Mathematical Summary of pyMacroIO: Dynamic Disequilibrium Input-Output Model

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This document gives an overview of the simple plain-vanilla single-region Dynamic Disequilibrium Input-Output (IO) model. This follows closely descriptions in [1,2,3]. The model is discrete-time and sector-level. It includes Leontief (traditional or adapted), linear, or CES production; inventories; labour hiring and firing; a Muellbauer consumption function; and two exogenous shocks: the consumption shock and the input-availability shock. Base-year example data are taken from a single regional data set. The Python implementation is provided in the `pyMacroIO` module; the default data file is `data/example_data.pkl` [4].

Model and document version: `pyMacroIO` 1.0.

Contents

1 Model Structure

1.1 Basic Setup

The model is defined over N sectors and T time periods within a single region only. Gross output at time t is denoted by $\mathbf{x}_t \in \mathbb{R}^N$, the technical coefficient matrix by $\mathbf{A} \in \mathbb{R}^{N\times N}$, total demand by $\mathbf{d}_t \in \mathbb{R}^N$, and final demand by $\mathbf{f}_t \in \mathbb{R}^N$. No regional index is used, since the system is strictly single-region.

1.2 Economic Balance Constraint

The fundamental static balance is

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}, \tag{1}$$

from which

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{L}\mathbf{f} \tag{2}$$

is obtained, where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ is the Leontief inverse. In the dynamic model, this structure underlies calibration and the identification of essential inputs; within-period equilibrium is enforced by the sequential determination of labour, demand, production, intermediate consumption, and inventory update.

2 Data and Parameter Calibration

2.1 Technical Coefficient Matrix

The technical coefficient matrix $\mathbf{A}$ is constructed from base-year intermediate flows $\mathbf{Z}_0$ and base-year gross output $\mathbf{x}_0$. The gross output $\mathbf{x}_0$ is obtained from the row identity: the output of each sector equals the sum of its intermediate sales plus total final demand. Then

$$A_{ij} = \frac{Z_{0,ij}}{x_{j,0}}, \tag{3}$$

with the convention that each column sum of intermediate inputs is less than output:

$$\sum_{i=1}^{N} A_{ij} < 1 \quad \forall j. \tag{4}$$

2.2 Essential Inputs Identification

For the adapted Leontief and CES production functions, a binary importance matrix $\mathbf{A}_{\text{essential}}$ is built. The primary method is the combined Linkage Method; a value-based fallback is used when the Leontief inverse cannot be computed [5].

2.2.1 Combined Linkage Method (Primary)

With $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ and $\mathbf{G} = (\mathbf{I} - \mathbf{A}^T)^{-1}$:

$$\text{Backward Linkage}_j = \sum_{i=1}^{N} L_{ij}, \tag{5}$$

$$\text{Forward Linkage}_i = \sum_{j=1}^{N} G_{ij}. \tag{6}$$

The linkages are normalised by their means (with a small constant $\varepsilon > 0$ to avoid division by zero):

$$\text{Norm Backward}_j = \frac{\text{Backward Linkage}_j}{\overline{\text{Backward Linkage}} + \varepsilon}, \tag{7}$$

$$\text{Norm Forward}_i = \frac{\text{Forward Linkage}_i}{\overline{\text{Forward Linkage}} + \varepsilon}. \tag{8}$$

The combined linkage is Combined Linkage$_{ij}$ = Norm Forward$_i$ × Norm Backward$_j$. The binary indicator is

$$A_{\text{essential},ij} = \begin{cases} 1 & \text{if Combined Linkage}_{ij} > \theta \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where θ is a (threshold) parameter.

2.2.2 Value-Based Fallback

If matrix inversion fails, essential inputs are identified by value share:

$$A_{\text{essential},ij} = \begin{cases} 1 & \text{if } \frac{A_{ij}}{\sum_{k=1}^{N} A_{kj} + \varepsilon} \geq \theta_{\text{value}} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where θ_{value} is a value-share threshold.

2.3 Value Added Decomposition

Value added in the base year is decomposed as

$$\text{VA}_j = L_j + \text{cap}_j + \text{tax}_j + \text{other}_j + \Pi_j, \tag{11}$$

where L_j is labour cost, cap_j capital cost, tax_j taxes, other_j other costs, and Π_j profits. Value added is taken from the column identity (output minus domestic intermediate inputs minus imports). Base-year labour $\ell_{j,0}$, capital shares λ_j^{cap}, tax shares λ_j^{tax}, and import shares λ_j^{imp} are used, so that profits are computed as

$$\Pi_{j,t} = x_{j,t} - \sum_{i=1}^{N} Z_{ij,t} - \ell_{j,t} - (\lambda_j^{\text{cap}} + \lambda_j^{\text{tax}} + \lambda_j^{\text{imp}})\, x_{j,t}. \tag{12}$$

3 Dynamic Model Equations

3.1 Time Step and Consumption Persistence

Two time frequencies are considered: daily and quarterly. The persistence parameter ρ_1 (and $\rho_0 = 1 - \rho_1$) in the consumption function is set from the time step length dt, which is determined by the chosen frequency. For daily frequency, $\rho_1 = 1 - (1 - \bar{\rho})\, dt$ and $\rho_0 = 1 - \rho_1$ for a baseline $\bar{\rho}$; for quarterly or other frequencies, dt and (ρ_0, ρ_1) are set accordingly. The parameter dt is used only for this calibration and does not appear in the dynamic equations.

3.2 Labour Adjustment Mechanism

Labour is adjusted towards capacity and demand/input constraints via sector-specific hiring and firing speeds $\gamma_{\text{hire},j}$ and $\gamma_{\text{fire},j}$. When hiring and firing are switched off, $\ell_{j,t} = \ell_{j,t-1}$ is imposed. Otherwise the following quantities are used:

$$
\begin{aligned}
\ell_{\text{cap},j,t} &= \ell_{j,0} \cdot \max\bigl(\min(1 - \delta_{j,t}, 1), \underline{\delta}\bigr), && (13)\\
\ell_{\text{cap,max},j} &= \overline{\varsigma}\, \ell_{j,0}, \quad \ell_{\text{cap,min},j} = \underline{\varsigma}\, \ell_{j,0}, && (14)\\
\ell^{\text{new}}_{\text{cap},j,t} &= \max\bigl(\max(\ell_{\text{cap},j,t}, \ell_{\text{cap,min},j}), \ell_{j,t-1}\bigr), && (15)\\
\text{share}_j &= \frac{\ell_{j,0}}{x_{j,0}}, && (16)\\
\text{potential}_j &= \min(\text{input\_constraint}_j, \text{demand}_j), \quad \Delta_j = \text{potential}_j - \text{capacity}_j. && (17)
\end{aligned}
$$

The adjustment uses $\gamma_{\text{hire},j}$ when $\Delta_j > 0$ and $\phi\, \gamma_{\text{fire},j}$ when $\Delta_j \le 0$:

$$\ell_{j,t} = \text{clip}\Bigl(\ell_{j,t-1} + \gamma_j \cdot \text{share}_j \cdot \Delta_j,\ \ell_{\text{cap,min},j},\ \ell^{\text{new}}_{\text{cap},j,t}\Bigr), \tag{18}$$

where $\gamma_j = \gamma_{\text{hire},j}$ if $\Delta_j > 0$ and $\gamma_j = \phi\, \gamma_{\text{fire},j}$ otherwise, and $\phi \in (0, 1]$ is a firing-speed damping factor. The bounds on δ and the capacity scale factors $\underline{\varsigma}$, $\overline{\varsigma}$ are model parameters.

3.3 Consumption Function (Muellbauer)

Aggregate consumption demand C_t is updated according to

$$C_t = \exp\left(\rho_1 \log C_{t-1} + \frac{\rho_0}{2} \log(\text{MPC} \cdot L^{\text{adj}}_t) + \frac{\rho_0}{2} \log(\text{MPC} \cdot L_0 \cdot \xi_t)\right), \tag{19}$$

with floor $C_t \ge \alpha_C\, \bar{C}$, where $\bar{C} = \max(\text{MPC} \cdot L_0, 1)$ and $\alpha_C \in (0, 1]$ is a floor ratio. The log arguments are floored by fractions of baseline consumption (model parameters). Here

$$
\begin{aligned}
&\rho_0 + \rho_1 = 1, && (20)\\
&\text{MPC} = \frac{\sum_{j=1}^{N} c_{j,0}}{\sum_{j=1}^{N} \ell_{j,0}}, \quad L_t = \sum_{j=1}^{N} \ell_{j,t}, \quad L^{\text{adj}}_t = \beta \cdot L_0 + (1 - \beta) \cdot L_t, && (21)
\end{aligned}
$$

with $\beta \in [0, 1]$ the benefits parameter. Sectoral consumption demand is

$$c^d_{j,t} = \theta_{j,t} \cdot C_t \cdot (1 - \epsilon_t), \tag{22}$$

with $\theta_{j,t} = c_{j,0} / \sum_k c_{k,0}$ fixed from base-year shares.

3.4 Inventory Management and Orders

Target inventories are proportional to base-year intermediate flows and a sector-specific coverage n_j (in days):

$$S_{\text{target},ij} = Z_{0,ij} \cdot n_j = A_{ij}\, x_{j,0}\, n_j. \tag{23}$$

Orders for intermediate inputs are

$$O_{ij,t} = A_{ij}\, d_{j,t-1} + \frac{S_{\text{target},ij} - S_{ij,t-1}}{\tau_j}, \quad O_{ij,t} = \max(0, O_{ij,t}), \tag{24}$$

where τ_j is the inventory-adjustment speed of the receiving sector j. Each τ_j is sector-specific and lies in an interval $[\underline{\tau}, \overline{\tau}]$ prescribed by the model.

3.5 Production Functions

Capacity is determined by labour: $x_{\text{cap},j,t} = (\ell_{j,t}/\ell_{j,0})\, x_{j,0}$, and may be further limited by an exogenous output constraint $x^{\text{out}}_{\text{cap},j,t}$. The effective capacity is $\min(x_{\text{cap},j,t}, x^{\text{out}}_{\text{cap},j,t})$. Production is given by the minimum of capacity, input-determined output, and demand. The model admits four production types: traditional Leontief, adapted Leontief (with essential inputs), linear, and CES.

3.5.1 Traditional Leontief

$$x_{j,t} = \min\left(\text{capacity}_{j,t},\ \min_{i:A_{ij}>0} \frac{S_{ij,t-1}}{A_{ij}},\ d_{j,t}\right). \tag{25}$$

3.5.2 Adapted Leontief with Essential Inputs

Essential inputs are those with $A_{\text{essential},ij} > \theta_{\text{ess}}$; important inputs are those with $A_{\text{essential},ij} = \theta_{\text{ess}}$, for a threshold θ_{ess}. They are treated separately:

$$x^{\text{ess}}_{\text{input},j} = \min_{i:A_{\text{essential},ij}>\theta_{\text{ess}},\ A_{ij}>0} \frac{S_{ij,t-1}}{A_{ij}}, \quad x^{\text{imp}}_{\text{input},j} = \min_{i:A_{\text{essential},ij}=\theta_{\text{ess}},\ A_{ij}>0} \left(\omega \frac{S_{ij,t-1}}{A_{ij}} + (1-\omega)\frac{x_{j,0}}{2}\right). \tag{26}$$

The input constraint is $x_{\text{input},j} = \min(x^{\text{ess}}_{\text{input},j}, x^{\text{imp}}_{\text{input},j})$ (with the usual convention that an empty set gives $+\infty$). Then

$$x_{j,t} = \min(\text{capacity}_{j,t},\ x_{\text{input},j},\ d_{j,t}). \tag{27}$$

The weight $\omega \in (0,1]$ in the important-input term is a model parameter. With the default (combined-linkage) method, $\mathbf{A}_{\text{essential}}$ is binary $\{0,1\}$; only essential inputs ($A_{\text{essential},ij} > \theta_{\text{ess}}$) are used in production. The "important" classification ($A_{\text{essential},ij} = \theta_{\text{ess}}$) is defined for consistency with a possible future ternary or continuous importance but is unused when the matrix is binary.

3.5.3 Linear Production

$$x_{j,t} = \min\left(\text{capacity}_{j,t},\ \frac{\sum_{i=1}^{N} S_{ij,t-1}}{\sum_{i=1}^{N} A_{ij}},\ d_{j,t}\right). \tag{28}$$

3.5.4 CES Production

For the CES specification, constraints from essential inputs ($A_{\text{essential},ij} > \theta_{\text{ess}}$), important inputs ($A_{\text{essential},ij} = \theta_{\text{ess}}$), and aggregated non-essential inputs ($A_{\text{essential},ij} = 0$) are combined. The essential constraint is $\min_i S_{ij}/A_{ij}$ over essential i; the important constraint is $\min_i(S_{ij}/A_{ij} + x_{\text{cap},j,0})\,\omega$ over important i; and the non-essential constraint is $\sum_{i\in\text{NC}} S_{ij}/\sum_{i\in\text{NC}} A_{ij}$ when the non-essential share is positive. Output is the minimum of capacity, the minimum of those constraints, and demand. As for adapted Leontief, with a binary $\mathbf{A}_{\text{essential}}$ from the default method only the essential and non-essential constraints apply; the important branch is unused.

3.6 Rationing

Absent shocks, industries can always meet total demand, i.e. $x_{j,t} = d_{j,t}$. However, in the presence of production capacity and/or input bottlenecks, industries' output may be smaller than total demand (i.e., $x_{j,t} < d_{j,t}$), in which case output is rationed across customers. Simple proportional rationing is assumed. The final delivery from industry i to industry j is the share of orders received,

$$Z_{ij,t} = O_{ij,t}\,\frac{x_{i,t}}{d_{i,t}}. \tag{29}$$

A share of household demand is received,

$$c_{i,t} = c^d_{i,t}\,\frac{x_{i,t}}{d_{i,t}}, \tag{30}$$

and the realised final consumption of agents with exogenous final demand is

$$f_{i,t} = f^d_{i,t}\,\frac{x_{i,t}}{d_{i,t}}. \tag{31}$$

(When the model disaggregates final demand into government, investment, exports, etc., the same sector-level share $x_{i,t}/d_{i,t}$ applies to each component.) In the implementation, deliveries and consumption are further capped by available inventories where relevant.

3.7 Intermediate Consumption

Desired intermediate use is given by orders scaled by a supply/demand factor. Under proportional rationing, when $x_{i,t} < d_{i,t}$, the supply/demand share $x_{i,t}/d_{i,t}$ for the supplying sector i yields the rationing formulae above. Two allocation rules are implemented:

- **By recipient:** $s_j = x_{j,t}/d_{j,t}$ per sector j; then $Z_{ij,t} = \min(O_{ij,t}\,s_j,\ S_{ij,t-1})$.
- **By supplier:** Per supplier i, $s_i = \min(1,\ x_{i,t}/\sum_{k=1}^{N} O_{ik,t})$; then $Z_{ij,t} = \min(O_{ij,t}\,s_i,\ S_{ij,t-1})$.

In both cases, factors are clipped to $[0, 1]$.

3.8 Final Consumption (Realised)

Under the by-recipient rule, realised consumption is $s_j = x_{j,t}/d_{j,t}$ per sector j, so $c_{j,t} = c^d_{j,t} \cdot s_j$. Under the by-supplier rule, a residual supply/demand ratio is applied: for each sector i,

$$s_i = \frac{x_{i,t} - \sum_{k=1}^{N} Z_{ik,t}}{d_{i,t} - \sum_{k=1}^{N} Z_{ik,t}}, \quad \text{clipped to } [0,1], \quad c_{i,t} = c^d_{i,t} \cdot s_i, \tag{32}$$

i.e. output net of intermediate deliveries over demand net of those deliveries.

3.9 Inventory Update

The inventory of input i held by sector j is updated according to

$$S_{ij,t} = \max\bigl(0,\ S_{ij,t-1} + Z_{ij,t} - A_{ij}\, x_{j,t}\bigr). \tag{33}$$

In a Leontief production function, where every input is critical, the maximum operator would be redundant since production could never continue once inventories are run down. It is necessary for other production functions since industries can produce even after inventories of one or more non-critical inputs are depleted; thus the maximum operator avoids inventories becoming negative.

3.10 Profit and GDP

Profits are computed using labour and value-added shares (capital, tax, import) from the base year:

$$\Pi_{j,t} = x_{j,t} - \sum_{i=1}^{N} Z_{ij,t} - \ell_{j,t} - (\lambda_j^{\text{cap}} + \lambda_j^{\text{tax}} + \lambda_j^{\text{imp}})\, x_{j,t}. \tag{34}$$

GDP (value added, output minus intermediate consumption, with import adjustment) is

$$\text{GDP}_t = \sum_{j=1}^{N} \left(x_{j,t} - \sum_{i=1}^{N} Z_{ij,t} - \lambda_j^{\text{imp}}\, x_{j,t} \right). \tag{35}$$

3.11 Savings

Aggregate savings are computed as

$$S_t = \sum_{j=1}^{N} \Pi_{j,t} + \sum_{j=1}^{N} \ell_{j,t} - \sum_{j=1}^{N} c_{j,t} - E_t, \tag{36}$$

where

$$E_t = \frac{\nu}{1-\nu} \sum_{j=1}^{N} c_{j,t} \tag{37}$$

and $\nu \in [0,1)$ is the extra-expenditure coefficient.

3.12 Final Demand and Aggregate Demand

Total demand for sector j in period t is

$$d_{j,t} = c^d_{j,t} + \sum_{k=1}^{N} O_{jk,t} + f_{\text{gov},j,t} + f_{\text{inv},j,t} + f_{\text{invnt},j,t} + f_{\text{exp},j,t} + f_{\text{other},j,t}, \tag{38}$$

where $c^d_{j,t}$ is consumption demand from the Muellbauer block, $\sum_k O_{jk,t}$ is intermediate demand for sector j's output (orders received by sector j from all sectors k), and f_{gov}, f_{inv}, f_{invnt}, f_{exp}, f_{other} are government, investment, inventory (final demand), exports, and any other final demand. In the single-region model one such vector per period is used.

4 Consumption Shock (Simple Example)

In this simple example, the consumption shock is active in the example run. For $t \in [t_{\text{start}}, t_{\text{start}} + \text{duration})$, the shock intensity $\epsilon_t \in [0, 1)$ is set (e.g. $\epsilon_t = 0.2$ for a 20% cut). Sectoral consumption demand is scaled down via

$$c^d_{j,t} = \theta_{j,t}\, C_t\, (1 - \epsilon_t). \tag{39}$$

The default example uses intensity 0.2, duration 3 periods, and start period 2. When demand is the binding constraint, all production functions yield identical results.

5 Input-Availability Shock (Simple Example)

An input-availability shock reduces the effective availability of a chosen input sector for a given duration. For each period t in the shock window, the usable stock of that sector's output (as an input to other sectors) is scaled by $(1 - \zeta)$, where $\zeta \in [0, 1)$ is the reduction fraction. Thus, in the production step, effective inventories $S^{\text{eff}}_{ij,t-1}$ are used in place of $S_{ij,t-1}$ when computing the input constraint: for the shocked sector i^*, $S^{\text{eff}}_{i^*,j,t-1} = (1 - \zeta)\, S_{i^*,j,t-1}$; for all other sectors $i \neq i^*$, $S^{\text{eff}}_{ij,t-1} = S_{ij,t-1}$. Output is then determined by the chosen production function (Leontief, adapted Leontief, linear, or CES) using S^{eff} instead of S. When the shocked sector is a key supplier (e.g. the sector with the largest forward supply), the input constraint can bind and production functions differ: Leontief is most constrained (each input essential), linear can remain near baseline (aggregate input, perfect substitution), and adapted Leontief and CES (same essential-input bottleneck from $\mathbf{A}_{\text{essential}}$) can coincide. The default example uses a key supplier chosen by largest row sum of $\mathbf{A}$, reduction 0.6, duration 3 periods, and start period 2.

6 Monte Carlo Uncertainty Analysis

Parameters may be drawn from specified distributions; for each draw the model is solved and GDP, aggregate consumption, and gross output by sector and time are stored.

6.1 Parameter Distributions

Typical choices are uniform distributions over prescribed intervals for:

1. **Consumption persistence** ρ_1 (with $\rho_0 = 1 - \rho_1$),
2. **Labour hiring speed** $\gamma_{\text{hire},j}$ per sector,
3. **Labour firing speed** $\gamma_{\text{fire},j}$ per sector,
4. **Inventory adjustment** τ_j per sector,
5. **Savings rate** s (used to adjust initial consumption via $c_0 = \ell_0\,(1 - s)$).

6.2 Uncertainty Metrics

For each stored variable Y and time t, the mean, standard deviation, and selected quantiles are computed over valid simulations (invalid or failed draws are excluded):

$$\mu_{Y,t} = \frac{1}{M}\sum_{i=1}^{M} Y_{i,t}, \quad \sigma_{Y,t} = \sqrt{\frac{1}{M-1}\sum_{i=1}^{M}(Y_{i,t}-\mu_{Y,t})^2}, \tag{40}$$

$$q_{Y,t}^{(p)} = \text{percentile}(Y_{1,t}, \ldots, Y_{M,t},\ p). \tag{41}$$

Selected quantiles are used for reporting and visualisation.

6.3 Interpretation of the Bands

The reported quantiles and bands are intended to reflect parameter uncertainty (uncertainty in the structural parameters listed above). Uncertainty about the shock itself (timing, size, or duration) is not included. Because the same parameters govern the dynamics in every period, the spread across simulations may persist or widen after the shock has ended; the bands do not neccessarily narrow once the shock is removed.

7 Model Assumptions and Consistency

7.1 Economic Balance

The row identity of the base-year accounts ($\mathbf{x}_0 = \mathbf{Z}_0\mathbf{1} + \mathbf{f}_0$) and the column identity (output equals domestic intermediate inputs plus imports plus value added) are assumed to hold. Simulated flows are consistent with these accounting identities by construction.

7.2 Parameter Bounds

Sector parameters are required to satisfy

$$\underline{\gamma}_{\text{hire}} \le \gamma_{\text{hire},j} \le \overline{\gamma}_{\text{hire}}, \quad \underline{\gamma}_{\text{fire}} \le \gamma_{\text{fire},j} \le \overline{\gamma}_{\text{fire}}, \quad \underline{\tau} \le \tau_j \le \overline{\tau} \quad \forall j, \tag{42}$$

for model-given bounds $\underline{\gamma}_{\text{hire}}, \overline{\gamma}_{\text{hire}}, \underline{\gamma}_{\text{fire}}, \overline{\gamma}_{\text{fire}}, \underline{\tau}, \overline{\tau}$. Consumption parameters satisfy $\rho_0 + \rho_1 = 1$. The benefits parameter β is in $[0, 1]$.

8 Order of Operations (Single Period)

The model is solved as a discrete-time sequential simulation. For each period $t = 1, \ldots, T-1$, the order of operations is:

1. **Labour:** $\ell_{j,t}$ is updated from $\ell_{\cdot,t-1}$, $x_{\cdot,t-1}$, $\delta_{\cdot,t}$, and the previous period's capacity, input, and demand constraints, using the hiring/firing rule above.
2. **Capacity:** $x_{\text{cap},j,t} = (\ell_{j,t}/\ell_{j,0})\, x_{j,0}$; output constraints are applied if present.
3. **Consumption demand:** C_t and $c^d_{j,t}$ are computed from persistence, labour income, expectations ξ_t, and the consumption shock ϵ_t.
4. **Orders:** $O_{ij,t}$ is computed from $\mathbf{A}$, $d_{\cdot,t-1}$, τ, S_{target}, and $S_{\cdot,t-1}$.
5. **Aggregate demand:** $d_{j,t} = c^d_{j,t} + \sum_k O_{jk,t} + f_{\text{gov},j,t} + f_{\text{inv},j,t} + f_{\text{invnt},j,t} + f_{\text{exp},j,t} + f_{\text{other},j,t}$.

6. **Production:** $x_{j,t}$ is determined from capacity, input constraints, and demand, according to the chosen production type.

7. **Intermediate consumption:** $Z_{ij,t}$ is determined from $O_{\cdot,t}$, $d_{\cdot,t}$, $x_{\cdot,t}$, the allocation rule (by recipient or by supplier), and $S_{\cdot,t-1}$. When rationing shocks are active for period t, deliveries from rationed suppliers are scaled proportionally.

8. **Realised consumption:** $c_{j,t}$ is determined from $c^d_{\cdot,t}$, $d_{\cdot,t}$, $x_{\cdot,t}$, $Z_{\cdot,t}$, and the allocation rule. When rationing shocks are active for period t, consumption from rationed suppliers is scaled proportionally.

9. **Inventory:** $S_{ij,t} = \max(0,\ S_{ij,t-1} + Z_{ij,t} - A_{ij}\, x_{j,t})$.

10. **Accounting:** $\Pi_{j,t}$, S_t, and GDP_t are computed from the formulae above.

9 References

1. Pichler, A., Pangallo, M., del Rio-Chanona, R. M., Lafond, F., & Farmer, J. D. (2022). Forecasting the propagation of pandemic shocks with a dynamic input-output model. Journal of Economic Dynamics and Control, 144, 104527. `https://doi.org/10.1016/j.jedc.2022.104527`

2. Ross, A. G., McGregor, P. G., & Swales, J. K. (2024). Labour market dynamics in the era of technological advancements: The system-wide impacts of labour augmenting technological change. Technology in Society, 77, 102539. `https://doi.org/10.1016/j.techsoc.2024.102539`

3. Raseta, M., Ross, A. G., & Voegele, S. (2025). Macro-level implications of the energy system transition to net-zero carbon emissions: Identifying quick wins amid short-term constraints. Economic Analysis and Policy, 85, 1065-1078. `https://doi.org/10.1016/j.eap.2025.01.011`

4. Stadler, K., Wood, R., Bulavskaya, T., Södersten, C.-J., Simas, M., Schmidt, S., Usubiaga, A., Acosta-Fernández, J., Kuenen, J., Bruckner, M., Giljum, S., Lutter, S., Merciai, S., Schmidt, J. H., Theurl, M. C., Plutzar, C., Kastner, T., Eisenmenger, N., Erb, K.-H., ... Tukker, A. (2025). EXIOBASE 3 (3.9.6) [Data set]. Zenodo. `https://doi.org/10.5281/zenodo.15689391`

5. Miller, R. E., & Blair, P. D. (2009). Input-output analysis: foundations and extensions. Cambridge university press.