Review of calculus: part 2

From the Mean Value Theorem for Derivatives we learned that if we are given a function $g(x)$ is defined on $[a.b]$ and a real number C then there is at most one differentiable function $F(x)$ such that $F'(x) = g(x)$ and $F(a) = C$. Is there always exactly one solution? How would we find such a solution? In basic calculus we answered these questions with Riemann integrals and the Fundamental Theorem of Calculus: if $g(x)$ is continuous then

$$F(x) = C + \int_a^x g(t)\,dt$$

satisfies the given initial value problem.

Riemann integrals

Recall that **derivative**, **slope**, **velocity**, and **rate of change** are all essentially synonyms. We use different words in different contexts, but mathematically they are all the same concept. We can use this to motivate our solution the simple initial problem

$$F'(x) = g(x), \; F(a) = C.$$

Here, $g(x)$ and C are given; $F(x)$ is the unknown.

The idea is to think of $F(x)$ as the position of a particle moving along a line with velocity $g(x)$ at time x. In this interpretation C represents the position at time a. If the velocity $g(x)$ is continuous then on a very small time interval, say from t_1 to t_2, the velocity is approximately constant, roughly equal to $g(t_1)$, and so we expect that on the interval $[t_1, t_2]$ we would compute the change in F as follows:

$$\Delta F \approx g(t_1)\,\Delta t$$

where $\Delta t = t_2 - t_1$. This is the familiar "distance = rate × time" formulation, which works when the rate is constant. Now if we divide the interval $[a, x]$ into a series of small subintervals $[t_1, t_2]$, $[t_2, t_3]$, ... then the new position at time x is approximately equal to the starting position C plus the sum of the changes ΔF over all of the subintervals:

$$F(x) \approx C + \sum_{i=1}^{n} g(t_i)\,\Delta t.$$

Riemann's Theorem is that the error in this approximation vanishes as the length of all the subintervals shrinks to 0. More precisely, if $g(x)$ is continuous then the limit

$$\int_a^x g(t)\,dt = \lim_{\max|\Delta t|\to 0} \sum_{i=1}^{n} g(t_i)\,\Delta t$$

does not depend on how you divide up the interval $[a, x]$ into subintervals now how you choose the representative point t_i from the i-th subinterval. No matter how you make these choices, the limit always exists, and always gives the same result, provided only that the lengths of all these subintervals shrink to 0. This limit is called the **Riemann integral** of g on the interval $[a.x]$.

There are many ways to interpret the Riemann integral, and hence many applications. If we draw a graph of g then we see that its integral also measures the area between the graph and the horizontal axis, at least where $g(t) > 0$: if $g(t) > 0$ then $g(t)\,\Delta t$ is the area of a rectangle of height $g(t)$ and width Δt. If $g(t) < 0$ then this differential measures the negative of the area.

The Mean Value Theorem for Integrals

Another important interpretation of Riemann integral is as an **averaging** process. It is simplest to explain this when we divide up the interval $[a, b]$ into n equal-size subintervals, hence of size $(b-a)/n$. Since the common factor $(b-a)$ is constant, we can factor it out of the Riemann sum and also out of the limit:

$$\int_a^b g = (b-a) \cdot \lim_{n\to\infty} \sum_{i=1}^{n} g(t_i).$$

Since the sum inside the limit is simply the mean (average) value of the $g(t_i)$, it is reasonable to interpret

$$\frac{1}{b-a}\int_a^b g = \textbf{mean value of } g \textbf{ on } [a,b].$$

The mean value of the $g(t_i)$ lies between the minimum and maximum values of g on $[a,b]$. By the Intermediate Value Theorem we deduce that if g is continuous then g takes its mean value somewhere in $[a,b]$. That is, if g is continuous on $[a,b]$ then for some c in $[a,b]$ we have that

$$f(c) = \frac{1}{b-a}\int_a^b g.$$

The Fundamental Theorem of Calculus

We now have all of the tools we need to see that the Riemann integral does indeed solve our simple initial value problem — that is, that $\int_a^x g$ is an antiderivative of g. To see this we compute the difference quotient and apply the Mean Value Theorem for Integrals:

Fundamental Theorem of Calculus If g is continuous and we define $F(x) = C + \int_a^x g$ then

$$\frac{dF}{dx} = g(x),\ F(a) = C.$$

proof.

$$\frac{dF}{dx} = \lim_{\Delta x\to 0}\frac{\Delta F}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_x^{x+\Delta x} g(t)\,dt = \lim_{\Delta x\to 0} g(c) = g(x).$$

This last equality is because of the **Squeeze Law**: since c is between x and $x+\Delta x$ and since $\Delta x \to 0$ we conclude that $c \to x$.

There are two ways (at least!) that we apply the Fundamental Theorem: to define functions as solutions to initial value problems, and to evaluate integrals when we know an antiderivative by some other means (usually rote memory). Let's look at examples of each of these applications.

None of the basic rules for derivatives — Power Rule, Chain Rule, Product Rule, ... — combine to give a formula for the antiderivative of $1/x$. But $1/x$ is continuous, at least when $x \neq 0$, and so the Fundamental Theorem guarantees that it has an antiderivative. Since this antiderivative is important in applications, we give it a name: **logarithm**. (The word and the definition are due to John Napier, who lived before Calculus was invented!)

More precisely,

$$\log(x) = \ln(x) = \int_1^x \frac{dt}{t}.$$

We use the notation $\log(x)$ and $\ln(x)$ interchangeably. The natural logarithm is in essence the only logarithm: all other logarithms are scalar multiples of it. So, whenever we say "logarithm" in this class we mean the natural logarithm, unless we explicitly say otherwise.

So, the natural logarithm is the unique solution to the initial value problem

$$\frac{d}{dx}\log(x) = \frac{1}{x},\ \log(1) = 0.$$

Hence it is an increasing function on $(0,+\infty)$. Therefore it has an inverse, which we call the exponential function:

$$\exp(\log(x)) = x,\ \log(\exp(x)) = x.$$

If we apply the Chain Rule to this second relation then we obtain the derivative of the exponential function:

$$\frac{1}{\exp(x)}\cdot\frac{d}{dx}\exp(x) = 1,$$

whence

$$\frac{d}{dx}\exp(x) = \exp(x).$$

We sometimes use the notation e^x for $\exp(x)$. This is because it obeys the usual laws for exponents. To see this, we start with the identities

$$\log(ab) = \log(a) + \log(b),\ \log(a^n) = n\ \log(a).$$

The first can be proved by replacing b by x and then observing that both sides have the same derivative and the same value at 1. By the Mean Value Theorem for Derivatives they must be the same everywhere. The second relation (for positive integers n) is simply repeated application of the first.

If we translate these relations using the definition of $\exp(x)$ as the inverse function for $\log(x)$ we find that

$$e^x\, e^y = e^{x+y},\ e^{nx} = (e^x)^n.$$

More generally, we define logarithms and exponentials to any positive base, other than 1, by the rules

$$\log_b(x) = \log(x)/\log(b),\ b^x = \exp(x\ \log(b)).$$

Note that $\log(2^n) = n\ \log(2) \to +\infty$ as $n \to +\infty$. Since $\log(x)$ is strictly increasing we conclude that $\log(x) \to +\infty$ as $x \to +\infty$. Similarly $\log(x) \to -\infty$ as $x \to 0^+$; $e^x \to +\infty$ as $x \to +\infty$; and $e^x \to 0$ as $x \to -\infty$.

Sometimes we know the antiderivative already, and want to evaluate the integral. The Fundamental Theorem tells us how to do this: if $F'(x) = g(x)$ then

$$\int_a^b g = F(b) - F(a).$$

For example, if $n \neq -1$ then

$$\int_a^b x^n\, dx = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

We can apply this to find a power series representation for $\log(x)$. If $|r| < 1$ then

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k.$$

This is the **geometric series**. If we take $r = 1 - t$ in the definition of $\log(x)$ we find that

$$\begin{aligned}
\log(x) &= \int_1^x \frac{dt}{t} \\
&= \int_1^x \frac{dt}{1-(1-t)} \\
&= \int_1^x \sum_{k=0}^{\infty} (1-t)^k\, dt \\
&= \int_1^x \sum_{k=0}^{\infty} (-1)^k (t-1)^k\, dt \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(x-1)^{k+1} - 0^{k+1}}{k+1} \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots
\end{aligned}$$

In particular,

$$\ln(\frac{1}{2}) \approx -\sum_{k=0}^{10} \frac{1}{(k+1)\cdot 2^{k+1}} = 0.6931\ldots$$

Homework problems: due Wednesday, 18 January

7 Draw the graphs of the following integrands and evaluate the given integrals by interpreting the integral as area. (Do *not* find antiderivatives!)

a $\displaystyle\int_{3}^{7} (2t+5)\,dt$

b $\displaystyle\int_{-4}^{4} (2t+5)\,dt$

c $\displaystyle\int_{-3}^{2} |t|\;dt$

d $\displaystyle\int_{-1}^{1} \sqrt{1-t^t}\,dt$

8 Draw a graph of $f(x) = \sin(3x)$ on $[-\pi, \pi]$, and use this to find the mean value of f on this interval. (Do *not* find antiderivatives!)

9 Find a power series representation of $f(x) = \arctan(x)$ around the point $x = 0$, and use this to approximate $\pi/4$.

10 Find a power series representation of e^x near $x = 0$, and use this to approximate e.