Part 1: Tropical Plane Curves

Especially: A tropical Bézout-Bernstein theorem.
(an illustration of tropical plane curves)

everything I say today can be generalised
my goal is to illustrate the ideas, not to be comprehensive.

K = a non-archimedian valued field.
(possibly with trivial valuation)

$= \overline{K}$

Based on my expository paper "Tropical Curves" (available on my website).

The classical Bézout-Bernstein theorem:

$P_1, P_2 \in K[x^{\pm}, y^{\pm}]$

C_1, C_2 resulting curves in the torus $\mathbb{T}^2 = \operatorname{Spec} K[x^{\pm}, y^{\pm}]$

N_1, N_2 = the Newton polygons of P_1, P_2

N_1 = convex hull of (i,j) st. P_1 has a $x^i y^j$ term

Thm For general P_1, P_2,

Minkowski sum

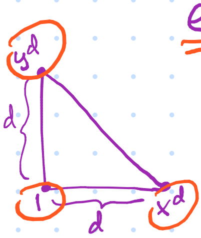
$$\#(C_1 \cap C_2) = \text{mixed volume } V(N_1, N_2)$$
$$= \text{area}(N_1 + N_2) - \text{area}(N_1) - \text{area}(N_2).$$

(Reference: eg. §5.5 of Fulton's "Toric Varieties") eg. 1) general curves in $\mathbb{P}^2$

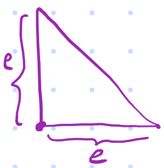
p_1 = general degree d poly. in $K[x,y]$

p_2 = " $\underline{\underline{e}}$ " .

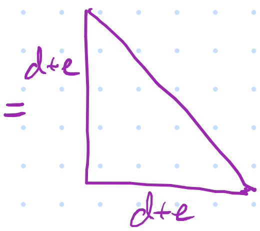
$N_1 = d$



$N_2 = e$

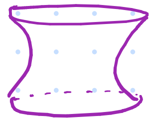


$N_1 + N_2 =$

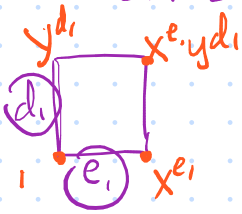


$$V(P_1, P_2) = \tfrac{1}{2}(d+e)^2 - \tfrac{1}{2}d^2 - \tfrac{1}{2}e^2 = \underline{de}.$$

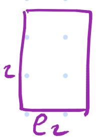
2) general curves in $\mathbb{P}^1 \times \mathbb{P}^1$



$N_1 = d_1$



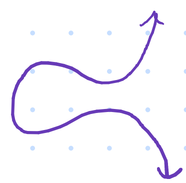
$N_2 = d_2$



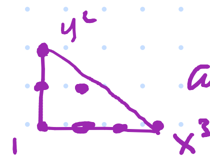
$$V(N_1,N_2) = \underline{(d_1+d_2)(e_1+e_2)} - \underline{d_1e_1} - \underline{d_2e_2}$$
$$= \underline{\underline{d_1e_2 + d_2e_1}}$$

3) Elliptic curve in Weierstrass form:

$p_1(x,y) = y^2 - x^3 - Ax - B$



$N_1 =$



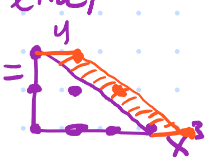
area 3

→ 3a) $p_2(x,y) = \underline{x-1}$ (vertical line)

$N_2 =$

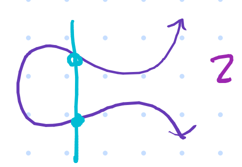
area 0

$N_1 + N_2 =$



area 5

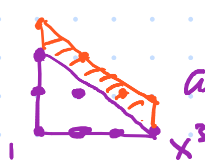
$V(N_1,N_2) = 5-3-0 =$



3b) $p_2(x,y) = y+1$ (horizontal line)

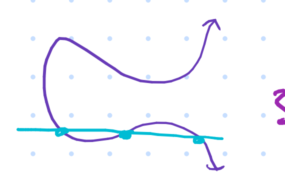
$N_2 =$

$N_1 + N_2 =$



area 3

$V(N_1,N_2) = 6-3-0$



3c) $p_2(x,y) = x+y+1$ //exercise.

How to prove Bézout-Bernstein?

Option 1 (usual way): intersection theory on toric surfaces.

compactify $\mathbb{T}$ in a way that curves w/ Newton polygon N_1 (or N_2) form a divisor class.

Option 2: Tropical Geometry!

Formulate & prove a tropical version ←

& deduce the classical theorem from it.

The Tropical Bézout-Bernstein theorem

(First appeared in Richter-Gebert, Sturmfels, & Theobald: "First Steps in Tropical Geometry".)

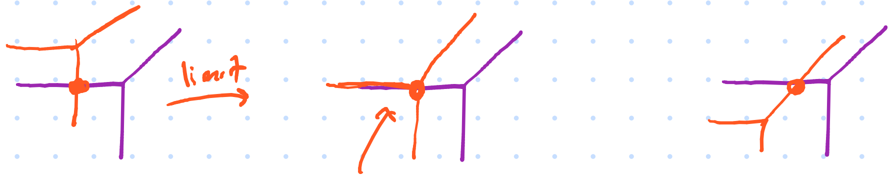
Thm Let $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^2$ be tropical plane curves w/ Newton polygons N_1, N_2. Then the stable intersection $\Gamma_1 \cap_{stab} \Gamma_2$, counted with multiplicity, has $V(N_1, N_2)$ points. (mixed volume).

must define the underlined terms!

// To recover classical Bézout-Bernstein, one must also prove a correspondence theorem.

$V(\Delta, \Delta) = \Delta - \Delta - \Delta$
$= 1$

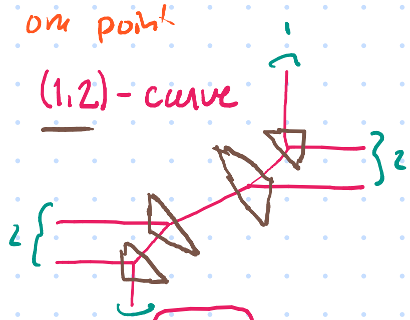
eg 1) Stable intersection of two tropical lines.

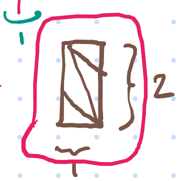


stable int. is this one point

2) Line and (1,2)-curve

$N_1 =$ 

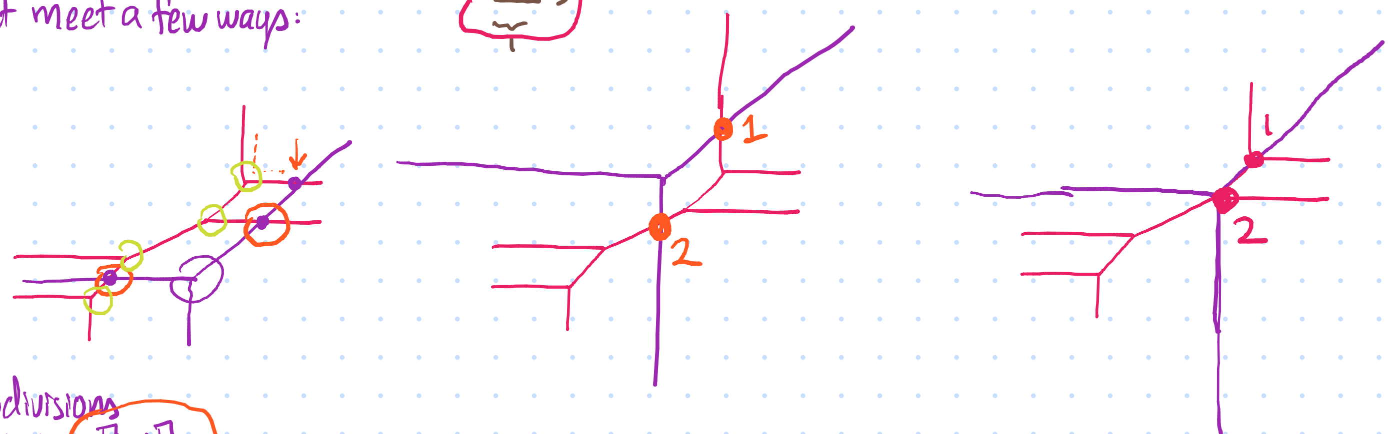


$N_2 =$ 

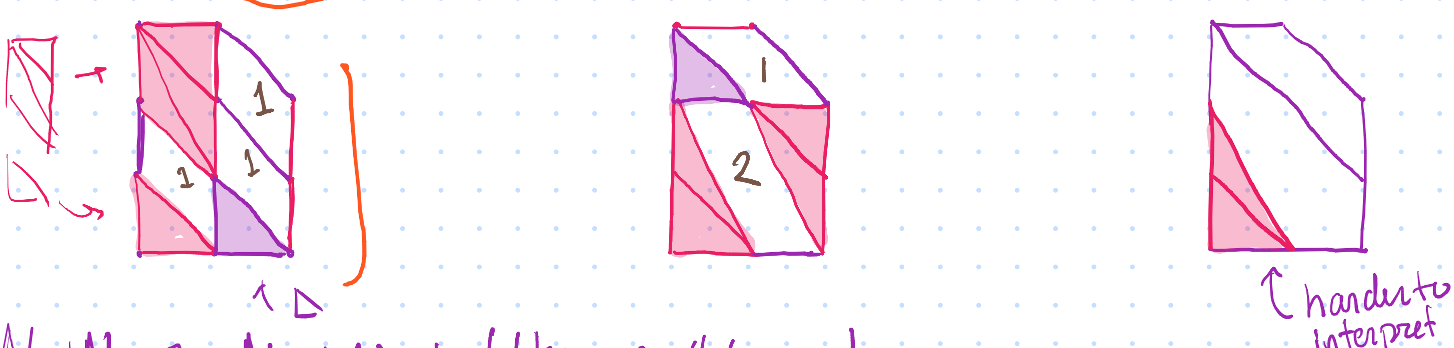
mixed volume:

$V(\Delta, \square) = 5\frac{1}{2} - \frac{1}{2} - 2 = 3$

might meet a few ways:



Newton subdivisions of the union $T_1 \cup T_2$:



$N_1 \cup N_2 \sim N_1 + N_2 + (\text{three parallelograms})$

Stable Intersection (of tropical curves)

Defn Set-theoretically,

$$\left.\begin{array}{l}\Gamma_1 \cap_{stab} \Gamma_2 = \text{the bend locus of } P_2 \\ \quad \text{when } \underline{\text{restricted to}}\ \Gamma_1.\end{array}\right\}$$ (!) Always discrete!

Exercise: same as bend locus of P_1 restricted to Γ_2.

eg.

Compare: For C_1, C_2 algebraic curves on a surface S,

$$C_1 \cdot C_2 = \deg\big(\underbrace{\mathcal{O}_S(C_1) \otimes \mathcal{O}_{C_2}}_{\text{AKA } \mathcal{O}_S(C_1)|_{C_2}.}\big) = \deg(\mathcal{O}_S(C_2) \otimes \mathcal{O}_{C_1}).$$

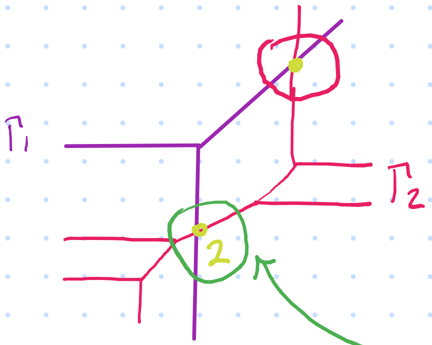
$\mathcal{O}_S(C_1)$ has a distinguished section, "1", vanishing on C_1.
$C_1 \cap C_2$ with multiplicities is given by restricting this section to C_2 & taking the associated divisor.

Tropical intersection multiplicities:

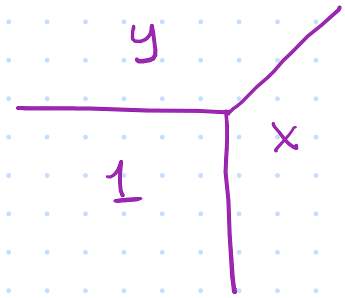
For $q \in \Gamma_1 \cap_{stab} \Gamma_2$, multiplicity of q is

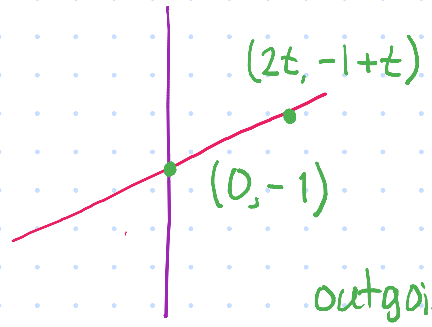
$$\sum_{\text{rays}} \left(\text{outgoing slope of } P_1|_{\Gamma_2}\right) \cdot (\text{multiplicity of ray})$$

eg.



$p_1(x,y) = \max(1, x, y)$





$p_1|_{\Gamma_2}(2t, -1+t)$

$= \max(1, \underline{\underline{2t}})$

outgoing slopes 0 & 2

$\Rightarrow$ multiplicity 2.

<u>Exercise</u> For two transverse line segments, this multiplicity is determinant of the two primitive integer vectors.

eg. above: $\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2.$

Proof of tropical Bézout-Bernstein:

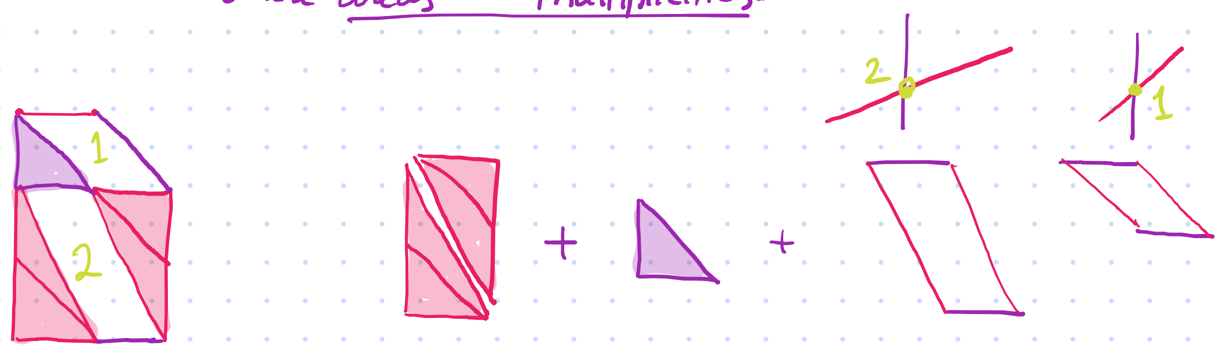
1) By the balancing condition,

$$\sum_{\Gamma_1 \cap \operatorname{star}_0 \Gamma_2} (\text{multiplicity}) = \sum_{\substack{\text{infinite rays} \\ \text{of } \Gamma_2}} (\text{outgoing slope}) \cdot (\text{multiplicity of ray}).$$

This depends only on the Newton polygon N_1.

2) Perturbing Γ_1 if necessary, Newton subdivision of $\Gamma_1 \cup \Gamma_2$ contains

   1) polygons from subdiv. of N_1 (area(N_1))
   2) " " N_2 (area(N_2))
   3) parallelograms for each pt. of $\Gamma_1 \cap \Gamma_2$, whose areas = multiplicities.

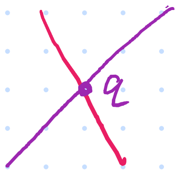


area(N_1+N_2) = area(N_1) + area(N_2) + intersection multiplicities

A Correspondence theorem

To deduce classical Bézout-Bernstein:

Thm If $C_1, C_2 \subseteq \mathbb{T}^2$ & $\Gamma_1 = \mathrm{Trop}(C_1)$, $\Gamma_2 = \mathrm{Trop}(C_2)$, then at any $q \in \Gamma_1 \cap \Gamma_2$ where two segments meet transversely,



there are exactly (multiplicity of q) points of $C_1 \cap C_2$.

Sketch: q defines an "initial degeneration" of $\mathbb{T}^2(K)$ to $\mathbb{T}^2(k)$.

The number of pts. in the initial degen. of $C_1 \cap C_2$ is the multiplicity of q.

A form of Hensel's lemma shows that all of these lift to K-points of $C_1 \cap C_2$.