CS5800 – ALGORITHMS

MODULE 7. GREEDY ALGORITHMS - II

Lesson 2: Cycles & Cuts

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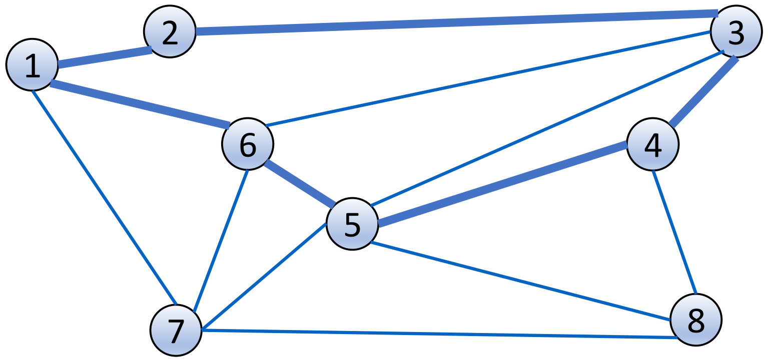
Topics

- Cycles, cuts and cycle-cut intersection
- Cut property & proof
- Cycle property & proof
- Summary

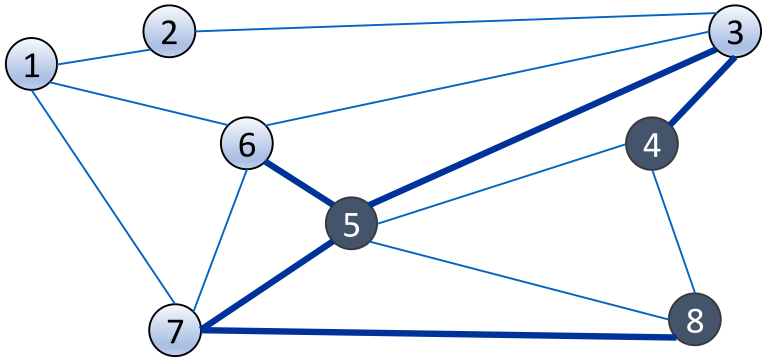
Cycles and Cuts

- Cycle. Set of edges the form a-b, b-c, c-d, …, y-z, z-a.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1

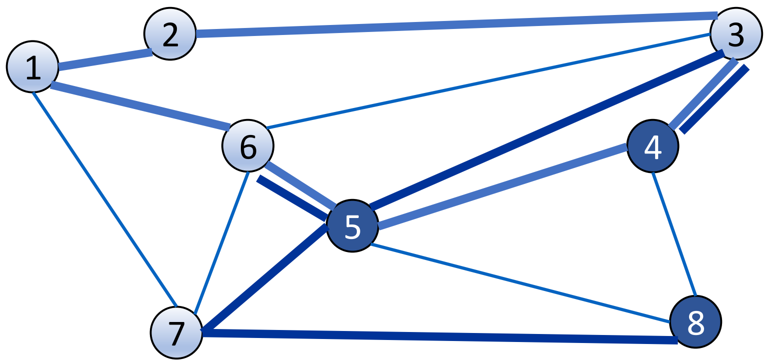
- Cutset. A cut is a subset of nodes S. The corresponding cutset D is the subset of edges with exactly one endpoint in S.



Cut S = { 4, 5, 8 }
Cutset D = 5-6, 5-7, 3-4, 3-5, 7-8

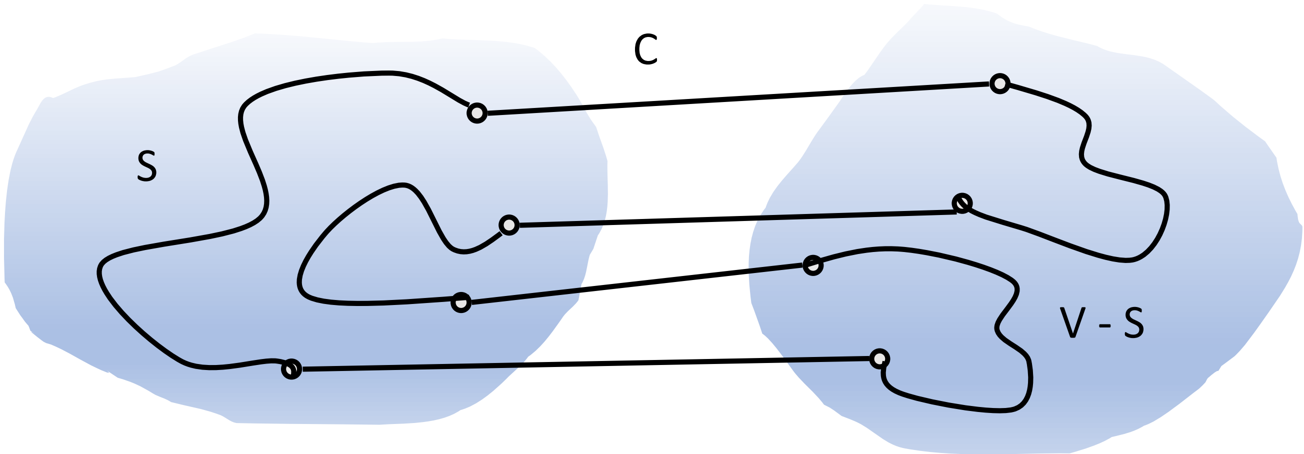
Cycle-Cut Intersection

- Lemma. A cycle and a cutset intersect in an even number of edges.



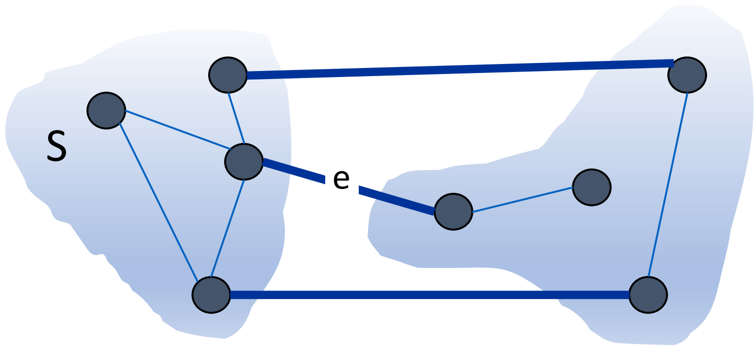
Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1
Cutset D = 3-4, 3-5, 5-6, 5-7, 7-8
Intersection = 3-4, 5-6

- Proof. By picture

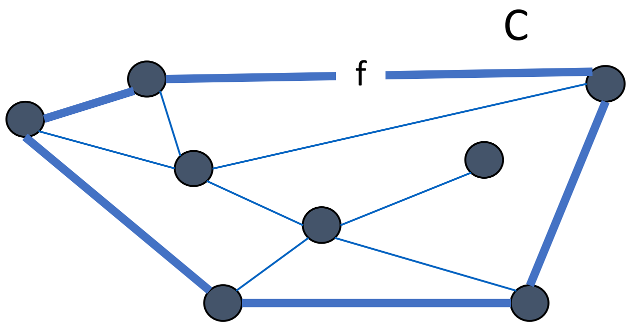


Key properties

- Simplifying assumption. All edge costs c_e are distinct.
- Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the (any) MST contains e.
- Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the (any) MST does not contain f.



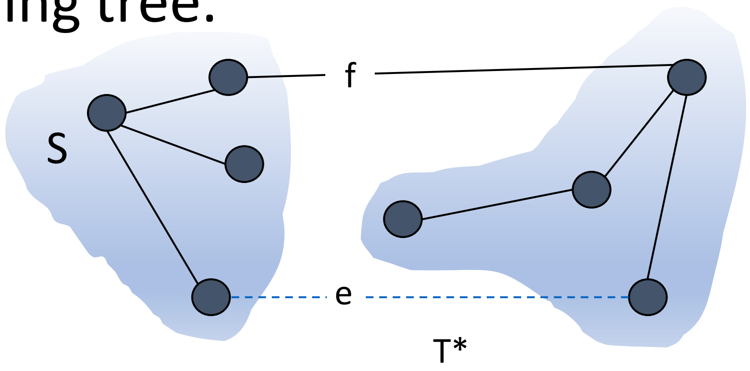
e is in the MST



f is not in the MST

Cut Property

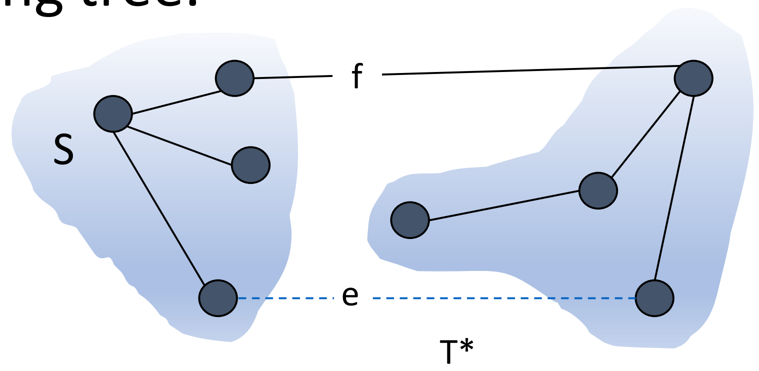
- Cut property: Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST T* contains e.
- Proof. (exchange argument)
  - Suppose e does not belong to T*.
  - Adding e to T* creates a cycle C in T*.
  - Edge e is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say f, that is in both C and D.
  - T' = T* $\cup$ { e } - { f } is also a spanning tree.
  - Since $c_e < c_f$, cost(T') < cost(T*).
  - This is a contradiction. ▪



Cycle Property

- Cycle property: Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

- Proof. (exchange argument)
  - Suppose f belongs to T*.
  - Deleting f from T* creates a cut S in T*.
  - Edge f is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say e, that is in both C and D.
  - T' = T* $\cup$ { e } - { f } is also a spanning tree.
  - Since $c_e < c_f$, cost(T') < cost(T*).
  - This is a contradiction. ▪



Summary

- Cuts & cycles are important concepts for spanning trees
- A spanning tree must cross every cut
- A spanning tree must not contain any cycle
- Main Takeaway: An MST must contain the lightest edge crossing a cut and must avoid the heaviest edge in any cycle