

Let $\pi \in S_n$ be a permutation. Define $D(\pi) = |\{1 \leq i < n \mid \pi(i+1) < \pi(i)\}|$ to be the number of descents in π . Viewing D_n as a random variable on uniformly distributed permutations, a central limit theorem is known and we establish a local limit theorem.

As another way of viewing a permutation π , consider a sequence a_1, \dots, a_n with $1 \leq a_i \leq n - i + 1$. To get a permutation π from such a sequence, start with $S = \{1, \dots, n\}$. For $i = 1, \dots, n$, let b_i be the a_i th remaining element of S , let $\pi(i) = b_i$, and remove b_i from S . Then $\pi(i+1) < \pi(i)$ if and only if $a_{i+1} < a_i$, so the problem of counting descents in π is equivalent to the problem of counting descents in $\{a_i\}$. This is nice since the a_i are distributed independently.

Now, define the random variable X_i for $1 \leq i < n$ as the indicator for the i th potential descent, that is, 1 if $a_{i+1} < a_i$ and 0 otherwise. We can write $D_n = X_1 + \dots + X_n$. Since X_i depends only on a_i and a_{i+1} , X_i is independent of all other X_j except for X_{i-1} and X_{i+1} . We calculate

$$\begin{aligned}\mathbb{P}(X_i = 1 \mid X_{i-1} = 1, X_{i+1} = 1) &= 1/6 \\ \mathbb{P}(X_i = 1 \mid X_{i-1} = 1, X_{i+1} = 0) &= 1/2 \\ \mathbb{P}(X_i = 1 \mid X_{i-1} = 0, X_{i+1} = 1) &= 1/2 \\ \mathbb{P}(X_i = 1 \mid X_{i-1} = 0, X_{i+1} = 0) &= 5/6.\end{aligned}$$

To bound the characteristic function of D_n , we first fix the values of X_i for odd i . For simplicity, assume n is even. (When n is odd, one must also observe that $\mathbb{P}(X_i = 1 \mid X_{i-1} = 1) = 1/3$ and $\mathbb{P}(X_i = 1 \mid X_{i-1} = 0) = 2/3$ are constant.) The even X_i are independent of each other, so conditioned on the odd X_i , D_n is a sum of independent random variables. The values of the odd X_i can be viewed as a binary string, and since the distribution of the even X_i depend only on the adjacent odd X_{i-1} and X_{i+1} , the distribution of the sum of the even X_i depends entirely on the number of occurrences of each length 2 substring in the binary string.

Write N_{11} for the number of substrings 11, N_{10} for the number of substrings 10, and the same for N_{01} and N_{00} . Let $B_j^{(p)}$ denote an independent Bernoulli random variable with expectation p . We compute

$$\begin{aligned}|\phi_n(t)| &= \left| \mathbb{E} \left[e^{itD_n/\sigma} \right] \right| = \left| \mathbb{E} \left[\exp \left(it \left(\sum_{j=1}^{N_{11}} B_j^{(\frac{1}{6})} + \sum_{j=1}^{N_{10}} B_j^{(\frac{1}{2})} + \sum_{j=1}^{N_{01}} B_j^{(\frac{1}{2})} + \sum_{j=1}^{N_{00}} B_j^{(\frac{5}{6})} \right) / \sigma \right) \right] \right| \\ &= \left| \mathbb{E} \left[e^{it \sum_{j=1}^{N_{11}} B_j^{(\frac{1}{6})} / \sigma} \right] \mathbb{E} \left[e^{it \sum_{j=1}^{N_{10}} B_j^{(\frac{1}{2})} / \sigma} \right] \mathbb{E} \left[e^{it \sum_{j=1}^{N_{01}} B_j^{(\frac{1}{2})} / \sigma} \right] \mathbb{E} \left[e^{it \sum_{j=1}^{N_{00}} B_j^{(\frac{5}{6})} / \sigma} \right] \right|\end{aligned}$$

Using an argument similar to that of Gilmer-Kopparty [1], we bound each expectation separately.

$$\begin{aligned}\left| \mathbb{E} \left[e^{it \sum_{j=1}^N B_j^{(p)} / \sigma} \right] \right| &= \left| \prod_{j=1}^N \mathbb{E} \left[e^{it B_j^{(p)} / \sigma} \right] \right| \\ &\leq \left(1 - 8p(1-p) \left\| \frac{t}{2\pi\sigma_n} \right\|^2 \right)^N \\ &= \left(1 - 8p(1-p) \left(\frac{t}{2\pi\sigma_n} \right)^2 \right)^N \quad (\text{when } t < \pi\sigma_n) \\ &\leq \left(1 - C \left(\frac{t}{2\pi\sigma_n} \right)^2 \right)^N \quad (\text{taking } C = \min \{8p(1-p)\} = 80/9)\end{aligned}$$

Since $N_{11} + N_{10} + N_{01} + N_{00} = \frac{n}{2} - 1$, we get

$$\begin{aligned}|\phi_n(t)| &\leq \left(1 - C \left(\frac{t}{2\pi\sigma_n} \right)^2 \right)^{(\frac{n}{2}-1)} \\ &= (1 - \Theta(t^2/n))^{(\frac{n}{2}-1)} \\ &\leq e^{-\Theta(t^2)}.\end{aligned}$$

Now we obtain the final bound for the local limit theorem. As shown in [2], we have a central limit theorem

$$\left| \mathbb{P}\left(\frac{D_n - \mu}{\sigma} < t\right) - \mathbb{P}(Z < t) \right| \leq \frac{C}{\sqrt{n}}$$

from which we get

$$\left| \phi_n(t) - e^{-t^2/2} \right| \leq \frac{C}{\sqrt{n}}.$$

For any $\varepsilon > 0$, we compute

$$\begin{aligned} \int_{-\pi\sigma}^{\pi\sigma} \left| \phi_n(t) - e^{-t^2/2} \right| dt &\leq \int_{-n^\varepsilon}^{n^\varepsilon} \left| \phi_n(t) - e^{-t^2/2} \right| dt + \int_{n^\varepsilon < |t| < \pi\sigma} (|\phi_n(t)| + |e^{-t^2/2}|) dt \\ &\leq \int_{-n^\varepsilon}^{n^\varepsilon} \left| \phi_n(t) - e^{-t^2/2} \right| dt + \int_{n^\varepsilon < |t| < \pi\sigma} e^{-\Theta(t^2)} dt \\ &= O(n^{-\frac{1}{2} + \varepsilon}). \end{aligned}$$

References

- [1] Justin Gilmer and Swastik Kopparty. A local central limit theorem for the number of triangles in a random graph. *ArXiv e-prints*, November 2014.
- [2] Jason Fulman. Stein's Method and Non-Reversible Markov Chains. *ArXiv e-prints*, December 1997.