Integration

Richardson Extrapolation and Romberg Integration

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Fall 2020

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We can represent the error of the Trapezoidal rule by

$$I = I(h) + E(h)$$

where I is the exact integral, $I(h)$ is the integral approximation using the Trapezoidal rule, and $E(h)$ is the truncation error

$$E \cong -\frac{b-a}{12} h^2 \bar{f}''$$

when $h = (b-a)/n$.

Richardson Extrapolation

If we make two seperate estimates using segment sizes h_1 and h_2, we can rewrite

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If we assume that $\bar{f}''$ is constant for any h, we have the relation

$$\frac{E(h_1)}{E(h_2)} \cong \frac{-\frac{b-a}{12}h_1^2\bar{f}''}{-\frac{b-a}{12}h_2^2\bar{f}''} \cong \frac{h_1^2}{h_2^2}$$

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Solving for $E(h_1)$,

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Which we can plug back into the above equation so that the integral approximate is only dependent on $E(h_2)$.

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Solving for $E(h_2)$ we have

$$E(h_2) = \frac{I(h_1) - I(h_2)}{1 - \frac{h_1^2}{h_2^2}}$$

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For the special case where the interval is halved, $h_2 = h_1/2 = \frac{b-a}{2n}$, we can simplify our approximate to be

$$I = I(h_2) + \frac{I(h_1) - I(h_2)}{1 - \frac{h_1^2}{(h_1/2)^2}} \quad = I(h_2) + \frac{I(h_1) - I(h_2)}{1 - 4} = I(h_2) + \frac{I(h_2)}{3} - \frac{I(h_1)}{3}$$

$$= \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$$

Richardson Extrapolation Example

Use Richardson Extrapolation to evaluate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

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First we will use the trapezoid rule on one segment

$$h = (0.8 - 0)/1 = 0.8$$

$$f(0.8) = 0.232, \quad f(0) = 0.2$$

$$I \approx h\frac{f(a) + f(b)}{2} = 0.8 * \frac{0.232 + 0.2}{2}$$
$$= 0.1728$$

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Next we will halve the interval $h = (0.8 - 0)/2 = 0.4$, and apply the Trapezoidal rule

$$\begin{aligned} I &\approx h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} \\ &= 0.4\frac{0.2 + 2.456}{2} + 0.4\frac{2.456 + f(0.232)}{2} \\ &= 1.0688 \end{aligned}$$

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Use Richardson Extrapolation to evaluate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

Now we can plug these values into our integral approximate

$$\begin{aligned} I &= \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) \\ &= \frac{4}{3} 1.0688 - \frac{1}{3} 0.1728 \\ &\approx 1.367467 \end{aligned}$$

Which is closer to the true value, 1.640533 than either Trapezoidal rule approximate.

Richardson Extrapolation Example

In the same manner, we can use estimates for 2 and 4 segments.

For 4 segments

$$h = (0.8 - 0)/4 = 0.2$$

$$x_0 = 0, \qquad x_1 = 0.2, \qquad x_2 = 0.4,$$

$$x_3 = 0.6, \qquad x_4 = 0.8$$

Plugging these values into our trapezoid rule,

$$I \approx \frac{h}{2}(f(x_0) + f(x_4)) + h\sum_{i=2}^{4} f(x_i) = 1.4848$$

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Substituting this and the two segment approximate into

$$\begin{aligned} I &= \frac{4}{3}1.4848 - \frac{1}{3}1.0688 \\ &\approx 1.623467 \end{aligned}$$

Which is even more accurate than the last approximation.

Richardson Extrapolation Example

These two improved integrals can, in turn, be combined to yield an even better value with $\mathcal{O}(h^6)$. For the special case where the original trapezoidal estimates are based on successive halving of the step size, the equation used for $\mathcal{O}(h^6)$ accuracy is

$$I = \frac{16}{15} I_m - \frac{1}{15} I_l$$

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For our example, we have

$$\begin{aligned} I &= \frac{16}{15}(1.623467) - \frac{1}{15}(1.367467) \\ &= 1.640533 \end{aligned}$$

Which is the exact integral.

Romberg Integration

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Notice the coefficients for

$$I = \frac{4}{3} I_m - \frac{1}{3} I_l$$

and

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add to 1.

You may think of these coefficients as weights. We place a larger weight on the more accurate integral approximation.

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If we let $k = 1, 2, ...n$ correspond to the order of accuracy where

k	L
1	$\mathcal{O}(h^2)$
2	$\mathcal{O}(h^4)$
3	$\mathcal{O}(h^6)$
$\vdots$	$\vdots$
n	$\mathcal{O}(h^{2n})$

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We can generalize the form of our integral approximate to be

$$I_k = \frac{4^{k-1} I_{m,k-1} - I_{l,k-1}}{4^{k-1} - 1}$$

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Applying this function iteratively is known as *Romberg Integration*. This method has the error

$$E_a = \left| \frac{I_k - I_{m,k-1}}{I_k} \right|$$

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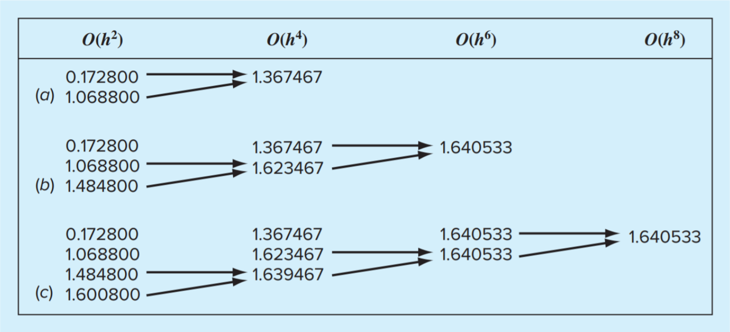
Graphical depiction of the sequence of integral estimates generated using Romberg integration. (a) First iteration, (b) second iteration, (c) third iteration.



$\mathcal{O}(h^4)$ Romberg Method Pseudocode

Download the `TrapezoidalRule.cpp` file from Canvas. We are going to modify it to contain the Romberg Method pseudocode.

```
define function that takes in a, b, and tol
declare/define error
declare/define n = 1
declare/define Il = TrapezoidalRule(a,b,n)
declare/define Ik = 0, Im = 0
declare/define k = 2

loop while error is greater than the tolerance
   update n for next trapezoid rule call, n*=2
   Find Im by calling Trapezoidal rule for new n,
   Im = TrapezoidalRule(a,b,n)
   Compute Ik = (pow(4,k-1)*Im-Il)/(pow(4,k-1)-1)
   Compute the error = abs((Ik-Im)/Ik)
   Update Il = Im
```