Semantic Preserving Bijective Mappings for Representations of Special Functions between Computer Algebra Systems and Word Processors

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Abstract

We present a translation tool between representations of semantically enriched mathematical formulae in a Word Processor (WP) and its corresponding representations in a Computer Algebra System (CAS). We chosen Maple and Mathematica for the CAS, and LaTeXas our WP. Bruce Miller in the U.S. has developed a set of semantic LaTeX macros for Orthogonal Polynomials and Special Functions (OPSF). The National Institute of Standards and Technology (NIST) Digital Library of Mathematical Functions (DLMF) uses these semantic macros to provide a semantically enhanced mathematical online compendium. These semantic macros usally provide exclusive access to the semantic information of the functions. However, even if the semantics of a representation of one formula is unique, the semantics in another representation for the same formula may differ, such as in domains of variables. While some distinctions are obvious, such as syntactical characteristics, others are more difficult to examine, such as differences in definitions, which leads to error-prone manual translations. We discuss difficulties related to an automatic translation system, and suggest possible solutions. Furthermore, this paper introduces new evaluation approaches for the developed translation tool. With the help of our automatic translation tool, the evaluation experiments were able to discover errors in the DLMF and in Maple.

Keywords: LaTeX, Computer Algebra System (CAS), Translation, Presentation to Computation (P2C)

1 Introduction

A typical workflow of a scientist who writes a scientific publication is to use a Word Processor (WP) to write the paper and one or more Computer Algebra System (CAS) for verification, analysis and visualization. Especially in the Science, Technology, Engineering and Mathematics (STEM) literature, LaTeX[1] has become the de facto standard for writing scientific publications over the past 30 years (Knuth, 1997; Knuth, 1998, p. 559; Alex, 2007). LaTeX enables printing of mathematical formulae in a structure similar to handwritten style. For example, consider the specific Jacobi polynomial (F. Olver et al., 2017, 18.3 in table 1)

$$P_n^{(\alpha,\beta)}(\cos(a\Theta)), \tag{1}$$

where $\alpha, \beta > -1$, and n is a nonnegative integer. This mathematical expression is written in LaTeX as

```
P_n^{(\alpha,\beta)}(\cos(a\Theta)).
```

While LaTeX focuses on displaying mathematics, a CAS concentrates on computations and user friendly syntax. Especially important for a CAS is to embed unambiguous semantic information within the input. Each system uses different representations and syntax in consequence. Hence, a writer needs to constantly translate mathematical expressions from one representation to another and back again. Table 1 shows four different representations for (1).

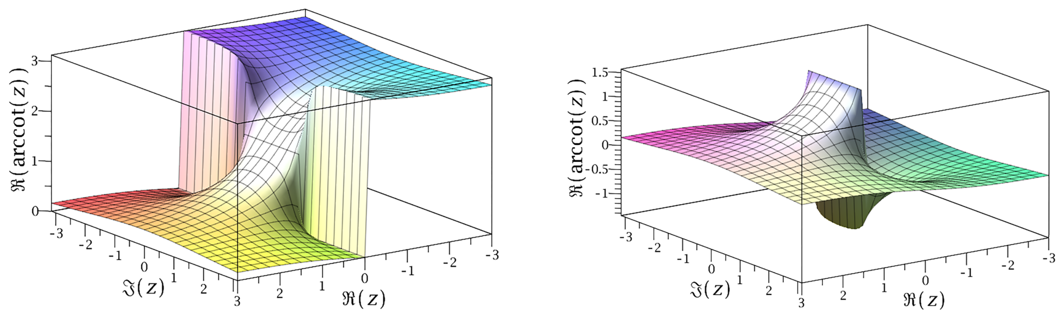
Systems	Representations
Generic LaTeX	`P_n^{(\alpha,\beta)}(\cos(a\Theta))`
Semantic LaTeX	`\JacobiP{\alpha}{\beta}{n}@{\cos@{a\Theta}}`
Maple	`JacobiP(n,alpha,beta,cos(a*Theta))`
Mathematica	`JacobiP[n,\[Alpha],\[Beta],Cos[a \[CapitalTheta]]]`

Table 1: Different representations for (1). Generic LaTeX is the default LaTeX expression; semantic LaTeX uses special semantic macros to embed semantic information; and CAS representations are unique to themselves.

Translations from generic LaTeX to CAS are difficult to realize since the full semantic information is not easily constructed from the input. Bruce Miller at the National Institute of Standards and Technology (NIST) has created a set of semantic LaTeX macros (Miller and Youssef, 2003). Each macro ties specific character sequences to a well-defined mathematical object and is linked with the corresponding definition in the Digital Library of Mathematical Functions (DLMF). The Digital Repository of Mathematical Formulae (DRMF) is an outgrowth of the DLMF with the goal to facilitate interaction among a community of mathematicians and scientists (Cohl, McClain, et al., 2014; Cohl, Schubotz, McClain, et al., 2015). The DRMF extends the set of semantic macros. These macros embed necessary semantic information into LaTeX expressions. One example of such a macro is given Table 1 for the semantic LaTeX representation for the Jacobi polynomial. The macros provide isolated access to important parts of the mathematical function, such as the arguments.

[1]Note that technically LaTeX is not a WP (`https://www.latex-project.org/about/`, seen 07/2017) like *Microsoft Word*. However, since LaTeX (an extension of TeX) is the de facto standard for writing articles in STEM, and we only focus on mathematical writing in this paper, we have categorize LaTeX as a WP as well.

Even with embedded semantic information, a translation between the systems can be difficult. A typical example of complex problems occurs for multivalued functions (Davenport, 2010). A CAS usually defines *branch cuts* to compute principal values of multivalued functions (England et al., 2014), which makes the implementation of a theoretically continues function to a discontinued presentation of it. In general, positioning branch cuts follows conventions, but can be positioned arbitrarily in many cases. Communicate and explain the decision of defined branch cuts is a critical point for CAS and can vary between the systems (Corless et al., 2000). Figure 1 illustrates two examples of different branch cut positioning for the inverse trigonometric arccotangent function. While Maple defines the branch cut at $[-\infty i, -i]$, $[i, \infty i]$ (figure 1a), Mathematica defines the branch cut at $[-i, i]$ (figure 1b).



(a) The real part of arccotangent with a branch cut at $[-\infty i, -i]$, $[i, \infty i]$.

(b) The real part of arccotangent with a branch cut at $[-i, i]$.

Figure 1: Two plots for the real part of the arccotangent function with a branch cut at $[-\infty i, -i]$, $[i, \infty i]$ in figure (a) and at $[-i, i]$ in figure (b), respectively. (Plotted with Maple 2016)

Hence, a CAS user needs to fully understand the properties and special definitions (such as the position of branch cuts) in the CAS to avoid mistakes during a translation (England et al., 2014). In consequence, a manual translation process is not only laborious, but also prone to errors. Note that this general problem has been named to automatic Presentation-To-Computation (P2C) conversion (Youssef, 2017).

This article presents a new approach for automatic P2C and vice versa conversions. Translations from presentational to computational (computational to presentational) systems are called forward (backward) translations. A forward translation is denoted with an arrow with the target system language above the arrow. For example,

$$t \overset{\mathfrak{M}_{aple}}{\mapsto} c,$$

where t is an expression in the LaTeX language and c is an element of the Maple language $\mathfrak{M}_{aple}$. As we will see later in this article, we need to compare mathematical concepts between systems. This is impossible from a mathematical point of view. Consider the irrational mathematical constant e, known as Euler's number. The theoretical construct for this symbol cannot be mathematically equivalent to the value `exp(0)` in Maple, caused by computational and implementational limitations. Instead of using the term *equivalent*, we introduce a *appropriate* and *inappropriate* translations. We call a translation such as

$$\texttt{\textbackslash cos@\{z\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{cos(z)} \tag{2}$$

as *appropriate*, while a translation such as

$$\texttt{\textbackslash cos@\{z\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{sin(z)} \tag{3}$$

is called *inappropriate*. Note that it is not always as easy as in this example to decide if a translation is appropriate or not. Therefore, this article also presents several validation techniques to automatically verify if a translation is appropriate or inappropriate. In addition to this terminology, we introduce *direct translations*. Later in the paper, we will explain that a translation from one specific mathematical object to its *appropriate* counterpart in the other system is not always possible. We call a translation to the *appropriate* counterpart *direct*. For example, the translation (2) is *direct*, while a translation to the definition of the cosine function

$$\texttt{\textbackslash cos@\{z\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{(exp(I*z)+exp(-I*z))/2}$$

is not a *direct* translation.

Note that partial results of this paper have been published in (Cohl, Schubotz, Youssef, et al., 2017).

2 Related Work

Since LaTeX became the de facto standard for writing papers in mathematics, most of the CAS provide simple functions to import and export mathematical LaTeX expressions[2]. Those tools have two essential problems. They are only able to import simple mathematical expressions, where the semantics are unique. For example, the internal LaTeX macro `\frac` always indicates a fraction. However, for more complex expressions, e.g., the Jacobi polynomial in table 1, the import functions fail. The second problem appears in the export tools. Mathematical expressions in CAS are fully semantic. Otherwise the CAS wouldn't be able to compute or evaluate the expressions. During the export process, the semantic information gets lost, because generic LaTeX is not able to carry semantic information. In consequence of these two problems, an exported expression cannot be imported to the same system again in most cases (except those simple expressions described above). Our tool should solve these problems and provide round-trip translations between LaTeX and CAS.

[2]The selected CAS Maple, Mathematica, Matlab, and SageMath provide import and/or export functions for LaTeX:
Maple, `http://www.maplesoft.com/support/help/Maple/view.aspx?path=latex` seen 06/2017;
Mathematica, `https://reference.wolfram.com/language/tutorial/GeneratingAndImportingTeX.html` seen 06/2017;
Matlab, `https://www.mathworks.com/help/symbolic/latex.html` seen 06/2017;
SageMath, `http://doc.sagemath.org/html/en/tutorial/latex.html` seen 06/2017

The semantics must be well known before an expression can be translated. There are two main approaches to solve that problem: (1) someone could specify the semantic information during the writing process (pre-defined semantics) or (2) the translator can find out the right semantic information in general mathematical expressions before it translates the expression. So-called *interactive documents*[3], Such as the Computable Document Format (CDF) (Research, 2011) by Wolfram Research or the *worksheets* by Maple, try to solve this problem with the approach (2) and allow to embed semantic information into the input. Those complex document formats require specialized tools to show and work with the documents (Wolfram CDF Player, or Maple for the *worksheets*). The `JOBAD` architecture (Giceva, Lange, and Rabe, 2009) is able to create web-based interactive documents and uses Open Mathematical Documents (OMDoc) (Kohlhase, 2006) to carry semantics. The documents can be viewed and edited in the browser. Those `JOBAD`-documents also allow to perform computations during CAS. This gives the opportunity to calculate, compute and change mathematical expressions directly in the document. The translation performs in the background, invisible to the user. Similar to the `JOBAD` architecture other interactive web documents exist, such as *MathDox* (Cuypers et al., 2008) and *The Planetary System* (Kohlhase et al., 2011). All of them demonstrate the potential of the education system.

Another approach tries to avoid translation problems by allow computations directly via the LaTeX compiler, e.g., *LaTeXCalc* (Churchill and Boyd, 2010). Those packages are limited to the possibilities of the compiler and therefore not as powerful as CAS. A workaround for this case is *sagetex* (Drake, 2009), which is a LaTeX package interface for the open source CAS *sage*[4]. This package allows *sage* commands in TeX-files and uses *sage* in the background to compute the commands. In this scenario, a writer still needs to manually translate expressions to the syntax of *sage*.

There exist two approaches that allow embedding semantic information within LaTeX expressions by using custom macros. Namely, sTeX (Kohlhase, 2008) developed by Kohlhase and the DLMF/DRMF LaTeX macros developed by Miller (Miller and Youssef, 2003). This paper shows that it is possible to develop a context-free translation tool using the semantic macros introduced by these two projects. sTeX aims to embed semantic information for general mathematics with a comprehensive set of macros. The macro set developed by Miller introduces new macros for special functions, orthogonal polynomials, and mathematical constants. Each of these macros ties specific character sequences to a well-defined mathematical object and is linked with the corresponding definition in the DLMF or DRMF. Therefore, we call these semantic macros DLMF/DRMF LaTeX macros. These semantic macros are internally used in the DLMF and the DRMF. Because of the linked definitions to the DLMF and the special macros for functions and mathematical constants this paper using the DLMF/DRMF LaTeX macros for performing translations to CAS instead of using sTeX.

[3]There is no adequate definition what interactive documents are. However, this name is widely used to describe electronic document formats that allow interactivity to change the content in real time.

[4]An abbreviation for *SageMath*.

3 Translation Problems

There are several potential problems to perform translations between systems that embed semantic information in the input. Those problems vary from simple cases, e.g., a function is not defined in the system, too complex cases, e.g., different positioning of branch cuts for multivalued functions. This section will discuss the problems and workarounds.

If a function is defined in one system but not in the other, we can translate the definition of the mathematical function. For example, the *Gudermannian* (F. Olver et al., 2017, eq. 4.23.10) $\mathrm{gd}(x)$ function is defined by

$$\mathrm{gd}(x) := \arctan(\sinh x) \quad x \in \mathbb{R}. \tag{4}$$

and linked with the semantic macro `\Gudermannian` in the DLMF but does not exist in Maple. We can perform a translation for the definition 4 instead of macro itself

$$\texttt{\textbackslash Gudermannian\{x\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{arctan(sinh(x))}. \tag{5}$$

Since those translations are nonintuitive, describing explanations become necessary for the translation process. A special logging function takes care of each translation and provide details after a successful translation process. Section 5 explains this task further.

Providing detailed information also solves the problem for multiple alternative translations. In some cases, a semantic macro has two alternative representations in the CAS or vice versa. In such cases, the translator picks one of the alternatives and informs the user about the decision.

In case of differences between defined branch cuts we can also use alternative translations to solve the problems. Consider the mentioned case of the arccotangent function (Corless et al., 2000) that has different positioned branch cuts in Maple compared to the DLMF or Mathematica definitions. Alternative but mathematically equivalent translations are

$$\begin{aligned} \texttt{\textbackslash acot@\{z\}} &\overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{arccot(z)}, & (6)\\ &\overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{arctan(1/z)}, & (7)\\ &\overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{I/2*ln((z-I)/(z+I))}. & (8) \end{aligned}$$

The direct translation (6) has the branch cut issue, while the alternative translations (7) and (8) using other functions instead of the arccotangent function. The arctangent function (7) and the natural logarithm (8) have the same positioned branch cuts in the DLMF and in Maple. In consequence, translation (7) solves the issue, as long as the user do not evaulate the function at $z = 0$, while translation (8) solves the issue except for $z = -i$.

Other problematic cases for translations are the DLMF/DRMF LaTeX macros itself. In some cases, they do not provide sufficient semantic information to perform translations. One example is the *Wronskian* symbol. For two differentiable functions the *Wronskian* is defined as (F. Olver et al., 2017, eq. 1.13.4)

$$\mathscr{W}\left\{w_1(z), w_2(z)\right\} = w_1(z)w_2'(z) - w_2(z)w_1'(z).$$

In semantic LaTeX it is used with

$$\texttt{\textbackslash Wronskian@\{w\_1(z), w\_2(z)\}.} \tag{9}$$

A translation become infeasible, because the macro do not explicitely defined the variable of the functions w_1 and w_2. For a correct translation, the CAS needs to be aware of the used variable z. We solved this issue by creating a new macro

$$\texttt{\textbackslash Wron\{z\}@\{w\_1(z)\}\{w\_2(z)\}.} \tag{10}$$

This example shows that the DLMF/DRMF LaTeX macros are still work in progress and are getting constantly updated.

A similar problem is multiplications since they are rarely explicitly marked in LaTeX expressions, e.g., scientists using whitespaces to indicate multiplications rather than using `\cdot` or similar symbols. For such problems, we introduced a new macro `\idot` for an invisible multiplication symbol (this macro will not be rendered). Since this macro is newly introduced and automatic conversion of existing equations is difficult, none of the equations in the DLMF uses this macro yet. As a consequence, the translator has some simple rules to perform translations without explicitly marked translations with `\idot`.

Even we only allow the special dialect of LaTeX using semantic macros not all expressions are unambiguous. In table 2 are four examples of ambiguous expressions. These expressions are unambiguous for the LaTeX compiler since it only considers the very next token for power and subscripts. Our translator following the same rules to solve these issues.

Ambiguous Input	LaTeX Output
`n^m!`	$n^m!$
`a^bc^d`	a^bc^d
`x^y^z`	Double superscript error
`x_y_z`	Double subscript error

Table 2: Ambiguous LaTeX expressions and how LaTeX displays them.

Another more questionable translation decision is alphanumerical expressions. As explained in table 6, the Part-of-Math (PoM)-tagger handles strings of letters and numbers differently, depending on the order of the symbols. The reason is, that an expression such as '4*b*' is usually considered to be a multiplication of 4 and '*b*,' while '*b*4' looks like indexing '*b*' by 4. While the first example produces two nodes, namely 4 and '*b*', the second example '*b*4' produces just a single alphanumerical node in the PoM-Parsed Tree (PPT). The translator interprets alphanumerical expressions as multiplications for two reasons. (1) We would assume that the inputs '4*b*' and '$b4$' are mathematically equivalent and (2) it is more common in mathematics to use single letter names for variables (Cajori, 1994). Therefore we define the following definitions

$$\begin{aligned} \texttt{4b} &\overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{4*b}, \\ \texttt{b4} &\overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{b*4}, \\ \texttt{energy} &\overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{e*n*e*r*g*y}. \end{aligned}$$

In general, the translator is drafted to solve ambiguous expressions or automatically find a work-around to disambiguate the expression. Only if there is no way to solve the ambiguity with the defined rules, the translation process stops.

4 The Translator

All translations are defined by a library (Comma-Separated Values (CSV) and JavaScript Object Notation (JSON) files) that defines translation patterns for each function and symbol. The pattern uses `$i` as placeholders to define the positions of the arguments. For example, the translation patterns for the Jacobi polynomial are illustrated in table 3.

Forward Translation:	
Maple	`JacobiP($2, $0, $1, $3)`
Mathematica	`JacobiP[$2, $0, $1, $3]`
Backward Translation from Maple/Mathematica:	
Semantic LaTeX	`\JacobiP{$1}{$2}{$0}@{$3}`

Table 3: Forward and backward translation patterns of the Jacobi polynomial. The pattern for the backward translation is the same for Maple and Mathematica.

These placeholders causes trouble when the CAS uses the symbol `$` for other reasons, e.g., the differentiation in Maple is defined by

```
diff(f, [x$n]),
```

where f as an algebraic expression or an equation, x the name of the differentiation variable and n for the n-th order differentiation. A translation for $\frac{d^2x^2}{dx^2}$ should be like this

$$\texttt{\textbackslash deriv[2]\{x\^{}2\}\{x\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{diff(x\^{}2, [x\$2])}$$

but would end up as

$$\texttt{\textbackslash deriv[2]\{x\^{}2\}\{x\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{diff(x\^{}2, [xx])}.$$

We can solve this issue by using paranthesis in such cases, e.g., `diff($1, [$2$($0)])`.

The DLMF/DRMF LaTeX macros also allow to specify optional arguments to distinguish between standard and another version of functions. The Legendre and associated Legendre function of the first kind are examples for such cases. The library that defines translations for each macro using the macro name as a primary key to identify the translation. The Legendre and associated Legendre function of the first kind both using the same macro `\LegendreP`. To distinguish such cases, we are using a special syntax such cases, shown in table 4.

Semantic Macro Entry	Maple Entry
`\LegendreP{\nu}@{x}`	`LegendreP($0, $1)`
`X1:\LegendrePX\LegendreP[\mu]{\nu}@{x}`	`LegendreP($1, $0, $2)`

Table 4: Example entries of the Legendre and associated Legendre function in the translation library. The prefix notation X<d>:<name>X defines the translation for <name> with <d>-number of optional arguments.

The translator using the PoM-Tagger (Youssef, 2017)[5] to parse LaTeX expressions into a parsed tree. The PoM-Tagger is an LL-Praser defined by a context-free grammar in Backus-Naur Form (BNF). Each token will be tagged by meta information defined in lexicon files. We extend the lexicon files to provide also the information that is necessary for the translation process. An example of an entry of the lexicon file is given in table 5.

- Symbol: `\sin`
  - Feature Set: dlmf-macro
    - DLMF: `\sin@@{z}`
    - DLMF-Link: dlmf.nist.gov/4.14# E1
    - Meanings: Sine
    - Number of Parameters: 0
    - Number of Variables: 1
    - Number of Ats: 2
    - Maple: `sin($0)`
    - Maple-Link: www.maplesoft.com/support/help/maple/view.aspx?path=sin
    - Mathematica: `Sin[$0]`
    - Mathematica-Link: reference.wolfram.com/language/ref/Sin.html

Table 5: The entry of the sine function in the lexicon file.

[5] Named according to the Part-of-Speech-Taggers in Natural Language Processing (NLP).

5 Forward Translations

An abstract translator class analyzes each node of the parsed tree and delegates them to specialized subtranslators. In this process, an object called Translated Expression Object (TEO) is built. This TEO is needed to rebuild a string representation of the mathematical expression after all nodes of the parsed tree were translated.

The parsed tree generated by the PoM-tagger is not a mathematical expression tree. The PoM project aims to disambiguate mathematical LaTeX expressions and generates an expression tree. However, in the current state, many expressions cannot be disambiguated yet. In consequence, the PoM-tagger generates a raw parsed tree where each token in the LaTeX expression is a node in the tree. We call this parsed tree the PPT.

The overall forward translation process is explained in figure 2. All translation patterns and related information are stored in the DLMF/DRMF tables. These tables are converted by the `lexicon-creator` to the `DLMF-macros-lexicon` lexicon file. Together with the `global-lexicon` file, the PPT will be created by the PoM-tagger. The `latex-converter` takes a string representation of a semantic LaTeX expression and uses the PoM engine as well as our `Translator` to create an appropriate string representation for a specified CAS.

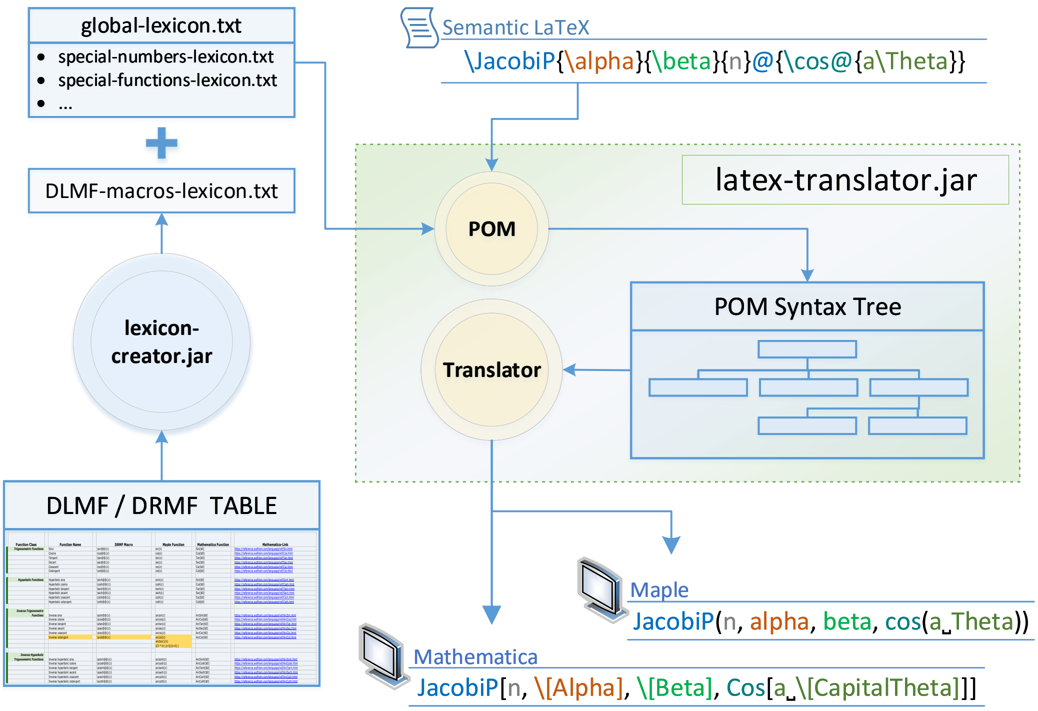


Figure 2: Process diagram of a forward translation process. The PoM-tagger generates the PPT based on lexicon and JSON files. The PPT will be translated to different CAS.

5.1 *Analyzing the PoM-Parsed Tree*

Since the BNF does not define rules for semantic macros, each argument of the semantic macro and each @ symbol are following siblings of the semantic macro node. That is the reason, why we stored the number of parameters, variables and @ symbols in the lexicon files. Otherwise, the translator could not find the end of a semantic macro in the PPT.

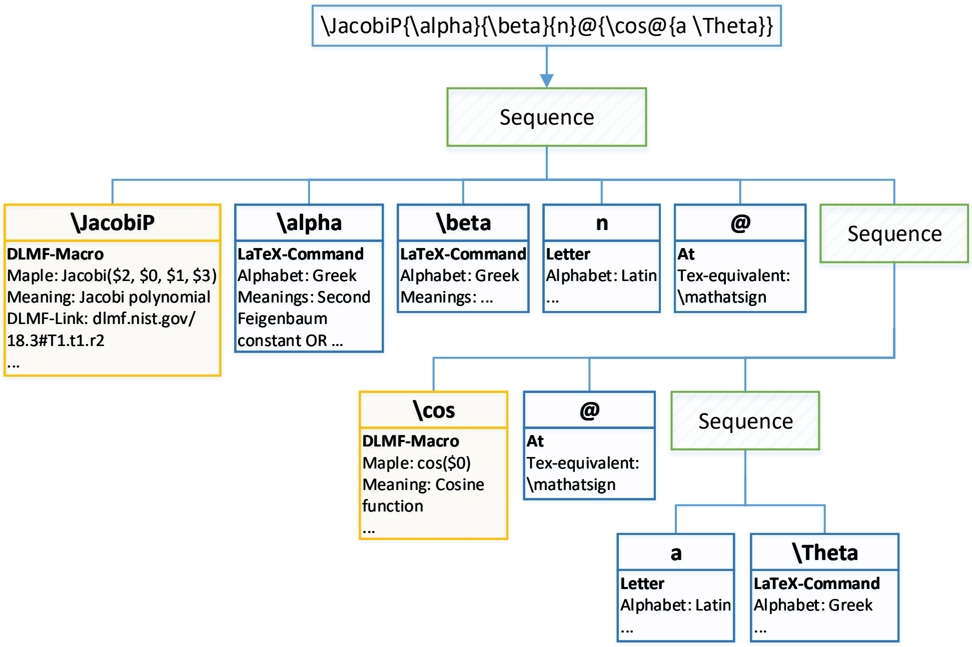


Figure 3: pom-pt for a Jacobi polynomial using the DLMF/DRMF LaTeX macro. Each leaf contains information from the lexicon files.

Figure 3 visualizes the PPT of the Jacobi polynomial example from table 1. Because of the differences to expression trees, a backward conversion of the PPT to a string representation can be difficult, especially for finding necessary or unnecessary parentheses. Therefore we create the TEO.

With these tools, we can translate a LaTeX expression by translating the PPT node by node and perform group or reordering operations for some special cases. The algorithm is realized in a simple recursive structure. The algorithm 1 explains the translation process. Whenever the algorithm finds a leaf, it can translate this single term. If the node is not a leaf, it

Algorithm 1 Simple translation algorithm for the POM-Parse Tree

Input: Root r of the parse tree T

procedure TRANSLATE(r)
    if r is leaf **then**
        TRANSLATE_LEAF(r);
    else
        for all children v_n of r **do**
            TRANSLATE(v_n);
        end for
    end if
end procedure

starts to translate all children of the node recursively.

Algorithm 2 Abstract translation algorithm to translate MLP-Parse trees.

Input: Root r of a POM-Parse tree T. List *following_siblings* with the following siblings of r. The list can be empty.

```
 1: procedure ABSTRACT_TRANSLATOR(r, following_siblings)
 2:     if r is leaf then
 3:         TRANSLATE_LEAF(r, following_siblings);
 4:     else
 5:         siblings = r.getChildren();                    ▷ siblings is a list of children
 6:         ABSTRACT_TRANSLATOR(siblings.removeFirst(), siblings);
 7:     end if
 8:     if following_siblings is not empty then
 9:         r = following_siblings.removeFirst();
10:         ABSTRACT_TRANSLATOR(r, following_siblings);
11:     end if
12: end procedure
```

This approach is not completely feasible since the algorithm needs to look ahead and check the following siblings in some cases, e.g., in the case of a semantic macro with arguments (see figure 3). The `TRANSLATE_LEAF` function needs to know the following siblings of the current node. Algorithm 2 explains a more detailed but abstract version of the final algorithm of the presented translation tool.

If the root r is a leaf, it still can be translated as a leaf. Eventually, some of the following siblings are needed to translate r. The list of *following_siblings* in line 3 might be reduced to avoid multiple translations for one node. If r is not a leaf, it contains one or more children. Therefore, we can call the `ABSTRACT_TRANSLATOR` recursively for the children. Once we have translated r, we can go a step further and translate the next node. Line 8 checks if there are following siblings left and calls the `ABSTRACT_TRANSLATOR` recursively in that case. Translated expressions are stored by the TEO object. The detailed translation process is not included in the algorithm.

Algorithm 2 is a simplified version of the translator process. The main progress happens in lines 3 and 6. There are several cases what r can be. Table 6 on page 14 gives an overview of all different types in a PPT. A more detailed explanation of the types can be found in (Youssef, 2017).

The BNF grammar defines some basic grammatical rules for generic LaTeX macros, such as for `\frac`, `\sqrt`. Therefore, there is a hierarchical structure for those symbols, similar to the structure in expression trees. As already mentioned, some of these types can be translated directly, such as Greek letters, while others are more complex, such as semantic LaTeX macros. Therefore, the translators delegate the translation to specialized subtranslators. This delegation process is implemented in lines 3 and 6 of algorithm 2. The Subsection 5.2 discusses these classes in more detail.

This approach seems to work for most of semantic LaTeX expressions. However, there is one problem with this straight forward approach which was mentioned shortly mentioned in section 3. There are many different types of notations used to represent formulae. Table 4 illustrates the simple expression $(a + b)x$ in different notations. The Normal Polish Notation (NPN)[6] (hereafter called prefix notation) places the operator to the left of/before its operands. The Reverse Polish Notation (RPN)[7] (hereafter called postfix notation) does the opposite and places the operator to the right of/after its operands. The infix notation is commonly used in arithmetic and places the operator between their operands, which only makes sense as long as the operator is a binary operator.

Notation	Expression
Infix	$(a + b) \cdot x$
Prefix	$\cdot + a\ b\ x$
Postfix	$a\ b + x\ \cdot$
Functional	$\cdot(+(a, b), x)$

Figure 4: The mathematical expression '$(a + b) \cdot x$' in infix, prefix, postfix and functional notation.

In mathematical expressions, notations are mostly mixed, depending on the case and number of operands. For example, infix notation is common for binary operators ($+, -, \cdot, \mod$ etc.), while functional notations are conveniently used for any kind of functions (sin, cos, etc.). Sometimes the same symbol is used in different notations to distinguish different meanings. For example, the '$-$' as a unary operator is used in prefix notation to indicate the negative value of its operand, such as in '-2'. Of course, $-$ can also be the binary operator for subtraction, which is commonly used in infix notation. An example for the postfix notation is factorial, such as '2!'.

[6] Also known as *Warsaw Notation* or *prefix notation*

[7] Also known as *postfix notation*

	Node type	Explanation	Example
r has children	Sequence	Contains a list of expressions.	$a + b$ is a sequence with three children (a, + and b).
	Balanced Expression	Similar to a sequence. But in this case the sequence is wrapped by `\left` and `\right` delimiters.	`\left(`$a + b$`\right)` is a balanced expression with three children (a, + and b).
	Fraction	All kinds of fractions, such as `\frac`, `\ifrac`, etc.	`\ifrac{a}{b}` is a fraction with two children (a and b).
	Binomial	Binomials	`\binom{a}{b}` has two children (a and b).
	Square Root	The square root with one child.	`\sqrt{a}` has one child (a).
	Radical with a specified index	n-th root with two children.	`\sqrt[a]{b}` has two children (a and b).
	Underscore	The underscore '_' for subscripts.	The sequence a_b has two children (a and '_'). The underscore itself '_' has one child (b).
	Caret	The caret '^' to for superscripts or exponents. Similar to the underscore.	The sequence $a\hat{}b$ has two children (a and '^'). The caret itself '^' has one child (b).
r is a leaf	DLMF/DRMF LaTeX macro	A semantic LaTeX macro	`\JacobiP`, etc.
	Generic LaTeX macro	All kinds of LaTeX macros	`\Rightarrow`, `\alpha`, etc.
	Alphanumerical Expressions	Letters, numbers and general strings.	Depends on the order of symbols. $ab3$ is alphanumerical, while $4b$ are two nodes (4 and b).
	Symbols	All kind of symbols	'@', '$*$', '+', '!', etc.

Table 6: A table of all kinds of nodes in a PoM syntax tree. Note that this table groups some kinds for a better overview. For a complete list and a more detailed version see (Youssef, 2017).

Most programming languages (and CAS as well) internally use prefix or postfix notation and do not mix the notations in one expression, since it is more convenient to parse expressions in uniform notations. However, the common practice in science is to use mixed notations in expressions. Since the PoM has rarely implemented mathematical grammatical rules yet, it takes the input as it is and does not build an expression tree. Therefore, it parses all four examples from table 4 to four different PPTs rather than to one unique expression tree. In general, this is not a problem for our translation process since most CAS are familiar with the most common notations. Therefore, the translator does not need to know that a and b are the operands of the binary operator '+' in '$a + b$.' The translator could simply translate the symbols in '$a + b$' in the same order as they appear in the expression and the CAS would understand it. However, this simple approach generates two problems.

1. The translated expression is only syntactically correct if the input expression was syntactically correct.
2. We cannot translate expressions to a CAS which uses a non-standard notation.

Problem 1 should be obvious. Since we want to develop a translation tool and not a verification tool for mathematical LaTeX expressions, we can assume syntactically correct input expressions and produce errors otherwise. Problem 2 is more difficult to solve. If a user wants to support a CAS that uses prefix or postfix notation by default, the translator would fail in its current state. Supporting CAS with another notation would be a part of future work.

Nonetheless, changing a notation could also solve ambiguities in some situations. Consider the two ambiguous examples in table 7. While a scientist would probably just ask for the right interpretation of the first example, Maple automatically computes the first interpretation. On the other hand, LaTeX automatically disambiguate the first example by only recognizing the very next element (single symbols or sequence in curly brackets) for the superscript and therefore displays the second interpretation. The second example is already interpreted as the double factorial function of n, since this notation is the standard interpretation in science. We wrote the second interpretation as the standard way in science to make it even more obvious. However, surprisingly, Maple computes the first interpretation again rather than the common standard interpretation.

	Text Format Expression	First Interpretation	Second Interpretation
1:	4 ^ 2!	$4^{2!}$	$4^2!$
2:	$n!!$	$(n!)!$	$(n)!!$

Table 7: Ambiguous examples of the factorial and double factorial function. One expression in a text format can be interpreted in different ways.

In most cases, parentheses can be used to disambiguate expressions. We used them in table 7 to clarify the different interpretations in example 2. But sometimes, even parentheses cannot solve a mistaken computation. For example, there is no way to add parentheses to force Maple to compute $n!!$ as the double factorial function. Even $(n)!!$ will be interpreted as $(n!)!$. Rather than using the exclamation mark in Maple, one could also use the functional notation. For example, the interpretations $(2!)!$ and $(2)!!$ can be distinguished in Maple by using `factorial(factorial(2))` and `doublefactorial(2)` respectively. We define the translations as follows:

$$\texttt{n!} \overset{\mathfrak{M}aple}{\mapsto} \texttt{factorial(n)},$$
$$\texttt{n!!} \overset{\mathfrak{M}aple}{\mapsto} \texttt{doublefactorial(n)}.$$

Algorithm 2 does not allow this translation right now. It has no access to previously translated nodes in its current state. This problem is solved by the TEO that stores and groups translated objects like lists. This allows to access the latest translated expression and use it as the argument for the factorial function. Table 8 shows three example for the TEO list that groups some tokens.

Input Expression	TEO List
$a+b$	`[a, +, b]`
$(a+b)$	`[(a+b)]`
$\frac{a}{b}-2$	`[(a)/(b), -, 2]`

Table 8: How the TEO-list groups subexpressions.

5.2 *Subtranslators*

A `SequenceTranslator` translates the *sequence* and *balanced expressions* in the PPT. If a node n is a leaf and the represented symbol an open bracket (parentheses, square brackets and so on) the following nodes are also taken as a *sequence*. Hence, combined with the recursive translation approach, the `SequenceTranslator` also checks balances of parentheses in expressions. An expression such as '$(a]$' is producing a mismatched parentheses error. On the other hand, this is a problem for interval expressions such as '$[a, b)$'. In the current version, the program cannot distinguish between mismatched parentheses and half-opened, half-closed intervals. Whether an expression is an interval or another expression is difficult to decide and can depend on the context. Also, the parenthesis checker could simply be deactivated to allow mismatched parentheses in an expression. But this functionality is not implemented yet.

The `SequenceTranslator` also handle positions of multiplication symbols. There are a couple of obvious choices to translate multiplication signs. The most common symbol for multiplications is still the white space or none symbol at all, as explained previously. Consider the simple expression $2n\pi$. The PPT generates a sequence node with three children, namely 2, n and π. This sequence should be interpreted as a multiplication of the three elements. The `SequenceTranslator` checks the types of the current and next nodes in the tree to decide if it should add a multiplication symbol or not. For example, if the current or next node is an operator, a relation symbol or an ellipsis, there will be no multiplication symbol added. However, this approach implies an important property. The translator interprets all sequences of nodes as multiplications as long as it is not defined otherwise. This potentially produces strange effects. Consider an expression such as $f(x)$. Translate this to Maple will be `f*(x)`. But we do not consider this translation to be wrong, because there is a semantic macro to represent functions. In this case, the user should use `\f{f}@{x}` instead of `f(x)` to distinguish between f as a function call and f as a symbol.

Algorithm 3 The translate function of the MacroTranslator. This code ignores error handling.

Input:

macro - node of the semantic macro.
args - list of the following siblings of *macro*.
lexicon - lexicon file

Output:

Translated semantic macro.

```
procedure TRANSLATE_MACRO(macro, args, lexicon)
    info = lexicon.getInfo(macro);
    argList = new List();          ▷ create a sorted list for the translated arguments.
    next = args.getNextElement();
    if next is caret then
        power = translateCaret(next);
        next = args.getNextElement();
    end if
    while next is [ do   ▷ square brackets starts a balanced sequence of optional arguments.
        optional = TRANSLATE_UNTIL_CLOSED_BRACKET(args);
        argList.add(optional);
        next = args.getNextElement();
    end while
    argList.add( TRANSLATE_PARAMETERS(args, info) );        ▷ number is given in info.
    SKIP_AT_SIGNS( args, info );                           ▷ number is given in info.
    argList.add( TRANSLATE_VARIABLES(args, info) );         ▷ number is given in info.
    pattern = info.getTranslationPattern();
    translatedMacro = pattern.fillPlaceHolders(argList);
    if power is not null then
        translatedMacro.add(power);
    end if
    return translatedMacro;
end procedure
```

Algorithm 3 explains the `MacroTranslator` without error handling. It extracted necessary information from the PPT, such as how many arguments this function has. In consequence, it also processes the following siblings to translate the arguments. There are some special cases about the next node right after the macro occurs. This node can be

- an exponent, such as for '`^2`';
- an optional parameter in square brackets;
- a parameter in curly brackets (a *sequence* node in the PPT) if none of the above;

- an @ symbols if none of the above;
- or a variable in curly brackets (a *sequence* node) if none of the above.

An exponent will be translated and shifted to the end of the translated semantic macro, since it is more common to write the exponent after the arguments of a function in CAS. Therefore, the function translates and stores the exponent in line 5. One could ask what happens when there is an exponent given before and after the arguments. The `MacroTranslator` only translates the following siblings until each argument is translated. The first exponent will be shifted to the end. If right after the translated macro (with all arguments) follows another exponent, we interpret it as another exponent for the whole previous expression. In that case, it would be the macro with the first translated exponent. Table 9 shows an example for the trigonometric cosine function with multiple exponents.

Displayed As	$\cos^2(x)^2$
Semantic LaTeX	`\cos^2@{x}^2`
Translated Maple Expression	`((cos(x))^(2))^2`

Table 9: A trigonometric cosine function example with exponents before and after the argument.

As you can see, the input expression in semantic LaTeX is interpreted as

$$\left(\cos(x)^2\right)^2. \tag{11}$$

6 Maple to Semantic LaTeX Translator

Instead of writing a custom Maple syntax parser, we using Maple's internal data structure to get a syntax tree of the input[8]. Maple allow different input styles. The `1D` input is mainly used for programming purposes and therefore also used to perform our translations. Internally, Maple using Directed Acyclic Graph (DAG) as syntax trees.

Each node in the DAG stores its children and has a header, which defines the type and the length of the node. Consider the polynomial x^2+x. Figure 5 illustrates the internal DAG representation with headers and arguments.

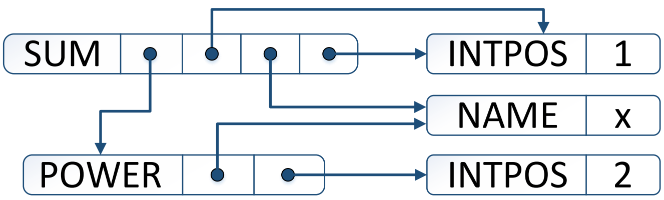


Figure 5: The internal Maple DAG representation of x^2+x.

One can access the internal data structure of expressions via the `ToInert` command. The re-

[8]In consequence, a license of Maple is mandatory to perform backward translations. The translator is using the version Maple 2016.

turned `InertForm` is a nested list representation as a tree equivalent[9] of the internal DAG for the given expression. Some of the important types for the nodes are specified in table 10.

Type	Explanation
SUM	Sums. Internally stored with factors for each summand, i.e., 'x+y' would be stored as '$x \cdot 1 + y \cdot 1$'.
PROD	Products.
EXPSEQ	Expression sequence is a kind of list. The arguments of functions are stored in such sequences.
INTPOS	Positive integers.
INTNEG	Negative integers.
COMPLEX	Complex numbers with real and imaginary part.
FLOAT	Float numbers are stored in the scientific notation with integer values for the exponent n and the significand m in $m \cdot 10^n$.
RATIONAL	Rational numbers are fractions stored in integer values for the numerator and positive integers for the denominator.
POWER	Exponentiation with expressions as base and exponent.
FUNCTION	Function invocation with the name, arguments and attributes of the function.

Table 10: A subset of important internal Maple data types. See (Bernardin et al., 2016) for a complete list.

The translator using the `OpenMaple` (Bernardin et al., 2016, §14.3) Application Programming Interface (API) for interacting with Maple's kernel and get access to the `InertForm` of inputs.

6.1 Automatic Changes of Inputs in Maple

Maple evaluates inputs automatically and changes the input into an internal representation. This internal representation might look a bit different to the input. One example has already been given with figure 5, where each summand of a sum is stored with a factor. Here is a list of all internal changes that occur for inputs.

- Maple evaluates input expressions immediately.
- There is no data type to represent square roots such as $\sqrt{x}$ (or n-th roots). Therefore, Maple stores roots as an exponentiation with a fractional exponent. For example, $\sqrt{x}$ is stored as $x^{\frac{1}{2}}$.
- There is no data type for subtractions, only for sums. Negative terms are changed to absolute values times '-1'. For example, $x - y$ is stored as $x + y \cdot (-1)$.

[9] A tree equivalent of a DAG splitting nodes with multiple parents into multiple nodes so that each node has only one parent node.

- Floating point numbers are stored in the scientific notation with a mantissa and an exponent in the base 10. For example, 3.1 is internally represented as $31 \cdot 10^{-1}$.
- There is only a data type for rational numbers (fractions with integer numerator and positive denominator), but not for general fractions, such as $\frac{x+y}{z}$. This will be automatically changed to $(x + y) \cdot z^{-1}$.

There are unevaluation quotes implemented to avoid evaluations on input expression. Table 11 gives an example how those unevaluation quotes work.

	Without unevaluation quotes	With unevalation quotes
Input expression:	`sin(Pi)+2-1`	`'sin(Pi)+2-1'`
Stored expression:	1	`sin(Pi)+1`

Table 11: Example of unevaluation quotes for `1D` Maple input expressions.

Since we want to keep a translated expression similar to the input expression, we implemented some cosmetic rules for the backward translations that solve or reduce the effects from the list of changes above.

- We use unevaluation quotes to suppress evaluations of the input.
- We perform reordering of factors and summands so that negative factors appear in front of the summand. This gives us the opportunity to translate $x - y$ to $x - y$ instead of $x + y \cdot (-1)$.
- We introduced the new internal data types `MYFLOAT` and `DIVIDE` to translate floats and fractions in more convenient notations.

The translation process then follows the same principle as for the forward translations. Since the syntax tree of Maple is an expression tree, we do not need to implement special reordering or grouping algorithms to perform backward translations. Translations for functions are also realized via patterns and placeholders. Figure 6 illustrates the backward translation process for the Jacobi polynomial example from table 1.

7 Evaluation

We implemented three approaches to evaluate whether a translation was *appropriate* or *inappropriate*.

1. **Round Trip Tests**: translates expressions back and forth and analyzing the changes.
2. **Function Relation Tests**: translate mathematically proven equivalent expressions from one system to a CAS and evaluate whether the relation remains valid
   (a) via equivalence checks of the CAS or
   (b) via numerical evaluation tests.

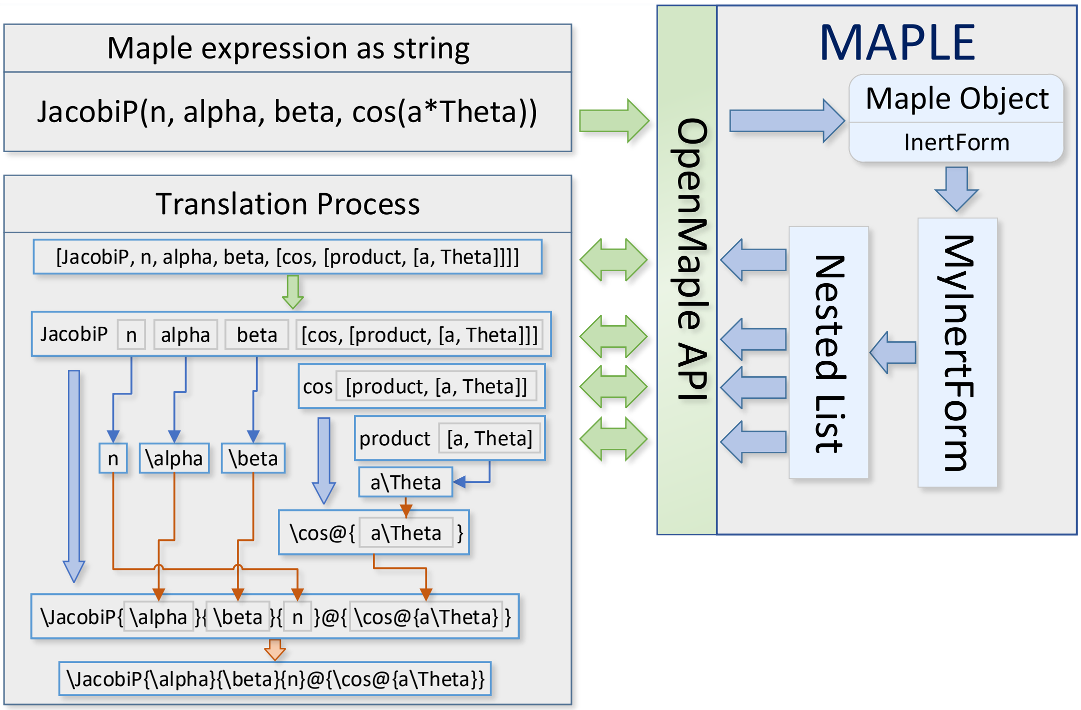


Figure 6: A scheme of the backward translation process from Maple for the Jacobi polynomial $P_n^{(\alpha,\beta)}(\cos(a\Theta))$. The input string is converted by the Maple kernel into the nested list representation. This list is translated by subtranslators (blue and red arrows). A function translation (bold blue arrows) is again realized by translation patterns to define the position of the arguments (red arrows).

7.1 Round Trip Tests

A round trip test always starts with a valid expression either in semantic LaTeX or in Maple. A translation from one system to another is called **a step**. A complete round trip translation (two steps) is called **one cycle**.

Definition 7.1: *(Fixed Point Representation)*
*A **fixed point representation** (or short fixed point) in a round trip translation process is a string representation that is identical to all string representations in the following cycles.*

Table 12 illustrates an example of a round trip test which reaches a fixed point. The test formula is

$$\frac{\cos(a\Theta)}{2}. \tag{12}$$

Steps	semantic LaTeX/Maple representations
0	`\frac{\cos@{a\Theta}}{2}`
1	`(cos(a*Theta))/(2)`
2	`\frac{1}{2}\idot\cos@{a\idot\Theta}`
3	`(1)/(2)*cos(a*Theta)`
4	`\frac{1}{2}\idot\cos@{a\idot\Theta}`

Table 12: A round trip test reaching a fixed point.

Step 4 is identical to step 2, and since the translator is a deterministic algorithm it can be easily shown that step 2 and step 3 are fixed-point representations for semantic LaTeX and Maple.

There is currently only one exception known where a round trip test does not reach a fixed point representation: Legendre's incomplete elliptic integrals (F. Olver et al., 2017, eq. 19.2.4-7) are defined with the amplitude ϕ in the first argument in the DLMF, while Maple takes the trigonometric sine of the amplitude as the first argument. Therefore, the forward and backward translations are defined as

$$\texttt{\textbackslash EllIntF@\{\textbackslash phi\}\{k\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{EllipticF(sin(phi),k)}, \tag{13}$$

$$\texttt{\textbackslash EllIntF@\{\textbackslash asin@\{\textbackslash phi\}\}\{k\}} \overset{\mathfrak{M}_{aple}}{\hookleftarrow} \texttt{EllipticF(phi,k)}, \tag{14}$$

And round-trip translations will produce infinite chains of sine and inverse sine calls. Since there are no evaluations involved, these chains will not be reduced during the translations. The round trip tests are very successful, but they only detect errors in string representations. However, because of the simplification techniques of fixed points, we are able to at least detect logical errors in one system: Maple. On the other hand, these tests cannot determine logical errors in the translations between the two systems. Consider we mistakenly defined an *inappropriate* forward and backward translation for the sine function

$$\texttt{\textbackslash sin@\{\textbackslash phi\}} \overset{\mathfrak{M}_{aple}}{\leftrightarrow} \texttt{cos(phi)}, \tag{15}$$

$$\texttt{\textbackslash cos@\{\textbackslash phi\}} \overset{\mathfrak{M}_{aple}}{\leftrightarrow} \texttt{sin(phi)}. \tag{16}$$

In that case the round trip test would not detect any errors and reach a fixed point representation, because the simplification techniques only simplify two representations in the same system but cannot compare the representation in one system to those in the other.

7.2 Function Relation Tests

The DLMF is a compendium of special functions and orthogonal polynomials and lists several relations between the functions and polynomials. The idea of this evaluation approach is to translate an entire relation and test whether the relation remains valid after performing the translations.

With this technique we can detect translation errors such as in (15 and 16). Consider the DLMF equation for the sine and cosine function (F. Olver et al., 2017, eq 4.21.2)

$$\sin(u+v) = \sin u \cos v + \cos u \sin v. \tag{17}$$

Assume the translator would forward translate the expression based on (15, 16). Than

$$\texttt{\textbackslash sin@\{u + v\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{cos(u + v)}, \tag{18}$$

$$\texttt{\textbackslash sin@@\{u\}\textbackslash cos@@\{v\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{cos(u)*sin(v)}, \tag{19}$$

$$\texttt{\textbackslash cos@@\{u\}\textbackslash sin@@\{v\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{sin(u)*cos(v)}. \tag{20}$$

This produces the equation in Maple

$$\cos(u+v) = \cos u \sin v + \sin u \cos v, \tag{21}$$

which is wrong. Since the expression is correct before the translation, we conclude an error during the translation process.

However, there are two essential problems with this approach. Testing the mathematical equivalence of expressions is hard to solve and CAS have trouble to test even simple equations. Furthermore, this approach only checks forward translations because there is no way to check equivalence of expressions in LaTeX automatically (again this could become feasible with our translator). We use Maple's *simplify* function to check if the difference of the left-hand side and the right-hand side of the equation is equal to zero. In addition, we use *simplify* and check if the division of the right-hand side by the left-hand side returns a numerical value or not. This simplification function is the most powerful function to check the equivalence in Maple. However, there are several cases where the simplification fails. Because of implementation details, there are some techniques that helps Maple to find possible simplifications. For example we can force Maple to convert the formula

$$\sinh x + \sin x \tag{22}$$

to an equivalent representation using the exponential representations

$$\frac{1}{2}e^{x} - \frac{1}{2}e^{-x} - \frac{1}{2}i\left(e^{ix} - e^{-ix}\right). \tag{23}$$

With such pre-conversions we are able to improve the simplification process in Maple. However, the limitations of the *simplify* function are still the weakest part of this verification approach. Consider the complex example (F. Olver et al., 2017, eq. 12.7.10)

$$U(0,z) = \sqrt{\frac{z}{2\pi}}K_{\frac{1}{4}}\left(\frac{1}{4}z^2\right), \tag{24}$$

where $U(0,z)$ is the parabolic cylinder function and $K_\nu(z)$ the modified Bessel function of the second kind. Both functions are well-defined in both systems and we can define a *direct* translation for (24). The modified Bessel function of the second kind has its branch cut in Maple and in the DLMF at $z < 0$. However, the argument of K contains z^2. If $|\mathrm{ph}(z)| \in \left(\frac{\pi}{2}, \pi\right)$ the value of the right-hand side of (24) would be no longer on the principal branch. However, Maple will still compute the principal values independently of the value of z. Hence, a translation

$$\texttt{\textbackslash BesselK\{\textbackslash frac\{1\}\{4\}\}@\{\textbackslash frac\{1\}\{4\}z\^{}2\}} \overset{\mathfrak{M}_{aple}}{\mapsto} \texttt{BesselK(1/4,(1/4)*z\^{}2)} \tag{25}$$

is incorrect if $|\mathrm{ph}(z)| \in \left(\frac{\pi}{2}, \pi\right)$ and one has to use analytic continuation for the right-hand side of equation (24).

To evaluate such complex cases, the equivalence checks of CAS are insufficient. Therefore we implement numerical tests as an additional step.

7.3 Numerical Tests

Consider the differences of the left- and right-hand side of equation (24)

$$D(z) := U(0,z) - \sqrt{\frac{z}{2\pi}}K_{\frac{1}{4}}\left(\frac{1}{4}z^2\right). \tag{26}$$

Table 13 presents four computations for $D(z)$, one value for each quadrant in the complex plane.

Considering machine accuracy and the default precision of 10 significant digits, we can regard the first and last values as zero differences. While this evaluation is very powerful, it has a significant problem. Even when all tested values return a value differ from zero, it does not proof the equivalence of (24). However, when the values are different from zero, it does proof an error in one of the four cases (S. Cohl, Greiner-Petter, and Schubotz, 2018)

z	$D(z)$
$1+i$	$2 \cdot 10^{-10} - 2 \cdot 10^{-10} i$
$-1+i$	$2.222121916 - 1.116719816i$
$-1-i$	$2.222121916 + 1.116719816i$
$1-i$	$2 \cdot 10^{-10} + 2 \cdot 10^{-10} i$

Table 13: Four computations of $D(z)$ in Maple.

1. the numerical engine tests invalid combinations of values;
2. the translation is incorrect;
3. there may be an error in the DLMF source; or
4. there may be an error in Maple.

7.4 Results

There are 685 DLMF/DRMF LaTeX macros[10] in total, and 665 of them were implemented in the translator engine. We defined forward translations to Maple for 201 and backward translations from Maple for 195 functions.

The DLMF provides a dataset of LaTeX expressions with semantic macros. We extracted 4.087 equations from DLMF and apply our round-trip and relation tests on them. The translator was able to translate 2.405 (58.8[11]%) of the extracted equations without errors. 660 (27.4%) of the successfully translated expressions were verified by the simplification techniques of Maple. We apply additional numerical tests for the remaining 1.745 equations. For 418 (24%) cases the numerical tests were valid. More detailed results for numerical and symbolical tests were presented in (S. Cohl, Greiner-Petter, and Schubotz, 2018).

The evaluation techniques have proven to be very powerful also for evaluating CAS and online mathematical compendia such as the DLMF. During the evaluations, we were able to detect several errors in the translation and evaluation engine. However, even errors in the DLMF were discovered.

The numerical test engine was able to discover a sign error in equation (F. Olver et al., 2017, eq. 14.5.14)[12]

$$\mathsf{Q}_{\nu}^{-1/2}(\cos\theta) = -\left(\frac{\pi}{2\sin\theta}\right)^{1/2} \frac{\cos\left(\left(\nu+\frac{1}{2}\right)\theta\right)}{\nu+\frac{1}{2}}. \tag{27}$$

The error can be found on (F. W. Olver et al., 2010, p. 359) and has been fixed in the DLMF with version 1.0.16. The same engine also identified a missing comma in the constraint of (F.

[10]The macros are still work in progress, and therefore the total number is constantly changing.

[11]All percentages are approximately calculated.

[12]The equation had originally been stated as shown in equation (27). The error were reported on 10th April 2017.

Olver et al., 2017, eq. 10.16.7). The original constraint was given by $2\nu \neq -1, -2-3, \ldots$, with a missing comma after -2.

8 Conclusion & Future Work

The results and conclusions of the translations of the formula set from the DLMF are diverse. Some test cases have shown us that the set of semantic macros is not perfect yet and needs to be improved, seen in the example of the macro definition for the Wronskian symbol. However, simply adding more and more macros to improve the set is not the right decision. In that case, we would end up in something similar to sTeX, which is overloaded and too complicated to use. If the system is no longer handy, it will not be used.

On the other hand, the concept of the translator has been proven to discovering errors in the online compendia DLMF. The test cases have also shown how difficult it is to validate a translated expression and have uncovered the problems of translations between two systems with different sets of supported functions. Our validation techniques also assume the correctness of the simplification and computational algorithms in CAS. However, combining those techniques and automatically running translation checks can potentially not only discover errors in mathematical compendia but also detect errors in simplifications or computations in CAS.

The tasks for future work are diverse. The main task to improve the translator by implementing more functions and features. For example, for the current state, only translations to Maple's standard function library were implemented. Maple allows to load extra packages dynamically and therefore support several more functions. This feature would drastically increase the number of possible translations. With such improvements, further work on evaluation techniques become worthwhile to evaluate DLMF and CAS.

Furthermore, the translator was designed to be easily extendable. This allows to implement translations for other CAS without much effort.

The biggest weakness of the translator is still the semantic macros. Currently, we are working on mathematical information retrieval techniques to allow an extension for the translator for generic LaTeX inputs.

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