

Lecture 16

Optimization: Parabolic Interpolation

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ISC3313 Fall 2021

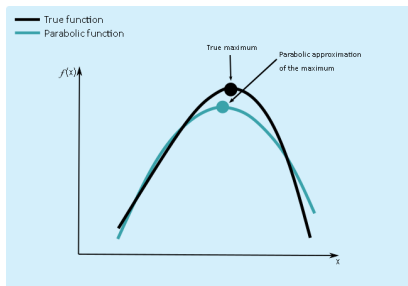


Parabolic Interpolation



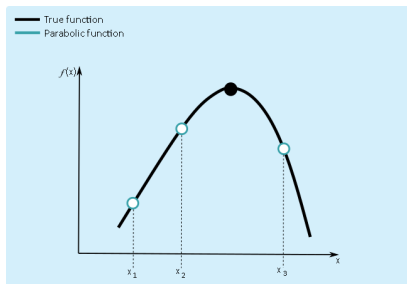
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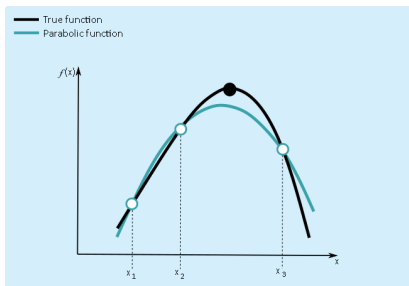


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Parabolic Interpolation

Parabolic interpolation takes advantage of the fact that a second-order polynomial often provides a good approximation to the shape of $f(x)$ near an optimum.



In this method, we choose three points that jointly bracket an optimum. If $x_2 < x_1$ and x_3 or $x_2 > x_1$ and x_3 , we can fit a parabola to the points using Lagrange interpolation.



Parabolic Interpolation

$q(x)$ is the parabolic function based on the three points chosen:

$$q(x) = f(x_1) \frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2) \frac{(x_4 - x_3)(x_4 - x_1)}{(x_2 - x_3)(x_2 - x_1)} + f(x_3) \frac{(x_4 - x_1)(x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$



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To find the optimum of this function, we need to take it's derivative and set it to 0.

$$\begin{aligned} q'(x) = & f(x_1) \frac{(x_4 - x_2) + (x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} + f(x_2) \frac{(x_4 - x_3) + (x_4 - x_1)}{(x_2 - x_3)(x_2 - x_1)} \\ & + f(x_3) \frac{(x_4 - x_1) + (x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)} = 0 \end{aligned}$$



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Now we need to solve for our unknown point, x_4

$$x_4 = x_2 - \frac{1}{2} \frac{(x_2 - x_1)^2[f(x_2) - f(x_3)] - (x_2 - x_3)^2[f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$$

where x_1 , x_2 , and x_3 are the initial guesses, and x_4 is the value of x that corresponds to the optimum value of the parabolic fit to the guesses.

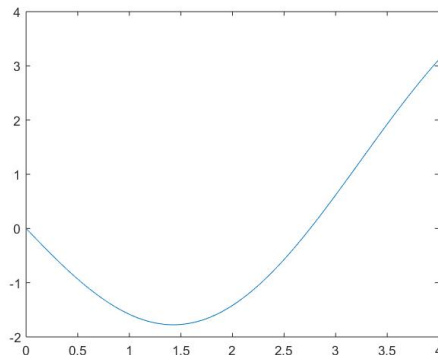


Parabolic Interpolation Example

Let's revisit our example from the previous lectures

$$f(x) = \frac{x^2}{10} - 2\sin(x)$$

Remembering our plot:



And let's start with initial guesses of $x_1 = 0$, $x_2 = 1$, and $x_3 = 4$.



Parabolic Interpolation Example

The first thing we'll do is calculate x_4 .

$$x_4 = 1 - \frac{1}{2} \frac{(1-0)^2[-1.5829 - 3.1136] - (1-4)^2[-1.5829 - 0]}{(1-0)[-1.5829 - 3.1136] - (1-4)[-1.5829 - 0]} = 1.5055$$



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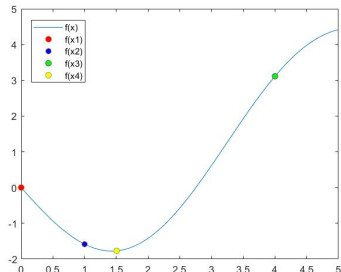
Next, a strategy similar to the golden-section search can be employed to determine which point should be discarded.

$$f(x_1) = 0$$

$$f(x_2) = -1.5829$$

$$f(x_3) = 3.1136$$

$$f(x_4) = -1.7691$$



Parabolic Interpolation Example

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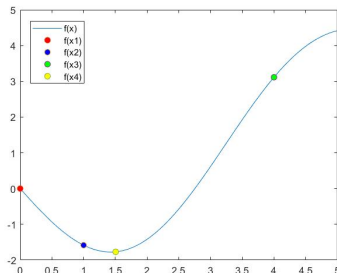
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Since $x_2 < x_4$

$$\begin{aligned} x_1 &\leftarrow x_2 \\ x_2 &\leftarrow x_4 \\ x_3 &= x_3 \end{aligned}$$



Parabolic Interpolation Example

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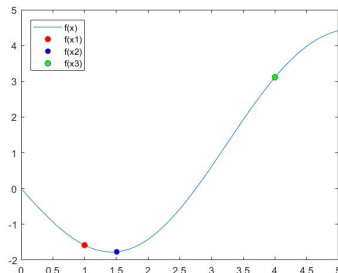
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Parabolic Interpolation Example

Now we need to recalculate x_4 .

$$\begin{aligned}x_4 &= 1.5055 - \frac{1}{2} \frac{(1.5055 - 1)^2[-1.7691 - 3.1136] - (1.5055 - 4)^2[-1.7691 - (-1.5829)]}{(1.5055 - 1)[-1.7691 - 3.1136] - (1.5055 - 4)[-1.7691 - (-1.5829)]} \\&= 1.4903\end{aligned}$$



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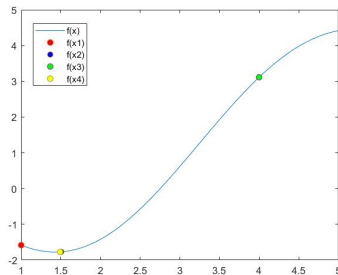
And then determine which point should be discarded.

$$f(x_1) = -1.5829$$

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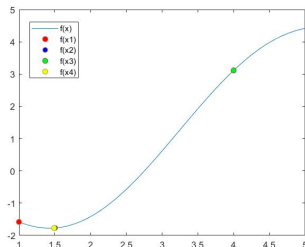
$$f(x_4) = -1.7714$$

Since $x_2 > x_4$

$$x_1 = x_1$$

$$x_3 \leftarrow x_2$$

$$x_2 \leftarrow x_4$$



Parabolic Interpolation Example

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$$= 1.4903$$

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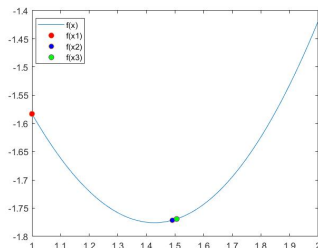
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$$x_2 \leftarrow x_4$$



Keep iterating until $error = \left| \frac{x_4 - x_{4old}}{x_4} \right| < tol$



Parabolic Interpolation Pseudocode

```

declare tol, x4old
declare/define error

define/declare  $x_4 = x_2 - \frac{1}{2} \frac{(x_2 - x_1)^2 [f(x_2) - f(x_3)] - (x_2 - x_3)^2 [f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$ 

while error > tol
  x4old = x4;
  if x2 > x4
    x3 = x2;
    x2 = x4;
  else
    x1 = x2;
    x2 = x4;
     $x_4 = x_2 - \frac{1}{2} \frac{(x_2 - x_1)^2 [f(x_2) - f(x_3)] - (x_2 - x_3)^2 [f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$ 
    error = abs((x4 - x4old) / x4);
return x4;
```

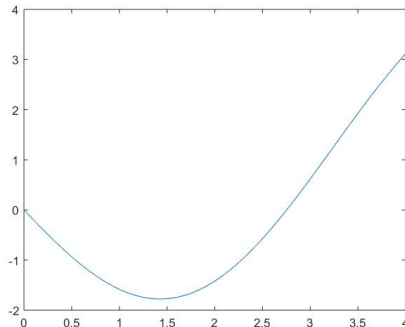


Let's code it!

We are going to write this function and apply it to

$$f(x) = \frac{x^2}{10} - 2\sin(x)$$

With an initial $x_1 = 0, x_2 = 1, x_3 = 4$. Remembering our plot:



When you are finished, post your code to the **Parabolic Interpolation Code** discussion board.

