Hence

$S = \{(h, k) \in \gamma : h > \frac{5}{6} \text{ or } -\frac{9}{10} < h < \frac{9}{16}\} \cap \{(h, k) \in \gamma : k > \frac{1}{10} \text{ or } -\frac{5}{2} < k < -\frac{7}{8}\} \Rightarrow$

$$S = \{(h, -\tfrac{74h-27}{40h-42}) : -\tfrac{9}{10} < h < \tfrac{9}{16} \text{ or } h > 3\}$$

3.1.2 One of the Tangents is Horizontal or Vertical

We now prove a result similar to Theorem 11 when one of the given lines, say T_3, is horizontal. Similar results can be proved if one of the lines is vertical by solving for x in terms of y in the equations of the T_j.

Theorem 12 *Let T_j, $j = 1, 2, 3$ be three distinct non–vertical lines with equations $y = m_j x + c_j$, and assume that $0 \neq m_1^2 \neq m_2^2 \neq 0$ and $m_3 = 0$. Assume also that the T_j do not have a common intersection point. Let (x_1, c_3) equal the intersection point of T_2 and T_3, (x_2, c_3) equal the intersection point of T_1 and T_3, and (x_3, y_3) equal the intersection point of T_1 and T_2. Let $D = m_1 m_2 (m_2 - m_1)$ and let γ be the curve with equation $Dhk + a_1 h + a_2 k + a_3 = 0$, where $a_1 = -m_2 m_1 (-m_1 c_2 + m_2 c_1)$,*

$a_2 = (c_1 - c_3) m_2^2 - (c_2 - c_3) m_1^2$, and $a_3 = (-c_1^2 m_2^2 + c_2^2 m_1^2 + (m_2^2 - m_1^2) c_3^2)/2$. Write equation (11) in the form $k = f(h)$ or $h = g(k)$. Let $v_j = f(x_j)$, $j = 1, 2, 3$, and let $q_3(k) = \beta(k - v_1)(k - v_2)(k - v_3)$, where $\beta = \dfrac{x_2 - x_3}{m_3 - m_2}$. Let S denote the set of admissible centers (h, k). Then

$$S = \{(h, k) \in \gamma : q_3(k) > 0 \text{ and } h \neq g(c_3)\}$$

For each $(h, k) \in S$, if E is the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ which is tangent to each of the T_j, then E is unique, and a^2 and b^2 are given by (18) and (19), with $b_j = m_j h + c_j - k$.

Finally, given any $0 \leq e_0 < 1$, there is an $(h, k) \in S$ such that E has eccentricity e_0.

Proof. We sketch the proof, since most of the details follow exactly as in the exactly as in the proof of Theorem 11. In particular, letting $m_3 = 0$ in (11) gives the curve γ above. $p_3(h)$, $p_1(h)$, $q_3(k)$, and $s_1(k)$ are the same, with $\frac{b_2^2 - b_1^2}{m_2^2 - m_1^2} = \frac{p_3(h)}{p_1(h)} = \frac{q_3(k)}{(s_1(k))^2}$, $k = f(h)$. However $m_3 = 0$ implies that $y_1 = y_2 = c_3$, and it follows that $\frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2} = \frac{s_3(k)}{s_1(k)}$, where $s_3 = (k - c_3)^2 (k - y_3)$ and $s_1(k) = k + \frac{a_1}{D}$.

If $k = f(h)$, then

$\frac{s_3(k)}{s_1(k)} = \frac{s_3(f(h))}{s_1(f(h))} =$

$\frac{1}{4}(m_2 c_1 - m_1 c_2 - m_2 c_3 + m_1 c_3)^2 \frac{(m_2 c_1 + 2 m_2 m_1 h - m_2 c_3 + m_1 c_2 - m_1 c_3)^2}{\left(m_2^2 m_1 h - m_2 m_1^2 h + c_1 m_2^2 - c_2 m_1^2 - c_3 m_2^2 + c_3 m_1^2\right)^2} = \frac{r_2(h)}{\left(h + \frac{a_2}{D}\right)^2}$,

where $r_2(h) = \alpha(h - w)^2$, $\alpha = m_1^2 (x_3 - x_2)^2$. Since $r_2(h) > 0$ for $h \neq w$, the set of admissible centers is given by S. Finally, to prove that there is

$(h,k) \in S$ such that E has eccentricity e_0: First, it is not hard to show that $w \neq x_j$ for any j, $w \neq -\frac{a_2}{D}$, $p_1(w) \neq 0$, and $p_3(w) \neq 0$. The rest of the proof follows exactly as in the proof of Theorem 11. ■

Example $T_1 : y = x + 1$, $T_2 : y = 2x - 2$, $T_3 : y = 0$. The intersection points of the T_j are

$(1,0), (-1,0), (3,4)$, and γ has equation $h(k-4) + 3k = 0$. $q_3(k) = 4(k-1)(k-2)(k+2)$ and $r_2(h) = 16h^2$. It follows that $S = \{(h,k) \in \gamma : -2 < k < 1 \text{ or } k > 2,\ k \neq 0\}$. See Figure 3.1 for a plot of γ, the tangents, and the particular ellipse $\frac{(x-3/7)^2}{30/49} + \frac{(y-1/2)^2}{1/4} = 1 (h = 3/7)$ tangent to all three given lines.

Finally, it is interesting to consider the case when T_1 is vertical and T_2 is horizontal.[6]

Theorem 13 *Given the lines $T_1 : x = c_1, T_2 : y = c_2$, and $T_3 : y = m_3 x + c_3$, with $m_3 \neq 0$, let S be the set of admissible centers. Then*

$$S = \{(h,k) : 2m_3hk - 2m_3(m_3c_1 + c_3)h + 2(c_3 - c_2)k = c_3^2 - c_2^2 - m_3^2c_1^2, h \neq c_1, k \neq c_2\}$$

Also, if E is the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ which is tangent to each of the T_j, then $a = c_1 - h$ and $b = c_2 - k$. Finally, given any $0 \leq e_0 < 1$, there is an $(h,k) \in S$ such that E has eccentricity e_0.

Proof. Given (h,k), write $T_1 : x - h = b_1, T_2 : y - k = b_2$, and $T_3 : y - k = m_3(x-h) + b_3$. Clearly, $a = b_1$ and $b = b_2$ is both necessary and sufficient for E to be tangent to T_1 and T_2. Applying Theorem 4(with $m_2 = 0$) to T_2 and to T_3, respectively, implies the equations $\frac{b_3^2 - b_2^2}{m_3^2} = b_1^2$ and $\frac{b_2^2 m_3^2}{m_3^2} = b_2^2$, and hence we have the one condition $b_3^2 - b_2^2 = b_1^2 m_3^2$. Using $b_1 = c_1 - h, b_2 = c_2 - k$, and $b_3 = m_3h + c_3 - k$ yields the above equation defining S. Note that $h \neq c_1$ and $k \neq c_2$ implies that $b_1 \neq 0 \neq b_2$

We omit the proof that the eccentricity can be specified in advance. ■

3.2 Two of the T_j have slopes equal in absolute value

Theorem 14 *Let T_j, $j = 1, 2, 3$ be three distinct non–vertical lines with equations $y = m_jx + c_j$, and assume that $m_1^2 = m_2^2 \neq m_3^2$. Assume also that*

[6] We do not discuss the case when *two* of the given lines are horizontal or vertical, which is trivial.

the T_j do not have a common intersection point. Let S denote the set of admissible centers (h,k).

Case 1: $m_2 = -m_1 \neq 0$

Let $h_0 = \dfrac{c_2 - c_1}{2m_1}$, $k_0 = \dfrac{c_1 + c_2}{2}$. Then there exist intervals I_1 and I_2 of the form $(-\infty, s)$ and/or (t, ∞) such that $S = \{(h_0, k) : k \in I_1\} \cup \{(h, k_0) : h \in I_2\}$

Case 2: $m_2 = m_1$

Let (x_1, y_1) and (x_2, y_2) be the intersection points of $\{T_1, T_3\}$ and $\{T_2, T_3\}$, respectively. [7] *Let L be the line with equation $y = m_1 x + \frac{1}{2}c_1 + \frac{1}{2}c_2$.*

(1) If $\dfrac{m_3 - m_1}{m_3 + m_1} > 0$, then assume, without loss of generality, that $x_1 < x_2$.[8]*. If $m_1 \neq 0$, let $u_j = \dfrac{y_j - \frac{1}{2}(c_1 + c_2)}{m_1}$, $j = 1, 2$.*

If $m_1 m_3 > 0$, then $S = \{(h,k) \in L : u_1 < h < x_1 \text{ or } x_2 < h < u_2\}$.

If $m_1 m_3 < 0$, then $S = \{(h,k) \in L : u_2 < h < x_1 \text{ or } x_2 < h < u_1\}$.

If $m_1 = 0$, then $S = \{(h, \frac{1}{2}c_1 + \frac{1}{2}c_2) : h < x_1\} \cup \{(h, \frac{1}{2}c_1 + \frac{1}{2}c_2) : h > x_2\}$.

(2) If $\dfrac{m_3 - m_1}{m_3 + m_1} < 0$, then assume, without loss of generality, that $y_1 < y_2$.[9]*. Let $v_j = m_1 x_j + \frac{1}{2}c_1 + \frac{1}{2}c_2$, $j = 1, 2$.*

If $m_1 m_3 > 0$, then $S = \{(h,k) \in L : v_1 < k < y_1 \text{ or } y_2 < h < v_2\}$.

If $m_1 m_3 < 0$, then $S = \{(h,k) \in L : v_2 < k < y_1 \text{ or } y_2 < h < v_1\}$.

For both case 1 and case 2, for each $(h,k) \in S$, if E is the ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ which is tangent to each of the T_j, then E is unique, and a^2 and b^2 are given by (26) and (27) below, with $b_j = m_j h + c_j - k$.

Finally, given any $0 \leq e_0 < 1$, there is an $(h,k) \in S$ such that E has eccentricity e_0.

Proof. Given $C = (h,k)$, write T_j in the form $y - k = m_j(x - h) + b_j$, $j = 1, 2, 3$. By Theorem 5, in order for an ellipse $E : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ to exist which is tangent to T_1 and T_2, we must have $b_1^2 = b_2^2$, and in that case

$$a^2 m_j^2 + b^2 = b_j^2,\ j = 1, 2 \tag{25}$$

[7] This notation differs slightly from that used in Theorem 11.

[8] If $x_1 = x_2$, then the T_j have a common intersection point.

[9] If $y_1 = y_2$, then the T_j have a common intersection point.

Again, by Theorem 3, since we need the a^2 and b^2 obtained from the pairs of tangents $\{T_1, T_3\}$ and $\{T_2, T_3\}$ to be equal, we want

$$a^2 = \frac{b_3^2 - b_2^2}{m_3^2 - m_2^2} = \frac{b_3^2 - b_1^2}{m_3^2 - m_1^2} \tag{26}$$

and

$$b^2 = \frac{b_1^2 m_3^2 - b_3^2 m_1^2}{m_3^2 - m_1^2} = \frac{b_2^2 m_3^2 - b_3^2 m_2^2}{m_3^2 - m_2^2} \tag{27}$$

Using (26) and (27), and substituting into (25), with $j = 2$, yields

$$\left(\frac{b_3^2 - b_2^2}{m_3^2 - m_2^2}\right) m_2^2 + \frac{b_1^2 m_3^2 - b_3^2 m_1^2}{m_3^2 - m_1^2} = b_2^2 \tag{28}$$

It is easy to show that (26), (27), and (28), along with $m_1^2 = m_2^2$, is equivalent to the one condition $b_1^2 = b_2^2$.

Note: The condition $(b_2^2 - b_3^2)m_1^2 + (b_3^2 - b_1^2)m_2^2 + (b_1^2 - b_2^2)m_3^2 = 0$ from Section 3.1 still holds, though for Section 3.2, of course, it takes the simpler form $b_1^2 = b_2^2$.

Also, using (26) to define a^2 and (27) to define b^2 implies that (25) holds.

The condition $b_1^2 = b_2^2$ implies $(m_1 h + c_1 - k)^2 - (m_2 h + c_2 - k)^2$, which implies(using $m_1^2 = m_2^2$) that

$$(2m_2 - 2m_1)hk + (2m_1 c_1 - 2m_2 c_2)h + (2c_2 - 2c_1)k + c_1^2 - c_2^2 = 0 \tag{29}$$

Recall that (x_1, y_1) and (x_2, y_2) are the intersection points of $\{T_1, T_3\}$ and $\{T_2, T_3\}$, respectively. By (16),

$$x_1 = \frac{c_3 - c_1}{m_1 - m_3},\ x_2 = \frac{c_3 - c_2}{m_2 - m_3},\ y_1 = \frac{m_1 c_3 - m_3 c_1}{m_1 - m_3},\ y_2 = \frac{m_2 c_3 - m_3 c_2}{m_2 - m_3}$$

3.3 *Proof of Case 1:* $m_2 = -m_1 \neq 0$

We shall show that the set of admissible centers is the union of an open, semi–infinite vertical line with an open, semi–infinite horizontal line.

(29) reduces to

$-4m_1 hk + 2m_1(c_1 + c_2)h + 2(c_2 - c_1)k + c_1^2 - c_2^2$[10] $= -(2k - c_1 - c_2)(2m_1 h - c_2 + c_1) = 0 \Rightarrow k = \frac{c_1 + c_2}{2}$ or $h = \frac{c_2 - c_1}{2m_1}$. Note that $\left(\frac{c_2 - c_1}{2m_1}, \frac{c_1 + c_2}{2}\right)$ is the intersection point

[10] One can also obtain this equation by letting $m_2 = -m_1$ in (11).

of T_1 and T_2 since $m_2 = -m_1$. Hence the admissible centers must lie on $\gamma = \{(h,k) : h = \frac{c_2-c_1}{2m_1}$ or $k = \frac{c_1+c_2}{2}\}$. We must determine the points on γ where $\frac{b_3^2-b_1^2}{m_3^2-m_1^2}$ and $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2}$ are positive.

Recall that $b_1 = m_1h + c_1 - k$, $b_3 = m_3h + c_3 - k$. Then $\frac{b_3^2-b_1^2}{m_3^2-m_1^2} =$ $((m_3-m_1)h + c_3 - c_1)\left((m_1+m_3)h - 2k + c_3 + c_1\right)/(m_3^2 - m_1^2) =$

$$(h - x_1)(h - \frac{2k - c_3 - c_1}{m_3 + m_1}) \qquad (30)$$

and $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2} = ((m_1-m_3)k - m_1c_3 + m_3c_1)\left((m_1+m_3)k - 2m_3m_1h - m_1c_3 - m_3c_1\right)/(m_1^2 - m_3^2) =$

$$(k - y_1)(k - \frac{2m_3m_1h + m_1c_3 + m_3c_1}{m_3 + m_1}) \qquad (31)$$

Recall that $h_0 = \frac{c_2-c_1}{2m_1}$, $k_0 = \frac{c_1+c_2}{2}$

(i) Suppose that $h = h_0$: Then, by (30), $\frac{b_3^2-b_1^2}{m_3^2-m_1^2}$ is a linear function of k,

$$L(k) = (h_0 - x_1)(h_0 - \frac{2k - c_3 - c_1}{m_3 + m_1})$$

and it is not hard to show, using (30), that $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2}$ is given by

$$P(k) = (k - y_1)(k - y_2)$$

Now T_1, T_2, T_3 have a common intersection point if and only if $x_1 = h_0$. Also, $L(k)$ is identically zero if and only if $(m_3 - m_1)h + c_3 - c_1 = 0$, which holds if and only if $h = x_1$. Since we assumed that T_1, T_2, T_3 do **not** have a common intersection point, L is not identically the zero function and thus has degree 1.

Now L has one real zero, while P is positive on $(-\infty, y_1) \cup (y_2, \infty)$. Hence $\frac{b_3^2-b_1^2}{m_3^2-m_1^2}$ and $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2}$ are both positive on an open interval(infinite), I, of k values.

(ii) Suppose that $k = k_0$: Then it is not hard to show, using (31), that $\frac{b_3^2-b_1^2}{m_3^2-m_1^2}$ is given by

$$Q(h) = (h - x_1)(h - x_2)$$

Also, by (31), $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2}$ is a linear function of h,

$$R(h) = (k_0 - y_1)(k_0 - \frac{2m_3m_1h + m_1c_3 + m_3c_1}{m_3 + m_1})$$

R has one real zero(if $m_3 \neq 0$), while Q is positive on $(-\infty, x_1) \cup (x_2, \infty)$. Again, arguing as above, since T_1, T_2, T_3 do **not** have a common intersection point, R is not identically the zero function and thus has degree 1. Hence $\frac{b_3^2-b_1^2}{m_3^2-m_1^2}$ and $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2}$ are both positive on an open interval(infinite), J, of h values.

If $m_3 = 0$, then $R(h)$ equals the constant function $(c_3 - k_0)^2$, which equals 0 if and only if $c_3 = k_0$. If $c_3 = k_0 = \frac{c_1+c_2}{2}$, then it is not hard to show that T_1, T_2, T_3 all intersect at the point $(\frac{c_2-c_1}{2m_1}, \frac{c_1+c_2}{2})$, which violates the assumption that the T_j do not have a common intersection point. Hence $R(h) > 0$ for all h, and it follows that $\frac{b_3^2-b_1^2}{m_3^2-m_1^2}$ and $\frac{b_1^2m_3^2-b_3^2m_1^2}{m_3^2-m_1^2}$ are both positive on the union of two open intervals(infinite) of h values. Hence the set of admissible centers has the form $\{(h, k_0) : h \in (-\infty, x_1) \cup (x_2, \infty)\}$.

We also need to choose (h, k) so that $b_1 \neq 0$. Towards that end, it is not hard to show that

$$(h_0 - x_1)(h_0 - \tfrac{2k_0-c_3-c_1}{m_3+m_1}) = \tfrac{(-c_2m_1+c_2m_3-c_1m_1-c_1m_3+2m_1c_3)^2}{4m_1^2(m_3+m_1)(m_3-m_1)} \tag{32}$$

and

$$(k_0 - y_1)\left(k_0 - \tfrac{2m_3m_1h_0+m_1c_3+m_3c_1}{m_3+m_1}\right) = -\tfrac{(-c_2m_1+c_2m_3-c_1m_1-c_1m_3+2m_1c_3)^2}{4(m_3+m_1)(m_3-m_1)} \tag{33}$$

A simple argument shows that if $h_0 - \frac{2k_0-c_3-c_1}{m_3+m_1} = 0$ or $k_0 - \frac{2m_3m_1h_0+m_1c_3+m_3c_1}{m_3+m_1} = 0$, then T_1, T_2, T_3 have a common intersection point. Hence, using (32) and (40) proves that $Q(h_0)R(h_0) < 0$ and $L(k_0)P(k_0) < 0$.

But if $h = h_0$, then $b_1 = m_1h_0 + c_1 - k$ equals 0 only when $k = k_0$, and

if $k = k_0$, then $b_1 = m_1h + c_1 - k_0$ equals 0 only when $h = h_0$. Hence $\{(h_0, k) : k \in I\}$ and $\{(h, k_0) : h \in J\}$ cannot contain (h_0, k_0).

Summarizing, we have $S = S_1 \cup S_2$, where $S_1 = \{(h_0, k) : k \in I\}$ and $S_2 = \{(h, k_0) : h \in J\}$ if $m_3 \neq 0$, and $S_2 = \{(h, k_0) : h \in (-\infty, x_1) \cup (x_2, \infty)\}$if $m_3 = 0$. The uniqueness in each case follows from Theorem 3.

Finally, to prove that there is $(h, k) \in S$ such that E has eccentricity e_0:

(i) Suppose that $h = h_0$: Then, by (30), $\frac{b^2}{a^2} = \frac{P(k)}{L(k)}$, where

$L(k) = (h_0 - x_1)(h_0 - \frac{2k-c_3-c_1}{m_3+m_1})$ and $P(k) = (k - y_1)(k - y_2)$. The proof follows exactly as in the proof of Theorem 11, if we can show that $L(y_j) \neq 0,\ j = 1, 2$, and $P(k_1) \neq 0$, where $k_1 = \frac{1}{4}\frac{m_3c_2+m_1c_2-m_3c_1+m_1c_1+2m_1c_3}{m_1}$ is the zero of L.

Now $L(y_1) = -\frac{1}{4}\left(-m_3c_2 + m_1c_2 + m_3c_1 + m_1c_1 - 2m_1c_3\right)^2 \frac{m_1-m_3}{m_1^2(m_3+m_1)^3} = \frac{m_3-m_1}{m_1^2(m_3+m_1)}(y_1 - y_3)^2$,

$L(y_2) = -\frac{1}{4}\frac{(-m_3c_2+m_1c_2+m_3c_1+m_1c_1-2m_1c_3)^2}{(m_1-m_3)(m_3+m_1)m_1^2} = \frac{m_3-m_1}{m_1^2(m_3+m_1)}(y_2 - y_3)^2$.

Since $y_1 \neq y_2$ and $y_1 \neq y_3$, $L(y_j) \neq 0,\ j = 1, 2$. Finally,

$P(k_1) = \frac{1}{16}\frac{(-m_3c_2+m_1c_2+m_3c_1+m_1c_1-2m_1c_3)^2}{m_1^2} = \frac{(m_1+m_3)^2}{4m_1^2}(y_1 - y_3)^2 \neq 0$

(ii) Suppose that $k = k_0$: Then, using (31), $\frac{b^2}{a^2} = \frac{R(h)}{Q(h)}$, where $Q(h) = (h - x_1)(h - x_2)$ and $R(h) = (k_0 - y_1)(k_0 - \frac{2m_3m_1h+m_1c_3+m_3c_1}{m_3+m_1})$. Again, the proof follows exactly as in the proof of Theorem 11, if we can show that $R(x_j) \neq 0,\ j = 1, 2$, and $Q(h_1) \neq 0$, where $h_1 = \frac{1}{4}\frac{-m_3c_1+m_1c_1+m_3c_2+m_1c_2-2m_1c_3}{m_3m_1}$ is the zero of R. Now

$R(x_1) = \frac{1}{4}\left(-m_3c_2 + m_1c_2 + m_3c_1 + m_1c_1 - 2m_1c_3\right)^2 \frac{m_1-m_3}{(m_3+m_1)^3} = \frac{m_1-m_3}{m_3+m_1}(y_1 - y_3)^2 \neq 0$ and

$R(x_2) = \frac{1}{4}\frac{(-m_3c_2+m_1c_2+m_3c_1+m_1c_1-2m_1c_3)^2}{(m_1-m_3)(m_3+m_1)} = \frac{m_1-m_3}{m_3+m_1}(y_2 - y_3)^2 \neq 0$.

Finally, $Q(h_1) = \frac{1}{16}\frac{(-m_3c_2+m_1c_2+m_3c_1+m_1c_1-2m_1c_3)^2}{m_1^2m_3^2} = \frac{(m_1+m_3)^2}{4m_1^2m_3^2}(y_1 - y_3)^2 \neq 0$. ■

3.4 Example

$T_1 :\ y = 3x+2,\ T_2 :\ y = -3x+1,\ T_3 : y = 2x$. The intersection points of the T_j are $(\frac{1}{5}, \frac{2}{5}), (-2, -4), (-\frac{1}{6}, \frac{3}{2})$. Letting $h = -\frac{1}{6}$, we get $L(k) = \frac{77}{180} - \frac{11}{15}k$ and $P(k) = \frac{1}{5}(k + 4)(5k - 2)$. Letting $k = \frac{3}{2}$, we get $Q(h) = \frac{1}{5}(h + 2)(5h - 1)$ and $R(h) = -\frac{66}{5}h + \frac{77}{20}$. It follows that $S = \{(-\frac{1}{6}, k) : k < -4\} \cup \{(h, \frac{3}{2}) : h < -2\}$

3.5 Proof of Case 2: $m_2 = m_1$

Note that then $c_1 \neq c_2$ since T_1 and T_2 are distinct. Now (29) reduces to $2hm_1(c_1 - c_2) + 2k(c_2 - c_1) + c_1^2 - c_2^2 = 0$[11] $\Rightarrow$

$$2hm_1 - 2k + c_1 + c_2 = 0 \tag{34}$$

Recall that L is the line with equation $y = m_1 x + \frac{1}{2}c_1 + \frac{1}{2}c_2$. Note that by (34), an admissible center must lie on L, and L is parallel to, and lies exactly halfway between, T_1 and T_2.

Solving (34) for k yields

(i) $k = m_1 h + \frac{1}{2}c_1 + \frac{1}{2}c_2$

That gives

$$\frac{b_3^2 - b_1^2}{m_3^2 - m_1^2} = \frac{((m_1 - m_3)h - c_3 + c_2)((m_1 - m_3)h - c_3 + c_1)}{m_3^2 - m_1^2} = \left(\frac{m_3 - m_1}{m_3 + m_1}\right)(h - x_1)(h - x_2) \equiv Q(h) \tag{35}$$

Solving (34) for h yields(if $m_1 \neq 0$)

(ii) $h = -\frac{1}{2}\frac{-2k + c_1 + c_2}{m_1}$

That gives

$$\frac{b_1^2 m_3^2 - b_3^2 m_1^2}{m_3^2 - m_1^2} = \frac{((m_3 - m_1)k + c_3 m_1 - m_3 c_1)((m_1 - m_3)k + m_3 c_2 - c_3 m_1)}{m_3^2 - m_1^2} = -\left(\frac{m_3 - m_1}{m_3 + m_1}\right)(k - y_1)(k - y_2) \equiv R(k) \tag{36}$$

Now $x_2 - x_1 = \frac{c_3 - c_2}{m_1 - m_3} - \frac{c_3 - c_1}{m_1 - m_3} = \frac{c_2 - c_1}{m_3 - m_1} \neq 0$. Also, since T_3 passes through (x_1, y_1) and (x_2, y_2), $m_3 = \frac{y_2 - y_1}{x_2 - x_1}$. There are several cases to consider. We prove the following case,

$$m_3 > 0 \text{ and } \frac{m_3 - m_1}{m_3 + m_1} > 0$$

, the other cases following in a similar fashion. Then $y_2 - y_1 > 0$ and $c_1 < c_2$. The latter implies that $T_1(x) < L(x) < T_2(x)\ \forall x$, and in particular

$$y_1 = T_1(x_1) < L(x_1) \text{ and } L(x_2) < T_2(x_2) = y_2$$

Clearly $Q(h) > 0$ on $(-\infty, x_1) \cup (x_2, \infty)$ and $R(k) > 0$ on (y_1, y_2). Recall that if $m_1 \neq 0$, then $u_j = \frac{y_j - \frac{1}{2}(c_1 + c_2)}{m_1}$, $j = 1, 2$, which implies that $L(u_j) = y_j$. We consider three subcases,

[11] One can also obtain this equation by letting $m_2 = m_1$ in (11).

(a) $m_1 > 0$: Then $L(u_1) = y_1 < L(x_1) \Rightarrow u_1 < x_1$, and $L(u_2) = y_2 > L(x_2) \Rightarrow u_2 > x_2$

Now $y_1 < k < y_2 \Leftrightarrow y_1 < L(h) < y_2 \Leftrightarrow L(u_1) < L(h) < L(u_2) \Leftrightarrow u_1 < h < u_2$. Hence by (35) and (36), $\frac{b_3^2-b_1^2}{m_3^2-m_1^2} > 0$ and $\frac{b_1^2 m_3^2 - b_3^2 m_1^2}{m_3^2-m_1^2} > 0$ on $(u_1, x_1) \cup (x_2, u_2)$.

(b) $m_1 < 0$: Then $L(u_1) = y_1 < L(x_1) \Rightarrow u_1 > x_1$, and $L(u_2) = y_2 > L(x_2) \Rightarrow u_2 < x_2$.

Now $y_1 < k < y_2 \Leftrightarrow y_1 < L(h) < y_2 \Leftrightarrow L(u_1) < L(h) < L(u_2) \Leftrightarrow u_2 < h < u_1$. Hence by (35) and (36), $\frac{b_3^2-b_1^2}{m_3^2-m_1^2} > 0$ and $\frac{b_1^2 m_3^2 - b_3^2 m_1^2}{m_3^2-m_1^2} > 0$ on $(u_2, x_1) \cup (x_2, u_1)$.

(c) $m_1 = 0$: (h, k) now lies on the horizontal line with equation $y = \frac{1}{2}c_1 + \frac{1}{2}c_2$. Since $y_1 = c_1$ and $y_2 = c_2$, $k = \frac{1}{2}c_1 + \frac{1}{2}c_2 \Rightarrow y_1 < k < y_2$. The set of admissible centers now equals

$\{(h, \frac{1}{2}c_1 + \frac{1}{2}c_2) : h < x_1\} \cup \{(h, \frac{1}{2}c_1 + \frac{1}{2}c_2) : h > x_2\}$.

The *uniqueness* follows for each case from Theorem 3.

The proof that there is $(h, k) \in S$ such that E has eccentricity e_0 differs somewhat from the proof of Theorem 11, since the T_j do not enclose a triangle, and thus there is no incenter. However, if we let

$h_0 = \left(\pm(1/2)\sqrt{(1+m_3^2)/(1+m_1^2)}(c_2 - c_1) - (c_1+c_2)/2 + c_3\right)/(m_1 - m_3)$

and $k_0 = m_1 h_0 + (c_1+c_2)/2$, then there is a circle with center (h_0, k_0) which is tangent to the T_j. Assume that $m_1 \neq 0$, the case $m_1 = 0$ following in a similar fashion. Then the admissible centers lie on the curve γ with equation $k = f(h)$ or $h = g(k)$, where $f(h) = m_1 h + \frac{1}{2}c_1 + \frac{1}{2}c_2$ and $g(k) = \frac{k - \frac{1}{2}c_1 + \frac{1}{2}c_2}{m_1}$. By (35) and (36), the set of admissible centers $= \{(h, k) \in \gamma : Q(h) > 0$ and $R(k) > 0\}$, and $\frac{b^2}{a^2} = \frac{R(k)}{Q(h)} = \frac{R(f(h))}{Q(h)} = \frac{R(k)}{Q(g(k))}$ if $(h, k) \in \gamma$. Now

$$Q(g(y_1)) = Q(g(y_2)) = \frac{1}{4}\frac{(c_1 - c_2)^2}{m_1^2} \neq 0 \tag{37}$$

and

$$R(f(x_1)) = R(f(x_2)) = \frac{1}{4}(c_1 - c_2)^2 \neq 0 \tag{38}$$

Case 1: $m_3 < m_2$

Let $s = \inf\{h < h_0 : R(f(h)) > 0 \text{ and } Q(h) > 0\}$. Since

$(h_0 - x_1)(h_0 - x_2) = \frac{1}{4}\left(\sqrt{\left(\frac{1+m_3^2}{1+m_2^2}\right)} - 1\right)(-c_2 + c_1)^2 \frac{\sqrt{\left(\frac{1+m_3^2}{1+m_2^2}\right)}+1}{(m_2-m_3)^2} < 0,\ h_0$ is between x_1 and x_2, and hence $-\infty < s$. Note that $\frac{R(f(h_0))}{Q(h_0)} = 1$. First, if $Q(s) = 0$, then $s = x_j \Rightarrow R(f(s)) \neq 0$ by (38), and thus $\lim_{h\to s^+} \frac{R(f(h))}{Q(h)} = \infty$. In that case $\frac{a^2}{b^2}$ varies from 0 to 1 on (s, h_0), and the equation $1 - \frac{a^2}{b^2} = e_0^2$ has a solution (h, k) which is an admissible center. Second, if $R(f(s)) = 0$, then $f(s) = y_j \Rightarrow s = g(y_j) \Rightarrow Q(s) \neq 0$ by (37), and thus $\lim_{h\to s^+} \frac{R(f(h))}{Q(h)} = 0$. In that case $\frac{b^2}{a^2}$ varies from 0 to 1 on (s, h_0), and again the equation $1 - \frac{b^2}{a^2} = e_0^2$ has a solution (h, k) which is an admissible center.

Case 2: $m_3 > m_2$

Let $s = \inf\{k < k_0 : R(k) > 0 \text{ and } Q(g(k)) > 0\}$. Since

$(k_0 - y_1)(k_0 - y_2) = \frac{1}{4}(-c_2 + c_1)^2 \frac{m_2^2 - m_3^2}{(1+m_2^2)(m_2-m_3)^2}$, k_0 is between y_1 and y_2, and hence $-\infty < s$. The rest of the proof follows as in case 1.

■

3.5.1 Example

$T_1 : y = x + 1,\ T_2 : y = x,\ T_3 : y = -2x$; The intersection points of T_1 and T_3, and of T_2 and T_3, respectively, are $(-1/3, 2/3)$ and $(0, 0)$. Hence $x_1 = -\frac{1}{3}$, $x_2 = 0$, $u_1 = \frac{1}{6}$, $u_2 = -\frac{1}{2}$. The admissible centers lie on the line L with equation $k = h + \frac{1}{2}$. Since $m_1 m_3 > 0$, $S = \{(h, h + \frac{1}{2}) : -\frac{1}{2} < h < -\frac{1}{3}$ or $0 < h < \frac{1}{6}\}$. See Figure 3.2 for a plot of the T_j and the ellipse $\frac{48(x+5/12)^2}{5} + \frac{48(y-1/12)^2}{7} = 1$.

3.6 3. All Three of the T_j have slopes equal in absolute value

In this case the set of admissible centers consists of two points. For each such center, there are infinitely many ellipses tangent to the T_j.

Theorem 15 *Let T_j, $j = 1, 2, 3$ be three distinct non–vertical lines with equations $y = m_j x + c_j$, and assume that $m_1^2 = m_2^2 = m_3^2$. Assume also that the T_j do not have a common intersection point, and let (x_1, y_1) and (x_2, y_2) be the intersection points of $\{T_1, T_3\}$ and $\{T_2, T_3\}$, respectively. Let S denote the set of admissible centers (h, k). Then S consists of the two points $(\frac{c_3 - c_2}{2m_1}, \frac{c_1 + c_3}{2}) = (x_2, y_1)$ and $(\frac{c_3 - c_1}{2m_1}, \frac{c_2 + c_3}{2}) = (x_1, y_2)$.*

Let a^2 and b^2 be any positive solutions of $a^2m_1^2 + b^2 = \frac{1}{4}(c_1 - c_2)^2$. Then, for each $(h,k) \in S$, the ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is tangent to each of the T_j. Finally, given any $0 \le e_0 < 1$ and any $(h,k) \in S$, one can choose a^2 and b^2 such that E has eccentricity e_0.

Proof. Suppose that $m_1^2 = m_2^2 = m_3^2$. Given $C = (h,k)$, write T_j in the form $y - k = m_j(x-h) + b_j$, $j = 1,2,3$. By Theorem 5, in order for an ellipse $E : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ to exist which is tangent to T_1, T_2, and T_3, we must have

$$b_1^2 = b_2^2 = b_3^2 \tag{39}$$

along with

$$a^2m_j^2 + b^2 = b_j^2,\ j = 1,2,3 \tag{40}$$

Clearly, if $m_1 = m_2 = m_3$, then there is no ellipse tangent to the T_j. Hence we can assume, without loss of generality, that $m_1 = m_2 = -m_3$.

Arguing as in subsection 2(see (29)), $(2m_j - 2m_1)hk + (2m_1c_1 - 2m_jc_j)h + (2c_j - 2c_1)k + c_1^2 - c_j^2 = 0$ for $j = 2,3$, which yields the two simultaneous equations

$$2hm_1 - 2k + c_1 + c_2 = 0,\ (2k - c_1 - c_3)(2m_1h - c_3 + c_1) = 0$$

with solutions $(h,k) = (\frac{c_3-c_2}{2m_1}, \frac{c_1+c_3}{2}) = (x_2, y_1)$ and $(\frac{c_3-c_1}{2m_1}, \frac{c_2+c_3}{2}) = (x_1, y_2)$. Note that those points are distinct since $c_1 \neq c_2$(else $T_1 = T_2$). Using $b_j = m_jh + c_j - k$, it is not hard to show that

$(h,k) = (\frac{c_3-c_2}{2m_1}, \frac{c_1+c_3}{2})$ if and only if $b_1 = \frac{1}{2}(c_1 - c_2)$, $b_2 = -b_1$, and $b_3 = -b_1$, and

$(h,k) = (\frac{c_3-c_1}{2m_1}, \frac{c_2+c_3}{2})$ if and only if $b_1 = \frac{1}{2}(c_1 - c_2)$, $b_2 = -b_1$, and $b_3 = b_1$.

For each choice of (h,k) above, the b_j are nonzero with $b_1^2 = b_2^2 = b_3^2$. As a^2 varies between 0 and $\frac{b_1^2}{m_1^2}$, one obtains infinitely many positive solutions a^2, b^2 of the equations in (40), and $\frac{b^2}{a^2}$ varies from ∞ to 0. Thus one can also find an ellipse with any preassigned eccentricity. ∎

Remark 6 *One can easily prove a rotated version of Theorems 11, 12, 14, and 15 above. The statement of the results is more complicated, but the outline of the proof is the same. One would have to consider separately cases where one of the T_j is parallel to a coordinate axis in the new coordinate system, as we did with horizontal or vertical lines when $\alpha = 0$. Given α,* *one would just replace* m_j *by* $\frac{m_j+\tan\alpha}{1-m_j\tan\alpha}$ *and* b_j *by* $\frac{b_j\sec\alpha}{1-m_j\tan\alpha}$ *throughout. For example, in Theorem 11, each of the variables* a_j, d, β, *and* τ *depend on* α, *but the intersection points of the* T_j *do not, of course, depend on* α.

Remark 7 *For Theorems 11, 12, 14, and 15, it is not hard to show that one can specify the area in advance instead of the eccentricity.*

4 Four Given Tangents

Given four lines in the plane, such that no three of the lines are parallel or have a common intersection point, we want to prove that there is some ellipse E which is tangent to each of the four lines. Of course, simple examples show that this result is only true in general for *rotated* ellipses(see the example below). First we need the following results.

Lemma 16 $f(s,t) = s^2 - 4s + 6st + 4 - 4t + t^2$ *is* ***positive*** *on* $\{(s,t) : s > 0$ *and* $t > 0\}$.

Proof. The implicit curve $C : s^2 - 4s + 6st + 4 - 4t + t^2 = 0$ C has horizontal tangents at $(2,0), (-1,1)$, and vertical tangents at $(0,2), (1,-1)$, and hence is tangent to the s–axis at $(2,0)$ and tangent to the t–axis at $(0,2)$. Since the second derivative is never zero on C, it then follows easily that C never touches

$\{(s,t) : s > 0$ and $t > 0\}$. Since $f(1,1) = 4 > 0$, the lemma follows from the Intermediate Value Theorem.

Proposition 17 *Suppose that we are given four lines in the plane, such that no two of the lines are parallel or have a common intersection point. Then the four lines must form the boundary of at least one four sided convex polygon* D.

Proof. Pick a line L. The other lines intersect L at three distinct points since no three lines have a common intersection point. Let M be the line which intersects L between the other two intersection points. The interior of the triangle T formed by the three lines not equal to M is cut by M. The two regions into which M divides T are each convex, since they are the intersections of convex regions(a triangle and a half plane). One of these regions is D.

Theorem 18 *Let T_1, T_2, T_3, T_4 be four given lines in the plane, such that no three of the T_j are parallel or have a common intersection point. Then there is an ellipse E which is tangent to each of the T_j.*

Proof. We prove the case when no **two** of the lines are parallel, the proof of the other cases being similar. By Proposition 17, the T_j form the boundary of a four sided convex polygon D. Choose any three vertices P_1, P_2, P_3 of D, and let A be a non-singular affine map which sends the P_j to the points $(0,0), (1,0)$, and $(0,1)$. By properties of non-singular affine maps, $A(D)$ must be a convex quadrilateral,with no two sides parallel. Hence A sends the other vertex of D to the point (s,t), where $s > 0, t > 0$, $s \neq 1 \neq t$, and $s + t \neq 1$. Let L_j, $j = 1,2,3,4$ denote the lines which make up the boundary of $A(D)$. We shall find an ellipse $E : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$ tangent to each of the L_j. Then the preimage of E under the map A is an ellipse tangent to each of the T_j. Write $L_1 : y = 0, L_2 : x = 0, L_3 : y = 1 + \left(\frac{t-1}{s}\right) x, L_4 : y = \frac{t}{s-1}(x-1)$

To prove the existence of E, we shall apply Theorem 13 to L_1, L_2, and L_3, and to L_1, L_2, and L_4. Let γ_1 denote the curve $2m_3hk - 2m_3(m_3c_1 + c_3)h + 2(c_3 - c_2)k = c_3^2 - c_2^2 - m_3^2c_1^2$, with $m_3 = \frac{t-1}{s}$ and $c_3 = 1$, and let γ_2 denote the same curve, with $m_3 = \frac{t}{s-1}$ and $c_3 = -\frac{t}{s-1}$. The key is showing that γ_1 and γ_2 have at least one intersection point. The equations for γ_1 and γ_2 are

$2\left(\frac{t-1}{s}\right) hk - 2\left(\frac{t-1}{s}\right) h + 2k = 1 \Rightarrow$

$$k = \frac{1}{2}\frac{(2t-2)h + s}{(t-1)h + s} \tag{41}$$

and

$2\left(\frac{t}{s-1}\right) hk + 2\left(\frac{t}{s-1}\right)^2 h - 2\left(\frac{t}{s-1}\right) k = \left(\frac{t}{s-1}\right)^2 \Rightarrow$

$$k = -\frac{1}{2}t\frac{2h-1}{(h-1)(s-1)} \tag{42}$$

Note that if $(t-1)h + s = 0$, then $2\left(\frac{t-1}{s}\right) hk - 2\left(\frac{t-1}{s}\right) h + 2k = 2 \neq 1$. Hence the denominator in (41) cannot be zero. Also, if $h = 1$, then $2\left(\frac{t}{s-1}\right) hk + 2\left(\frac{t}{s-1}\right)^2 h - 2\left(\frac{t}{s-1}\right) k = 2\left(\frac{t}{s-1}\right)^2 \neq \left(\frac{t}{s-1}\right)^2$ since $t \neq 0$. Hence the denominator in (42) is well defined. Now we set the expressions for k in (41) and (42) equal to one another. This yields $\frac{1}{2}\frac{(2t-2)h+s}{(t-1)h+s} + \frac{1}{2}t\frac{2h-1}{(h-1)(s-1)} = \frac{1}{2}(s-1+t)\frac{(2t-2)h^2+(s+2-t)h-s}{((t-1)h+s)(h-1)(s-1)} = 0$, which holds if and only if $(2t-2)h^2 + (s+2-t)h - s = 0$. The discriminant is

$(s+2-t)^2+4s(-2+2t)=s^2-4s+6st+4-4t+t^2$, which is positive on

$\{(s,t): s>0 \text{ and } t>0\}$ by Lemma 16. Hence γ_1 and γ_2 have two points of intersection (h_1,k_1) and (h_2,k_2). Finally, to apply Theorem 13, we must show that $h_j \neq c_1$ and $k_j \neq c_2$, where $c_1 = c_2 = 0$. It follows easily that $h=0$ in γ_1 and/or $k=0$ in γ_2 implies that $s+t=1$. Hence there are actually two ellipses tangent to each of the L_j. ■

Remark 8 *Similar methods show that Theorem 18 also holds for hyperbolas.*

Remark 9 *Simple examples show that in general one cannot specify the eccentricity in advance in Theorem 18.*

4.1 Example

$T_1: y=x+2, T_2: y=3x, T_3: y=-2x-1, T_4: y=-4x+1$. The T_j enclose the convex quadrilateral D with vertices $\{(-\frac{1}{5},-\frac{3}{5}),(-1,1),(-\frac{1}{5},\frac{9}{5}),(\frac{1}{7},\frac{3}{7})\}$. Let A be the affine transformation with $A(-1,1)=(0,0)$, $A(-\frac{1}{5},-\frac{3}{5})=(1,0)$, and $A(-\frac{1}{5},\frac{9}{5})=(0,1)$. Then $A(\frac{1}{7},\frac{3}{7})=(\frac{5}{7},\frac{5}{7}) \Rightarrow s=t=\frac{5}{7}$. The two solutions of the equations (41) and (42) are $(\frac{7}{4}\pm\frac{1}{4}\sqrt{29},\frac{7}{4}\pm\frac{1}{4}\sqrt{29}) \approx (3.0963, 3.0963)$ and $(.40371, .40371)$. Mapping back to D gives two ellipses tangent to the given T_j with centers $\approx (-.35406, .67703)$ and $(3.9541, -1.477)$. One of the ellipses is inscribed in D(see Figures 4.1 and 4.2). [12]

5 Discussion

Finding general conics tangent to a given set of lines is discussed in books on projective geometry, where the duality between points and lines is used. The dual of finding a conic passing thru given points is finding a conic tangent to given lines. Salmon([3]), Liming([1], page 242) and Fishback([2]) discuss, in particular, how to fit a a conic to *five* given tangents. It is not clear, however, whether duality would more easily yield specific results about fitting less than five tangents to an ellipse, given certain information about the ellipse(such as the center, orientation of the axes, or the eccentricity), or determining

[12]Using Theorem 11, applied to T_1, T_2, T_3, and then T_2, T_3, T_4, it is not hard to show that there is no *non-rotated* ellipse tangent to all four lines!

when a given set of lines is the tangent to such an ellipse. We have not seen any results give necessary and sufficient conditions on when a finite set of lines is tangent to a certain type of conic, such as the family of ellipses with a specified center we discussed in this paper. Salmon does discuss briefly the locus of centers of a conic tangent to four lines, or to three lines with another condition on the sum of the squares of the axes. However, his results do not imply Theorem 18 or any of the Theorems in section 3.

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[5] Cox, Litle, and O'Shea, Ideals, Varieties, and Algorithms, 1992, Springer-Verlag, New York.

Fig. 2.1

Fig. 2.2

Fig. 3.1

Fig. 3.2

Fig. 4.1

Fig. 4.2