Autonomous Differential Equations

An *autonomous* differential equation is one where there is no explicit occurrence of the independent variable:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)}).$$

In particular, a first-order autonomous equation has the form

$$y' = f(y).$$

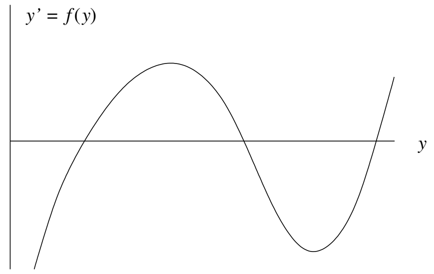
Note that a first-order autonomous equation is separable, since we can rewrite it in the form

$$\frac{dy}{f(y)} = dt.$$

Autonomous equations arise fairly frequently in science and engineering, since they reflect "timeless" laws of nature.

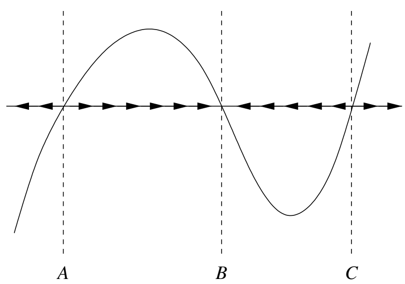
Since the equation is autonomous, the slopes in the slope field are constant as we scan left-to-right — they only change with y, not t. For this reason it is often more revealing to plot y' versus y, on a separate graph, and then use this graph to plot the slope field.

For example, suppose that the graph of $f(y)$ is as pictured.



This is *not* a picture of the slope field nor of an integral curve. This picture does not involve t explicitly. It shows how y' depends on y. We view y as the independent variable in this picture. When the graph is above the (*horizontal!*) y-axis, $y' > 0$ and hence y is increasing. Since y is represented in this picture along the horizontal axis, we can draw horizontal arrows indicating the tendency of y, based on the graph of $f(y)$.

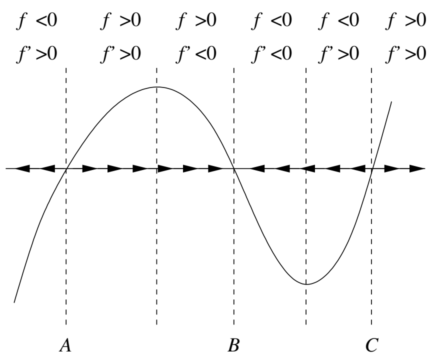
If k is a root of the equation $f(y) = 0$, then the constant function $\phi(t) = k$ is a solution of the differential equation $y' = f(y)$. Such constant solutions are called *equilibrium* values, since at these points the system is in balance. An equilibrium can be stable or unstable, depending on whether nearby values move away from or towards the equilibrium. In our example, the equilibria A and C are unstable, while the equilibrium B is stable. Can you explain why?

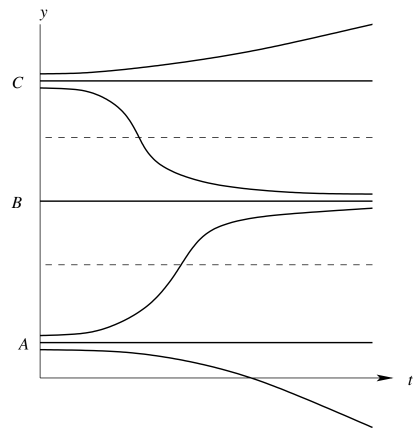


We can even deduce the concavity from this picture. If $y' = f(y)$ then Chain Rule tells us that

$$y'' = \frac{d}{dt}(y') = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy}\, f(y).$$

Thus, $y'' > 0$ if either both f and df/dy are positive or when both are negative. If one is positive and the other is negative, $y'' < 0$. At the inflection points, $df/dy = 0$. Putting all of this information together we can now sketch a few representive integral curves.





Further reading

Read section 2.5. Good practice problems are 1–6, 16–17, and 20, pages 88–90. For each of these, analyze the dynamics using the graph of y' against y, just as above. Use this analysis to determine the slope field. Identify and classify the equilibria. Next, solve the differential equation, and graph several solutions on your slope field. Summarize the behavior of the solutions, paying particular attention to the domain of validity and the asymptotic behavior. How does this behavior depend on the initial values?

Reading quiz

1. What is an autonomous equation? Give both the general form and some examples.
2. What is an equilibrium value for an autonomous equation?
3. What does it mean for an equilibrium value to be stable?
4. Describe how to find the regions of concavity for an autonomous equation.

Assignment 6: due Monday, 13 February

1. Suppose instead we wanted to model a system with an autonomous equation having stable equilibria at -1 and 3, and unstable equilibria at 0 and 5. Write down a suitable differential equation, then solve and analyze your equation as above.
2. For each of the following equations, analyze the graph of y' versus y, as above. Make sure to find and classify the equilibria and the regions of concavity. Use this information to draw the slope field and several integral curves. Finally, solve the equation, if possible.

   (a) $y' = y - y^3$.
   (b) $y' = 1 - e^{-y}$.
   (c) $y' = \tan(y)$.
   (d) $y' = (y-1)/(y+1)$.
   (e) $y' = y\ \ln(y)$.