Definitions

Let V be a vector space over $\mathbb{R}$. The dual space of V is V^*, the set of linear functions $V \to \mathbb{R}$. Let $0 \le k < \dim(V)$. The tensor product $(V^*)^{\otimes k}$ is the set of k-linear functions $V^k \to \mathbb{R}$. The exterior k-forms $\Lambda^k(V^*)$ are a subset of $(V^*)^{\otimes k}$, specifically the asymmetric functions.

Let M be a smooth manifold. Then T_pM is the tangent plane of M at the point $p \in M$. Also, TM is informally used as a function $p \mapsto T_pM$. The dual space $(T_pM)^*$ is written T_p^*M. Similarly, T^*M refers to $p \mapsto T_p^*M$. (Apparently TM is called the tangent bundle and T^*M is called the cotangent bundle.) Note that $T_p\mathbb{R}^n$ is $\mathbb{R}^n$ itself for any $p \in \mathbb{R}^n$.

Many of these linear algebraic objects can be parameterized by a point on a manifold. For example, vector spaces are generalized by smooth vector fields. For a smooth manifold M, define $\mathfrak{X}(M)$ as the set of smooth functions that take each point $p \in M$ to a vector in T_pM. Restricting to a single point p, we get a vector space (T_pM).
Differential forms generalize exterior forms in the same way. For $0 \le k < \dim(V)$, $\Omega^k(M)$ is the set of "smooth" functions that take each point $p \in M$ to a k-form in $\Lambda^k(T_p^*M)$. Restricting to a single point p, all we get a space of k-forms $(\Lambda^k(T_p^*M))$. The differential forms $\Omega^k(M)$ were defined in class as $\{\sum_I a_I \mathrm{d}x_I \mid a_I \text{ smooth}\}$. This is essentially equivalent to the definition here, except "smooth" is in quotes since we never defined what it means for a function like this to be smooth, but it is the same idea. Note that $\Omega^0(M)$ is just $C^\infty(M, \mathbb{R})$, the set of all smooth functions $M \to \mathbb{R}$.

d

The symbol d has many meanings. Suppose $f : M \to N$ is a smooth function between smooth manifolds M and N. Then for $p \in M$, $\mathrm{d}f|_p : T_pM \to T_{f(p)}N$ is linear. Explicitly, with $v \in T_pM$, $\mathrm{d}f|_p(v) = \sum \frac{\partial f}{\partial x_i} v_i$.

$\mathrm{d} : \Omega^k \to \Omega^{k+1}$

Pullbacks

Let $f : M \to N$ be smooth and $\omega \in \Omega^k(N)$, so for $q \in N$, $\omega|_q \in \Lambda^k(T_q^*N)$. We want the pullback to be $f^*\omega \in \Omega^k(M)$, so for $p \in M$, $(f^*\omega)|_p \in \Lambda^k(T_p^*M)$. Using the fact that exterior k-forms are k-linear functions, and for $p \in M$ and $v_1, \ldots, v_k \in T_pM$, the pullback is defined as

$$((f^*\omega)|_p)(v_1, \ldots, v_k) = (\omega|_{f(p)})(\mathrm{d}f|_p(v_1), \ldots, \mathrm{d}f|_p(v_k)).$$

The reason $\mathrm{d}f|_p$ is involved is because it maps from T_pM to $T_{f(p)}N$. In fact, the pullback $(f^*\omega)|_p$ is really the dual of the linear map $\mathrm{d}f|_p$, namely $(\mathrm{d}f|_p)^*(\omega|_{f(p)})$. Less precisely, pullbacks can be written by excluding all mentions of p and making the pointwise definition implicit:

$$f^*\omega(v_1, \ldots, v_k) = \omega(\mathrm{d}f(v_1), \ldots, \mathrm{d}f(v_k)).$$