Lecture 17

Integration and Differentiation: Unit Overview and Integration with the Trapezoid Rule

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ISC3313 Fall 2021

Overview

Derivatives and Integration

- In high school or during your first year of college, you were introduced to differential and integral calculus.
- There you learned techniques to obtain analytical or exact derivatives and integrals.
- Mathematically, the derivative represents the rate of change of a dependent variable with respect to an independent variable.
  - For example, if we are given a function y(t) that specifies an object's position as a function of time, differentiation provides a means to determine its velocity, as in

$$\frac{dy(t)}{dt} = v(t)$$

- Integration is the inverse of differentiation.
  - Just as differentiation uses differences to quantify an instantaneous process, integration involves summing instantaneous information to give a total result over an interval.
  - if we are provided with velocity as a function of time, integration can be used to determine the distance traveled:

$$\int_0^t v(t)dt = y(t)$$

- In short, think of a derivative as a slope and an integral as a summation.

Methods we will cover in this unit

- Integration
  - Trapezoidal Rule
  - Simpson's 1/3 Rule
  - Simpson's 3/8 Rule
  - Richardson Extrapolation
  - Gauss Quadrature
- Differentiation
  - Finite Difference Methods
  - Richardson Extrapolation

Integration

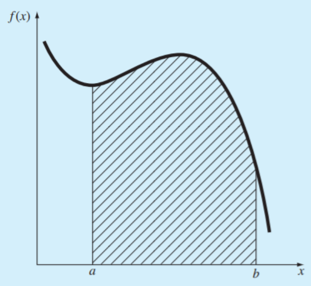
What is integration

Mathematically, definite integration is represented by

$$I = \int_a^b f(x)dx$$

which stands for the integral of the function $f(x)$ with respect to the independent variable x evaluated between the limits $x = a$ and $x = b$.

The integral is the summation of $f(x)dx$ over the range of $x = a$ to $x = b$.



Fun fact: the symbol $\int$ is an elongated letter S to stand for Summa, Latin for sum.

Integration

Recall that the velocity of a free-falling bungee jumper as a function of time can be computed as

$$v(t) = \sqrt{\frac{gm}{c_d}} tanh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

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Suppose that we would like to know the vertical distance y the jumper has fallen after a certain time t. This distance can be evaluated by integration:

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Solving this integral with calculus we get the equation for position as a function of time

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Although a closed form solution can be developed for this case, there are other functions that cannot be integrated analytically.

Newton-Cotes formulas

Newton-Cotes Formulas

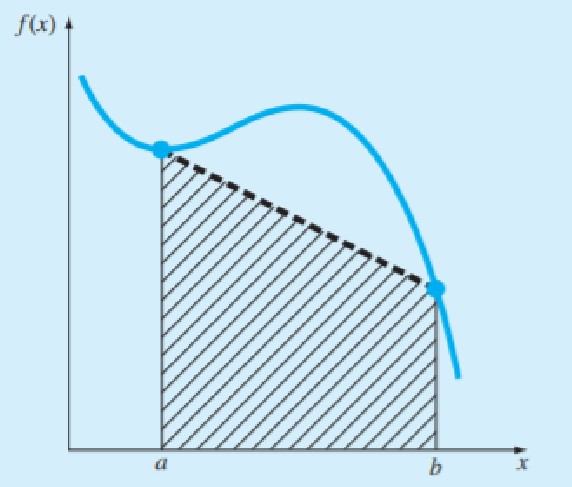
The *Newton-Cotes formulas* are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with a polynomial that is easy to integrate:

$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

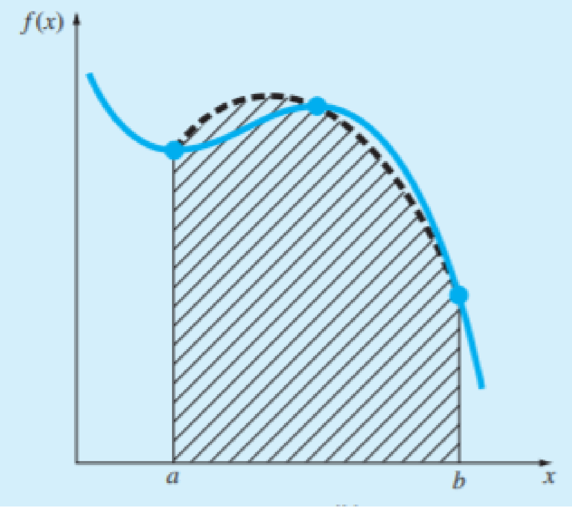
where $f_n(x)$ is an n^{th} order polynomial of the form:

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} + a_n x^n$$

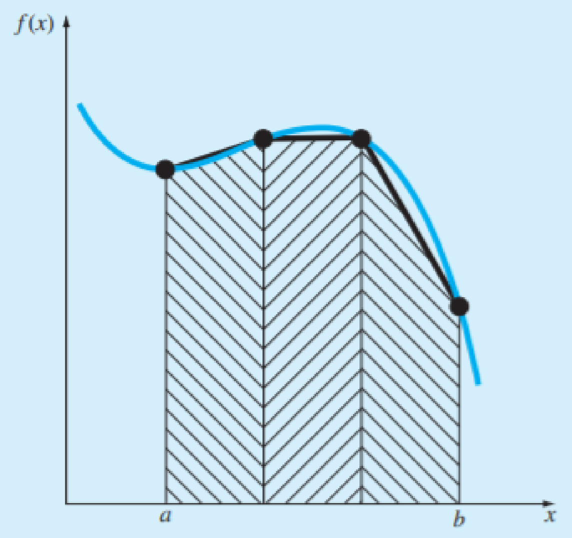
First order approximate



2nd order approximate



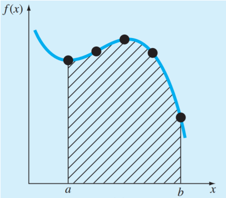
Piecewise function approximate



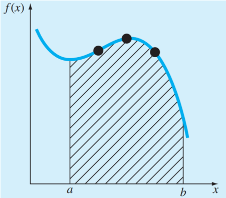
Newton-Cotes formulas

Newton-Cotes formulas exist in 2 categories: *closed forms* or *open forms*.

The *closed forms* are those where the data points at the beginning and end of the limits of integration are known



The *open forms* have integration limits that extend beyond the range of the data



The Trapezoidal Rule

The Trapezoidal Rule

The trapezoidal rule is the first of the Newton-Cotes closed integration formulas. It is formed using the first order polynomial of type:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

which has the solution

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

Which is called the *trapezoid rule*.

The Trapezoidal Rule

Geometrically, the trapezoidal rule is equivalent to approximating the area of the trapezoid under the straight line connecting $f(a)$ and $f(b)$

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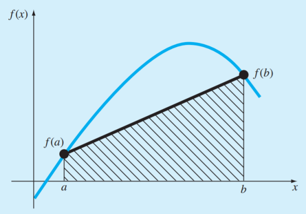

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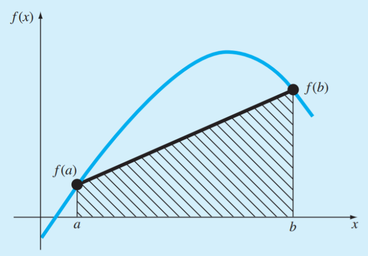
- Recall from geometry that the formula for computing the area of a trapezoid is the height times the average of the bases.



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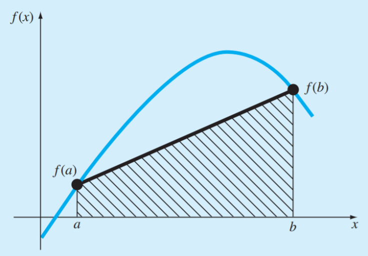
- In our case, the concept is the same but the trapezoid is on its side. Therefore, the integral estimate can be represented as

$$I = \text{width } x \text{ average height}$$



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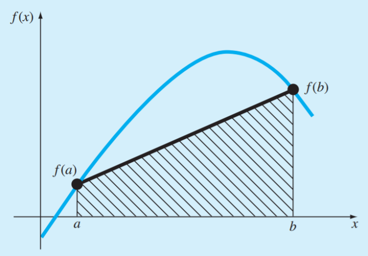
- The width is $(b - a)$ and the average height is the average of the function values at the end points, or $\frac{[f(a)+f(b)]}{2}$.



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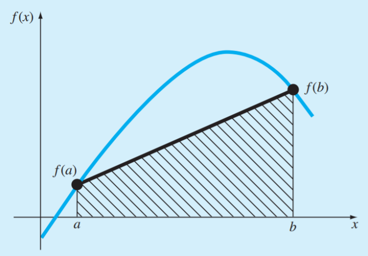
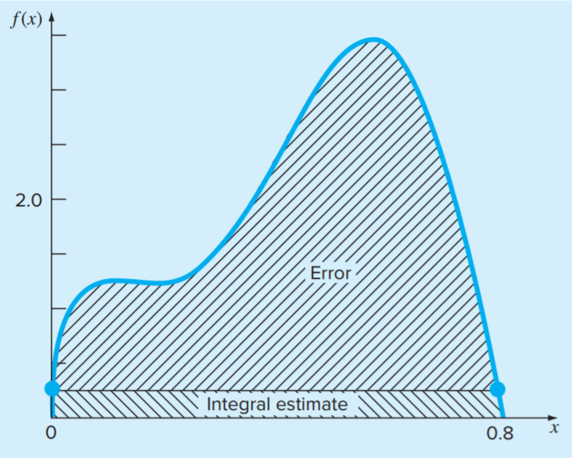


After substituting these values into the above equation, we get the trapezoid rule

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

Error of the Trapezoidal Rule

When we employ the integral under a straight-line segment to approximate the integral under a curve, we obviously can incur an error that may be substantial



Error of the Trapezoidal Rule

The type of error we will compute for this method is *local truncation error*. The local truncation error is the amount error caused by one iteration of the method.

An estimate for the local truncation error for the trapezoid rule is

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

Where ξ lies somewhere in the domain $[a, b]$.

Single Application of the Trapezoidal Rule

Let's use the trapezoid rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$

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$$I = (b-a)\frac{f(a)+f(b)}{2}$$

We can plug in the values for $f(0) = 0.2$ and $f(0,8) = 0.232$

$$I = (0.8-0)\frac{0.2+0.232}{2} = 0.1728$$

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Now, we can find this integral analytically, it has an exact value of 1.640533. We can compute the percent relative error

$$\epsilon_a = \frac{1.640533 - 0.1728}{1.640533} = 89.5\%$$

Single Application of the Trapezoidal Rule

Since we won't always know what the true solution is, we need to estimate the approximate error. We can do this using the local truncation error

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The function's second derivative can be found analytically

$$f''(x) = -400 + 4050x - 10800x^2 + 8000x^3$$

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To find $f''(\xi)$, we will take the average value of the second derivative which is computed as

$$\bar{f}''(x) = \frac{\int_0^{0.8} (-400 + 4050x - 10800x^2 + 8000x^3)dx}{0.8 - 0} = -60$$

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Which we can now substitute into our truncation error equation

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You may have noticed we changed the E_t to E_a, this is because for an interval of this size, the average second derivative is not necessarily an accurate approximation of $f''(\xi)$. We are denoting that this is an approximation using the subscript a.

The Composite Trapezoid Rule

One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from *a* to *b* into a number of segments and apply the method to each segment.

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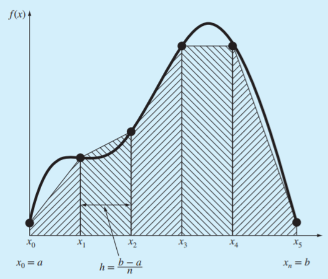

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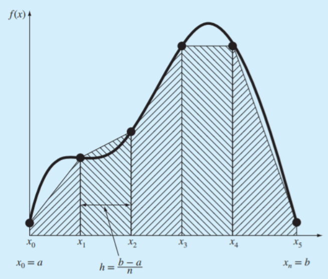

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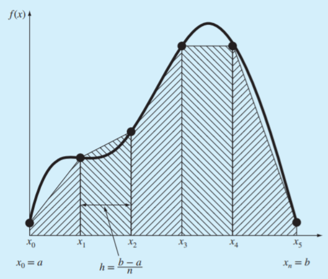
There are $n+1$ equally spaced base points $(x_0, x_1, x_2, ..., x_n)$ with n segments of equal width:

$$h = \frac{b-a}{n}$$

In this figure $n = 5$



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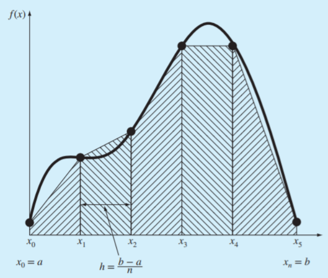

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If $a = x_0$ and $b = x_n$, the total integration can be represented by

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{n-1}}^{x_n} f(x)dx$$

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Substituting the trapezoidal rule for each integral yields

$$I = h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + ... + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

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Which can be re-written as

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Error of the The Composite Trapezoid Rule

An error for the composite trapezoidal rule can be obtained by summing the individual errors for each segment to give

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i+1}^{n} f''(\xi_i)$$

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Subbing this into our equation for E_t, we can approximate the error E_a

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

Where $\bar{f}''$ is found the same was as in the case of the single application of the trapezoidal rule.

Composite Trapezoidal Rule Example

Looking back at our previous problem, let's use the composite trapezoid rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$ with $n = 2$. Recalling our formula for the composite trapezoidal rule:

$$I = h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + ... + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

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$$f(0) = 0.2, \quad f(0.4) = 2.456, \quad f(0.8) = 0.232$$

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.

$$f(0) = 0.2, \quad f(0.4) = 2.456, \quad f(0.8) = 0.232$$

Substituting these into our equation for I

$$I = 0.4\frac{0.2 + 2.456}{2} + 0.4\frac{2.456 + 0.232}{2} = 1.0688$$

Composite Trapezoidal Rule Example

We can use the average second derivate from the single application of the trapezoidal rule

$$\bar{f}'' \cong -60$$

Therefore, our error is

$$E_a = -\frac{0.8^3}{12(2)^2}(-60) = 0.64$$

Which is a quarter of the error we found using a single application of the trapezoidal rule.

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This is because the error is inversly related to the square of n, so as we double n, we quarter the error.

Composite Trapezoid Rule Pseudocode

How would you code the composite trapezoid rule? Write your pseudocode and submit it to the canvas discussion board **Composite Trapezoidal Rule**.

NOTE: Unlike the other methods we have covered, this one does not need a cut-off condition. This code is looping over the sections of your interval to find the total area under the curve. In other words, you don't need to factor in error.