CS5800 – ALGORITHMS

MODULE 3. DIVIDE & CONQUER - I

Lesson 3: Integer Multiplication

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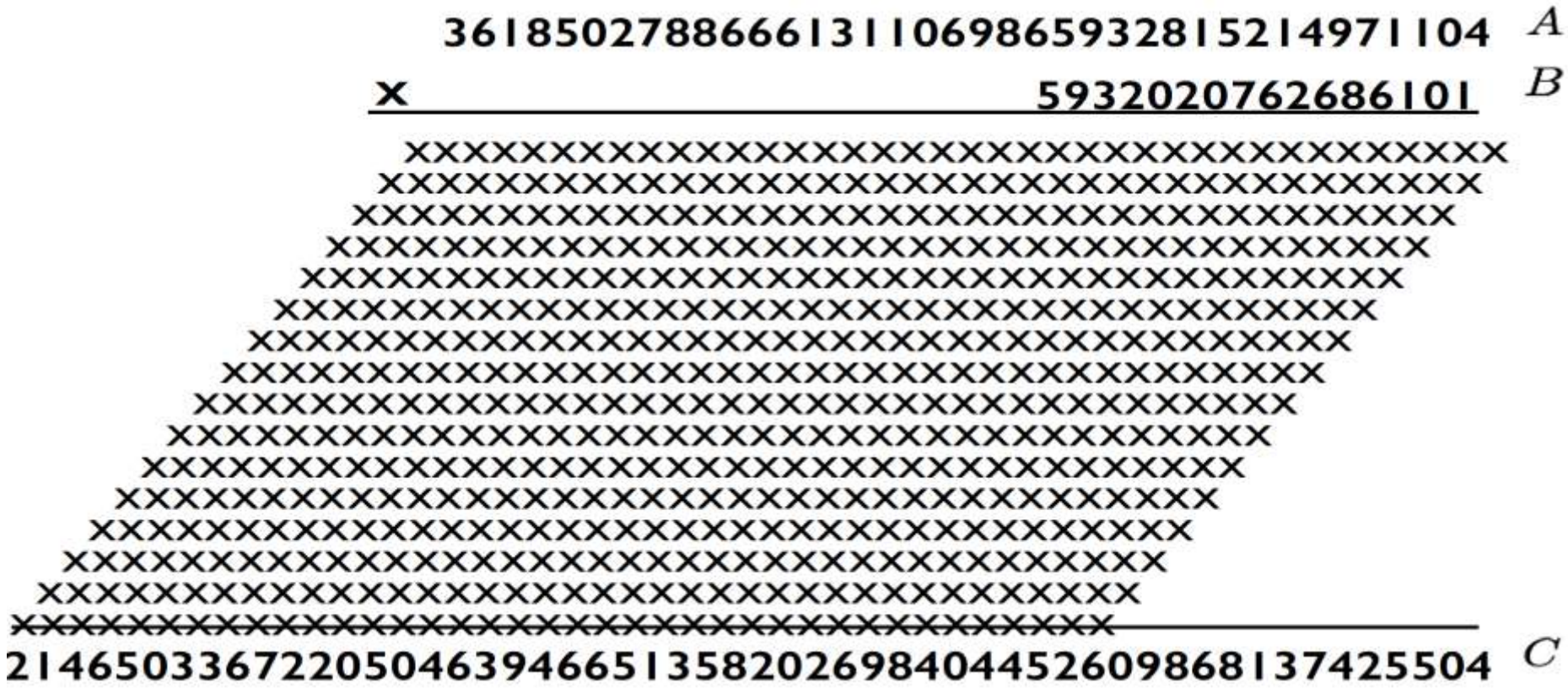
Topics

- Integer multiplication – grade school
- Naïve Divide & Conquer
- Karatsuba’s algorithm
- Summary

3. Multiplication over the integers



- Naïve (grade school) multiplication is quadratic time $\Theta(\lg A * \lg B)$
- If both A and B are n-bit numbers then (naïve) multiplication is $\Theta(n^2)$

Naïve Divide & Conquer

- Let X = | A | B | and Y = | C | D | where A,B,C and D are n/2 bit integers

- Simple Method: $XY = (2^{n/2}A+B)(2^{n/2}C+D)$

  $= 2^{n}A*C + 2^{n/2}(A*D+B*C) + B*D$

- Running Time Recurrence

  $T(n) < 4T(n/2) + 100n$

- Solution $T(n) = \theta(n^2)$

Karatsuba's Algorithm

- Famous Russian mathematician Andrey Kolmogorov conjectured that the grade school algorithm was asymptotically optimal
- Posed it as a conjecture in his seminar at Moscow State Univ in 1960
- 23 yr old Karatsuba solved it within a week and the seminar was terminated!

Karatsuba's Algorithm

- Let X = | A | B | and Y = | C | D | where A,B,C and D are n/2 bit integers

- Karatsuba:

  Compute A*C, B*D and use AD + BC = (A+B)*(C+D) – AC – BD

  $XY = 2^{n}AC + 2^{n/2}(AD + BC) + BD$

  Reduced from 4 to 3 multiplications.

  Remember - addition and subtraction are linear

Karatsuba's Algorithm

- Running Time Recurrence

$$T(n) < 3T(n/2) + 100n$$

- Solution: $T(n) = O(n^{\lg 3}) = O(n^{1.58})$

Summary

- Divide & Conquer is only an approach.
- Cleverness continues to play a very important role in how to realize the Divide and Conquer steps
- In a recent breakthrough (Harvey, van der Hoeven 2019) the complexity of integer multiplication and division of n-bit numbers is now at O(n log n)
- Main takeaway: An understanding of recurrence equations (Master Theorem) is crucial for designing fast algorithms.