Differentiation

Richardson Extrapolation

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Richardson Extrapolation

Richardson Extrapolation Revisited

So far, we've seen that we can increase the accuracy of our derivative approximation by decreasing the step size h.

We could also improve these estimates by using higher order taylor series to form our schemes

We're going to use Richardson Extrapolation. Like we did for integration, this method combines two derivative estimates to compute a third, more accurate approximation.

Richardson Extrapolation

Recall from the unit on integration

$$I = \frac{4}{3}I(h_2) - \frac{1}{3}I(h_1)$$

where $I(h_1)$ and $I(h_2)$ are the integral estimates for step sizes h_1 and h_2.

We can write a similar expression for derivatives

$$D = \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1)$$

where $D(h_1)$ and $D(h_2)$ are the derivative estimates for step sizes h_1 and h_2.

- For Forward or Backward difference approximations with $\mathcal{O}(h)$, the application of this formula will yield a new derivative estimate of $\mathcal{O}(h^2)$
- For Centered difference approximations with $\mathcal{O}(h^2)$, the application of this formula will yield a new derivative estimate of $\mathcal{O}(h^4)$

Richardson Extrapolation - Forward Difference

Use forward difference approximations to estimate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at $x = 0.5$, using a step size $h = 0.25$. Note that the derivative of this function is

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25, \qquad f'(0.5) = -0.9125$$

Starting again with the forward difference scheme.

We have when $h = 0.5$,

$$f'(0.5) \approx \frac{0.2 - 0.925}{0.5} = -1.45$$

The percent relative error is 58.9%. When $h = 0.25$,

$$f'(0.5) \approx \frac{0.63632813 - 0.925}{0.5} \approx -1.155$$

The percent relative error is 26.5%.

Richardson Extrapolation - Forward Difference

Plugging these values into out Richardson Extrapolation scheme

$$D = \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1)$$

We have

$$D = \frac{4}{3}(-1.155) - \frac{1}{3}(-1.45) = -1.05667$$

Which has a percent relative error of 15.8%, which is close to the error from the centered difference approximation.

Richardson Extrapolation - Centered Difference

Use a centered difference approximation to estimate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at $x = 0.5$. Note that the derivative of this function is

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25, \qquad f'(0.5) = -0.9125$$

When $h = 0.5$, applying the centered difference scheme we have

$$f'(0.5) \approx \frac{0.2 - 1.2}{1} = -1$$

The percent relative error is 9.6%. When $h = 0.25$, applying the centered difference scheme we have

$$f'(0.5) \approx \frac{0.63632813 - 1.10351563}{0.5} = -0.934$$

The percent relative error is 2.4%

Richardson Extrapolation - Centered Difference

Plugging these values into out Richardson Extrapolation scheme

$$D = \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1)$$

We have

$$D = \frac{4}{3}(-0.934) - \frac{1}{3}(-1) = -0.9125$$

which for the present case is exact.

- Recall the Richardson extrapolation is equivalent to fitting a high-order polynomial through data, in this case it's 4th order so it fits our 4th order polynomial $f(x)$ exactly.

Wrapping Up

Differentiation wrapping up

This section covered one-step methods for solving ordinary differential equations (ODEs).

If you continue taking courses in Scientific Computing, you will apply these methods and some of the other methods we've covered in this course to solve ODEs.