TWO IDENTITIES INVOLVING THE CUBIC PARTITION FUNCTION

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ABSTRACT. We give a new proof of Chan's identity involving the cubic partition function and we also give a new identity for the cubic partition function which is analogues to the Zuckerman's identity for the ordinary partition function.

1. INTRODUCTION

Let $p(n)$ be the number of partitions of n, defined by $\sum_{n \geq 0} p(n)q^n := \prod_{n=1}^{\infty}(1 - q^n)^{-1}$. In connection with his discovery of certain divisibility properties of $p(n)$ Ramanujan stated the identities:

$$(1.1) \qquad \sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_{\infty}^5}{(q;q)_{\infty}^6},$$

and

$$(1.2) \qquad \sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_{\infty}^3}{(q;q)_{\infty}^4} + 49q\frac{(q^7;q^7)_{\infty}^7}{(q;q)_{\infty}^8}.$$

Here and in the rest of the paper we follow the customary q-product notation: we set (for $|q| \leq 1$)

$$(a;q)_{\infty} = \prod_{n=0}^{\infty}(1 - aq^n).$$

Both Hardy and MacMahon considered 1.1 as Ramanujan's "Most Beautiful Identity". Darling [7] proved the first and Mordell [9], Watson [11] and Rademacher-Zuckerman [10] gave proofs for both identities. Recently, H.H. Chan and R.P. Lewis [4] also gave different proofs of both identities. All these proofs used the theory of modular functions. In another paper, Zuckerman [12] obtained the following identity (Zuckerman's identity):

$$\sum_{n=0}^{\infty} p(25n+24)q^n = 63 \cdot 5^2\frac{(q^5;q^5)_{\infty}^6}{(q;q)_{\infty}^7} \quad + \quad 52 \cdot 5^5 q\frac{(q^5;q^5)_{\infty}^{12}}{(q;q)_{\infty}^{13}} + 63 \cdot 5^7 q^2\frac{(q^5;q^5)_{\infty}^{18}}{(q;q)_{\infty}^{19}}$$

$$(1.3) \qquad\qquad + \quad 6 \cdot 5^{10} q^3\frac{(q^5;q^5)_{\infty}^{24}}{(q;q)_{\infty}^{25}} + 5^{12}q^4\frac{(q^5;q^5)_{\infty}^{30}}{(q;q)_{\infty}^{31}}.$$

In two recent papers, H.-C. Chan [2, 3] proved a generalization of 1.1 and 1.2 for a certain kind of partition function $a(n)$ which is defined by

$$(1.4) \qquad \sum_{n=0}^{\infty} a(n)q^n := \frac{1}{(q;q)_{\infty}(q^2;q^2)_{\infty}}.$$

Kim [8] noted that $a(n)$ can be interpreted the number of 2-color partitions of n with colors r and g subject to the restriction that the color b appears only in even parts, so he called $a(n)$ to be the cubic partition function owing to the fact that $a(n)$ is related to Ramanujan's cubic continued fraction. Using some identities for the cubic continued fraction, H.C.-Chan derived the following identity :

Theorem 1.1 ([2] Theorem 1)**.**

$$(1.5) \qquad \sum_{n=0}^{\infty} a(3n+2)q^n = \frac{3(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}.$$

In this note, we only use 3-dissections of functions $\frac{1}{\Phi(-q)}$ and $\frac{1}{\Psi(q)}$ to give a new elementary proof of 1.5 and we also give the following identity for the cubic partition function $a(n)$ which is similar to Zuckerman's identity 1.3 for the cubic partition function $a(n)$.

Theorem 1.2.

$$
\begin{aligned}
\sum_{n=0}^{\infty} a(9n+8)q^n &= 2\cdot 3^3 \frac{(q^3;q^3)_\infty^{30}}{(q;q)_\infty^{19}(q^2;q^2)_\infty^7(q^6;q^6)_\infty^6} + 8\cdot 3^3 q\frac{(q^3;q^3)_\infty^{21}(q^6;q^6)_\infty^3}{(q;q)_\infty^{16}(q^2;q^2)_\infty^{10}} \\
&\quad + 19\cdot 3^4 q^2 \frac{(q^3;q^3)_\infty^{12}(q^6;q^6)_\infty^{12}}{(q;q)_\infty^{13}(q^2;q^2)_\infty^{13}} - 64\cdot 3^3 q^3 \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^{21}}{(q;q)_\infty^{10}(q^2;q^2)_\infty^{16}} \\
(1.6) \qquad &\quad + 128\cdot 3^3 q^4 \frac{(q^6;q^6)_\infty^{30}}{(q;q)_\infty^7(q^2;q^2)_\infty^{19}(q^3;q^3)_\infty^6}.
\end{aligned}
$$

Hence $a(9n+8) \equiv 0 \pmod{27}$, which coincides with the result of Chan, he derived this result with different method.

2. PRELIMINARIES

We require a few definitions and lemmas. Let

$$\Phi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \quad \Psi(q) = \sum_{n=0}^{\infty} q^{(n^2+n)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$P(q) = \frac{(q^2;q^6)_\infty(q^4;q^6)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty}, \quad X(-q) = \frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}.$$

We will use the following necessaries in proving our main results. The following lemma are the 3-dissections of functions $\frac{1}{\Phi(-q)}$ and $\frac{1}{\Psi(q)}$.

Lemma 2.1 ([5], last line in the proof Theorem 1)**.**

$$\frac{1}{\Phi(-q)} = \frac{\Phi(-q^9)}{\Phi(-q^3)^4}(\Phi(-q^9)^2 + 2q\Phi(-q^9)X(-q^3) + 4q^2 X(-q^3)^2).$$

Lemma 2.2 ([6], Lemma 2.2)**.**

$$\frac{1}{\Psi(q)} = \frac{\Psi(q^9)}{\Psi(q^3)^4}(P(q^3)^2 - qP(q^3)\Psi(q^9) + q^2\Psi(q^9)^2).$$

Lemma 2.3.

$$\Phi(-q)\Psi(q) = (q;q)_\infty(q^2;q^2)_\infty, \quad X(-q)P(q) = (q^3;q^3)_\infty(q^6;q^6)_\infty.$$

Proof. We have

$$(2.1) \quad \Phi(-q)\Psi(q) = \frac{(q;q)^2_\infty}{(q^2;q^2)_\infty} \cdot \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q;q)^2_\infty}{(q;q^2)_\infty} = (q;q)_\infty(q^2;q^2)_\infty.$$

and

$$
\begin{aligned}
X(-q)P(q) &= \frac{(q;q)_\infty(q^6;q^6)^2_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty} \cdot \frac{(q^2;q^6)_\infty(q^4;q^6)_\infty(q^3;q^3)^2_\infty}{(q;q)_\infty} \\
&= \frac{(q^2;q^6)_\infty(q^4;q^6)_\infty(q^3;q^3)_\infty(q^6;q^6)^2_\infty}{(q^2;q^2)_\infty} \\
&= \frac{(q^2;q^6)_\infty(q^4;q^6)_\infty(q^3;q^3)_\infty(q^6;q^6)^2_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty(q^6;q^6)_\infty} \\
(2.2) \quad &= (q^3;q^3)_\infty(q^6;q^6)_\infty.
\end{aligned}
$$

$\square$

3. AN NEW PROOF OF IDENTITY 1.5

We begin with the proof of identity 1.5.

Proof. We note that

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)^2_\infty} \cdot \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{1}{\Phi(-q)\Psi(q)}$$

by using lemma 2.2. So we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} a(n)q^n &= \frac{1}{\Phi(-q)\Psi(q)} \\
&= \frac{\Phi(-q^9)}{\Phi(-q^3)^4}(\Phi(-q^9)^2 + 2q\Phi(-q^9)X(-q^3) + 4q^2X(-q^3)^2) \\
&\quad \frac{\Psi(q^9)}{\Psi(q^3)^4}(P(q^3)^2 - qP(q^3)\Psi(q^9) + q^2\Psi(q^9)^2) \\
&= \frac{\Phi(-q^9)\Psi(q^9)}{\Phi(-q^3)^4\Psi(q^3)^4}(\Phi(-q^9)^2P(q^3)^2 + 2q^3\Phi(-q^9)\Psi(q^9)^2X(-q^3) \\
&\quad - 4q^3\Psi(q^9)X(-q^3)^2P(q^3) + 2q\Phi(-q^9)X(-q^3)P(q^3)^2 - q\Phi(-q^9)^2\Psi(q^9)P(q^3) \\
&\quad + 4q^4\Psi(q^9)^2X(-q^3)^2 + q^2\Phi(-q^9)^2\Psi(q^9)^2 + 4q^2X(-q^3)^2P(q^3)^2 \\
&\quad - 2q^2\Phi(-q^9)\Psi(q^9)X(-q^3)P(q^3)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{n=0}^{\infty} a(3n+2)q^{3n+2} \\
= \frac{\Phi(-q^9)\Psi(q^9)}{\Phi(-q^3)^4\Psi(q^3)^4}(q^2\Phi(-q^9)^2\Psi(q^9)^2 + 4q^2X(-q^3)^2P(q^3)^2 \\
- 2q^2\Phi(-q^9)\Psi(q^9)X(-q^3)P(q^3)).
\end{aligned}
$$

Dividing by q^2 on both sides and replacing q^3 by q, we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} a(3n+2)q^n \\
= \frac{\Phi(-q^3)\Psi(q^3)}{\Phi(-q)^4\Psi(q)^4}(\Phi(-q^3)^2\Psi(q^3)^2 + 4X(-q)^2P(q)^2 - 2\Phi(-q^3)\Psi(q^3)X(-q)P(q))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4} + 4\frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty^4(q^2;q^2)_\infty^4}(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2 \\
&\quad - 2\frac{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^4}(q^3;q^3)_\infty(q^6;q^6)_\infty \\
&= 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4},
\end{aligned}
$$

by using Lemma 2.3. $\qquad\square$

4. PROOF OF THE IDENTITY 1.6

We will use the following lemma in proving the identity 1.6.

Lemma 4.1. *Let*
$$
L := \Phi(-q^9)^2 + 2q\Phi(-q^9)X(-q^3) + 4q^2X(-q^3)^2
$$
and
$$
M := P(q^3)^2 - q\Psi(q^9)P(q^3) + q^2\Psi(q^9)^2.
$$
Then all terms having the exponents of the form of $3n+2(n \geq 0)$ in powers of q in L^4M^4 are $A + B + C + D + E$, Where

$$
\begin{aligned}
A &= 40q^2P(q^3)^8X(-q^3)^2\Phi(-q^9)^6 - 32q^2P(q^3)^7X(-q^3)\Phi(-q^9)^7\Psi(q^9) \\
&\quad + 10q^2P(q^3)^6\Phi(-q^9)^8\Psi(q^9)^2, \\
B &= 512q^5P(q^3)^8X(-q^3)^5\Phi(-q^9)^3 - 1216q^5P(q^3)^7X(-q^3)^4\Phi(-q^9)^4\Psi(q^9) \\
&\quad + 1280q^5P(q^3)^6X(-q^3)^3\Phi(-q^9)^5\Psi(q^9)^2 \\
&\quad - 640q^5P(q^3)^5X(-q^3)^2\Phi(-q^9)^6\Psi(q^9)^3 \\
&\quad + 152q^5P(q^3)^4X(-q^3)^3\Phi(-q^9)^7\Psi(q^9)^4 - 16q^5P(q^3)^3\Phi(-q^9)^8\Psi(q^9)^5, \\
C &= 256q^8P(q^3)^8X(-q^3)^8 - 2048q^8P(q^3)^7X(-q^3)^7\Phi(-q^9)\Psi(q^9) \\
&\quad + 6400q^8P(q^3)^6X(-q^3)^6\Phi(-q^9)^2\Psi(q^9)^2 \\
&\quad - 8192q^8P(q^3)^5X(-q^3)^5\Phi(-q^9)^3\Psi(q^9)^3 \\
&\quad + 5776q^8P(q^3)^4X(-q^3)^4\Phi(-q^9)^4\Psi(q^9)^4 \\
&\quad - 2048q^8P(q^3)^3X(-q^3)^3\Phi(-q^9)^5\Psi(q^9)^5 \\
&\quad + 400q^8P(q^3)^2X(-q^3)^2\Phi(-q^9)^6\Psi(q^9)^6 - 32q^8P(q^3)X(-q^3)\Phi(-q^9)^7\Psi(q^9)^7 \\
&\quad + q^8\Phi(-q^9)^8\Psi(q^9)^8, \\
D &= -4096q^{11}P(q^3)^5X(-q^3)^8\Psi(q^9)^3 + 9728q^{11}P(q^3)^4X(-q^3)^7\Phi(-q^9)\Psi(q^9)^4 \\
&\quad - 10240q^{11}P(q^3)^3X(-q^3)^6\Phi(-q^9)^2\Psi(q^9)^5 \\
&\quad + 5120q^{11}P(q^3)^2X(-q^3)^5\Phi(-q^9)^3\Psi(q^9)^6 \\
&\quad - 1216q^{11}P(q^3)X(-q^3)^4\Phi(-q^9)^4\Psi(q^9)^7 + 128q^{11}X(-q^3)^3\Phi(-q^9)^5\Psi(q^9)^8
\end{aligned}
$$

and

$$
\begin{aligned}
E &= 2560q^{14}P(q^3)^2X(-q^3)^8\Psi(q^9)^6 - 2048q^{14}P(q^3)X(-q^3)^7\Phi(-q^9)\Psi(q^9)^7 \\
&\quad + 640q^{14}X(-q^3)^6\Phi(-q^9)^2\Psi(q^9)^8.
\end{aligned}
$$

Proof. We directly expand the expression of L^4M^4 and then extract the terms having exponents $3n+2$ in powers of q, then we can obtain the results above. $\quad\square$

Lemma 4.2.

$$
\begin{aligned}
A &= 2 \cdot 3^2 q^2 \frac{(q^6;q^6)_\infty^6(q^9;q^9)_\infty^{26}}{(q^3;q^3)_\infty^6(q^{18};q^{18})_\infty^{10}}, \\
B &= 8 \cdot 3^2 q^5 \frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^{17}}{(q^3;q^3)_\infty^3(q^{18};q^{18})_\infty},
\end{aligned}
$$

$$C = 19 \cdot 3^3 q^8 (q^9;q^9)_\infty^8 (q^{18};q^{18})_\infty^8,$$

$$D = -64 \cdot 3^2 q^{11} \frac{(q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty^{17}}{(q^6;q^6)_\infty^3 (q^9;q^9)_\infty},$$

(4.1)

$$E = 128 \cdot 3^2 q^{14} \frac{(q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^{26}}{(q^6;q^6)_\infty^6 (q^9;q^9)_\infty^{10}}.$$

Proof. By using the definitions of $P(q)$, $X(-q)$, the Lemma 2.1, Lemma 2.3 and the formulas

$$(q^6;q^6)_\infty = (q^6;q^{18})_\infty (q^{12};q^{18})_\infty (q^{18};q^{18})_\infty, \quad (q^9;q^9)_\infty = (q^9;q^{18})_\infty (q^{18};q^{18}),$$

we have

$$
\begin{aligned}
A &= 40q^2 \frac{(q^6;q^{18})_\infty^8 (q^{12};q^{18})_\infty^8 (q^9;q^9)_\infty^{16}(q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty^4 (q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^8 (q^6;q^6)_\infty^2 (q^9;q^9)_\infty^2 (q^{18};q^{18})_\infty^6} \\
&\quad - 32q^2 \frac{(q^6;q^{18})_\infty^7 (q^{12};q^{18})_\infty^7 (q^9;q^9)_\infty^{14}(q^3;q^3)_\infty (q^{18};q^{18})_\infty^2 (q^9;q^9)_\infty^{14}(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^7 (q^6;q^6)_\infty (q^9;q^9)_\infty (q^{18};q^{18})_\infty^7 (q^9;q^{18})_\infty} \\
&\quad + 10q^2 \frac{(q^6;q^{18})_\infty^6 (q^{12};q^{18})_\infty^6 (q^9;q^9)_\infty^{12}(q^9;q^9)_\infty^{16}(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^8 (q^9;q^{18})_\infty^2} \\
&= 18q^2 \frac{(q^6;q^6)_\infty^6 (q^9;q^9)_\infty^{26}}{(q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^{10}}.
\end{aligned}
$$

$$
\begin{aligned}
B &= 512q^5 \frac{(q^6;q^{18})_\infty^8 (q^{12};q^{18})_\infty^8 (q^9;q^9)_\infty^{16}(q^3;q^3)_\infty^5 (q^{18};q^{18})_\infty^{10}(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8 (q^6;q^6)_\infty^5 (q^9;q^9)_\infty^5 (q^{18};q^{18})_\infty^3} \\
&\quad - 1216q^5 \frac{(q^6;q^{18})_\infty^7 (q^{12};q^{18})_\infty^7 (q^9;q^9)_\infty^{14}(q^3;q^3)_\infty^4 (q^{18};q^{18})_\infty^8 (q^9;q^9)_\infty^8 (q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^7 (q^6;q^6)_\infty^4 (q^9;q^9)_\infty^4 (q^{18};q^{18})_\infty^4 (q^9;q^{18})_\infty} \\
&\quad + 1280q^5 \frac{(q^6;q^{18})_\infty^6 (q^{12};q^{18})_\infty^6 (q^9;q^9)_\infty^{12}(q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty^6 (q^9;q^9)_\infty^{10}(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty^6 (q^6;q^6)_\infty^3 (q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^5 (q^9;q^{18})_\infty^2} \\
&\quad - 640q^5 \frac{(q^6;q^{18})_\infty^5 (q^{12};q^{18})_\infty^5 (q^9;q^9)_\infty^{10}(q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty^4 (q^9;q^9)_\infty^{12}(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^5 (q^6;q^6)_\infty^2 (q^9;q^9)_\infty^2 (q^{18};q^{18})_\infty^6 (q^9;q^{18})_\infty^3} \\
&\quad + 152q^5 \frac{(q^6;q^{18})_\infty^4 (q^{12};q^{18})_\infty^4 (q^9;q^9)_\infty^8 (q^3;q^3)_\infty (q^{18};q^{18})_\infty^2 (q^9;q^9)_\infty^{14}(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty (q^9;q^9)_\infty (q^{18};q^{18})_\infty^7 (q^9;q^{18})_\infty^4} \\
&\quad - 16q^5 \frac{(q^6;q^{18})_\infty^3 (q^{12};q^{18})_\infty^3 (q^9;q^9)_\infty^6 (q^9;q^9)_\infty^{16}(q^{18};q^{18})_\infty^5}{(q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty^6 (q^9;q^{18})_\infty^5} \\
&= 8 \cdot 3^2 q^5 \frac{(q^6;q^6)_\infty^3 (q^9;q^9)_\infty^{17}}{(q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty}.
\end{aligned}
$$

$$
\begin{aligned}
C &= 256q^8 \frac{(q^6;q^{18})_\infty^8 (q^{12};q^{18})_\infty^8 (q^9;q^9)_\infty^{16}(q^3;q^3)_\infty^8 (q^{18};q^{18})_\infty^{16}}{(q^3;q^3)_\infty^8 (q^6;q^6)_\infty^8 (q^9;q^9)_\infty^8} \\
&\quad - 2048q^8 \frac{(q^6;q^{18})_\infty^7 (q^{12};q^{18})_\infty^7 (q^9;q^9)_\infty^{14}(q^3;q^3)_\infty^7 (q^{18};q^{18})_\infty^{14}(q^9;q^9)_\infty^2 (q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^7 (q^6;q^6)_\infty^7 (q^9;q^9)_\infty^7 (q^{18};q^{18})_\infty (q^9;q^{18})_\infty} \\
&\quad + 6400q^8 \frac{(q^6;q^{18})_\infty^6 (q^{12};q^{18})_\infty^6 (q^9;q^9)_\infty^{12}(q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^{12}(q^9;q^9)_\infty^4 (q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty^6 (q^6;q^6)_\infty^6 (q^9;q^9)_\infty^6 (q^{18};q^{18})_\infty^2 (q^9;q^{18})_\infty^2} \\
&\quad - 8192q^8 \frac{(q^6;q^{18})_\infty^5 (q^{12};q^{18})_\infty^5 (q^9;q^9)_\infty^{10}(q^3;q^3)_\infty^5 (q^{18};q^{18})_\infty^{10}(q^9;q^9)_\infty^6 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^5 (q^6;q^6)_\infty^5 (q^9;q^9)_\infty^5 (q^{18};q^{18})_\infty^3 (q^9;q^{18})_\infty^3} \\
&\quad + 5776q^8 \frac{(q^6;q^{18})_\infty^4 (q^{12};q^{18})_\infty^4 (q^9;q^9)_\infty^8 (q^3;q^3)_\infty^4 (q^{18};q^{18})_\infty^8 (q^9;q^9)_\infty^8 (q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4 (q^9;q^9)_\infty^4 (q^{18};q^{18})_\infty^4 (q^9;q^{18})_\infty^4} \\
&\quad - 2048q^8 \frac{(q^6;q^{18})_\infty^3 (q^{12};q^{18})_\infty^3 (q^9;q^9)_\infty^6 (q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty^6 (q^9;q^9)_\infty^{10}(q^{18};q^{18})_\infty^5}{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3 (q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^5 (q^9;q^{18})_\infty^5}
\end{aligned}
$$

$$+ \quad 400q^8 \frac{(q^6;q^{18})_\infty^2 (q^{12};q^{18})_\infty^2 (q^9;q^9)_\infty^4 (q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty^4 (q^9;q^9)_\infty^{12} (q^{18};q^{18})_\infty^6}{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^2 (q^9;q^9)_\infty^2 (q^{18};q^{18})_\infty^6 (q^9;q^{18})_\infty^6}$$

$$- \quad 32q^8 \frac{(q^6;q^{18})_\infty (q^{12};q^{18})_\infty (q^9;q^9)_\infty^2 (q^3;q^3)_\infty (q^{18};q^{18})_\infty^2 (q^9;q^9)_\infty^{14} (q^{18};q^{18})_\infty^7}{(q^3;q^3)_\infty (q^6;q^6)_\infty (q^9;q^9)_\infty (q^{18};q^{18})_\infty^7 (q^9;q^{18})_\infty^7}$$

$$+ \quad q^8 \frac{(q^9;q^9)_\infty^{16} (q^{18};q^{18})_\infty^8}{(q^{18};q^{18})_\infty^8 (q^9;q^{18})_\infty^8}$$

$$= \quad 19 \cdot 3^3 q^8 (q^9;q^9)_\infty^8 (q^{18};q^{18})_\infty^8.$$

$$D = -4096q^{11} \frac{(q^6;q^{18})_\infty^5 (q^{12};q^{18})_\infty^5 (q^9;q^9)_\infty^{10} (q^3;q^3)_\infty^8 (q^{18};q^{18})_\infty^{16} (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^5 (q^6;q^6)_\infty^8 (q^9;q^9)_\infty^8 (q^9;q^{18})_\infty^3}$$

$$+ \quad 9728q^{11} \frac{(q^6;q^{18})_\infty^4 (q^{12};q^{18})_\infty^4 (q^9;q^9)_\infty^8 (q^3;q^3)_\infty^7 (q^{18};q^{18})_\infty^{14} (q^9;q^9)_\infty^2 (q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^7 (q^9;q^9)_\infty^7 (q^{18};q^{18})_\infty (q^9;q^{18})_\infty^4}$$

$$- \quad 10240q^{11} \frac{(q^6;q^{18})_\infty^3 (q^{12};q^{18})_\infty^3 (q^9;q^9)_\infty^6 (q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^{12} (q^9;q^9)_\infty^4 (q^{18};q^{18})_\infty^5}{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^6 (q^9;q^9)_\infty^6 (q^{18};q^{18})_\infty^2 (q^9;q^{18})_\infty^5}$$

$$+ \quad 5120q^{11} \frac{(q^6;q^{18})_\infty^2 (q^{12};q^{18})_\infty^2 (q^9;q^9)_\infty^4 (q^3;q^3)_\infty^5 (q^{18};q^{18})_\infty^{10} (q^9;q^9)_\infty^6 (q^{18};q^{18})_\infty^6}{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^5 (q^9;q^9)_\infty^5 (q^{18};q^{18})_\infty^3 (q^9;q^{18})_\infty^6}$$

$$- \quad 1216q^{11} \frac{(q^6;q^{18})_\infty (q^{12};q^{18})_\infty (q^9;q^9)_\infty^2 (q^3;q^3)_\infty^4 (q^{18};q^{18})_\infty^8 (q^9;q^9)_\infty^8 (q^{18};q^{18})_\infty^7}{(q^3;q^3)_\infty (q^6;q^6)_\infty^4 (q^9;q^9)_\infty^4 (q^{18};q^{18})_\infty^4 (q^9;q^{18})_\infty^7}$$

$$+ \quad 128q^{11} \frac{(q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty^6 (q^9;q^9)_\infty^{10} (q^{18};q^{18})_\infty^8}{(q^6;q^6)_\infty^3 (q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^5 (q^9;q^{18})_\infty^8}$$

$$= \quad -64 \cdot 3^2 q^{11} \frac{(q^3;q^3)_\infty^3 (q^{18};q^{18})_\infty^{17}}{(q^6;q^6)_\infty^3 (q^9;q^9)_\infty}$$

$$E = 2560q^{14} \frac{(q^6;q^{18})_\infty^2 (q^{12};q^{18})_\infty^2 (q^9;q^9)_\infty^4 (q^3;q^3)_\infty^8 (q^{18};q^{18})_\infty^{16} (q^{18};q^{18})_\infty^6}{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^8 (q^9;q^9)_\infty^8 (q^9;q^{18})_\infty^6}$$

$$- \quad 2048q^{14} \frac{(q^6;q^{18})_\infty (q^{12};q^{18})_\infty (q^9;q^9)_\infty^2 (q^3;q^3)_\infty^7 (q^{18};q^{18})_\infty^{14} (q^9;q^9)_\infty^2 (q^{18};q^{18})_\infty^7}{(q^3;q^3)_\infty (q^6;q^6)_\infty^7 (q^9;q^9)_\infty^7 (q^{18};q^{18})_\infty (q^9;q^{18})_\infty^7}$$

$$+ \quad 640q^{14} \frac{(q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^{12} (q^9;q^9)_\infty^4 (q^{18};q^{18})_\infty^8}{(q^6;q^6)_\infty^6 (q^9;q^9)_\infty^6 (q^{18};q^{18})_\infty^2 (q^9;q^{18})_\infty^8}$$

$$= \quad 128 \cdot 3^2 q^{14} \frac{(q^3;q^3)_\infty^6 (q^{18};q^{18})_\infty^{26}}{(q^6;q^6)_\infty^6 (q^9;q^9)_\infty^{10}}.$$

$\square$

Now we prove the identity 1.6.

Proof. By Theorem 1.5, Lemma 4.1 and Lemma 4.2, we have

$$\sum_{n=0}^\infty a(9n+8)q^{3n+2} = \quad \text{all terms having the exponents of the form of } 3n+2$$

$$(n \geq 0) \text{ in powers of } q \text{ in } 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}$$

$$= \quad \text{all terms having the exponents of the form of } 3n+2$$

$$(n \geq 0) \text{ in powers of } q \text{ in } 3\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{\Phi(-q)^4 \Psi(q)^4}$$

$$\begin{aligned}
&= \quad \text{all terms having the exponents of the form of } 3n+2 \\
&\quad (n \geq 0) \text{ in powers of } q \text{ in } 3(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3 \\
&\quad \cdot \frac{\Phi(-q^9)^4\Psi(q^9)^4}{\Phi(-q^3)^{16}\Psi(q^3)^{16}} \cdot L^4 M^4 \\
&= \quad 3(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3 \cdot \frac{\Phi(-q^9)^4\Psi(q^9)^4}{\Phi(-q^3)^{16}\Psi(q^3)^{16}} \\
&\quad \cdot (A+B+C+D+E) \\
&= \quad 3\frac{(q^9;q^9)_\infty^4(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^{16}(q^6;q^6)_\infty^{16}} \cdot (A+B+C+D+E) \\
&= \quad 3\frac{(q^9;q^9)_\infty^4(q^{18};q^{18})_\infty^4}{(q^3;q^3)_\infty^{16}(q^6;q^6)_\infty^{16}} \cdot (18q^2\frac{(q^6;q^6)_\infty^6(q^9;q^9)_\infty^{26}}{(q^3;q^3)_\infty^6(q^{18};q^{18})_\infty^{10}} \\
&+ \quad 8 \cdot 3^2 q^5\frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^{17}}{(q^3;q^3)_\infty^3(q^{18};q^{18})_\infty} + 19 \cdot 3^3 q^8(q^9;q^9)_\infty^8(q^{18};q^{18})_\infty^8 \\
&- \quad 64 \cdot 3^2 q^{11}\frac{(q^3;q^3)_\infty^3(q^{18};q^{18})_\infty^{17}}{(q^6;q^6)_\infty^3(q^9;q^9)_\infty} \\
&+ \quad 128 \cdot 3^2 q^{14}\frac{(q^3;q^3)_\infty^6(q^{18};q^{18})_\infty^{26}}{(q^6;q^6)_\infty^6(q^9;q^9)_\infty^{10}})
\end{aligned}$$

If we divide q^2 on both sides and replace q^3 by q, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} a(9n+8)q^n &= \quad 2 \cdot 3^3 \frac{(q^3;q^3)_\infty^{30}}{(q;q)_\infty^{19}(q^2;q^2)_\infty^7(q^6;q^6)_\infty^6} + 8 \cdot 3^3 q\frac{(q^3;q^3)_\infty^{21}(q^6;q^6)_\infty^3}{(q;q)_\infty^{16}(q^2;q^2)_\infty^{10}} \\
&+ \quad 19 \cdot 3^4 q^2\frac{(q^3;q^3)_\infty^{12}(q^6;q^6)_\infty^{12}}{(q;q)_\infty^{13}(q^2;q^2)_\infty^{13}} - 64 \cdot 3^3 q^3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^{21}}{(q;q)_\infty^{10}(q^2;q^2)_\infty^{16}} \\
&+ \quad 128 \cdot 3^3 q^4\frac{(q^6;q^6)_\infty^{30}}{(q;q)_\infty^7(q^2;q^2)_\infty^{19}(q^3;q^3)_\infty^6},
\end{aligned}$$

which is the identity 1.6. $\qquad\square$

5. ACKNOWLEDGEMENT

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