Definitions

Let V be a vector space over $\mathbb{R}$. The dual space of V is V^*, the set of linear functions $V \to \mathbb{R}$. Let $0 \leq k < \dim(V)$. The tensor product $(V^*)^{\otimes k}$ is the set of k-linear functions $V^k \to \mathbb{R}$. The exterior k-forms $\Lambda^k(V^*)$ are a subset of $(V^*)^{\otimes k}$, specifically the asymmetric functions.

Let M be a smooth manifold. Then T_pM is the tangent plane of M at the point $p \in M$. Also, TM is informally used as a function $p \mapsto T_pM$. The dual space $(T_pM)^*$ is written T_p^*M. Similarly, T^*M refers to $p \mapsto T_p^*M$. (Apparently TM is called the tangent bundle and T^*M is called the cotangent bundle.) Note that $T_p\mathbb{R}^n$ is $\mathbb{R}^n$ itself for any $p \in \mathbb{R}^n$.

Many of these linear algebraic objects can be parameterized by a point on a manifold. For example, vector spaces are generalized by smooth vector fields. For a smooth manifold M, define $\mathfrak{X}(M)$ as the set of smooth functions that take each point $p \in M$ to a vector in T_pM. Restricting to a single point p, we get a vector space (T_pM).
Differential forms generalize exterior forms in the same way. For $0 \leq k < \dim(V)$, $\Omega^k(M)$ is the set of "smooth" functions that take each point $p \in M$ to a k-form in $\Lambda^k(T_p^*M)$. Restricting to a single point p, all we get a space of k-forms $(\Lambda^k(T_p^*M))$. The differential forms $\Omega^k(M)$ were defined in class as $\{\sum_I a_I \mathrm{d}x_I \mid a_I : M \to \mathbb{R} \text{ smooth}\}$, where $\{\mathrm{d}x_I\}$ forms a basis for $\Lambda^k(T_p^*M)$. This is essentially equivalent to the definition here, except "smooth" is in quotes since we never defined what it means for a function with a codomain of k-forms to be smooth, but it is the same idea. Note that $\Omega^0(M)$ is just $C^\infty(M, \mathbb{R})$, the set of all smooth functions $M \to \mathbb{R}$.

d

The symbol d has many meanings. Suppose $f : M \to N$ is a smooth function between smooth manifolds M and N. Then for $p \in M$, $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ is linear. It can also be written $\mathrm{d}f|_p$ instead of $\mathrm{d}f_p$.

If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, then for any $v \in T_pM$,

$$\mathrm{d}f_p(v) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_p v_i.$$

In matrix form, if $f(x) = (f_1(x), \ldots, f_n(x))$, $\mathrm{d}f_p$ has n rows and m columns, with ijth element $\frac{\partial f_i}{\partial x_j}\big|_p$. Equivalently,

$$\mathrm{d}f_p(v) = \lim_{h \to 0} \frac{1}{h}(f(p + hv) - f(p)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(p + tv).$$

The definition of $\mathrm{d}f$ on general manifolds is not necessary to include.

When written $\mathrm{d}f$, this implicitly means a function $p \mapsto \mathrm{d}f_p$. Similarly, if $X \in \mathfrak{X}(M)$, $\mathrm{d}f(X)$ implicitly means a function $p \mapsto \mathrm{d}f_p(X|_p)$ (see: pushforward).

Another meaning is $\mathrm{d} : \Omega^k \to \Omega^{k+1}$, from differential k-forms to differential $(k+1)$-forms. Let $\omega = \sum_I a_I \mathrm{d}x_I$, then $\mathrm{d}\omega = \sum_I \mathrm{d}a_I \wedge \mathrm{d}x_I$. Here, each a_I is a function $M \to \mathbb{R}$, so $\mathrm{d}a_I : T_pM \to T_{f(p)}\mathbb{R} = T_pM \to \mathbb{R} = T_p^*M$ is as above. At each point $p \in M$, $\mathrm{d}\omega|_p = \sum_I \mathrm{d}a_I|_p \wedge \mathrm{d}x_I$.

The operator D is defined by $D_v f(p) = \mathrm{d}f_p(v)$. It is the same thing with different placement of the arguments. Similarly, $Df(p) = \mathrm{d}f_p$ and $D_X f = \mathrm{d}f(X)$. The directional derivative of f in the direction of v at p is given by $D_v f(p)$. The directional derivative can also be written $D_v f(p) = \mathrm{d}f_p(v) = Df(p)v$. Another way to define $D_v f(p)$ is $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(p + tv)$.

Pushforward

Let $f : M \to N$ be smooth. The pushforward is written $\mathrm{d}f : \mathfrak{X}(M) \to \mathfrak{X}(N)$. This is the same as saying that for each $p \in M$, $\mathrm{d}f_p : T_pM \to T_{f(p)}N$, so the pushforward is nothing more than a pointwise application of $\mathrm{d}f$. For $X \in \mathfrak{X}(M)$, the pushforward $\mathrm{d}f(X)$ is also written Xf. This makes sense since, for example, if $X = \frac{\partial}{\partial x_i}$, then $\mathrm{d}f(X) = \frac{\partial f}{\partial x_i}$ since $\mathrm{d}f_p(X|_p) = \mathrm{d}f_p(e_i) = \frac{\partial f}{\partial x_i}\big|_p$ (in this example we are in Euclidean space).

Pullback

Let $f : M \to N$ be smooth and $\omega \in \Omega^k(N)$, so for $q \in N$, $\omega|_q \in \Lambda^k(T_q^* N)$. We want the pullback to be $f^*\omega \in \Omega^k(M)$, so for $p \in M$, $(f^*\omega)|_p \in \Lambda^k(T_p^* M)$. Using the fact that exterior k-forms are k-linear functions, and for $p \in M$ and $v_1, \ldots, v_k \in T_pM$, the pullback is defined as

$$((f^*\omega)|_p)(v_1, \ldots, v_k) = (\omega|_{f(p)})(\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_k)).$$

The reason $\mathrm{d}f|_p$ is involved is because it maps from T_pM to $T_{f(p)}N$. In fact, the pullback $(f^*\omega)|_p$ is really the dual of the linear map $\mathrm{d}f_p$, namely $(\mathrm{d}f_p)^*(\omega|_{f(p)})$. Less precisely, pullbacks can be written by excluding all mentions of p and making the pointwise definition implicit, with $X_i \in \mathfrak{X}(M)$:

$$f^*\omega(X_1, \ldots, X_k) = \omega(\mathrm{d}f(X_1), \ldots, \mathrm{d}f(X_k)).$$