Integration

Adaptive Quadrature

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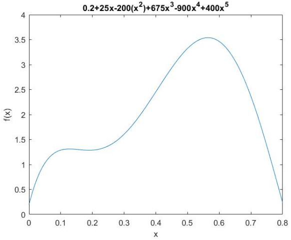

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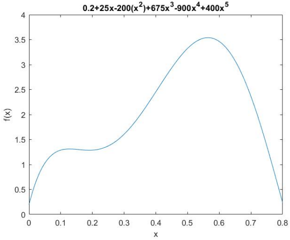

Adaptive quadrature methods remedy this situation by automatically adjusting the step size so that small steps are taken in regions of sharp variations and larger steps are taken where the function changes gradually.

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  - The total integral is then computed as the summation of the integral estimates for the subintervals.

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In theory, if we have the interval $x \in [a, b]$ with a width of $h_1 = b - a$, a first estimate with Simpson's 1/3 rule can be computed using

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As we did with Richardson Extrapolation, a more refined estimate can be computed by halving the interval, $h_2 = h_1/2 = (b - a)/2$

$$I(h_2) = \frac{h_2}{6}(f(a) + 4f(d) + 2f(c) + 4f(e) + f(b))$$

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where $d = (a + c)/2$ and $e = (c + b)/2$. Because both $I(h_1)$ and $I(h_2)$ are both estimates of the same integral, their difference can be used to measure the error of approximation

$$E \cong |I(h_2) - I(h_1)|$$

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In addition, the estimate and error associated with either application can be represented generally as

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Using a similar approach to what we did for Richardson extrapolation, we can derive an estimate in the error of the more refined estimate $I(h_2)$ as a function of the difference between the two integral estimates so that

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Which can be substituted into the general formula above,

$$I = I(h_2) + \frac{1}{15}\left(I(h_2) - I(h_1)\right)$$

This formula is known as *Boole's rule*.

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- Evaluate the initial application of Simpson's 1/3 rule by computing the two integral estimates

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- Estimate the error using the absolute difference between the two integral estimates, $|f(h_2) - f(h_1)|$.
- Depending on the error, one of two things will happen
  1. If the error is less than or equal to the tolerance, we estimate the integral using Boole's rule:
  $$I = I(h_2) + \frac{1}{15}(I(h_2) - I(h_1))$$
  2. If the error is larger than the tolerance, `AdaptiveQuadrature` is recursively called for each subinterval, $[a, c]$ and $[c, b]$.

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The beauty of this algorithm is in the two recursive calls. This recursion allows us to keep subdividing the intervals until our tolerance is met.

Adaptive Quadrature Pseudocode

```
Define function that takes in a, b, and tol
Declare/define h1 = b-a
Declare/define h2 = h1/2
Declare/define midpoints:
  c = (a+b)/2
  d = (a+c)/2
  e = (c+b)/2
Declare/define I

Declare/define Ih1 and Ih2
  Ih1 = (h1/6)*(f(a)+4*f(c)+f(b))
  Ih2 = (h2/6)*(f(a)+4*f(d) +2*f(c)+ 4*f(e) +f(b))

Declare/define the error
error = abs(Ih2-Ih1)
if error > tol
    Boole's Rule:
    I = Ih2 + (1/15)*(Ih2-Ih1)
else
    declare/define Ia = AdaptiveQuadrature(a,c,tol)
    declare/define Ib = AdaptiveQuadrature(c,b,tol)
    I = Ia + Ib
```

Let's code it!

Let's try this out on our problem

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$ with a tol of 1e-6.

If we've done it correctly, we should have the sum, $I = 1.6405333$