Slope Fields

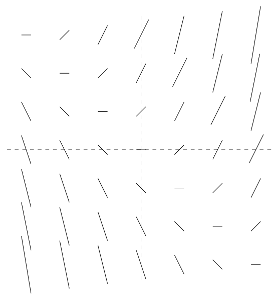
This semester we will learn some techniques for solving differential equations explicitly. However, as with integral in Calc II there are very few differential equations that can be solved explicitly. In fact, most of the techniques we learn apply only to linear equations and then only to those linear equations which have constant co-efficients. In practice you will find that many of the most important and useful differential equations are not linear.

What do we do when confronted with a differential equation that we cannot solve explicitly? There are many things we can do to analyze a differential equation that we cannot solve explicitly. One thing we can do is to use a computer to approximate the solution. This is one approach we will explore in detail, in chapter 8. However, often we do not need numerical values for the solution, we need an understanding of the behavior of the solution. (The systems engineer Richard Hamming used to say that the point of computation was understanding, not numbers.) One of the simplest yet most powerful techniques is graphical analysis. Let us look at one particular graphical technique: the slope field.[1]

Suppose we have a first-order differential equation $y' = f(t, y)$. This equation tells us that the the slope of y is determined by some function of t and y. In other words, if a solution to this differential equation happens to pass thru a particular point (t, y) then its slope there must be $f(t, y)$. Now even though we do not know y (yet) we are given f. So, one way to analyze the solutions is to draw little lines of slope $f(t, y)$ at various different points (t, y).

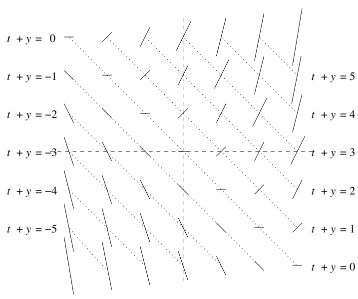
Let's look at a particular example. Let's suppose we have the equation $y' = t + y$. First, let us make a table of values of the right-hand side, for all the integer values of t and y between -3 and 3, using the fact that $y' = t + y$. Second, at each of these 49 points (t, y) we draw a small line of slope $t + y$.

y							
3	0	1	2	3	4	5	6
2	−1	0	1	2	3	4	5
1	−2	−1	0	1	2	3	4
0	−3	−2	−1	0	1	2	3
−1	−4	−3	−2	−1	0	1	2
−2	−5	−4	−3	−2	−1	0	1
−3	−6	−5	−4	−3	−2	1	0
	−3	−2	−1	0	1	2	3 t



This picture is called the *slope field.* The slope field conveys a clear sense of what direction any solution to the differential equation must flow. Start at various points in this picture and trace out a solution to the differential equation by following the "flow" of this slope field. These are the *integral curves.*

We don't have to generate the entire table of values in order to draw the slope field. There is an easier way: note that along the line where $t + y = 0$, all of the little lines we've drawn are horizontal. More generally along the line $t + y = k$ all of the slope field lines have same slope, namely k. (See the picture below.) Do not confuse the slopes of these dotted lines (all of which are -1) with the slopes of the solution curves. The curves where the slope function $y' = f(t, y)$ is constant are called *isoclines.* In our example the isoclines are the lines where $t + y = k$, for some constant k. Isoclines are not always straight lines; they can be almost kind of curve. Isoclines are usually not integral curves. In our example only one of the isoclines is also an integral curve — can you see which one?

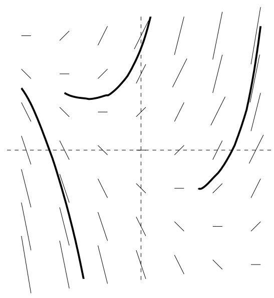


[1] Sometimes slope fileds are also called the direction field, but there is a slight differnce between the two notions, as we shall see when we study systems of differential equations.

Isoclines are convenient devices for drawing the slope field. To draw the slope field for the differential equation $y' = f(t, y)$ you should choose various values of k and then graph the equation $f(t, y) = k$ with a very faint dotted line. At points along this line sketch small (solid) lines of slope k. Which values of k should you choose? That depends very much on the differential equation. The most important value of k is 0. In calculus we called a point of zero slope a critical point. In differential equations the isocline $y' = 0$ is called the *critical curve.*

Let's return to our example and analyze a few integral curves in detail. First, consider a solution that begins at the point $(1.5, -1)$. At this point the curve starts out with a positive slope of 0.5, hence it moves up and to the right. As it does its slope continually increases. So the curve is always increasing.

Next, consider a curve that starts at $(-2, 1.5)$. This curve starts out with slope -0.5, and hence the graph is decreasing at that point. As the graph moves down (and to the right) its slope increases (gets less negative) until it flattens out ($y' = 0$), which must be at some point where $t + y = 0$. Thereafter, $t + y > 0$, and so the solution is increasing, as in the first case.



In the process of this analysis we found that the critical points of any solution, the points where $y' = 0$, must lie somewhere on the line where $t + y = 0$. Altho different solutions will hit this line at different points, we see that every critical point for every solution lies somewhere along this line. Once any solution hits this line its graph begins increasing — up and to the right. Hence the solution never hits this line again. So, each solution has at most one critical point.

What happens if we start at a point along the line $t + y = -1$? It appears that the solution hugs this line. In fact, we can check that the function $\phi(t) = -t - 1$ is a solution to the differential equation, since $\phi'(t) = -1 = t + \phi(t)$. Thus, the line $y = -t - 1$ does happen to be an integral curve.

We can go a bit further and analyze the concavity, for any integral curve. Since $y' = t+y$ we compute that $y'' = 1 + y' = 1 + t + y$. Thus, when a solution starts above the line $t + y = -1$, its slope is greater than -1 and it is concave up. Hence its slope will only increase, and so the the slope of the curve is always greater than -1. That is, a curve that starts above the line $t + y = -1$ stays above this line forever, and hence stays concave up forever. On the other hand, when the integral curve starts below the line $t + y = -1$ it is concave down, and hence its slope decreases further. This keeps the curve always below the line $t + y = -1$, and hence always concave down.

Let us look at a third detailed example. Consider a solution that starts at $(-3, 1.5)$. From our analysis we see that this curve starts out decreasing and concave down. That is, its slope is negative and becoming even more negative. Therefore the solution will head downward, forever decreasing and curving down.

As in Calc I, when we identify the regions of concavity we add a lot to our picture. Whenever possible you should compute y'', starting with the equation $y' = f(t, y)$ and computing the derivative of both sides, taking care to apply Chain Rule when appropriate. Once you have a formula for y'' locate the curve or curves where $y'' = 0$. When an integral curve meets a point where $y'' = 0$ there is likely to be a change in concavity. The curves where $y'' = 0$ cut the picture into regions where the concavity does not change. Identify and label each region as "up" or "down". One way to do this is to choose a point in the region and evaluate y''. When you draw your integral curves pay attention to the slope lines and to the regions of concavity.

Further reading

Read more about slope fields on pages 3–5 of your text (which calls them direction fields). Good practice problems are 1–6 on page 7 and 15–20 on pages 8–9.

Reading quiz

1. What is a slope field?
2. What is an isocline?
3. What is the critical curve?
4. What is the strategy for drawing a slope field?
5. How do we identify the regions of concavity in a slope field?

Assignment 3: due Friday, 3 February

Draw slope fields for the following differential equations. Use the strategy of finding the isoclines. Compute y'' and identify the regions of concavity. Draw several integral curves, using different color lines for each. For each curve describe its behavior as $t \to +\infty$.

1. $y' + 5y = t$.
2. $y' = ty$.
3. $y' = y^2 + 1$.
4. $y' = t - y^2$.