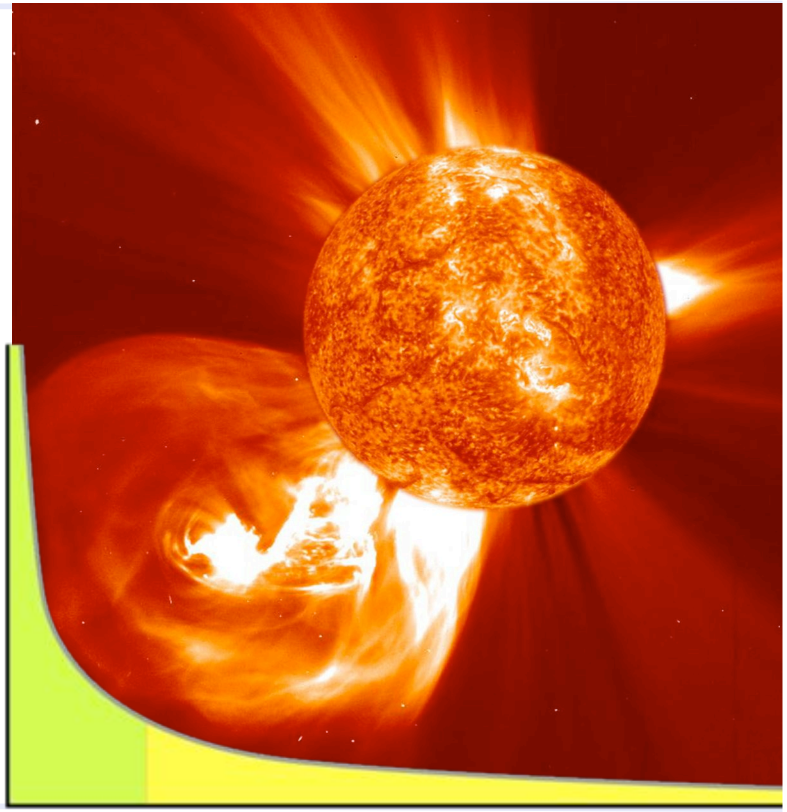
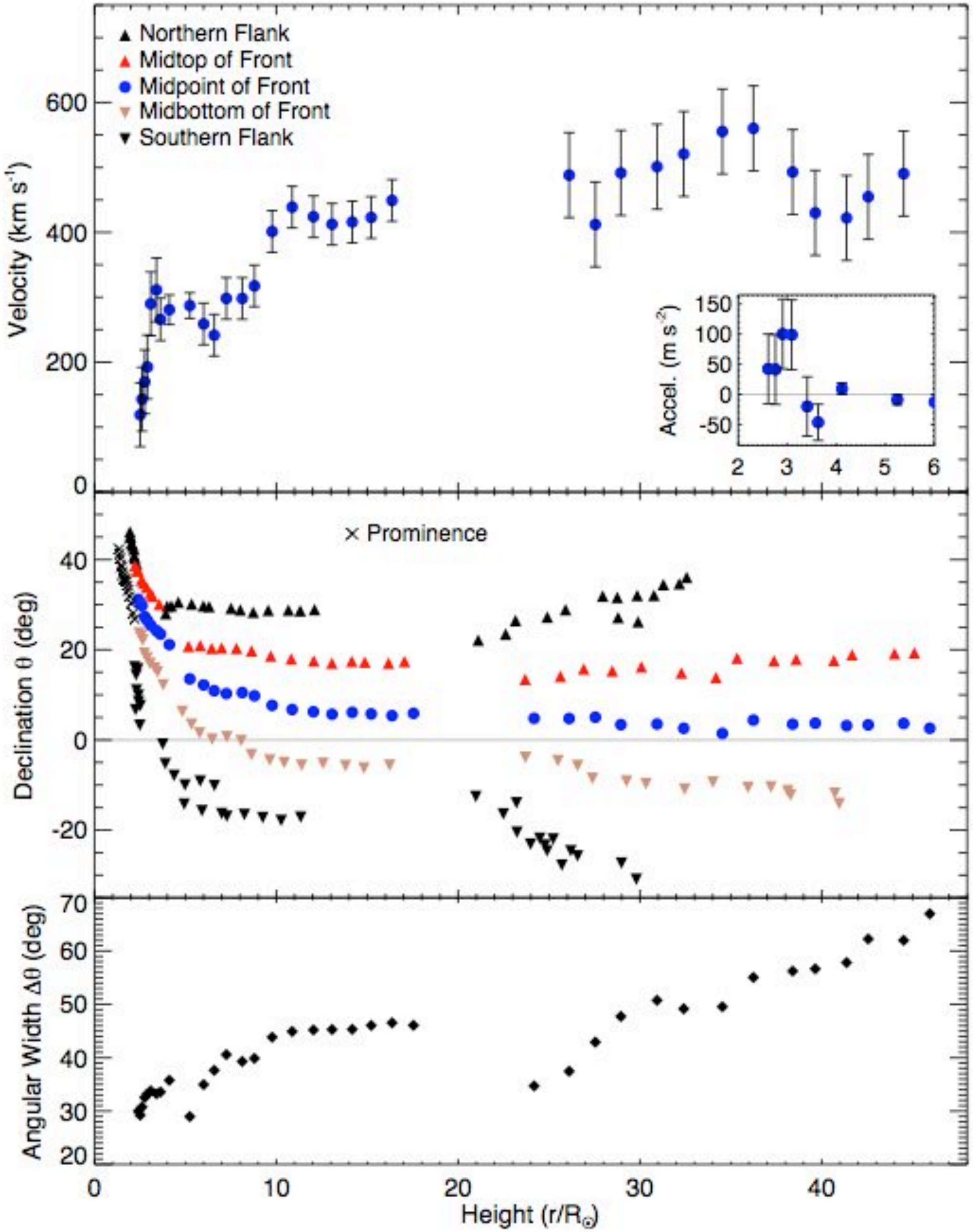
Investigating Power Laws: implications for CME dynamics

Jason P. Byrne

ARG Solar Group Meeting
Trinity College Dublin
April 2010





CME propagation:

Early acceleration phase.

Subsequent drag phase in the solar wind.

CME deflection:

Source region ~ 55˚N

Tends toward the ecliptic.

CME expansion:

Occulter effects apparent.

Power Law

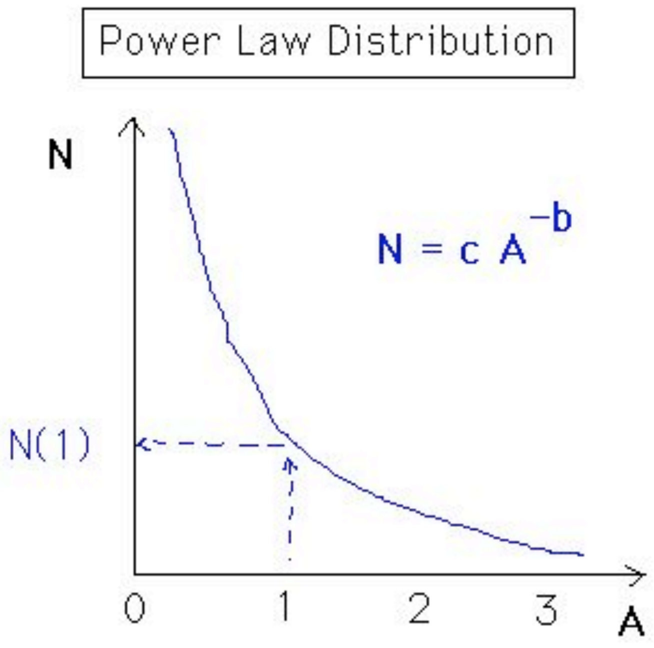
$$y \equiv f(x) = ax^b$$

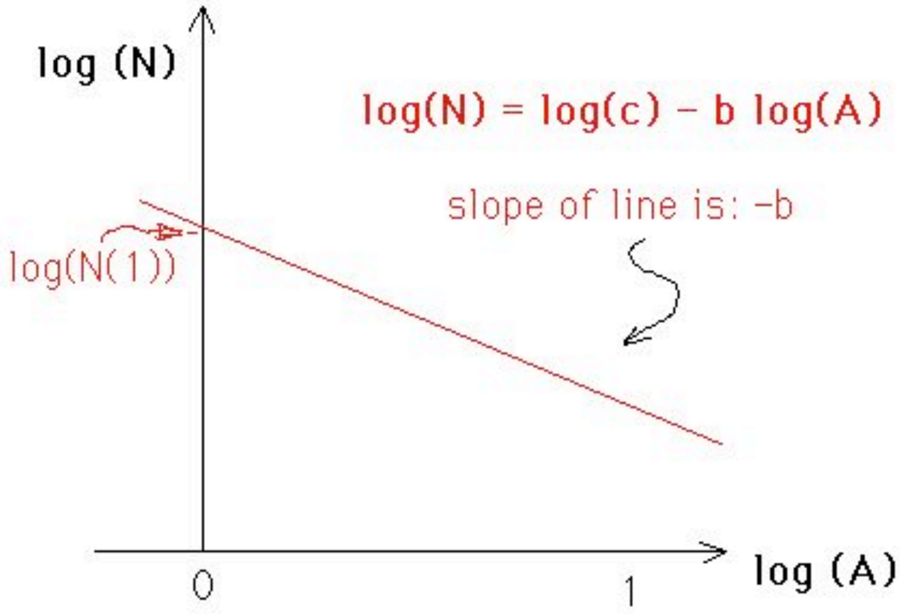
$$\log y = b \log x + \log a$$

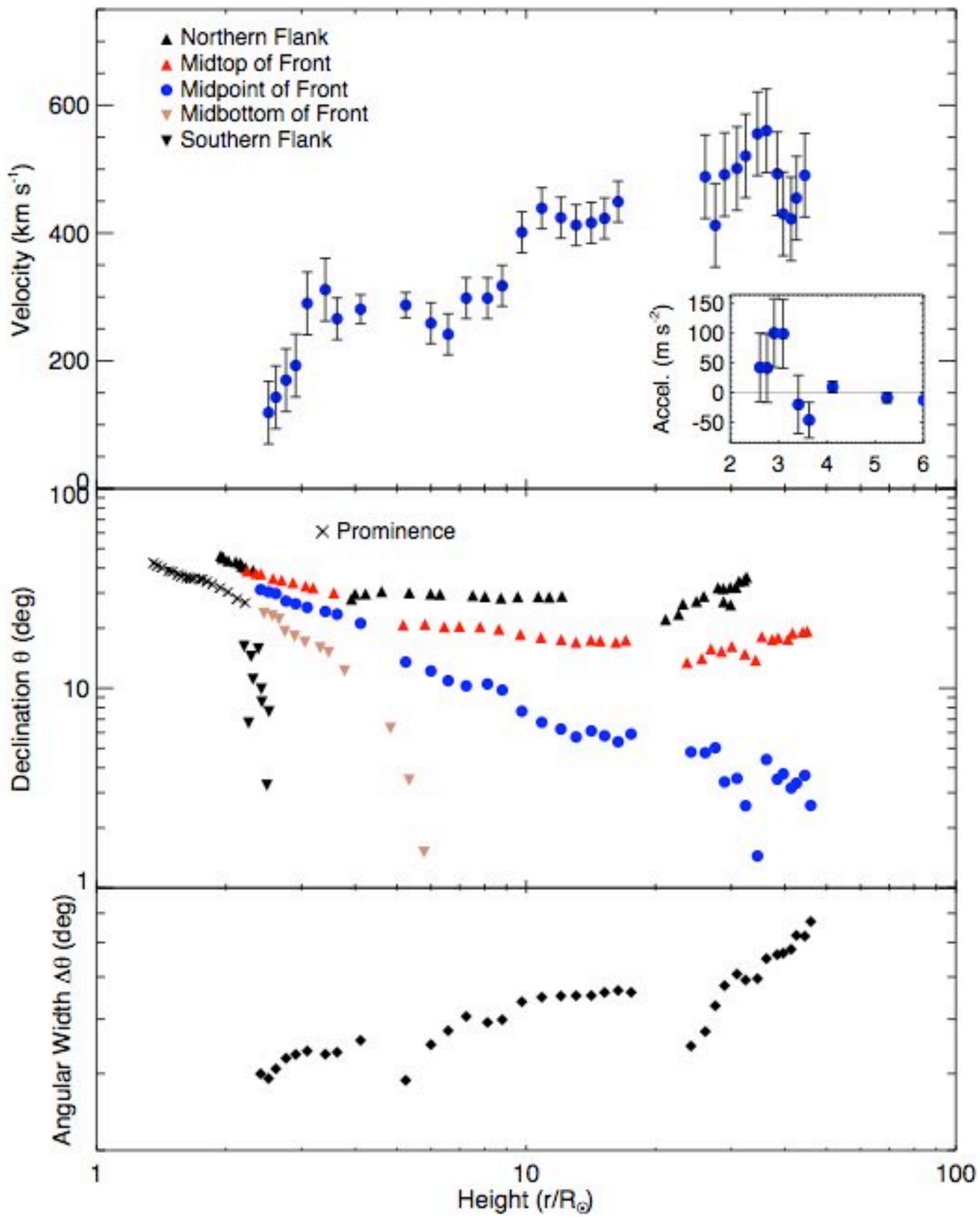
$$Y = mX + c$$

In the simplest model x_i are normally distributed about x and y_i about y.

But $\log(x_i)$ and $\log(y_i)$ are not necessarily normally distributed about X and Y.







Northern Flank
Midtop of Front
Midpoint of Front
Midbottom of Front
Southern Flank
Velocity (km s-1)
Accel. (m s-2)
× Prominence
Declination θ (deg)
Angular Width Δθ (deg)
Height (r/R⊙)

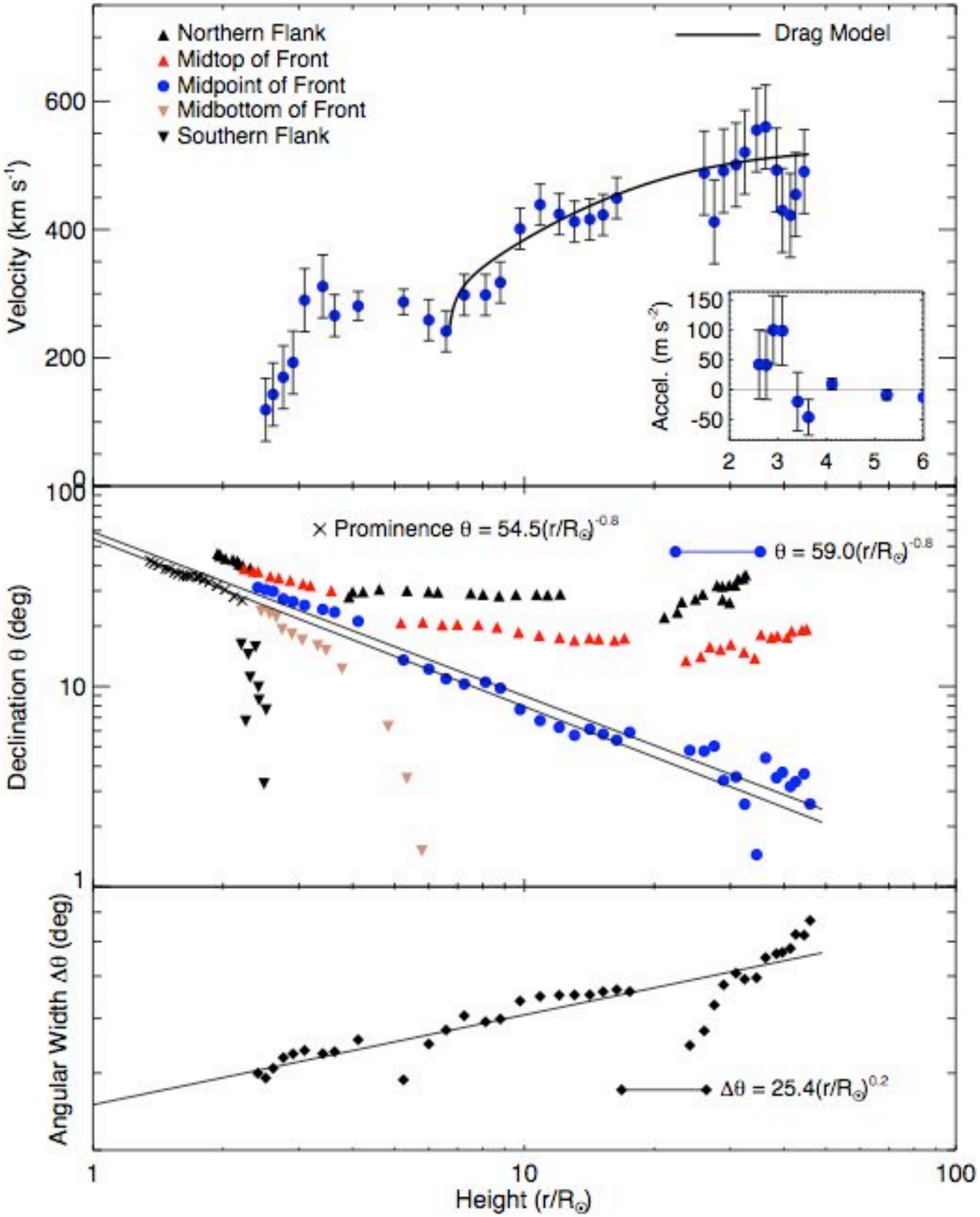
Power Law: least squares fit

$$y = Ax^B$$

$$\ln y = B \ln x + \ln A$$

$$b = \frac{n\sum_{i=1}^{n}(\ln x_i \ln y_i) - \sum_{i=1}^{n}(\ln x_i)\sum_{i=1}^{n}(\ln y_i)}{n\sum_{i=1}^{n}(\ln x_i)^2 - \left(\sum_{i=1}^{n}\ln x_i\right)^2} \qquad B \equiv b$$

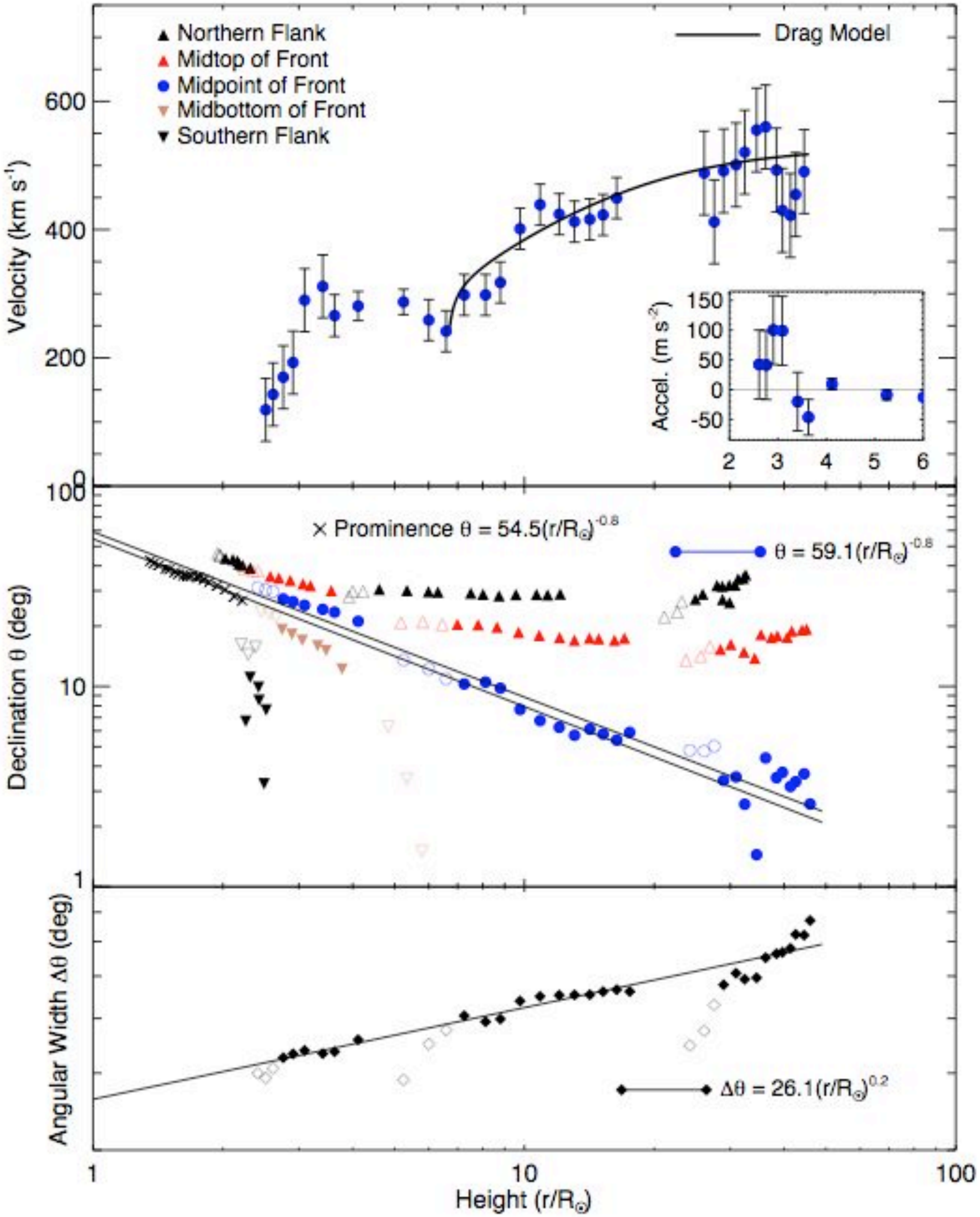
$$a = \frac{\sum_{i=1}^{n}(\ln y_i) - b\sum_{i=1}^{n}(\ln x_i)}{n} \qquad A \equiv e^a$$



$$\theta_{lsq}(R) = 59.0R^{-0.8}$$

$$\theta_{lsq}^{prom}(R) = 54.5R^{-0.8}$$

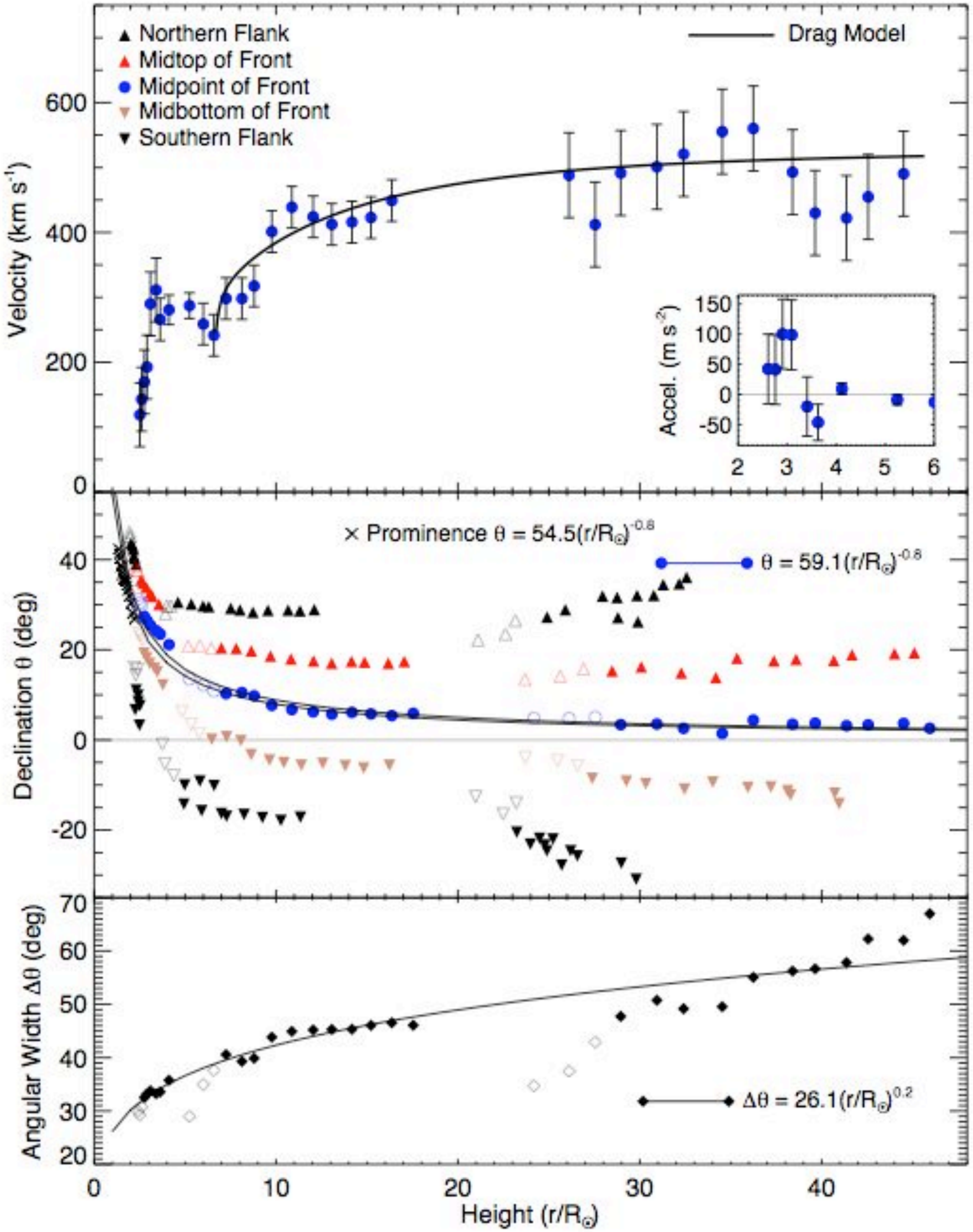
$$\Delta\theta_{lsq}(R) = 25.4R^{0.2}$$



$$\theta_{lsq}(R) = 59.1R^{-0.8}$$

$$\theta_{lsq}^{prom}(R) = 54.5R^{-0.8}$$

$$\Delta\theta_{lsq}(R) = 26.1R^{0.2}$$



$$\theta_{lsq}(R) = 59.1R^{-0.8}$$

$$\theta_{lsq}^{prom}(R) = 54.5R^{-0.8}$$

$$\Delta\theta_{lsq}(R) = 26.1R^{0.2}$$

Gradient Descent Method

The sum of the squares of the deviations of the expected 'fit' from the observables:

$$\chi^2(\vec{p}) = \sum_i \left(\frac{y(t_i) - \hat{y}(t_i;\vec{p})}{w_i} \right)^2$$

Matrix Notation:

$$= (y - \hat{y}(\vec{p}))^{\mathrm{T}} W (y - \hat{y}(\vec{p}))$$

Gradient of the function:

$$\frac{\partial}{\partial \vec{p}} \chi^2(\vec{p}) = (y - \hat{y}(\vec{p}))^{\mathrm{T}} W \frac{\partial}{\partial \vec{p}} (y - \hat{y}(\vec{p}))$$

$$= -(y - \hat{y}(\vec{p}))^{\mathrm{T}} W \left[\frac{\partial}{\partial \vec{p}} \hat{y}(\vec{p}) \right]$$

Jacobian:

$$= -(y - \hat{y}(\vec{p}))^{\mathrm{T}} W J$$

Perturbation in direction of steepest descent:

$$h_{gd} = \alpha J^{\mathrm{T}} W (y - \hat{y}(\vec{p}))$$

Gauss-Newton Method

The function with perturbed parameters may be locally approximated with a first order Taylor expansion:

$$\hat{y}(\vec{p}+h) \approx \hat{y}(\vec{p}) + \left[\frac{\partial \hat{y}}{\partial \vec{p}}\right] h$$

$$= \hat{y} + Jh$$

Subbing in to chi-squared:

$$\chi^2(\vec{p}) = (y - \hat{y}(\vec{p}))^{\mathrm{T}} W (y - \hat{y}(\vec{p}))$$

$$\chi^2(\vec{p}+h) = (y - (\hat{y} + Jh))^{\mathrm{T}} W (y - (\hat{y} + Jh))$$

Results in a quadratic in h:

$$= \ldots + \ldots - \ldots - (y - \hat{y})^{\mathrm{T}} WJh + h^{\mathrm{T}} J^{\mathrm{T}} WJh$$

The perturbation that minimises the chi-squared is found where the first derivative is zero:

$$\frac{\partial}{\partial h} \chi^2(\vec{p}+h) \approx -(y - \hat{y})^{\mathrm{T}} WJ + h^{\mathrm{T}} J^{\mathrm{T}} WJ$$

$$\left[J^{\mathrm{T}} WJ\right] h_{gn} = J^{\mathrm{T}} W (y - \hat{y})$$

Levenberg-Marquardt Algorithm

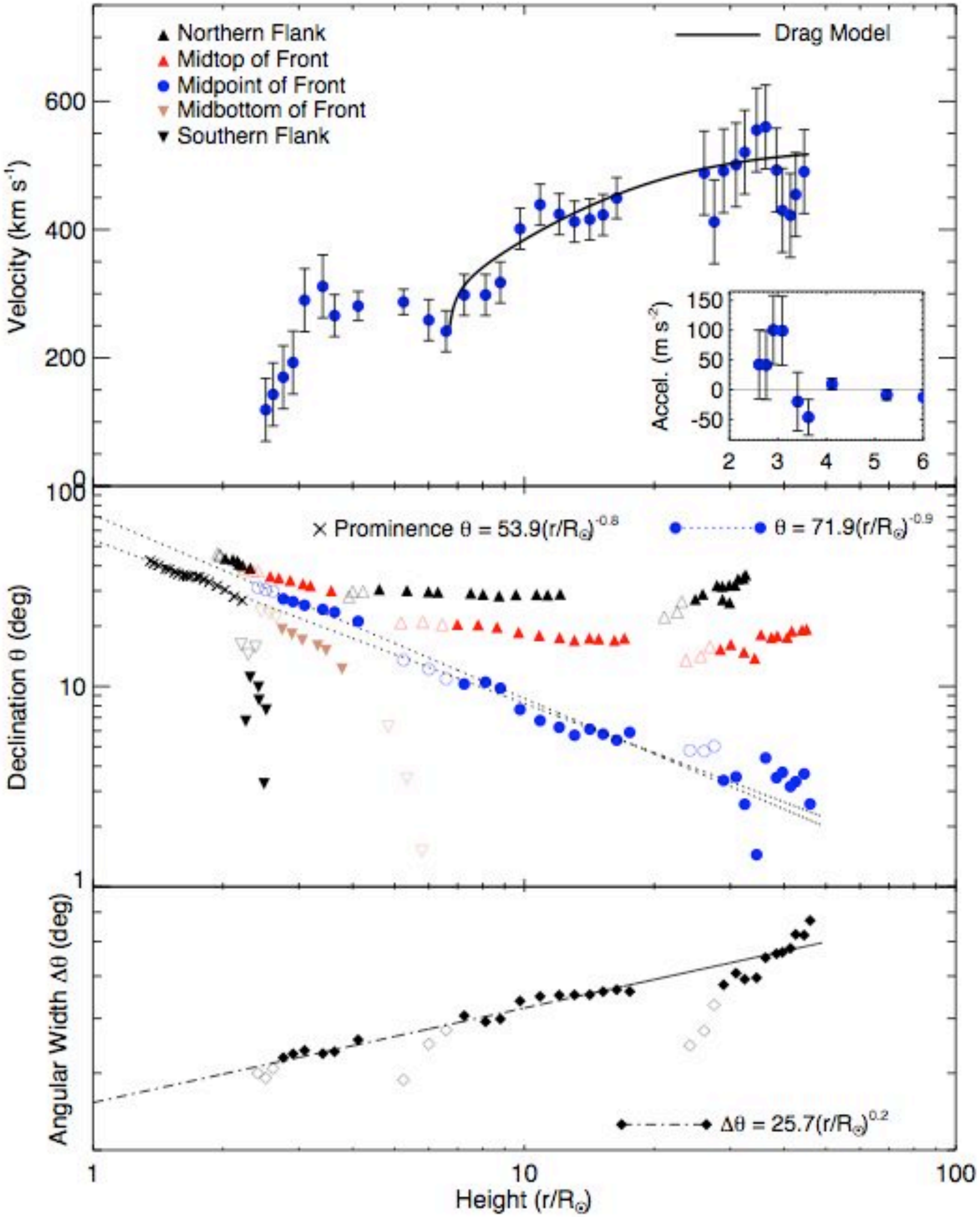
Gradient Descent: $$h_{gd} = \alpha J^{\mathrm{T}} W (y - \hat{y})$$

Gauss-Newton: $$\left[J^{\mathrm{T}} W J\right] h_{gn} = J^{\mathrm{T}} W (y - \hat{y})$$

Levenberg-Marquardt: $$\left[J^{\mathrm{T}} W J + \lambda I\right] h_{lm} = J^{\mathrm{T}} W (y - \hat{y})$$

Iterations proceed, varying between the two methods to update the parameters, converging on an optimum fit.

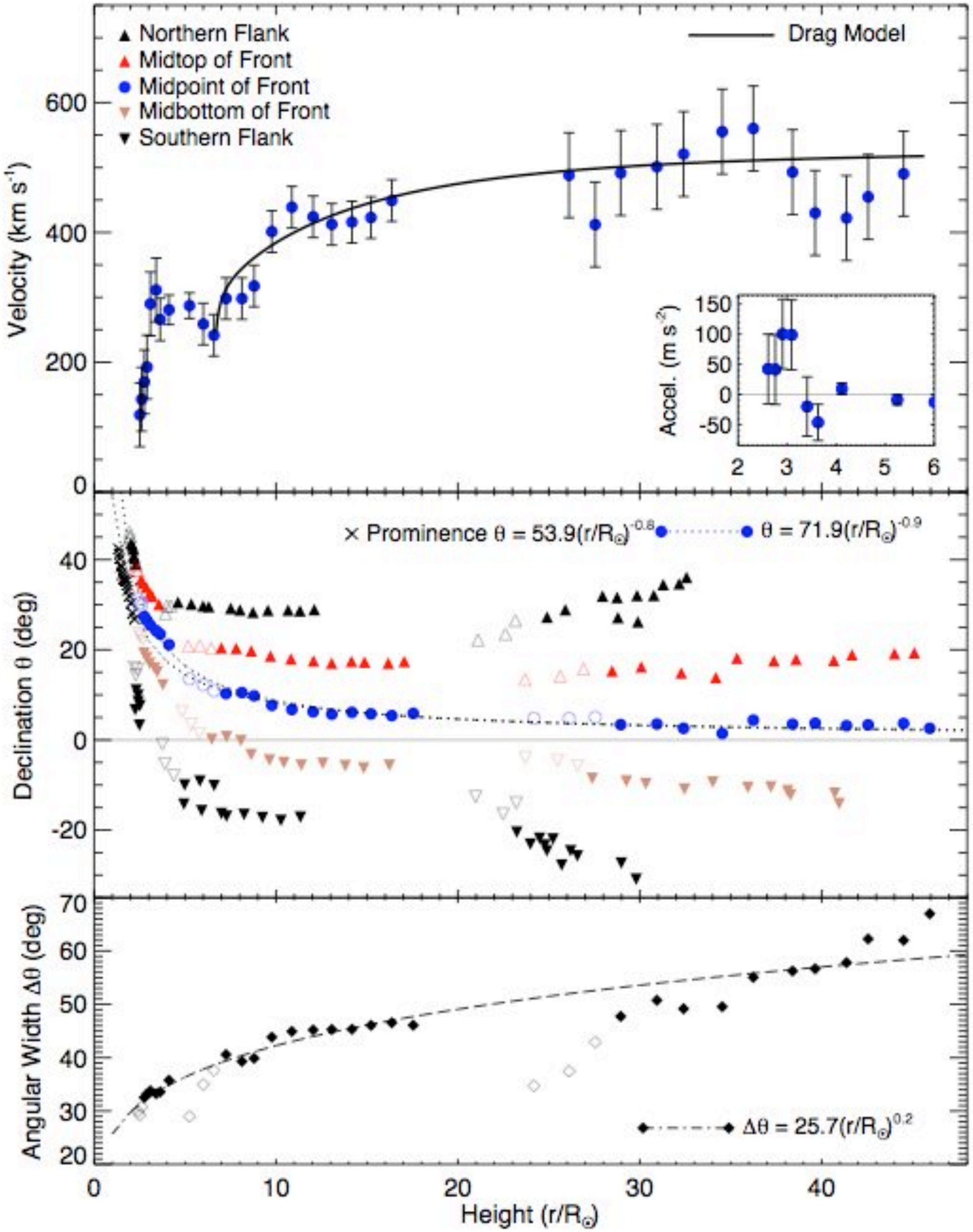
```
y = ‘ p[0] * x^p[1] ’

f = mpfitexpr(y, t, z)

model = f[0] * t^f[1]
```



$$\theta_{lma}(R) = 71.9R^{-0.9}$$

$$\theta_{lma}^{prom}(R) = 53.9R^{-0.8}$$

$$\Delta\theta_{lma}(R) = 25.7R^{0.2}$$



$$\theta_{lma}(R) = 71.9R^{-0.9}$$

$$\theta_{lma}^{prom}(R) = 53.9R^{-0.8}$$

$$\Delta\theta_{lma}(R) = 25.7R^{0.2}$$

Dipole Field

Magnetic dipole field:
(symmetric about the azimuth)

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{M}{r^3}(-2\sin\lambda\hat{e}_r + \cos\lambda\hat{e}_\lambda)$$

Field strength:

$$B = \frac{\mu_0}{4\pi}\frac{M}{r^3}(1 + 3\sin^2\lambda)^{1/2}$$

Arc length:

$$d\vec{s} \times \vec{B} = 0 \qquad \frac{dr}{B_r} = \frac{rd\lambda}{B_\lambda}$$

Subbing:

$$\frac{dr}{r} = \frac{B_r}{B_\lambda}d\lambda = \frac{-2\sin\lambda}{\cos\lambda}d\lambda = \frac{2d(\cos\lambda)}{\cos\lambda}$$

Integrating:

$$\int\frac{dr}{r} = \int\frac{2d(\cos\lambda)}{\cos\lambda}$$

$$\ln r = 2\ln\cos\lambda + c \qquad c = \ln r_{eq}$$

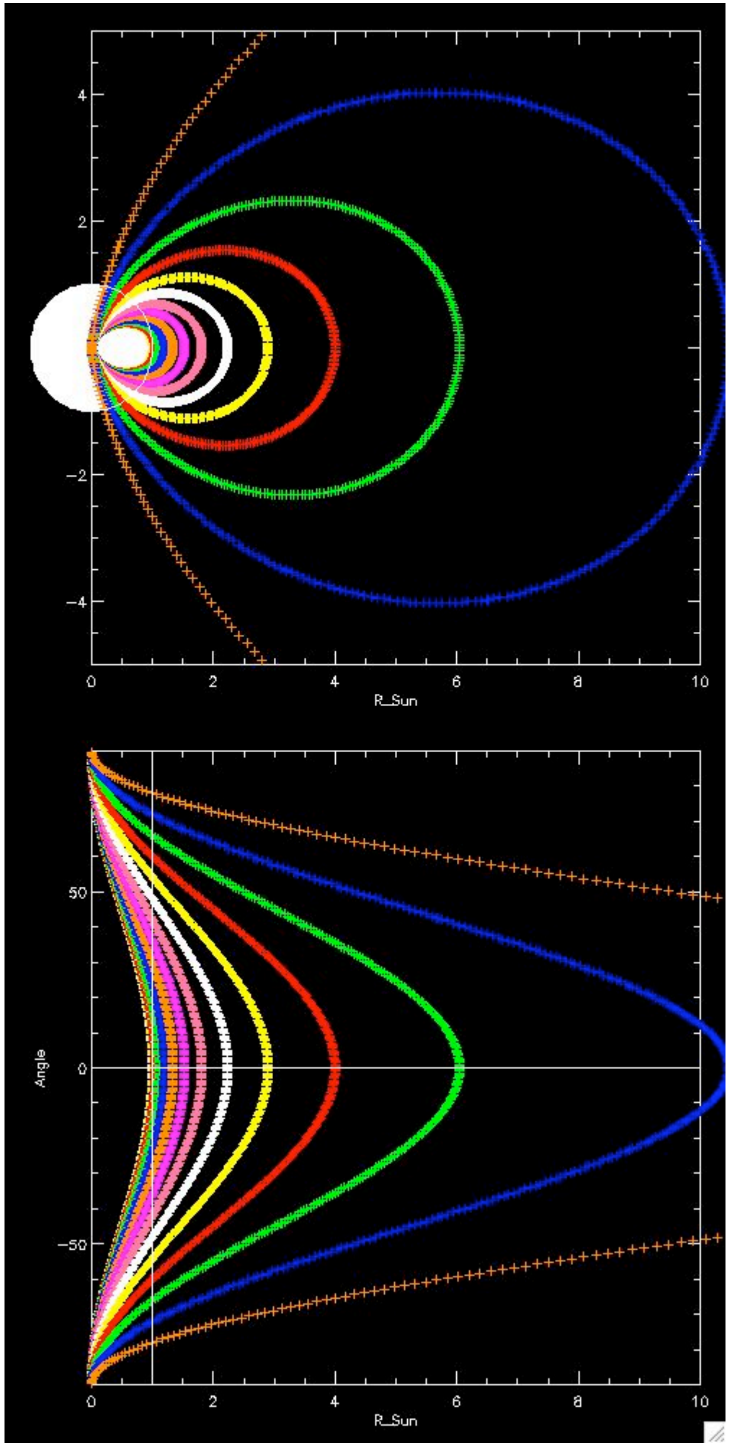
$$\therefore r = r_{eq}\cos^2\lambda \qquad r(\lambda = 0) = r_{eq}$$

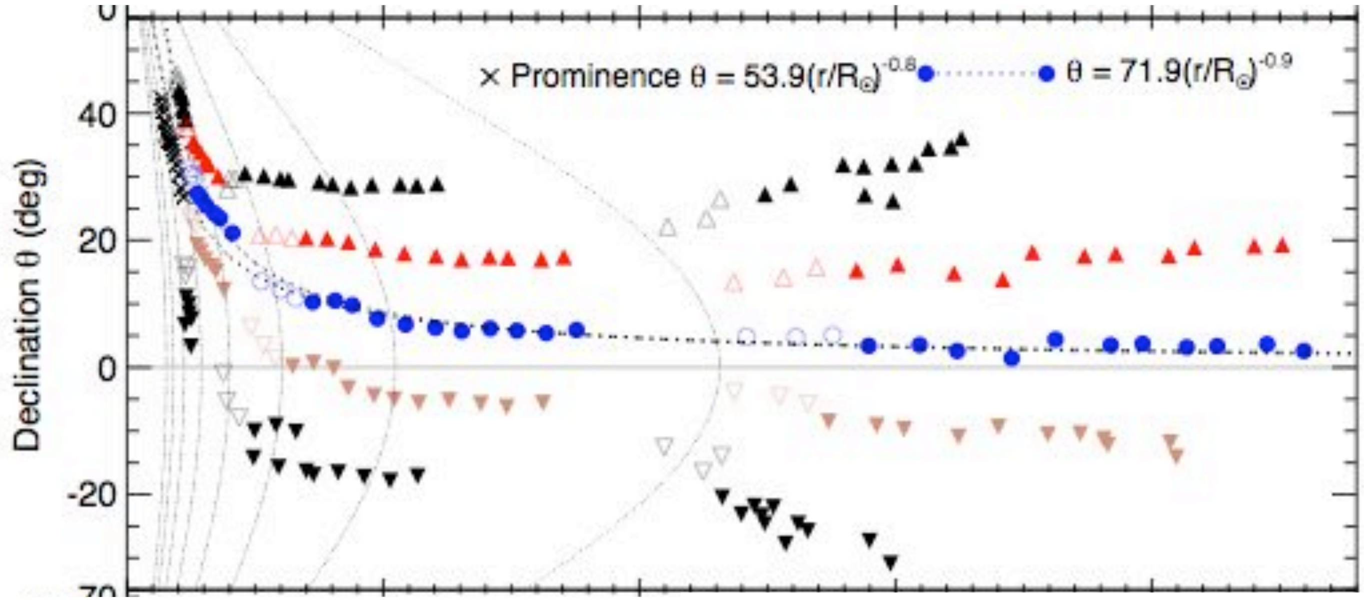
Model Solar Dipole field:

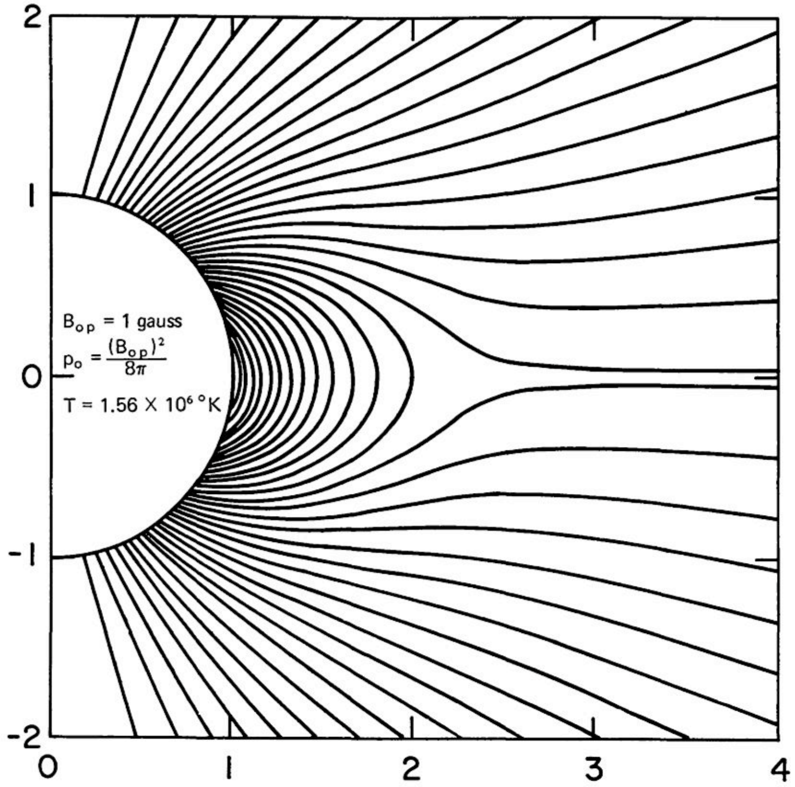
$$r = r_{eq} \cos^2 \lambda$$

Source is photosphere at 1 $R_\odot$

Field line at 40 deg. hits ecliptic at 1.7 $R_\odot$
Field line at 50 deg. hits ecliptic at 2.4 $R_\odot$
Field line at 60 deg. hits ecliptic at 4.0 $R_\odot$







"the pressure and inertial forces of the solar wind eventually dominate and distend the field outward into interplanetary space."

Pneumann & Kopp, 1971

Deflection from initial trajectory:

Power law fit:
$$f(x) = -x^{1.3}$$
$$f'(x) = -1.3x^{0.3}$$

Force ∝ accel:
$$f''(x) = -0.39x^{-0.7}$$

Magnetic dipole force downward:

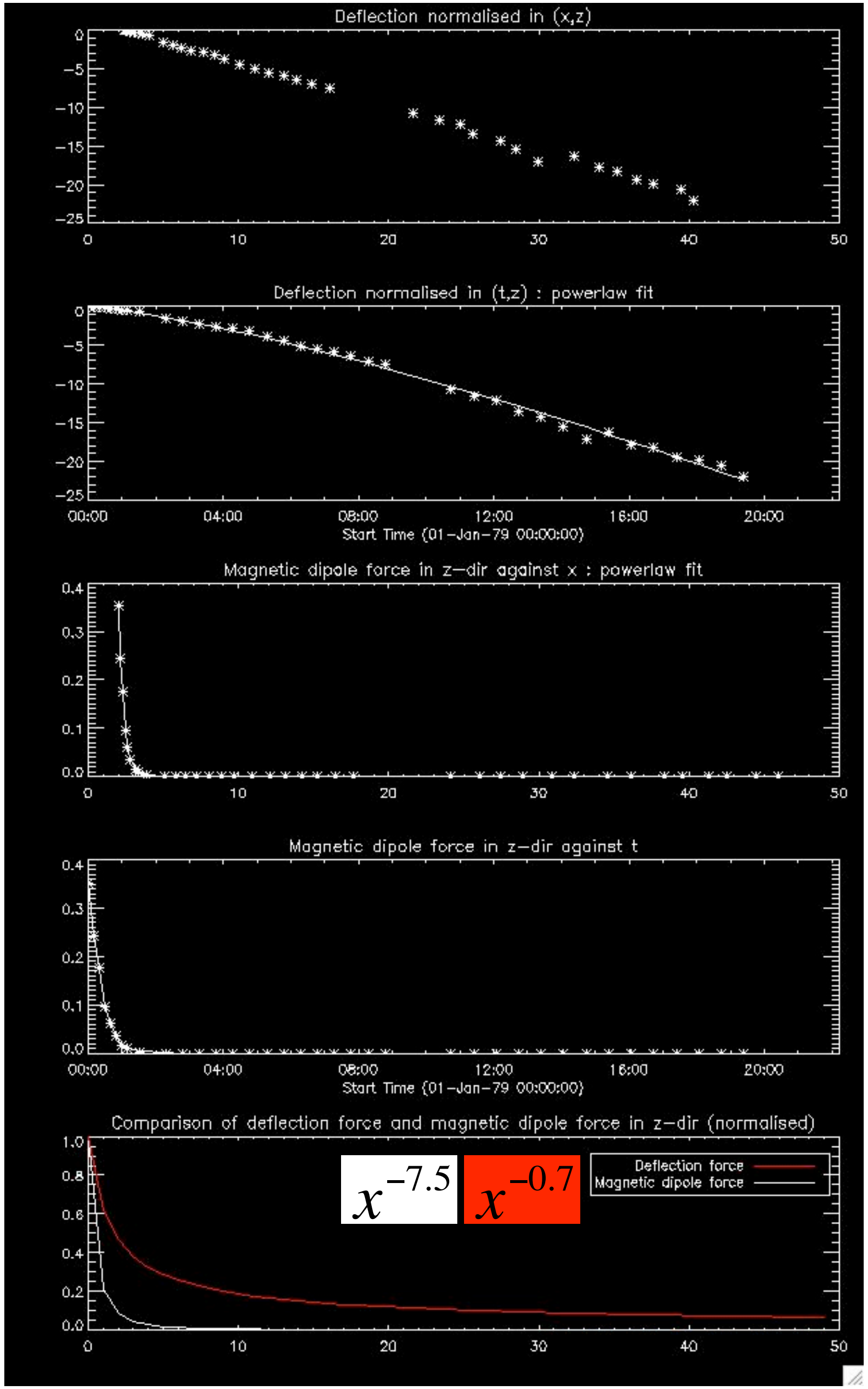
$$F = -\nabla P_B = -\nabla \frac{B^2}{8\pi}$$

$$B = \frac{\mu_0}{4\pi}\frac{M}{r^3}(1+3\sin^2\lambda)^{1/2}$$

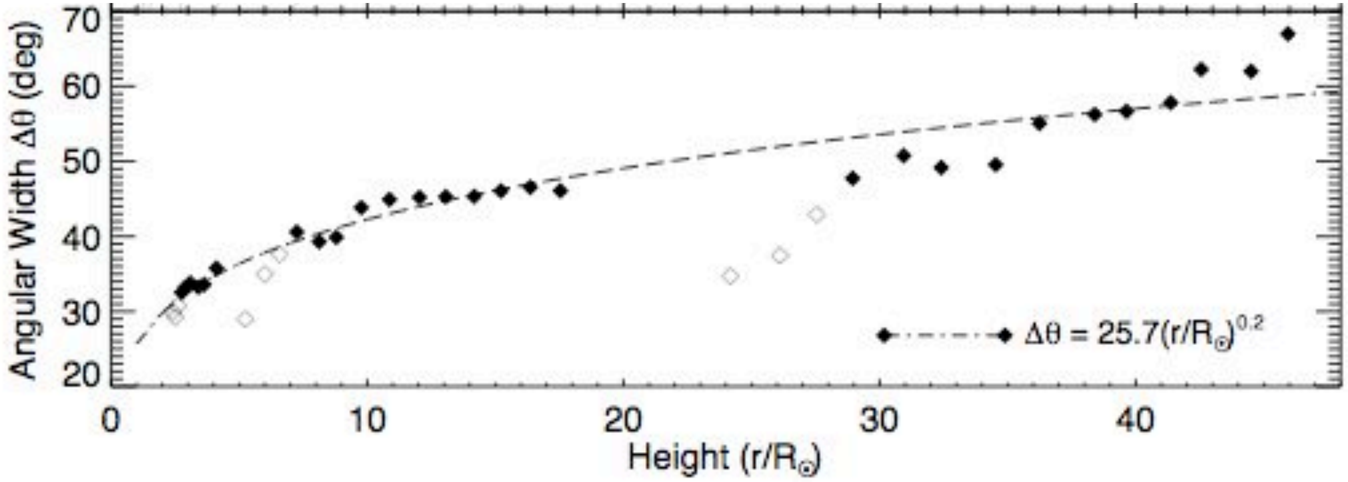
Cylindrical coords:

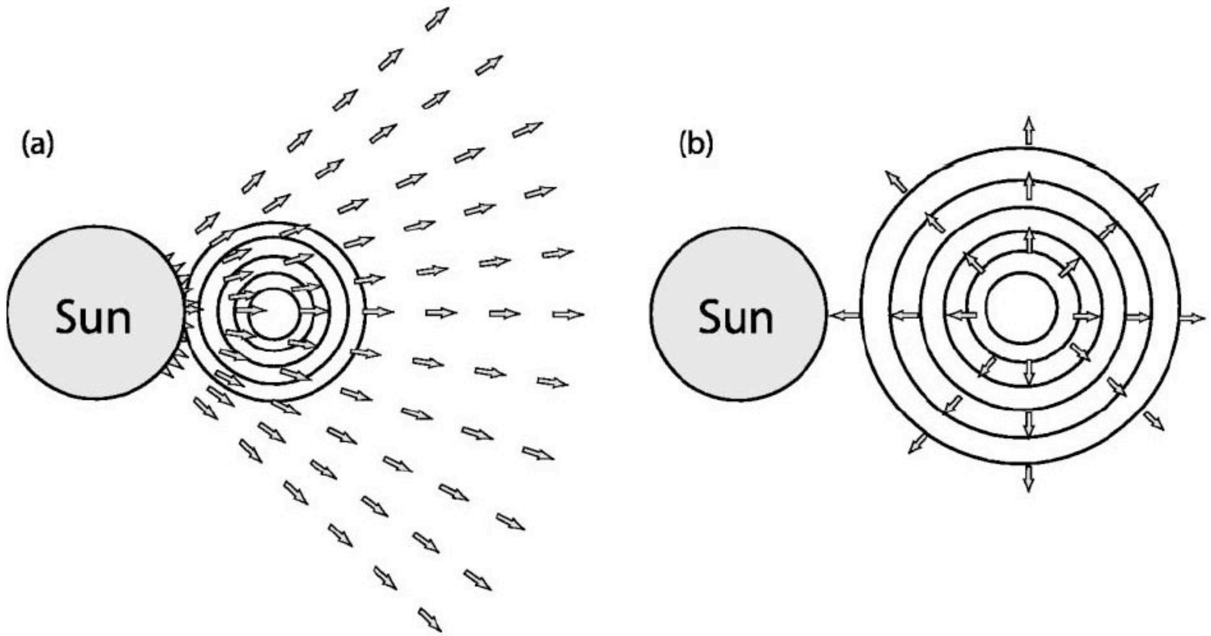
$$r^2 = z^2 + \rho^2 \qquad \lambda = \arcsin\left(\frac{z}{\sqrt{z^2+\rho^2}}\right)$$

$$F_z = -\left(\frac{\mu_0 M^2}{128\pi^3}\right)\frac{8z}{(z^2+\rho^2)^4}\left(1-\frac{4z^2+\rho^2}{z^2+\rho^2}\right)$$



CME Expansion





Polytropic Process

$$PV^n = C$$

$$P = K\rho^\gamma \quad \gamma = 1 + \frac{1}{n}$$

...solar wind ~ 1.46

explosion	$n < 0$
isobaric	$n = 0 \Rightarrow PV^0 = P$
isothermal	$n = 1 \Rightarrow PV = NkT$
adiabatic	$n = \gamma = \frac{c_P}{c_V}$
isochoric	$n = \infty$

Wang et al. (2009) suggest polytropic index of CMEs should be greater than 2/3 to ensure the flux-rope will finally approach a steady expansion and propagation state.