Lecture 14

Optimization

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ISC3313 Fall 2021

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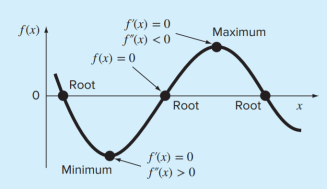
- The fundemental difference is that finding the location of a root involves searching for x where the $f(x) = 0$ while optimization involves searching for x where $f'(x) = 0$.



Optimization Example

Let's consider the following problem:

An object like a bungee jumper can be projected upward at a specified velocity. If it is subject to linear drag, its altitude as a function of time can be computed as

$$z = z_0 + \frac{m}{c_d}\left(v_0 + \frac{mg}{c_d}\right)\left(1 - e^{\frac{-c_d t}{m}}\right) - \frac{mg}{c_d}t$$

Where z is the distance from the Earth's surface, z_0 is the initial distance from the earths surface, v_0 is the initial velocity, m is the mass of the jumper, c_d is the drag coefficient, and t is time.

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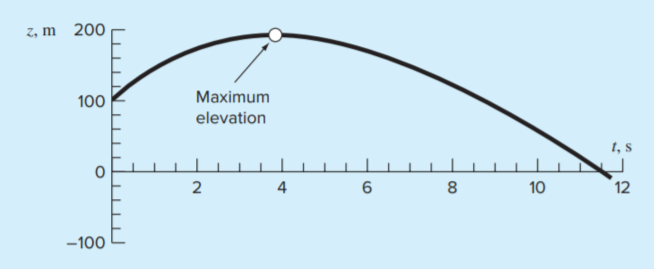
If we let $z_0 = 100$m, $m = 80$kg, $c_d = 15$kg/s, $v_0 = 55$m/s, and $g = 9.81\text{m/s}^2$, and plot z for $t = 0...12$



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To find the maximum altitude analytically, we have to find the time where $z'(t) = 0$.

$$\frac{dz}{dt} = v_0 e^{\frac{-c_d t}{m}} - \frac{mg}{c_d}\left(1 - e^{\frac{-c_d t}{m}}\right) = 0$$

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Again, letting $z_0 = 100$m, $m = 80$kg, $c_d = 15$kg/s, $v_0 = 55$m/s, and $g = 9.81\text{m/s}^2$

$$t \approx 3.83166\text{s}$$

Optimization Example

Now to find the maximum position, we plug this value for t and our other parameters back into

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Now if we want to verify that this is a maximum value, we need to evaluate $z''(x)$

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$$\frac{dz^2}{d^2z} = -\frac{c_d v_0}{m} e^{\frac{-c_d t}{m}} - g e^{\frac{-c_d t}{m}}$$

plugging in our variable values,

$$\frac{dz^2}{d^2z} \approx -9.81\mathrm{m/s}$$

The fact that the second derivative is negative tells us that we have a maximum.

What is optimization?

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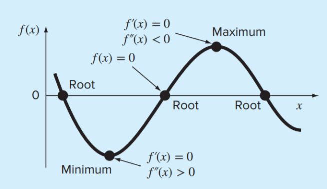
- The fundemental difference is that finding the location of a root involves searching for x where the $f(x) = 0$ while optimization involves searching for x where $f'(x) = 0$.



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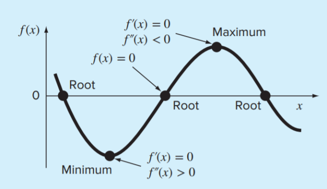

- Although such analytical solutions are sometimes feasible, most practical optimization problems require numerical, computer solutions.
- From a numerical standpoint, such optimization methods are similar in spirit to the root-location methods we discussed in the last section.
  - both involve guessing and searching for a location on a continuous function.

One Dimensional Optimization

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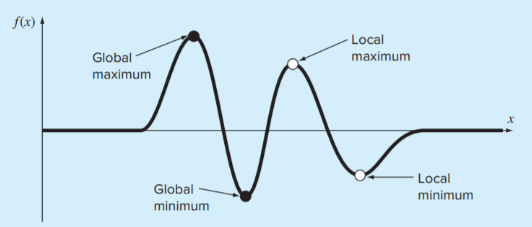

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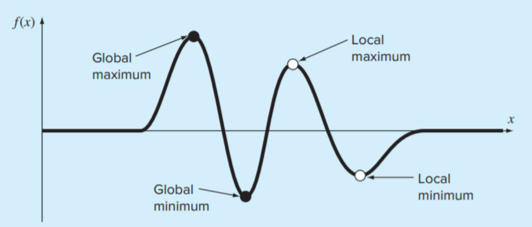

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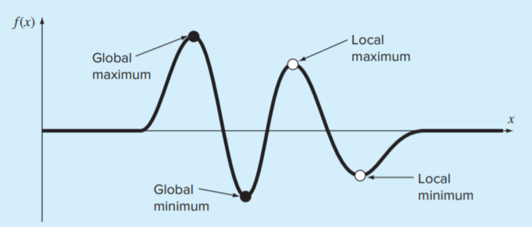

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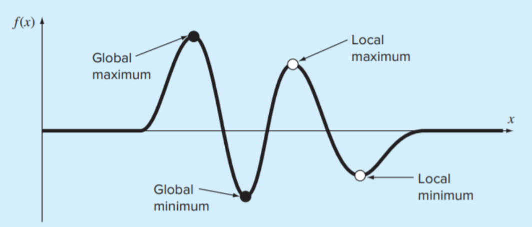

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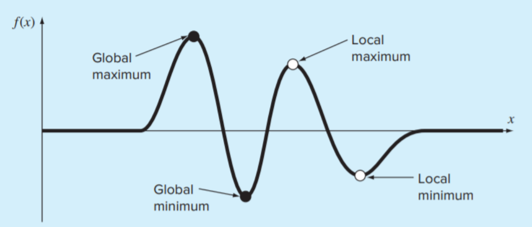

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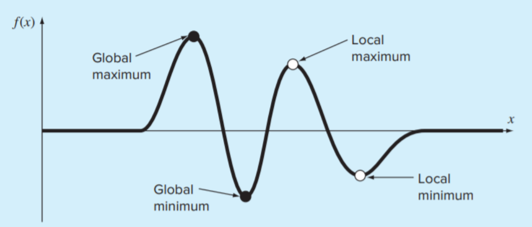

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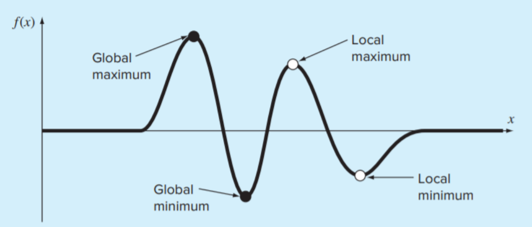

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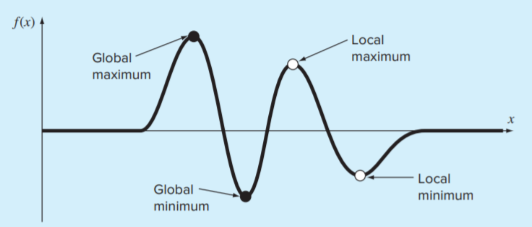

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- We will cover two methods for optimization - The *golden section-search method* and the *parabolic method*.
- Both of these methods can be considered *Bracketing methods*.

The Golden-Section Search Method

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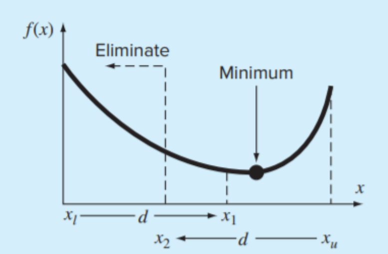
- We cannot use a single intermediate value, x_r, like we did for Bisection, instead we need two intermediate values to detect whether a optima occurred.
  - The key to making this approach efficient is the wise choice of the intermediate points.
  - For the golden-section search, the two intermediate points are chosen according to the golden ratio:
    $$x_1 = x_l + d$$
    $$x_2 = x_u - d$$
    Where
    $$d = (\Phi - 1)(x_u - x_l)$$
    and the golden ratio $\Phi = 1.61803398874989$.



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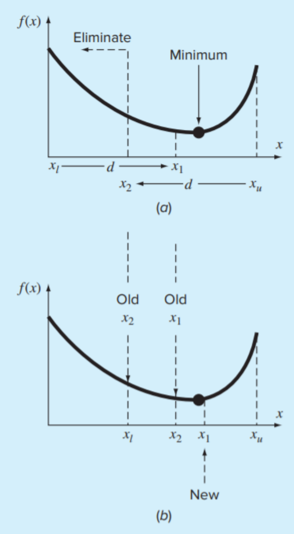
- The function is evaluated at these two interior points. Two results can occur:
  - If $f(x_1) < f(x_2)$, then $f(x_1)$ is the optima, and the domain of x to the left of x_2, from x_l to x_2, can be eliminated because it does not contain the optima. For this case, x_2 becomes the new x_l for the next round.
  - If $f(x_2) < f(x_1)$, then $f(x_2)$ is the optima and the domain of x to the right of x_1, from x_1 to x_u would be eliminated. For this case, x_1 becomes the new x_u for the next round.



Example

Let's find the minimum of

$$f(x) = \frac{x^2}{10} - 2\sin(x)$$

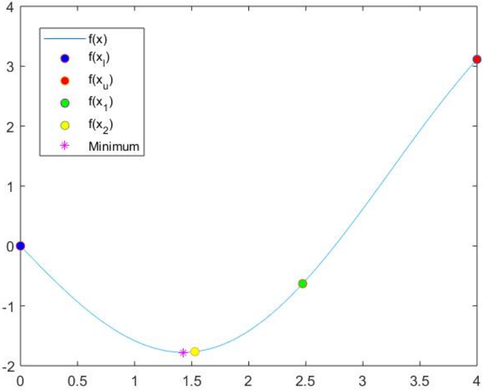
With and initial $x_l = 0$ and $x_u = 4$.

The first thing we need to do is calculate the golden ratio that we use to create the two interior points:

$$\begin{aligned} d &= (1.61803 - 1)(4 - 0) \\ &= 2.4721 \end{aligned}$$

Then our two interior points are

$$\begin{aligned} x_1 &= x_l + d = 0 + 2.4721 \\ &= 2.4721 \\ x_2 &= x_u - d \\ &= 4 - 2.4721 \\ &= 1.5279 \end{aligned}$$



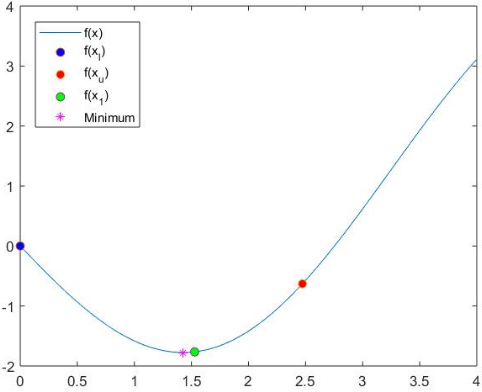
Example

Now we need to compute our function values and compare them to find our new domain

$$f(x_1) = \frac{2.4721^2}{10} - 2\sin(2.4721)$$
$$= -0.6300$$
$$f(x_2) = \frac{1.5279^2}{10} - 2\sin(1.5279)$$
$$= -1.7647$$

Now since $f(x_2) < f(x_1)$,

$$x_l = x_l$$
$$x_u \leftarrow x_1$$
$$x_1 \leftarrow x_2$$



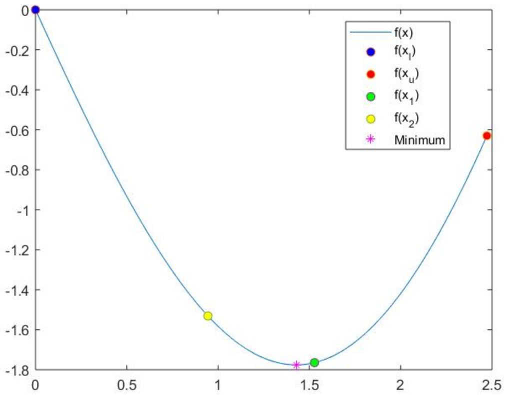
Example

We now need to recalculate d

$$\begin{aligned} d &= 0.61803(x_u - x_l) \\ &= 0.61803(2.4721 - 0) \\ &= 1.5279 \end{aligned}$$

And define a new x_2

$$\begin{aligned} x_2 &= x_u - d \\ &= 2.4721 - 1.5279 \\ &= 0.9443 \end{aligned}$$



Example

Next iteration: We need to compute our function values and compare them to find our new domain

$$f(x_1) = \frac{1.5279^2}{10} - 2\sin(1.5279) = -1.7647$$

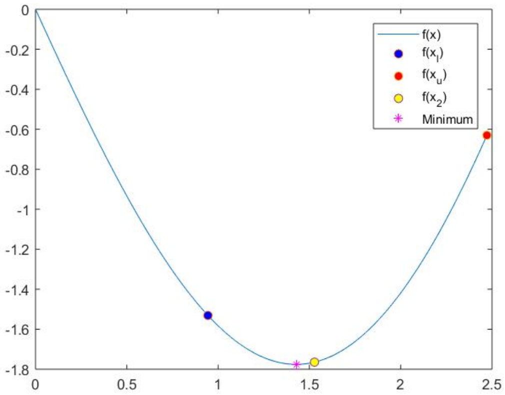
$$f(x_2) = \frac{0.9443^2}{10} - 2\sin(0.9443) = -1.5310$$

Now since $f(x_1) < f(x_2)$,

$$x_u = x_u$$

$$x_l \leftarrow x_2$$

$$x_2 \leftarrow x_1$$



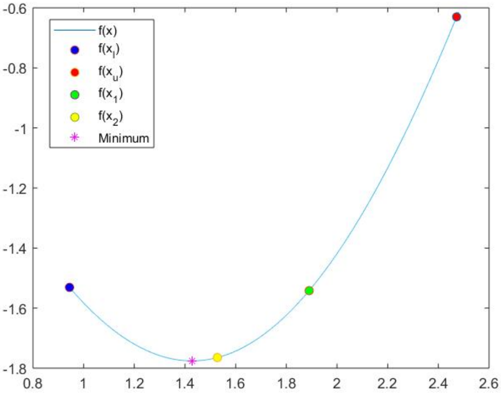
Example

We now need to recalculate d

$$\begin{aligned} d &= 0.61803(x_u - x_l) \\ &= 0.61803(2.4721 - 0.9443) \\ &= 0.9456 \end{aligned}$$

And define a new x_2

$$\begin{aligned} x_1 &= x_l + d \\ &= 0.9443 + 0.9456 \\ &= 1.8899 \end{aligned}$$



Keep iterating until

$$\epsilon_a = (2 - \Phi)\left|\frac{x_u - x_l}{x_{opt}}\right| < tol$$

Where x_{opt} is the new value of x_1 or x_2.

Sign-Off Activity

Now that you have an idea how this method works, write a pseudo code and post it to the **Golden-section search method** discussion board. You may sign-off once you've finished.