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Robust H_∞ observer design of linear time-delay systems with parametric uncertainty

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Abstract

This paper deals with the problem of H_∞ observer design for a class of uncertain linear systems with delayed state and parameter uncertainties. This problem aims at designing the linear state observers such that, for all admissible parameter uncertainties, the observation process remains robustly stable and the transfer function from exogenous disturbances to error state outputs meets the prespecified H_∞ norm upper bound constraint, independently of the time delay. The time delay is assumed to be unknown, and the parameter uncertainties are allowed to be norm-bounded and appear in all the matrices of the state-space model. An effective matrix inequality methodology is developed to solve the proposed problem. We derive the conditions for the existence of the desired robust H_∞ observers, and then characterize the analytical expression of these observers in terms of some free parameters. A numerical example demonstrates the validity and applicability of the present approach. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Linear systems; Parametric uncertainty; Time delay; Robust observer; H_∞ observer; Robust stabilization

1. Introduction

As is well known, the theory of state observers has long been one of the fruitful research areas, owing to its utility and its intimate connection with fundamental system concepts [6,27]. Various methods such as algebraic, geometric, inversion approaches, generalized inverse, singular-value decomposition, and the Kronecker canonical form techniques have been used in the observer design. Also, different types of state observers have been studied, such as reduced and minimal-order, full-order, unknown input, functional, disturbance decoupled, et al. The observer technique has shown its successful applications in not only system monitoring and regulation but also detecting as well as identifying failures in dynamical systems [11]. Furthermore, since the system uncertainties and exogenous disturbance input are unavoidable in modeling, the robust state observer

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design problem ([1,3,4,9,10,16,24,34]) as well as the H_∞ state observer design problem ([12,23]) have been studied for many years, in order to preserve the satisfactory observer action under system uncertainties and exogenous disturbances. It is well known that the H_∞ filtering problem is dual to the H_∞ control one for linear systems without uncertainty. However, this is not true for general uncertain delay systems, and thus the relevant robust and/or H_∞ filtering (state estimation) problems have recently received much attention (see for example [20,28,29,32,36,38]) after the publication of the classical DGKF paper [8].

On the other hand, since the delayed state is very often the cause for instability and poor performance of systems [15], increasing attention has recently been devoted to the robust and/or H_∞ controller (observer) design problems of the linear uncertain state delayed systems. For the controller design case, robust and/or H_∞ stabilization techniques, which are independent of the time delay and thus suitable for the systems with *unknown* time delay, have been developed in [7,18,25,30,33,37] by using a Riccati equation approach, and the delay-dependent design methods, which are suitable for systems with the time delay being of known size, have been proposed in [26] based on a Riccati equation approach and in [22] based on a linear matrix inequality (LMI) approach. However, for the observer design case, the relevant literature is relatively few for uncertain time delay systems (see e.g. [19,40]), and so far, little attention has been paid to the state observer design problems in the *simultaneous* presence of time delay, exogenous disturbance input and parametric uncertainty. This motivates the present research on designing robust H_∞ observers for linear systems against *unknown* time delay and admissible norm-bounded uncertainties.

This paper deals with the problem of robust H_∞ observer design for a class of uncertain linear systems with delayed state and parameter uncertainties. This problem aims at designing the linear state observers such that, for all admissible parameter uncertainties, the observation process remains robustly stable and the transfer function from exogenous disturbances to error state outputs meets the prespecified H_∞ norm upper bound constraint, independently of the time delay. The time delay is assumed to be *unknown*, and the parameter uncertainties are allowed to be norm-bounded and appear in all the matrices of the state-space model, and may be time varying. A new, simple, algebraic *parameterized* approach is exploited, which enables us to derive the existence conditions and *characterize* the set of expected robust H_∞ observers in terms of some *free* parameters, for a class of uncertain linear state delayed systems.

A special feature of the results obtained in the present paper is that the set of the desired observer gains, when it is not empty, must be very large because of the *free* design parameters in the expression of observer gains, and much *explicit* freedom is subsequently offered which gives the possibility for *directly* achieving further performance requirements on the observation process such as the transient property, H_2-norm constraint and reliability behavior (for the application of the freedom contained in the parameterization of a set of observers or filters see e.g. [21,39]). In this sense, the proposed *parametrized* approach is somehow different from the popular $\mathscr{D}$-scaled H_∞ filtering method (see e.g. [20] and references therein), and provides an alternative way to robust H_∞ state observer (filter) design. It is shown that a desired solution is related to two Riccati matrix equations, or two quadratic matrix inequalities (or two linear matrix inequalities) which are not difficult to solve.

The rest of the paper is arranged as follows. The robust H_∞ observer design problem is formulated in Section 2 for uncertain continuous time delay systems. The main results as well as detailed derivations are given in Section 3, including the existence conditions and the explicit expression of the desired robust H_∞ observers. We demonstrate the validity and applicability of the proposed theory in Section 4 by a simple example. In Section 5, some concluding remarks end the paper by pointing out possible extensions and future research directions.

2. Problem formulation and assumptions

We consider a linear uncertain continuous-time state delayed system described by

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t-h) + D_1 w(t), \tag{1}$$

$$x(t) = \phi(t), \quad t \in [-h, 0] \tag{2}$$

together with the measurement equation

$$y(t) = (C + \Delta C)x(t) + D_2 w(t), \tag{3}$$

where $x(t) \in R^n$ is the state, $w(t) \in R^r$ is the square-integrable exogenous disturbance, $y(t) \in R^p$ is the controlled output. For brevity, we have omitted known input terms in (1) and (3), since it is well known this does not affect the generality of the discussion on the observer design. $A, A_{\rm d}, D_1, C, D_2$ are known constant matrices with appropriate dimensions, h denotes the *unknown* state delay, $\phi(t)$ is a continuous vector-valued initial function. $\Delta A, \Delta A_{\rm d}, \Delta C$ are real-valued matrix functions representing norm-bounded parameter uncertainties and satisfy

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F N_1, \quad \Delta A_{\rm d} = M_1 F N_2, \tag{4}$$

where $F \in R^{i \times j}$, which may be time-varying, is a real uncertain matrix with Lebesgue measurable elements and meets

$$FF^{\rm T} \leqslant I \tag{5}$$

and M_1, M_2, N_1, N_2 are known real constant matrices of appropriate dimensions which specify how the uncertain parameters in F enter the nominal matrices $A, A_{\rm d}, C$. The uncertainties $\Delta A, \Delta A_{\rm d}, \Delta C$ are said to be admissible if both (4) and (5) are satisfied. The reason for assuming the structure of the uncertainties to be of the form given in (4), (5) can be found in many papers dealing with robust control and estimation problems, see [29,31,38] and references therein.

Throughout this paper, we will make the following assumptions.

Assumption 1. The system matrix A is asymptotically stable.

Assumption 2. The matrix M_2 is of full row rank.

It is noted that Assumption 2 does not lose any generality. In this paper, the full-order linear state observer under consideration is of the form

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ky(t), \tag{6}$$

where the constant matrices G and K are observer parameters to be designed.

Define the error state $e(t) = x(t) - \hat{x}(t)$, then it follows from (1)–(3) and (6) that

$$\dot{e}(t) = Ge(t) + [(A + \Delta A) - K(C + \Delta C) - G]x(t) + (A_{\rm d} + \Delta A_{\rm d})x(t-h) + (D_1 - KD_2)w(t). \tag{7}$$

Let

$$z(t) = Le(t) \tag{8}$$

stand for the output of error states where L is a known constant matrix. Now, by defining

$$x_{\rm f} := \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad A_{\rm f} := \begin{bmatrix} A & 0 \\ A - G - KC & G \end{bmatrix}, \quad A_{\rm df} := \begin{bmatrix} A_{\rm d} & 0 \\ A_{\rm d} & 0 \end{bmatrix}, \quad D_{\rm f} := \begin{bmatrix} D_1 \\ D_1 - KD_2 \end{bmatrix}, \tag{9}$$

$$M_{\rm f} := \begin{bmatrix} M_1 \\ M_1 - KM_2 \end{bmatrix}, \quad N_{\rm f} := \begin{bmatrix} N_1 & 0 \end{bmatrix}, \quad \Delta A_{\rm f} = M_{\rm f} F N_{\rm f}, \tag{10}$$

$$M_{\rm df} := \begin{bmatrix} M_1 \\ M_1 \end{bmatrix}, \quad N_{\rm df} := \begin{bmatrix} N_2 & 0 \end{bmatrix}, \quad \Delta A_{\rm df} := M_{\rm df} F N_{\rm df}, \quad C_{\rm f} := \begin{bmatrix} 0 & L \end{bmatrix} \tag{11}$$

and combining (1)–(3), (4) and (7), we obtain the following augmented system:

$$\dot{x}_\mathrm{f}(t) = (A_\mathrm{f} + \Delta A_\mathrm{f})x_\mathrm{f}(t) + (A_\mathrm{df} + \Delta A_\mathrm{df})x_\mathrm{f}(t-h) + D_\mathrm{f}w(t), \tag{12}$$

$$z(t) = C_\mathrm{f}x_\mathrm{f}(t). \tag{13}$$

The transfer function from disturbance $w(t)$ to error state output $C_\mathrm{f}x_\mathrm{f}(t)$ (or $Le(t)$) is given by

$$H_{zw}(s) = C_\mathrm{f}[sI - (A_\mathrm{f} + \Delta A_\mathrm{f}) - (A_\mathrm{df} + \Delta A_\mathrm{df})\mathrm{e}^{-sh}]^{-1}D_\mathrm{f}. \tag{14}$$

The objective of this paper is to design the observer parameters, G and K, such that for all admissible parameter uncertainties $\Delta A, \Delta A_\mathrm{d}, \Delta C$, the augmented system (12) and (13) is asymptotically stable and the following specified H_∞-norm upper bound constraint is simultaneously guaranteed:

$$\|H_{zw}(s)\|_\infty \leqslant \gamma \tag{15}$$

independently of the *unknown* time-delay h, where $\|H_{zw}(s)\|_\infty := \sup_{\omega\in R}\sigma_{\max}[H_{zw}(\mathrm{j}\omega)]$ and $\sigma_{\max}[\,\cdot\,]$ denotes the largest singular value of $[\,\cdot\,]$; and $\gamma < 1$ is a given positive constant. If such a design goal is achieved, the observer (6) is then said to be a robust H_∞ state observer of perturbed time delay system (1)–(3).

3. Main results and proofs

The following lemmas will be useful in designing an expected robust H_∞ observer for the uncertain linear time-delay system (1)–(3).

Lemma 1 (Wang et al. [35]). *For arbitrary positive scalar $\varepsilon_1 > 0$ and positive-definite matrix $P > 0$, we have*

$$(\Delta A_\mathrm{f})^\mathrm{T}P + P(\Delta A_\mathrm{f}) \leqslant \varepsilon_1 PM_\mathrm{f}M_\mathrm{f}^\mathrm{T}P + \varepsilon_1^{-1}N_\mathrm{f}^\mathrm{T}N_\mathrm{f}. \tag{16}$$

Lemma 2 (Wang et al. [35]). *Let a positive scalar $\varepsilon_2 > 0$ and a positive-definite matrix $Q > 0$ be such that $N_\mathrm{df}Q^{-1}N_\mathrm{df}^\mathrm{T} < \varepsilon_2 I$. Then*

$$(A_\mathrm{df} + \Delta A_\mathrm{df})Q^{-1}(A_\mathrm{df} + \Delta A_\mathrm{df})^\mathrm{T} \leqslant A_\mathrm{df}(Q - \varepsilon_2^{-1}N_\mathrm{df}^\mathrm{T}N_\mathrm{df})^{-1}A_\mathrm{df}^\mathrm{T} + \varepsilon_2 M_\mathrm{df}M_\mathrm{df}^\mathrm{T}. \tag{17}$$

Lemma 3. *For a given positive constant γ and a positive-definite matrix Q, if there exists a positive-definite matrix P satisfying the inequality*

$$(A_\mathrm{f} + \Delta A_\mathrm{f})^\mathrm{T}P + P(A_\mathrm{f} + \Delta A_\mathrm{f}) + P(A_\mathrm{df} + \Delta A_\mathrm{df})Q^{-1}(A_\mathrm{df} + \Delta A_\mathrm{df})^\mathrm{T}P + Q + C_\mathrm{f}^\mathrm{T}C_\mathrm{f} + \gamma^{-2}PD_\mathrm{f}D_\mathrm{f}^\mathrm{T}P < 0 \tag{18}$$

for all admissible parameter uncertainties $\Delta A, \Delta A_\mathrm{d}, \Delta C$, then the system (12) and (13) *is robustly asymptotically stable*; *and $\|H_{zw}(s)\|_\infty \leqslant \gamma$.*

Proof. Using the technique proposed in [18], this lemma can be proved by a standard manipulation of (18). □

Prior to stating the main results of this paper, we first give the following definitions for the sake of simplicity:

$$\Phi := A_\mathrm{d}(Q_1 - \varepsilon_2^{-1}N_2^\mathrm{T}N_2)^{-1}A_\mathrm{d}^\mathrm{T} + \varepsilon_2 M_1M_1^\mathrm{T}, \tag{19}$$

$$\hat{A} := A + \varepsilon_1 M_1M_1^\mathrm{T}P_1 + \Phi P_1 + \gamma^{-2}D_1D_1^\mathrm{T}P_1, \tag{20}$$

$$\hat{C} := C + \varepsilon_1 M_2M_1^\mathrm{T}P_1 + \gamma^{-2}D_2D_1^\mathrm{T}P_1, \tag{21}$$

$$R := \varepsilon_1 M_2M_2^\mathrm{T} + \gamma^{-2}D_2D_2^\mathrm{T}, \tag{22}$$

$$\Theta := \hat{C} + \varepsilon_1 M_2 M_1^{\mathrm{T}} P_2 + \gamma^{-2} D_2 D_1^{\mathrm{T}} P_2, \tag{23}$$

$$\Omega := \varepsilon_1 M_2 M_1^{\mathrm{T}} + \gamma^{-2} D_2 D_1^{\mathrm{T}}. \tag{24}$$

The following theorem offers the theoretical basis for achieving the desired design goal.

Theorem 1. *Let $\delta_1, \delta_2, \sigma$ be sufficiently small positive constants and $Q_1 > 0$ be a positive-definite matrix. If there exist positive scalars $\varepsilon_1, \varepsilon_2$ $(N_2 Q_1^{-1} N_2^{\mathrm{T}} < \varepsilon_2 I)$ and a matrix $S \in R^{n\times p}$ such that the following two Riccati equations have positive-definite solutions $P_1 > 0$ and $P_2 > 0$, respectively,*

$$A^{\mathrm{T}} P_1 + P_1 A + P_1(\varepsilon_1 M_1 M_1^{\mathrm{T}} + \gamma^{-2} D_1 D_1^{\mathrm{T}} + \Phi) P_1 + \varepsilon_1^{-1} N_1^{\mathrm{T}} N_1 + Q_1 + \delta_1 I = 0, \tag{25}$$

$$\begin{aligned}&(\hat{A} - \Omega^{\mathrm{T}} R^{-1} \hat{C})^{\mathrm{T}} P_2 + P_2(\hat{A} - \Omega^{\mathrm{T}} R^{-1} \hat{C}) + P_2(\varepsilon_1 M_1 M_1^{\mathrm{T}} + \gamma^{-2} D_1 D_1^{\mathrm{T}} + \Phi - \Omega^{\mathrm{T}} R^{-1} \Omega) P_2 \\ &\quad + L^{\mathrm{T}} L + S S^{\mathrm{T}} - \hat{C}^{\mathrm{T}} R^{-1} \hat{C} + (\sigma + \delta_2) I = 0,\end{aligned} \tag{26}$$

where the matrices $\Phi, \hat{A}, \hat{C}, R, \Theta, \Omega$, respectively, are defined in (19)–(24), *then the observer* (6) *with parameters*

$$K = P_2^{-1}[\Theta^{\mathrm{T}} R^{-1} + S U R^{-1/2}], \tag{27}$$

$$G = \hat{A} - K\hat{C}, \tag{28}$$

where $U \in R^{p\times p}$ is arbitrary orthogonal (i.e., $UU^{\mathrm{T}} = I$), *will be such that*, *independently of the* unknown *time-delay h*,
(1) *the augmented system* (12) *and* (13) *is asymptotically stable*,
(2) $\|H_{zw}(s)\|_\infty \leqslant \gamma$.

Proof. Since the matrix M_2 is of full row rank, R^{-1} then exists. In view of Lemmas 1 and 2, we have

$$\begin{aligned}&(A_{\mathrm{f}} + \Delta A_{\mathrm{f}})^{\mathrm{T}} P + P(A_{\mathrm{f}} + \Delta A_{\mathrm{f}}) + P(A_{\mathrm{df}} + \Delta A_{\mathrm{df}}) Q^{-1} (A_{\mathrm{df}} + \Delta A_{\mathrm{df}})^{\mathrm{T}} P \\ &\quad \leqslant A_{\mathrm{f}}^{\mathrm{T}} P + P A_{\mathrm{f}} + \varepsilon_1 P M_{\mathrm{f}} M_{\mathrm{f}}^{\mathrm{T}} P + \varepsilon_1^{-1} N_{\mathrm{f}}^{\mathrm{T}} N_{\mathrm{f}} + P[A_{\mathrm{df}}(Q - \varepsilon_2^{-1} N_{\mathrm{df}}^{\mathrm{T}} N_{\mathrm{df}})^{-1} A_{\mathrm{df}}^{\mathrm{T}} + \varepsilon_2 M_{\mathrm{df}} M_{\mathrm{df}}^{\mathrm{T}}] P.\end{aligned} \tag{29}$$

We set

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & \sigma I \end{bmatrix} > 0 \tag{30}$$

and consider definitions (9)–(11) and (19)–(24), then we have

$$\begin{aligned}\Sigma &:= A_{\mathrm{f}}^{\mathrm{T}} P + P A_{\mathrm{f}} + \varepsilon_1 P M_{\mathrm{f}} M_{\mathrm{f}}^{\mathrm{T}} P + \varepsilon_1^{-1} N_{\mathrm{f}}^{\mathrm{T}} N_{\mathrm{f}} + P[A_{\mathrm{df}}(Q - \varepsilon_2^{-1} N_{\mathrm{df}}^{\mathrm{T}} N_{\mathrm{df}})^{-1} A_{\mathrm{df}}^{\mathrm{T}} + \varepsilon_2 M_{\mathrm{df}} M_{\mathrm{df}}^{\mathrm{T}}] P \\ &\quad + Q + C_{\mathrm{f}}^{\mathrm{T}} C_{\mathrm{f}} + \gamma^{-2} P D_{\mathrm{f}} D_{\mathrm{f}}^{\mathrm{T}} P \\ &:= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^{\mathrm{T}} & \Sigma_{22} \end{bmatrix},\end{aligned} \tag{31}$$

where

$$\Sigma_{11} = A^{\mathrm{T}} P_1 + P_1 A + \varepsilon_1 P_1 M_1 M_1^{\mathrm{T}} P_1 + \varepsilon_1^{-1} N_1^{\mathrm{T}} N_1 + P_1 \Phi P_1 + Q_1 + \gamma^{-2} P_1 D_1 D_1^{\mathrm{T}} P_1, \tag{32}$$

$$\Sigma_{12} = (A - KC - G)^{\mathrm{T}} P_2 + \varepsilon_1 P_1 M_1 (M_1 - K M_2)^{\mathrm{T}} P_2 + P_1 \Phi P_2 + \gamma^{-2} P_1 D_1 (D_1 - K D_2)^{\mathrm{T}} P_2, \tag{33}$$

$$\begin{aligned}\Sigma_{22} &= G^{\mathrm{T}} P_2 + P_2 G + \varepsilon_1 P_2 (M_1 - K M_2)(M_1 - K M_2)^{\mathrm{T}} P_2 + P_2 \Phi P_2 + \sigma I \\ &\quad + \gamma^{-2} P_2 (D_1 - K D_2)(D_1 - K D_2)^{\mathrm{T}} P_2 + L^{\mathrm{T}} L.\end{aligned} \tag{34}$$

It follows from (25) that $\Sigma_{11}=-\delta_1 I<0$. By resorting to $G=\hat{A}-K\hat{C}$ and the definitions of R and Θ, we have

$$\begin{aligned}
\Sigma_{22} &= (\hat{A}-K\hat{C})^{\mathrm{T}}P_2+P_2(\hat{A}-K\hat{C})+\varepsilon_1 P_2(M_1M_1^{\mathrm{T}}-M_1M_2^{\mathrm{T}}K^{\mathrm{T}}-KM_2M_1^{\mathrm{T}}+KM_2M_2^{\mathrm{T}}K^{\mathrm{T}})P_2\\
&\quad +P_2\Phi P_2+\sigma I+\gamma^{-2}P_2(D_1D_1^{\mathrm{T}}-D_1D_2^{\mathrm{T}}K^{\mathrm{T}}-KD_2D_1^{\mathrm{T}}+KD_2D_2^{\mathrm{T}}K^{\mathrm{T}})P_2+L^{\mathrm{T}}L\\
&= \hat{A}^{\mathrm{T}}P_2+P_2\hat{A}+P_2(\varepsilon_1M_1M_1^{\mathrm{T}}+\gamma^{-2}D_1D_1^{\mathrm{T}})P_2+L^{\mathrm{T}}L+P_2\Phi P_2+\sigma I\\
&\quad -(P_2K)(\hat{C}+\varepsilon_1M_2M_1^{\mathrm{T}}P_2+\gamma^{-2}D_2D_1^{\mathrm{T}}P_2)-(\hat{C}+\varepsilon_1M_2M_1^{\mathrm{T}}P_2+\gamma^{-2}D_2D_1^{\mathrm{T}}P_2)^{\mathrm{T}}(P_2K)^{\mathrm{T}}\\
&\quad +(P_2K)(\varepsilon_1M_2M_2^{\mathrm{T}}+\gamma^{-2}D_2D_2^{\mathrm{T}})(P_2K)^{\mathrm{T}}\\
&= \hat{A}^{\mathrm{T}}P_2+P_2\hat{A}+P_2(\varepsilon_1M_1M_1^{\mathrm{T}}+\gamma^{-2}D_1D_1^{\mathrm{T}})P_2+L^{\mathrm{T}}L+P_2\Phi P_2+\sigma I\\
&\quad -(\hat{C}+\varepsilon_1M_2M_1^{\mathrm{T}}P_2+\gamma^{-2}D_2D_1^{\mathrm{T}}P_2)^{\mathrm{T}}(\varepsilon_1M_2M_2^{\mathrm{T}}+\gamma^{-2}D_2D_2^{\mathrm{T}})^{-1}\\
&\quad \cdot(\hat{C}+\varepsilon_1M_2M_1^{\mathrm{T}}P_2+\gamma^{-2}D_2D_1^{\mathrm{T}}P_2)+[(P_2K)(\varepsilon_1M_2M_2^{\mathrm{T}}+\gamma^{-2}D_2D_2^{\mathrm{T}})^{1/2}\\
&\quad -(\hat{C}+\varepsilon_1M_2M_1^{\mathrm{T}}P_2+\gamma^{-2}D_2D_1^{\mathrm{T}}P_2)^{\mathrm{T}}(\varepsilon_1M_2M_2^{\mathrm{T}}+\gamma^{-2}D_2D_2^{\mathrm{T}})^{-1/2}]\\
&\quad \cdot[(P_2K)(\varepsilon_1M_2M_2^{\mathrm{T}}+\gamma^{-2}D_2D_2^{\mathrm{T}})^{1/2}-(\hat{C}+\varepsilon_1M_2M_1^{\mathrm{T}}P_2+\gamma^{-2}D_2D_1^{\mathrm{T}}P_2)^{\mathrm{T}}\\
&\quad \cdot(\varepsilon_1M_2M_2^{\mathrm{T}}+\gamma^{-2}D_2D_2^{\mathrm{T}})^{-1/2}]^{\mathrm{T}}\\
&= \hat{A}^{\mathrm{T}}P_2+P_2\hat{A}+P_2(\varepsilon_1M_1M_1^{\mathrm{T}}+\gamma^{-2}D_1D_1^{\mathrm{T}})P_2+L^{\mathrm{T}}L+P_2\Phi P_2+\sigma I\\
&\quad -\Theta^{\mathrm{T}}R^{-1}\Theta+[(P_2K)R^{1/2}-\Theta^{\mathrm{T}}R^{-1/2}][(P_2K)R^{1/2}-\Theta^{\mathrm{T}}R^{-1/2}]^{\mathrm{T}}.
\end{aligned}\tag{35}$$

In the light of (27), it is easy to see that

$$[(P_2K)R^{1/2}-\Theta^{\mathrm{T}}R^{-1/2}][(P_2K)R^{1/2}-\Theta^{\mathrm{T}}R^{-1/2}]^{\mathrm{T}}=SS^{\mathrm{T}}.\tag{36}$$

Furthermore, it is noted that (26) can be rewritten as

$$\hat{A}^{\mathrm{T}}P_2+P_2\hat{A}+P_2(\varepsilon_1M_1M_1^{\mathrm{T}}+\gamma^{-2}D_1D_1^{\mathrm{T}}+\Phi)P_2+L^{\mathrm{T}}L+SS^{\mathrm{T}}-\Theta^{\mathrm{T}}R^{-1}\Theta+(\sigma+\delta_2)I=0\tag{37}$$

and thus (35)–(37) indicate that

$$\Sigma_{22}=-\delta_2 I<0.\tag{38}$$

Moreover, substituting (28) into (33) immediately yields $\Sigma_{12}=0$, and therefore $\Sigma<0$. Finally, it follows from Lemma 3 that system (12) and (13) is robustly asymptotically stable and $\|H_{zw}(s)\|_\infty\leqslant\gamma$. This proves Theorem 1. $\square$

Remark 1. The use of the sufficiently small positive scalars $\delta_1>0$, $\delta_2>0$ is just to ensure that $\Sigma_{11}<0$ and $\Sigma_{22}<0$ hold. As will be seen later, these two parameters can be removed when we use two quadratic matrix inequalities or two linear matrix inequalities to replace the Riccati-like matrix equations (25) and (26) and restate Theorem 1.

Remark 2. Theorem 1 shows that the robust H_∞ stability constraint on the uncertain state delayed system (1)–(3) can be guaranteed when two positive-definite solutions P_1, P_2, respectively, to the algebraic Riccati equations (25) and (26) are known to exist for some positive scalars $\varepsilon_1>0$, $\varepsilon_2>0$ and positive-definite matrix Q_1. It is seen that the existence of a positive-definite solution to (25) means that the system matrix A must be asymptotically stable, i.e., the Assumption 1 holds.

Remark 3. Note that Theorem 1 offers sufficient conditions for the existence of the expected robust H_∞ observer design problem for time-delay systems. The result may be conservative mainly due to the introduction of the inequalities (16) and (17). However, the conservativeness in Theorem 1 can be reduced over the design parameters $\varepsilon_1>0$, $\varepsilon_2>0$. A related discussion can be found in [31] and references therein. Also, when the time-delay h is a known constant, a delay-dependent algorithm (see e.g. [22]) has to be developed in order to reduce the relevant conservativeness, which gives one of the further research topics.

Remark 4. It is worth mentioning that, the problem in this paper essentially aims at designing a filter to estimate $Lx(t)$ based on the H_∞ norm constraint (15). The analysis result establishes $\Sigma < 0$ which is actually a constantly $\mathscr{D}$-scaled H_∞ norm condition. However, the corresponding synthesis problem (i.e., the parameterization problem of the filters) is more involved. As is well known, for systems without uncertainty, we only need one observer parameter to be designed. However, since the observer structure (6) is uncertainty-independent, we have two observer parameters G and K here. It can be seen from (9) to (13) that, unlike [13,17], it is impossible to treat (combine) the parameters G and K as a unified "compact factor". Therefore, the parameterization techniques developed in [13,17] cannot be directly applied to the problem addressed in this paper. Alternatively, in Theorem 1, we develop a new *parameterized* approach which enables us to derive the existence conditions and *characterize* the set of expected robust H_∞ observers in terms of some *free* parameters by solving two Riccati-like equations, or as will be shown in sequel, by solving two quadratic matrix inequalities or two linear matrix inequalities. Compared to the usual $\mathscr{D}$-scaled H_∞ filtering method, the principle advantage of the approach presented in Theorem 1 is that a parameterization of the set of desired filters can be given which is compact over some free design parameters such as the orthogonal matrix U ($U \in R^{p\times p}$), and thus make it possible to *directly* consider other performance requirements besides the robust H_∞ constraints. A further discussion on the use of design freedom is given in Remark 6.

The parameter-dependent Riccati equations (25) and (26), which play a key role in the design of expected observers, have the same type as those in [37,38], and thus in general they can be dealt with by using the approach proposed in [37,38]. Moreover, instead of the Riccati matrix equations (25) and (26), we shall restate Theorem 1 in terms of two quadratic matrix inequalities (QMIs) or LMIs in a clearer sense, and subsequently reduce the complexity of computation. To achieve such a goal, we first give a proposition as follows which can be easily proved.

Proposition 1. *For a given negative definite matrix* $\Upsilon < 0$ ($\Upsilon \in R^{n\times n}$), *there always exists a matrix* $S \in R^{n\times p}$ ($p \leqslant n$) *such that* $\Upsilon + SS^{\mathrm{T}} < 0$.

Theorem 2. *Let* σ *be a sufficiently small positive constant and* $Q_1 > 0$ *be a positive-definite matrix. If there exist positive scalars* ε_1, ε_2 ($N_2Q_1^{-1}N_2^{\mathrm{T}} < \varepsilon_2 I$) *such that the following QMIs*:

$$A^{\mathrm{T}}P_1 + P_1A + P_1(\varepsilon_1 M_1M_1^{\mathrm{T}} + \gamma^{-2}D_1D_1^{\mathrm{T}} + \Phi)P_1 + \varepsilon_1^{-1}N_1^{\mathrm{T}}N_1 + Q_1 < 0, \tag{39}$$

$$\begin{aligned}\Upsilon := {} & (\hat{A} - \Omega^{\mathrm{T}}R^{-1}\hat{C})^{\mathrm{T}}P_2 + P_2(\hat{A} - \Omega^{\mathrm{T}}R^{-1}\hat{C}) + P_2(\varepsilon_1 M_1M_1^{\mathrm{T}} + \gamma^{-2}D_1D_1^{\mathrm{T}} + \Phi - \Omega^{\mathrm{T}}R^{-1}\Omega)P_2 \\ & + L^{\mathrm{T}}L - \hat{C}^{\mathrm{T}}R^{-1}\hat{C} + \sigma I < 0,\end{aligned} \tag{40}$$

respectively, have positive-definite solutions $P_1 > 0$ *and* $P_2 > 0$, *where the matrices* Φ, $\hat{A}$, $\hat{C}$, R, Θ, Ω, *respectively, are defined in* (19)–(24), *then the observer* (6) *with parameters*

$$K = P_2^{-1}[\Theta^{\mathrm{T}}R^{-1} + SUR^{-1/2}], \tag{41}$$

$$G = \hat{A} - K\hat{C}, \tag{42}$$

where $U \in R^{p\times p}$ *is arbitrary orthogonal* (i.e., $UU^{\mathrm{T}} = I$), $S \in R^{n\times p}$ *is an arbitrary matrix meeting* $\Upsilon + SS^{\mathrm{T}} < 0$ *and* Υ *is defined in* (40), *will be such that, independently of the unknown time-delay* h,

(1) *the augmented system* (12) *and* (13) *is asymptotically stable*,
(2) $\|H_{zw}(s)\|_\infty \leqslant \gamma$.

Proof. The proof is a direct combination of Theorem 1 and Proposition 1. □

Remark 5. Theorem 2 gives a QMI approach to the design of robust H_∞ observers for linear uncertain time-delay systems. When we tackle with the QMIs (39) and (40), the local numerical searching algorithms suggested in [2,14] are effective for a relatively low-order model. A related discussion of the solving algorithm for quadratic matrix inequalities can also be found in [31]. Instead, in the case that

$\sigma > 0$, $Q_1 > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are fixed, we can easily convert the two QMIs (39) and (40) into two LMIs by using the well-known results on Schur complements of a partitioned symmetric matrix, and it follows that the design problem can be efficiently solved [5]. Furthermore, although we consider only the state observer design problem here, it is not difficult to directly extend the present design method to the filter design case.

Remark 6. It should be pointed out that, in the present design procedure of robust H_∞ observers for time-delay systems, there still exists much *explicit* freedom, such as the choices of the positive-definite matrix $Q_1 > 0$, the free parameters S ($S \in R^{n\times p}$ satisfies $\Upsilon + SS^{\mathrm{T}} < 0$) and orthogonal matrix U ($U \in R^{p\times p}$) in expressions (41) and (42), etc. This remaining freedom provides the possibility for considering more performance constraints (e.g., the transient requirement and reliability behavior on the observation process) which requires further investigations. Note that in [21,39], a similar freedom on an arbitrary orthogonal matrix in the parameterization of the set of filters was successfully employed to minimize the H_2 norm of the filtering error transfer function by solving an unconstrained parametric optimization problem over the set of filters.

4. Numerical example

In this section, we demonstrate the theory developed in this paper by means of a simple example.

Consider the linear continuous uncertain time-delay system (1) and (2) with parameters given by

$$A = \begin{bmatrix} -2 & -0.5 \\ 0.5 & -3 \end{bmatrix}, \quad A_{\mathrm{d}} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C = [1 \quad 0], \quad D_2 = [0.5 \quad 0.8], \quad M_1 = \begin{bmatrix} 0.1 & 0.05 \\ -0.02 & 0.1 \end{bmatrix}, \quad L = [0.1\ 0.1],$$

$$M_2 = [-0.2 \quad 0.8], \quad N_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.1 & 0.01 \\ 0.02 & 0.5 \end{bmatrix}, \quad |h| \leqslant 0.05,$$

where $0 < h < 0.05$ is an unknown positive scalar.

The purpose is to design the robust H_∞ observer being of the structure (6), which does not depend on both the uncertainties and time delay, such that for all admissible parameter perturbations, the observation process is asymptotically stable and the transfer function from exogenous disturbances to error state outputs meets the prespecified H_∞ norm upper bound constraint $\|H_{zw}(s)\|_\infty \leqslant 0.8$.

Subjected to the constraint $N_2 Q_1^{-1} N_2^{\mathrm{T}} < \varepsilon_2 I$, we choose

$$\varepsilon_1 = 0.1, \quad \delta_1 = 0.01, \quad \varepsilon_2 = 0.5, \quad \sigma = 10, \quad Q_1 = I_2$$

and then obtain the positive-definite solution to Riccati equation (25) and thus $\hat{A}$, $\hat{C}$, and R, respectively, as follows:

$$P_1 = \begin{bmatrix} 1.6511 & -0.0470 \\ -0.0470 & 1.1315 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} -1.5405 & -0.4983 \\ 0.4970 & -2.4094 \end{bmatrix},$$

$$\hat{C} = [1.1260\ 0.1472], \quad R = 1.4586.$$

Then, choose $L = [0.5\ 0.5]^{\mathrm{T}}$ and solve Riccati equation (6) to give

$$P_2 = \begin{bmatrix} 2.4162 & -0.0706 \\ -0.0706 & 1.7927 \end{bmatrix}, \quad \Theta = [1.3102 \quad 0.3807].$$

Finally, since the dimension of measurement output is $p = 1$, the arbitrary "matrix" U can be only chosen to be 1 or -1. Therefore, in these two cases, the desired filter parameters K_1, G_1 (for $U = 1$) and K_2, G_2

(for $U = -1$) can be obtained from (27) and (28), respectively, as the following:

$$K_1 = \begin{bmatrix} 0.5547 \\ 0.3984 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -2.1651 & -0.5800 \\ -0.0484 & -2.4680 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.1981 \\ -0.0776 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1.7636 & -0.5275 \\ 0.5843 & -2.3979 \end{bmatrix}.$$

It is not difficult to verify that the specified robust stability as well as H_∞ disturbance rejection constraints are achieved.

5. Concluding remarks

This paper has studied the problem of robust H_∞ state observer design for a class of continuous-time state delayed systems with parameter uncertainties in both state and measurement matrices. A linear observer structure has been adopted. A modified algebraic Riccati inequality (equation) approach has been developed to solve the above problem. Specifically, the conditions for the existence of the expected robust H_∞ observers have been derived in terms of two algebraic Riccati inequalities (equations). Also, the analytical expression of the desired observers has been characterized. A numerical example has shown the effectiveness of the present design approach.

It has been demonstrated that the desired robust H_∞ observers of time-delay systems, when they exist, are usually a large set, and the remaining freedom can be used to meet other expected performance requirements. The main results can also be extended to discrete-time systems and sampled-data systems, and systems with convex parameter uncertainties. These will be the subjects of further investigations.

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