Optimization

Parabolic Interpolation

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Fall 2020

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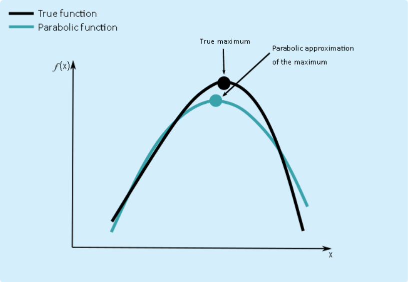

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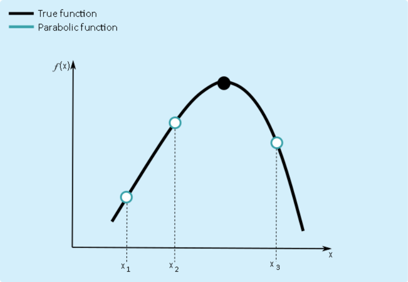

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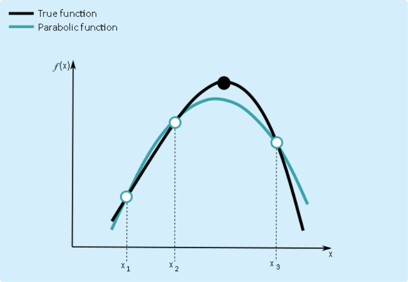

In this method, we choose three points that jointly bracket an optimum. If $x_2 < x_1 \text{ and } x_3$ or $x_2 > x_1 \text{ and } x_3$, we can fit a parabola to the points using Lagrange interpolation.

Parabolic Interpolation

$q(x)$ is the parabolic function based on the three points chosen:

$$q(x) = f(x_1)\frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2)\frac{(x_4 - x_3)(x_4 - x_1)}{(x_2 - x_3)(x_2 - x_1)} + f(x_3)\frac{(x_4 - x_1)(x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

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To find the optimum of this function, we need to take it's derivative and set it to 0.

$$\begin{aligned} q'(x) =& f(x_1)\frac{(x_4 - x_2) + (x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2)\frac{(x_4 - x_3) + (x_4 - x_1)}{(x_2 - x_3)(x_2 - x_1)} \\ &+ f(x_3)\frac{(x_4 - x_1) + (x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)} = 0 \end{aligned}$$

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Now we need to solve for our unknown point, x_4

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2 - x_1)^2[f(x_2) - f(x_3)] - (x_2 - x_3)^2[f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$$

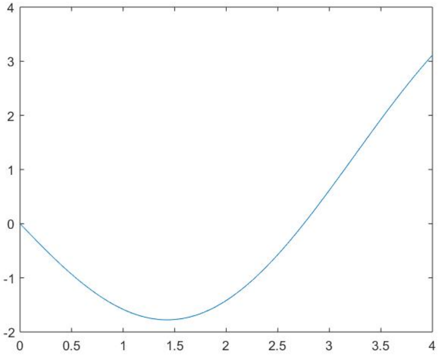
where x_1, x_2, and x_3 are the initial guesses, and x_4 is the value of x that corresponds to the optimum value of the parabolic fit to the guesses.

Parabolic Interpolation Example

Let's revisit our example from the previous lectures

$$f(x) = \frac{x^2}{10} - 2sin(x)$$

Remembering our plot:



And let's start with initial guesses of $x_1 = 0$, $x_2 = 1$, and $x_3 = 4$.

Parabolic Interpolation Example

The first thing we'll do is calculate x_4.

$$x_4 = 1 - \frac{1}{2}\frac{(1-0)^2[-1.5829 - 3.1136] - (1-4)^2[-1.5829 - 0]}{(1-0)[-1.5829 - 3.1136] - (1-4)[-1.5829 - 0]} = 1.5055$$

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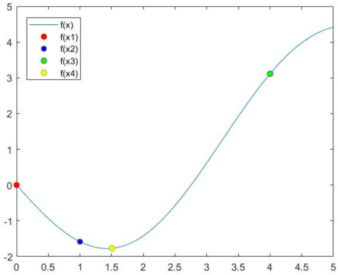
Next, a strategy similar to the golden-section search can be employed to determine which point should be discarded.

$$f(x_1) = 0$$
$$f(x_2) = -1.5829$$
$$f(x_3) = 3.1136$$
$$f(x_4) = -1.7691$$



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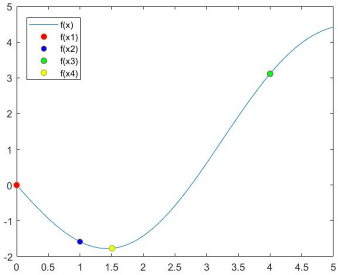
Since $x_2 < x_4$

$$x_1 \leftarrow x_2$$
$$x_2 \leftarrow x_4$$
$$x_3 = x_3$$



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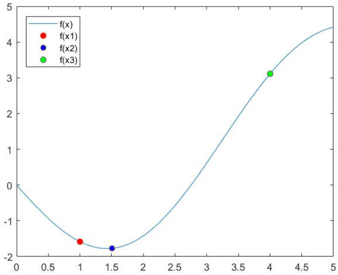

Parabolic Interpolation Example

Now we need to recalculate x_4.

$$x_4 = 1.5055 - \frac{1}{2}\frac{(1.5055-1)^2[-1.7691-3.1136]-(1.5055-4)^2[-1.7691-(-1.5829)]}{(1.5055-1)[-1.7691-3.1136]-(1.5055-4)[-1.7691-(-1.5829)]}$$
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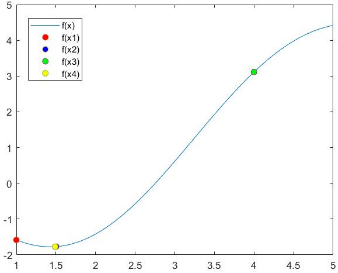

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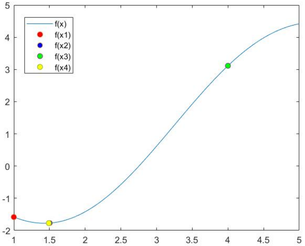

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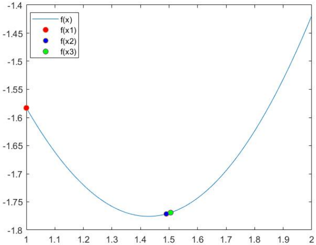

Since $x_2 > x_4$

$$x_1 = x_1$$
$$x_3 \leftarrow x_2$$
$$x_2 \leftarrow x_4$$

Keep iterating until $error = |\frac{x_4 - x_{4\,old}}{x_4}| < tol$

Parabolic Interpolation Pseudocode

```
declare tol, x4old
declare/define error
```

define/declare $x_4 = x_2 - \frac{1}{2}\frac{(x_2-x_1)^2[f(x_2)-f(x_3)]-(x_2-x_3)^2[f(x_2)-f(x_1)]}{(x_2-x_1)[f(x_2)-f(x_3)]-(x_2-x_3)[f(x_2)-f(x_1)]}$

```
while error>tol
    x4old =x4;
    if x2 > x4
        x3 = x2;
        x2 = x4;
    else
        x1 = x2;
        x2 = x4;
```

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2-x_1)^2[f(x_2)-f(x_3)]-(x_2-x_3)^2[f(x_2)-f(x_1)]}{(x_2-x_1)[f(x_2)-f(x_3)]-(x_2-x_3)[f(x_2)-f(x_1)]}$$

```
    error = abs((x4-x4old)/x4);
return x4;
```

Let's code it!

We are going to write this function and apply it to

$$f(x) = \frac{x^2}{10} - 2sin(x)$$

With an initial $x_1 = 0, x_2 = 1$, $x_3 = 4$. Remembering our plot: