Let $\pi \in S_n$ be a permutation. Define $D(\pi) = |\{1 \le i < n \mid \pi(i+1) < \pi(i)\}|$ to be the number of descents in π. Viewing D_n as a random variable on uniformly distributed permutations, a central limit theorem is known and we establish a local limit theorem.

As another way of viewing a permutation π, consider a sequence $a_1, \ldots, a_n$ with $1 \le a_i \le n - i + 1$. To get a permutation π from such a sequence, start with $S = \{1, \ldots, n\}$. For $i = 1, \ldots, n$, let b_i be the a_ith remaining element of S, let $\pi(i) = b_i$, and remove b_i from S. Then $\pi(i+i) < \pi(i)$ if and only if $a_{i+1} < a_i$, so the problem of counting descents in π is equivalent to the problem of counting descents in $\{a_i\}$. This is nice since the a_i are distributed independently.

Now, define the random variable X_i for $1 \le i < n$ as the indicator for the ith potential descent, that is, 1 if $a_{i+1} < a_i$ and 0 otherwise. We can write $D_n = X_1 + \ldots + X_n$. Since X_i depends only on a_i and a_{i+1}, X_i is independent of all other X_j except for X_{i-1} and X_{i+1}. We calculate

$$\begin{aligned}
\mathbb{P}(X_i = 1 \mid X_{i-1} = 1, X_{i+1} = 1) &= 1/6 \\
\mathbb{P}(X_i = 1 \mid X_{i-1} = 1, X_{i+1} = 0) &= 1/2 \\
\mathbb{P}(X_i = 1 \mid X_{i-1} = 0, X_{i+1} = 1) &= 1/2 \\
\mathbb{P}(X_i = 1 \mid X_{i-1} = 0, X_{i+1} = 0) &= 5/6.
\end{aligned}$$

To bound the characteristic function of D_n, we first fix the values of X_i for odd i. For simplicity, assume n is even. (When n is odd, one must also observe that $\mathbb{P}(X_i = 1 \mid X_{i-1} = 1) = 1/3$ and $\mathbb{P}(X_i = 1 \mid X_{i-1} = 0) = 2/3$ are constant.) The even X_i are independent of each other, so conditioned on the odd X_i, D_n is a sum of independent random variables. The values of the odd X_i can be viewed as a binary string, and since the distribution of the even X_i depend only on the adjacent odd X_{i-1} and X_{i+1}, the distribution of the sum of the even X_i depends entirely on the number of occurrences of each length 2 substring in the binary string.

Write N_{11} for the number of substrings 11, N_{10} for the number of substrings 10, and the same for N_{01} and N_{00}. Let $B_j^{(p)}$ denote an independent Bernoulli random variable with expectation p. We compute

$$\begin{aligned}
|\phi_n(t)| = \left|\mathbb{E}\left[e^{itD_n/\sigma}\right]\right| &= \left|\mathbb{E}\left[\exp\left(it\left(\sum_{j=1}^{N_{11}} B_j^{(\frac{1}{6})} + \sum_{j=1}^{N_{10}} B_j^{(\frac{1}{2})} + \sum_{j=1}^{N_{01}} B_j^{(\frac{1}{2})} + \sum_{j=1}^{N_{00}} B_j^{(\frac{5}{6})}\right)/\sigma\right)\right]\right| \\
&= \left|\mathbb{E}\left[e^{it\sum_{j=1}^{N_{11}} B_j^{(\frac{1}{6})}/\sigma}\right]\mathbb{E}\left[e^{it\sum_{j=1}^{N_{10}} B_j^{(\frac{1}{2})}/\sigma}\right]\mathbb{E}\left[e^{it\sum_{j=1}^{N_{01}} B_j^{(\frac{1}{2})}/\sigma}\right]\mathbb{E}\left[e^{it\sum_{j=1}^{N_{00}} B_j^{(\frac{5}{6})}/\sigma}\right]\right|
\end{aligned}$$

Using an argument similar to that of Gilmer-Kopparty [1], we bound each expectation separately.

$$\begin{aligned}
\left|\mathbb{E}\left[e^{it\sum_{j=1}^{N} B_j^{(p)}/\sigma}\right]\right| &= \left|\prod_{j=1}^{N} \mathbb{E}\left[e^{itB_j^{(p)}/\sigma}\right]\right| \\
&\le \left(1 - 8p(1-p)\left\|\frac{t}{2\pi\sigma_n}\right\|^2\right)^N \\
&= \left(1 - 8p(1-p)\left(\frac{t}{2\pi\sigma_n}\right)^2\right)^N \qquad \text{(when } t < \pi\sigma_n\text{)} \\
&\le \left(1 - C\left(\frac{t}{2\pi\sigma_n}\right)^2\right)^N \qquad \text{(taking } C = \min\{8p(1-p)\} = 80/9\text{)}
\end{aligned}$$

Since $N_{11} + N_{10} + N_{01} + N_{00} = \frac{n}{2} - 1$, we get

$$\begin{aligned}
|\phi_n(t)| &\le \left(1 - C\left(\frac{t}{2\pi\sigma_n}\right)^2\right)^{(\frac{n}{2}-1)} \\
&= \left(1 - \Theta(t^2/n)\right)^{(\frac{n}{2}-1)} \\
&\le e^{-\Theta(t^2)}.
\end{aligned}$$

Now we obtain the final bound for the local limit theorem. As shown in [2], we have a central limit theorem

$$\sup_{-\infty<x<\infty}\left|\mathbb{P}(\frac{D_n-\mu}{\sigma}<t)-\mathbb{P}(Z<t)\right|\leq\frac{12}{\sqrt{n}}$$

.

For $y<0$, $P(D_n-\mu\leq y\sigma)=\frac{1}{2}P(|D_n-\mu|\geq|y|\,\sigma)\leq\frac{1}{2y^2}$ by Chebyshev's Inequality. Similarly, $P(D_n-\mu\leq y\sigma)\leq\frac{1}{2y^2}$.

Thus,

$$\left|\mathbb{P}(\frac{D_n-\mu}{\sigma}<t)-\mathbb{P}(Z<t)\right|=\begin{cases}\frac{12}{\sqrt{n}}\\ \mathbb{P}(\frac{D_n-\mu}{\sigma}\leq y)+\mathbb{P}(Z\leq y)\leq\frac{1}{2y^2}+e^{-\Theta(y^2)} & \text{for } y<0\\ \mathbb{P}(\frac{D_n-\mu}{\sigma}\geq y)+\mathbb{P}(Z\geq y)\leq\frac{1}{2y^2}+e^{-\Theta(y^2)} & \text{for } y>0\end{cases}$$

.

Hence, we have

$$\begin{aligned}\left|\phi_n(t)-e^{-t^2/2}\right| &\leq|t|\int_R\left|\mathbb{P}(\frac{D_n-\mu}{\sigma}<y)-\mathbb{P}(Z<y)\right|dy &(1)\\ &\leq|t|\int_{|y|>k\sigma}\left|\mathbb{P}(\frac{D_n-\mu}{\sigma}<y)-\mathbb{P}(Z<y)\right|dy+|t|\int_{|y|\leq k\sigma}\left|\mathbb{P}(\frac{D_n-\mu}{\sigma}<y)-\mathbb{P}(Z<y)\right|dy &(2)\\ &\leq|t|\int_{|y|>k\sigma}\frac{1}{2y^2}+e^{-\Theta(y^2)}dy+|t|2k\sigma\sqrt{\frac{12}{n}}. &(3)\end{aligned}$$

If we take $k=O(n^{-\frac{1}{2}+\epsilon})$. Since $\sigma=O(n)$, we have $\left|\phi_n(t)-e^{-t^2/2}\right|\leq|t|O(n^{-\frac{1}{2}+\epsilon})$.

Thus, for any $\varepsilon>0$, we compute

$$\begin{aligned}\int_{-\pi\sigma}^{\pi\sigma}\left|\phi_n(t)-e^{-t^2/2}\right|\mathrm{d}t &\leq\int_{-n^\varepsilon}^{n^\varepsilon}\left|\phi_n(t)-e^{-t^2/2}\right|\mathrm{d}t+\int_{n^\varepsilon<|t|<\pi\sigma}(|\phi_n(t)|+|e^{-t^2/2}|)\mathrm{d}t\\ &\leq\int_{-n^\varepsilon}^{n^\varepsilon}\left|\phi_n(t)-e^{-t^2/2}\right|\mathrm{d}t+\int_{n^\varepsilon<|t|<\pi\sigma}e^{-\Theta(t^2)}\mathrm{d}t\\ &=O(n^{-\frac{1}{2}+\varepsilon}).\end{aligned}$$

References

[1] Justin Gilmer and Swastik Kopparty. A local central limit theorem for the number of triangles in a random graph. *ArXiv e-prints*, November 2014.

[2] Jason Fulman. Stein's Method and Non-Reversible Markov Chains. *ArXiv e-prints*, December 1997.