

# COMP 576 - Fall 2017

## Assignment 1

September 30, 2017

### Backpropagation in a Simple Neural Network

#### 1a) Dataset

#### 1b) Derivatives of Activation Functions

Sigmoid:

$$\begin{aligned}f(x) &= \frac{1}{1 + e^{-x}} = (1 + e^{-x})^{-1} \\ \frac{d(1 + e^{-x})^{-1}}{dx} &= (1 + e^{-x})^{-2}(e^{-x}) \\ &= \frac{e^{-x}}{(1 + e^{-x})^2} = \frac{1}{1 + e^{-x}} \frac{e^{-x}}{1 + e^{-x}} \\ &= \frac{1}{1 + e^{-x}} \frac{(1 + e^{-x}) - 1}{1 + e^{-x}} \\ &= \frac{1}{1 + e^{-x}} \left(1 - \frac{1}{1 + e^{-x}}\right) \\ \frac{d(1 + e^{-x})^{-1}}{dx} &= f(x)(1 - f(x))\end{aligned}$$

Tanh:

$$\begin{aligned}f(x) &= \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \\ \frac{d\left(\frac{\sinh(x)}{\cosh(x)}\right)}{dx} &= \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)} \\ &= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{\cosh^2(x)}{\cosh^2(x)} - \frac{\sinh^2(x)}{\cosh^2(x)} \\ \frac{d\left(\frac{\sinh(x)}{\cosh(x)}\right)}{dx} &= 1 - \tanh^2(x)\end{aligned}$$

ReLu:

$$f(x) = \max(0, x)$$

$$f(x) = \begin{cases} x, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f'(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

## 1c) Building the Neural Network

Three Layer Network

$$z^1 = W^1 x + b^1 \quad (1)$$

$$a^1 = \text{actFun}(z^1) \quad (2)$$

$$z^2 = W^2 a^1 + b^2 \quad (3)$$

$$a^2 = \hat{y} = \text{softmax}(z^2) \quad (4)$$

Mean Cross Entropy Loss of Batch

$$L(y, \hat{y}) = -\frac{1}{N} \sum_{n=1}^N \sum_{i=1}^C y_{n,i} \log \hat{y}_{n,i} \quad (5)$$

## 1d) Backward Pass - Backpropagation

Gradients:  $\frac{\partial L}{\partial W^2}, \frac{\partial L}{\partial b^2}, \frac{\partial L}{\partial W^1}, \frac{\partial L}{\partial b^1}$

$$\frac{\partial L}{\partial W^2} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z^2} \frac{\partial z^2}{\partial W^2}$$

$$\frac{\partial z^2}{\partial W^2} = \frac{\partial (W^2 a^1 + b^2)}{\partial W^2}$$

$$\frac{\partial z^2}{\partial W^2} = a^1$$

$$\frac{\partial \hat{y}_i}{\partial z_i^2} = \frac{\partial \left( \frac{e^{z_i^2}}{\sum_{j=1}^C e^{z_j^2}} \right)}{\partial z_i^2}$$

if  $j = i$

$$= \frac{e^{z_i^2} \sum_{j=1}^C e^{z_j^2} - e^{z_i^2} e^{z_i^2}}{\left( \sum_{j=1}^C e^{z_j^2} \right)^2} = \frac{e^{z_i^2} (\sum_{j=1}^C e^{z_j^2} - e^{z_i^2})}{\sum_{j=1}^C e^{z_j^2} \sum_{j=1}^C e^{z_j^2}}$$

$$\frac{\partial \hat{y}_i}{\partial z_i^2} = \hat{y}_i (1 - \hat{y}_i)$$

if  $j \neq i$

$$\frac{\partial \hat{y}_i}{\partial z_j^2} = \frac{0 - e^{z_i^2} e^{z_j^2}}{(\sum_{j=1}^C e^{z_j^2})^2}$$

$$\frac{\partial \hat{y}_i}{\partial z_j^2} = -\hat{y}_i \hat{y}_j$$

$$\frac{\partial L}{\partial \hat{y}_i} = -\frac{1}{N} \sum_{n=1}^N \sum_{i=1}^C \frac{\partial (y_i \log \hat{y}_i)}{\partial \hat{y}_i} = -\frac{1}{N} \sum_{n=1}^N \sum_{i=1}^C y_i \left( \frac{1}{\hat{y}_i} \right) \frac{\partial \hat{y}_i}{\partial z_i^2}$$

$\forall i$

$$= -\frac{1}{N} \sum_{n=1}^N \left[ \frac{y_i}{\hat{y}_i} (1 - \hat{y}_i) + \sum_{j \neq i}^C \frac{y_j}{\hat{y}_j} (-\hat{y}_j \hat{y}_i) \right]$$

$$= -\frac{1}{N} \sum_{n=1}^N \left[ y_i (1 - \hat{y}_i) - \sum_{j \neq i}^C y_j \hat{y}_i \right]$$

$$= \frac{1}{N} \sum_{n=1}^N \left[ -y_i + y_i \hat{y}_i + \sum_{j \neq i}^C y_j \hat{y}_i \right]$$

$$= \frac{1}{N} \sum_{n=1}^N \left[ -y_i + \sum_{j=1}^C y_j \hat{y}_i \right]$$

= since  $\mathbf{y}_j$  is one-hot encoded

$$\implies \sum_{j=1}^C y_j = 1$$

$$\therefore \frac{\partial L}{\partial z_i^2} = \frac{1}{N} \sum_{n=1}^N (\hat{y}_i - y_i)$$

Let  $\delta^3 = (\hat{y}_i - y_i)$

$$\frac{\partial L}{\partial W^2} = \frac{1}{N} \sum_{n=1}^N \delta^3 a^3$$

$$\frac{\partial L}{\partial W^2} = \frac{1}{N} \mathbf{a}^{1T} \delta^3$$

$$\frac{\partial L}{\partial b^2} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z^2} \frac{\partial z^2}{\partial b^2}$$

$$\frac{\partial z^2}{\partial b^2} = 1$$

$$\frac{\partial L}{\partial b^2} = \frac{1}{N} \sum_{n=1}^C \delta^3$$

$$\begin{aligned} \frac{\partial L}{\partial W^1} &= \underbrace{\frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z^2}}_{\frac{1}{N} \sum_{i=1}^N \delta^3} \underbrace{\frac{\partial z^2}{\partial a^1}}_{W^2} \underbrace{\frac{\partial a^1}{\partial z^1}}_{f'(z^1)} \underbrace{\frac{\partial z^1}{\partial W^1}}_x \\ &= \frac{1}{N} W^{2T} \delta^3 f'(z^1) \end{aligned}$$

$$\text{Let } \delta^2 = W^{2T} \delta^3 f'(z^1)$$

$$= \frac{1}{N} \delta^2 x$$

$$\frac{\partial L}{\partial W^1} = \frac{1}{N} x^T \delta^2$$

$$\frac{\partial L}{\partial b^1} = \underbrace{\frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z^2} \frac{\partial z^2}{\partial a^1} \frac{\partial a^1}{\partial z^1}}_{\frac{1}{N} \delta^2} \underbrace{\frac{\partial z^1}{\partial b^1}}_1$$

$$\frac{\partial L}{\partial b^1} = \frac{1}{N} \delta^2$$