Integration

Simpson's 1/3 Rule, Simpson's 3/8 Rule

Ashley Gannon

Fall 2020

The Composite Simpson's 1/3 Rule Continued

Let's think back to our formula

The total integration using Simpson's 1/3 Rule can be represented as

$$I = \frac{h}{3}(f(x_0) + f(x_n)) + \frac{h}{3}\left(4\sum_{i=1,3,5\dots}^{n-1} f(x_i) + 2\sum_{j=2,4,6,\dots}^{n-2} f(x_j)\right)$$

with error

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

.

Let's think back to our formula

The total integration using the composite Simpson's 1/3 Rule can be represented as

$$I = \frac{h}{3}(f(x_0) + f(x_n)) + \frac{h}{3}\left(4\sum_{i=1,3,5\ldots}^{n-1} f(x_i) + 2\sum_{j=2,4,6,\ldots}^{n-2} f(x_j)\right)$$

with error

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

Thinking back to the total integration using the composite trapezoid rule

$$I = \frac{h}{2} * (f(x_0) + f(x_n)) + h\sum_{i=1}^{n-1} f(x_i)$$

with error

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

Let's think back to our formula

The total integration using the composite Simpson's 1/3 Rule can be represented as

$$I = \frac{h}{3}(f(x_0) + f(x_n)) + \frac{h}{3}\left(4\sum_{i=1,3,5\dots}^{n-1} f(x_i) + 2\sum_{j=2,4,6,\dots}^{n-2} f(x_j)\right)$$

with error

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

Thinking back to the total integration using the composite trapezoid rule

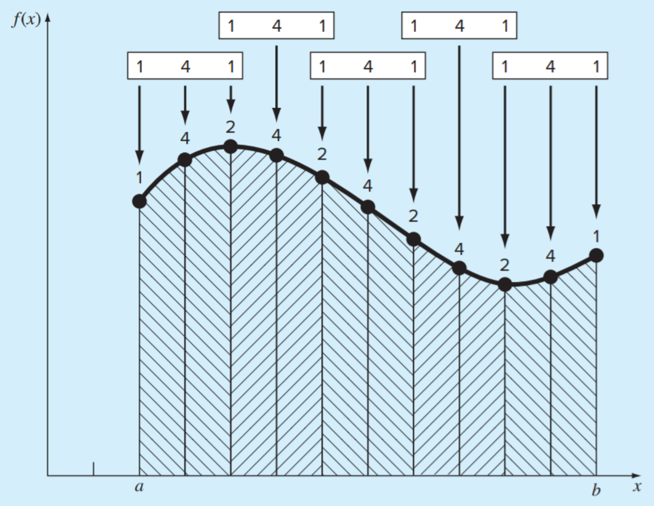
$$I = \frac{h}{2} * (f(x_0) + f(x_n)) + h\sum_{i=1}^{n-1} f(x_i)$$

with error

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

We notice that the only big difference between these methods is that the odd **index values** of x_i carry a different coefficient for the $f(x_i)$ values than they do when they are even. So we need to add this conditional to our for loop.

I bolded **index values** in the last slide because



The Modulo Operator, %

- One way to tell if a number is even is to use the modulo operator %.
- The modulo operator gives the remainder of a division between two values.
  - `x = 31 % 4` would result in `x` being equal to `3` since 31/4 = 7R3.
- Now, if a number `n` is even, we expect that in the case of `x = n % 2`, `x` would be equal to `0`.
- Knowing this information now, we can use this to form a conditional statement that evaluates whether a number is even
  - `if (x % 2 == 0)`

Recursive Simpson's 1/3 Rule Pseudocode

```
declare function that takes in a, b, n, f̄^(4), tolerance

declare/define the error -(b-a)^5/(180n^4) * f̄^(4)
if error > tolerance
This is where recursion comes in
   redefine n, n = 2*n for example
   return function(a,b,n,f̄^(4), tol)

declare/define h = (b-a)/n;
declare/define sum = (h/3)*(f(a)+f(b));
declare/define xi = a+h;

loop over i = 1,...,n-1
   if(i % 2 == 0)
      add 2*h/3*f(xi) to sum
   else
      add 4*h/3*f(xi) to sum
   update xi = xi +h;
```

Applying it to our problem

Let's apply our code to our problem from last class:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$, with $\bar{f}^{(4)} = -24000$, $n = 2$ and $tol = 0.00001$.

Applying it to our problem

Let's apply our code to our problem from last class:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$, with $\bar{f}^{(4)} = -24000$, $n = 2$ and $tol = 0.00001$.

We notice this method required a 16 times less segments to reach the specified tolerance.

- Composite Trapezoid: 512
- Composite Simpson's 1/3: 32

- The composite version of Simpson's 1/3 rule is considered superior to the trapezoidal rule for most applications.
- As mentioned previously, it is limited to cases where the values are equispaced.
- Further, it is limited to situations where there are an even number of segments and an odd number of points.
- Consequently, an odd-segment–even-point formula known as Simpson's 3/8 rule can be used in conjunction with the 1/3 rule to permit evaluation of both even and odd numbers of equispaced segments.

Simpson's 3/8 Rule

Simpson's 3/8 Rule

In a similar manner to the derivation of the trapezoidal and Simpson's 1/3 rule, a third-order Lagrange polynomial can be fit to four points and integrated to yield

$$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

where $h = (b - a)/3$. This equation is known as Simpson's 3/8 rule because h is multiplied by 3/8. It is the third Newton-Cotes closed integration formula.

Simpson's 3/8 rule has an error of

$$E_a = -\frac{(b-a)^5}{6480}\bar{f}^{(4)}$$

Because the denominator is larger than for Simpson's 1/3 Rule,

$$E_a = -\frac{(b-a)^5}{2880}\bar{f}^{(4)}$$

the 3/8 rule is somewhat more accurate than the 1/3 Rule.

When do we use it?

We've written the Trapezoid Rule and Simpson's 1/3 Rule to integrate functions, where we are able to use recursion to find the value of n that reduces our error below our tolerance.

What if now our problem is to integrate over an even set of data values where the number of segments is not divisible by three?

- We would have an odd number of segments, so we wouldn't be able to only use Simpson's 1/3 method alone since segments aren't divisible by 2
- and we wouldn't be able to use Simpson's 3/8 method alone since the number of segments isn't divisible by 3.
- We can evaluate this data set using a combination of Simpson's 1/3 Rule and Simpson's 3/8 Rule.

Example Problem

Let's say we've got a set of data we want to integrate

i	0	1	2	3	4	5	6	7
x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x_i)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

Example Problem

Let's say we've got a set of data we want to integrate

i	0	1	2	3	4	5	6	7
x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x_i)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

The first thing we need to do is divide the interval between the two methods.

- We'll reserve the last 3 segments for Simpson's 3/8 Rule.
- The remaining 4 segments (the first four segments) will be evaluated using Simpson's 1/3 Rule.
- This method is still 4th order accurate.

Example Problem

Let's say we've got a set of data we want to integrate

i	0	1	2	3	4	5	6	7
x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x_i)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

The first thing we need to do is divide the interval between the two methods.

1. Use Simpson's 1/3 rule on the interval [1.0,1.4]. From our data, $h = 1.1 - 1.0 = 0.1$.

$$\begin{aligned}\int_{1.0}^{1.4} f(x)dx &\approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)\\ &= \frac{0.1}{3}\left(1.543 + 4(1.669) + 2(1.811) + 4(1.971) + 2.151\right)\\ &= 0.729200\end{aligned}$$

Example Problem

Let's say we've got a set of data we want to integrate

i	0	1	2	3	4	5	6	7
x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x_i)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

The first thing we need to do is divide the interval between the two methods.

2. Use Simpson's 3/8 rule on the interval [1.4,1.7]. From our data, $h = 1.1 - 1.0 = 0.1$.

$$\begin{aligned}\int_{1.4}^{1.7} f(x)dx &\approx \frac{3h}{8}\Big(f(x_4) + 3f(x_5) + 3f(x_6) + f(x_7)\Big)\\ &= \frac{3(0.1)}{3}\Big(2.151 + 3(2.352) + 3(2.577) + 2.828\Big)\\ &= 0.741225\end{aligned}$$

Example Problem

Let's say we've got a set of data we want to integrate

i	0	1	2	3	4	5	6	7
x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x_i)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

The first thing we need to do is divide the interval between the two methods.

3. Add them together.

$$\int_{1.0}^{1.7} f(x)dx \approx \int_{1.0}^{1.4} f(x)dx + \int_{1.4}^{1.7} f(x)dx$$
$$= 1.47042$$

Example Problem

Let's say we've got a set of data we want to integrate

i	0	1	2	3	4	5	6	7
x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f(x_i)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828

The first thing we need to do is divide the interval between the two methods.

3. Add them together.

$$\int_{1.0}^{1.7} f(x)dx \approx \int_{1.0}^{1.4} f(x)dx + \int_{1.4}^{1.7} f(x)dx$$
$$= 1.47042$$

Alternatively, we could have used Simpson's 3/8 rule on the fist three segments, [1.0, 1.3], and Simpson's 1/3 rule on the remaining 4 segments, [1.3, 1.7]. We would obtain a slightly different approximation: 1.4704416. Either way works, they are both accurate up to the 4th decimal.

Things to consider before we code

We want to write a `DataSimpsonSRules` function that will evaluate both odd and even sets of data.

- Our function will have to take in arrays for x and $f(x)$ values.
- We will have to use the number of x values in these arrays to figure out how many segments,n, our domain has.
- We will need to set up different cases for when n is even, odd and divisible by 3, or odd and not divisible by 3 and evaluate the sum for the appropriate case.

Think about how you might approach this between now and next class.