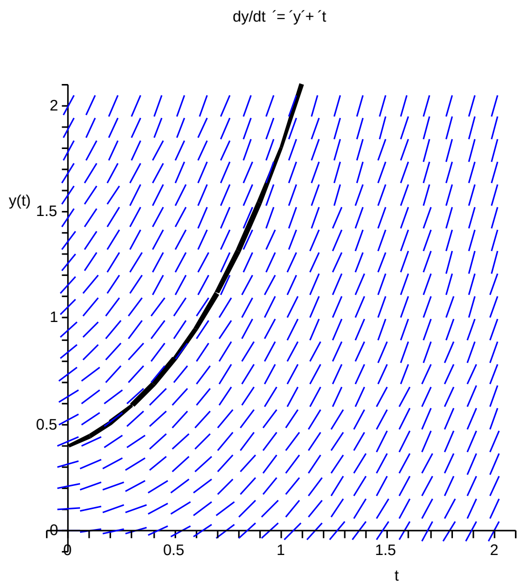
Euler's method

The Theorem on Existence and Uniqueness states that if the slope field is sufficiently smooth at each point, then there is a unique integral curve passing thru any given point. We see this when we look at a slope field: if we start at a point then follow the flow we construct the integral curve by "connecting the dots". Today we make this visual precise. We will end up with an approximation scheme called the Euler Method.

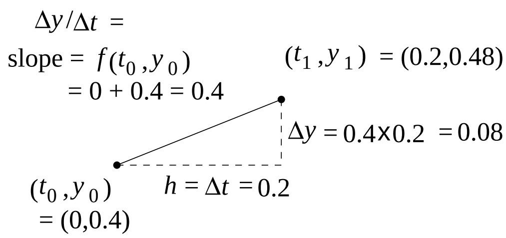
Let's start with an example:

$$y' = y + t,\ \ y(0) = 0.4. \tag{1}$$

Picture below is the slope field, for t in the interval $[0, 2]$.



So, if we start at the initial point and connect the dots we obtain Analytically what are we doing? We divide up the interval $[0, 2]$ by picking points $t_0, t_1, t_2, \ldots$, usually equally spaced. For example, we might take $t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, $\ldots$ $\ t_k = 0.2k$, $\ldots$. The length of the subintervals, 0.2 in this case, is called the *step size*, and typically is denoted h. So, in general, $t_k = t_0 + kh$. Now we start by taking y_0, the approximate value at t_0, to be the given initial value y_0. So, in our case we take $y_0 = 0.4$. Next we compute y_1, the approximate value at t_1, by extrapolating from the point (t_0, y_0). In other words, we know that the slope at (t_0, y_0) is supposed to be $f(t_0, y_0)$, and so we simply follow a line of this slope for an interval of length h, and take the approximate value at t_1 to be the value on this line.



The simple formula for y_1 is $y_1 = y_0 + hf(t_0, y_0)$. More generally, as we step from t_i to t_{i+1} we use the formula

$$y_{i+1} = y_i + hf(t_i, y_i). \tag{2}$$

In our example this becomes

$$y_{i+1} = y_i + 0.2(t_i + y_i) = 0.2t_i + 1.2y_i.$$

Here is the table of values:

t_i	y_i	$f(t_i, y_i)$	t_i	y_i	$f(t_i, y_i)$
0.0	0.40000000	0.40000000	1.2	1.98037760	3.18037760
0.2	0.48000000	0.68000000	1.4	2.61645312	4.01645312
0.4	0.61600000	1.01600000	1.6	3.41974374	5.01974374
0.6	0.81920000	1.41920000	1.8	4.42369249	6.22369249
0.8	1.10304000	1.90304000	2.0	5.66843099	7.66843099
1.0	1.48364800	2.48364800			

You can check that the exact solution is $y = 1.4e^t - t - 1$. [Exercise!] The correct value when $t = 2$ is 7.34467854, to 8 decimal places. Hence the absolute error at this point is approximately 1.67624755, or about 22.82% of the correct value. We say that the *relative* error is 20.3%. In other words, the relative error is the absolute error divided by the correct value, usually (but not always) converted to a percent.

As we use a smaller and smaller step size, the errors will decrease linearly. For example, here are the computed values at $t = 2$ for various step sizes.

h	$y(2)$	relative error
0.20000	5.66843099	22.82%
0.04000	6.94935668	5.38%
0.00800	7.26268651	1.12%
0.00160	7.32815789	0.22%
0.00032	7.34136948	0.05%

Each time we decrease the step size by a factor of 5, the relative error decreases by approximately a factor of 4 or 5.

Further reading

Euler's Method is first discussed in section 2.7, and in more detail in section 8.1. Study examples 1–3 in section 2.7, and 1–2 in section 8.1. Pay particular attention to the numbers in the "error" columns. How do the errors change as you read down the column? How do they change when you read across a row? For a given value of t, what happens to the error when you halve the stepsize h? Good practice problems are 1–4, page 108, and 1–6, page 449

Reading quiz

1. Why is Euler's Method also called the Tangent Line Method?
2. What is the stepsize? What role does it play in Euler's Method?
3. How does the error change as the stepsize changes?
4. What is the difference between absolute error and relative error?
5. What does Euler's Method produce? Be precise!

Assignment 10: due Friday, 3 March

On a piece of graph paper, sketch the slope field for the equation $y' = t - y$, focusing on the rectangle $[0, 2] \times [-2, 2]$. Use as much of the graph paper as possible. Shade or color the regions of concavity. Use a calculator to sketch slope lines at equally spaced points, 0.4 units apart. So, you should end up with a 6×11 grid of slope lines. Starting with $t_0 = 0$ and each of the initial values $y_0 = -1.6$, -0.8, 0.0, 0.8, 1.6, use the Euler Method, with a step size of 0.4, to approximate the integral curve that passes thru the point (t_0, y_0). Follow each curve until it leaves your rectangle. Sketch these 5 approximate curves on your slope field, as sequences of straight line segments. For each curve, determine whether the approximation appears to be an overestimate or an underestimate of the exact curve. Explain your reasoning. (Hint: think about the concavity!)