Data Structures and Algorithms – COMS21103

2015/2016

Dynamic Search Structures

Self-balancing Trees and Skip Lists

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University of Bristol

Dynamic Search Structures

A **dynamic search structure**,

stores a set of elements

Each element x must have a unique key - x.key

The following operations are supported:

INSERT(x, k) - inserts x with key $k = x$.key

FIND(k) - returns the (unique) element x with x.key $= k$

(or reports that it doesn't exist)

DELETE(k) - deletes the (unique) element x with x.key $= k$

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We would also like it to support (among others):

PREDECESSOR(k) - returns the (unique) element x with the largest key such that x.key $< k$

RANGEFIND(k_1, k_2) - returns every element x with $k_1 \leqslant x$.key $\leqslant k_2$

Using a Linked List as a Dynamic Search Structure

There are many ways in which we could implement a search structure...

but they aren't all efficient

Let n denote the number of elements stored in the structure

- our goal is to implement a structure with operations which scale well as n grows

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We could implement a Dynamic Search Structure using an unsorted linked list:

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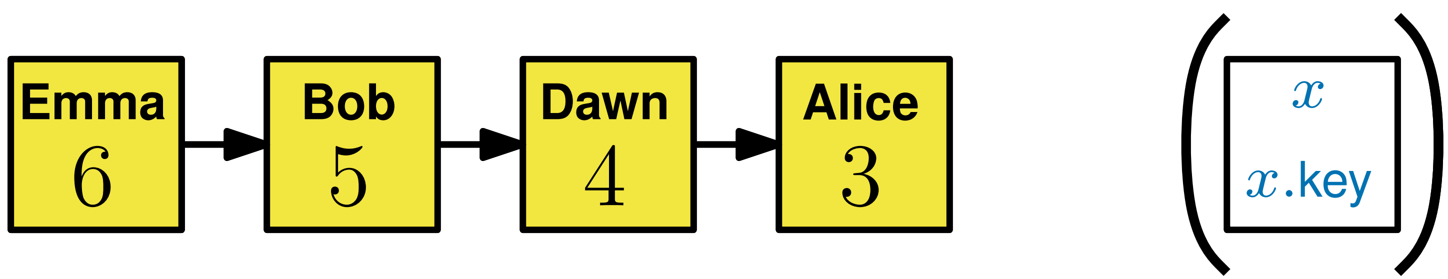

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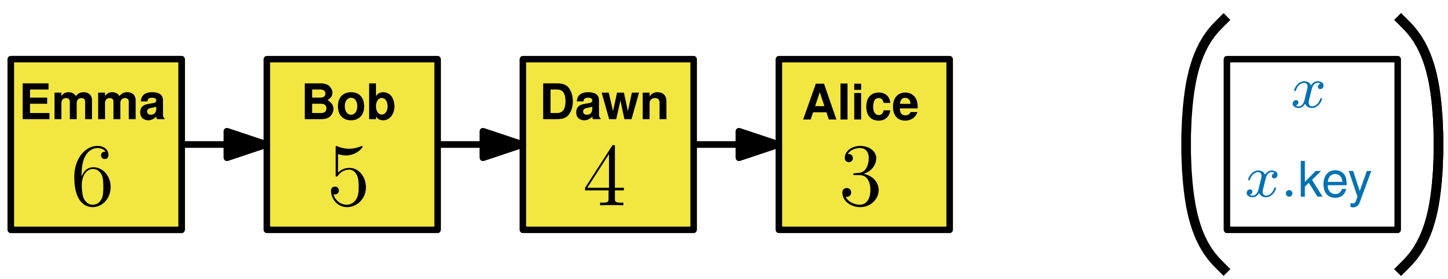

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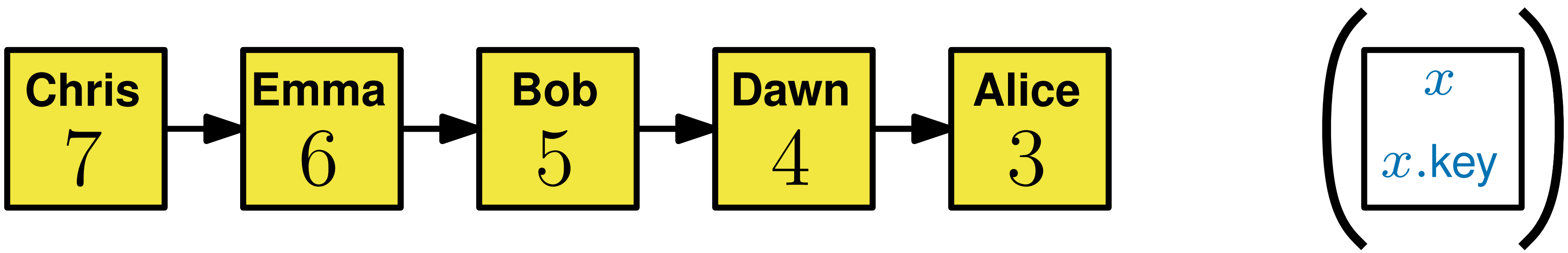

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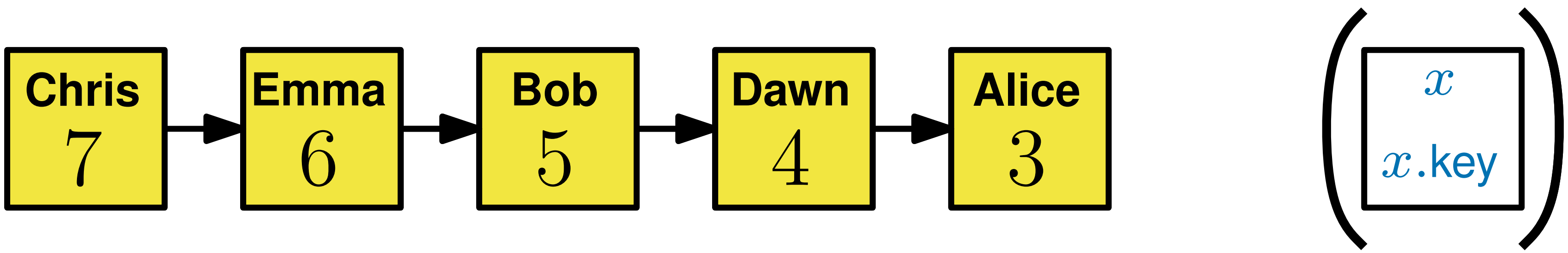

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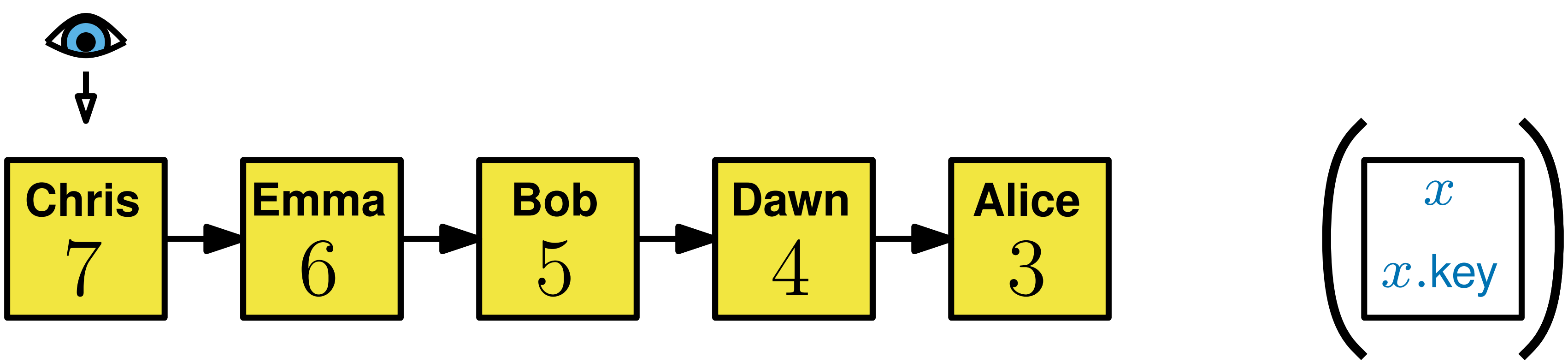

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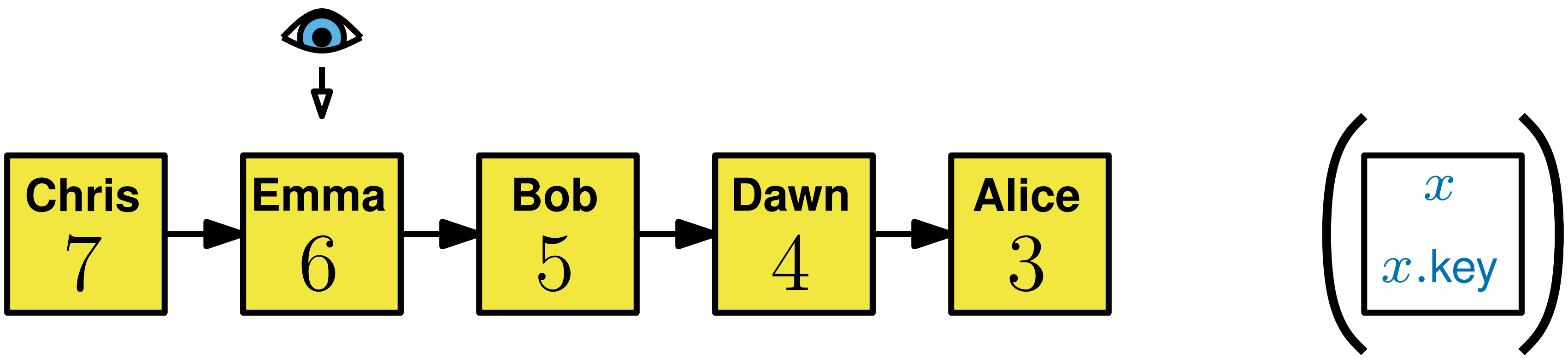

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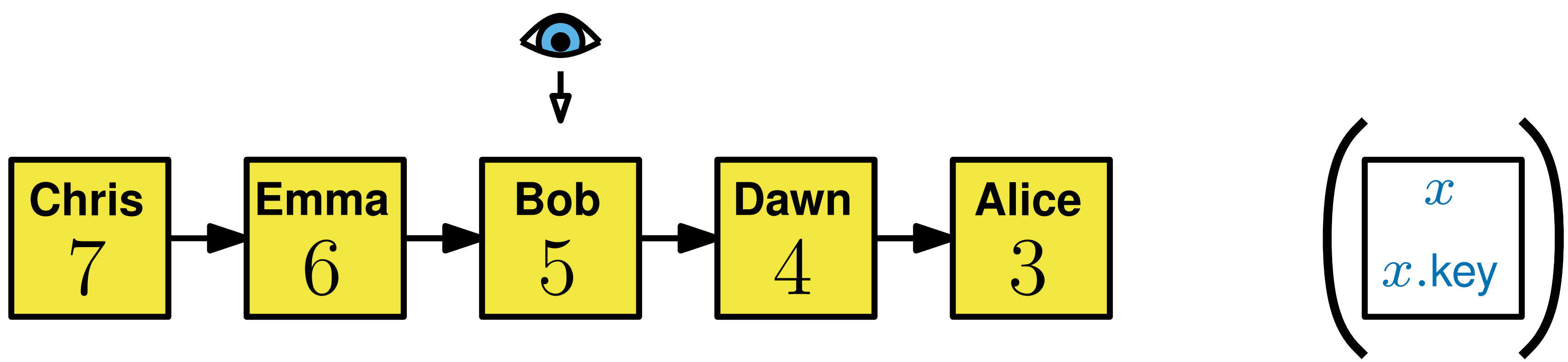

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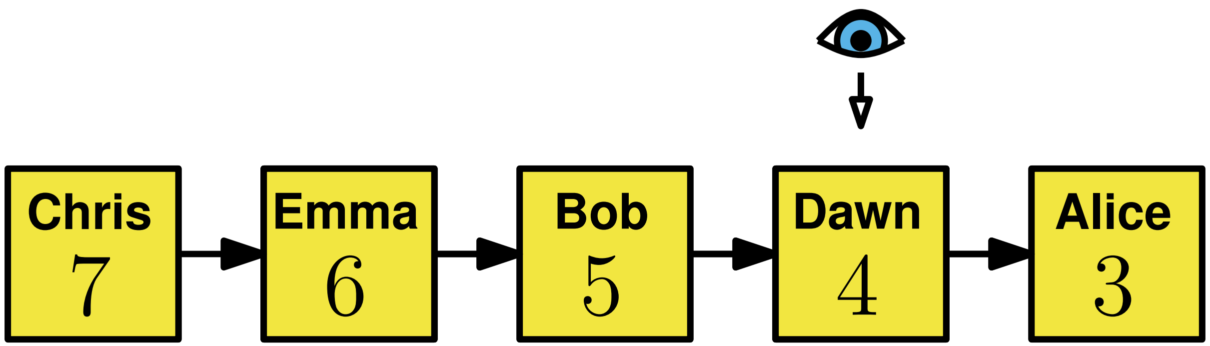

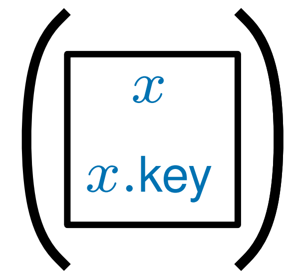

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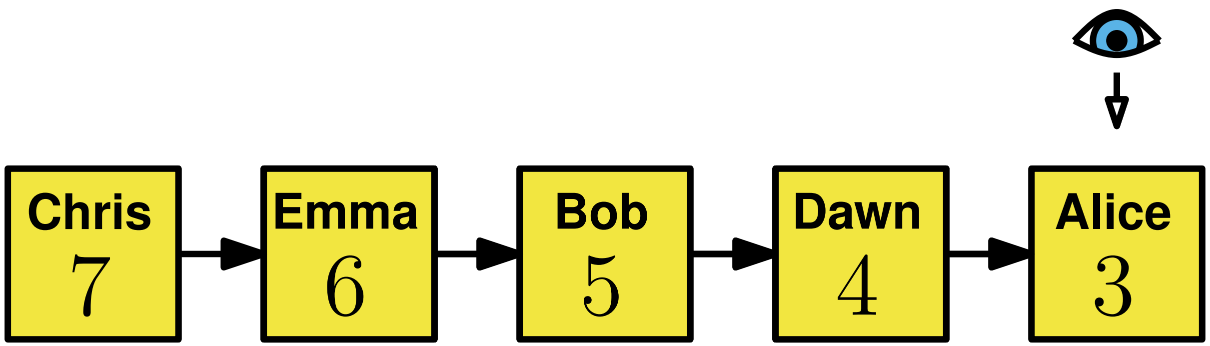

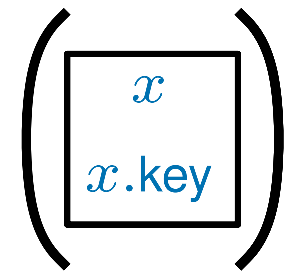

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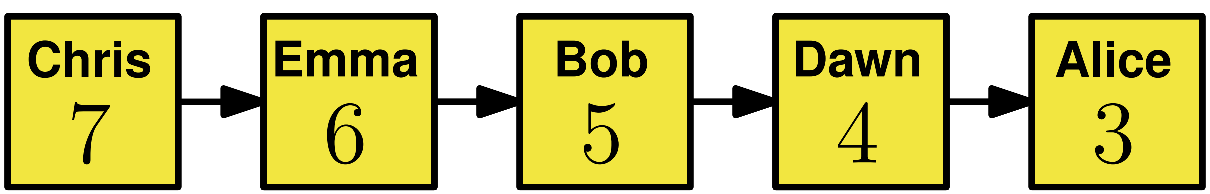

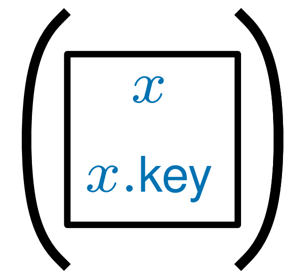

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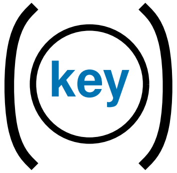

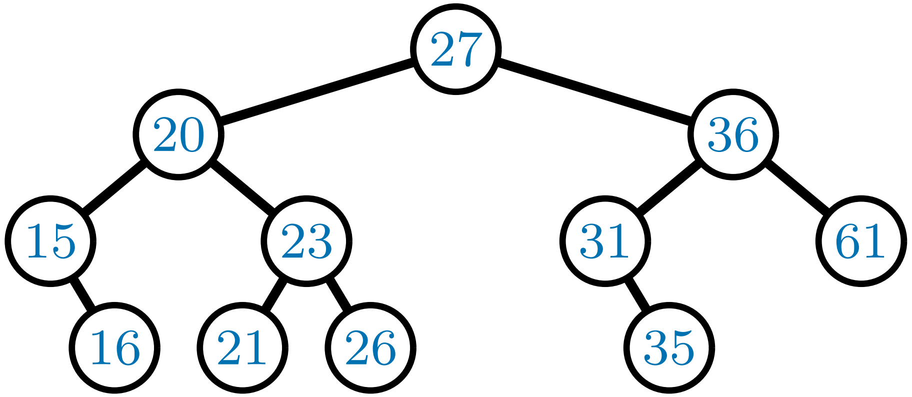

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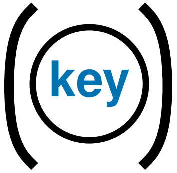

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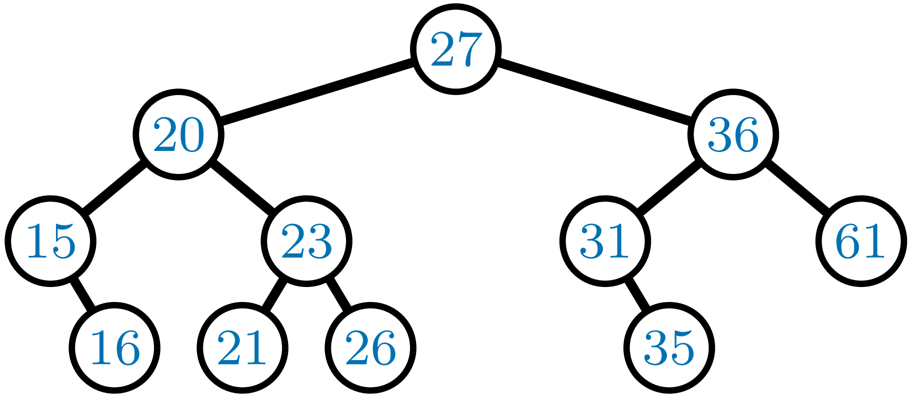

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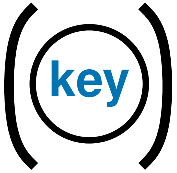

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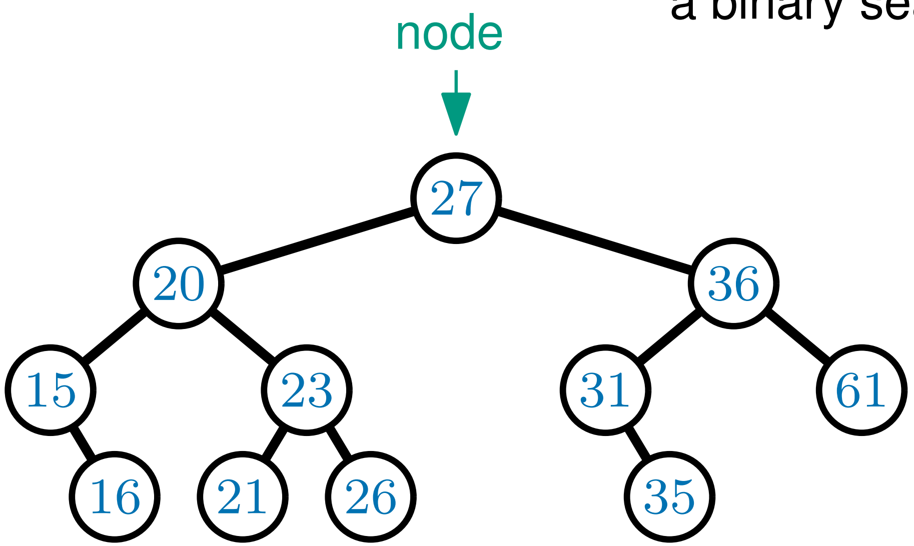

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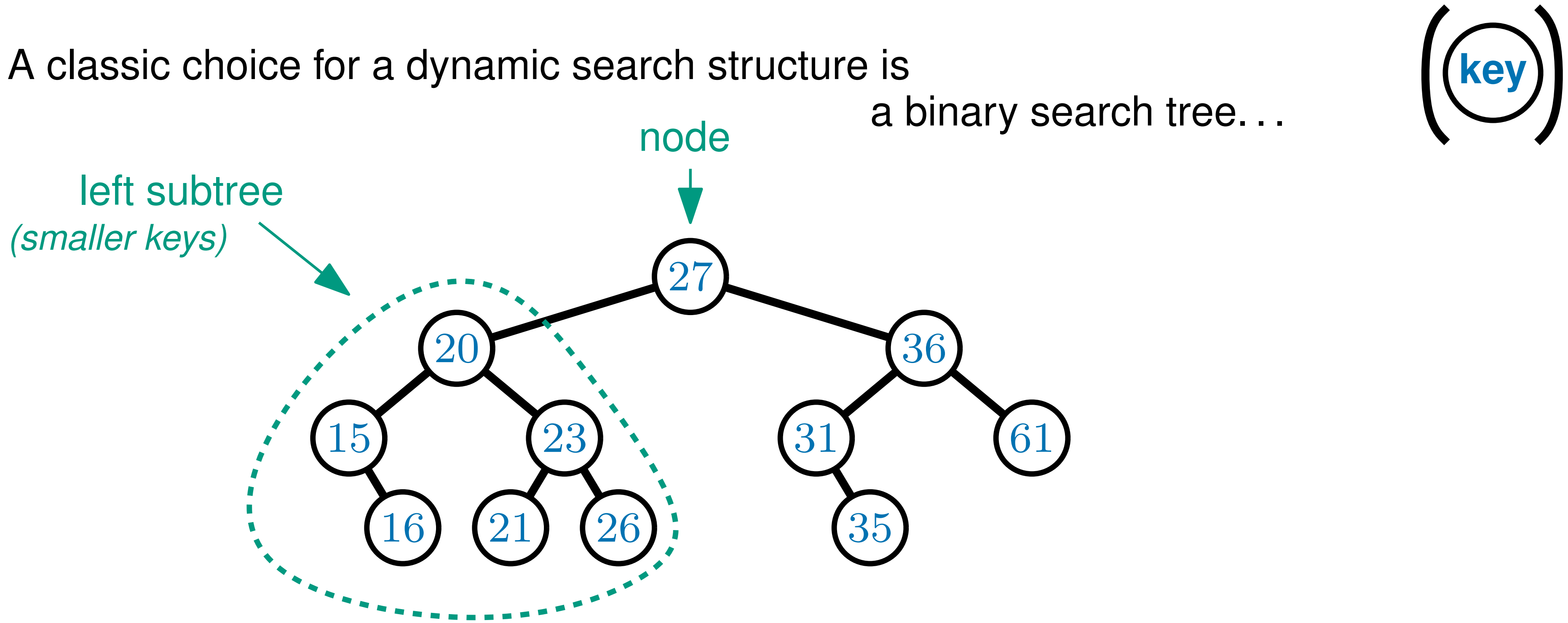

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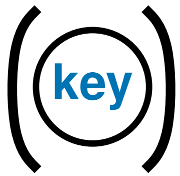

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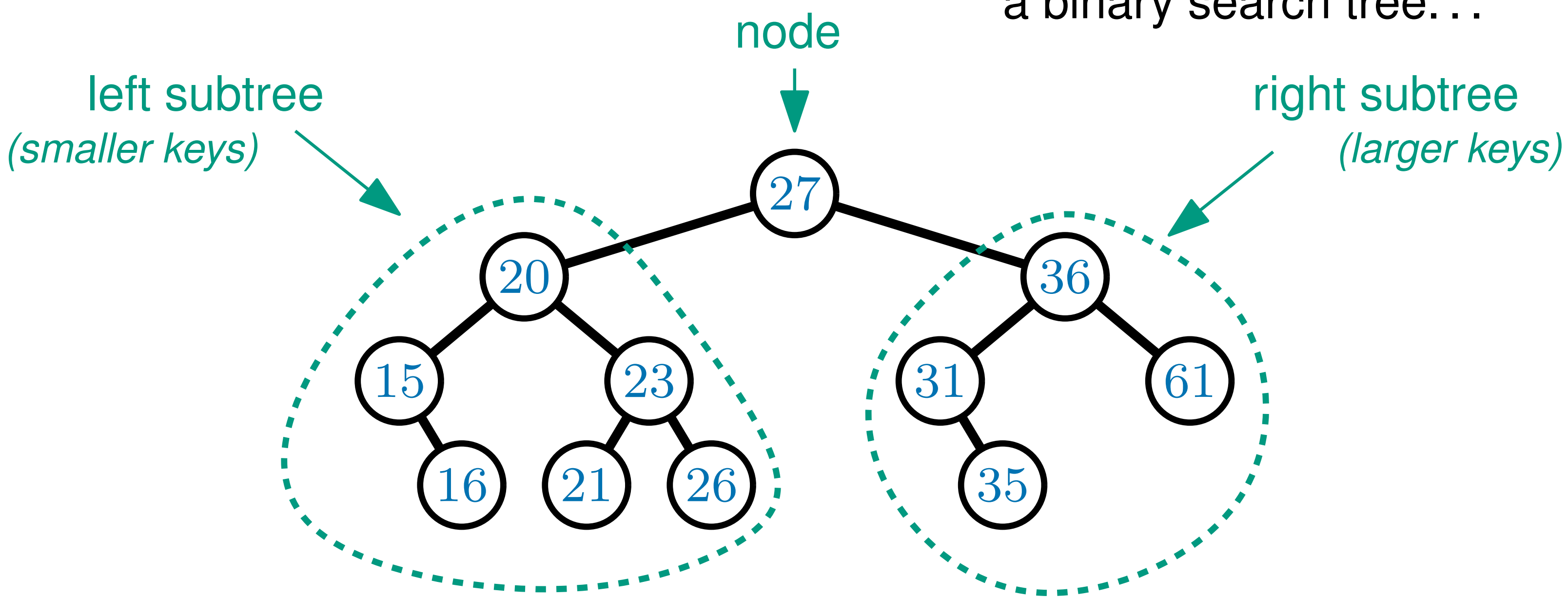

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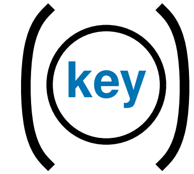

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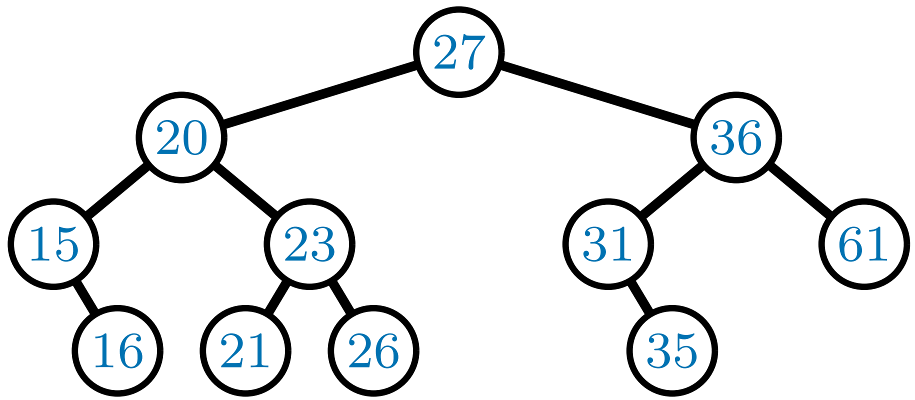

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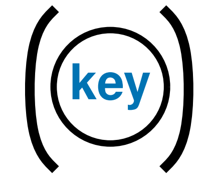

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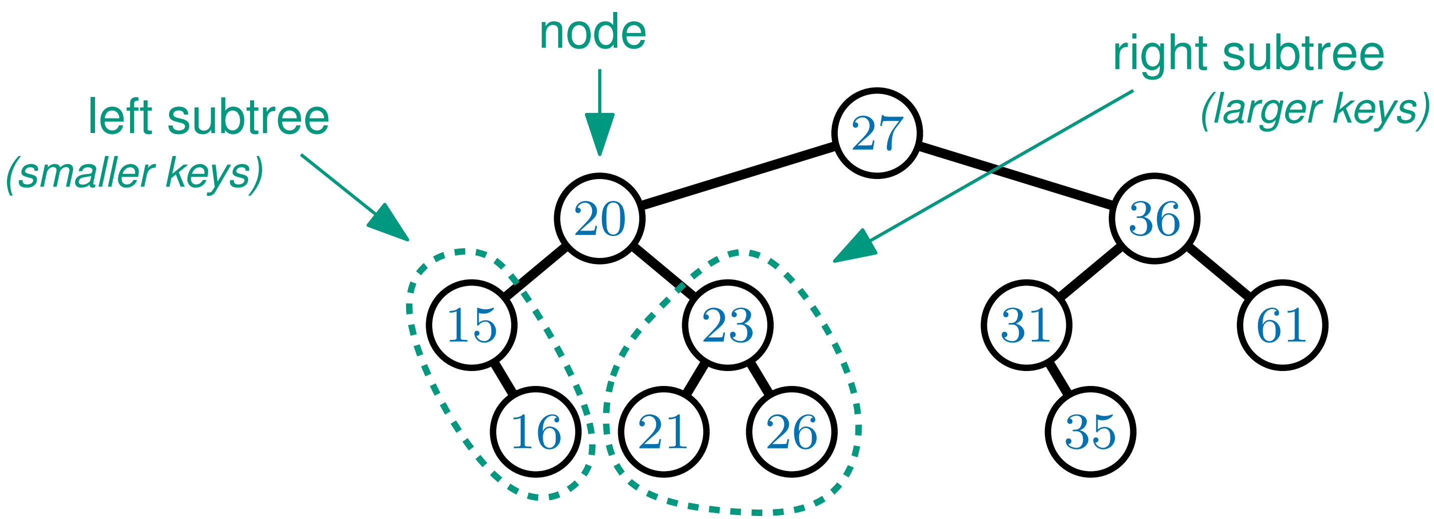

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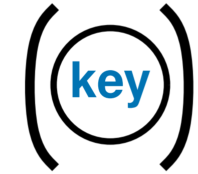

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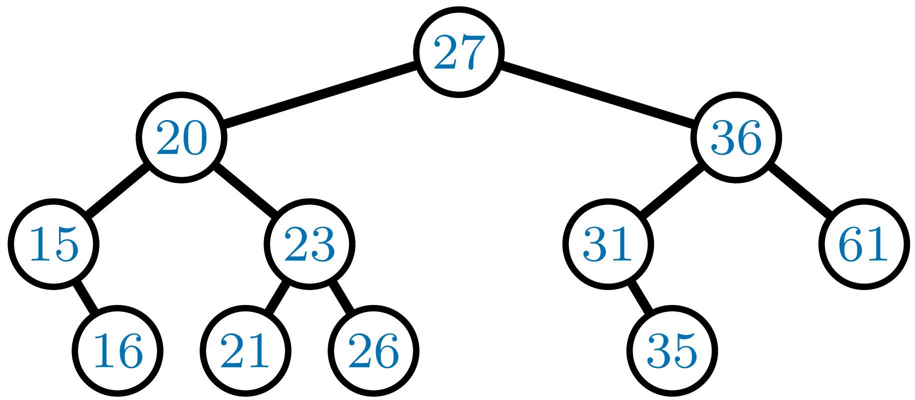

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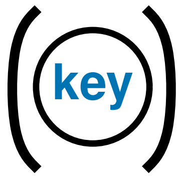

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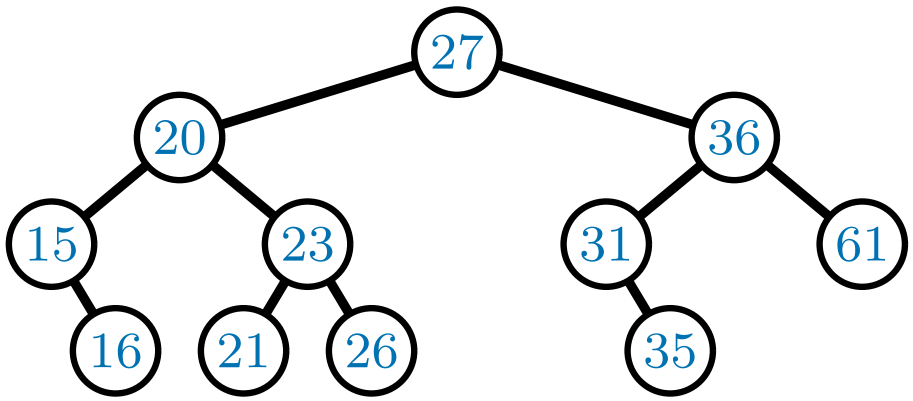

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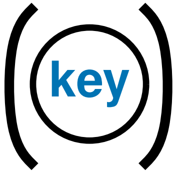

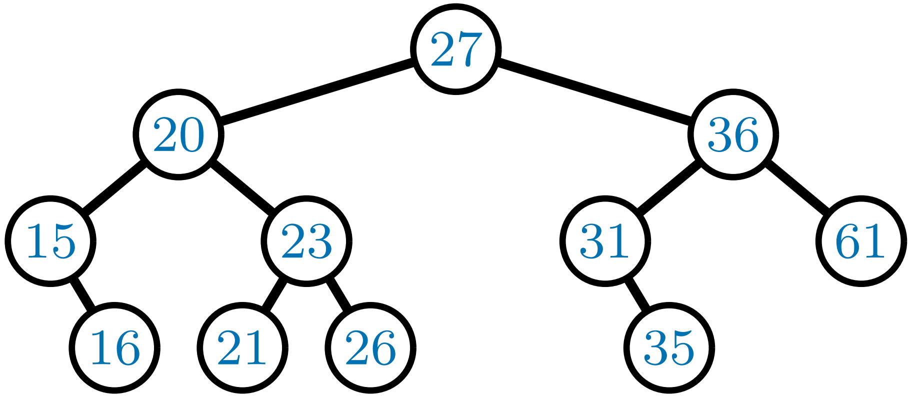

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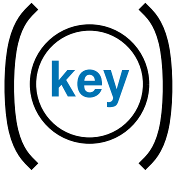

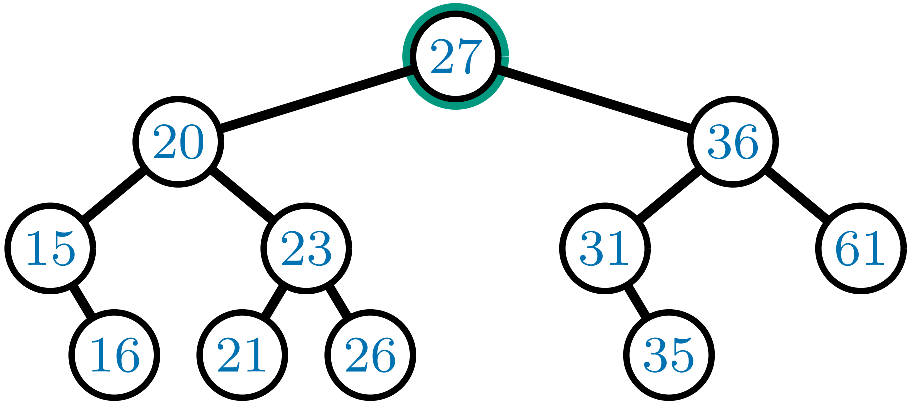

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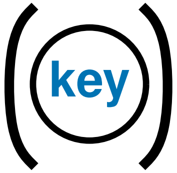

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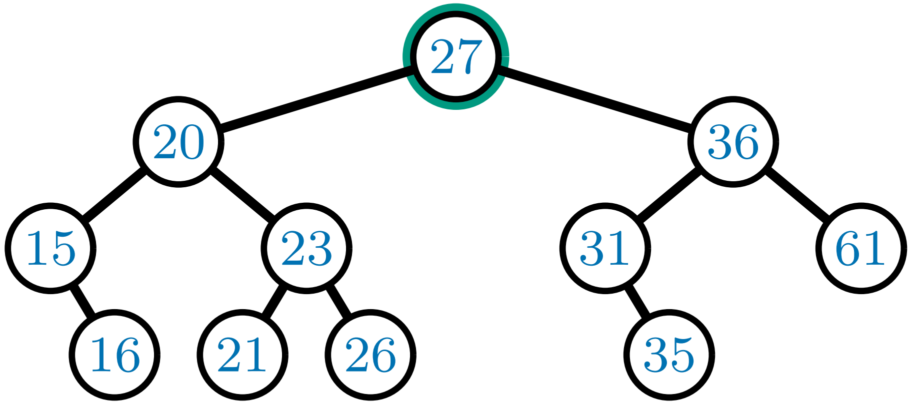

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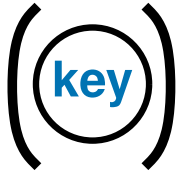

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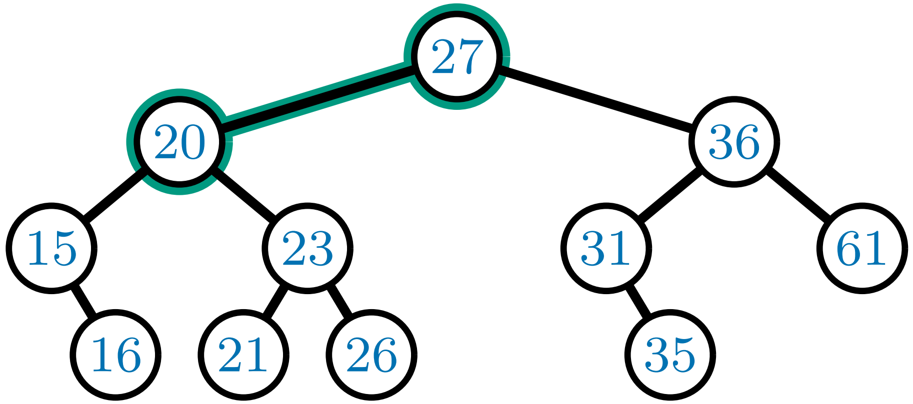

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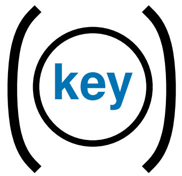
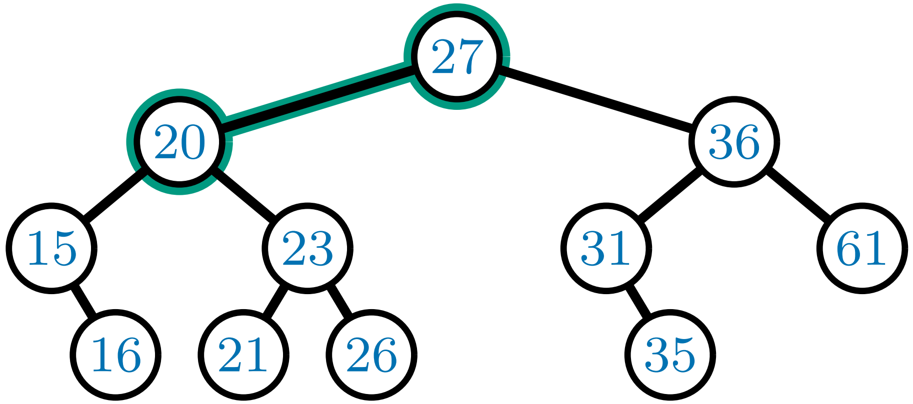


$21 < 27$ so go left
$21 > 20$ so go right



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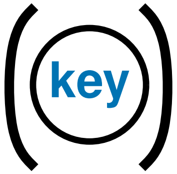

$21 < 27$ so go left
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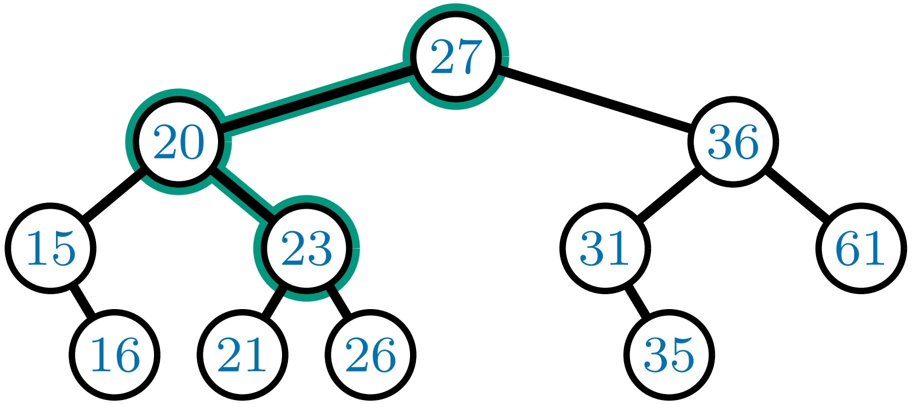

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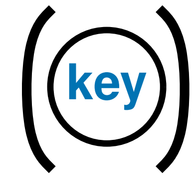

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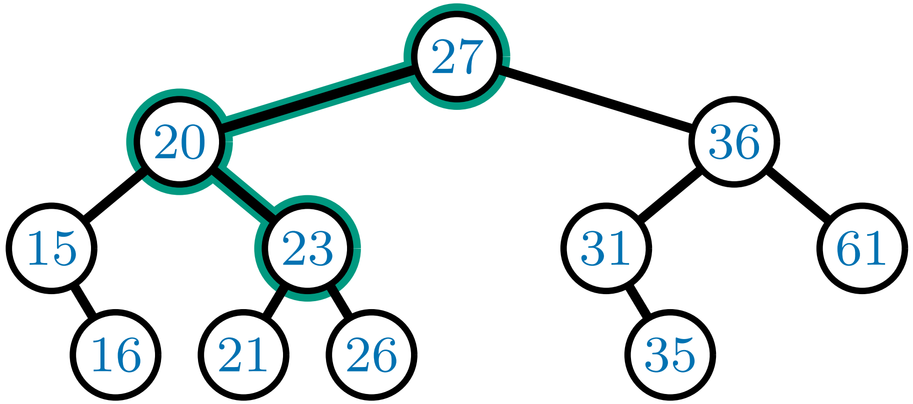

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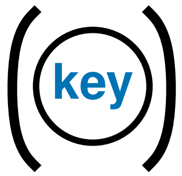

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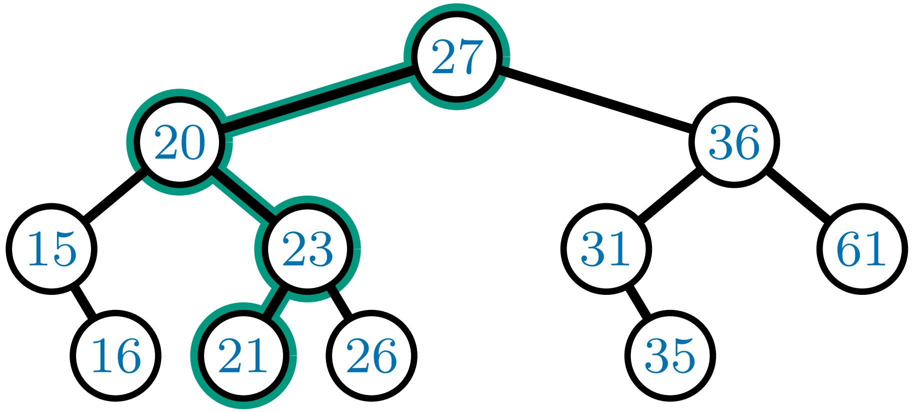

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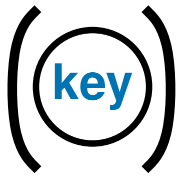

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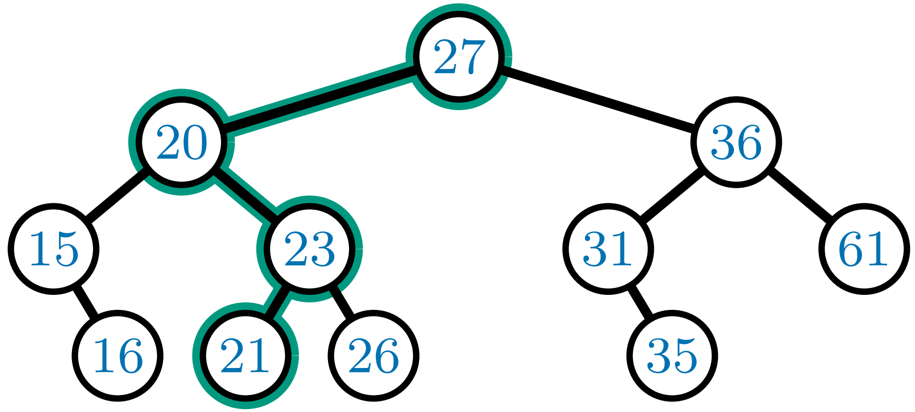

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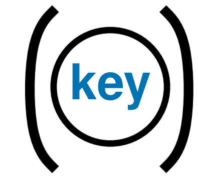

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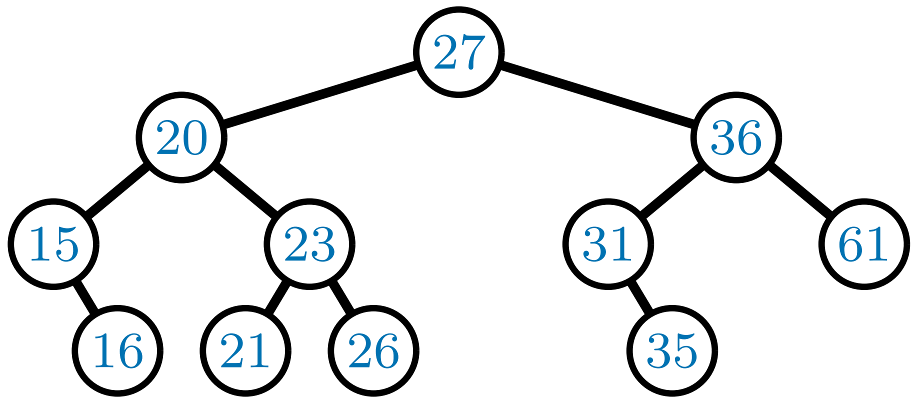

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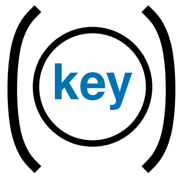

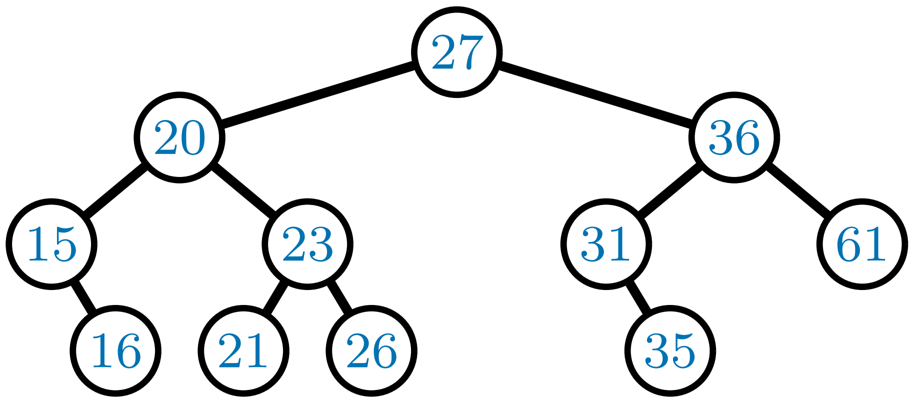

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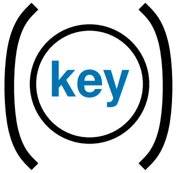

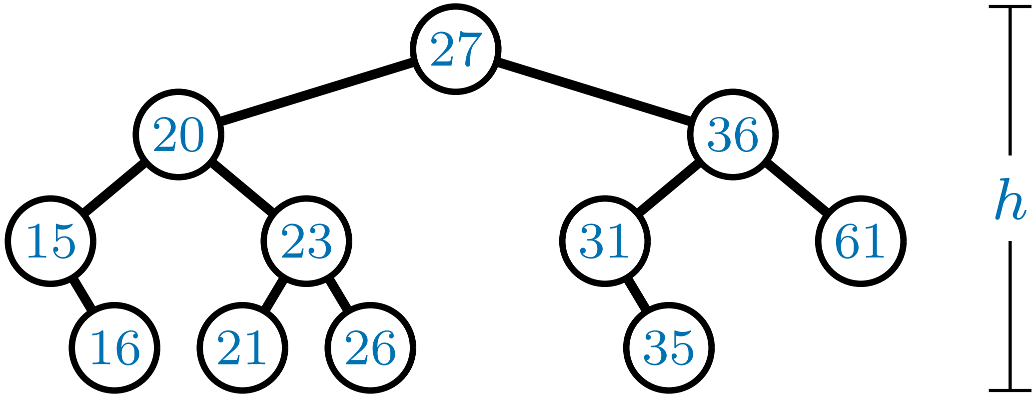

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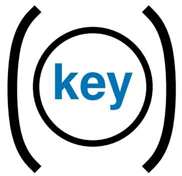

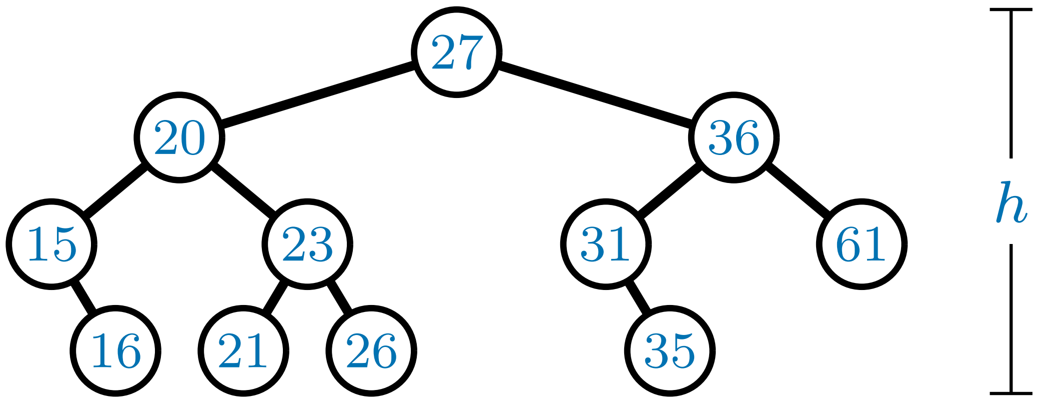

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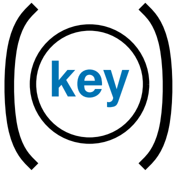

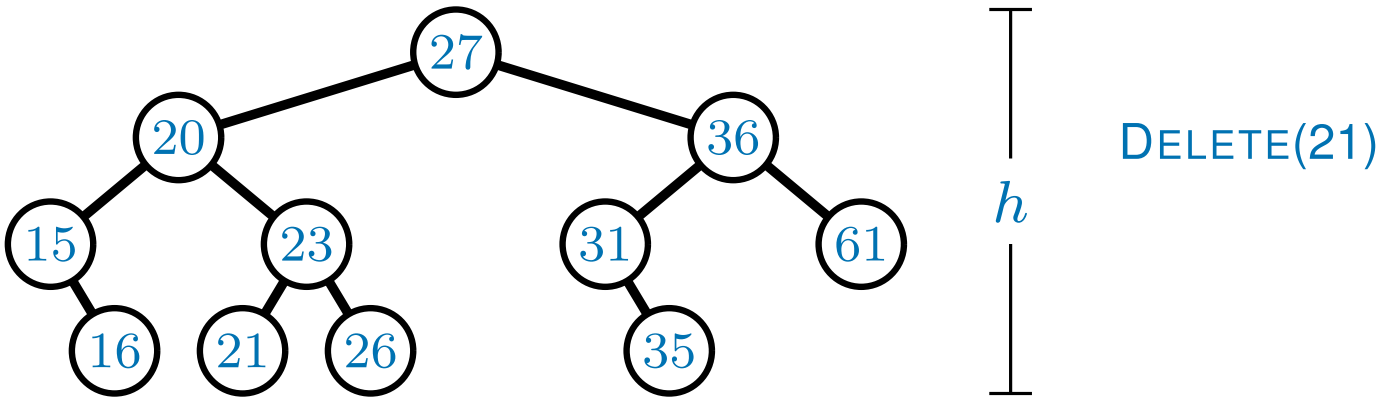

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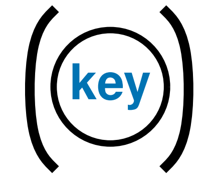

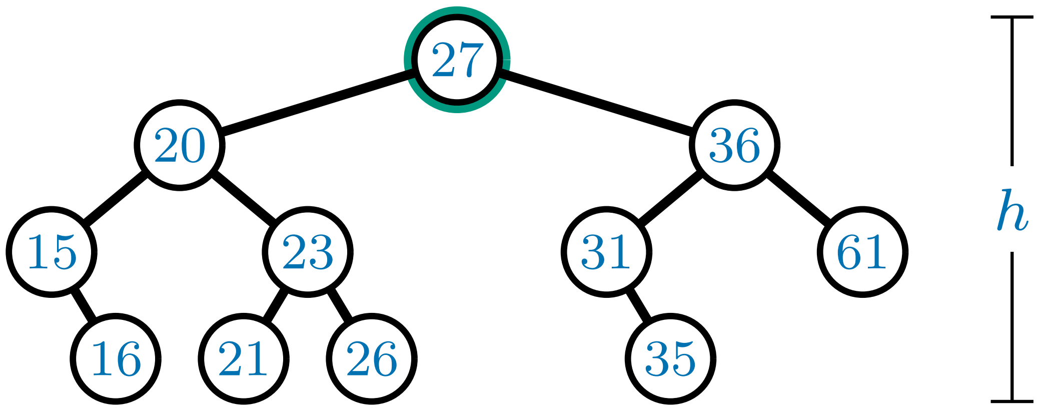

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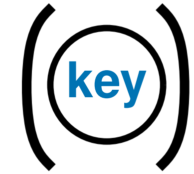

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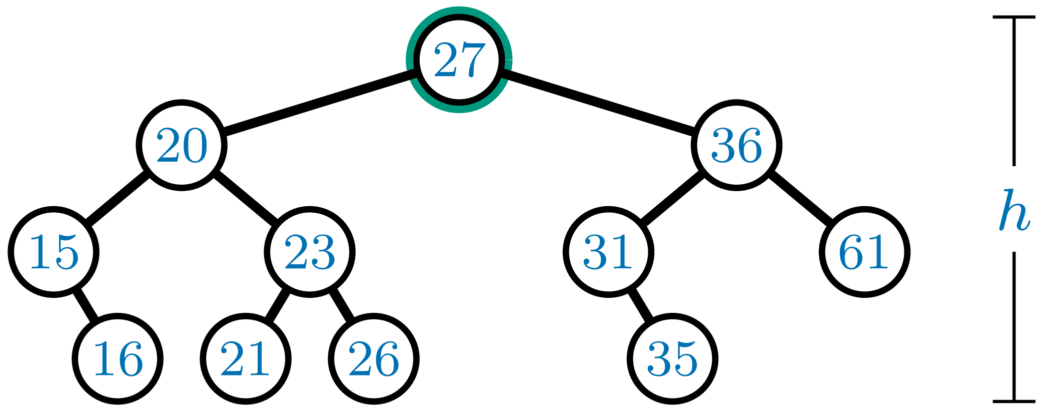

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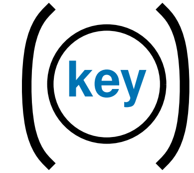

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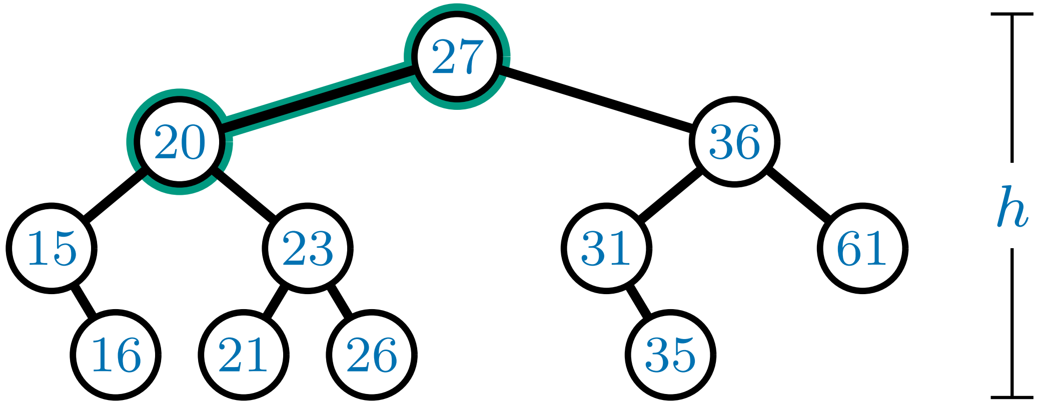

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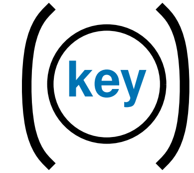

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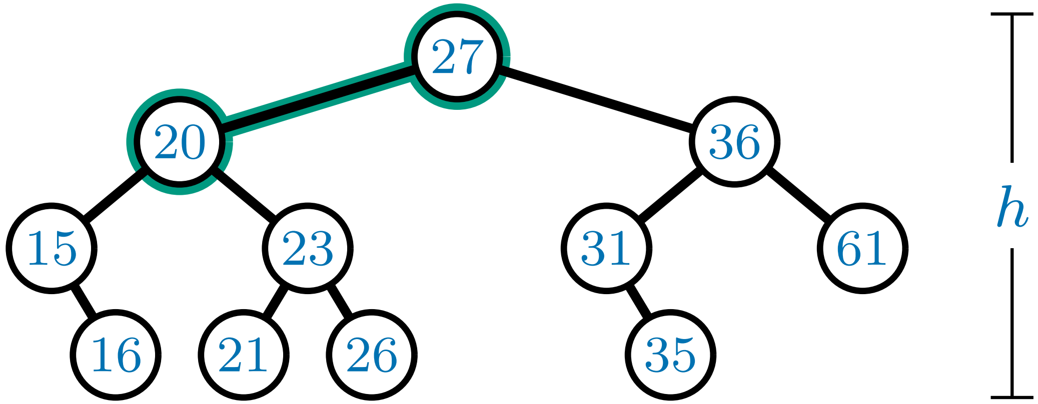

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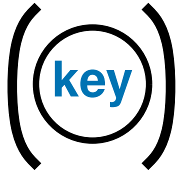

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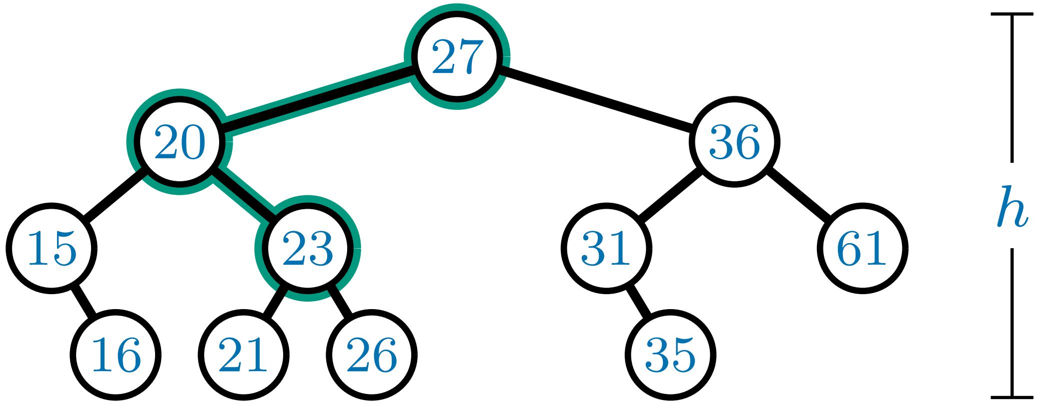

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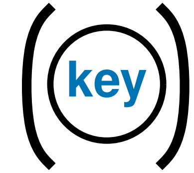

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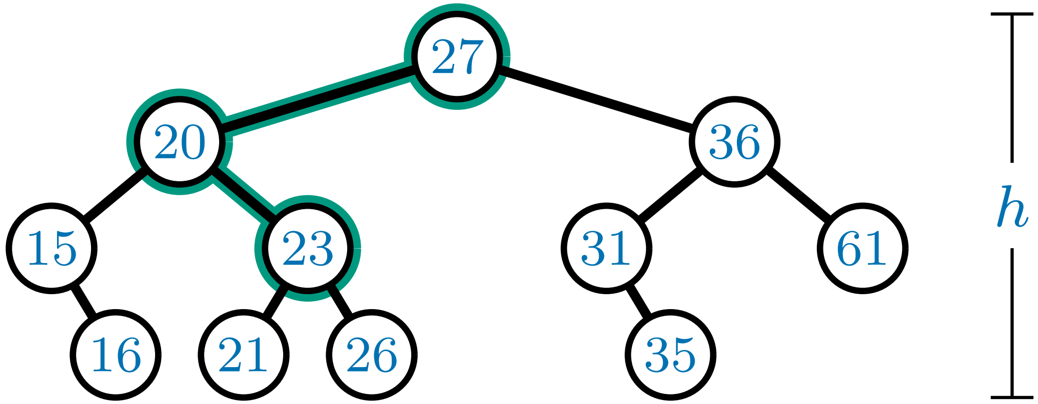

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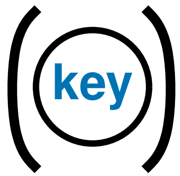

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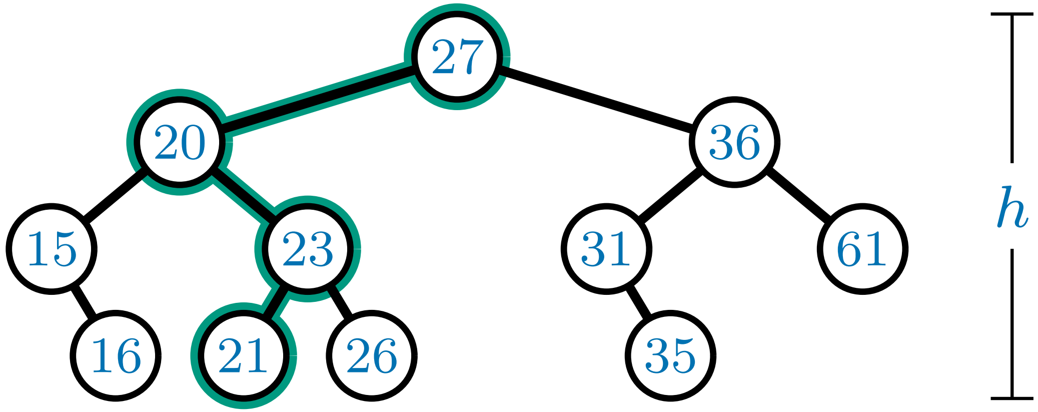

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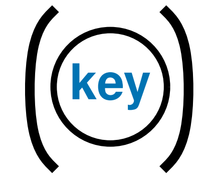

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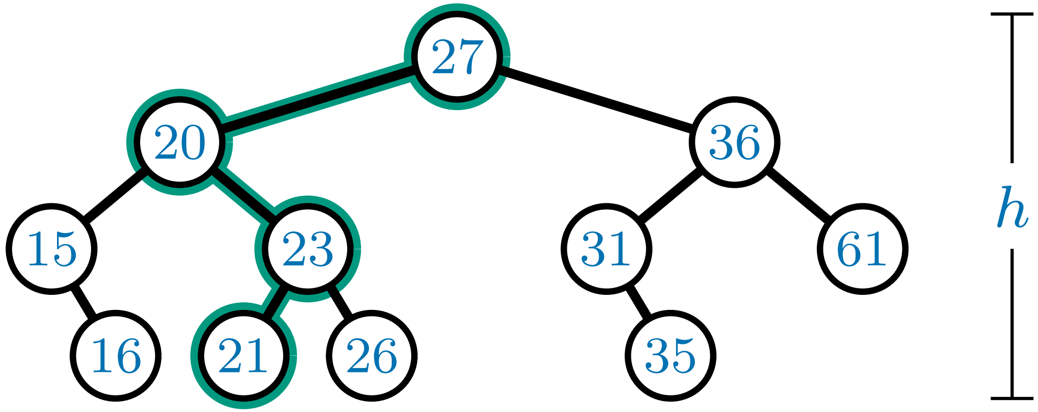

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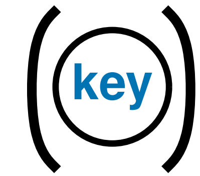

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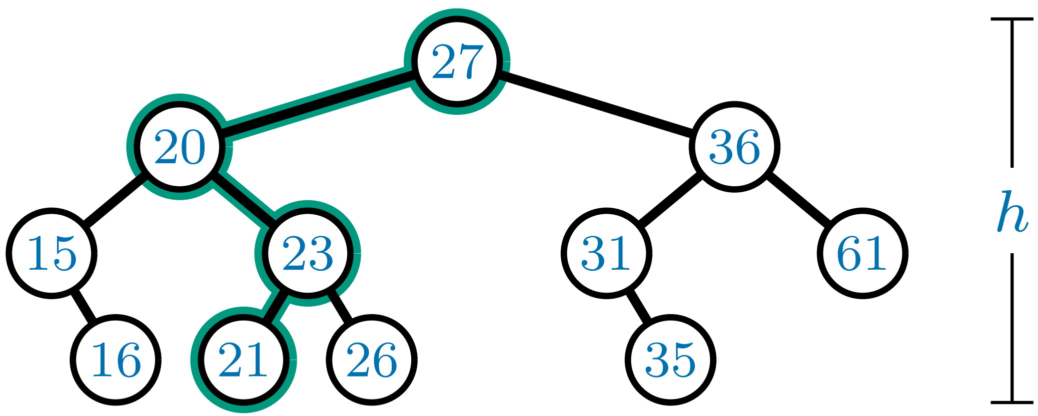

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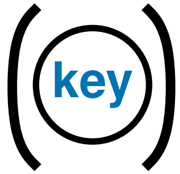

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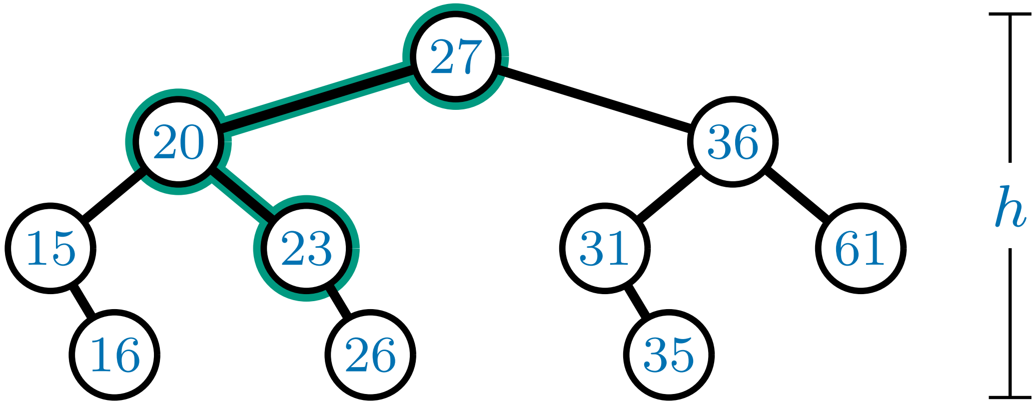

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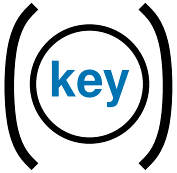

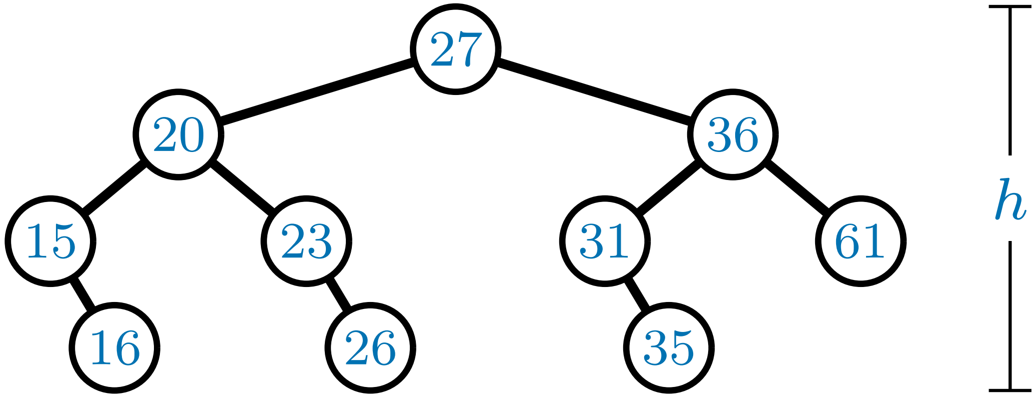

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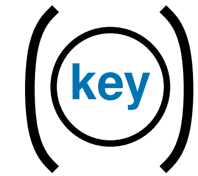

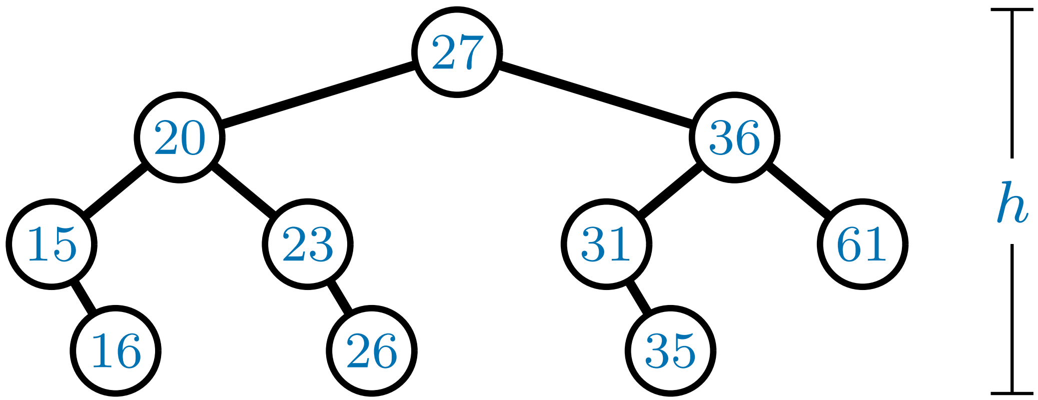

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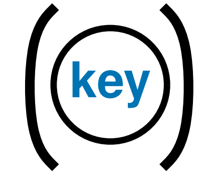

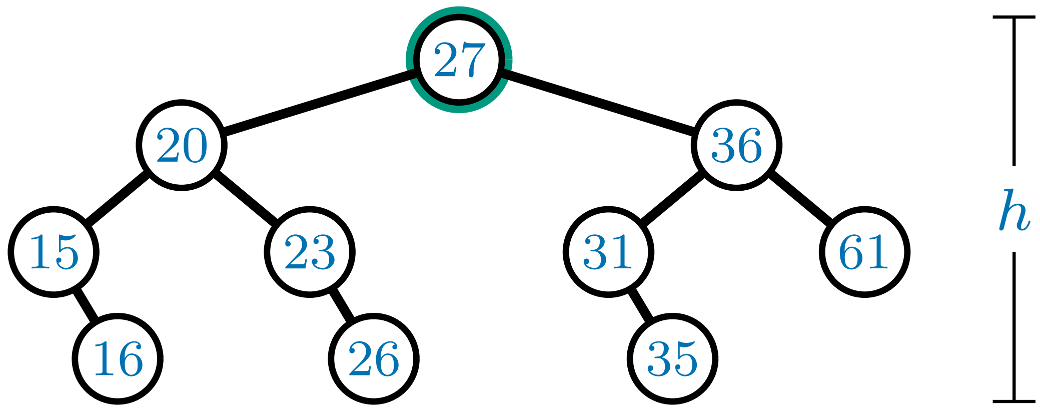

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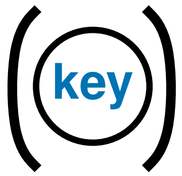

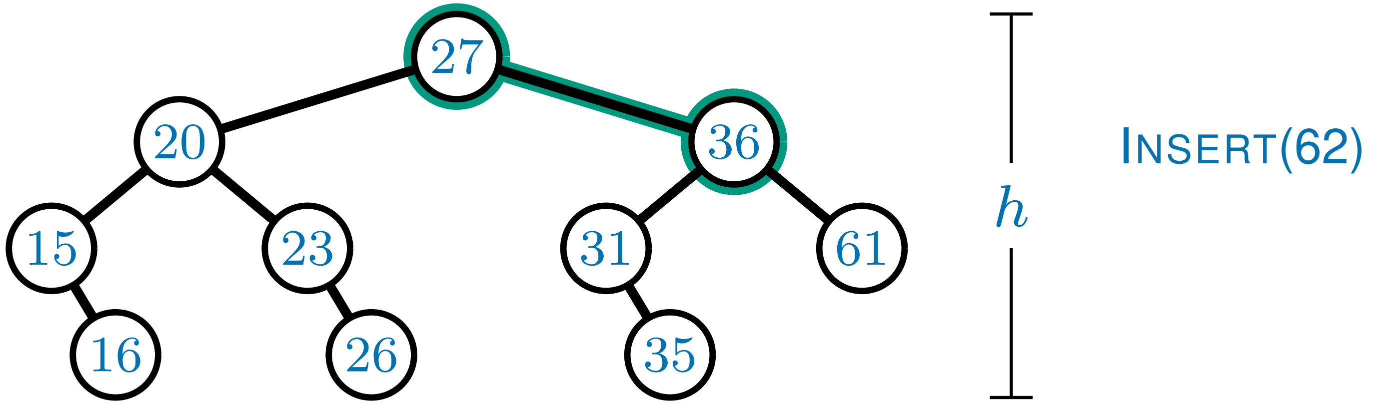

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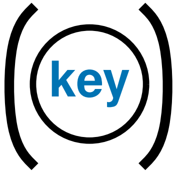

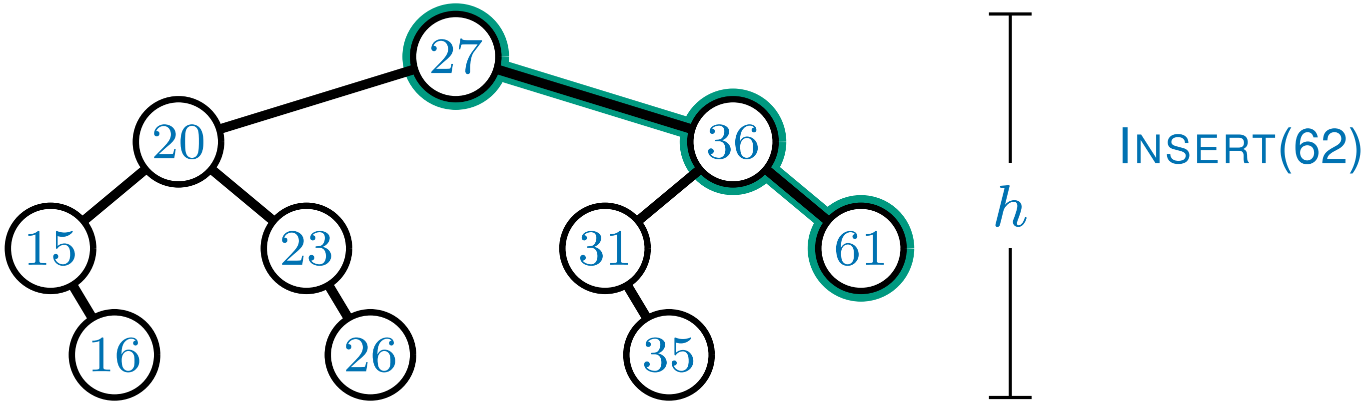

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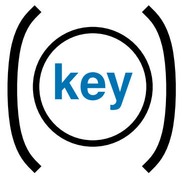

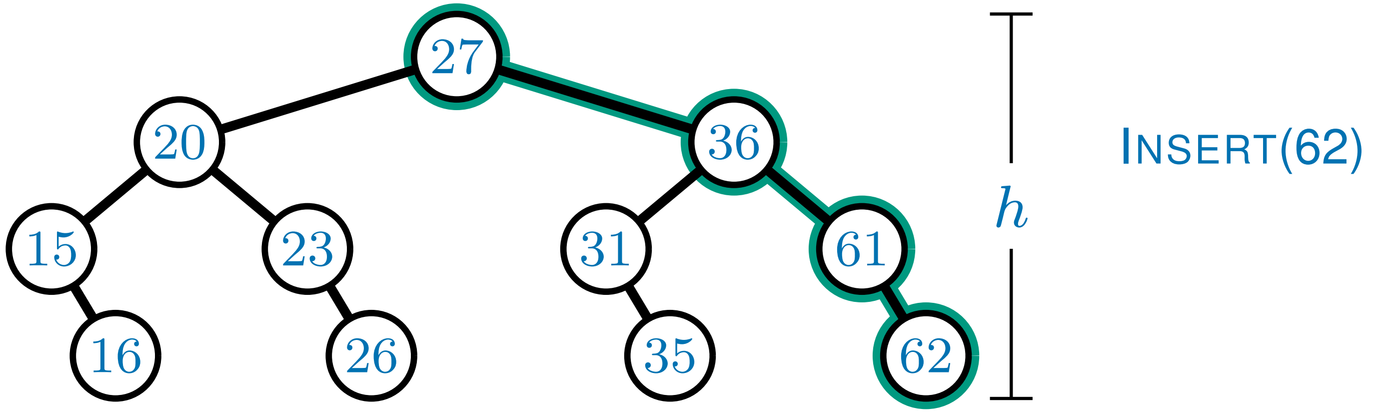

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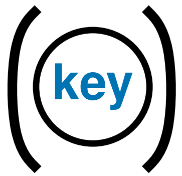

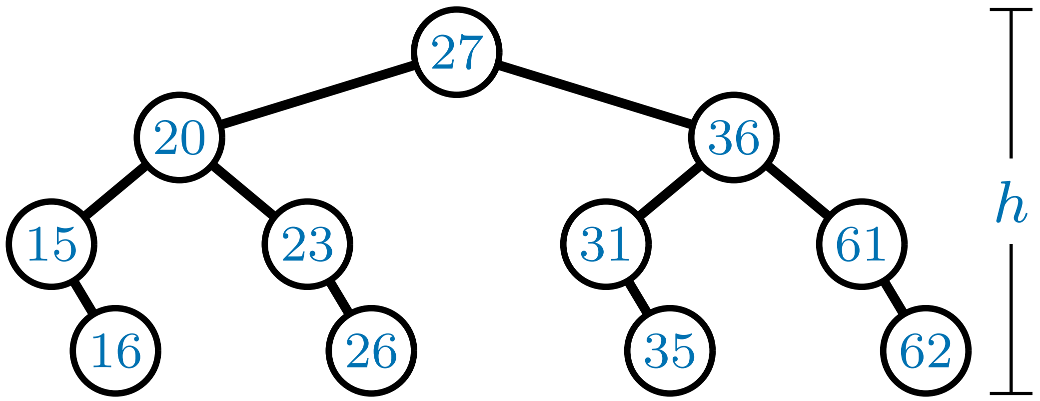

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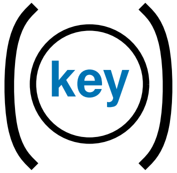

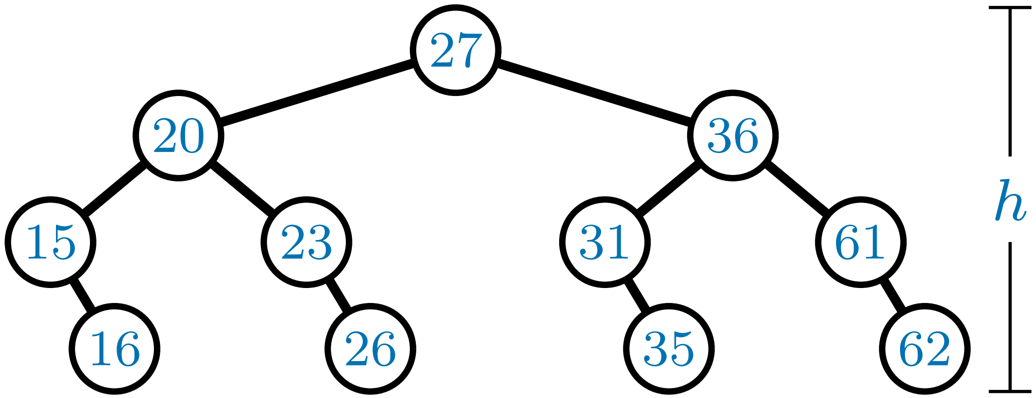

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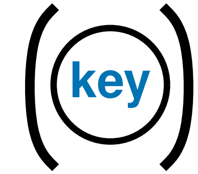

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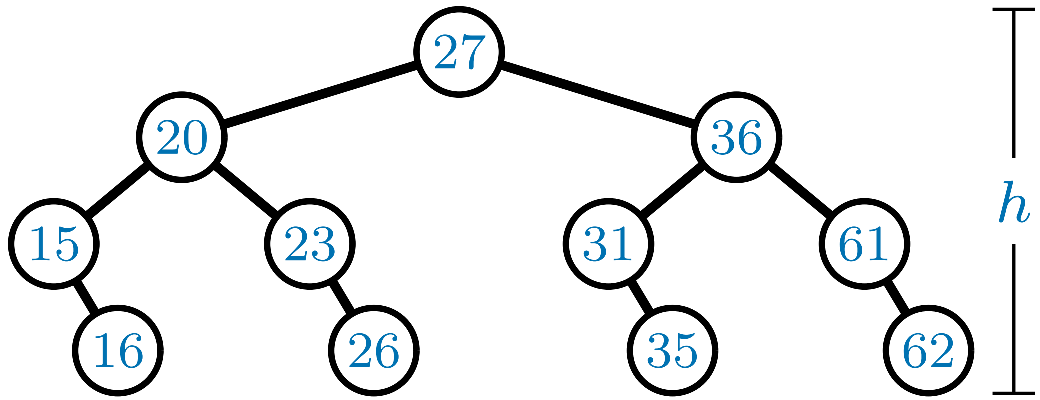

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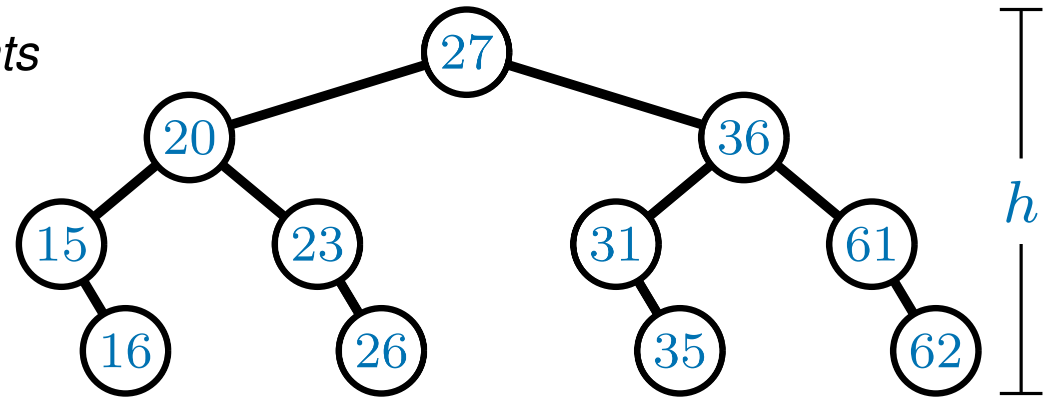

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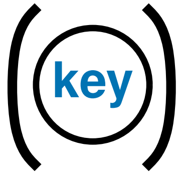

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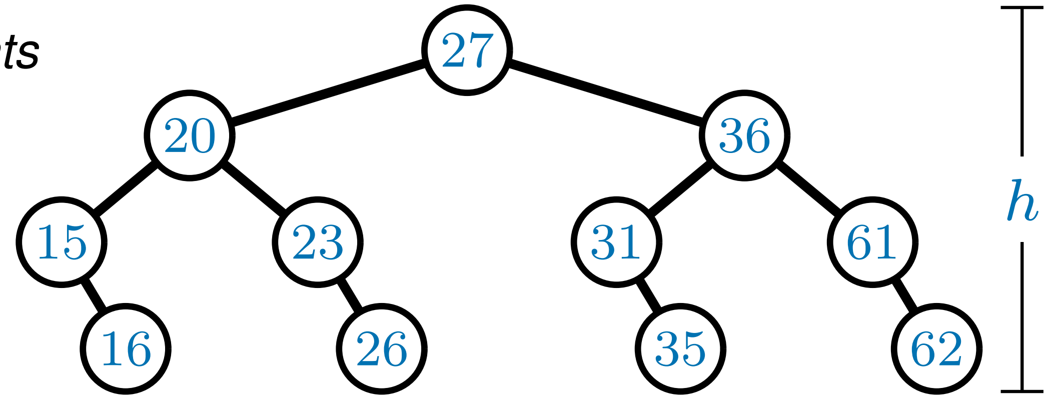

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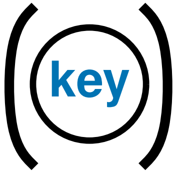

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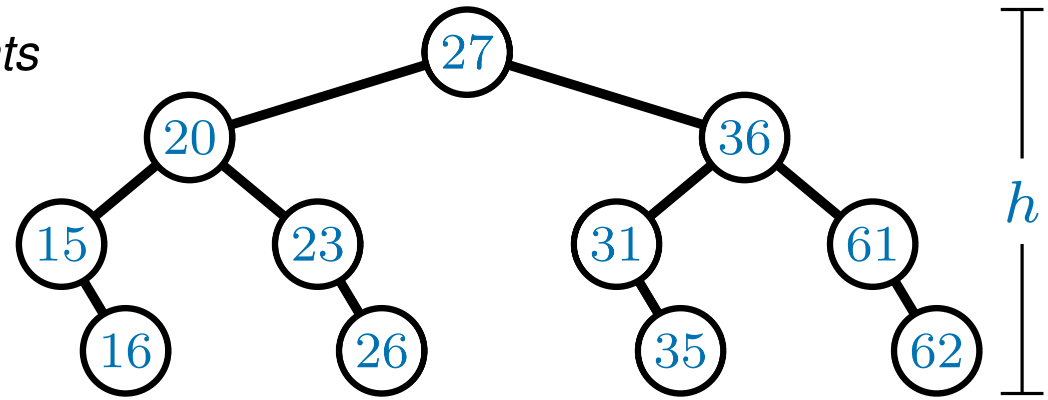

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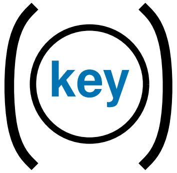

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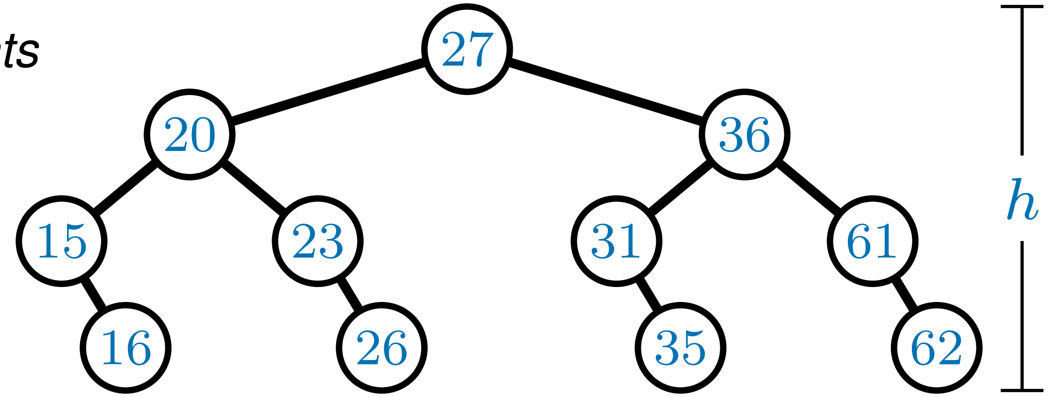

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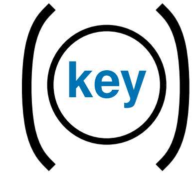

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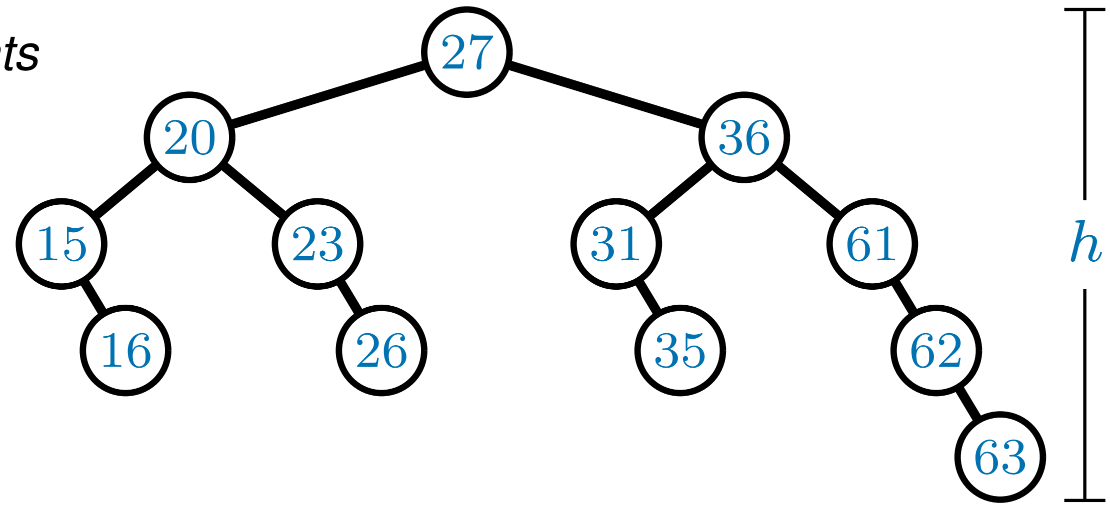

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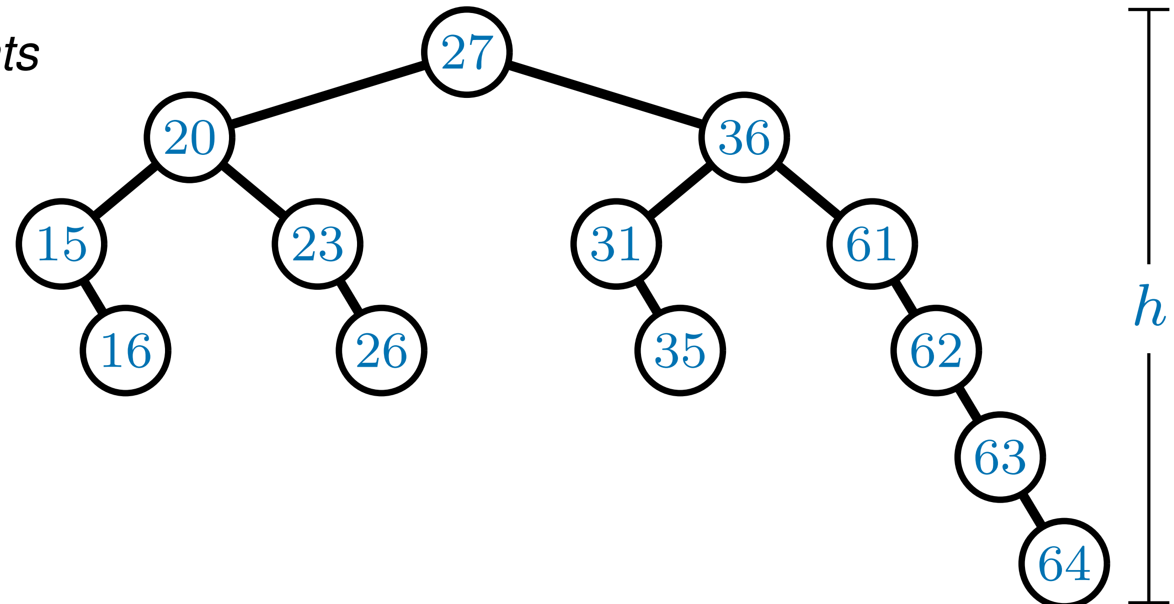

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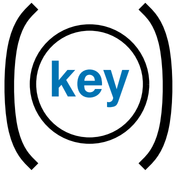

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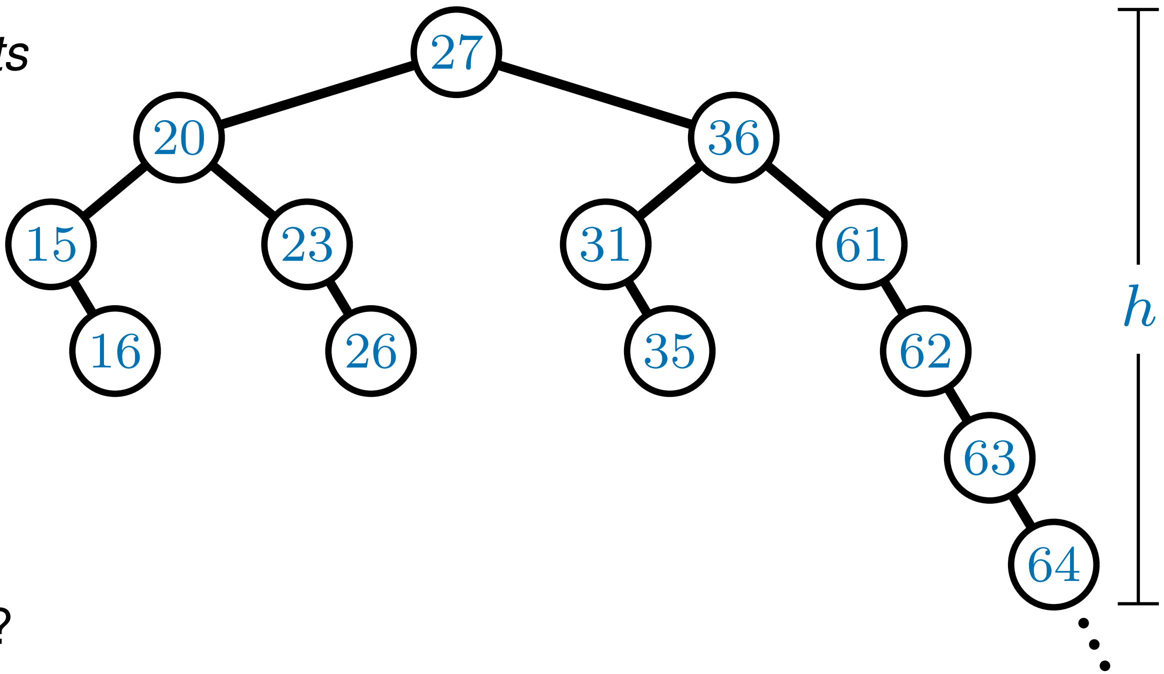

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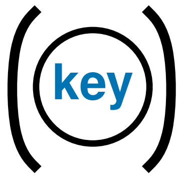

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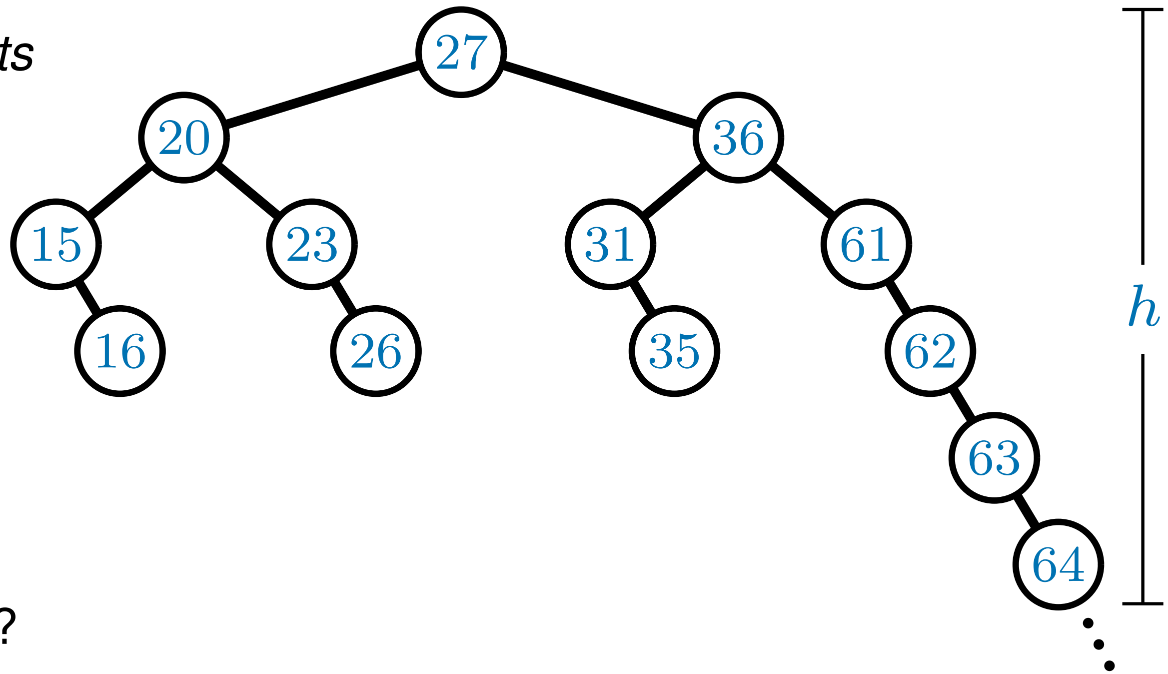

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how can we overcome this?

Part one

Self-balancing trees

inspired by slides by Inge Li Gørtz
in turn inspired by slides by Kevin Wayne

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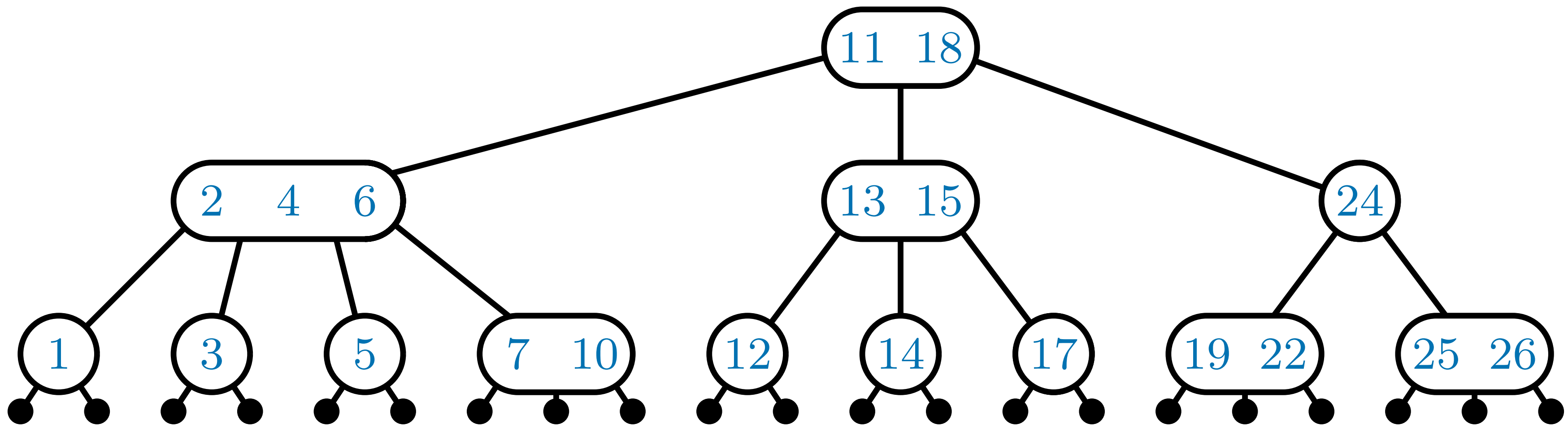
Key idea: Nodes can have between 2 and 4 children *(hence the name)*

Perfect balance - every path from the root to a leaf has the same length

(always, all the time)



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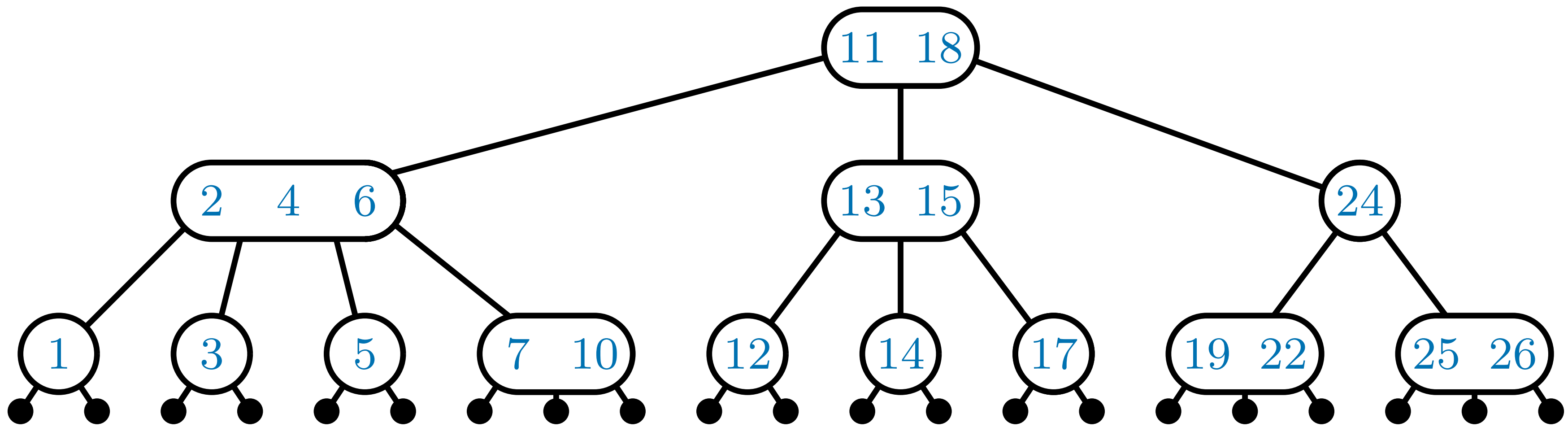

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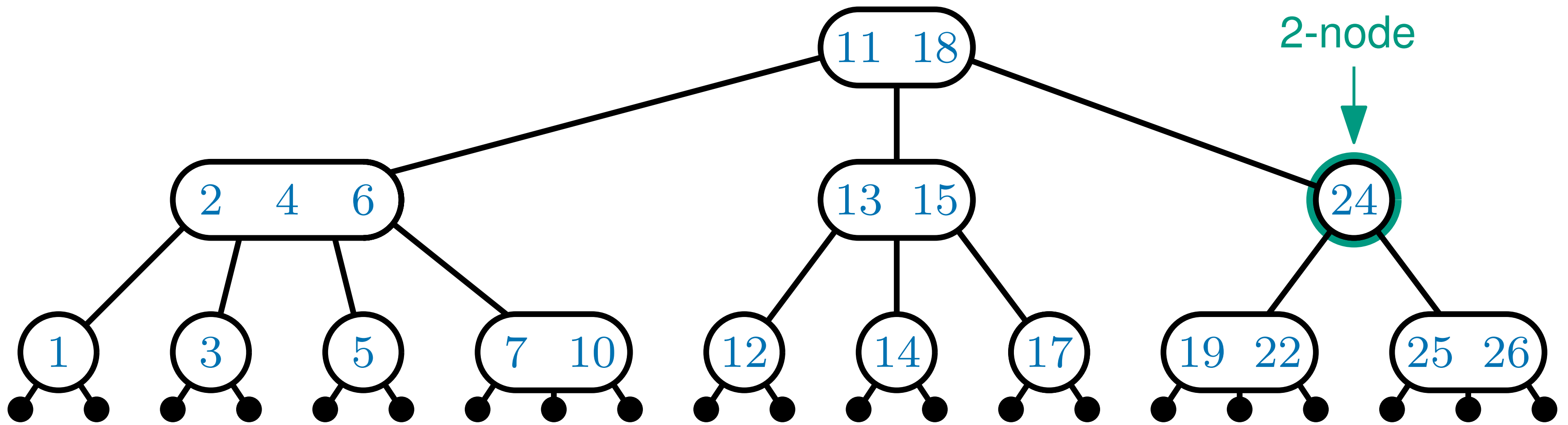

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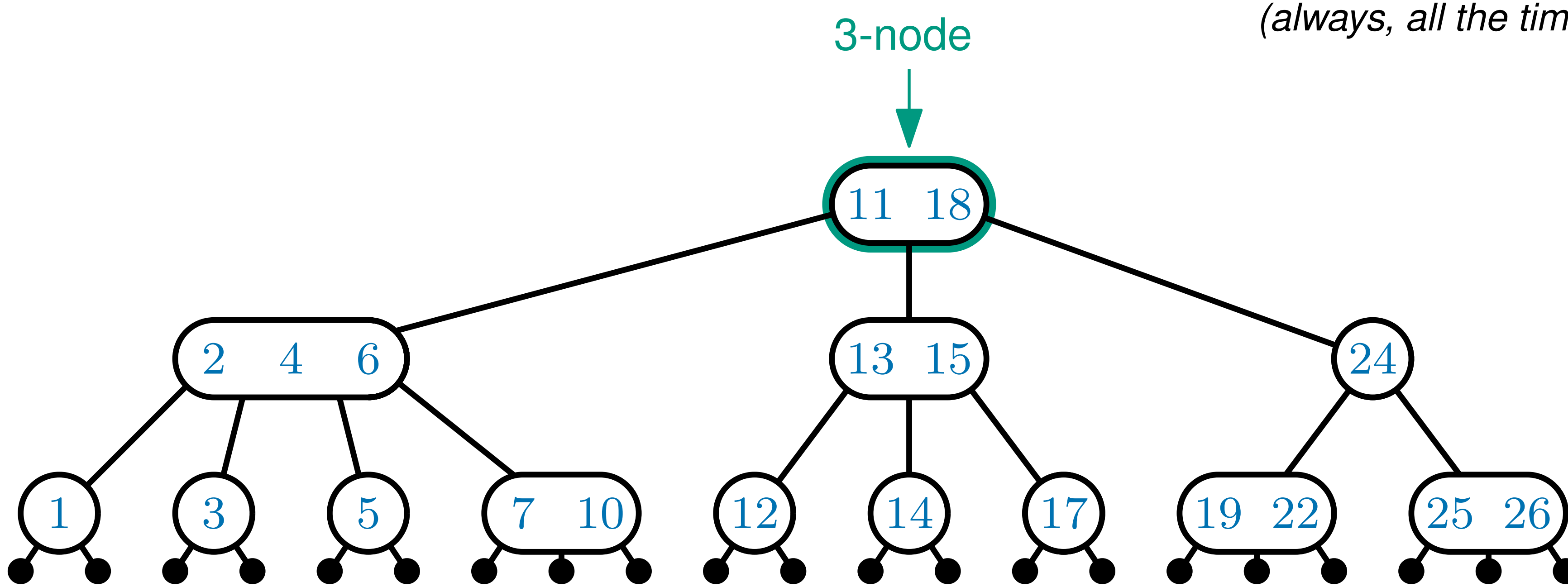

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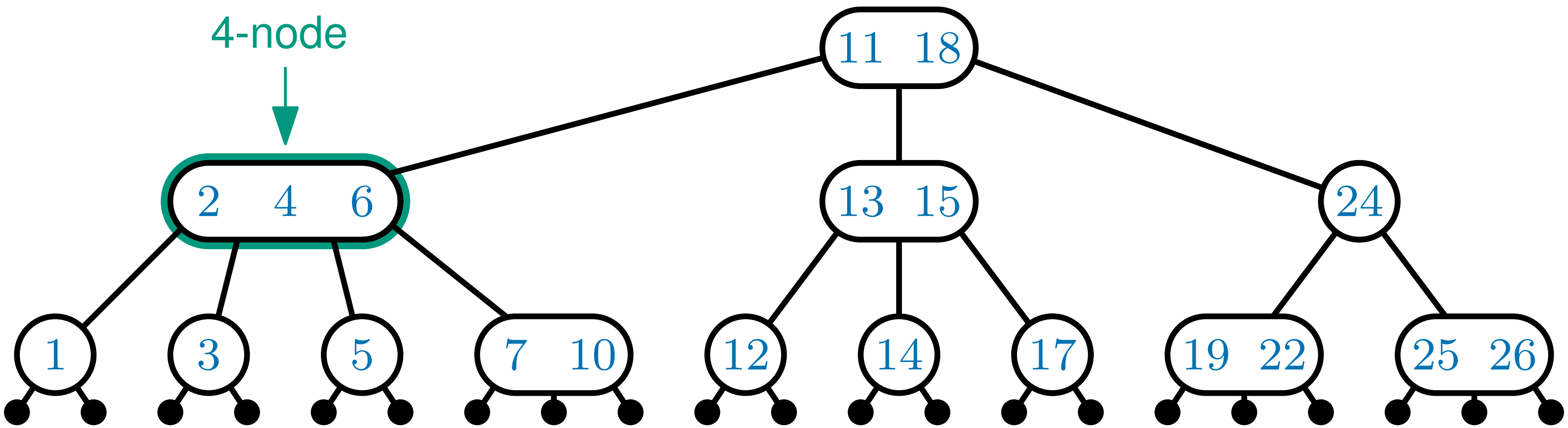

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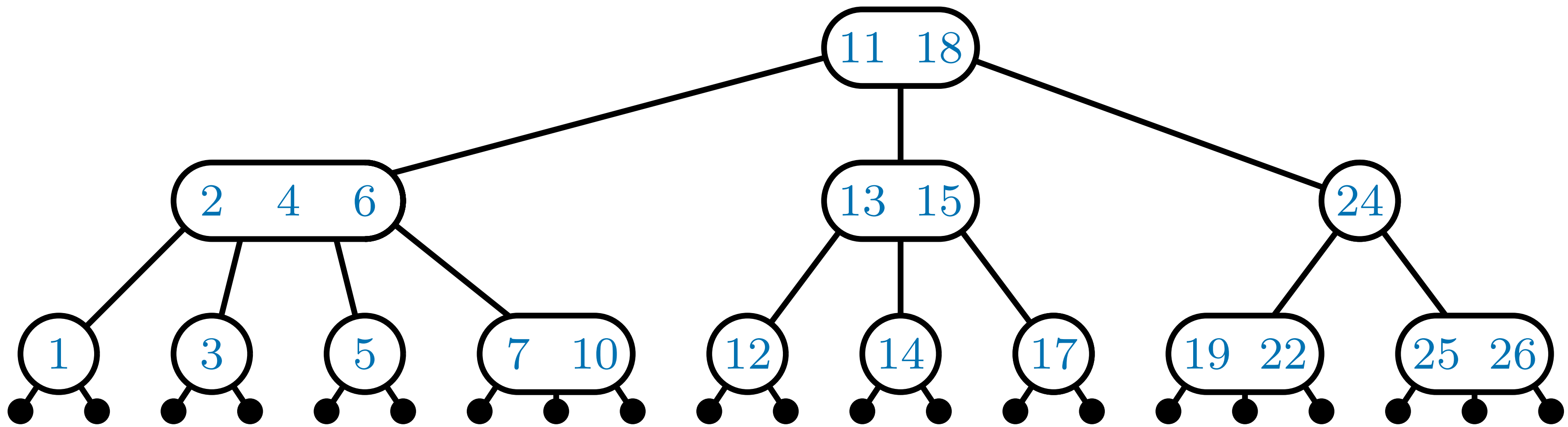

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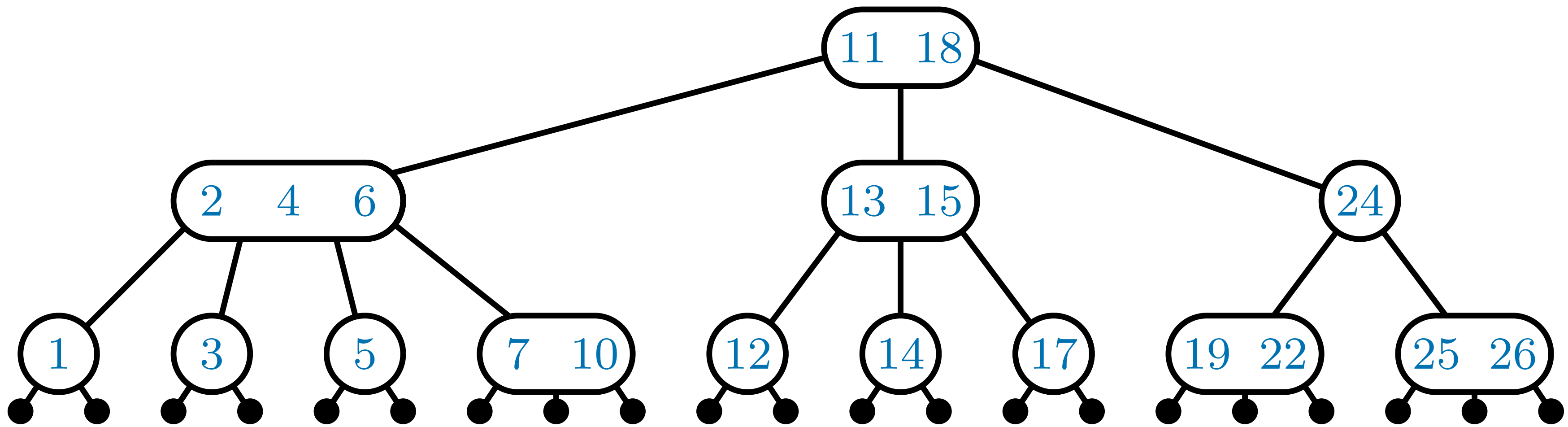

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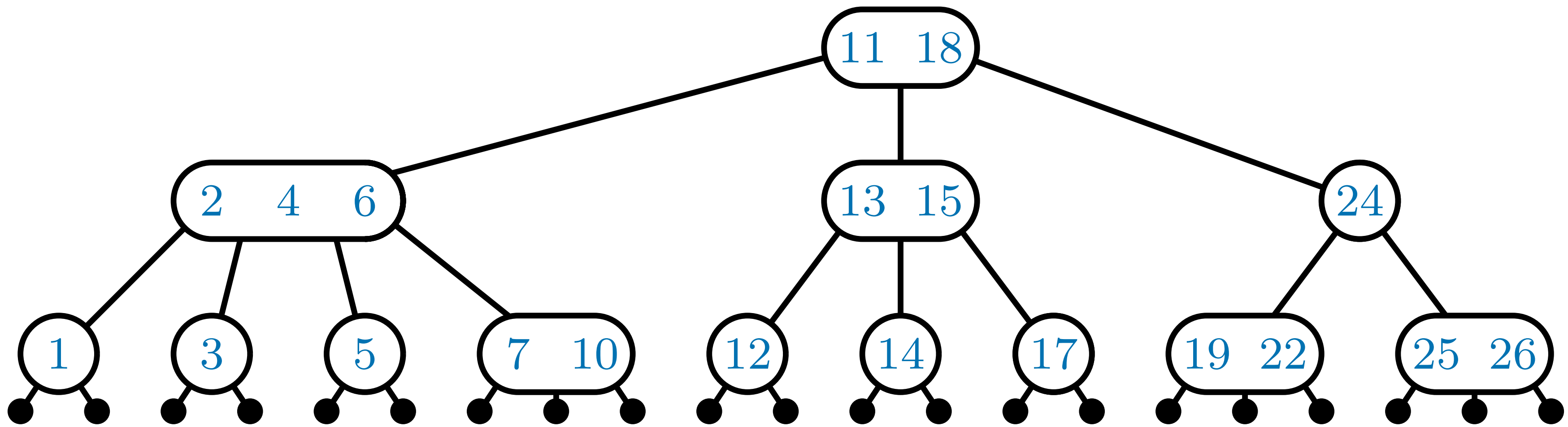

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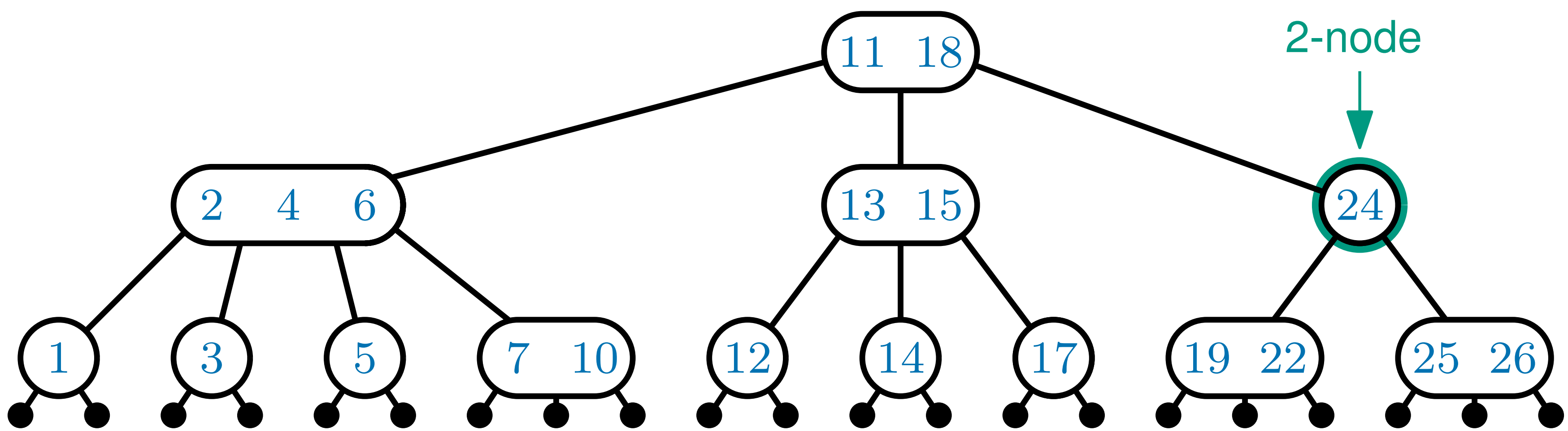

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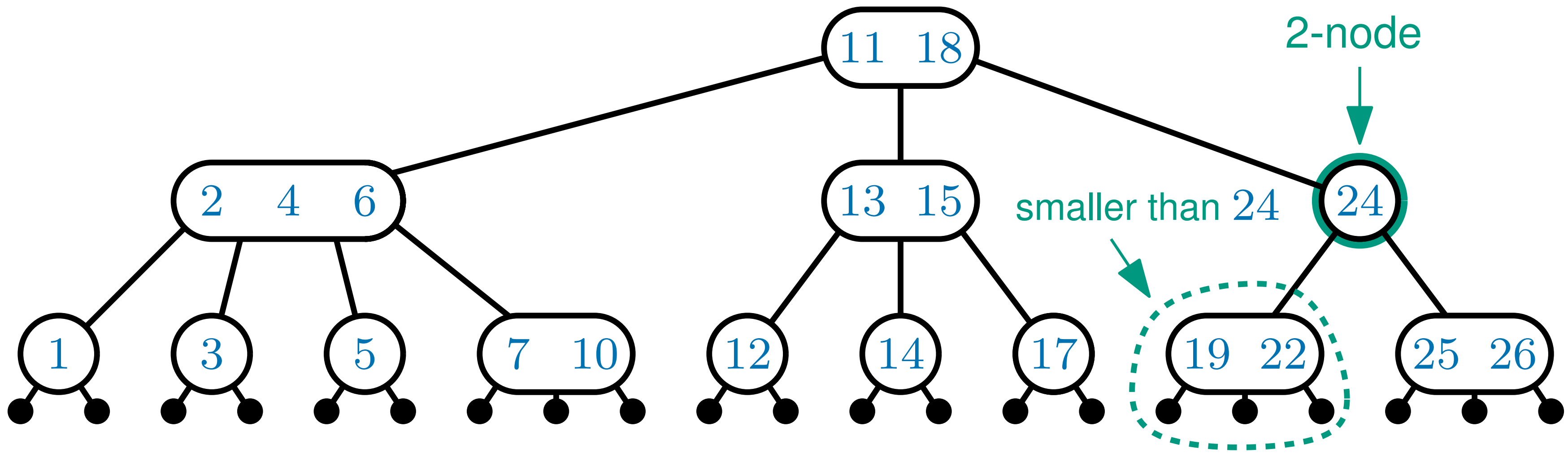

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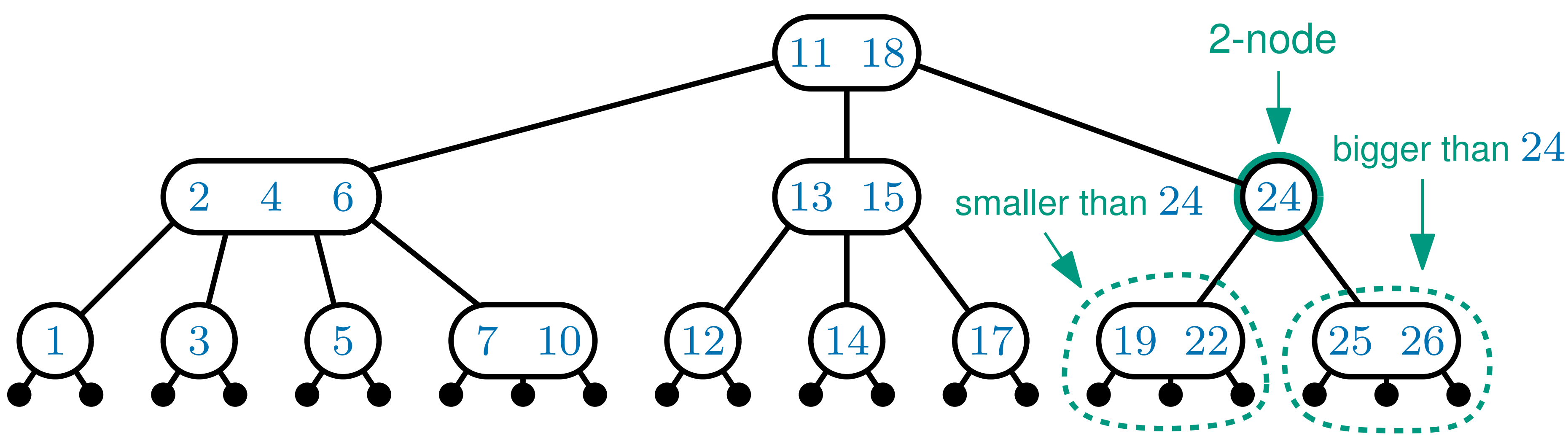

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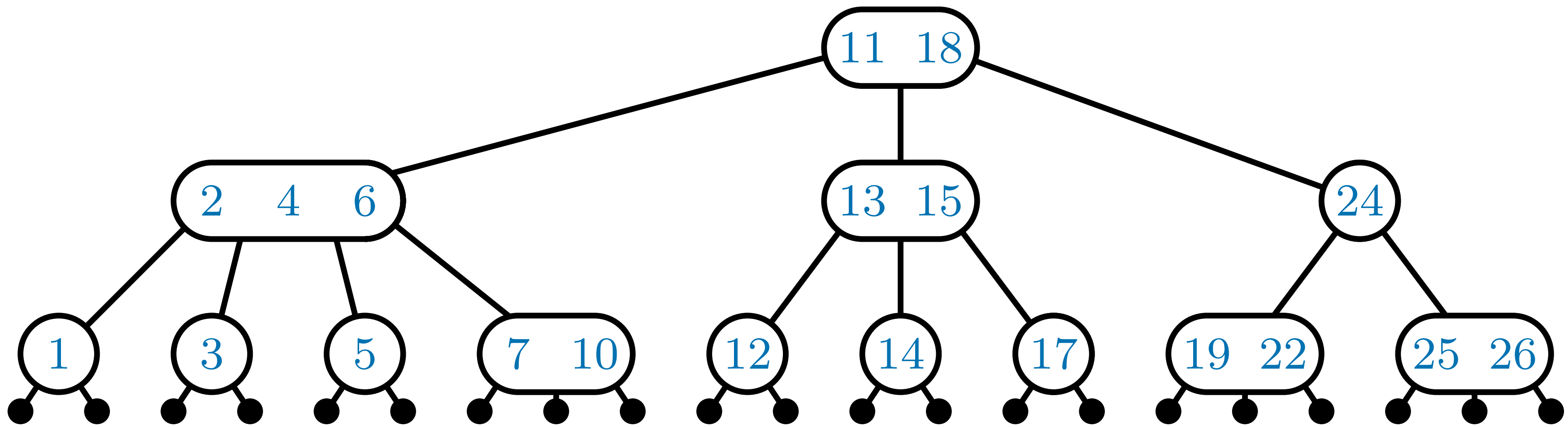

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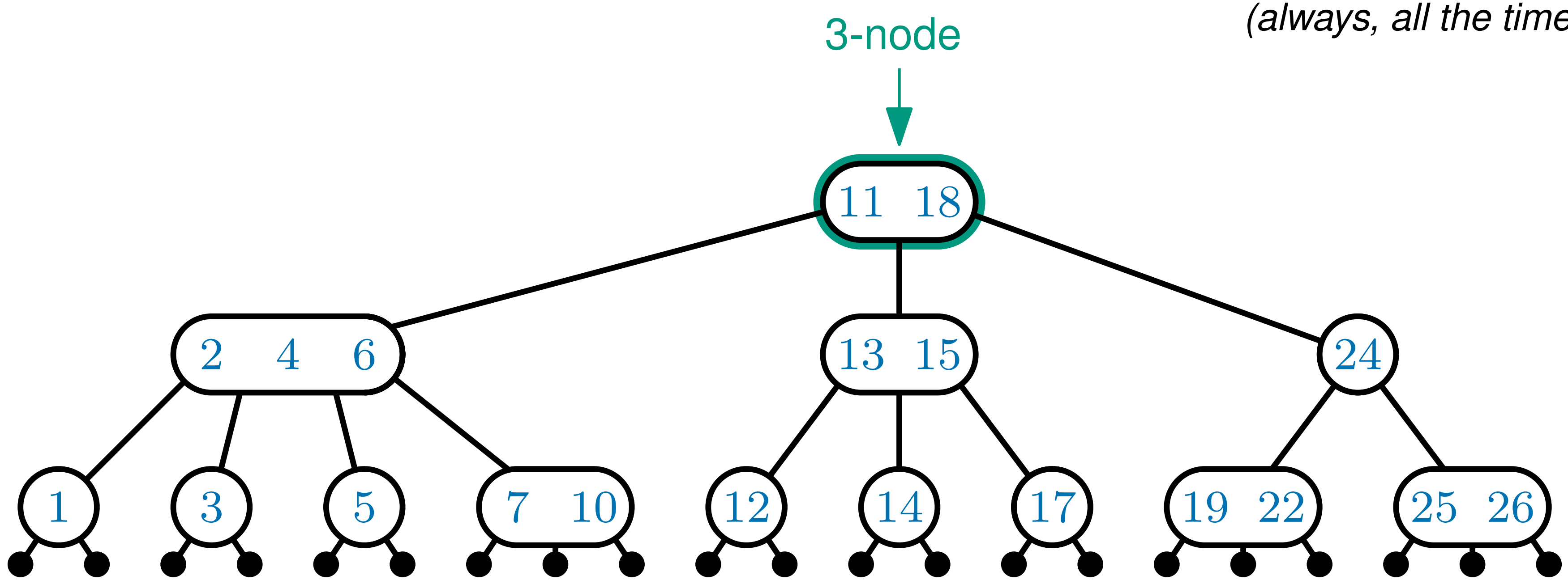

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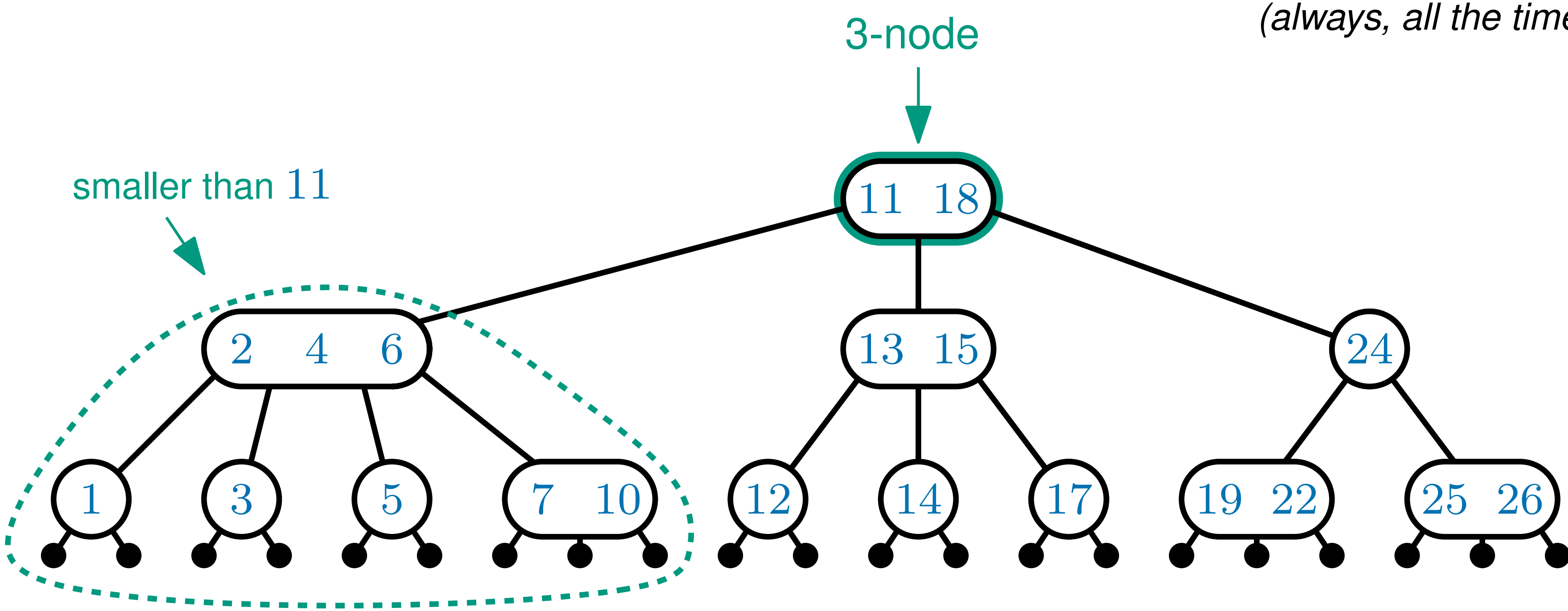

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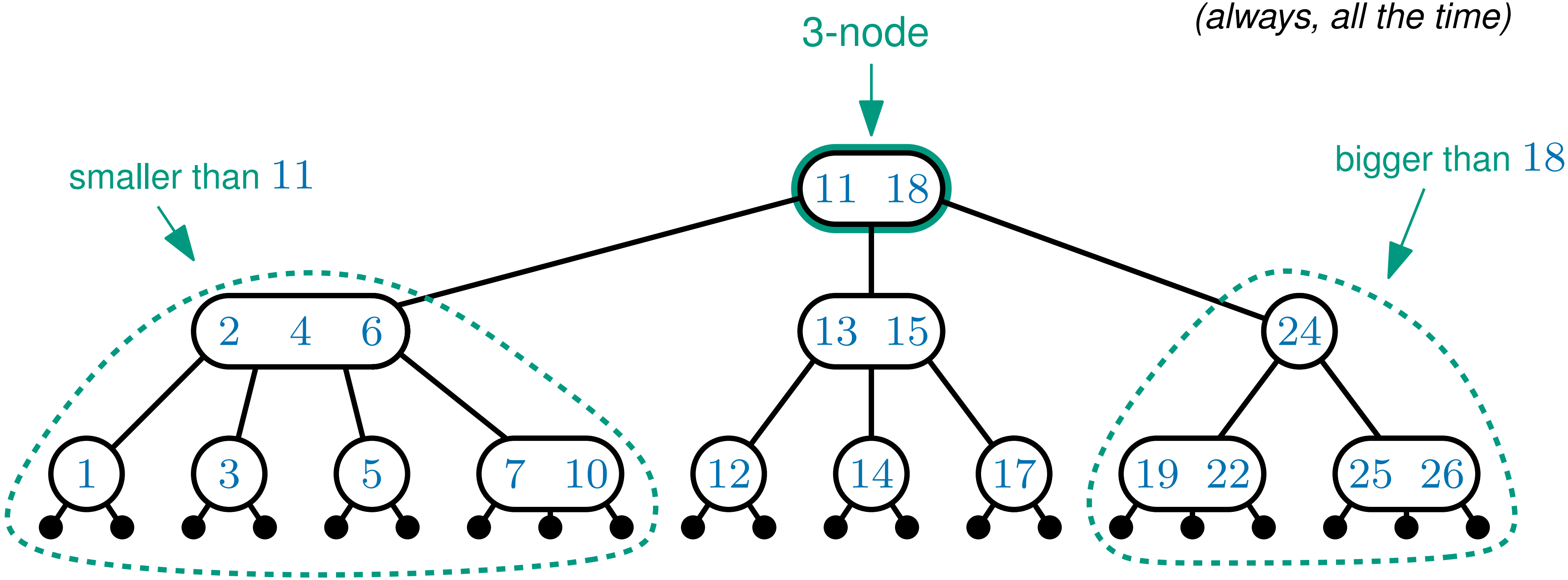

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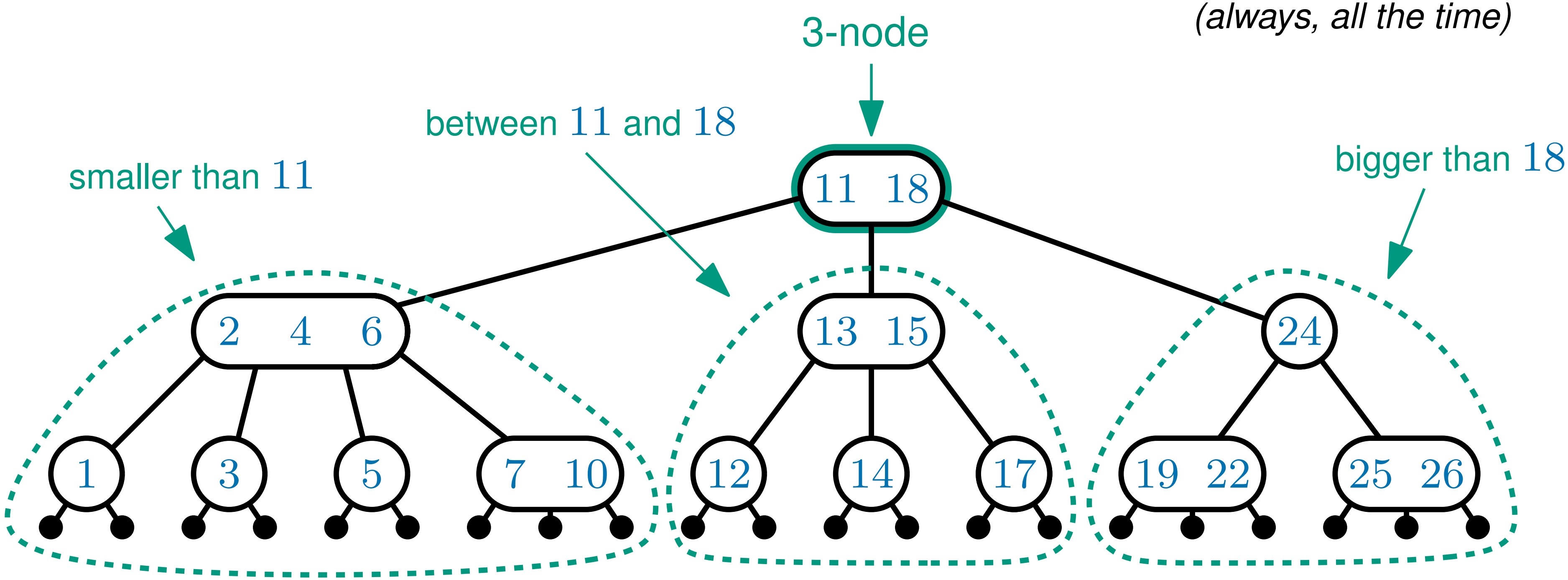

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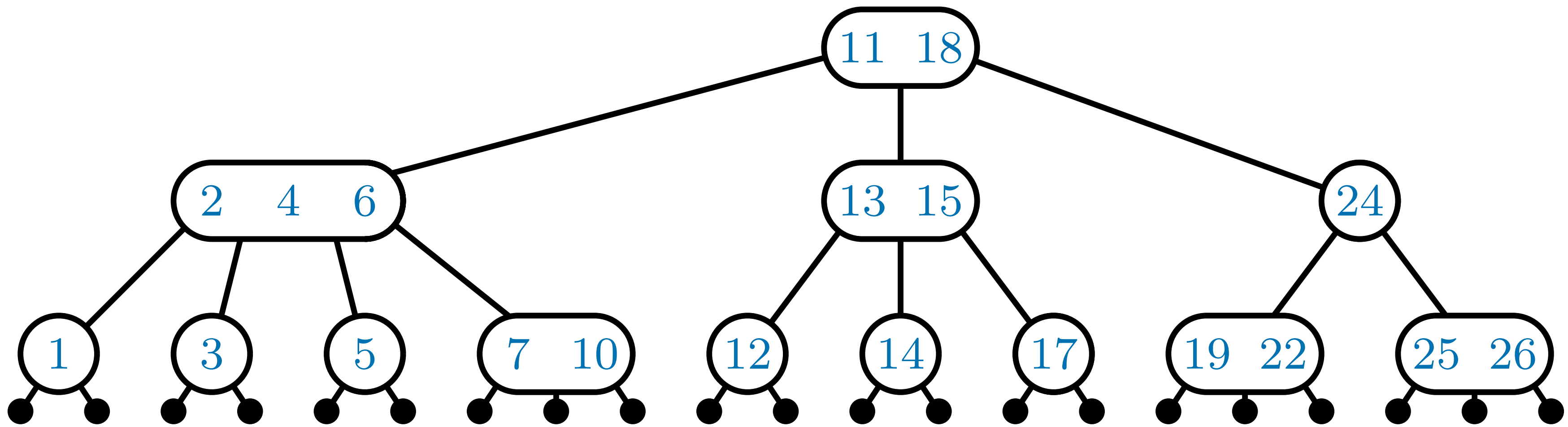

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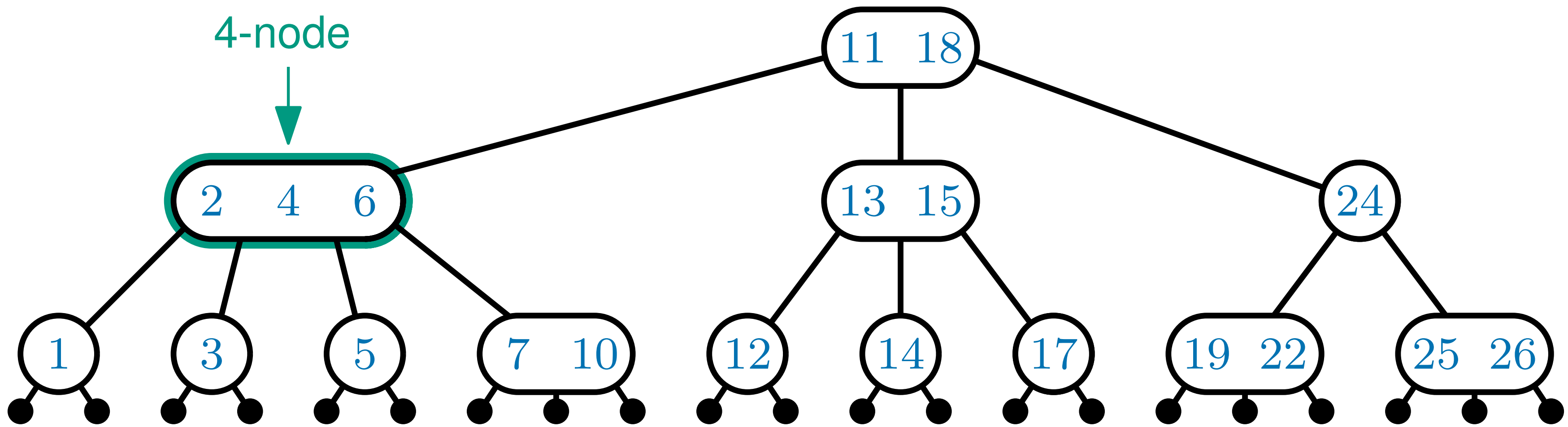

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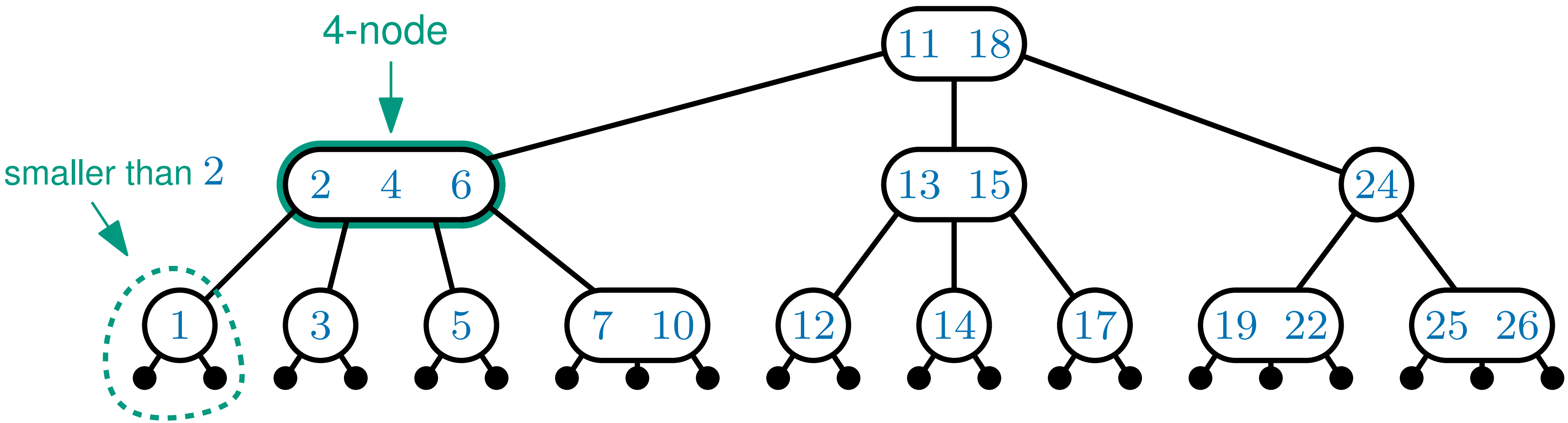

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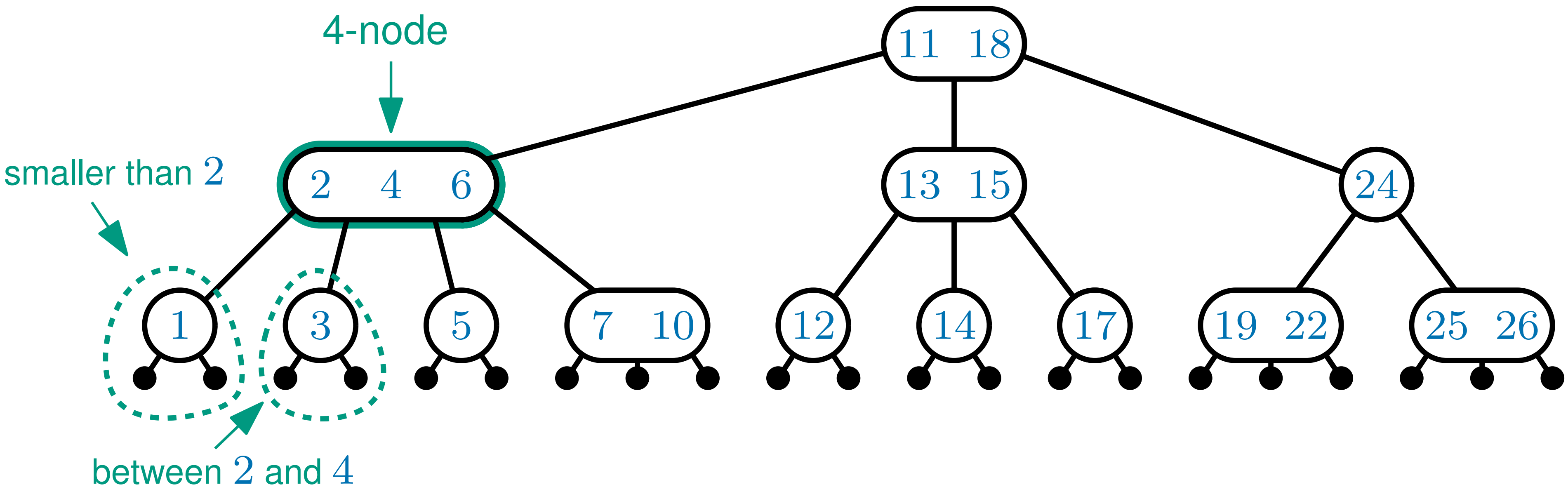

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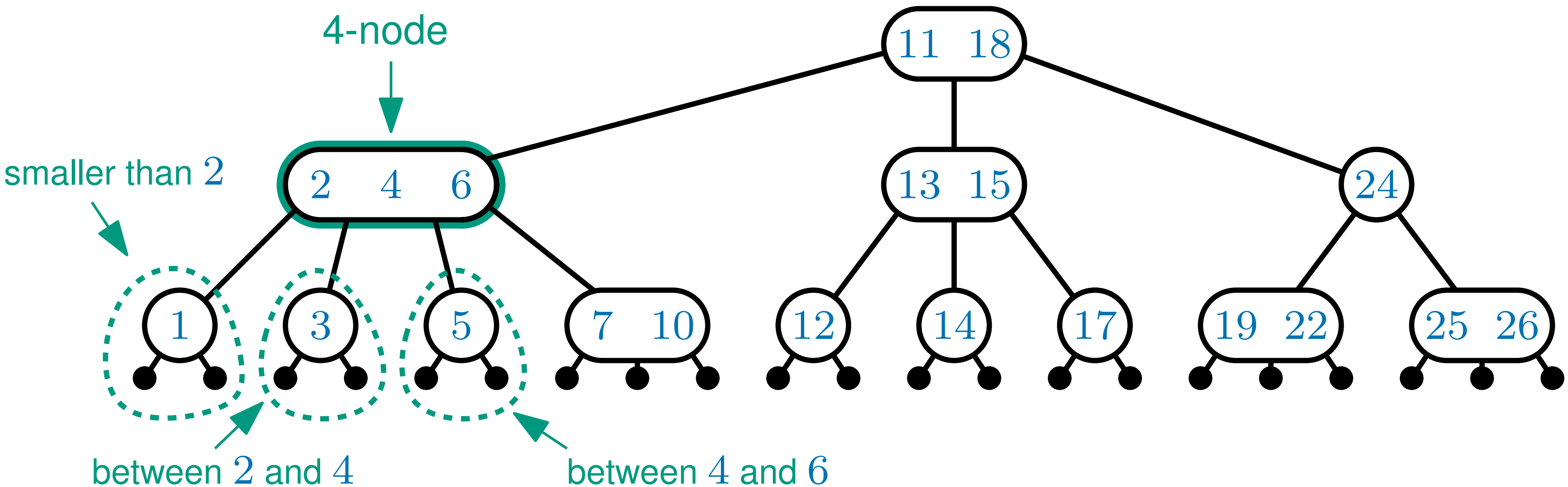

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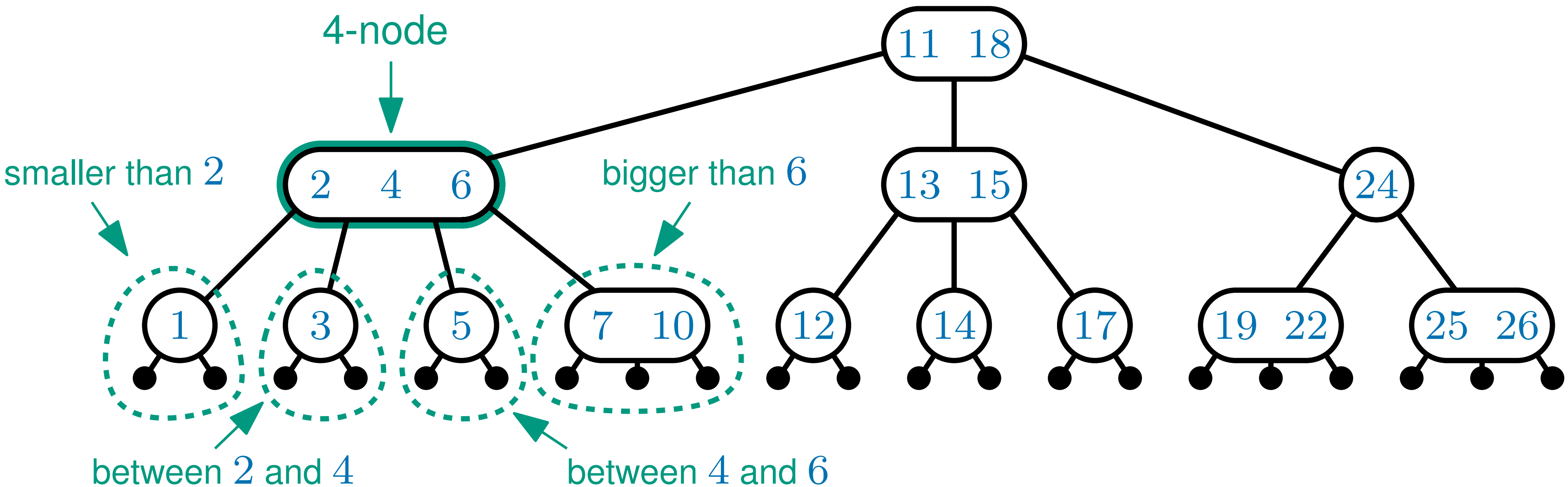

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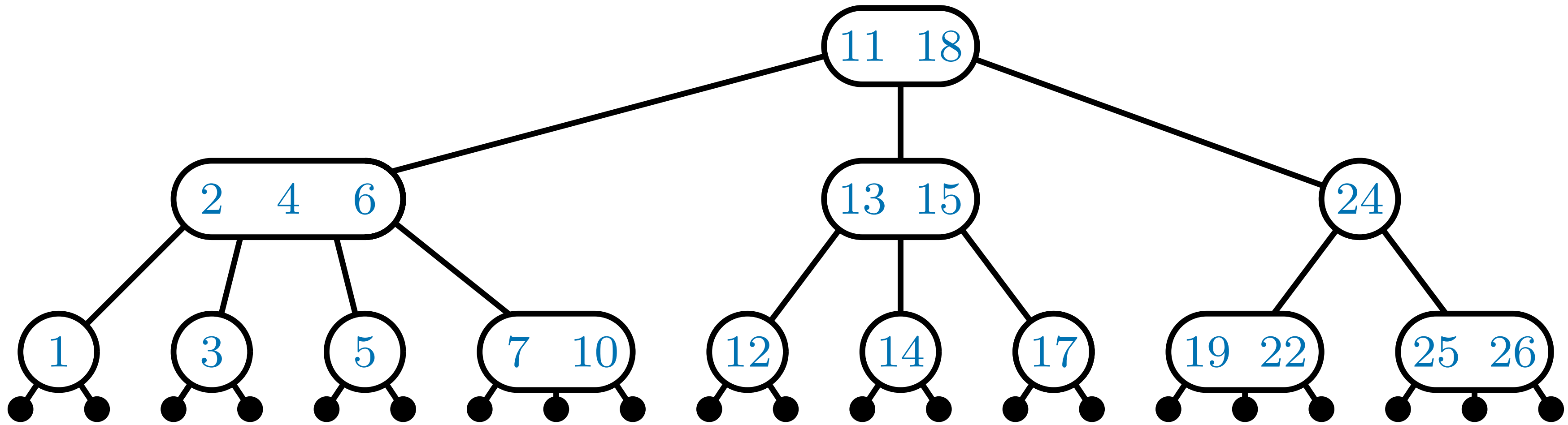

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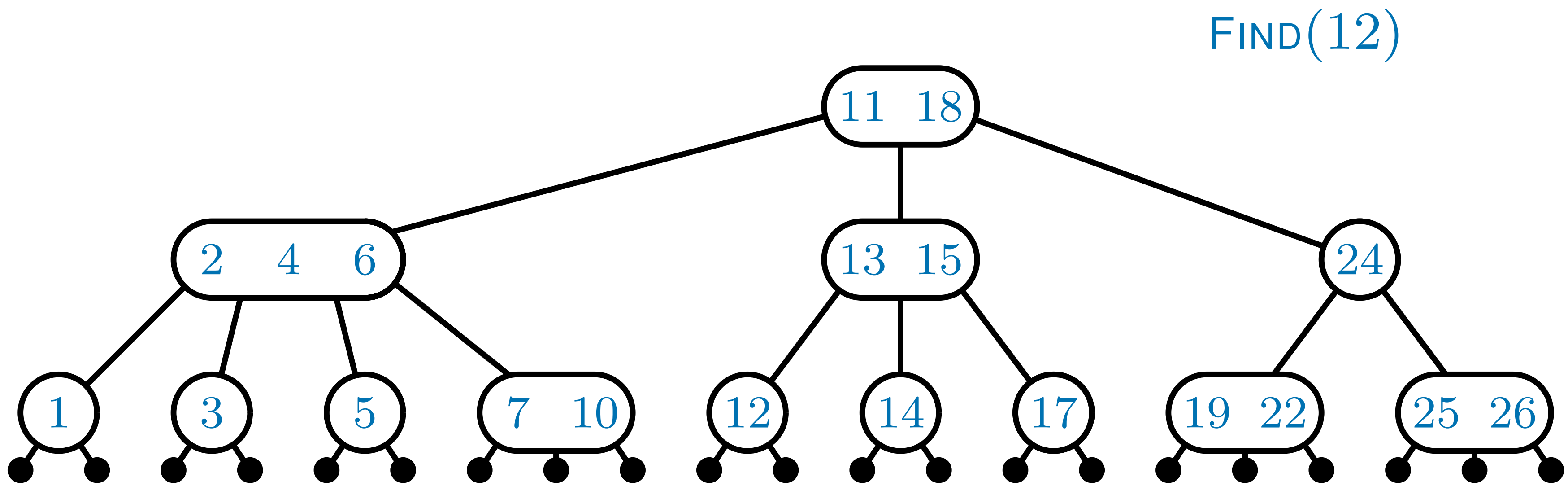

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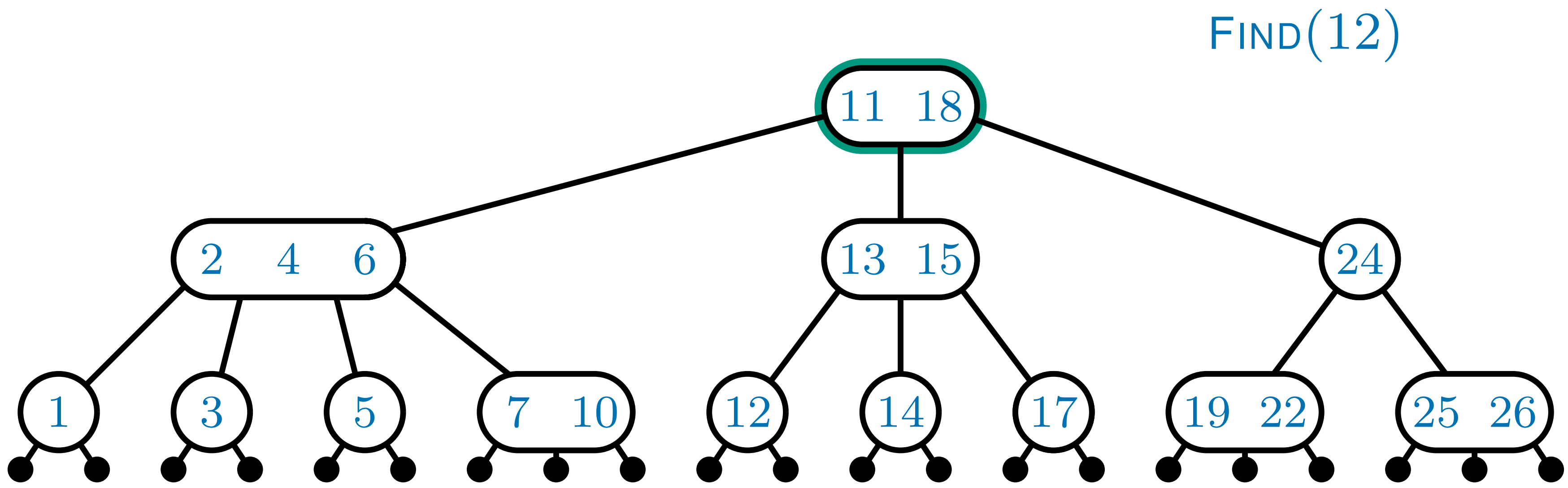

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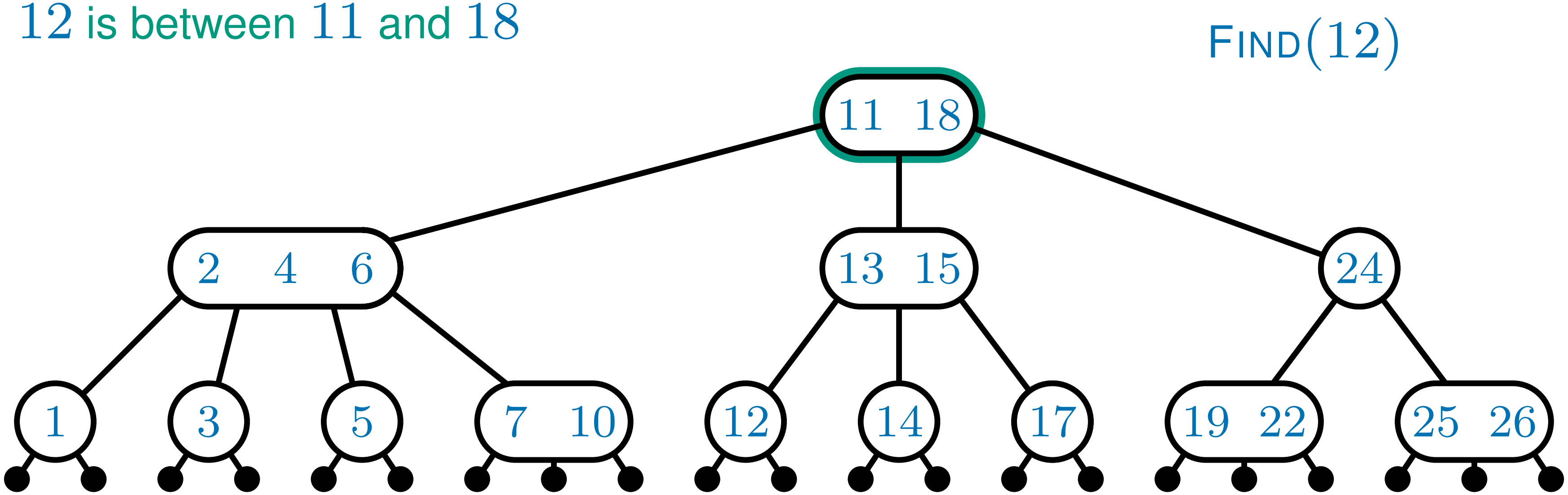

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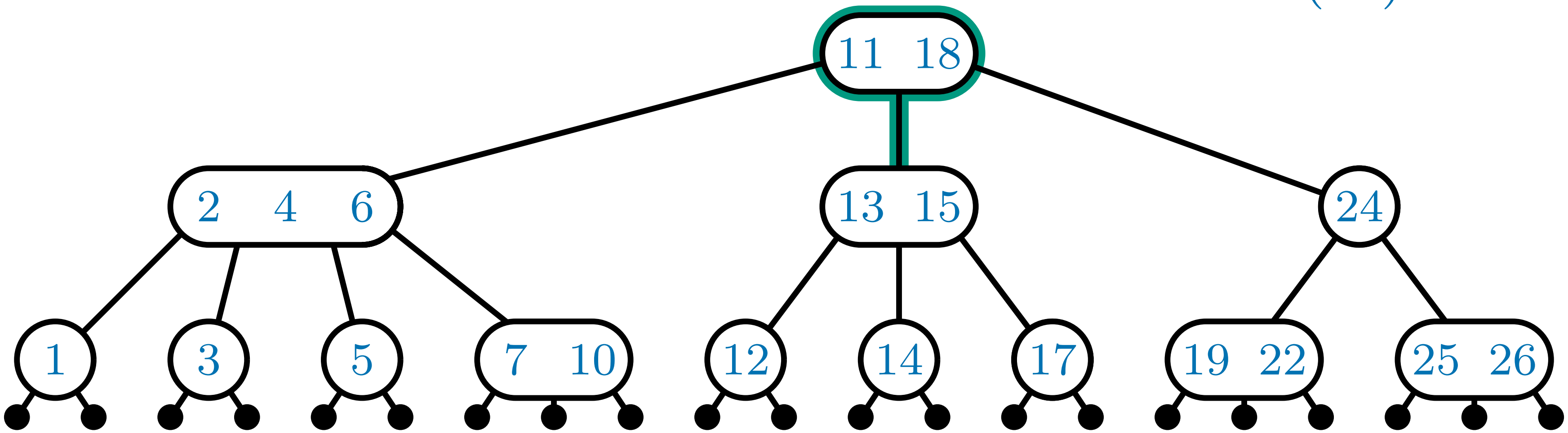

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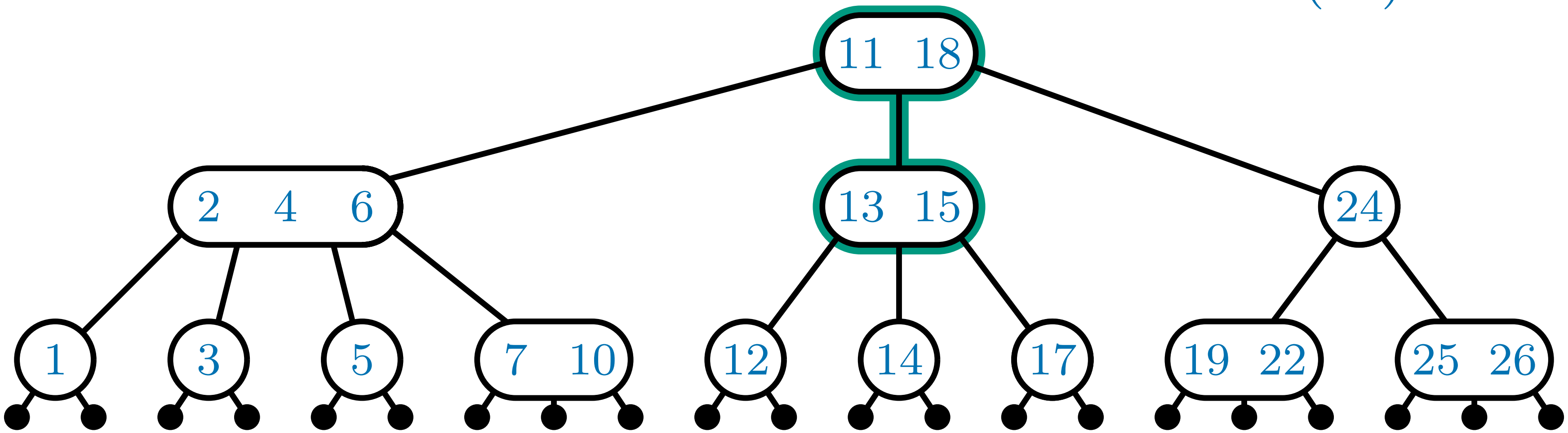

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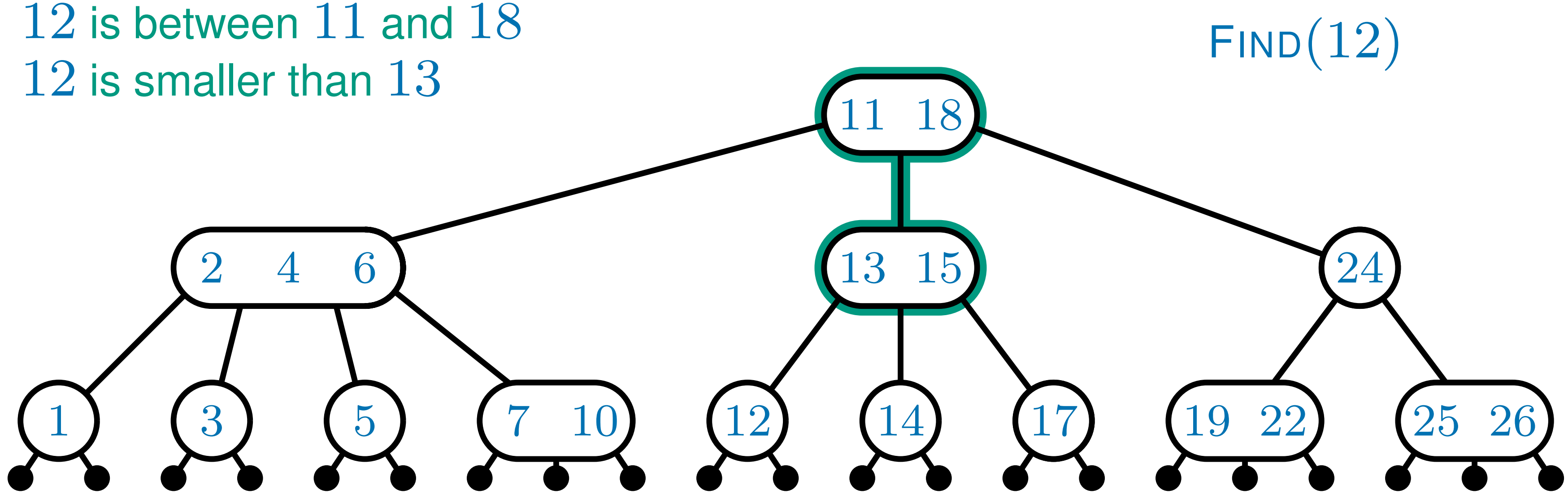
12 is between 11 and 18
12 is smaller than 13



descisions are made by inspecting the key(s) at the current node

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The FIND operation

Just like in a binary search tree,

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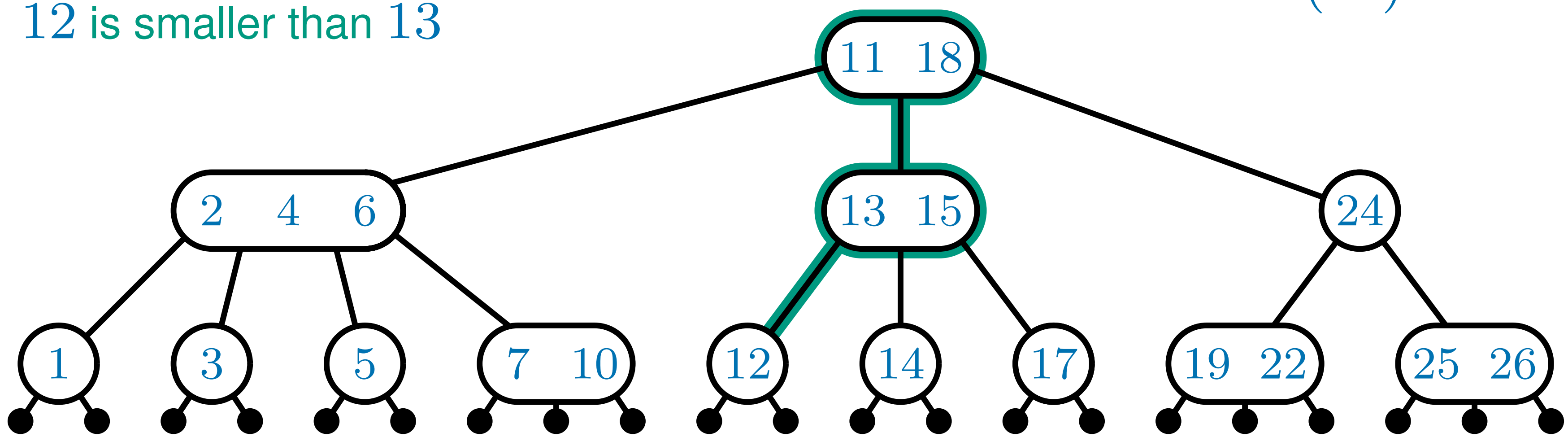
FIND(12)



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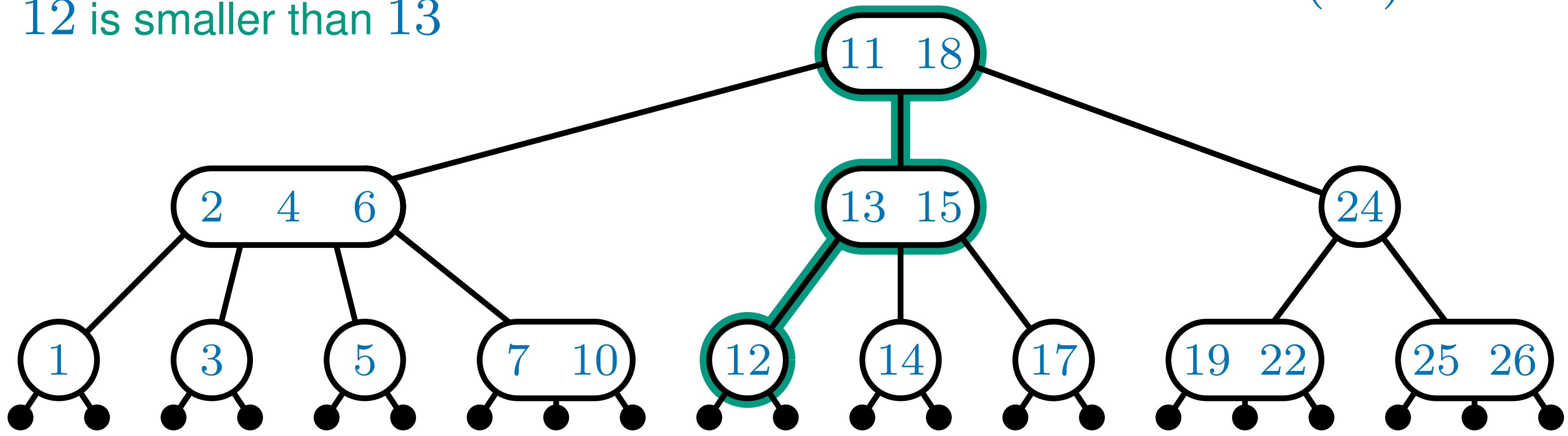

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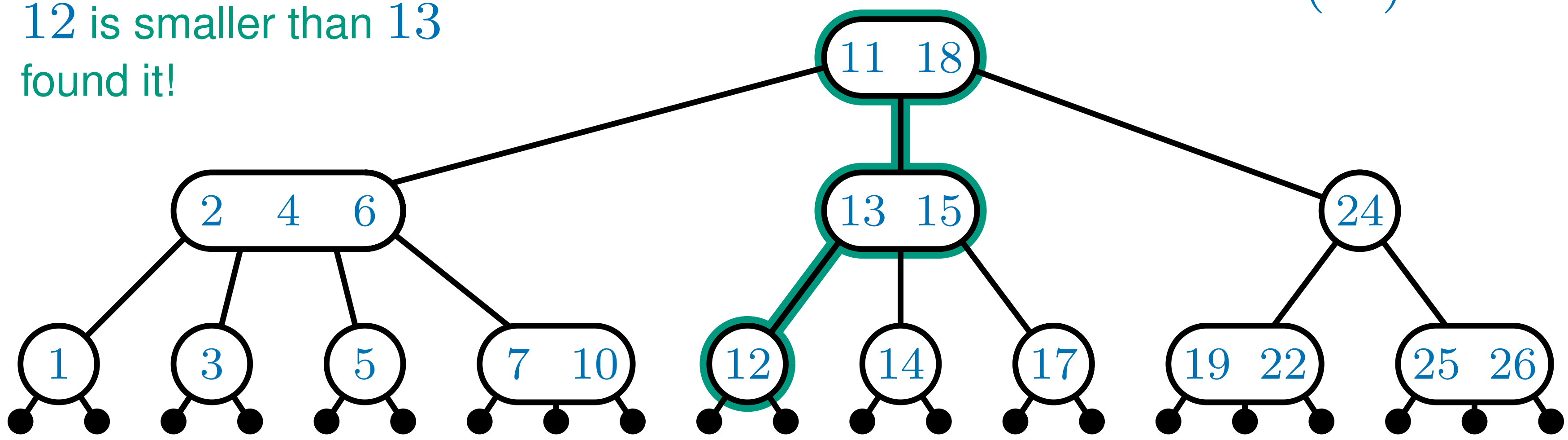

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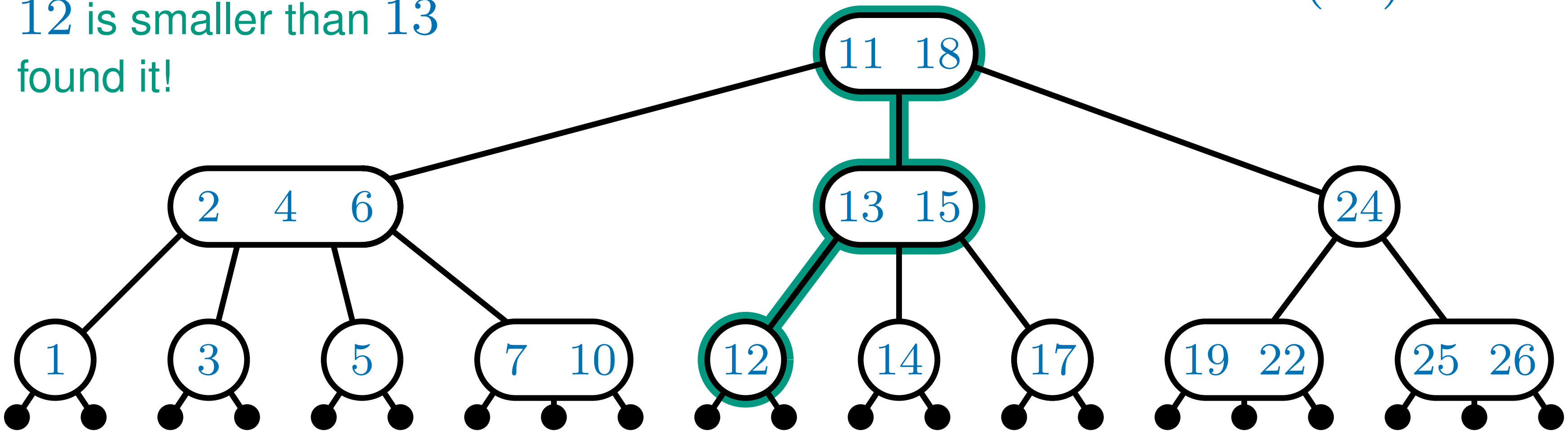

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What is the time complexity of the FIND *operation?*

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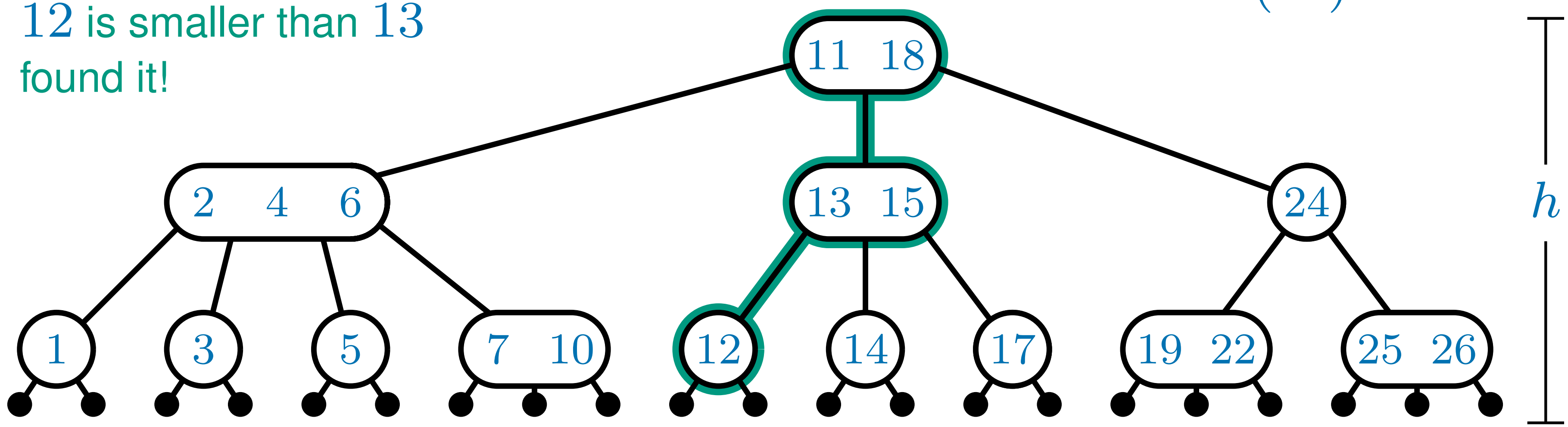

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It's $O(h)$ again

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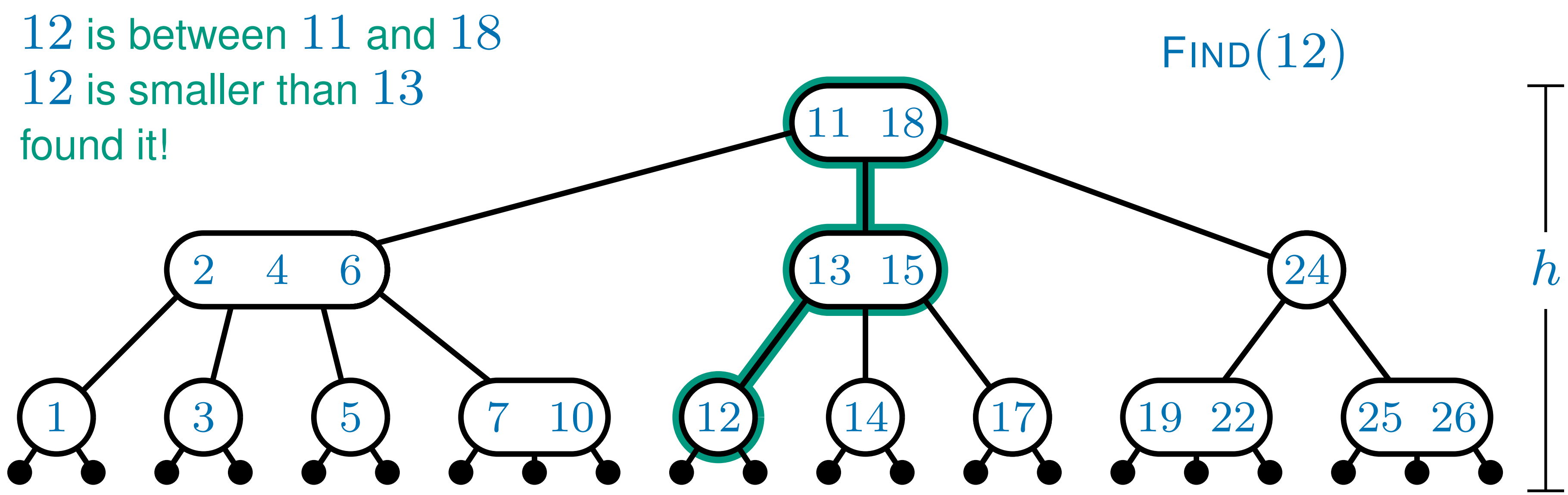

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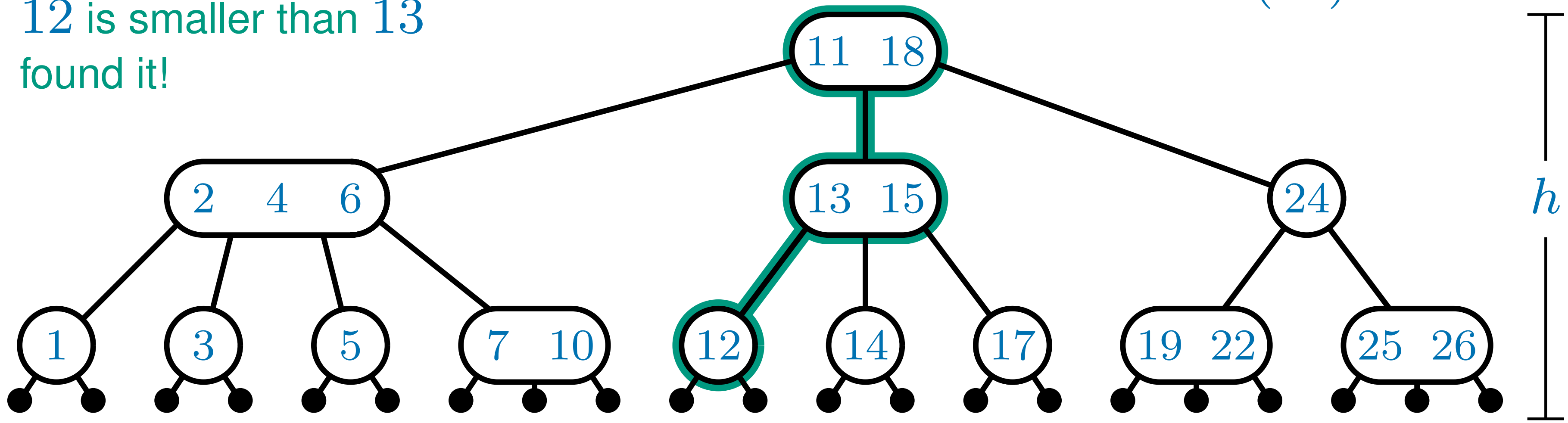

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What is the time complexity of the FIND *operation?*

It's $O(h)$ again

(each step down the path takes $O(1)$ time)

What is the height, h of a 2-3-4 tree?

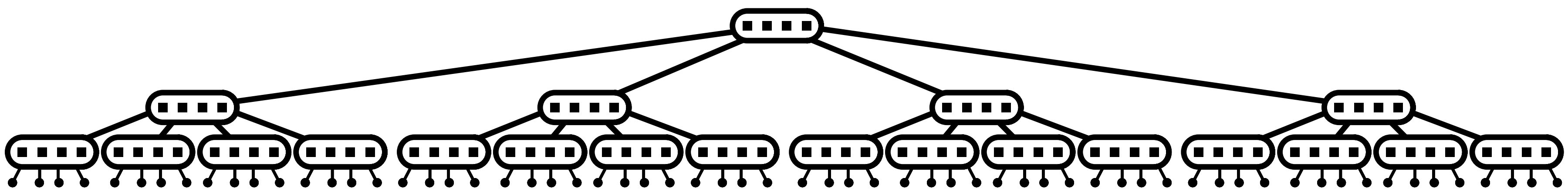
The height of a 2-3-4 tree

Perfect balance - every path from the root to a leaf has the same length

(we'll justify this as we go along)

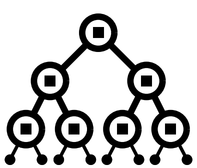
This implies that the height, h of a 2-3-4 tree with n nodes is

Best case: $\log_4 n = \frac{\log_2 n}{2}$ (all 4-nodes)



(■ is an element)

Worst case: $\log_2 n$ (all 2-nodes)



h is between 10 and 20 for a million nodes

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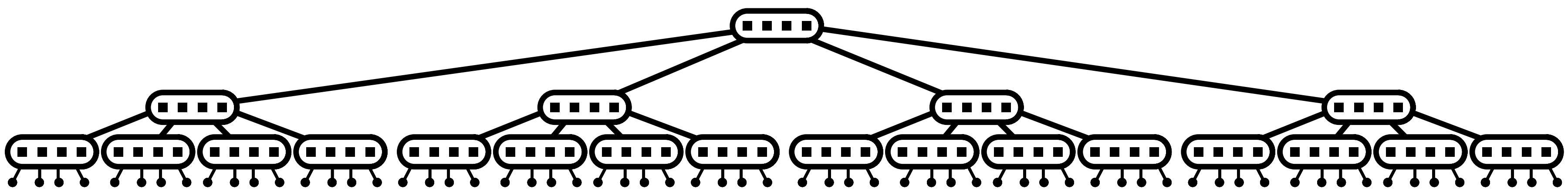

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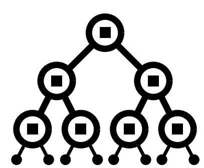

h is between 10 and 20 for a million nodes

The time complexity of the FIND operation is $O(h)$

The height of a 2-3-4 tree

Perfect balance - every path from the root to a leaf has the same length

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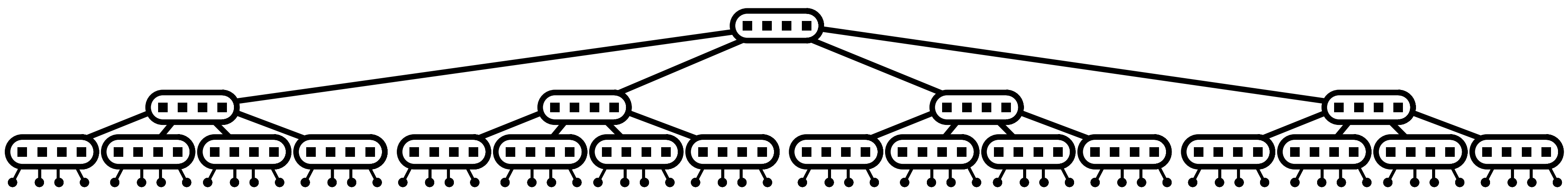

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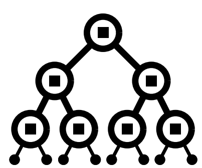
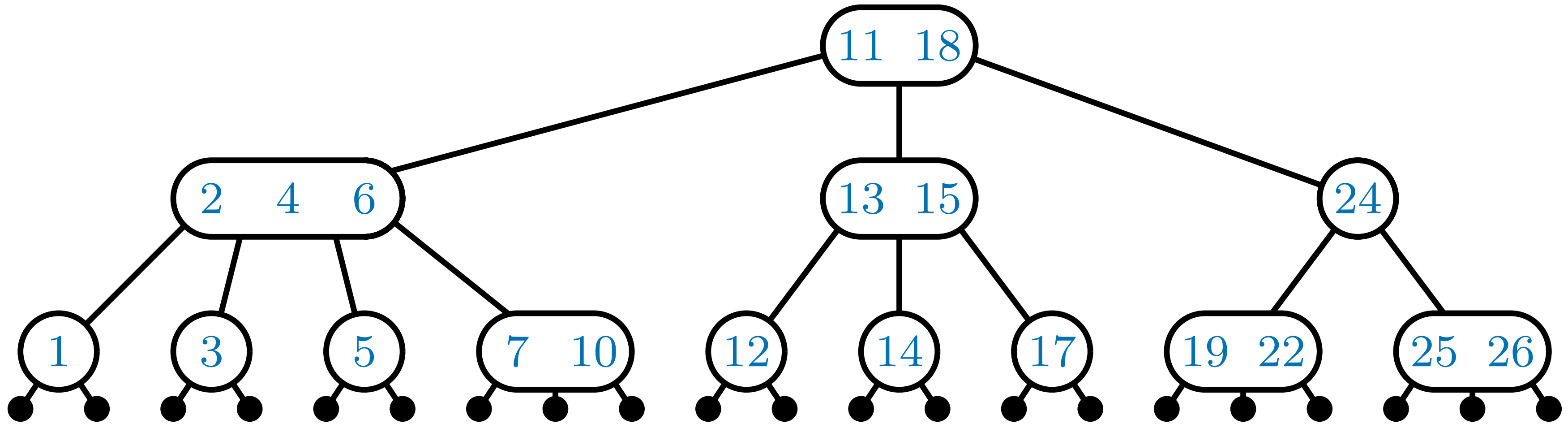


h is between 10 and 20 for a million nodes

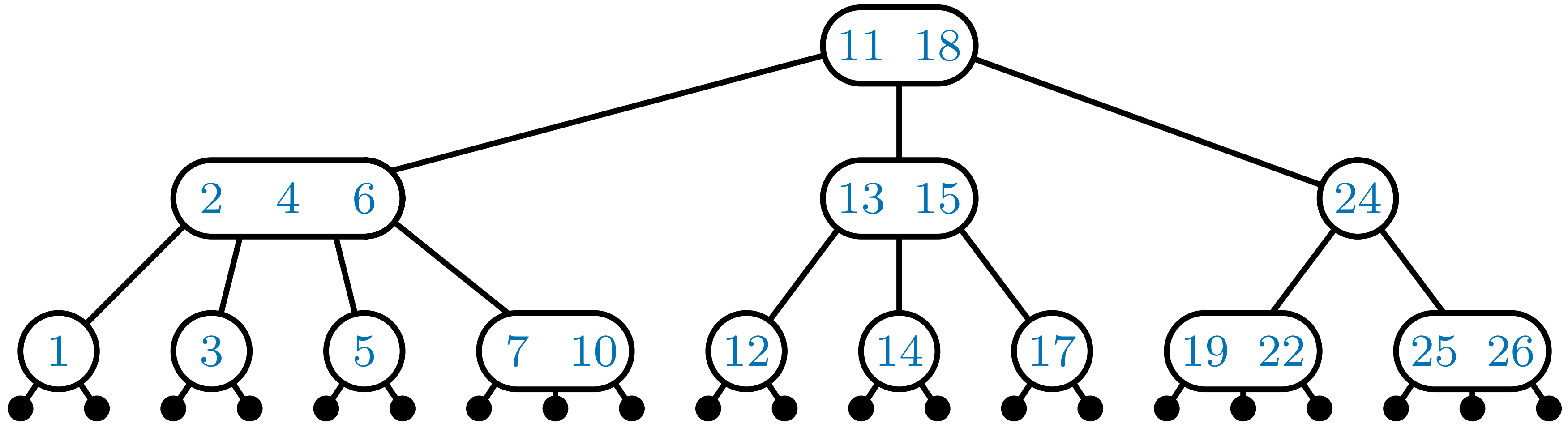
The time complexity of the FIND operation is $O(h) = O(\log n)$

The INSERT operation



To perform INSERT(x, k),

The INSERT operation



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Step 1: Search for the key k as if performing $\textsc{Find}(k)$.

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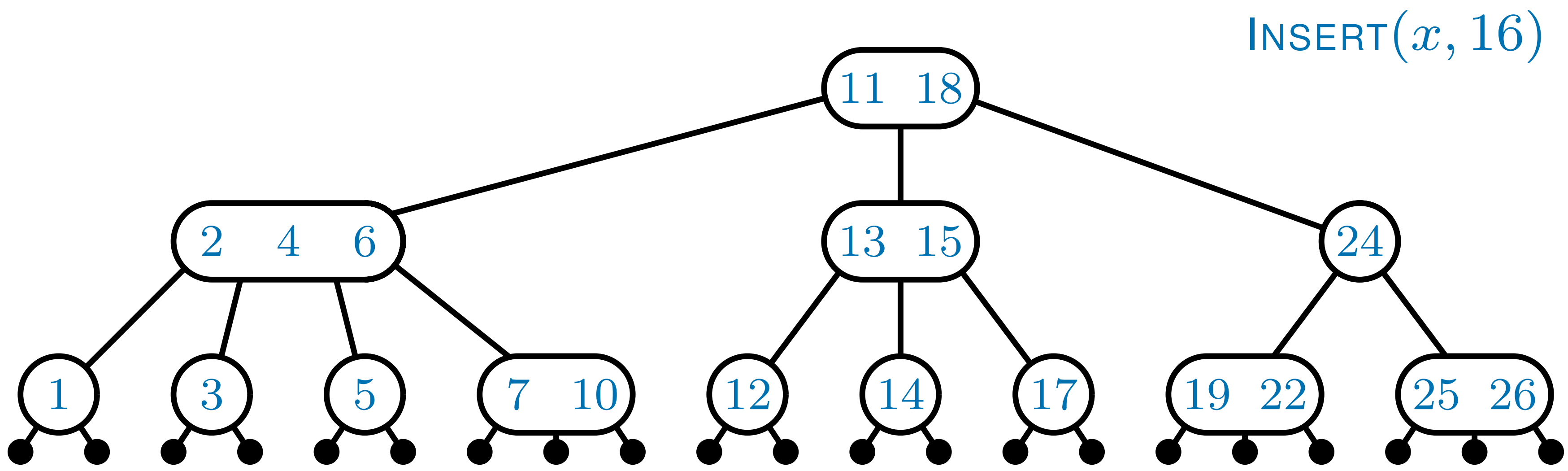

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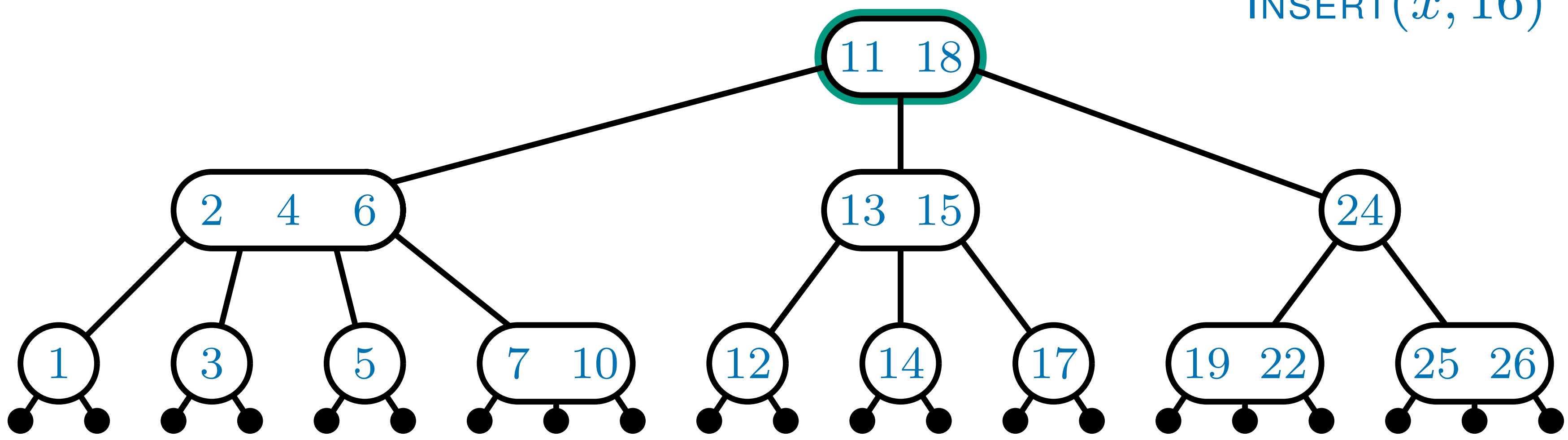

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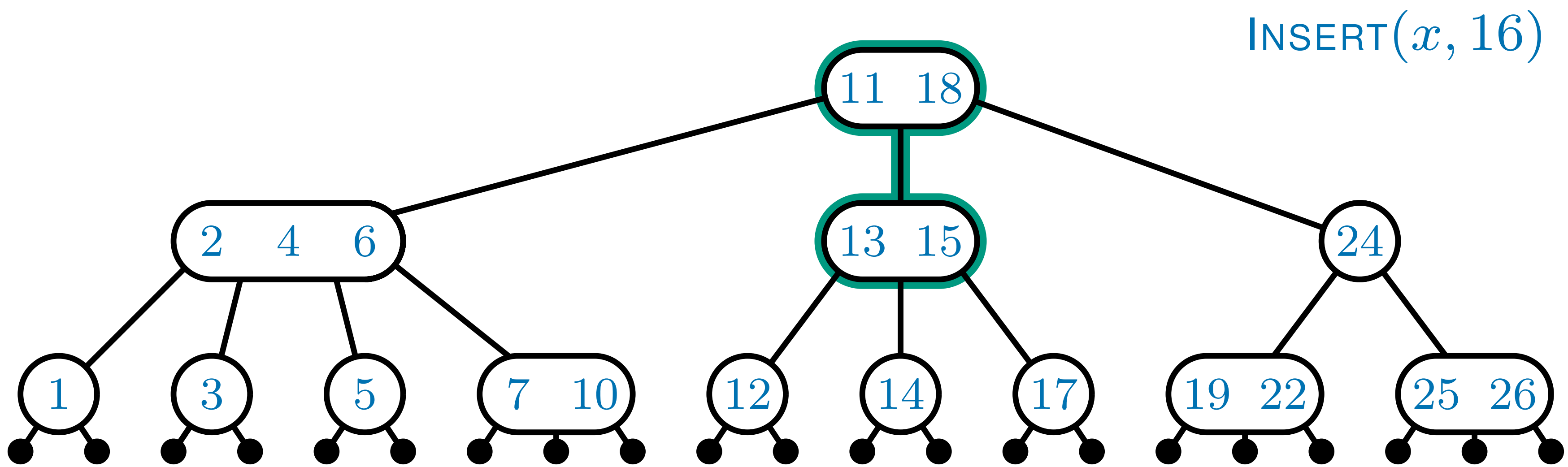

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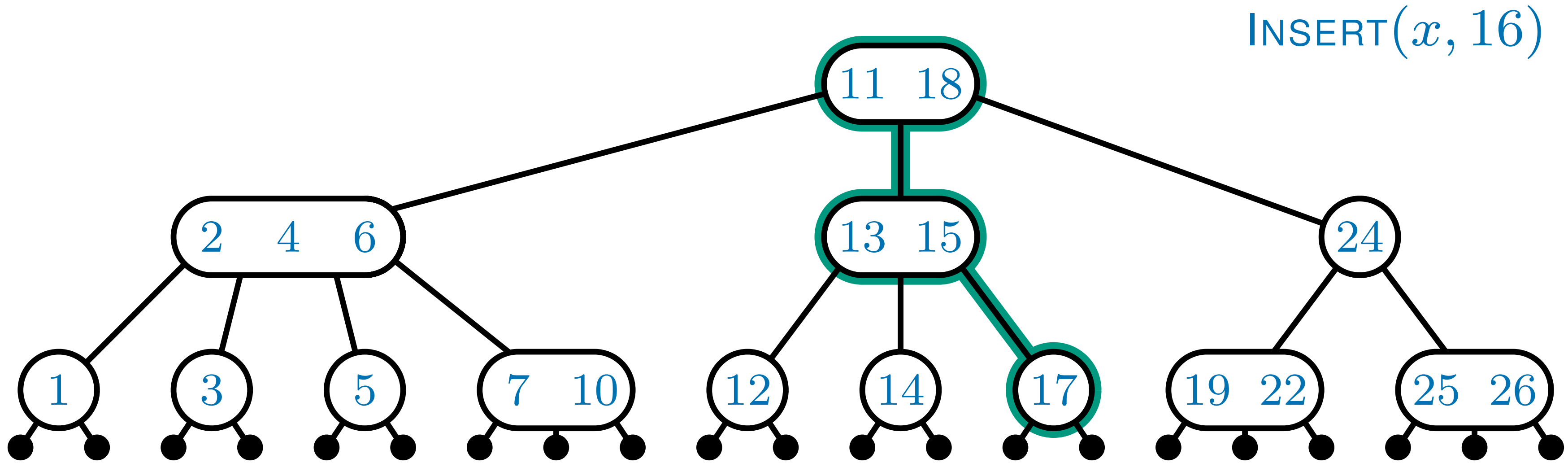

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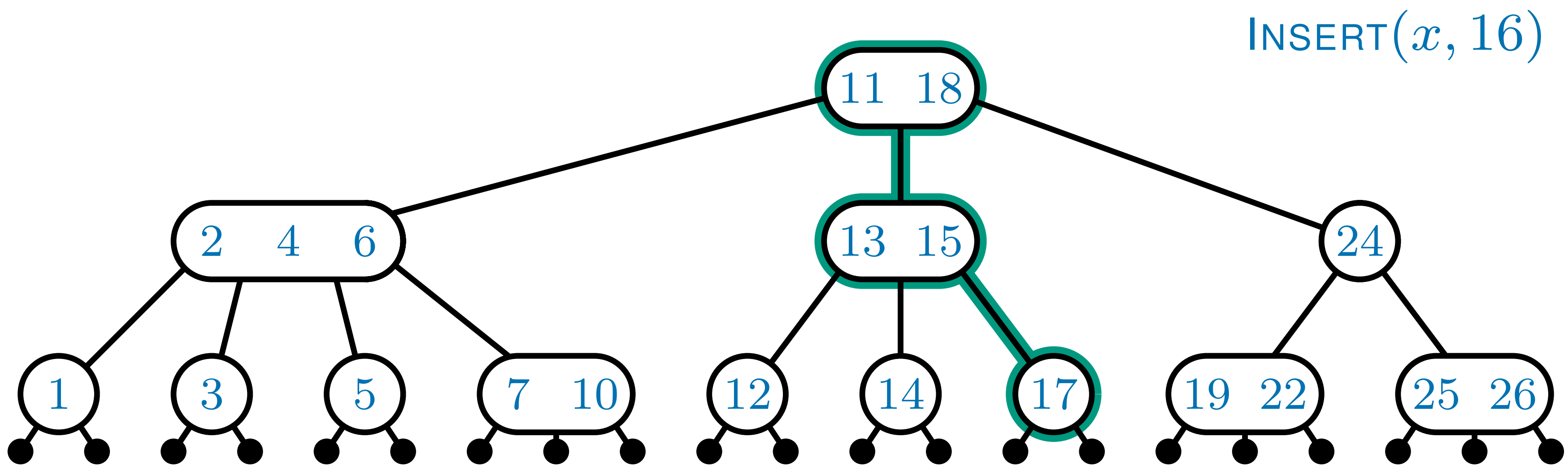

To perform INSERT(x, k),

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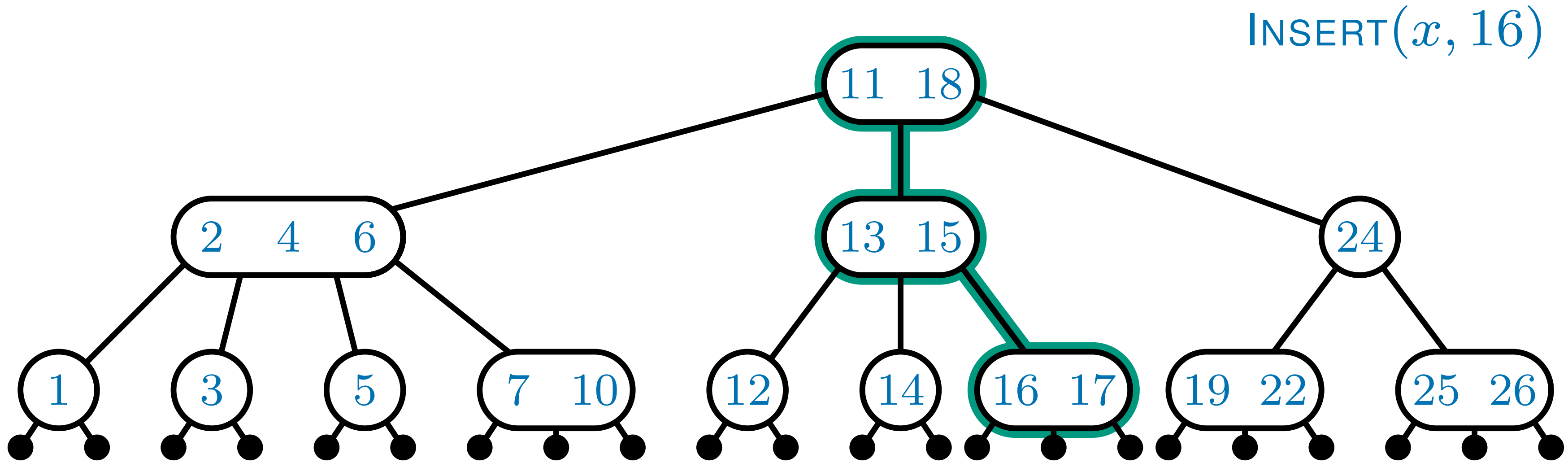

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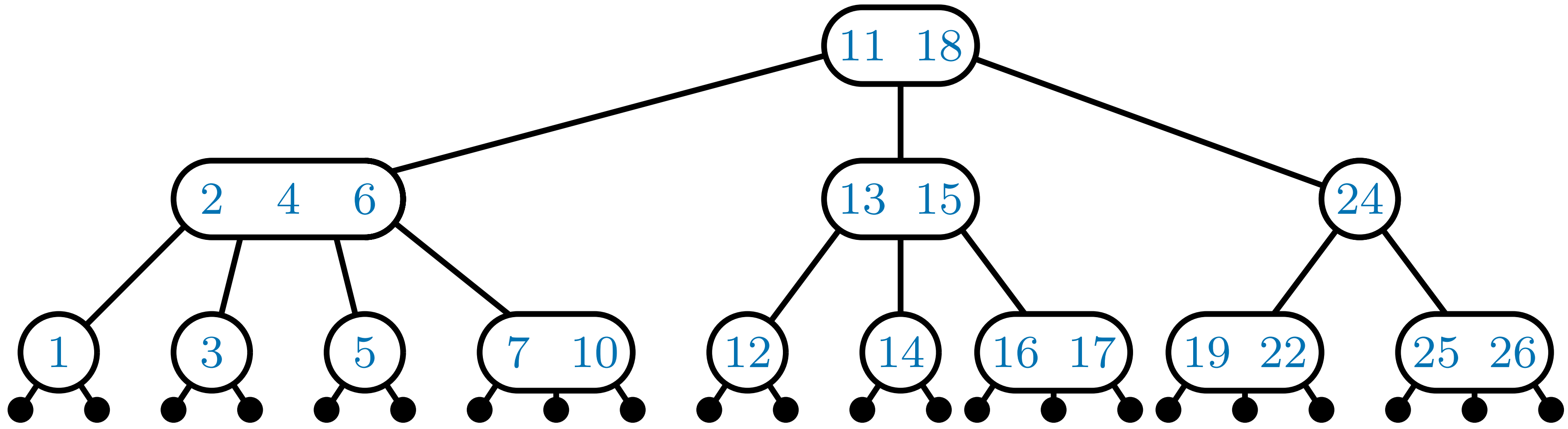

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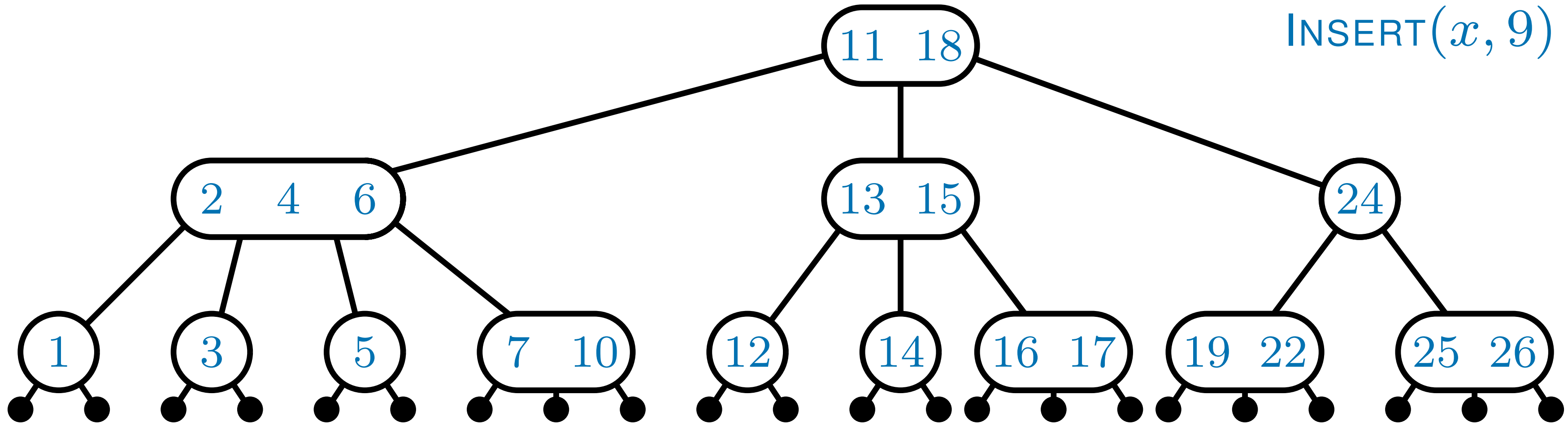

INSERT$(x, 9)$

To perform INSERT(x, k),

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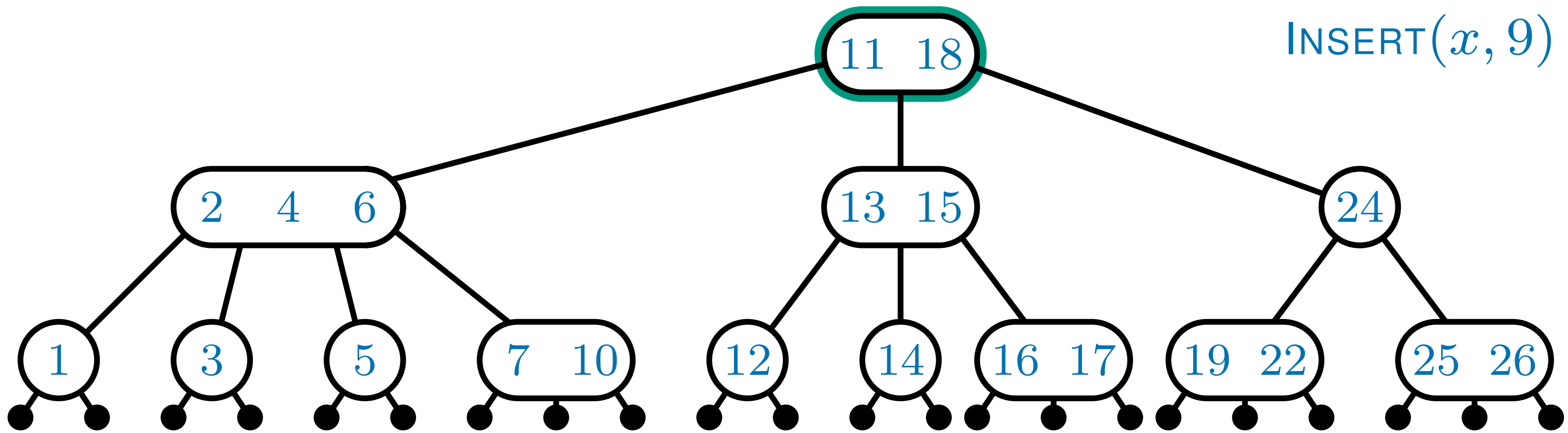
The INSERT operation



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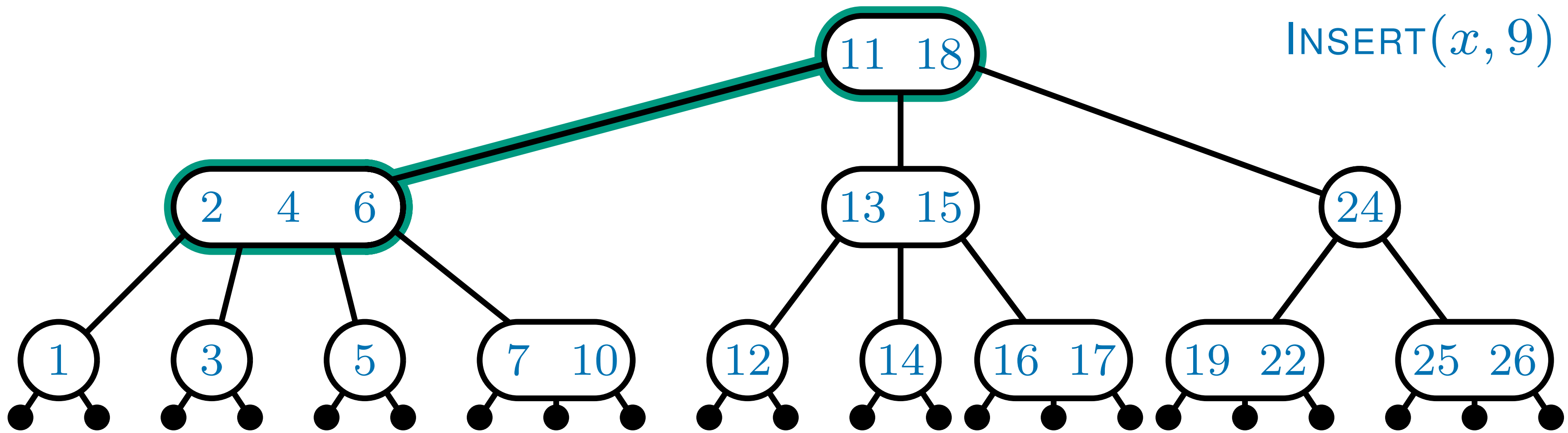

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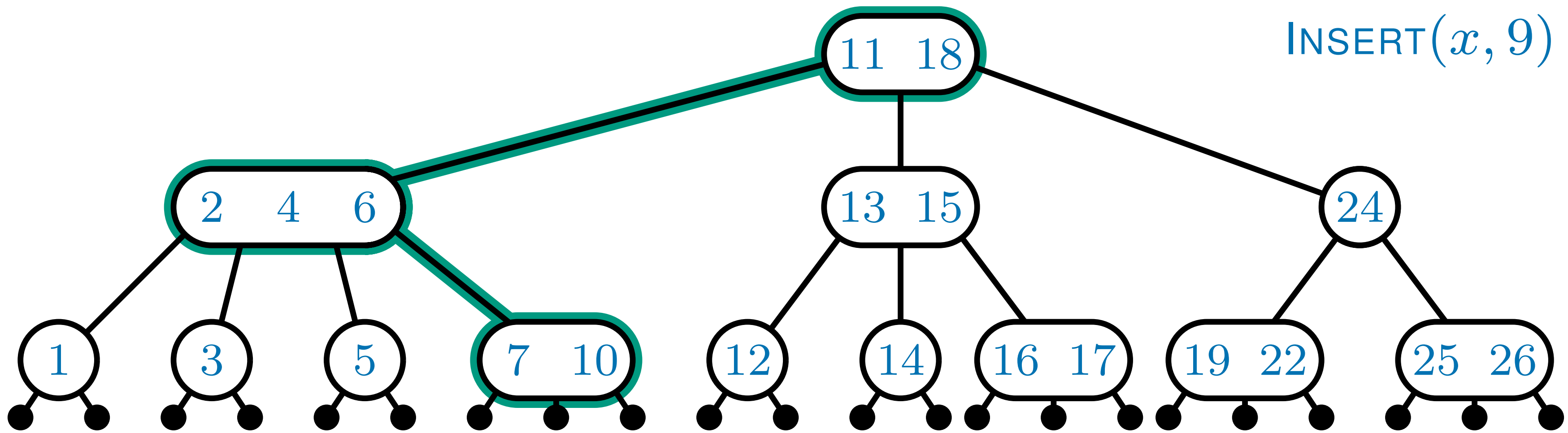

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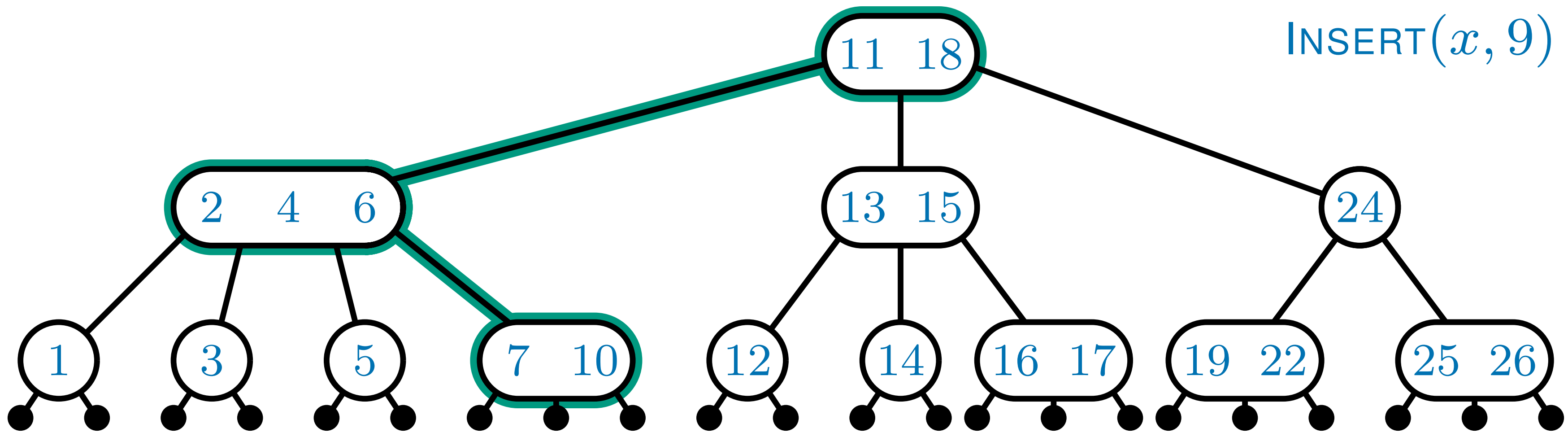

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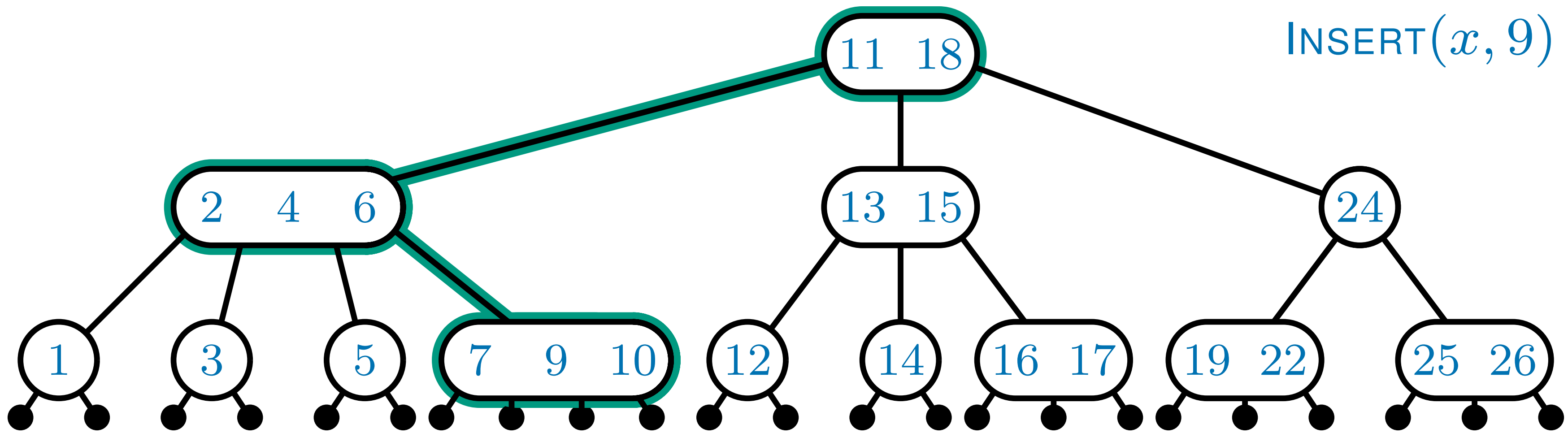

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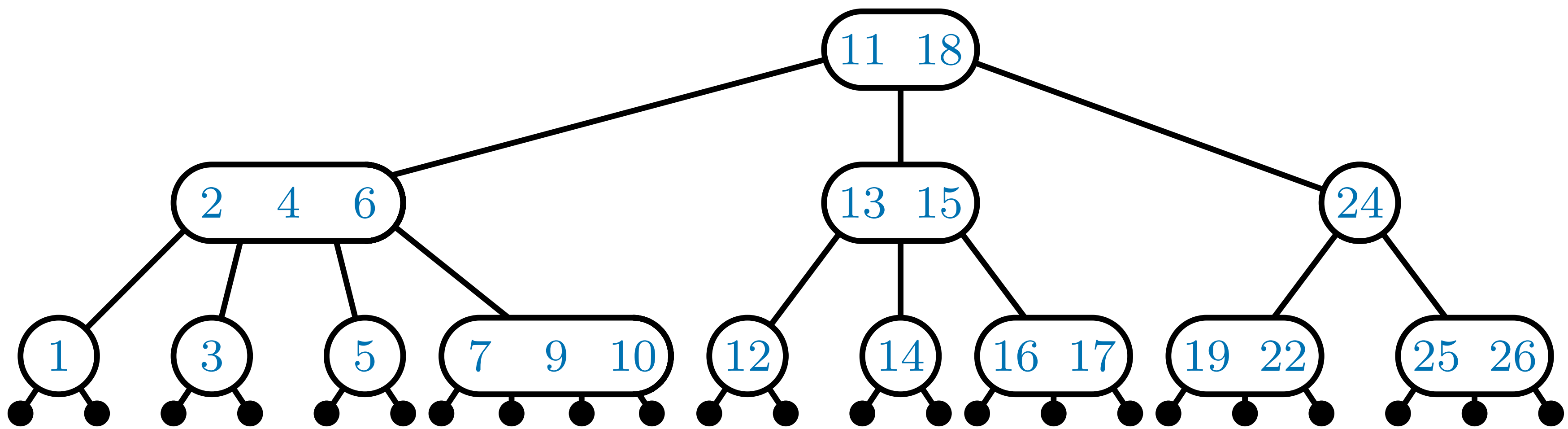

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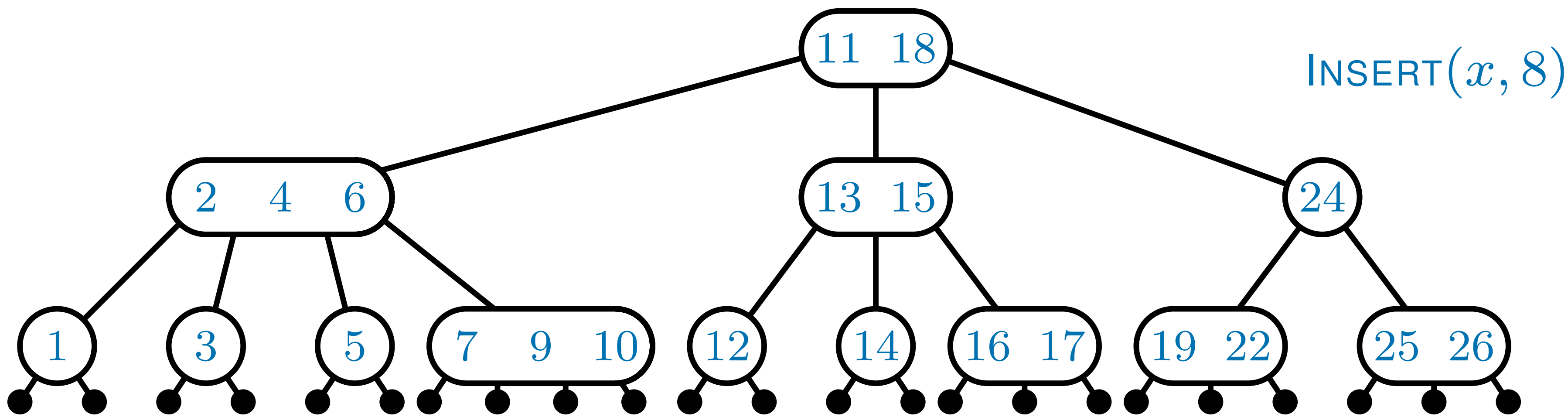

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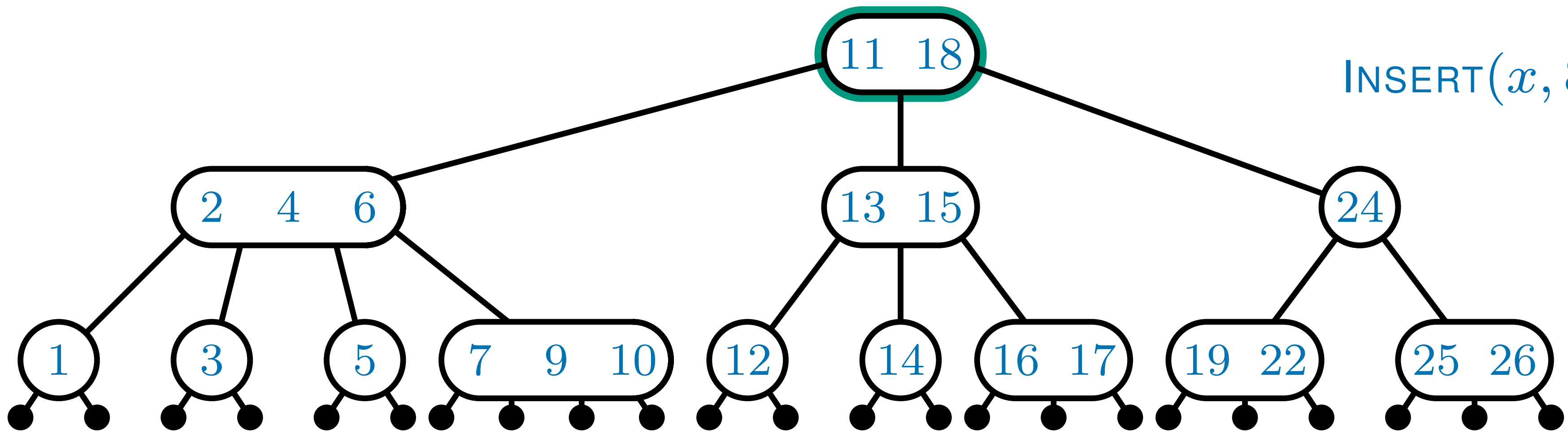

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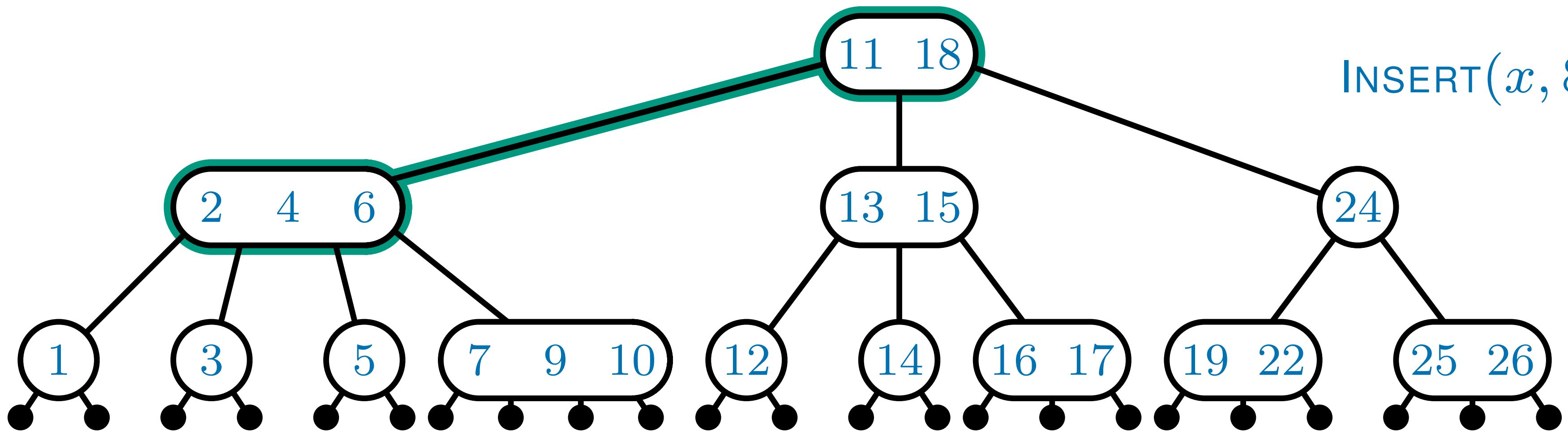

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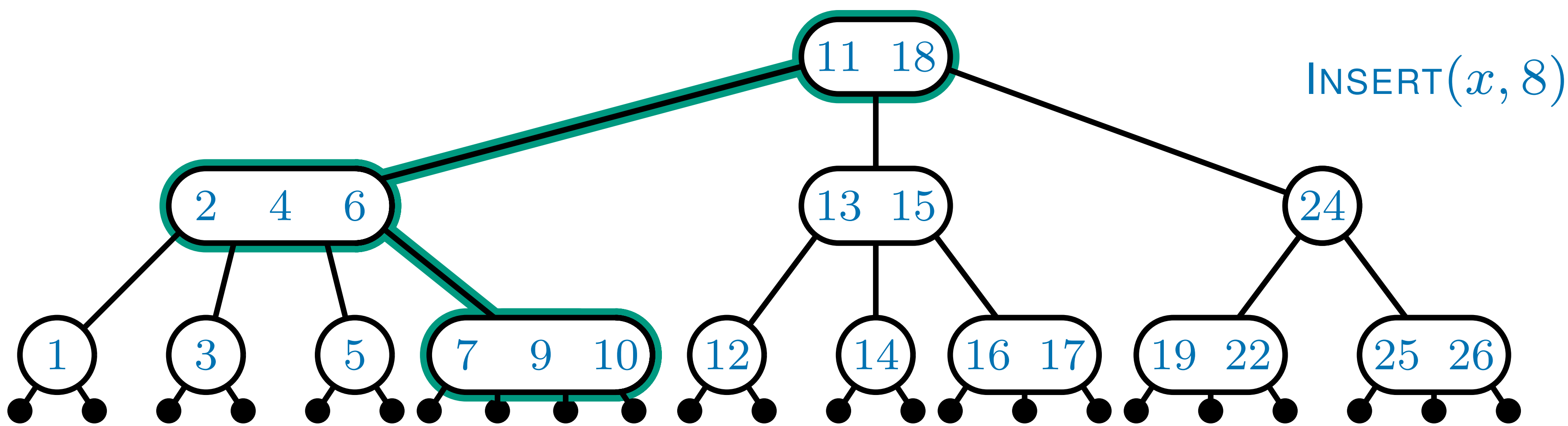

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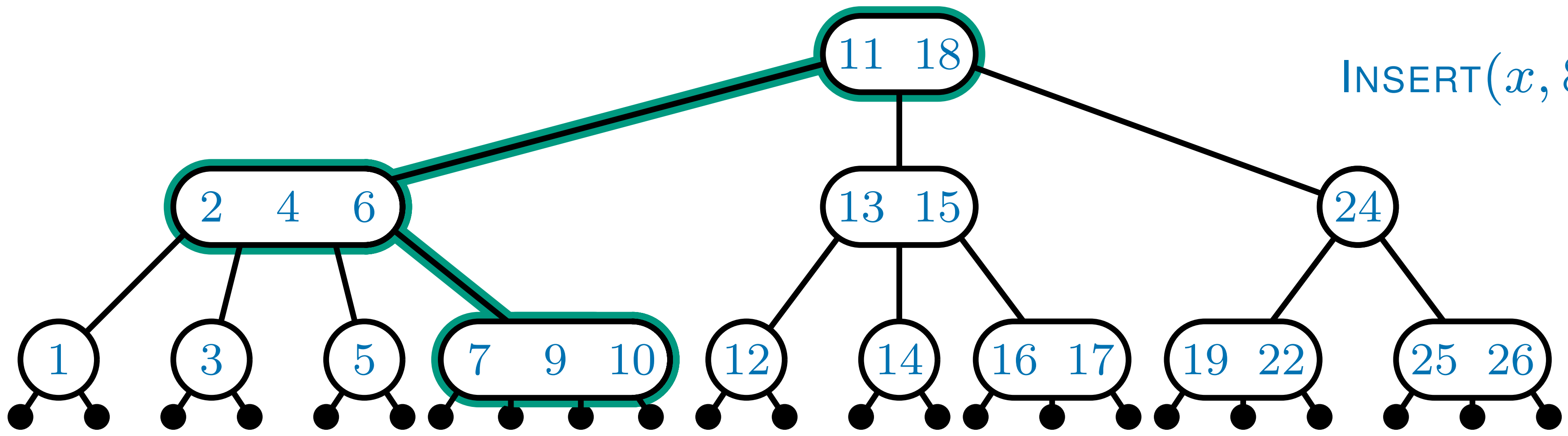

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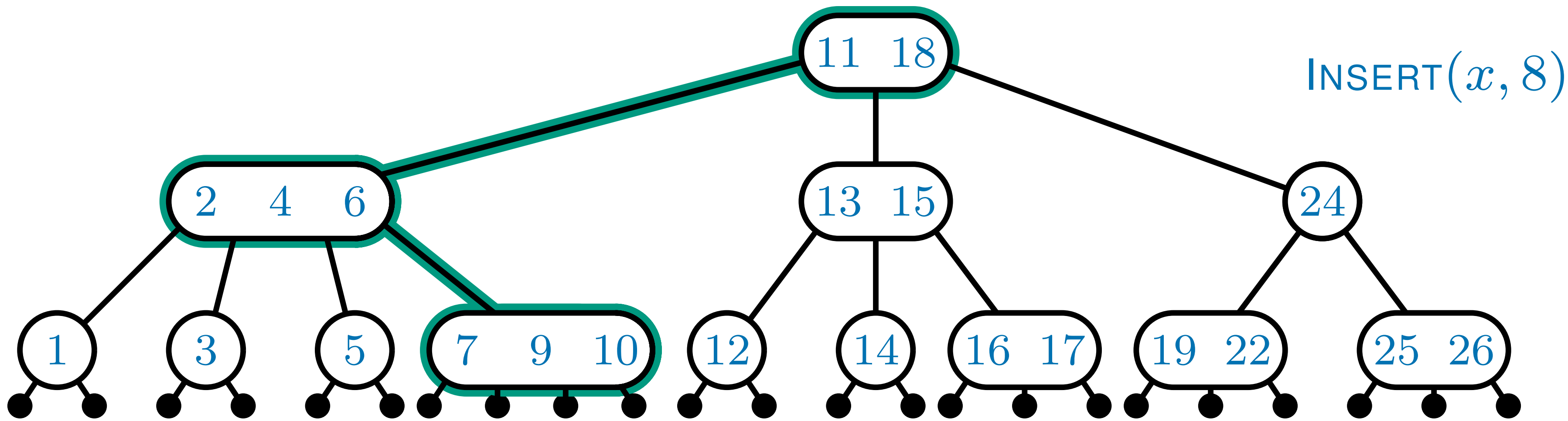
Step 4: If the leaf is a 4-node,

The INSERT operation



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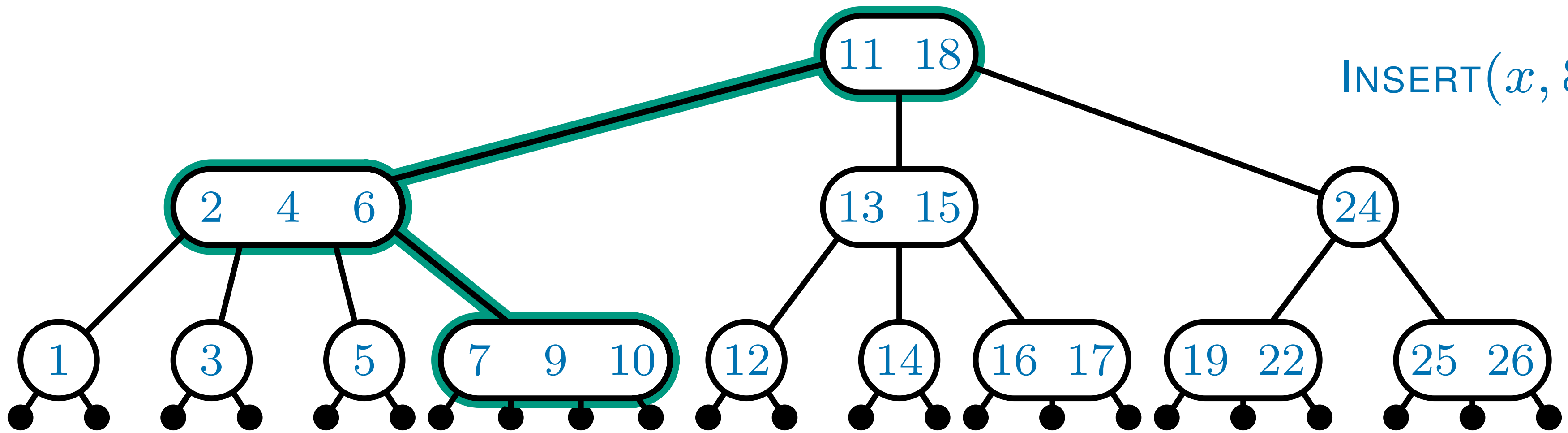
The INSERT operation



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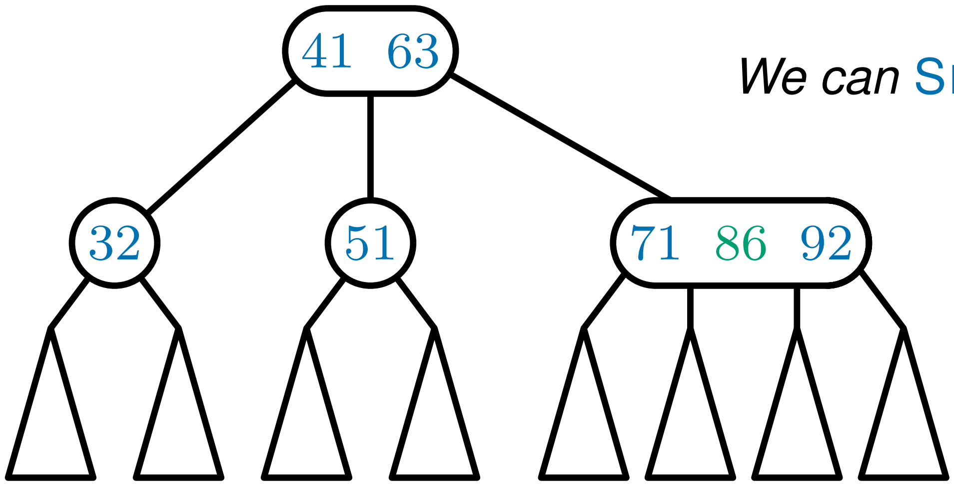
We will make sure this *never* happens

SPLITTING 4-nodes



We can SPLIT *any 4-node into two 2-nodes*
if it's parent isn't a 4-node

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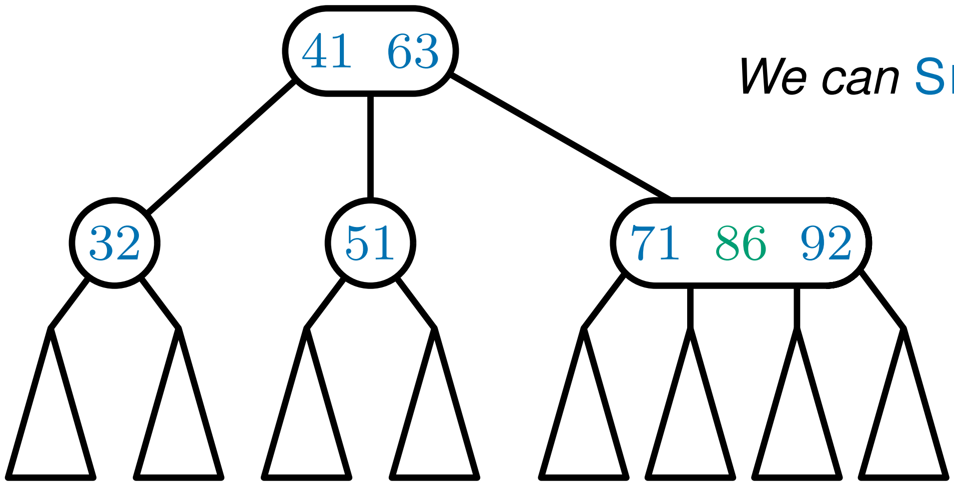
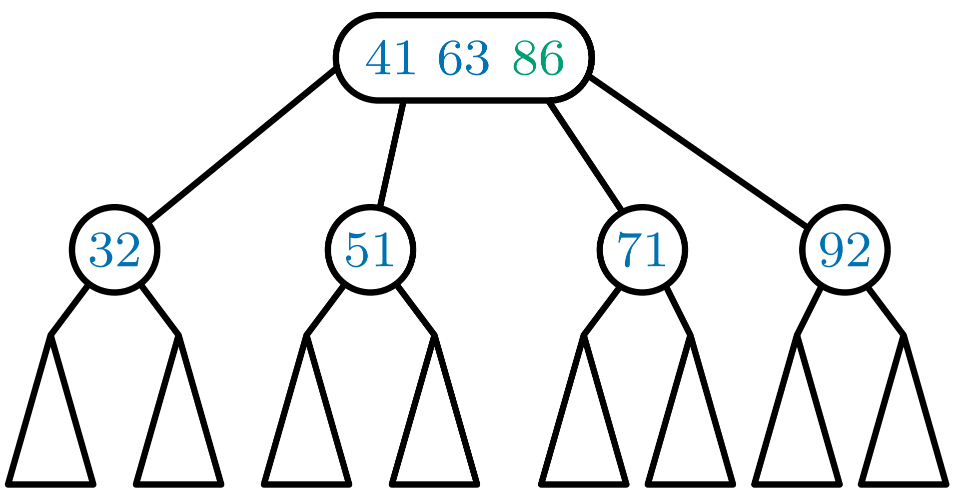


BEFORE

AFTER



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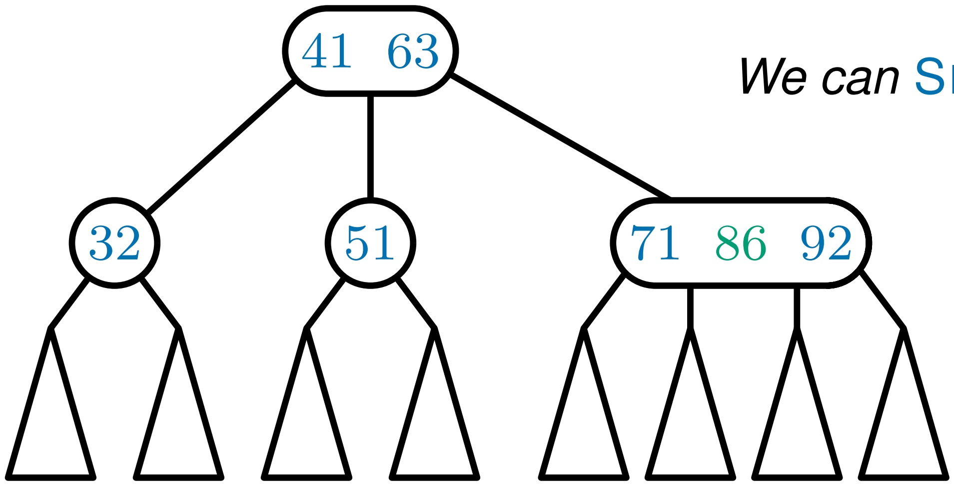

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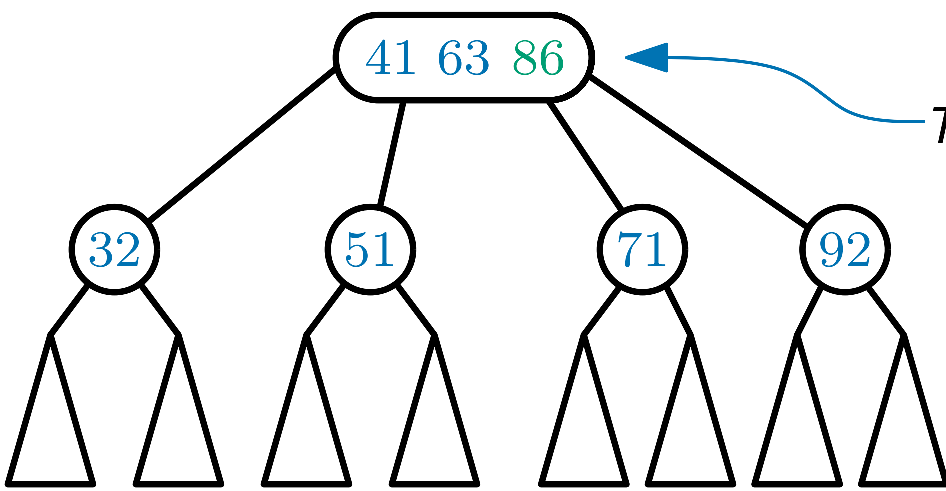
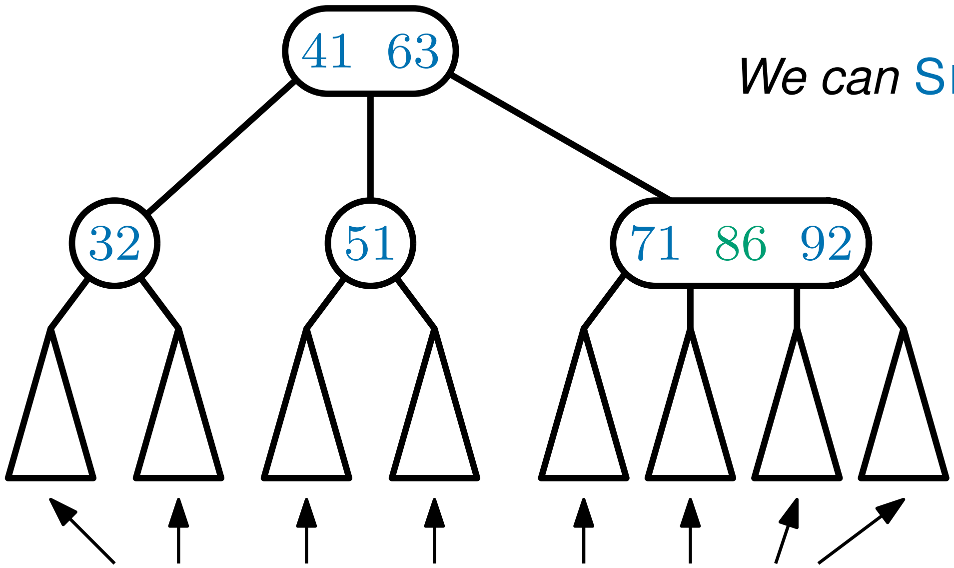


The extra key is pushed up to the parent

(so it won't work if the parent is a 4-node)

SPLITTING 4-nodes



We can SPLIT *any 4-node into two 2-nodes*
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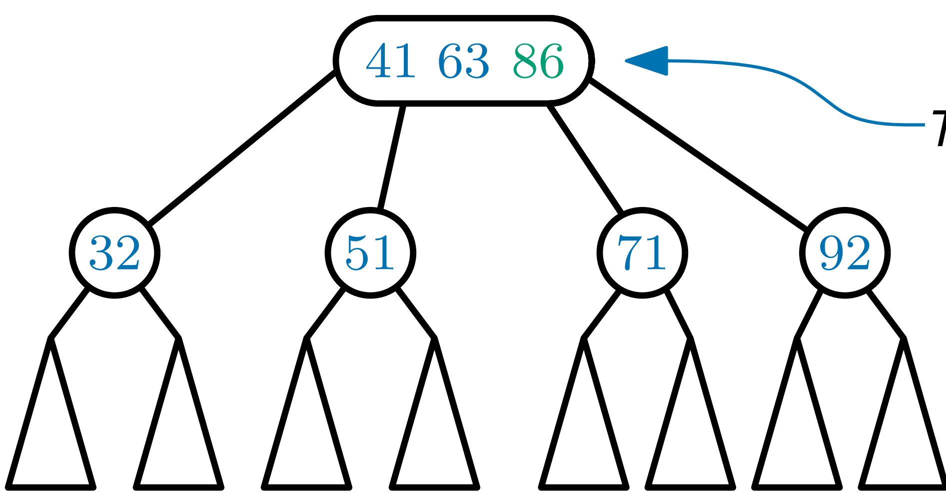
these subtrees could have any size

BEFORE

AFTER



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SPLITTING 4-nodes

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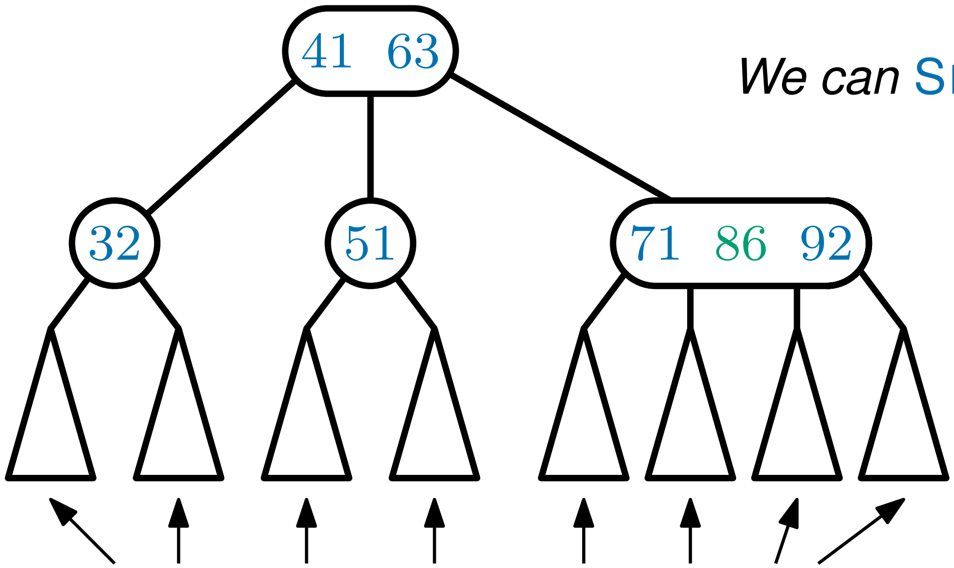

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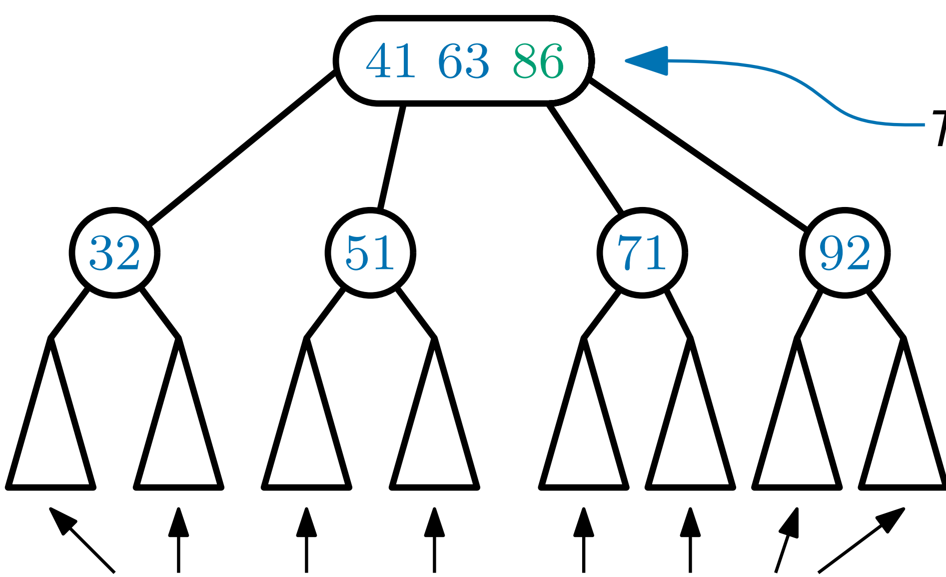

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these subtrees haven't changed

SPLITTING 4-nodes

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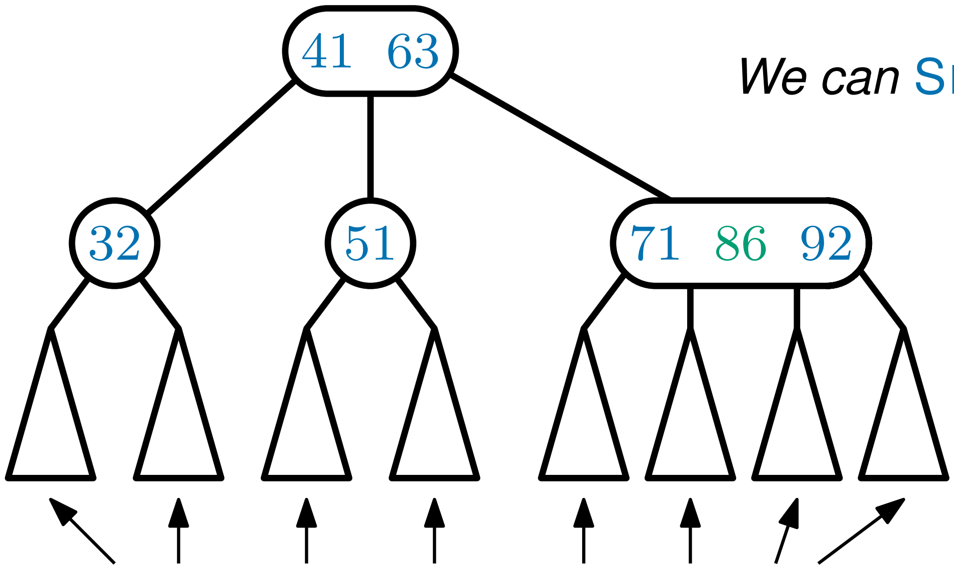

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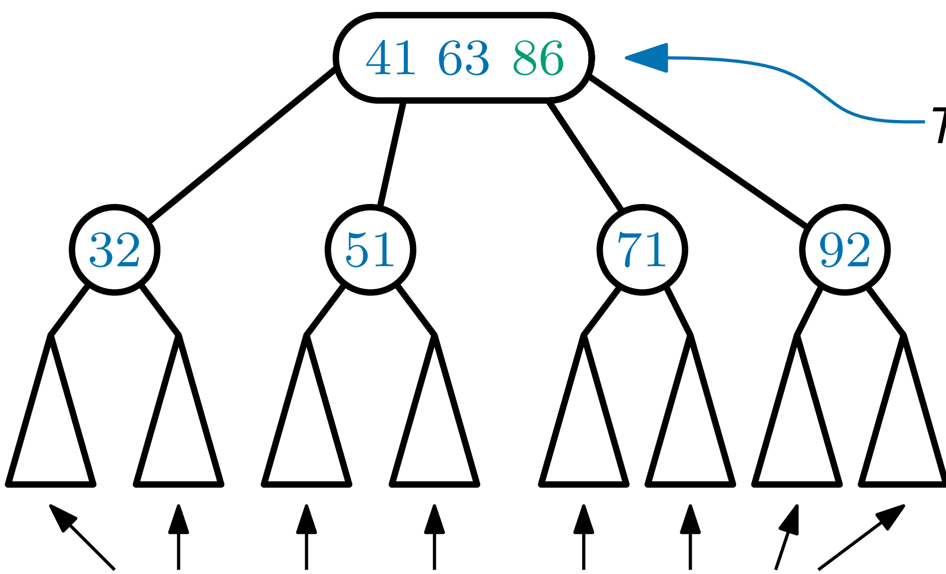
BEFORE

AFTER

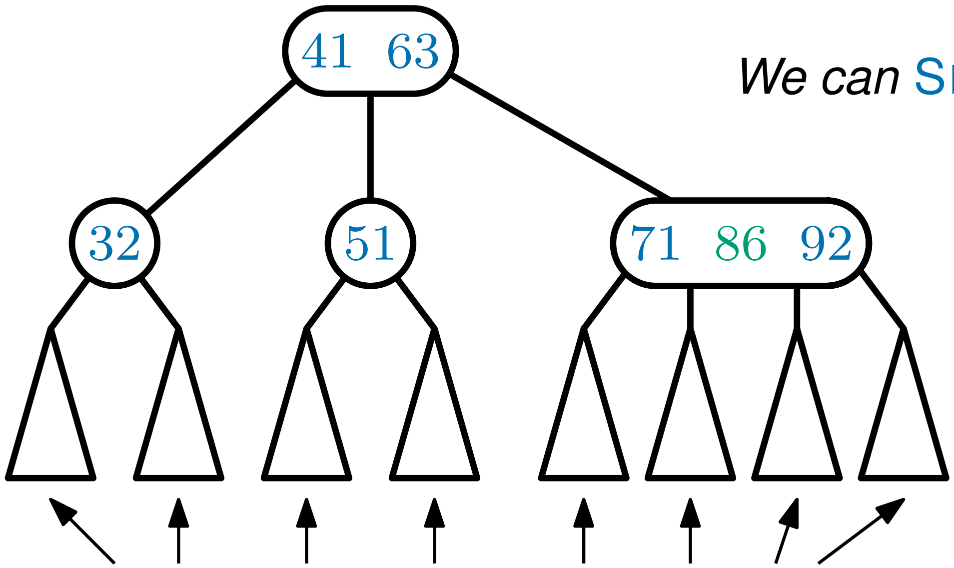


The extra key is pushed up to the parent
(so it won't work if the parent is a 4-node)

no path lengths have changed

these subtrees haven't changed

SPLITTING 4-nodes

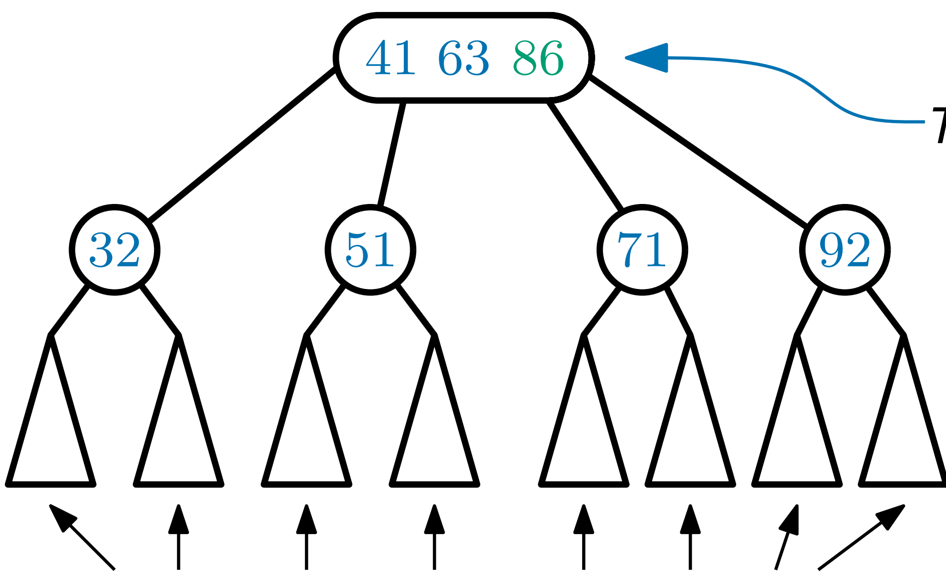


these subtrees could have any size

We can SPLIT *any 4-node into two 2-nodes*
if it's parent isn't a 4-node

BEFORE

AFTER



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The extra key is pushed up to the parent

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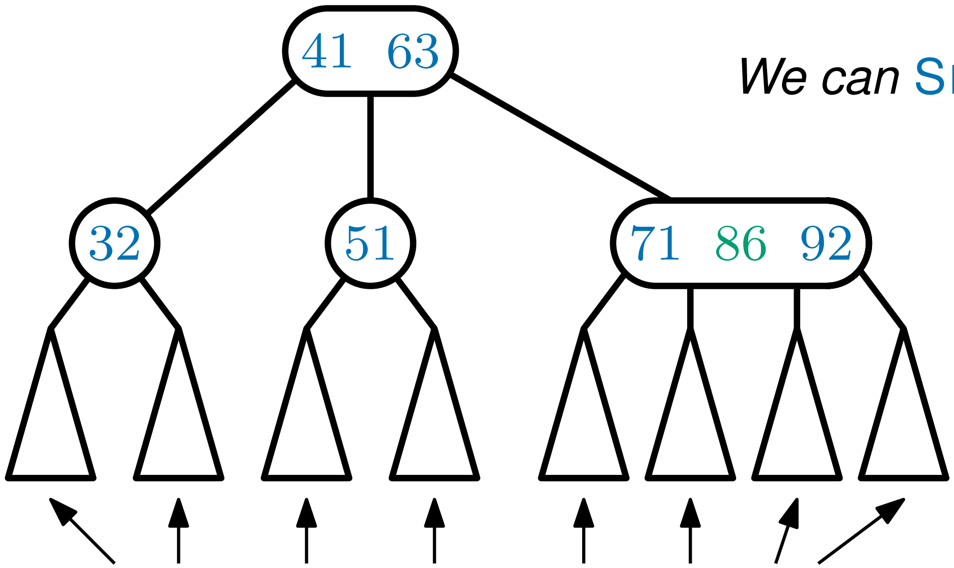
no path lengths have changed

(if it was **perfectly balanced**, *it still is)*

SPLITTING 4-nodes



these subtrees could have any size

We can SPLIT *any 4-node into two 2-nodes*
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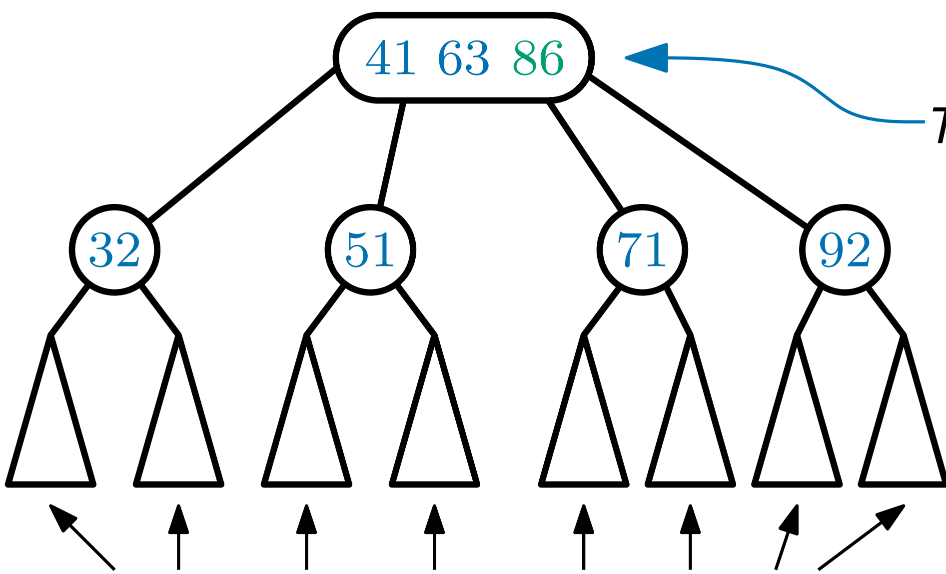
BEFORE

AFTER



these subtrees haven't changed

The extra key is pushed up to the parent

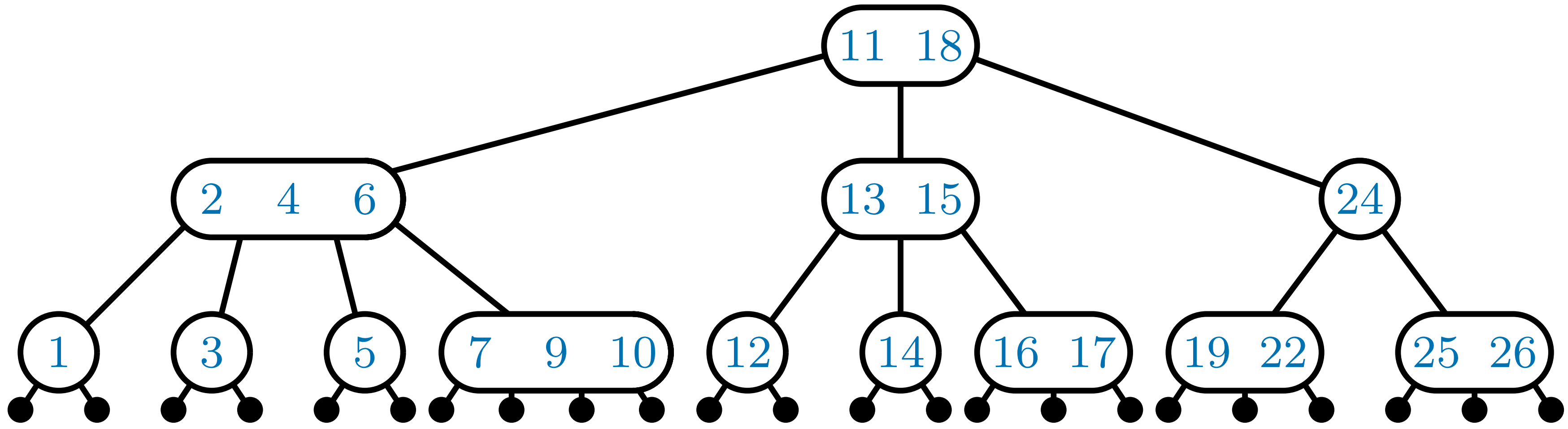
(so it won't work if the parent is a 4-node)

no path lengths have changed

(if it was **perfectly balanced**, *it still is)*

SPLIT takes $O(1)$ time

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

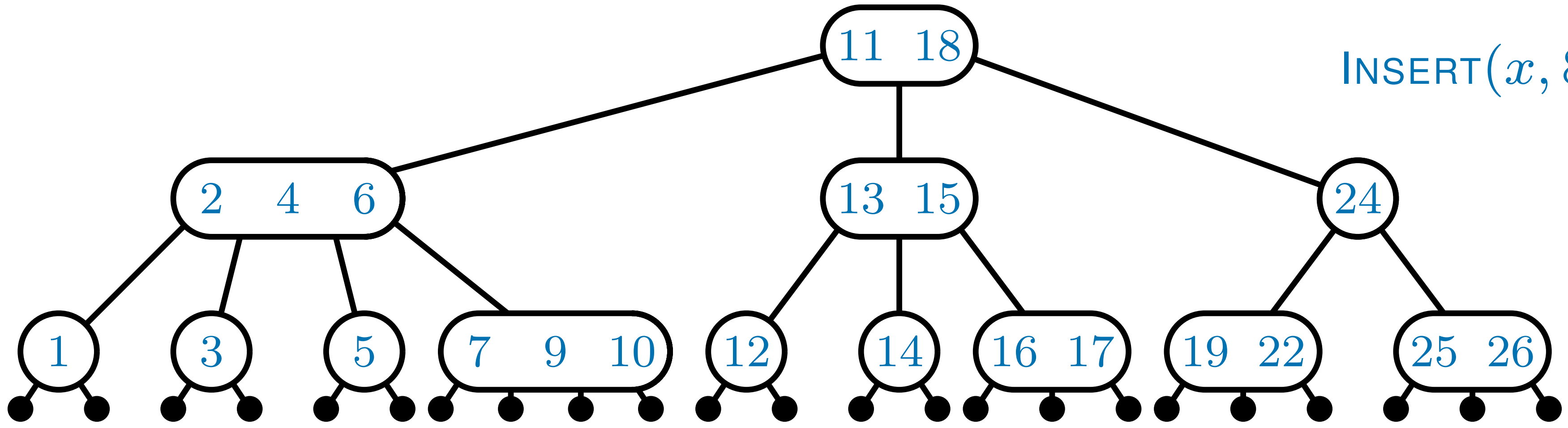
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



INSERT$(x, 8)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

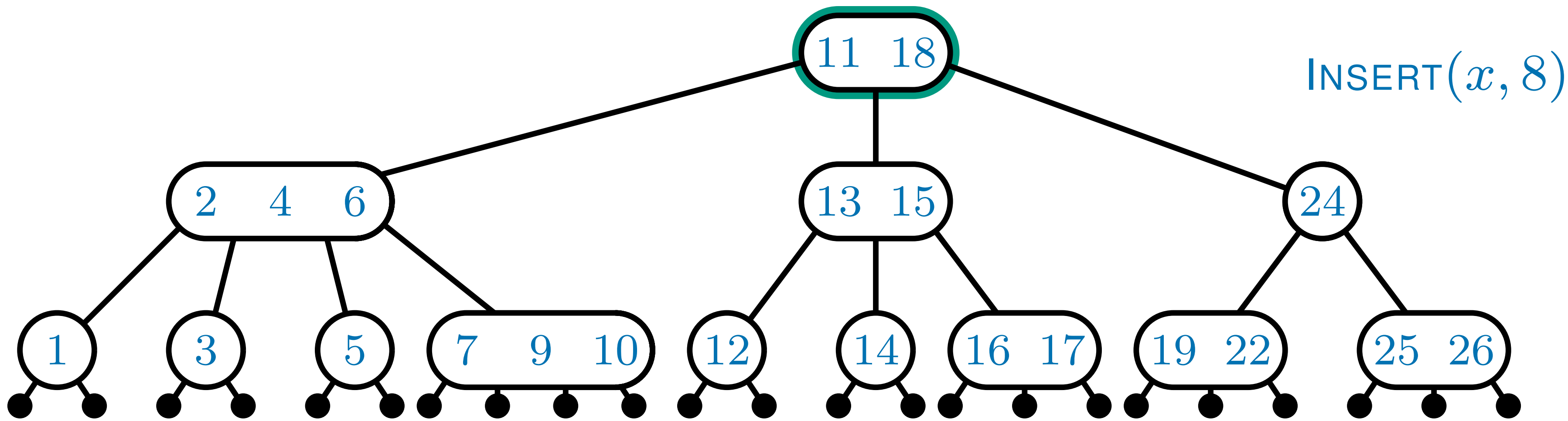
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

University of BRISTOL

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

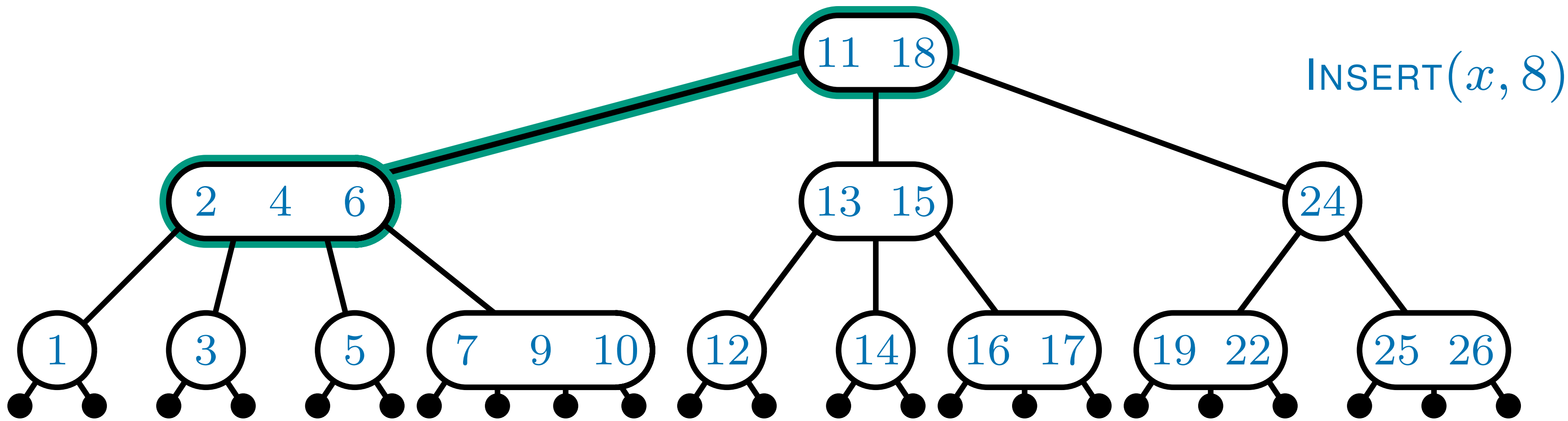
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

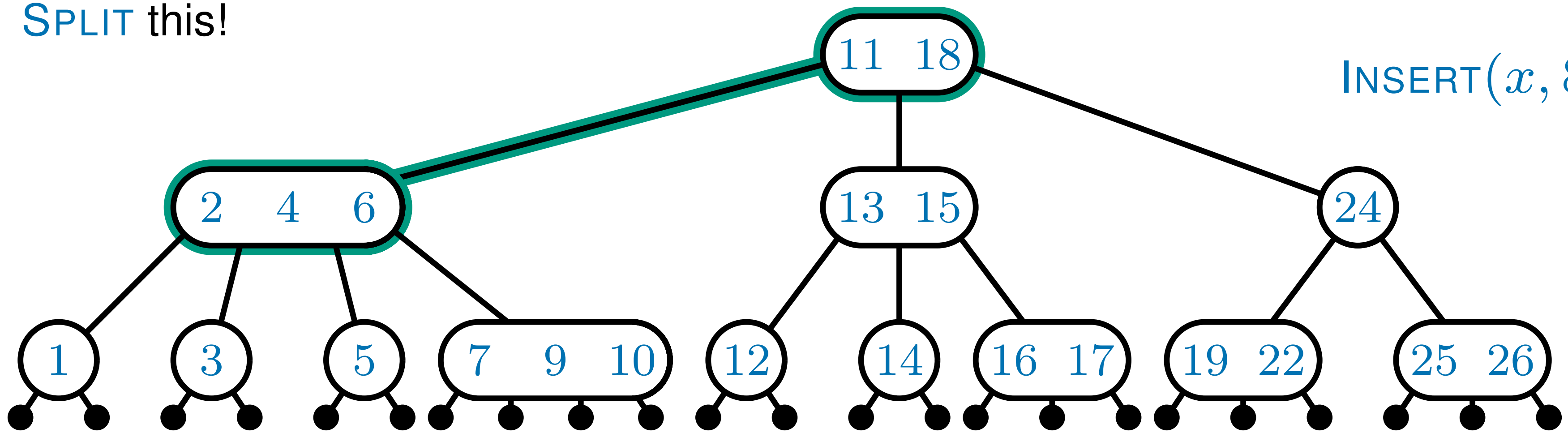
Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation

SPLIT this!

INSERT$(x, 8)$



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

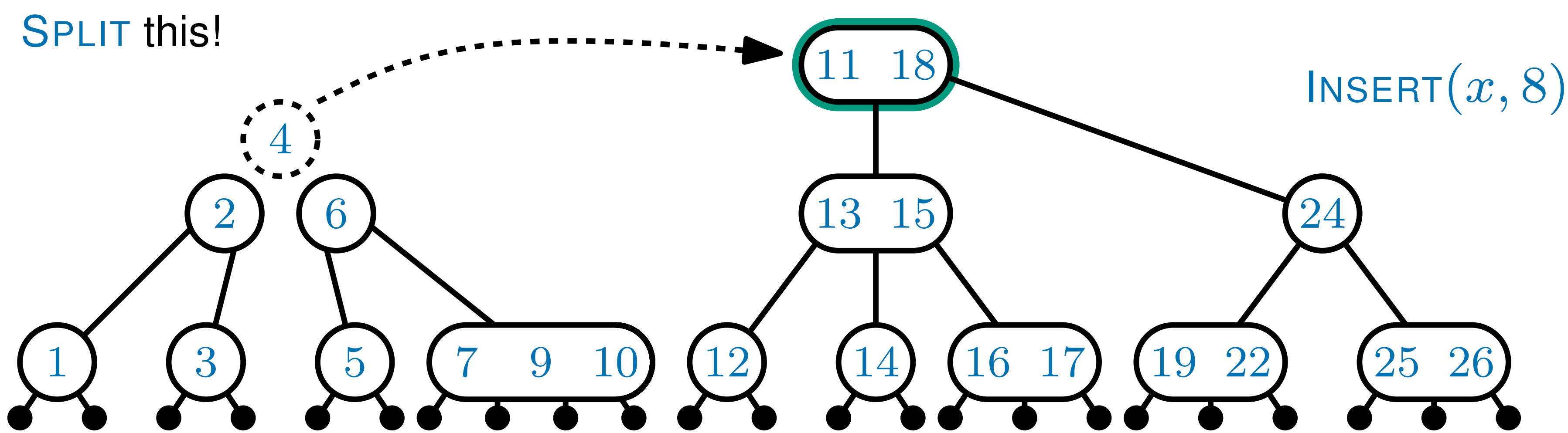
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

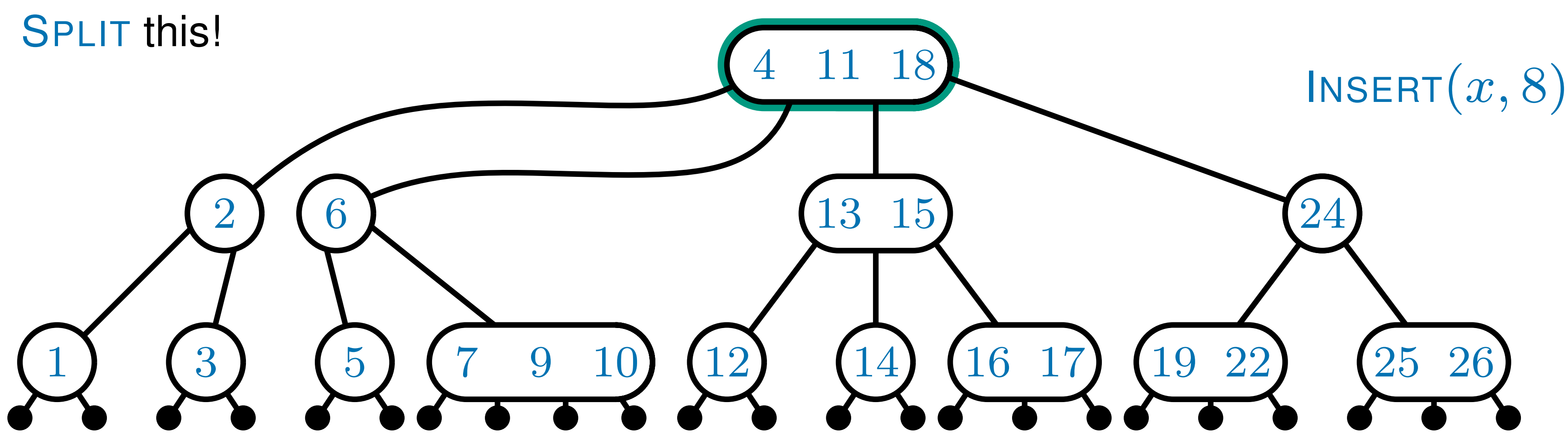
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

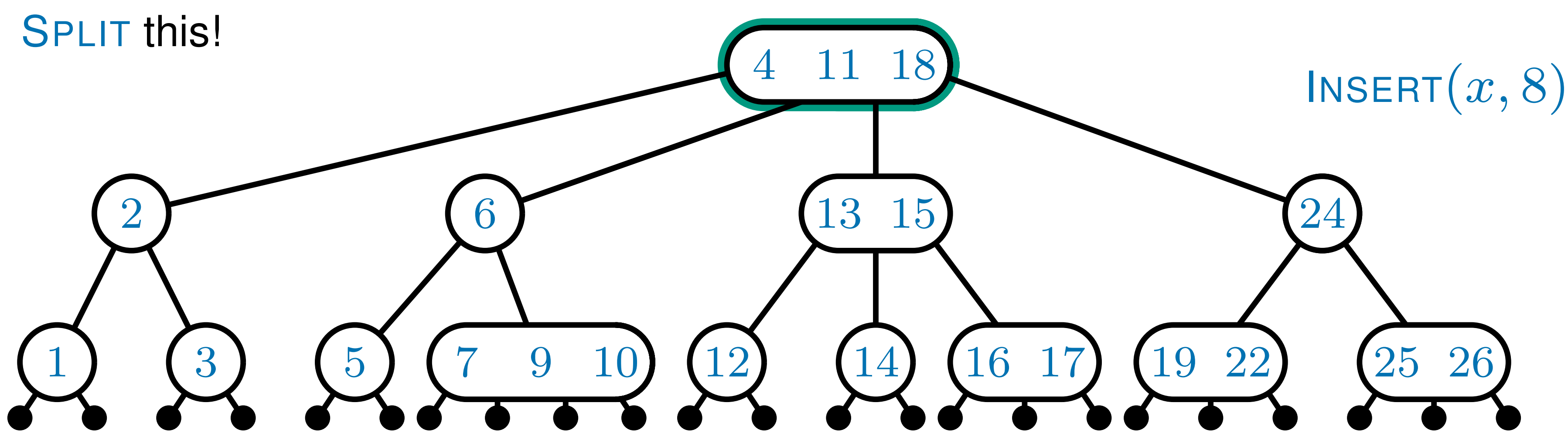
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



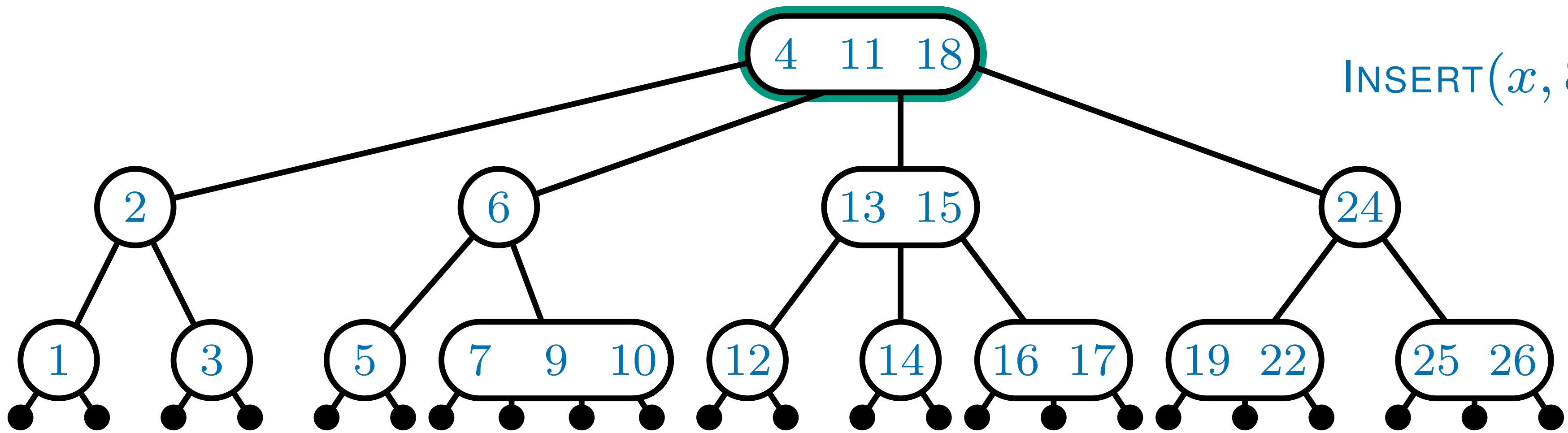
To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).
SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,
insert (x, k), converting it into a 4-node

The INSERT operation



INSERT$(x, 8)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

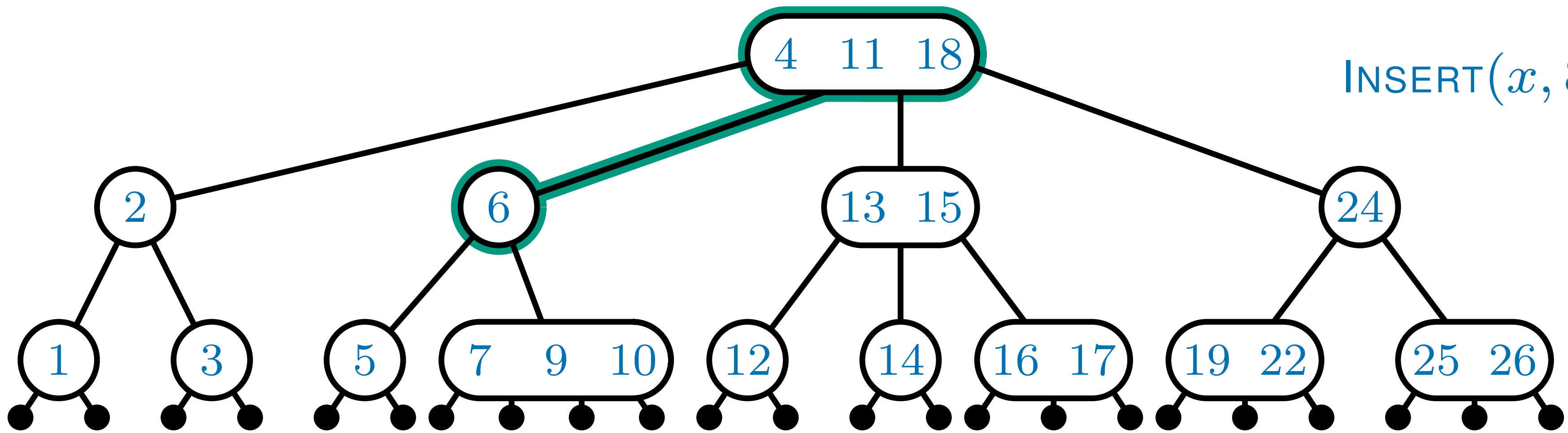
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



INSERT$(x, 8)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

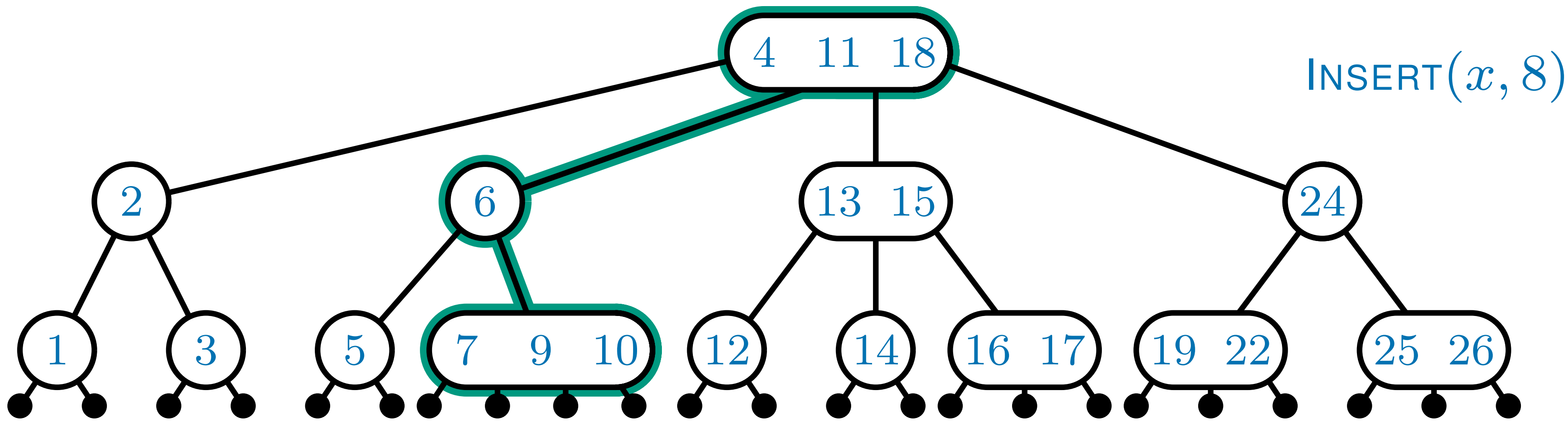
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

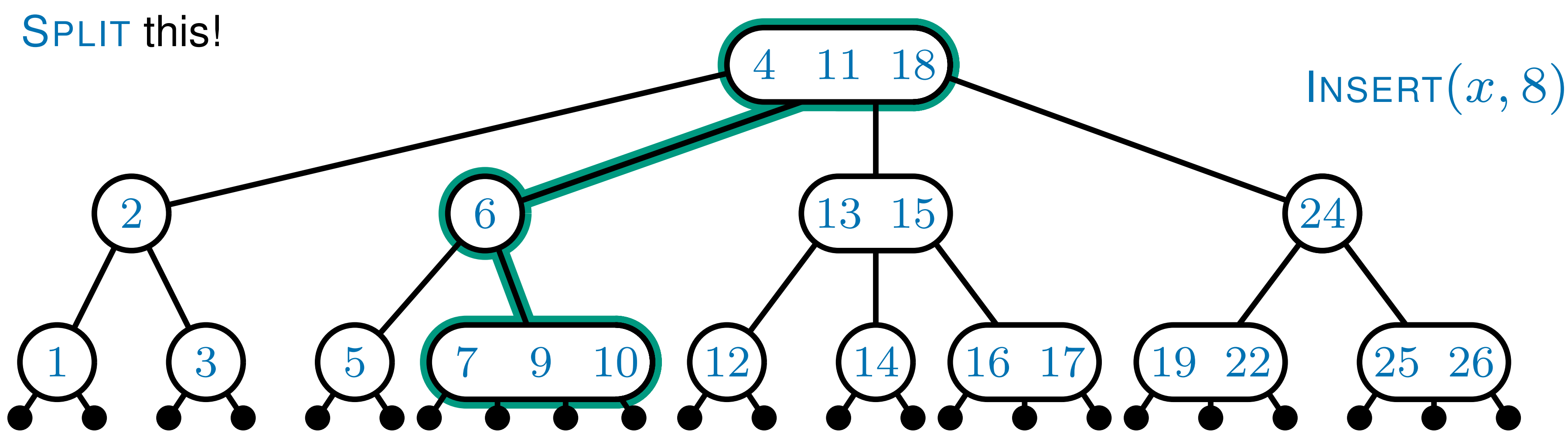
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The Insert operation



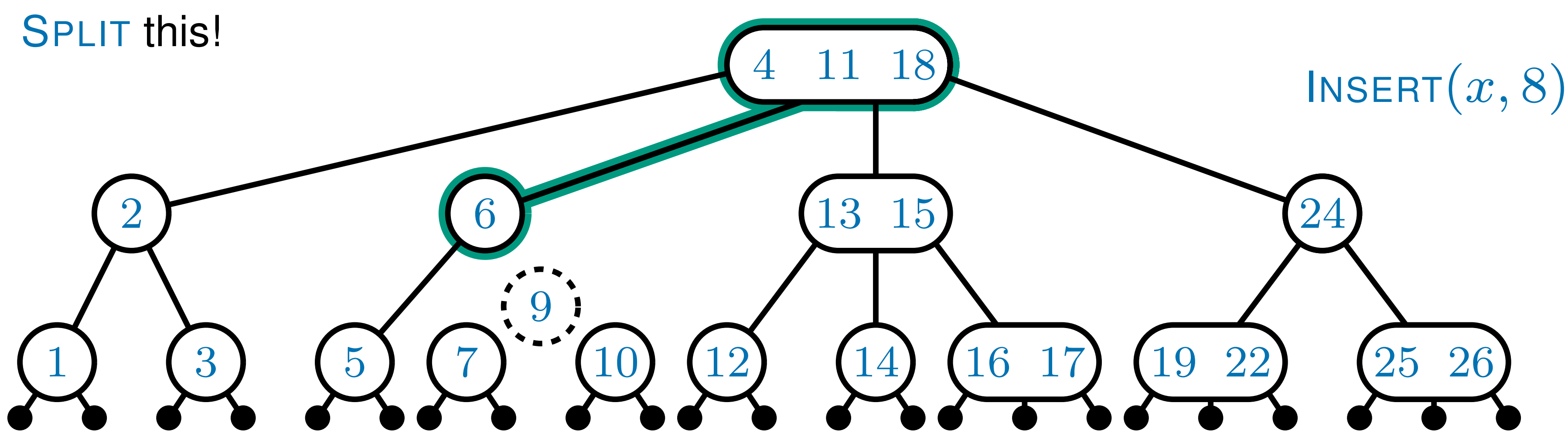
To perform Insert(x, k),

Step 1: Search for the key k as if performing Find(k).
Split *4-nodes as we go down*

Step 2: If the leaf is a 2-node,
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,
insert (x, k), converting it into a 4-node

The INSERT operation



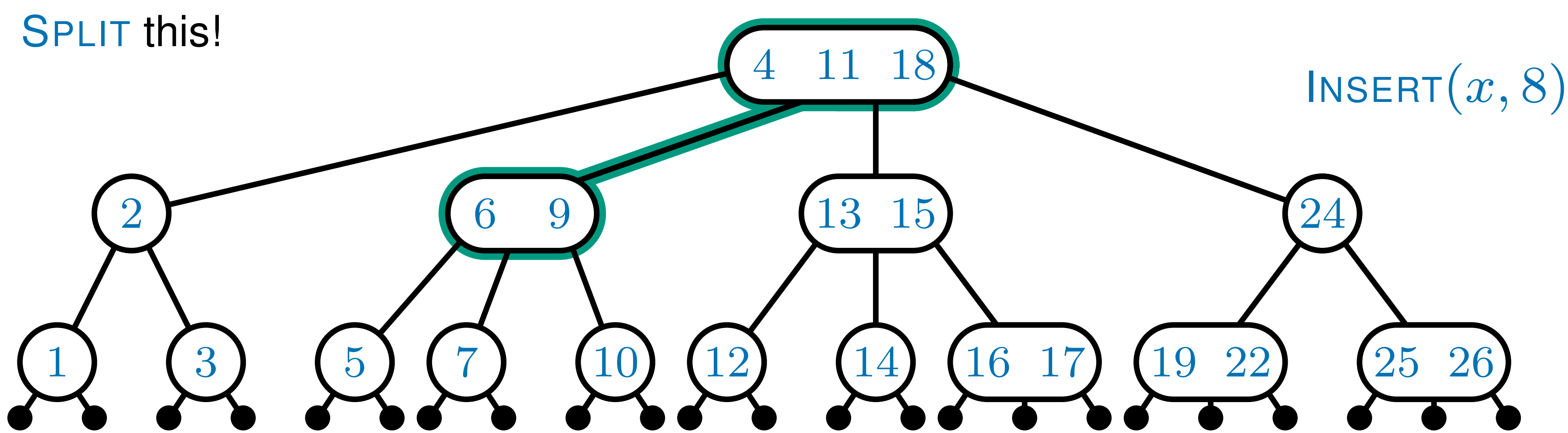
To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).
SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,
insert (x, k), converting it into a 4-node

The INSERT operation



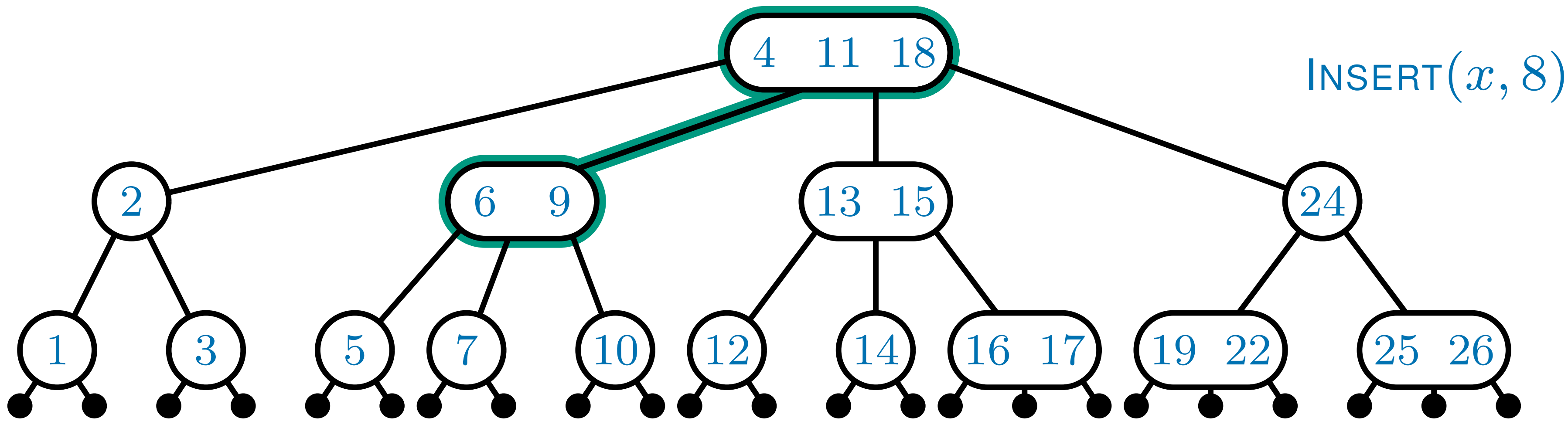
To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).
SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,
insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

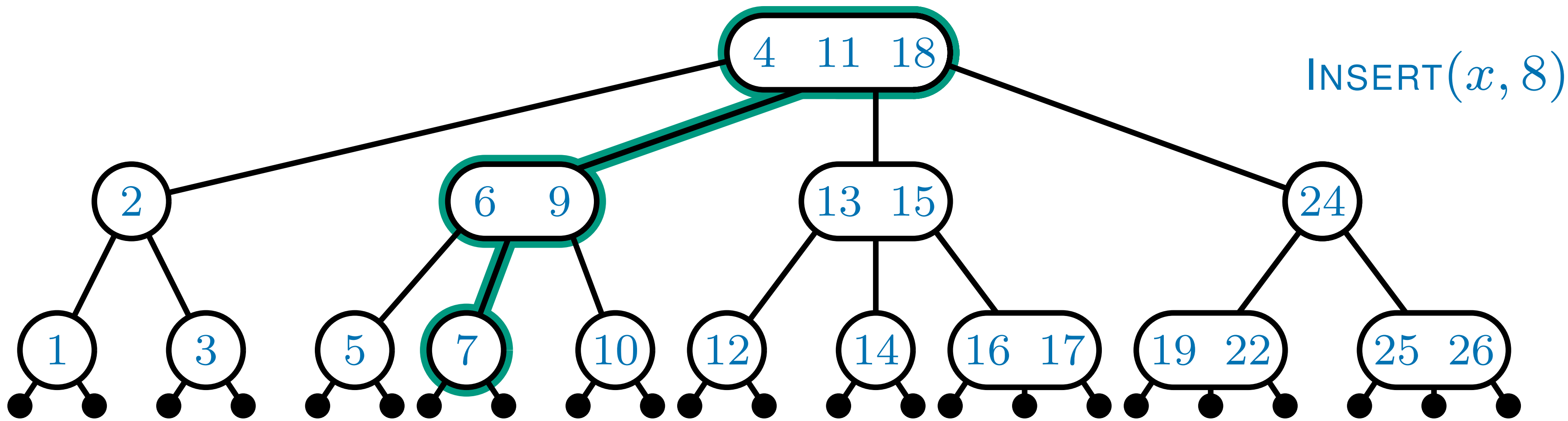
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

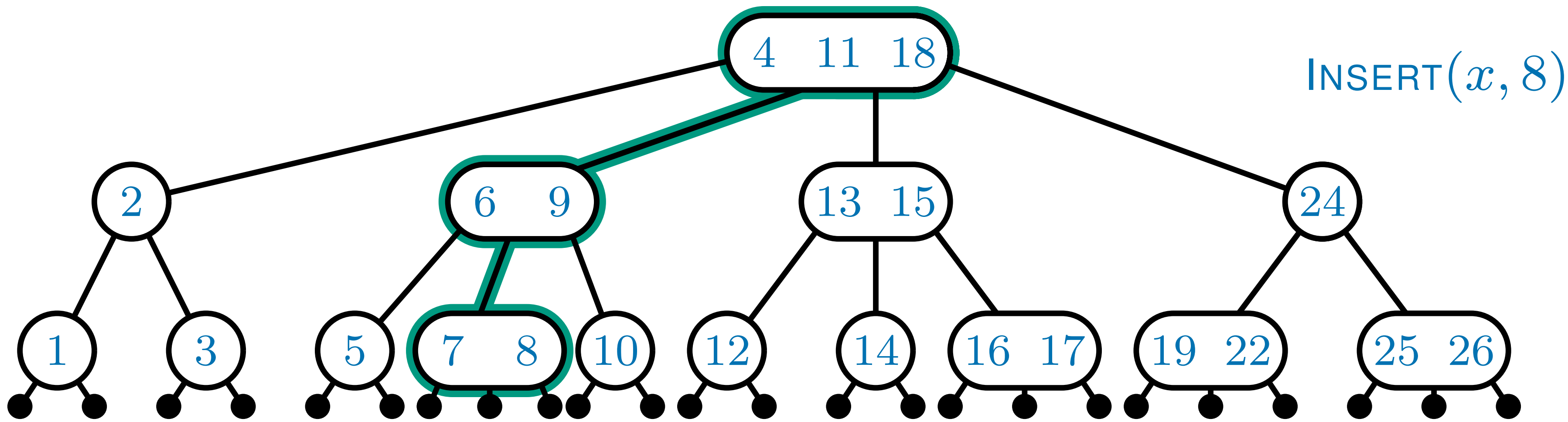
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

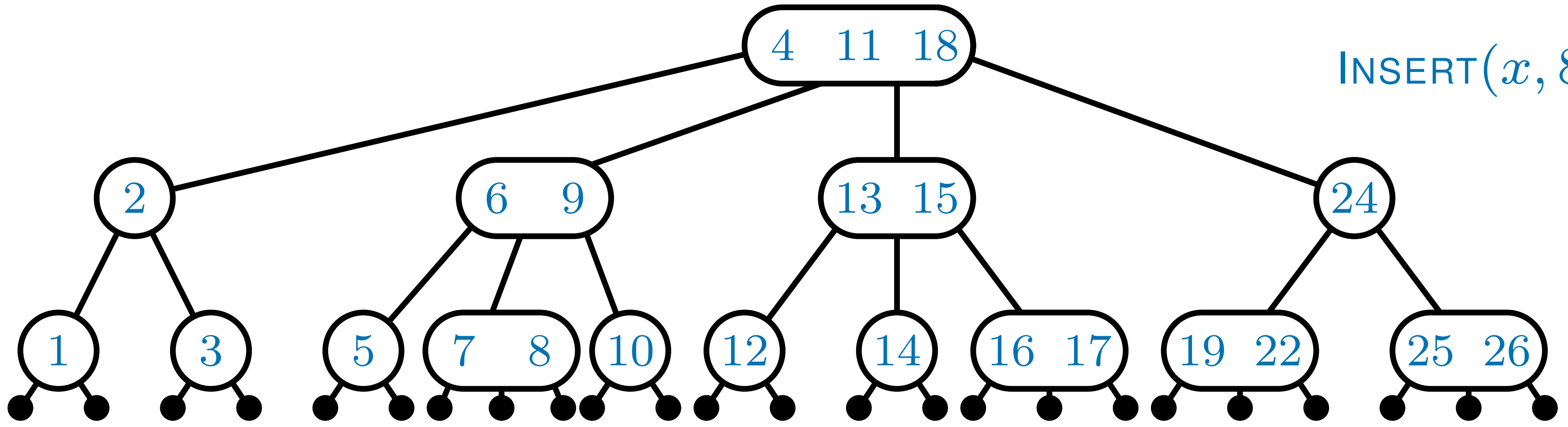
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The $\textsc{Insert}$ operation



$\textsc{Insert}(x, 8)$

To perform $\textsc{Insert}(x, k)$,

Step 1: Search for the key k as if performing $\textsc{Find}(k)$.

$\textsc{Split}$ *4-nodes as we go down*

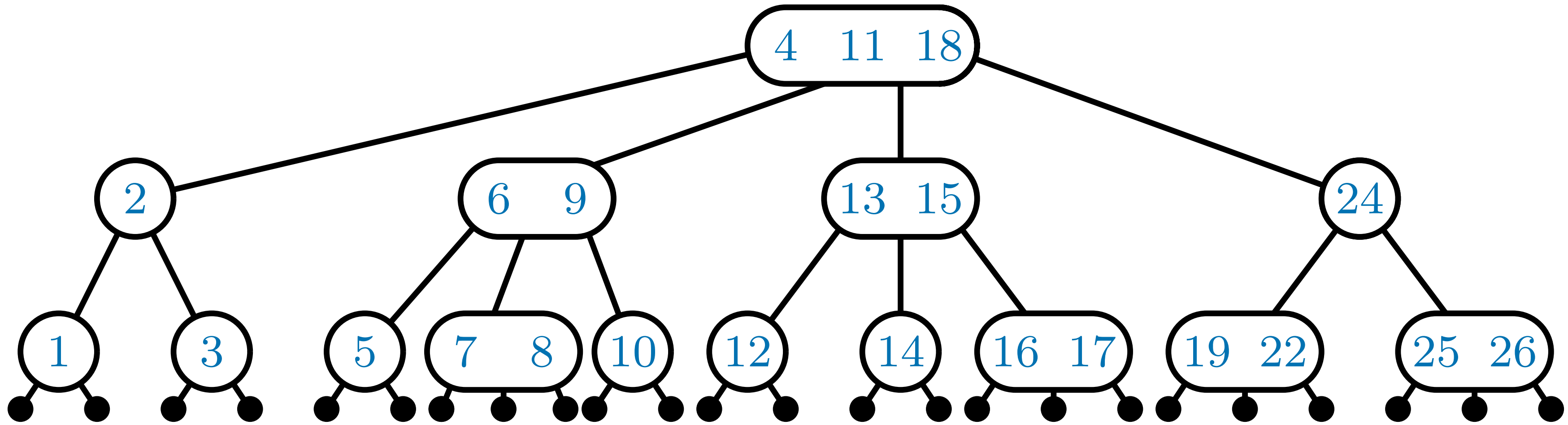
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

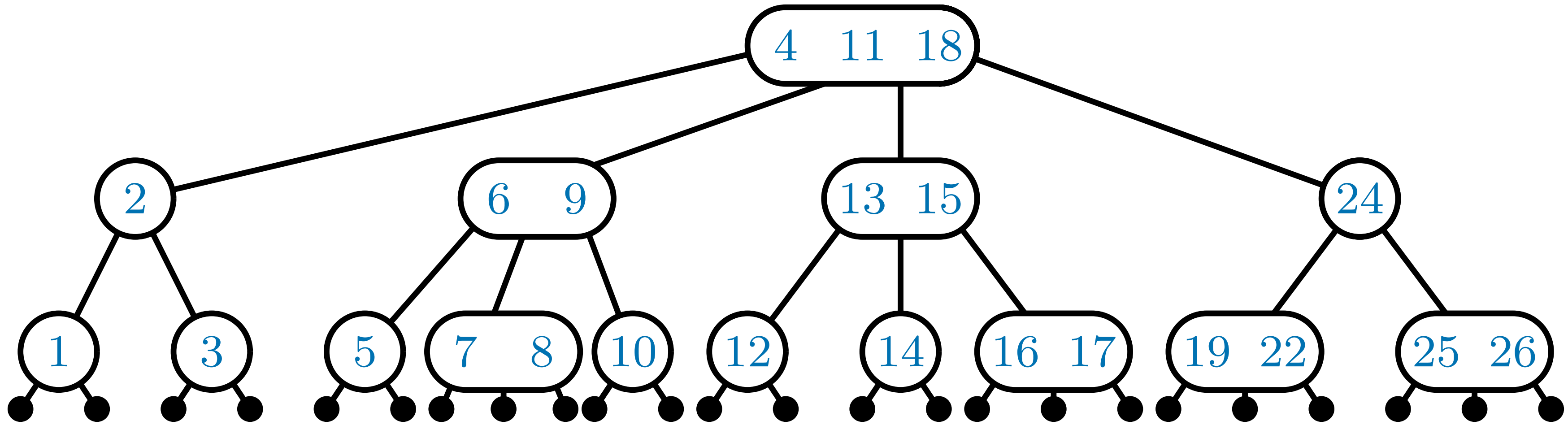
Step 2: If the leaf is a 2-node,

insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

The INSERT operation



To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

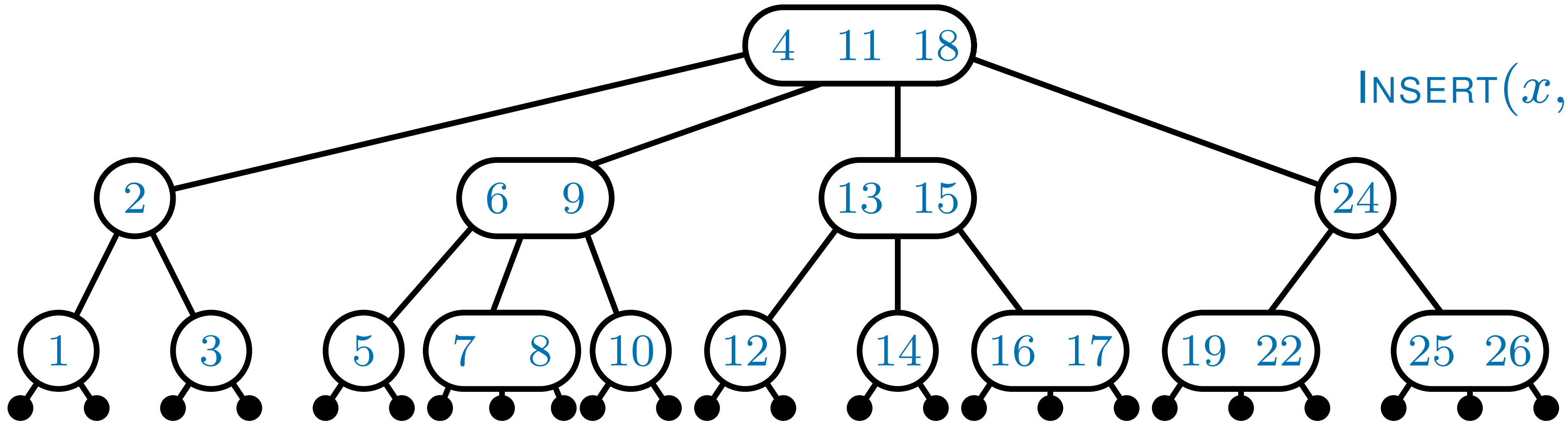
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

OK, one more thing. . .

The INSERT operation



INSERT$(x, 20)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

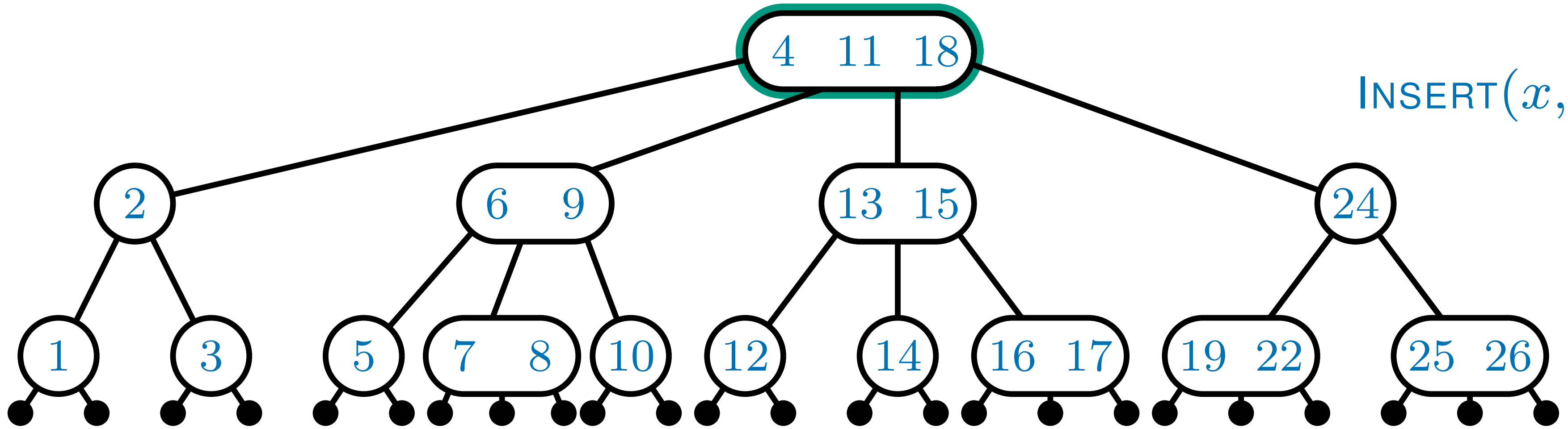
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

OK, one more thing. . .

The INSERT operation



INSERT$(x, 20)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

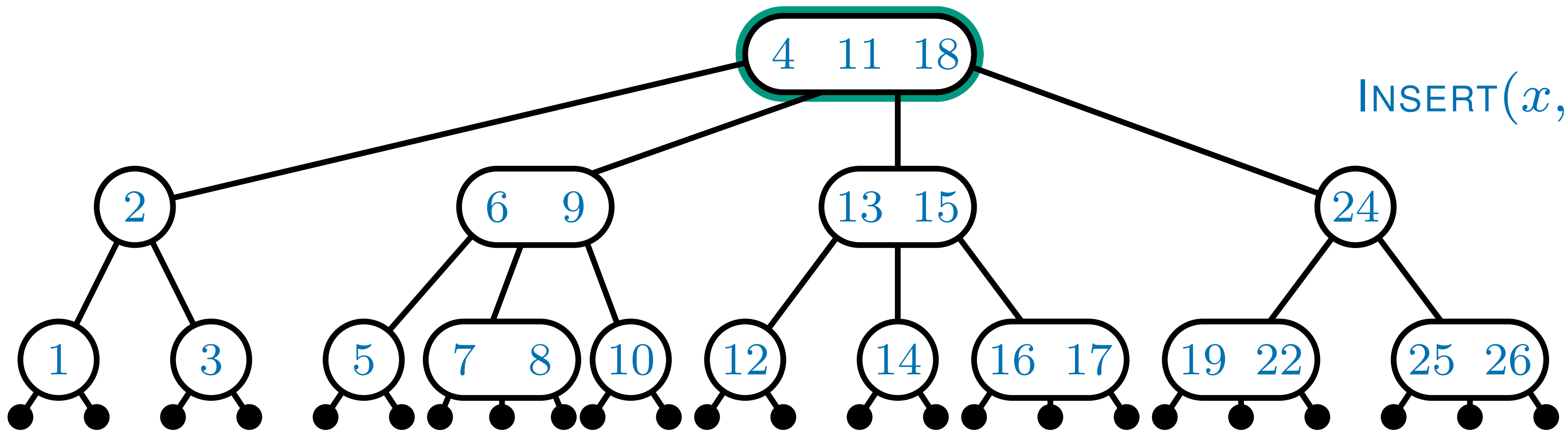
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

OK, one more thing. . .

The INSERT operation



INSERT$(x, 20)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

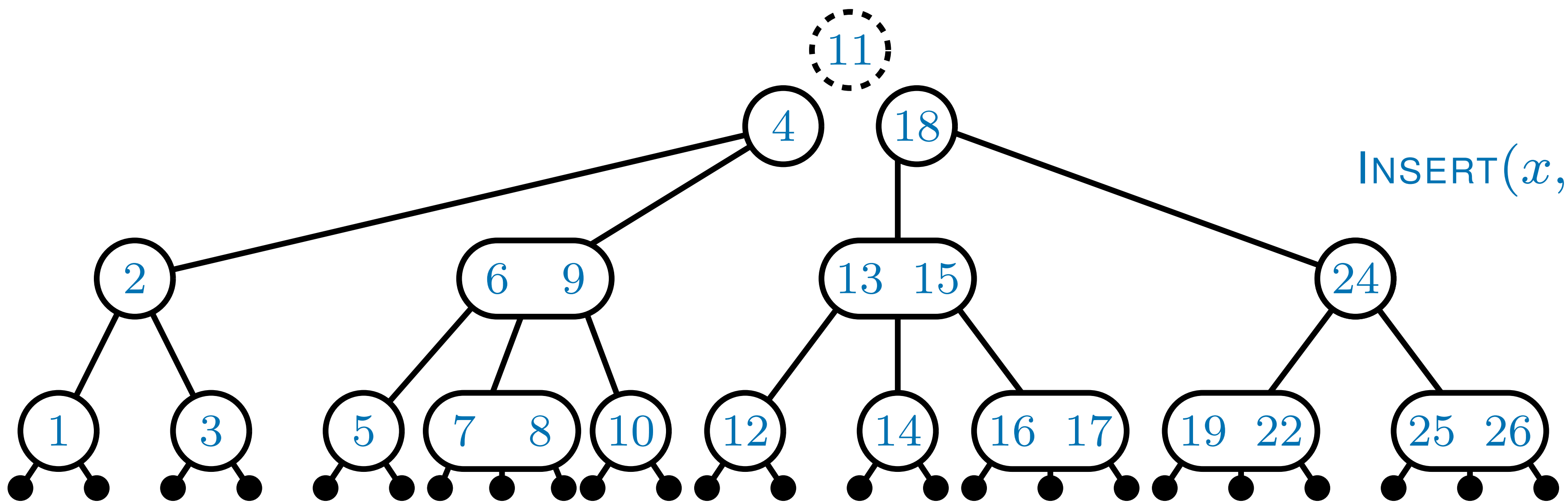
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

OK, one more thing... what happens when we SPLIT the root?

The INSERT operation



INSERT$(x, 20)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

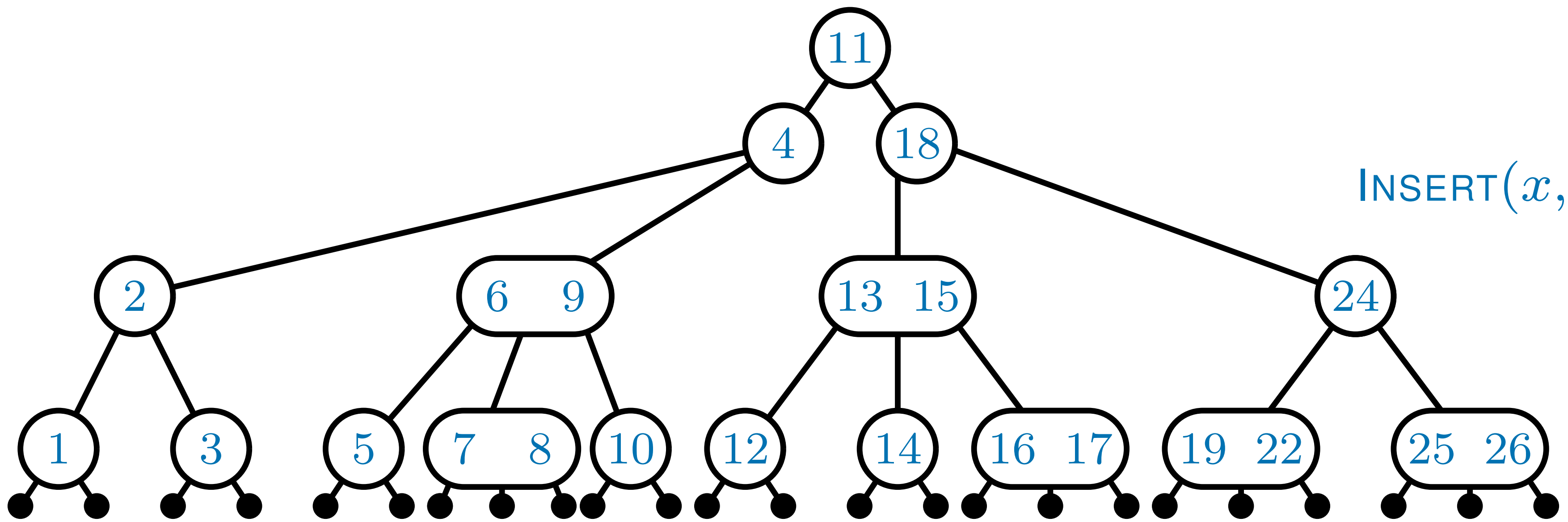
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

OK, one more thing... what happens when we SPLIT the root?

The INSERT operation



INSERT$(x, 20)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the leaf is a 2-node,

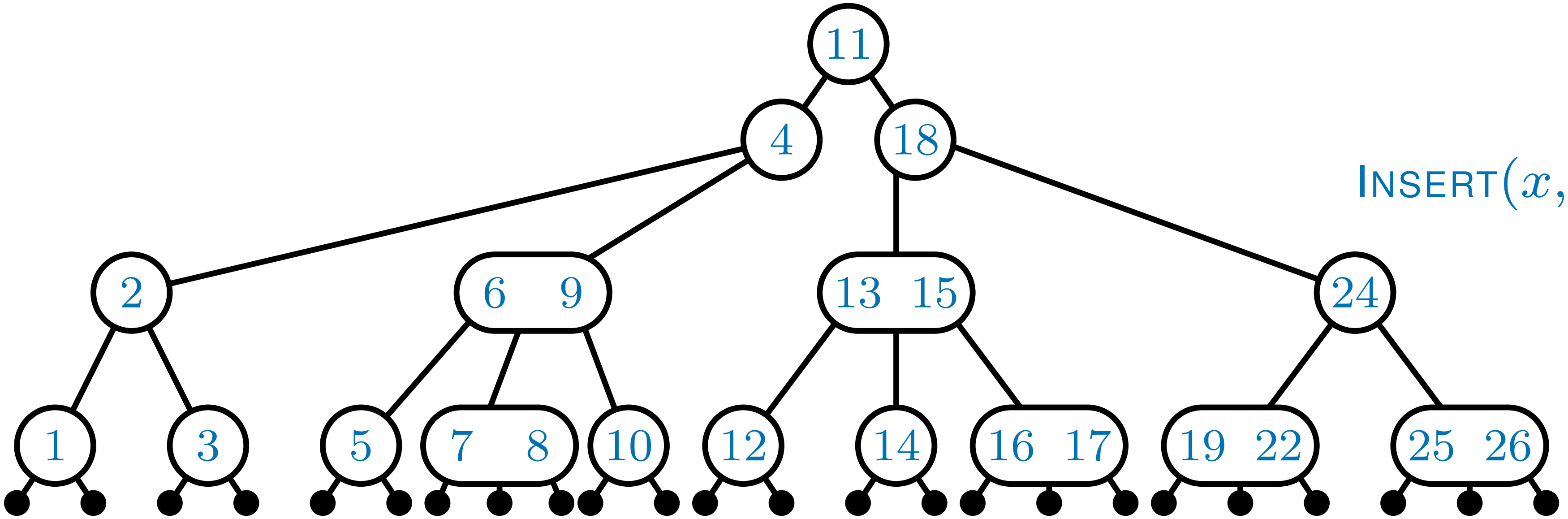
insert (x, k), converting it into a 3-node

Step 3: If the leaf is a 3-node,

insert (x, k), converting it into a 4-node

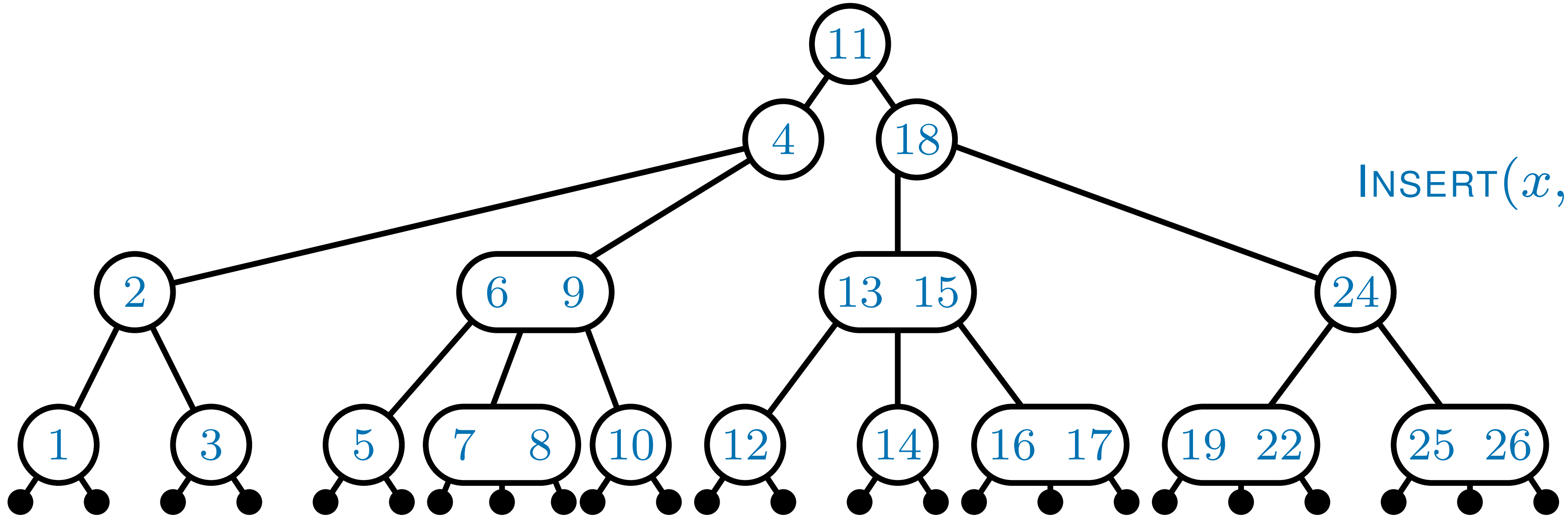
OK, one more thing... what happens when we SPLIT the root?

The INSERT operation



INSERT$(x, 20)$

The INSERT operation



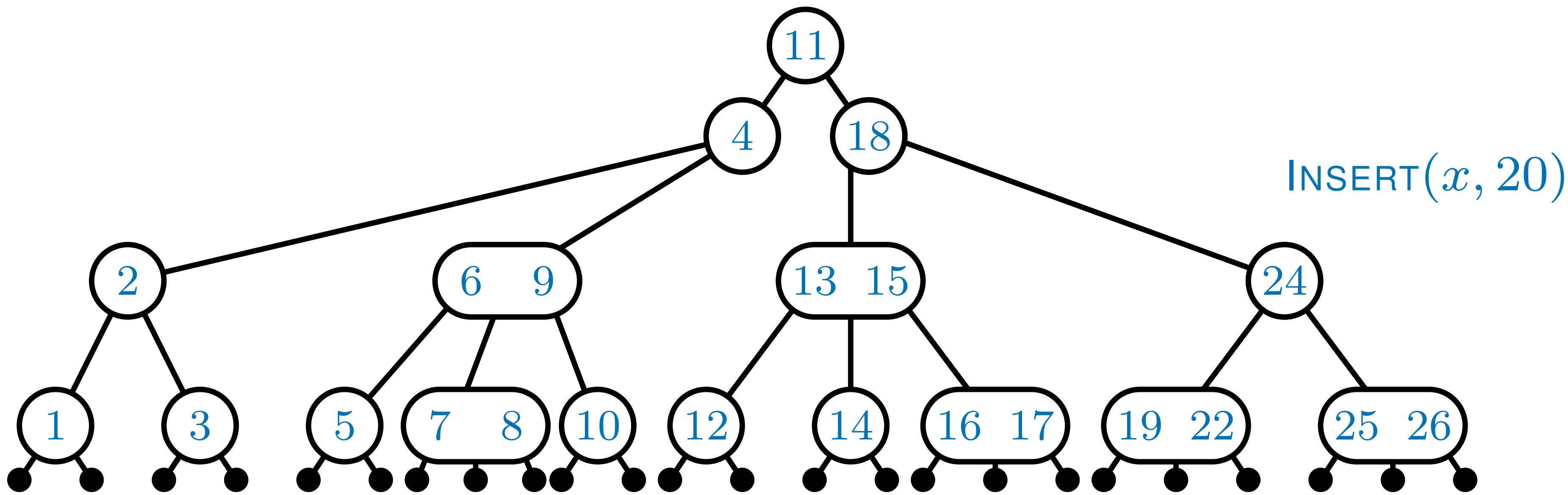
INSERT$(x, 20)$

SPLITTING the root increases the height of the tree
and increases the length of all root-leaf paths by one

So it maintains the **perfect balance** property

- i.e every path from the root to a leaf has the same length

The INSERT operation



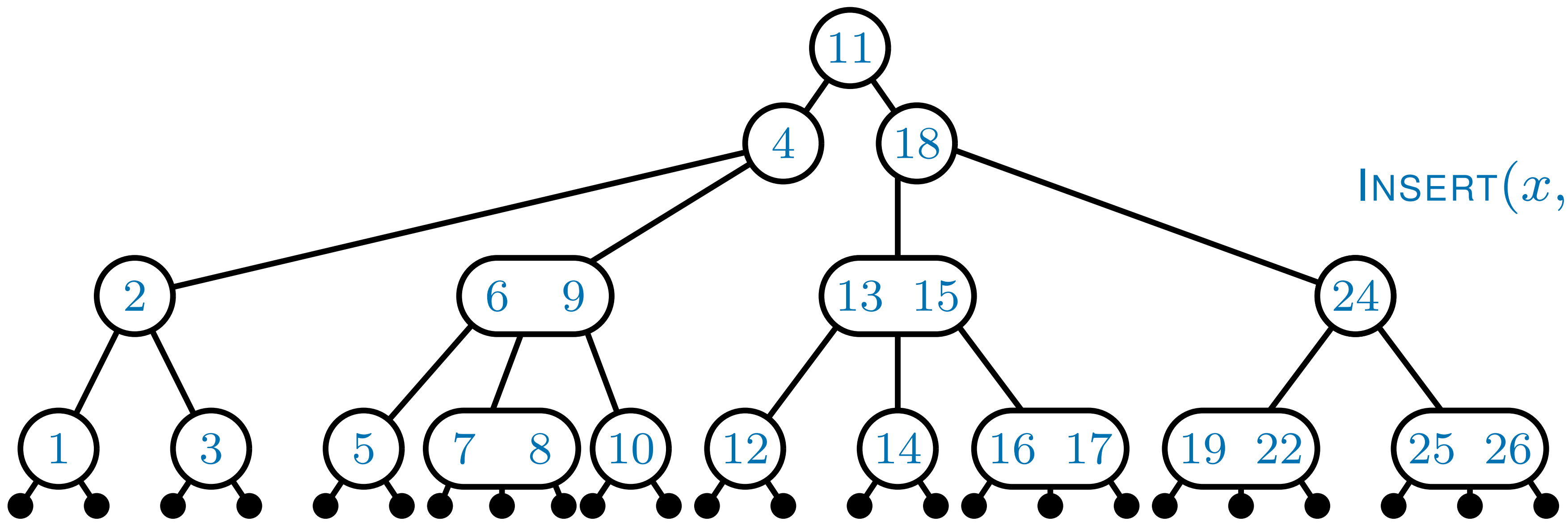
SPLITTING the root increases the height of the tree
and increases the length of all root-leaf paths by one

So it maintains the **perfect balance** property

- i.e every path from the root to a leaf has the same length

This is the only way INSERT can affect the length of paths
so it also maintains the **perfect balance** property

The INSERT operation



INSERT$(x, 20)$

SPLITTING the root increases the height of the tree
and increases the length of all root-leaf paths by one

So it maintains the **perfect balance** property

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This is the only way INSERT can affect the length of paths
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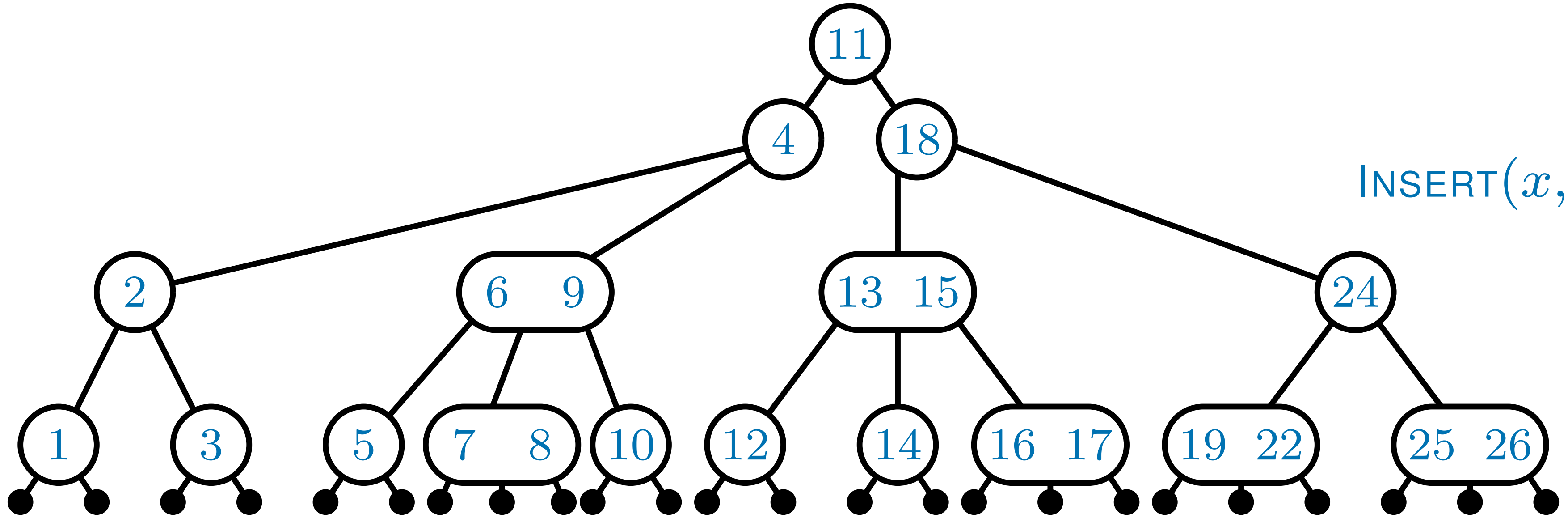
As each SPLIT takes $O(1)$ time, overall INSERT takes $O(\log n)$ time

The INSERT operation



INSERT$(x, 20)$

To perform INSERT(x, k),

Step 1: Search for the key k as if performing FIND(k).

SPLIT *4-nodes as we go down*

Step 2: If the bottom node is a 2-node,

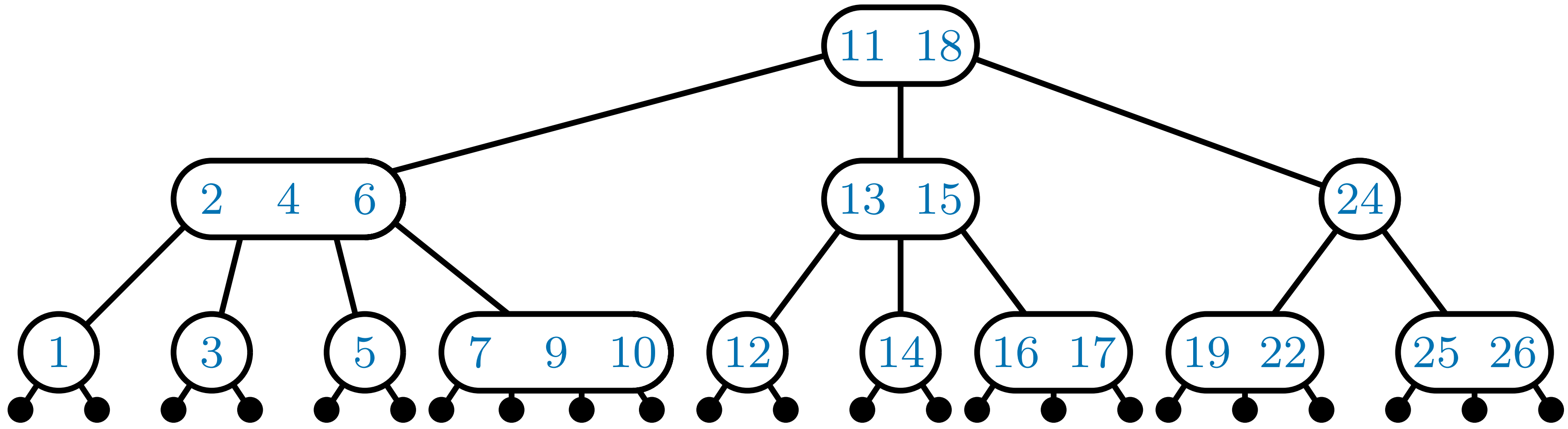
insert (x, k), converting it into a 3-node

Step 3: If the bottom node is a 3-node,

insert (x, k), converting it into a 4-node

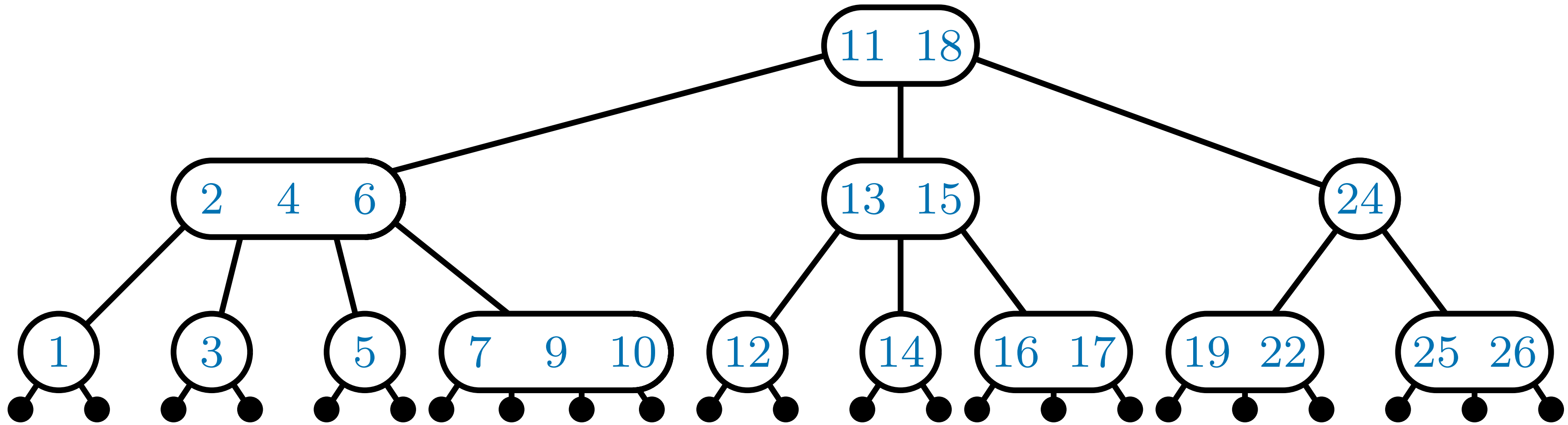
As each SPLIT takes $O(1)$ time, overall INSERT takes $O(\log n)$ time

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

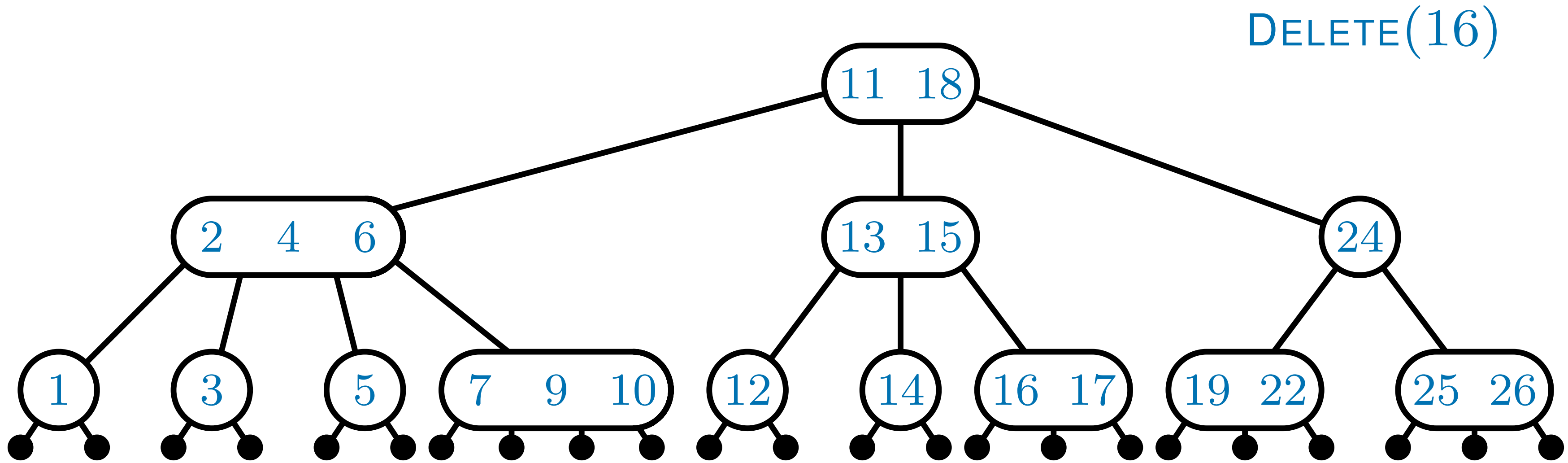
The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

The DELETE operation



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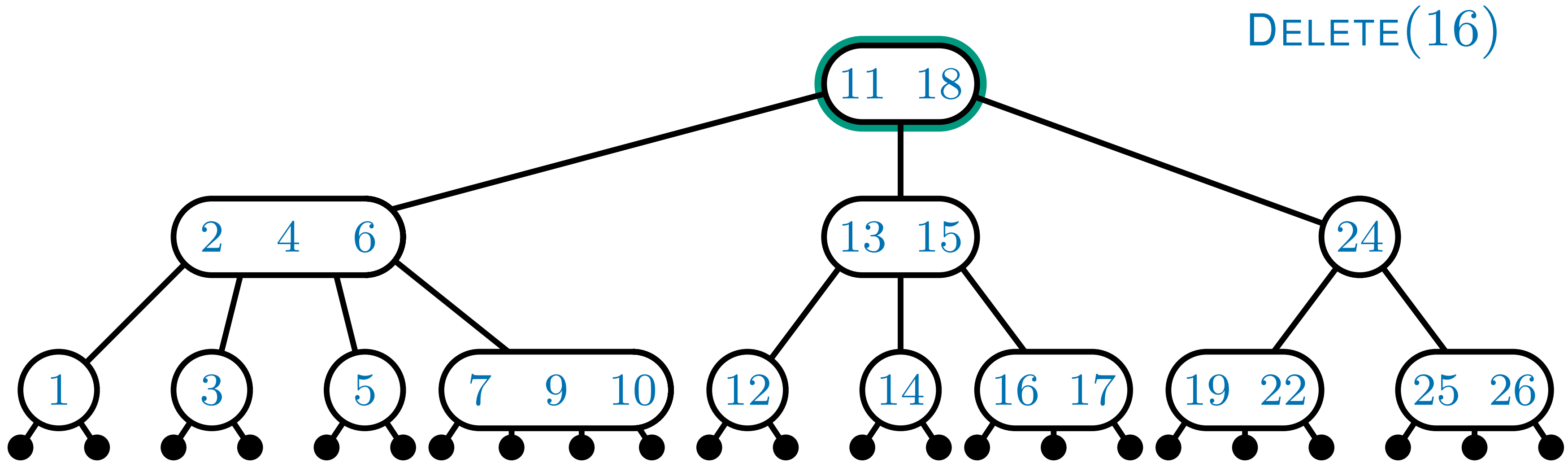
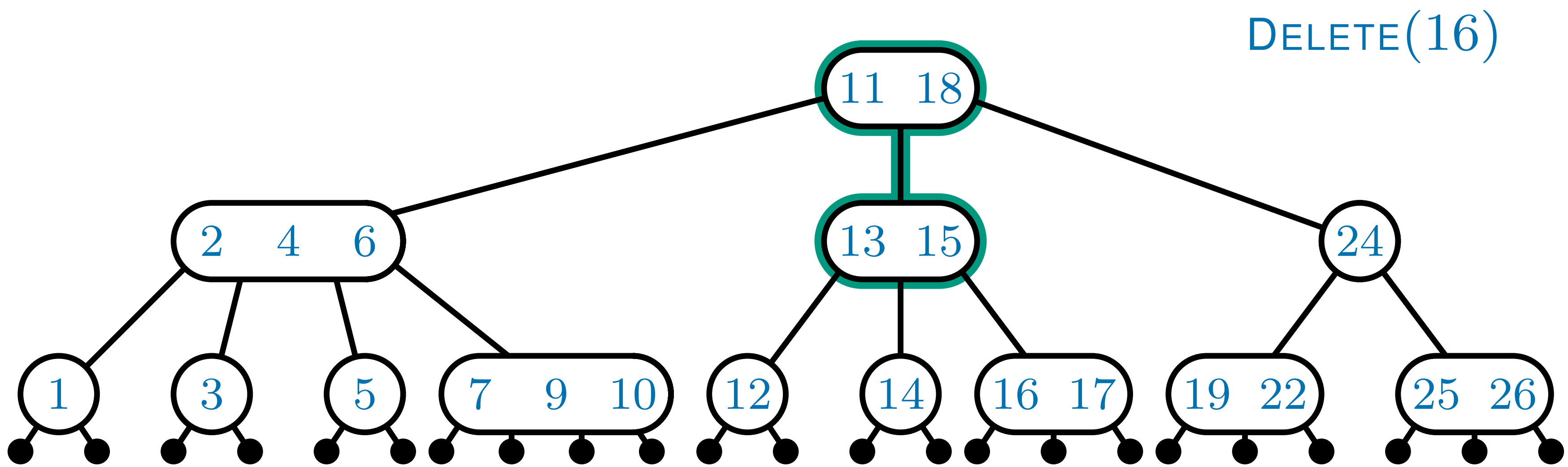
The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

The DELETE operation



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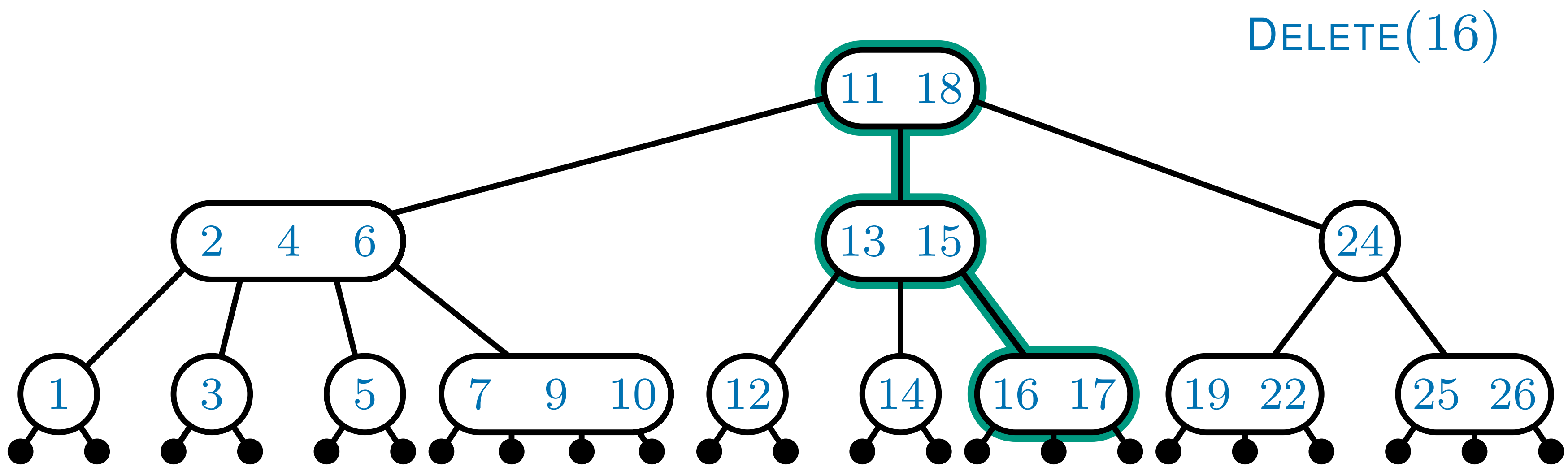
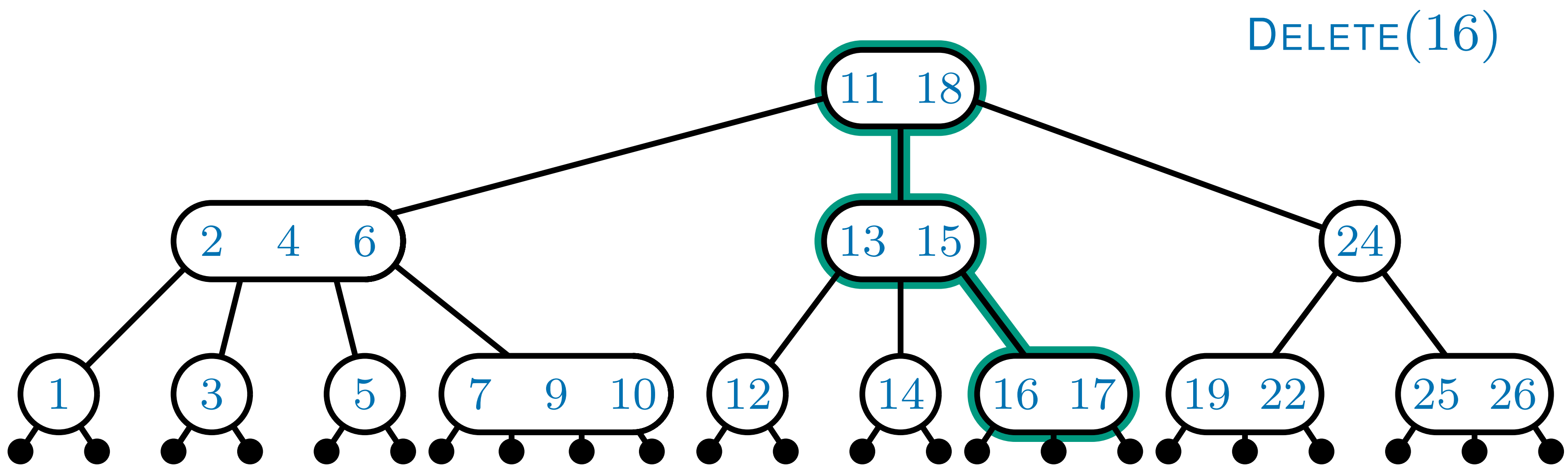
The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

The DELETE operation

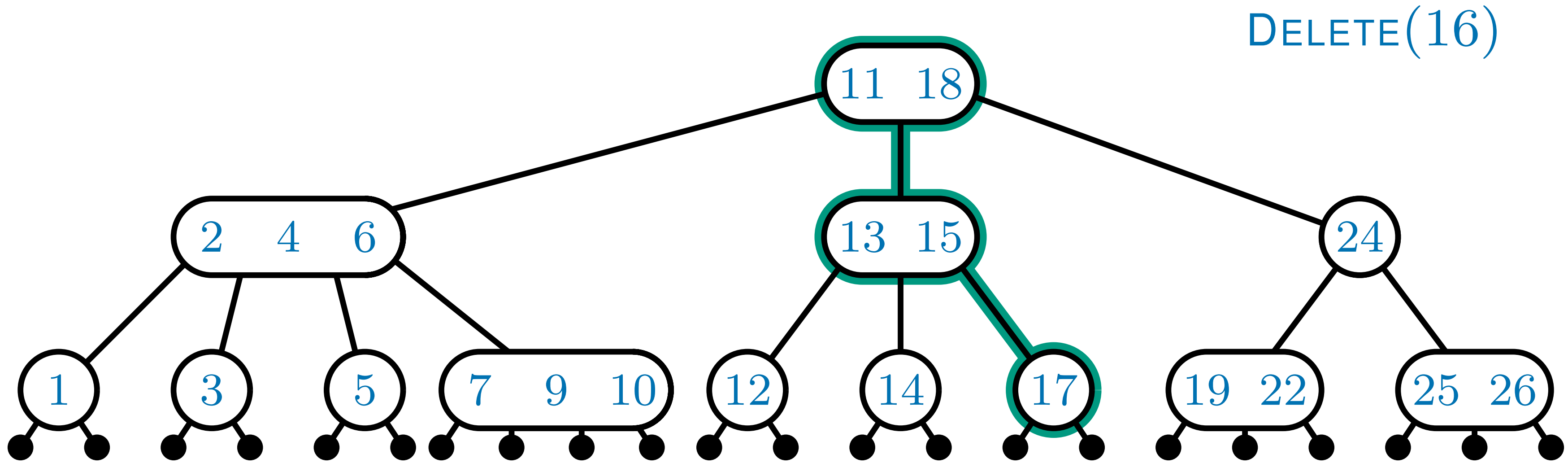


To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

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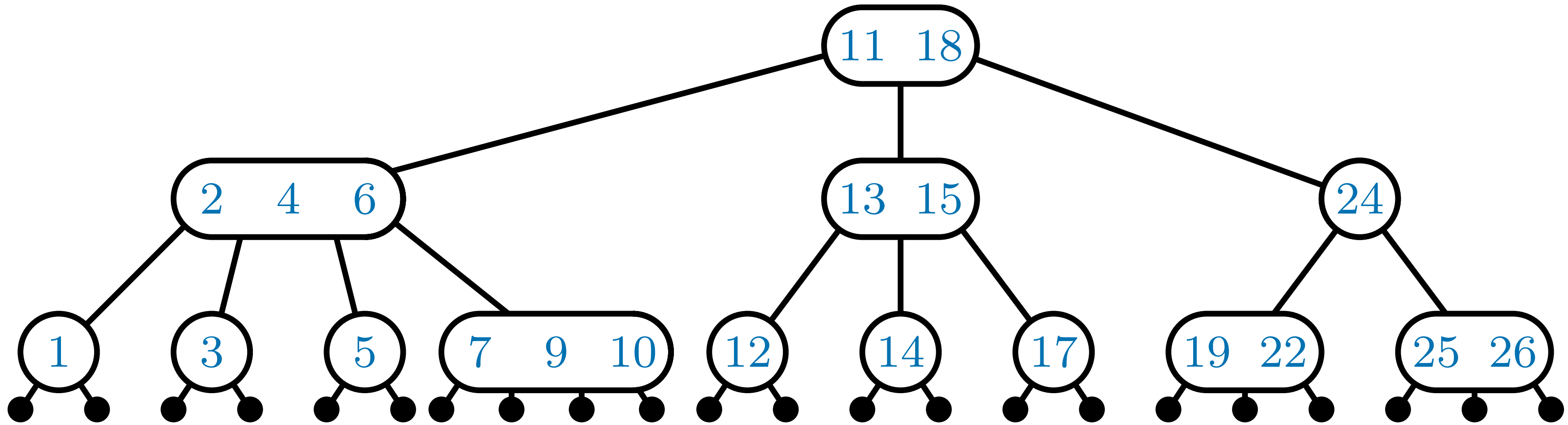
delete (x, k), converting it into a 2-node

The DELETE operation

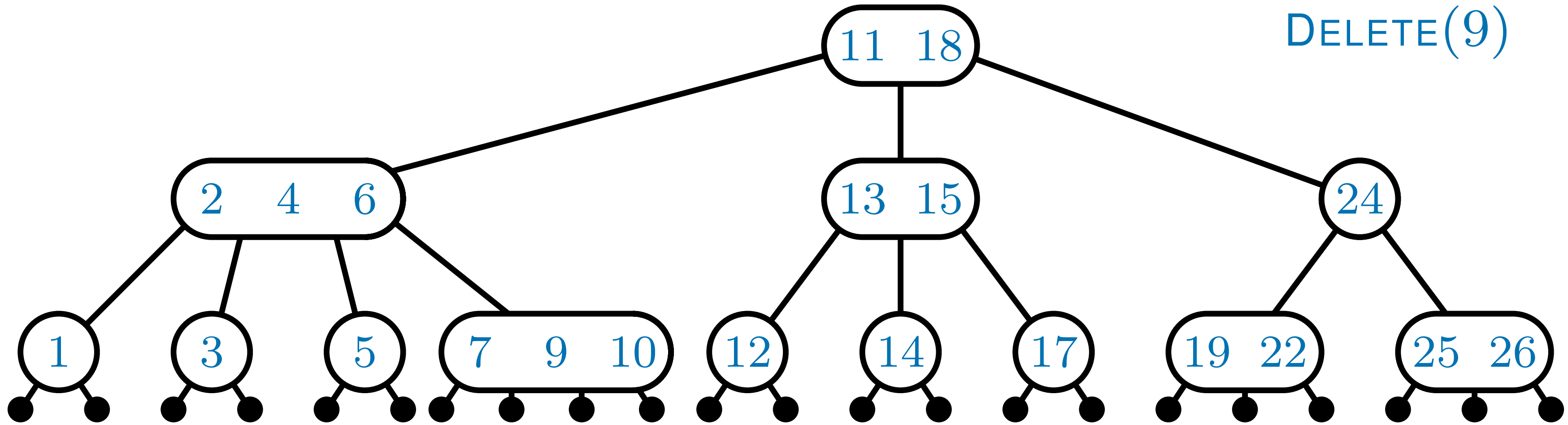


To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

The DELETE operation



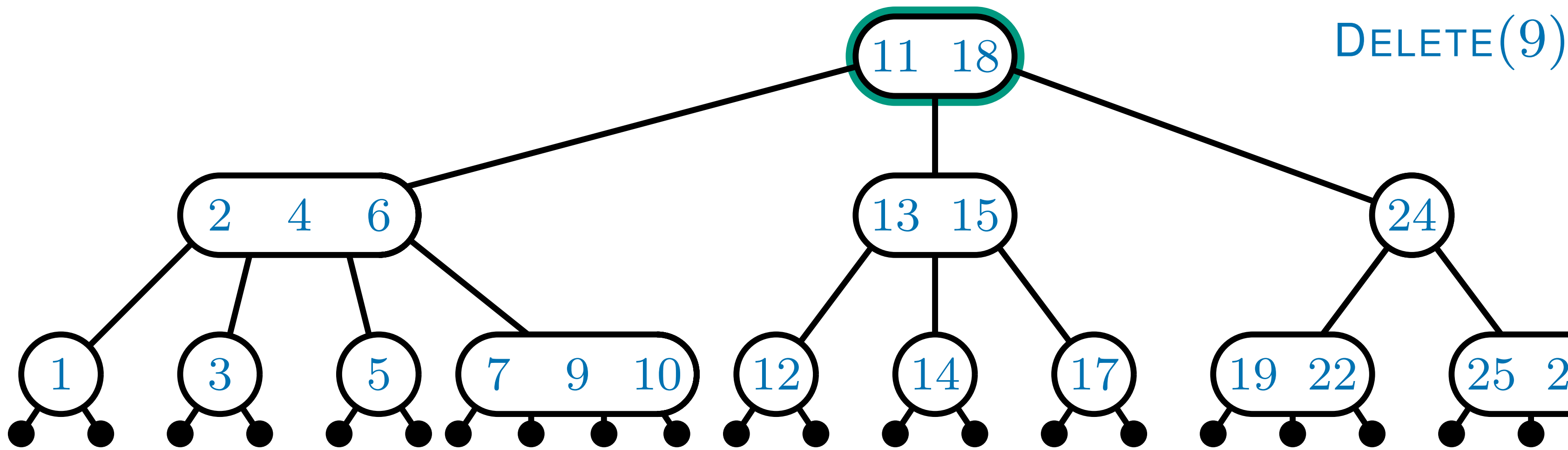
DELETE(9)

To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

The DELETE operation



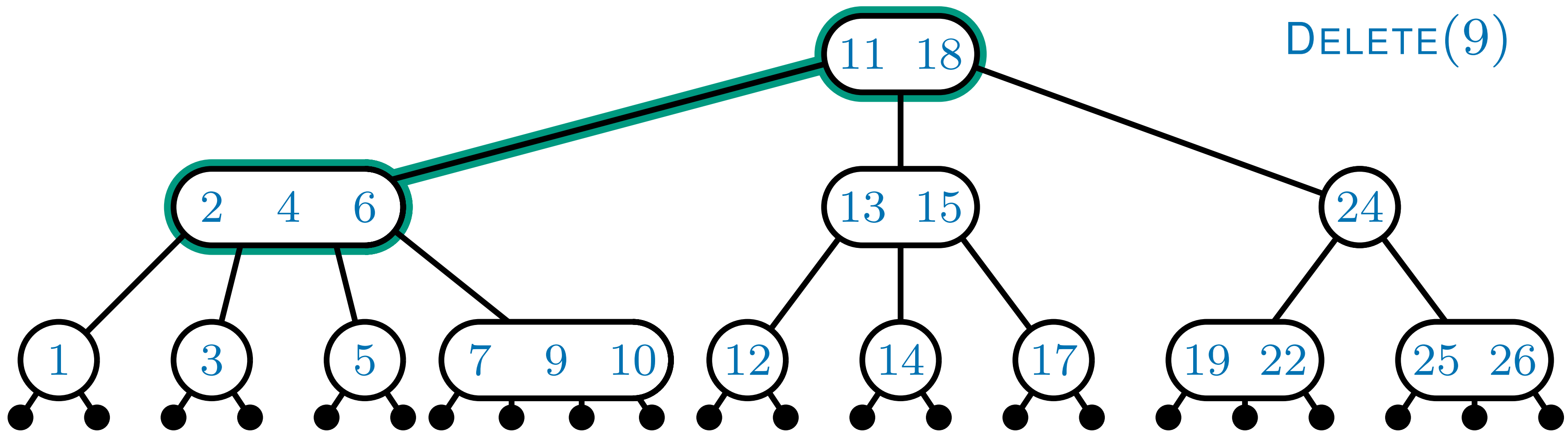
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,

delete (x, k), converting it into a 2-node

The DELETE operation



DELETE(9)

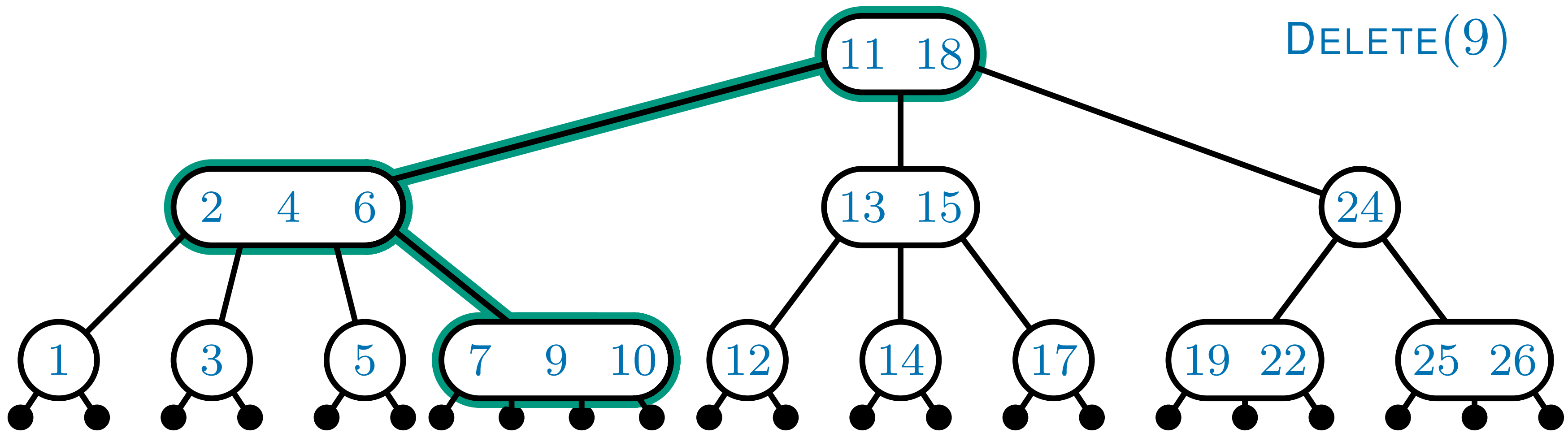
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,

delete (x, k), converting it into a 2-node

The DELETE operation



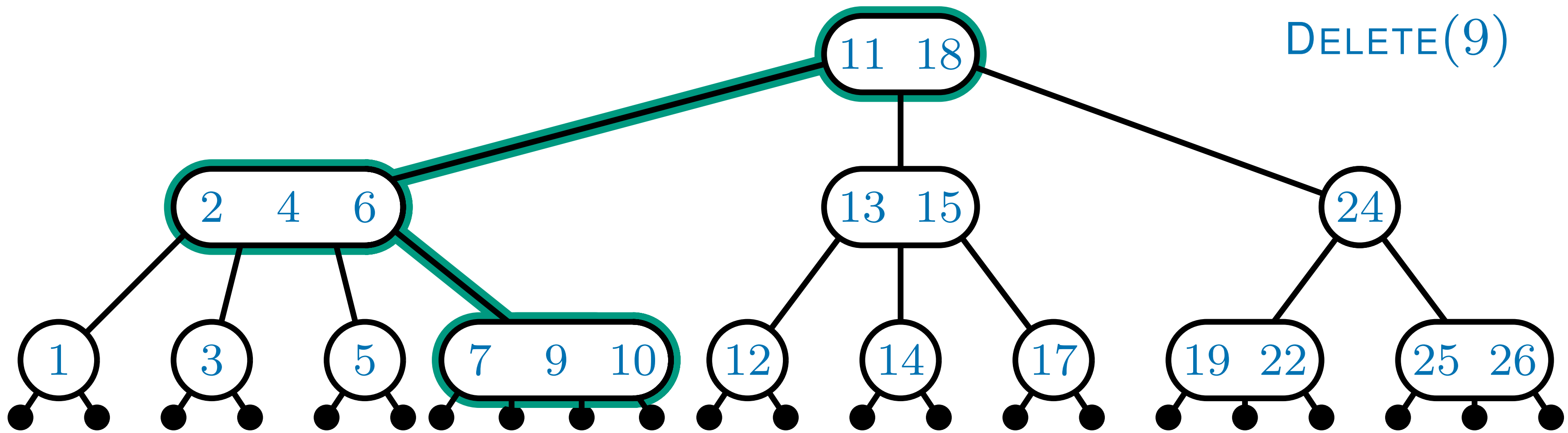
DELETE(9)

To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

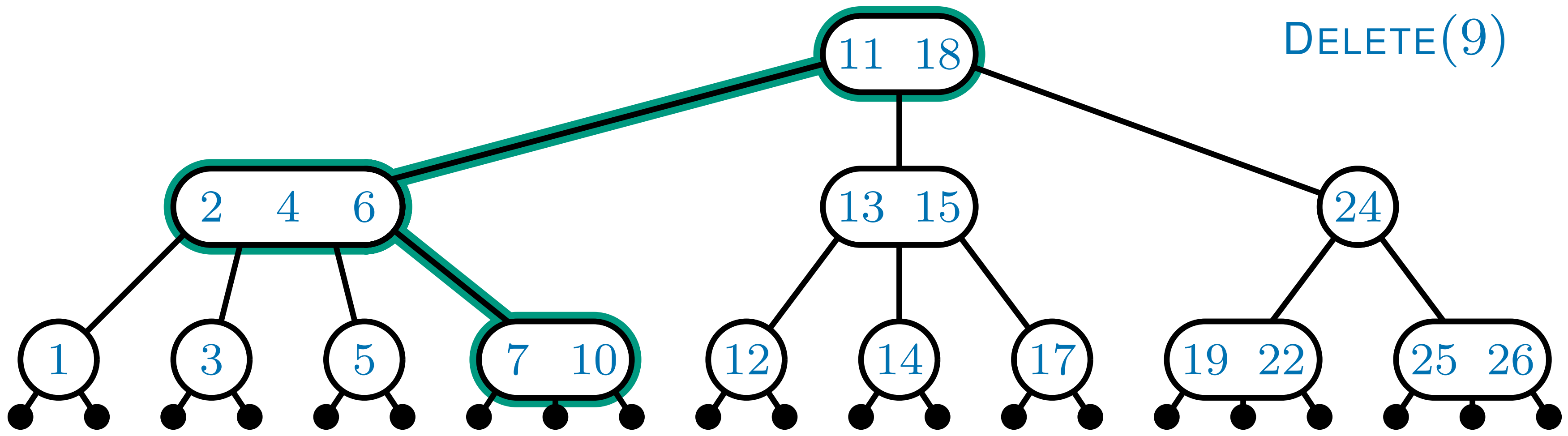
Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation

DELETE(9)



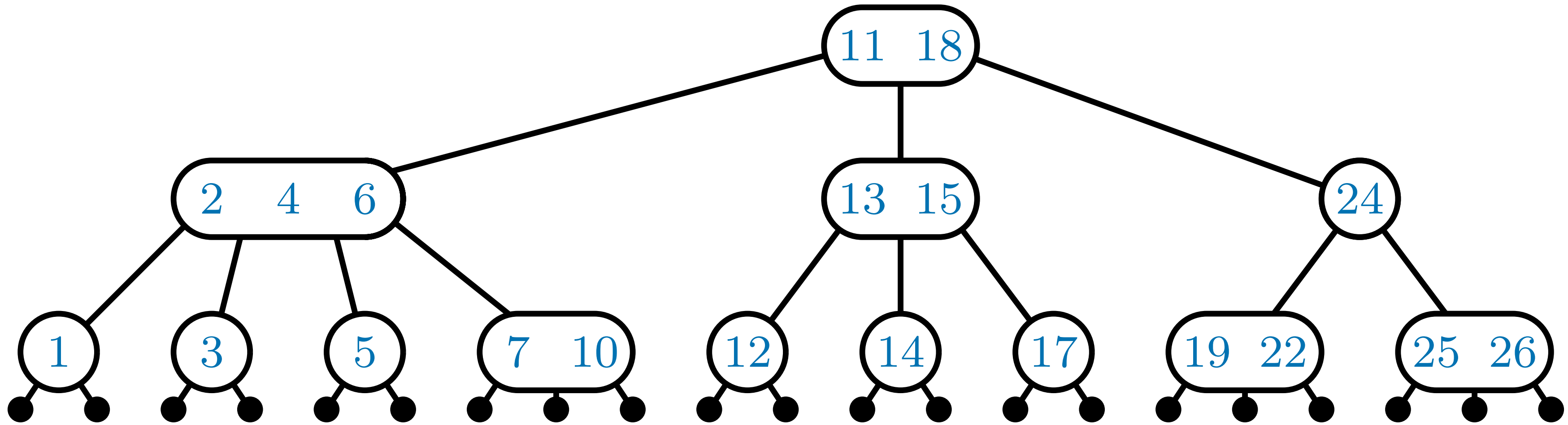
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

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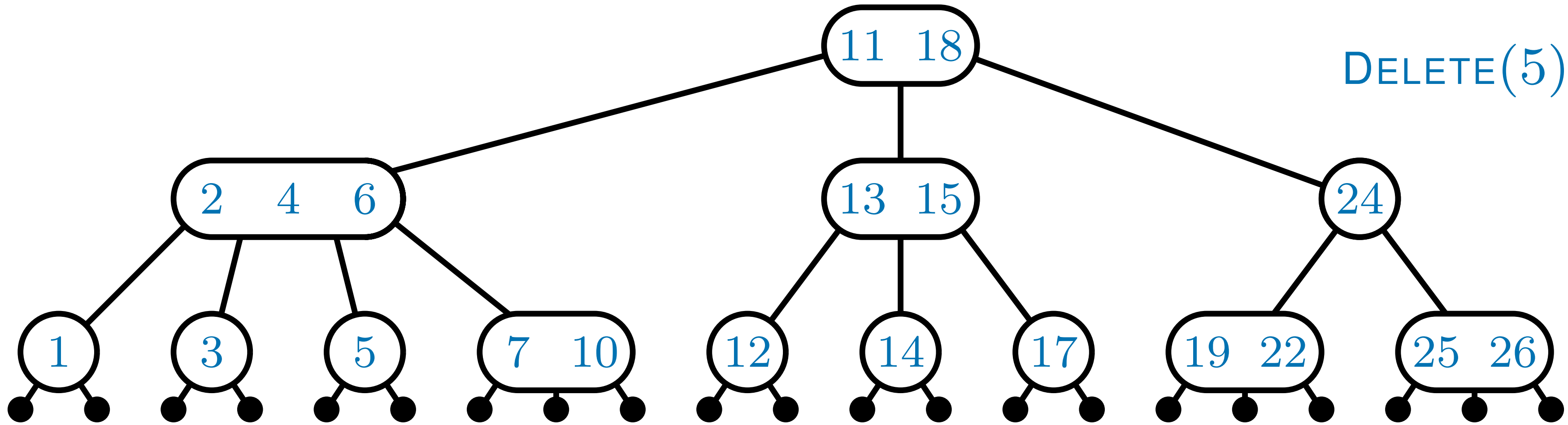
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



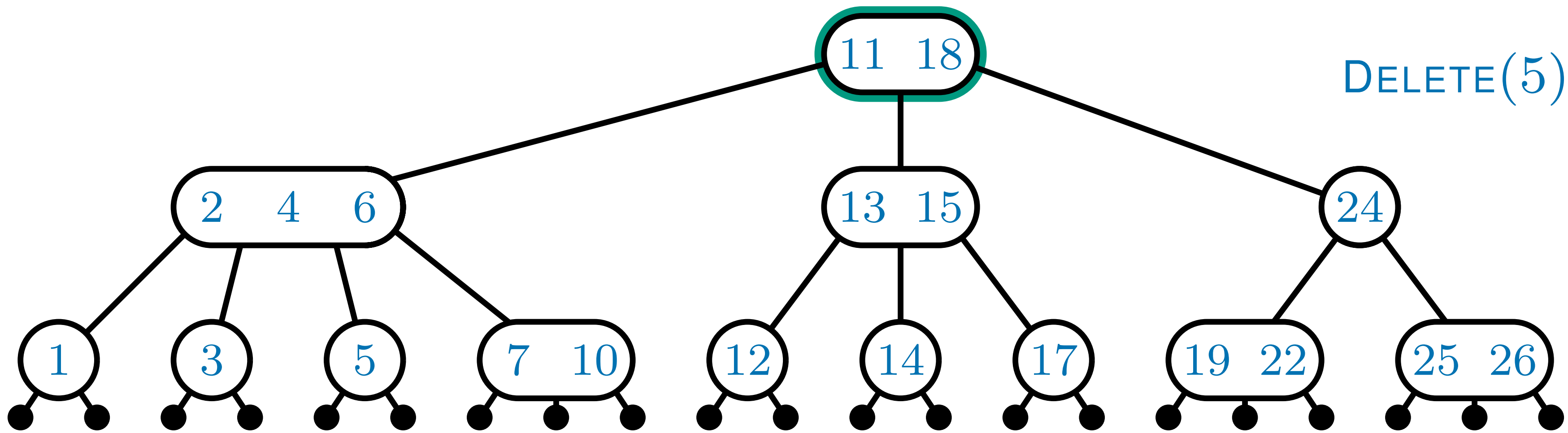
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



DELETE(5)

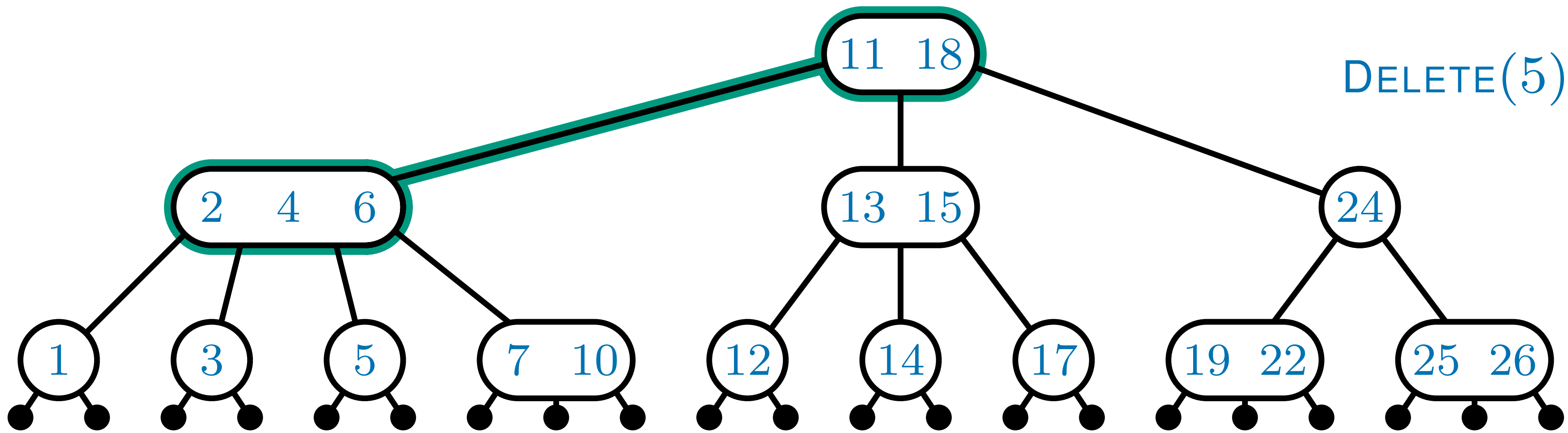
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



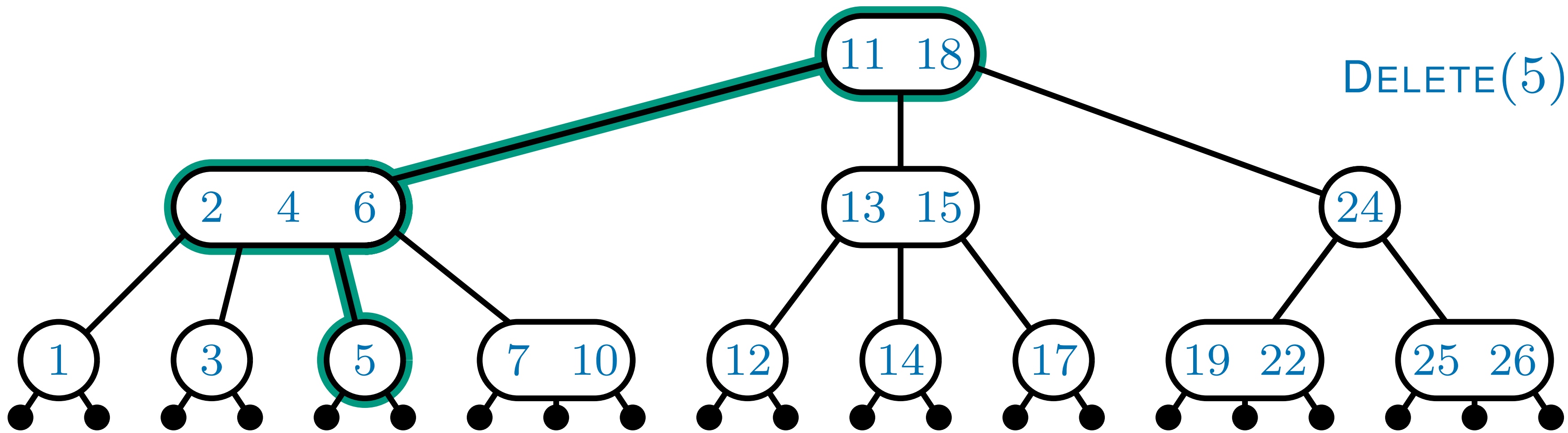
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



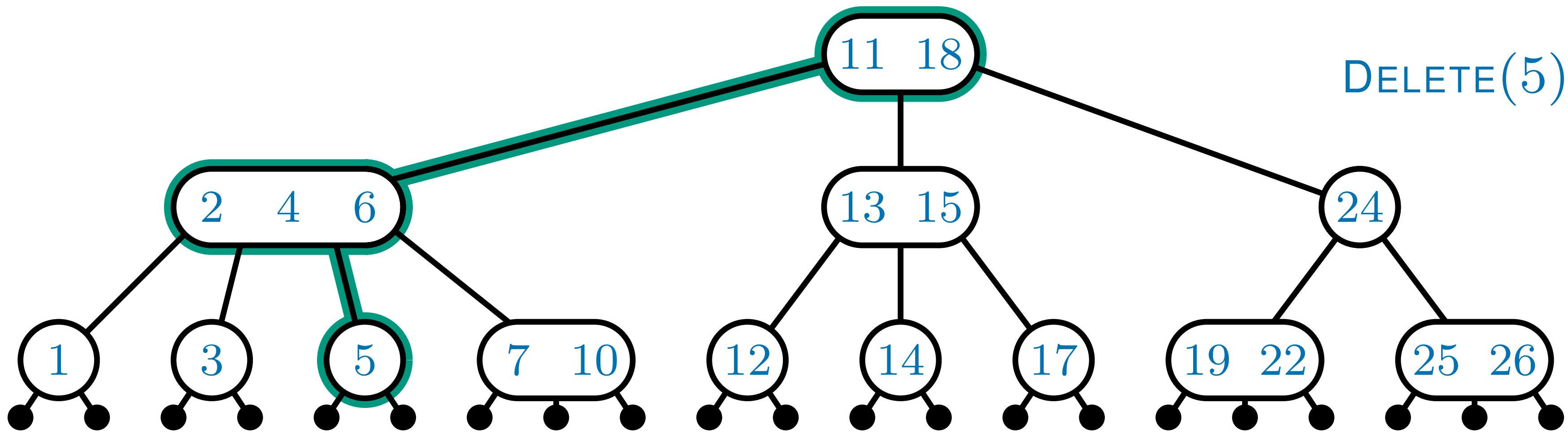
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

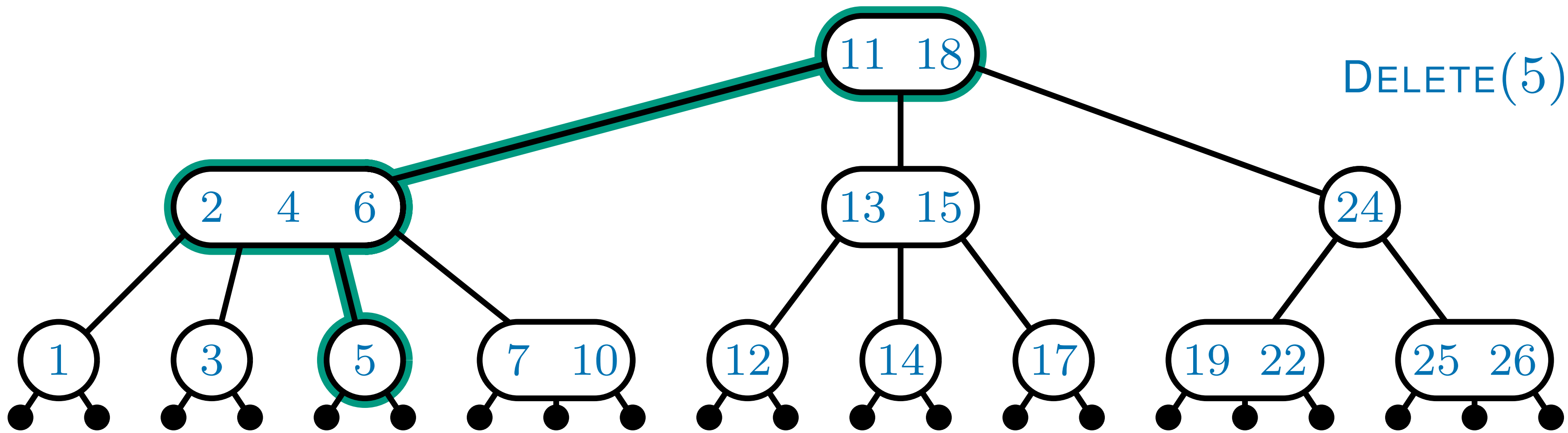
Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

Step 4: If the leaf is a 2-node,

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

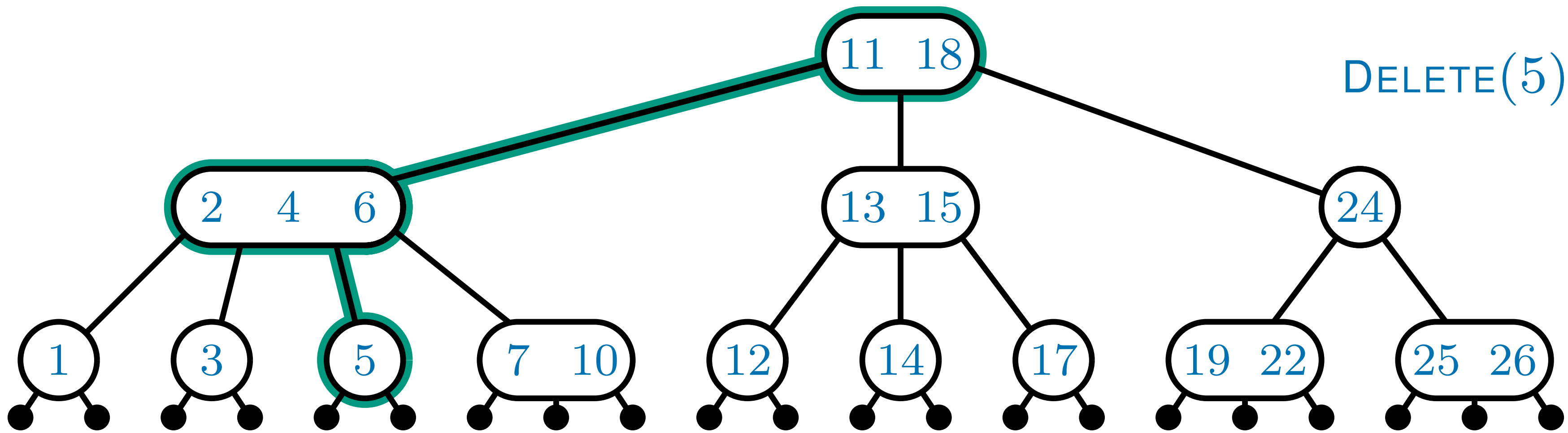
Step 1: Search for the key k using FIND(k).

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

Step 4: If the leaf is a 2-node, ???

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).

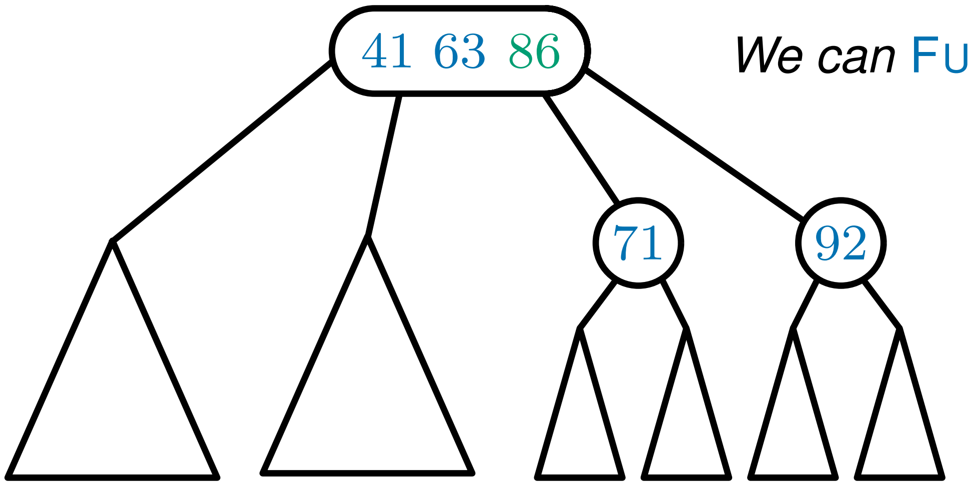
Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

Step 4: If the leaf is a 2-node, ???

We will make sure this *never* happens

FUSING 2-nodes



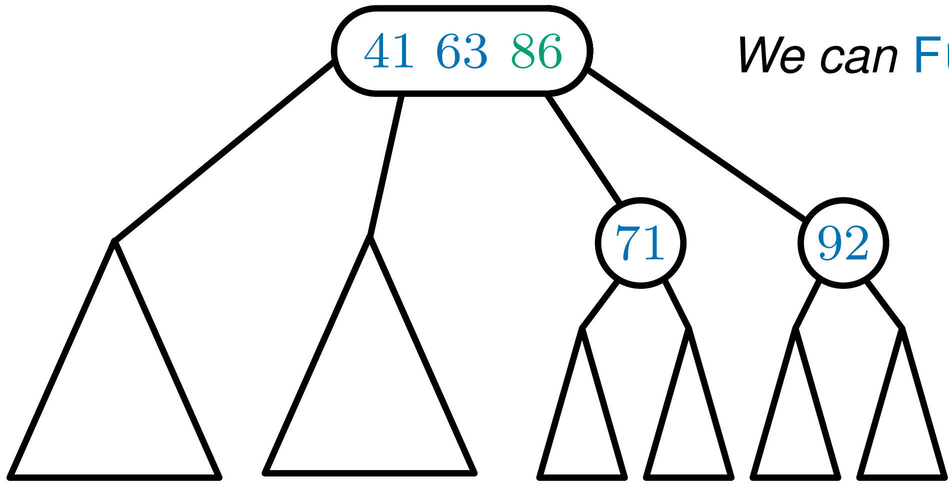
We can FUSE *two 2-nodes (with the same parent)*
into a 4-node

if that parent isn't a 2-node

FUSING 2-nodes

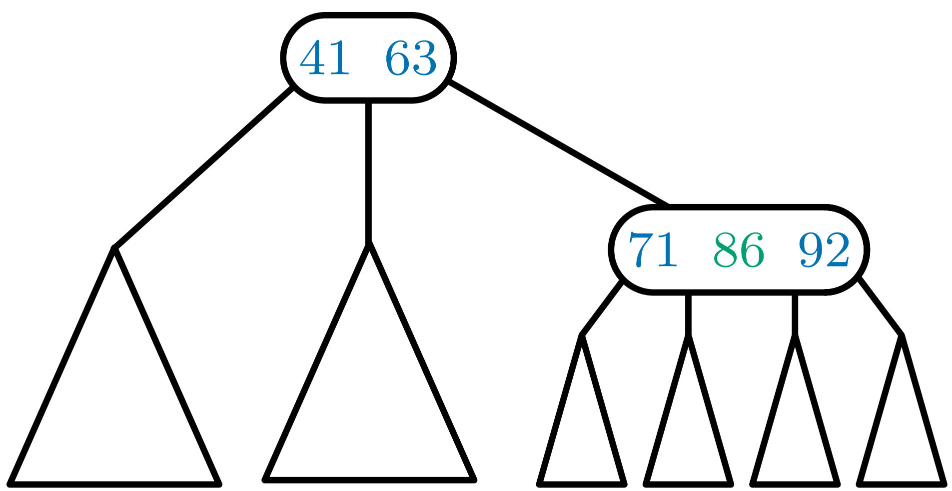
We can FUSE *two 2-nodes (with the same parent)*
into a 4-node

if that parent isn't a 2-node

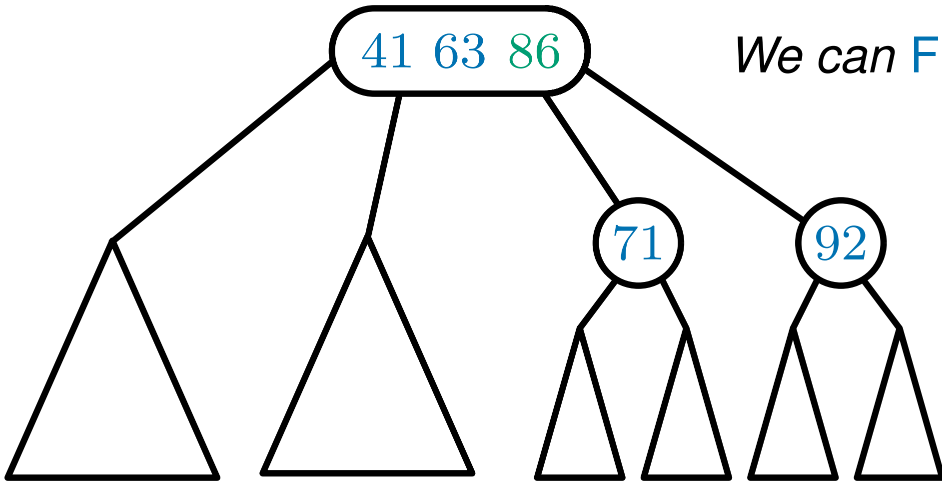


BEFORE

AFTER



FUSING 2-nodes

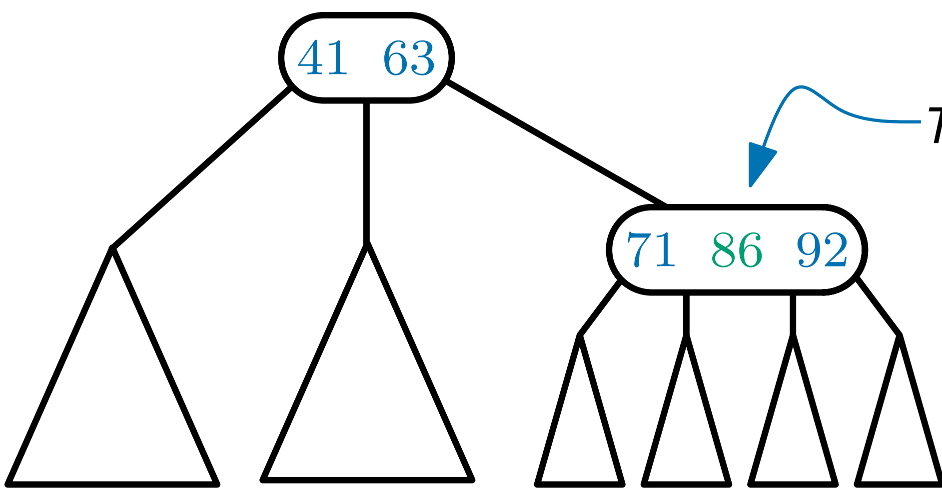


We can FUSE *two 2-nodes (with the same parent) into a 4-node*

if that parent isn't a 2-node

BEFORE

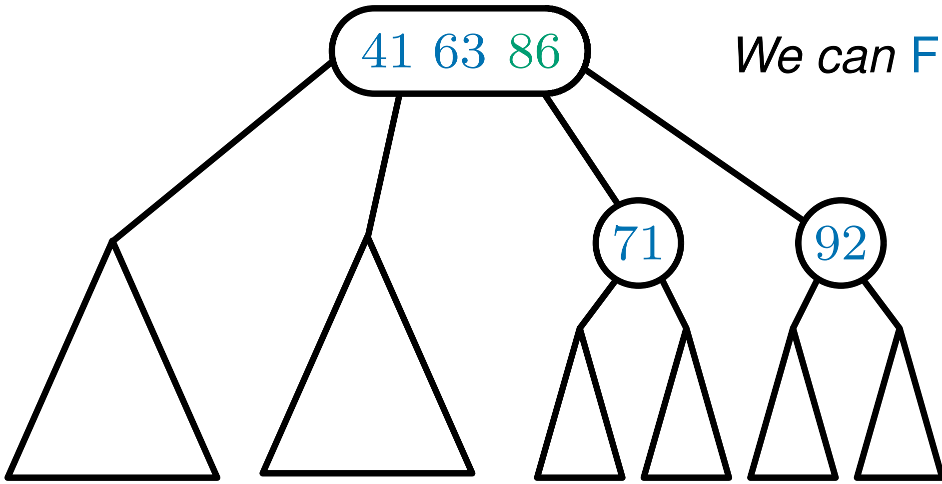
AFTER



The extra key is pulled down from the parent

(so it won't work if the parent is a 2-node)

FUSING 2-nodes



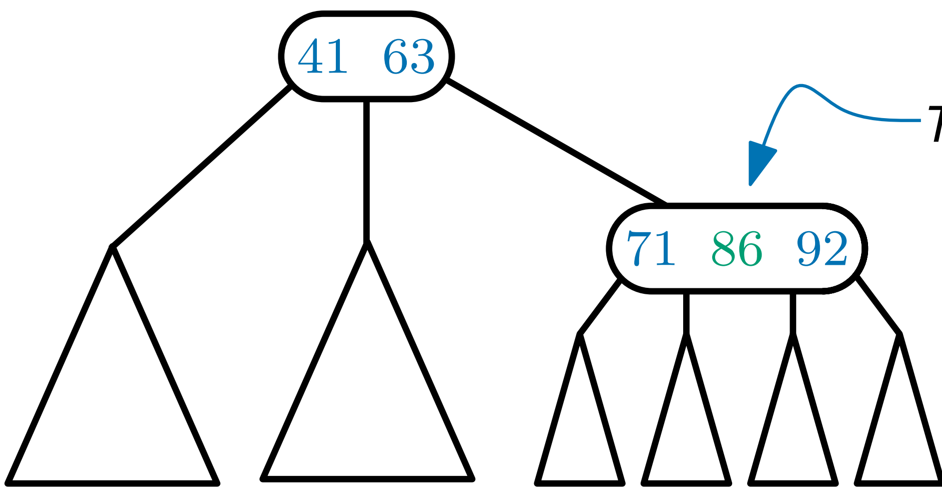
We can FUSE *two 2-nodes (with the same parent)*
into a 4-node

if that parent isn't a 2-node

This is the opposite of a
SPLIT operation

BEFORE

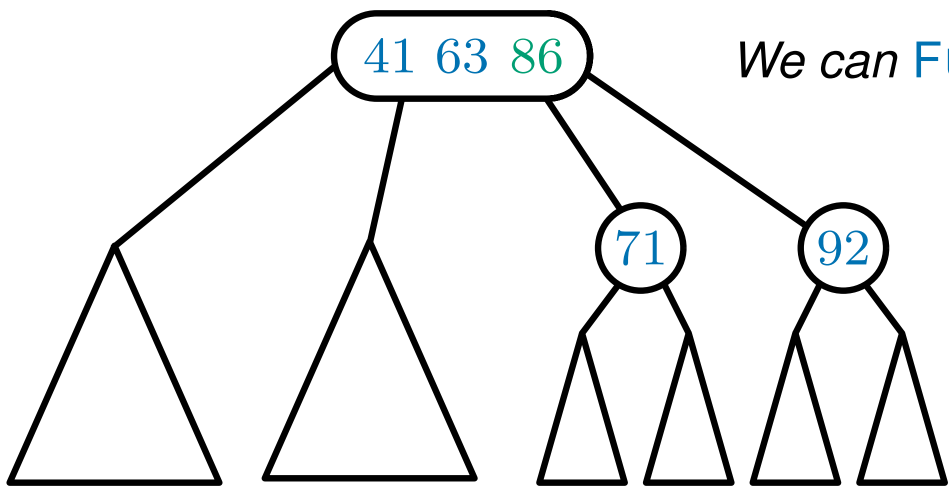
AFTER



The extra key is pulled down from the parent

(so it won't work if the parent is a 2-node)

Fusing 2-nodes



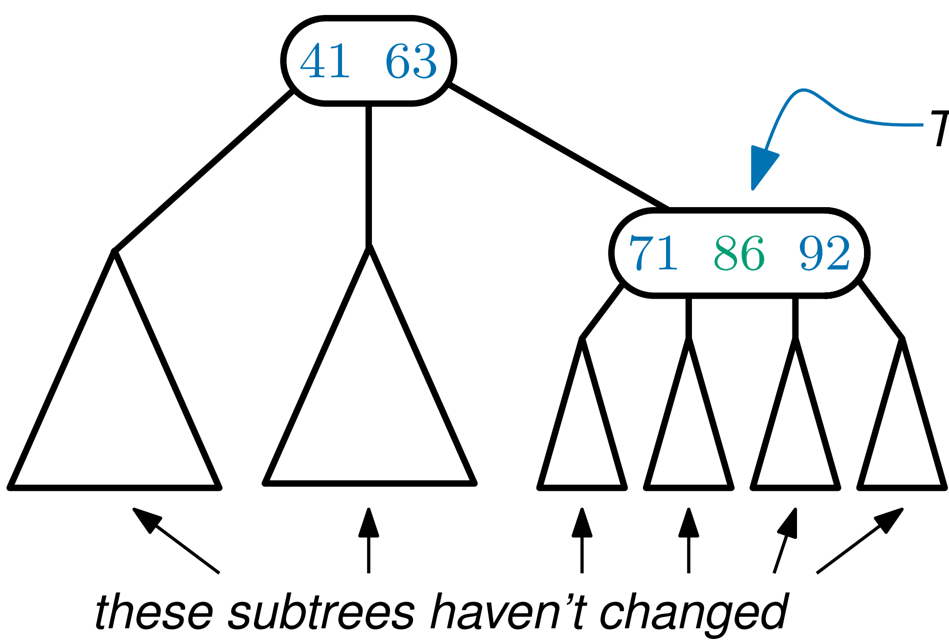
We can Fuse *two 2-nodes (with the same parent) into a 4-node*

if that parent isn't a 2-node

This is the opposite of a Split operation

BEFORE

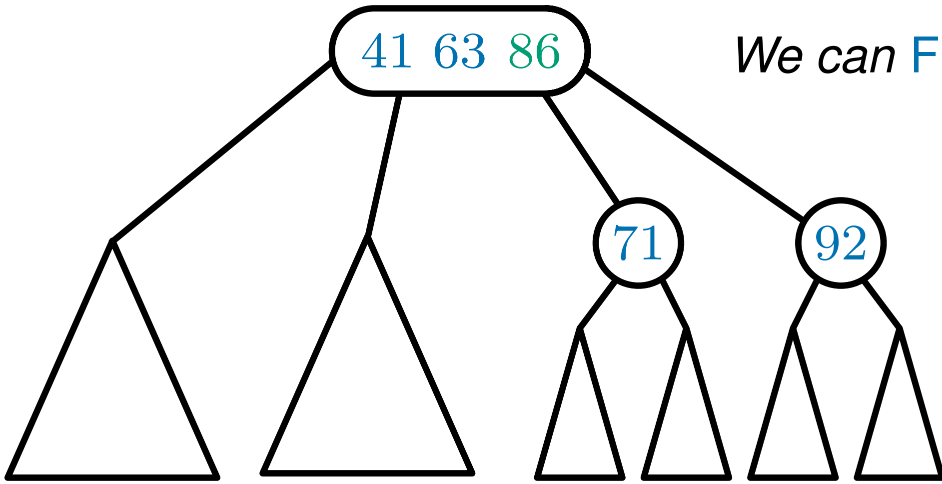
AFTER



The extra key is pulled down from the parent

(so it won't work if the parent is a 2-node)

FUSING 2-nodes



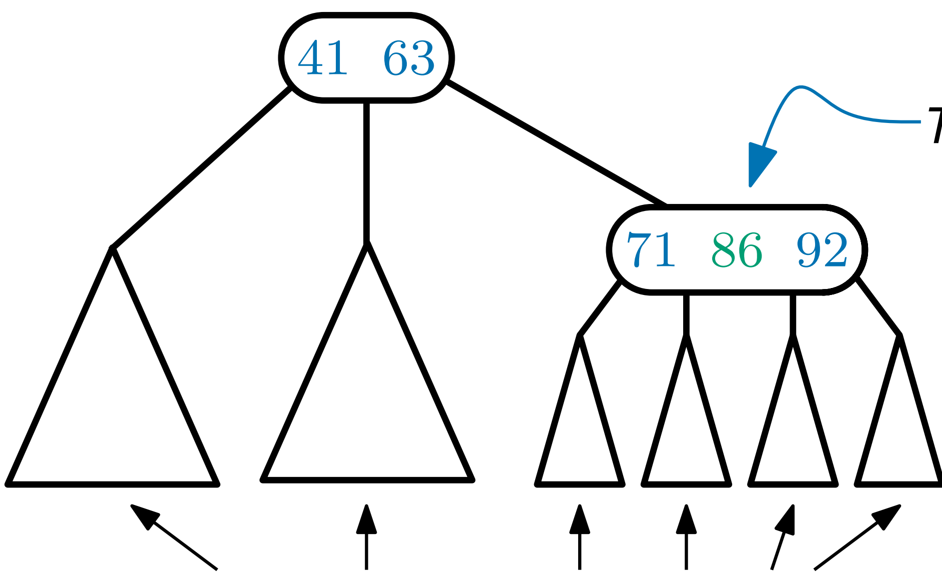
We can FUSE *two 2-nodes (with the same parent)*
into a 4-node

if that parent isn't a 2-node

This is the opposite of a SPLIT operation

BEFORE

AFTER



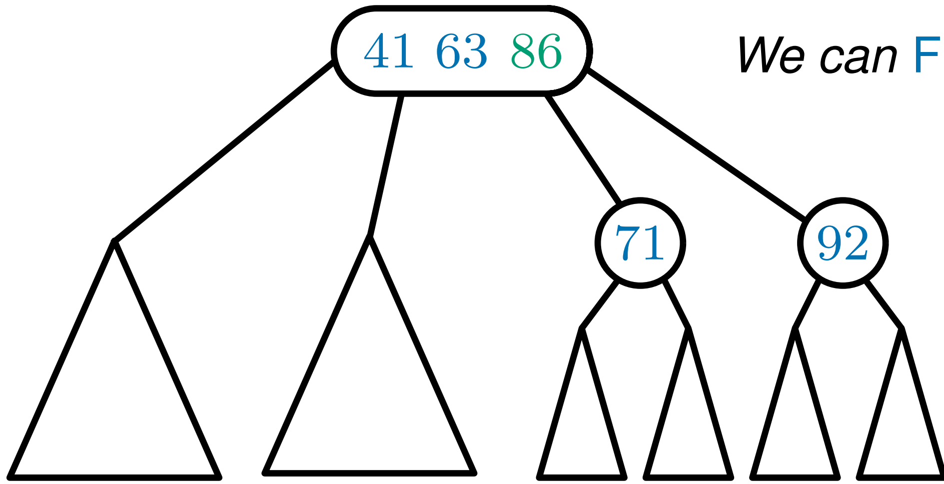
The extra key is pulled down from the parent

(so it won't work if the parent is a 2-node)

no path lengths have changed

these subtrees haven't changed

FUSING 2-nodes



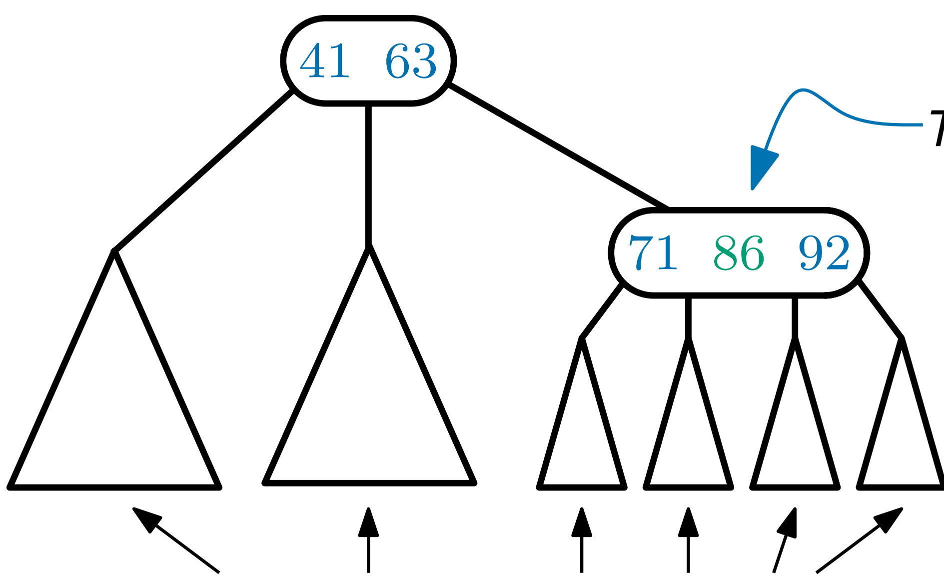
We can FUSE *two 2-nodes (with the same parent)*
into a 4-node

if that parent isn't a 2-node

This is the opposite of a
SPLIT operation

BEFORE

AFTER



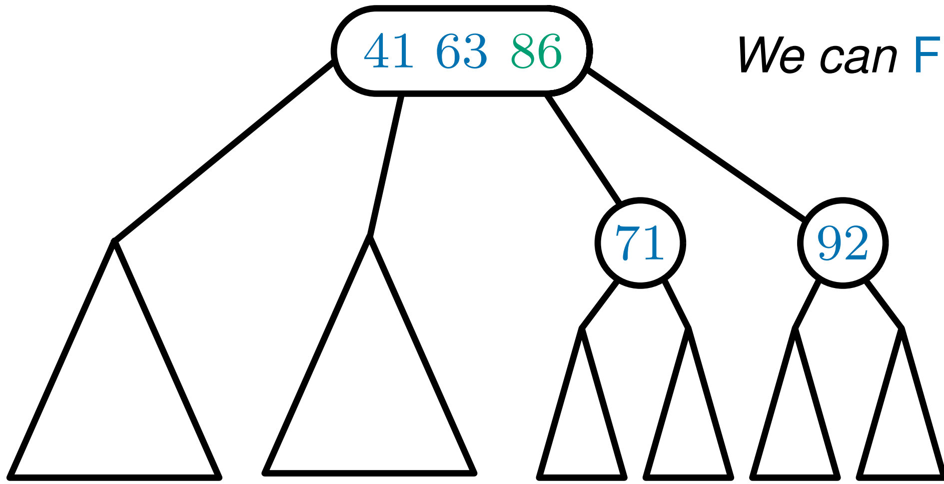
The extra key is pulled down from the parent
(so it won't work if the parent is a 2-node)

no path lengths have changed

(if it was **perfectly balanced**, *it still is)*

these subtrees haven't changed

FUSING 2-nodes



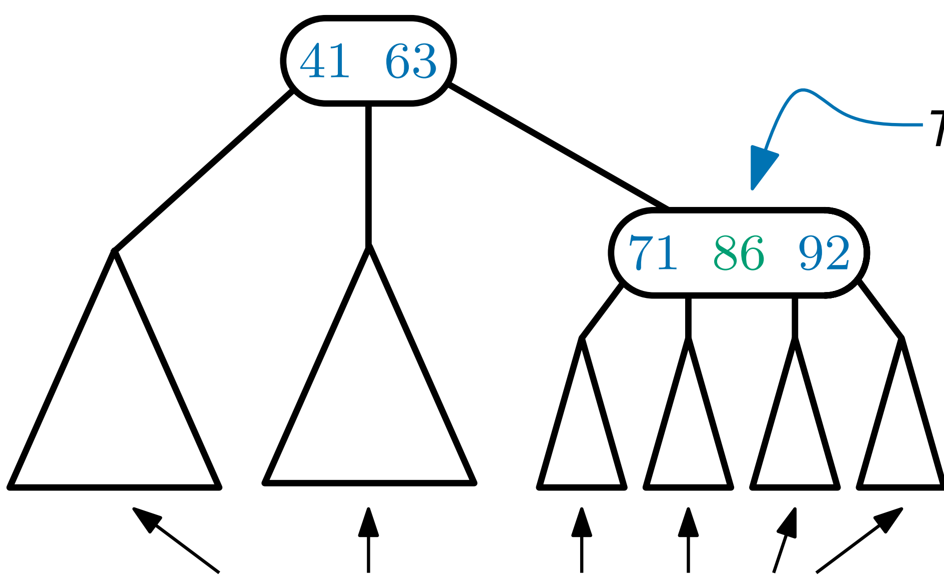
We can FUSE *two 2-nodes (with the same parent)*
into a 4-node

if that parent isn't a 2-node

This is the opposite of a
SPLIT operation

BEFORE

AFTER



The extra key is pulled down from the parent
(so it won't work if the parent is a 2-node)

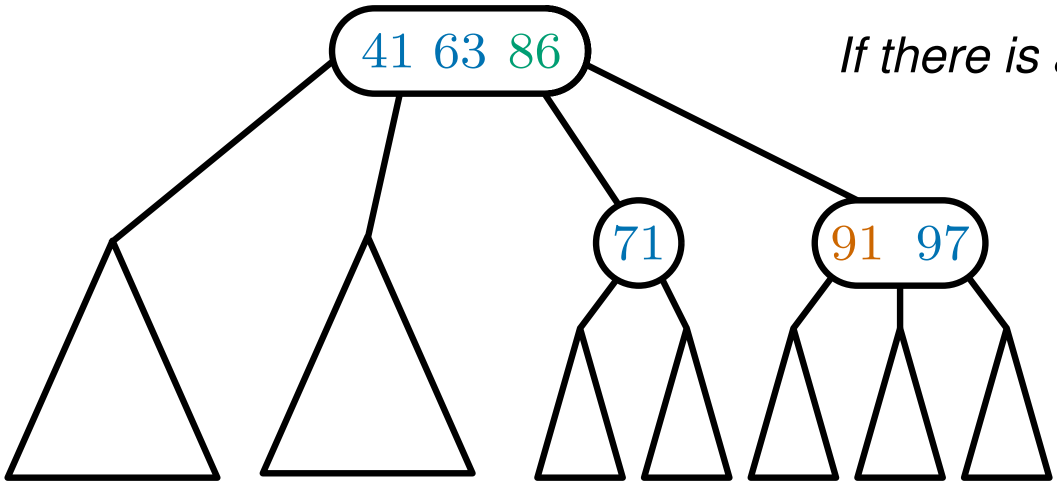
no path lengths have changed

(if it was **perfectly balanced***, it still is)*

FUSE takes $O(1)$ time

these subtrees haven't changed

TRANSFERING keys



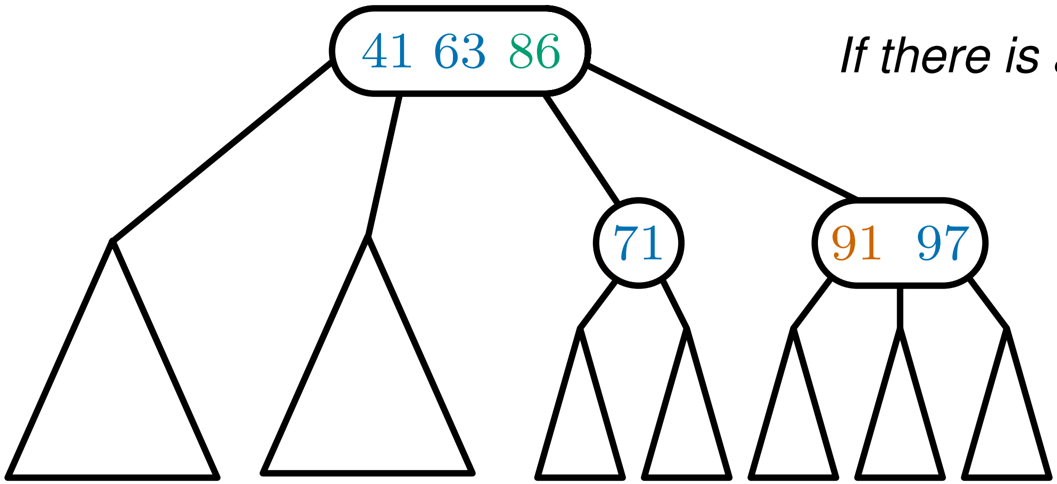
If there is a 2-node and a 3-node

(with the same parent)

we can perform a TRANSFER

(even if the parent is the root)

TRANSFERING keys



If there is a 2-node and a 3-node
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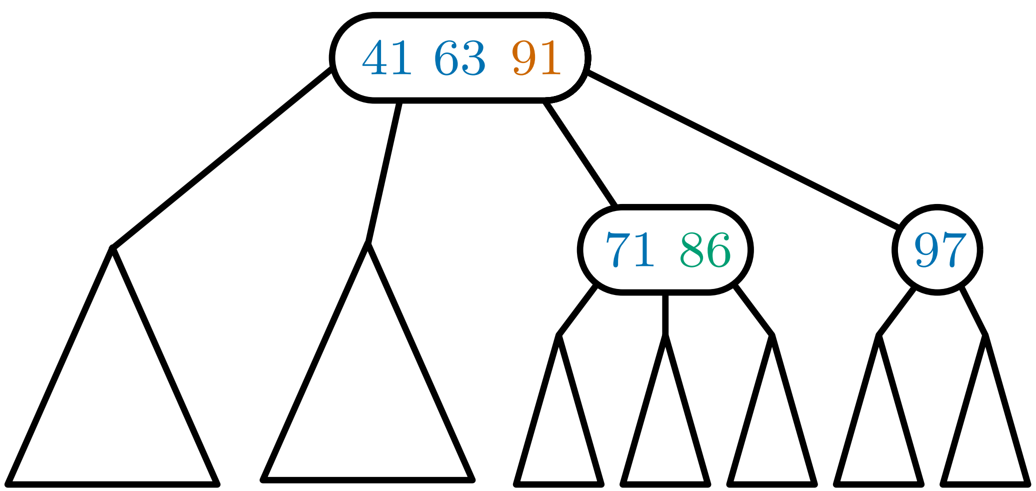
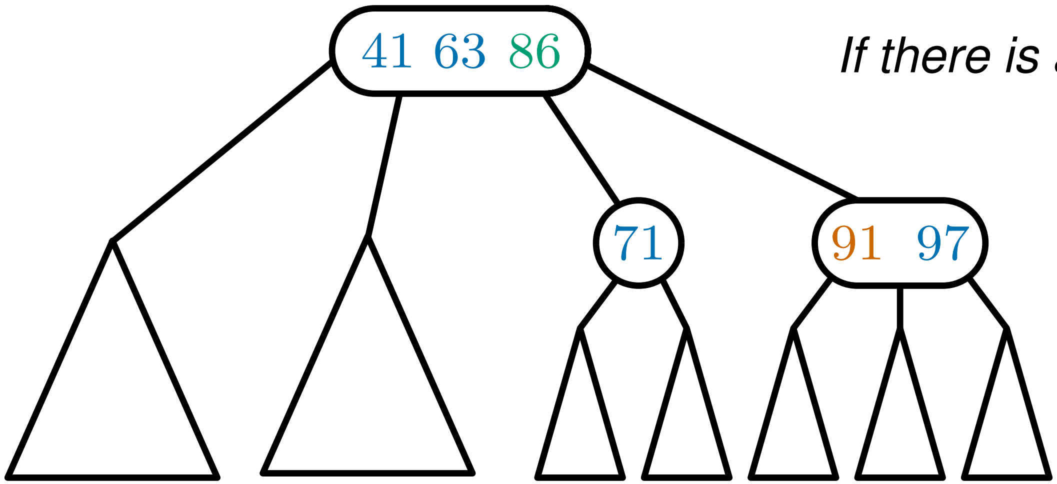
(even if the parent is the root)

BEFORE

AFTER



TRANSFERING keys

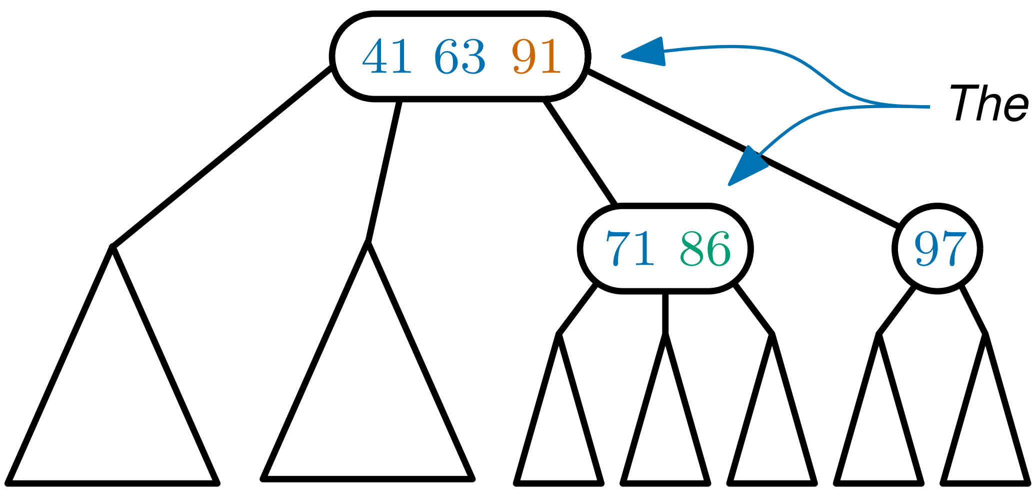


If there is a 2-node and a 3-node
(with the same parent)

we can perform a TRANSFER
(even if the parent is the root)

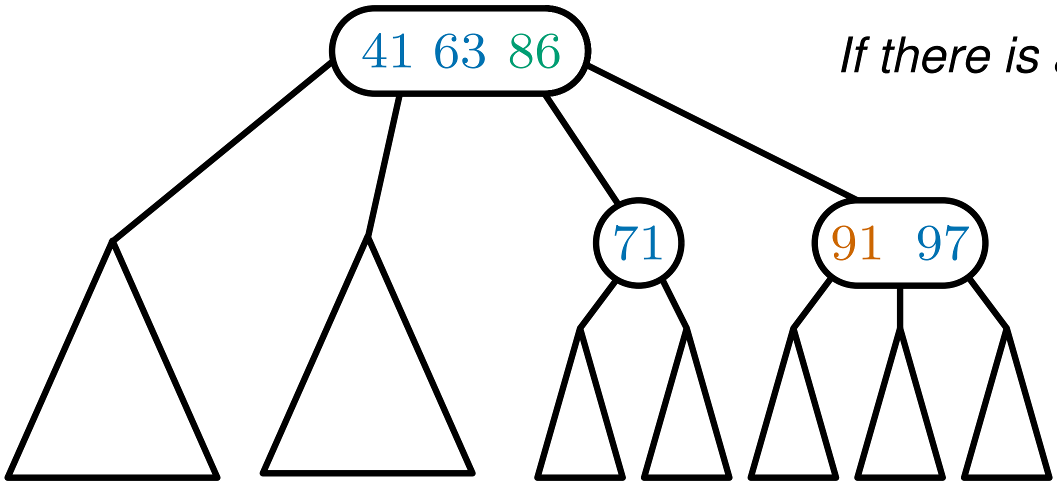
BEFORE

AFTER



The keys have been rearranged

TRANSFERING keys



If there is a 2-node and a 3-node

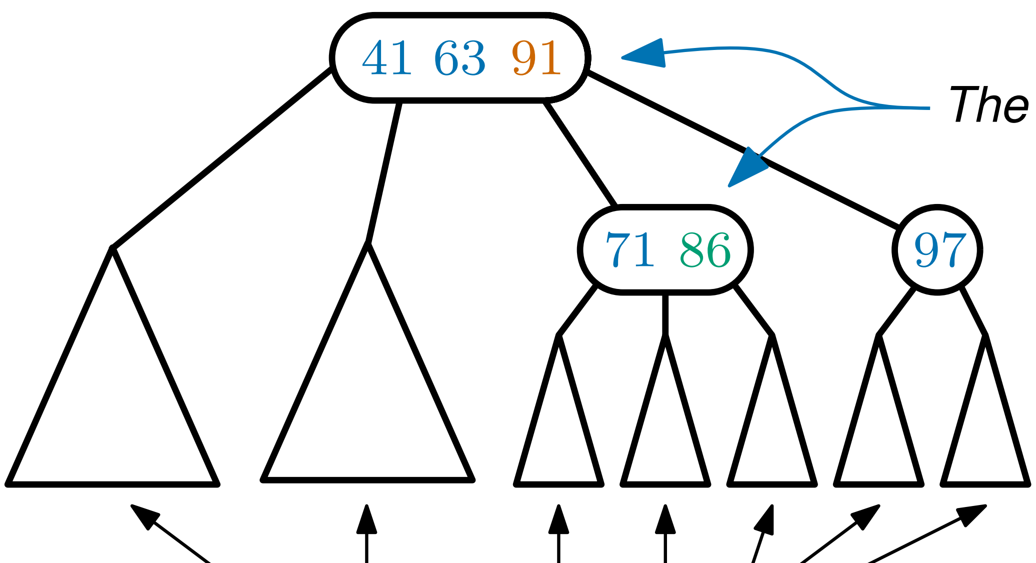
(with the same parent)

we can perform a TRANSFER

(even if the parent is the root)

BEFORE

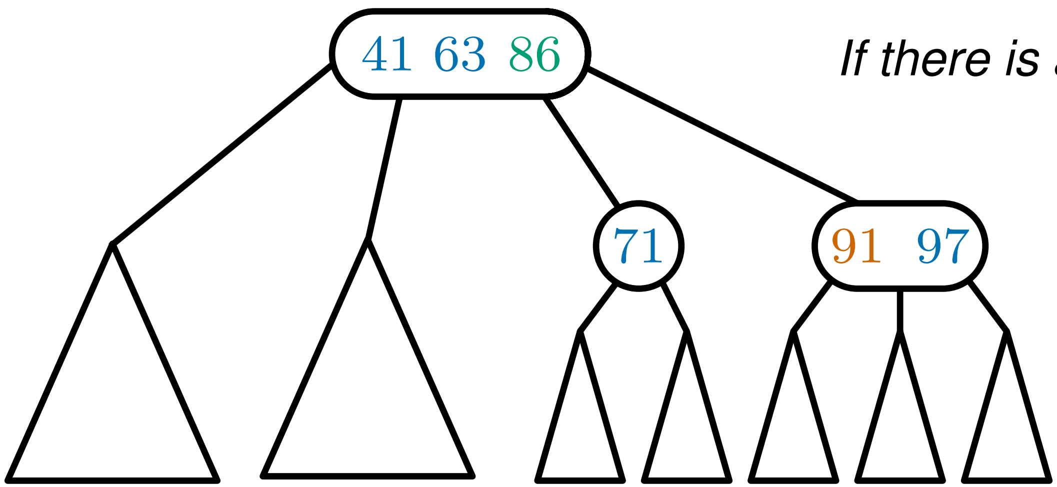
AFTER



The keys have been rearranged

these subtrees haven't changed

TRANSFERING keys



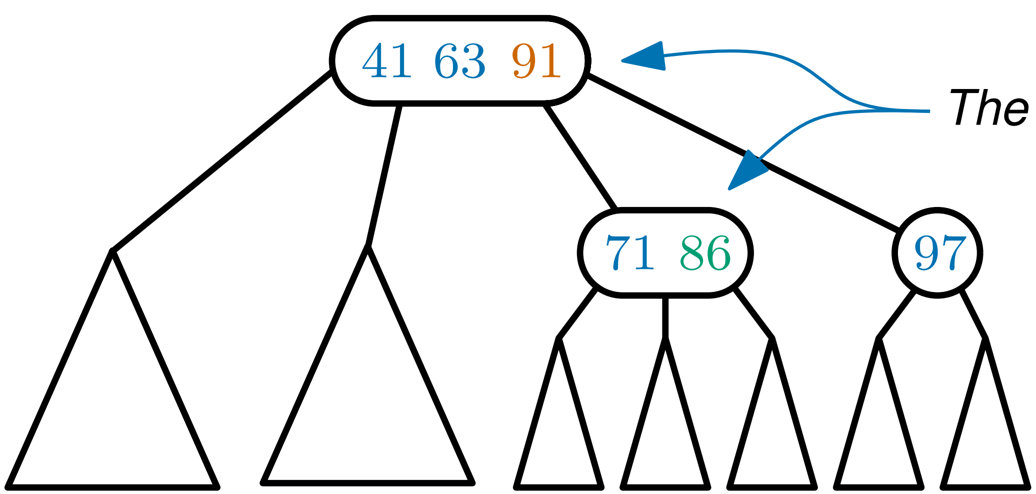
If there is a 2-node and a 3-node
(with the same parent)

we can perform a TRANSFER

(even if the parent is the root)

BEFORE

AFTER

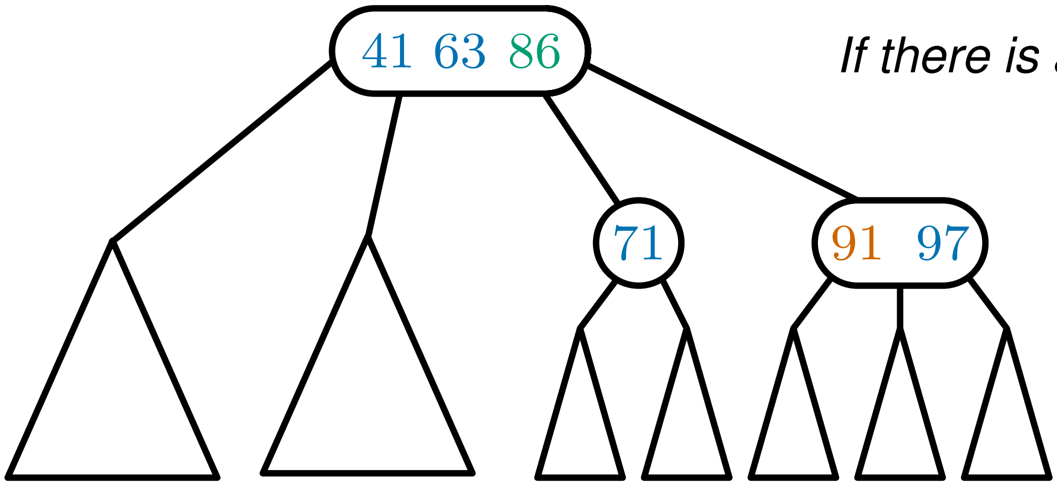


The keys have been rearranged

no path lengths have changed

these subtrees haven't changed

TRANSFERING keys

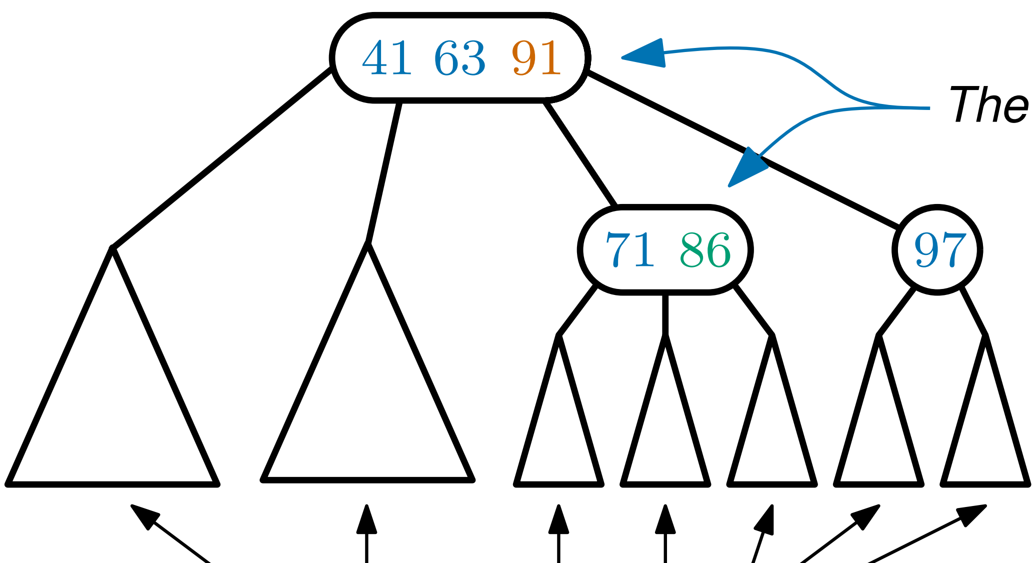


If there is a 2-node and a 3-node
(with the same parent)

we can perform a TRANSFER
(even if the parent is the root)

BEFORE

AFTER



The keys have been rearranged

no path lengths have changed

(if it was ***perfectly balanced****, it still is)*

these subtrees haven't changed

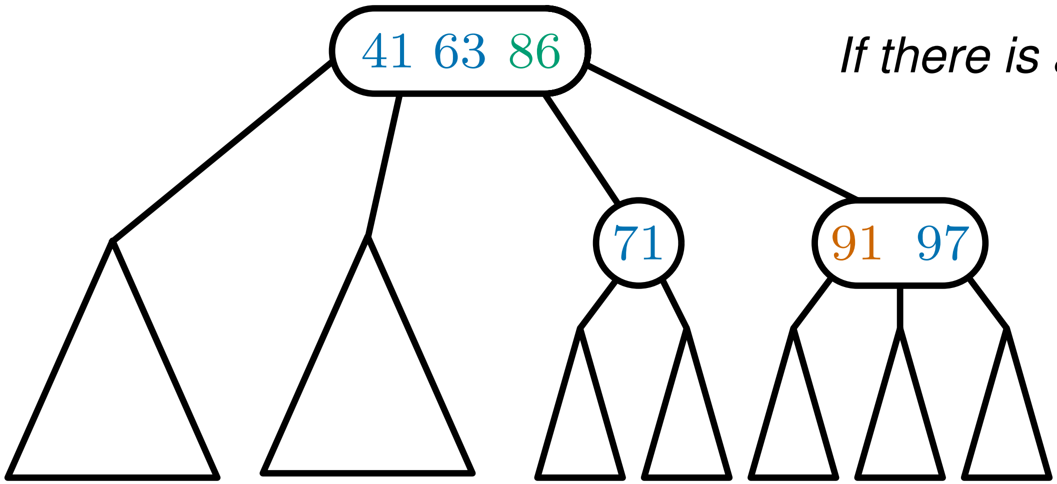
TRANSFERING keys

If there is a 2-node and a 3-node

(with the same parent)

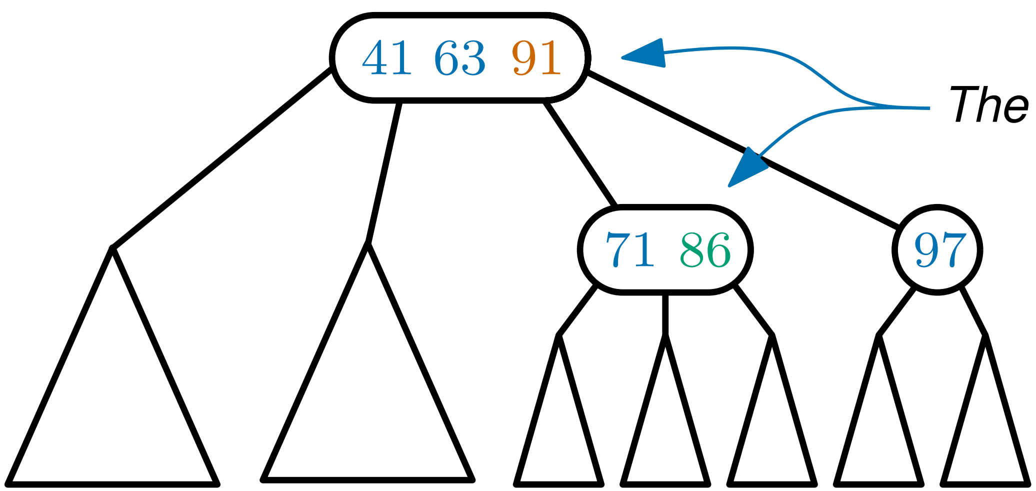
we can perform a TRANSFER

(even if the parent is the root)



BEFORE

AFTER



The keys have been rearranged

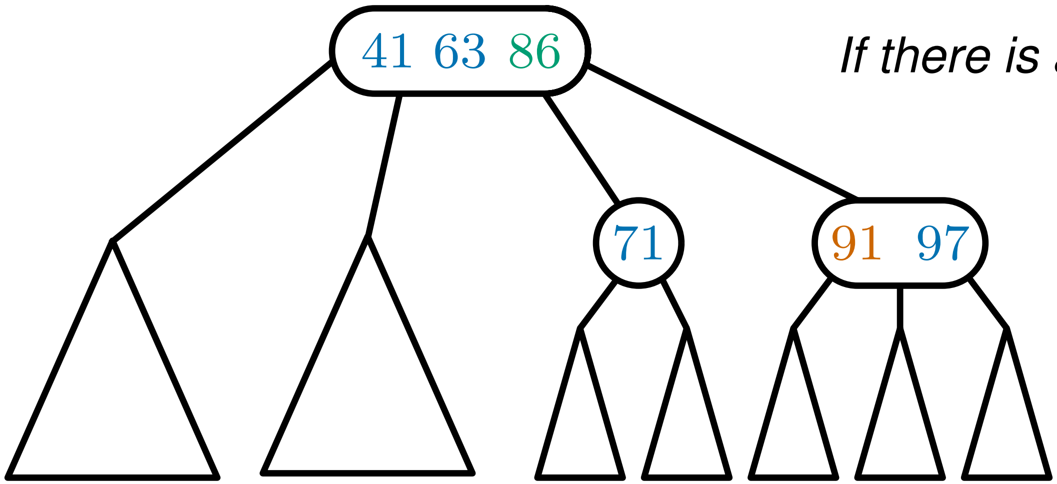
these subtrees haven't changed

no path lengths have changed

(if it was ***perfectly balanced****, it still is)*

TRANSFER takes $O(1)$ time

TRANSFERING keys



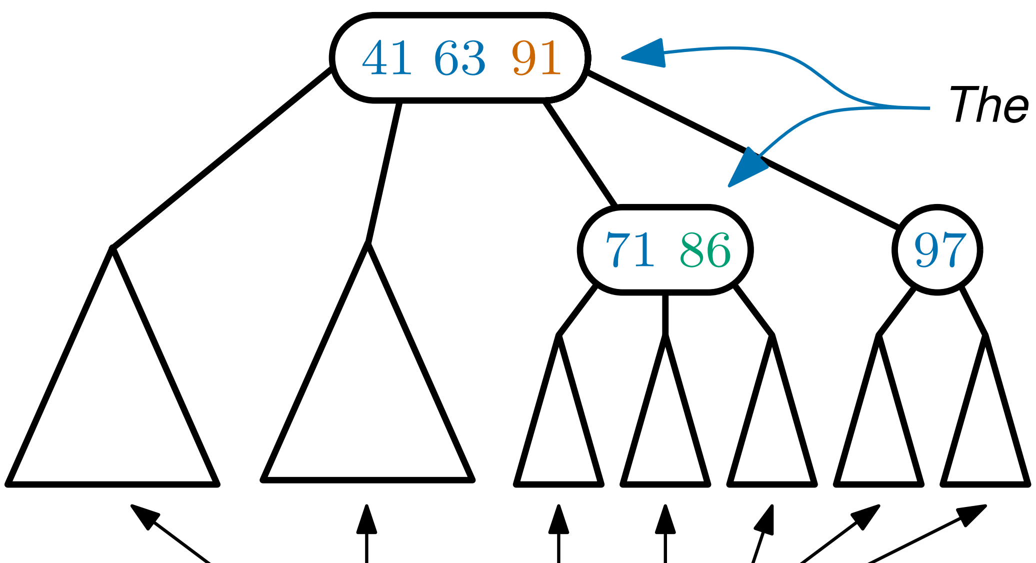
If there is a 2-node and a 3-node
(with the same parent)

we can perform a TRANSFER
(even if the parent is the root)

TRANSFER also works with a 2-node and a 4-node

BEFORE

AFTER



The keys have been rearranged

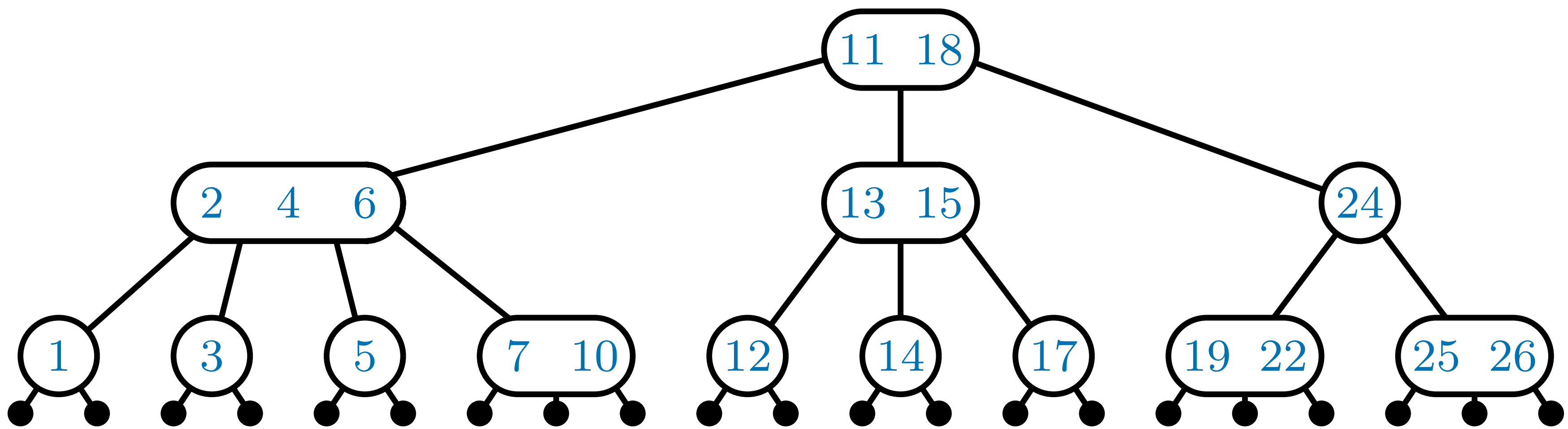
these subtrees haven't changed

no path lengths have changed

(if it was ***perfectly balanced****, it still is)*

TRANSFER takes $O(1)$ time

The DELETE operation



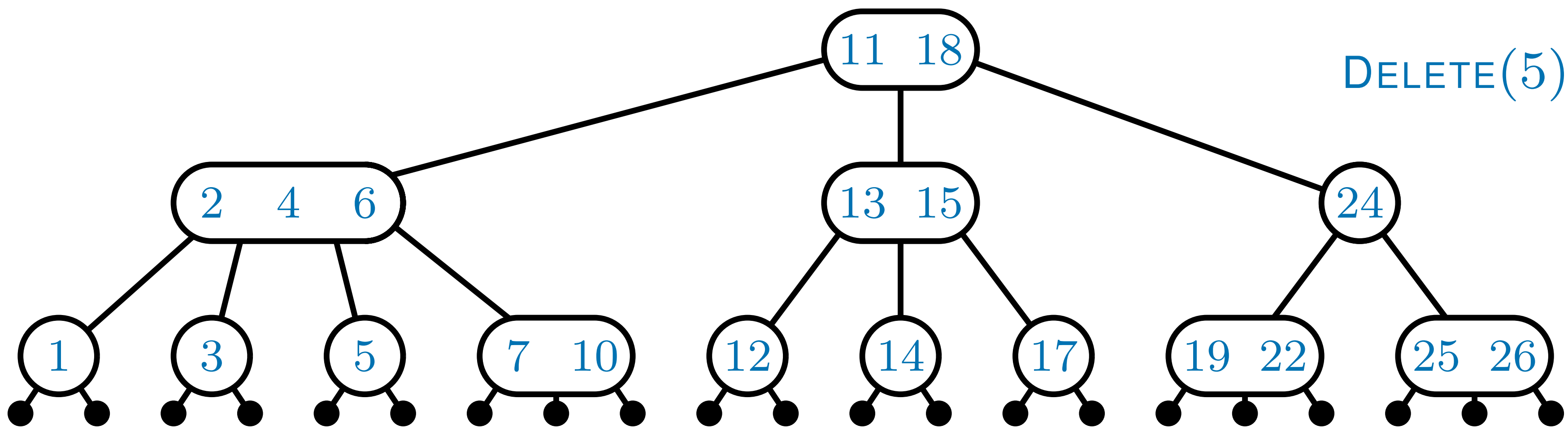
To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

Step 1: Search for the key k using FIND(k).
use FUSE and TRANSFER *to convert 2-nodes as we go down*

Step 2: If the leaf is a 3-node,
delete (x, k), converting it into a 2-node

Step 3: If the leaf is a 4-node,
delete (x, k), converting it into a 3-node

The DELETE operation



To perform DELETE(k) **on a leaf** *(we'll deal with other nodes later)*

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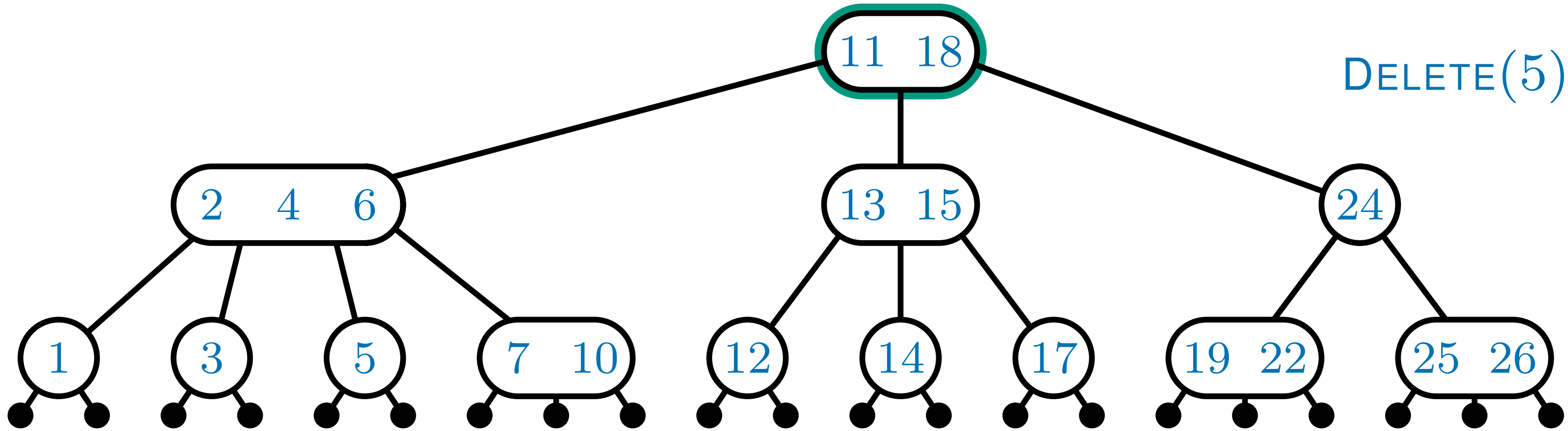

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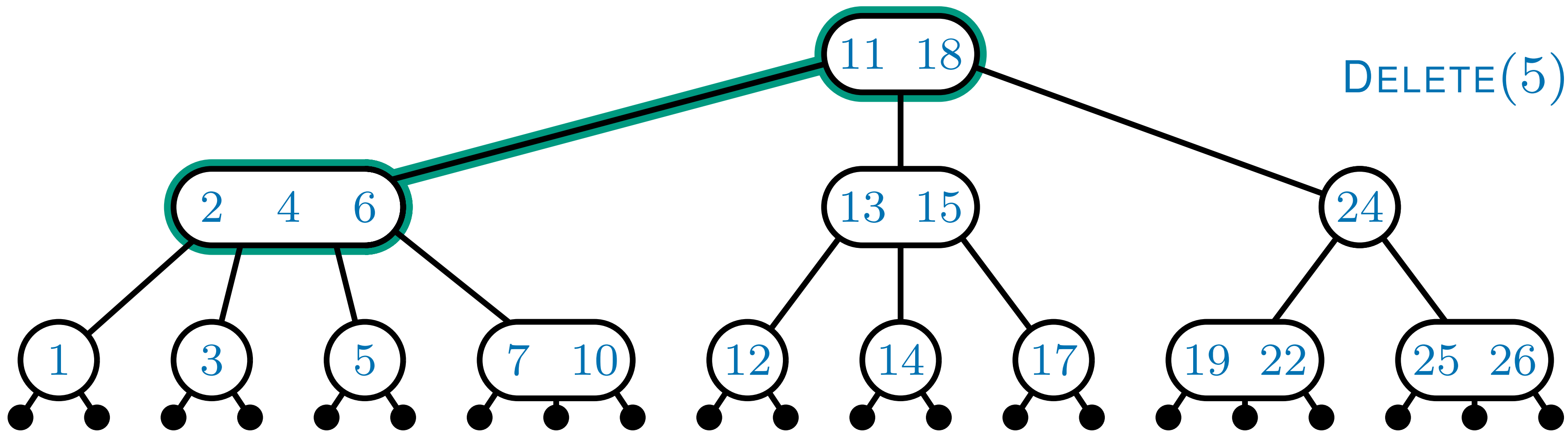

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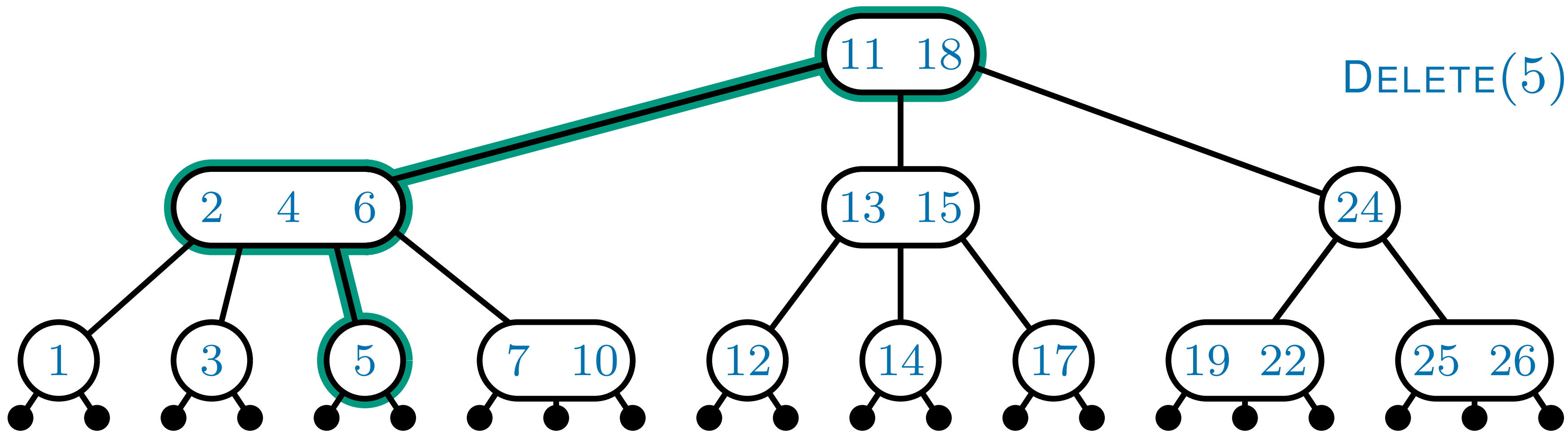

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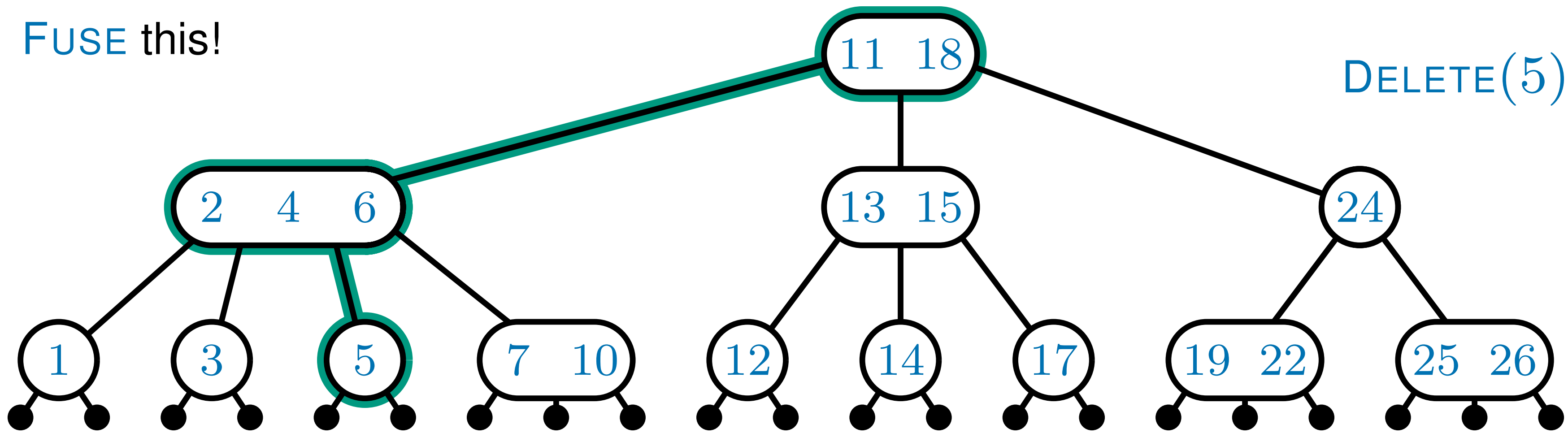

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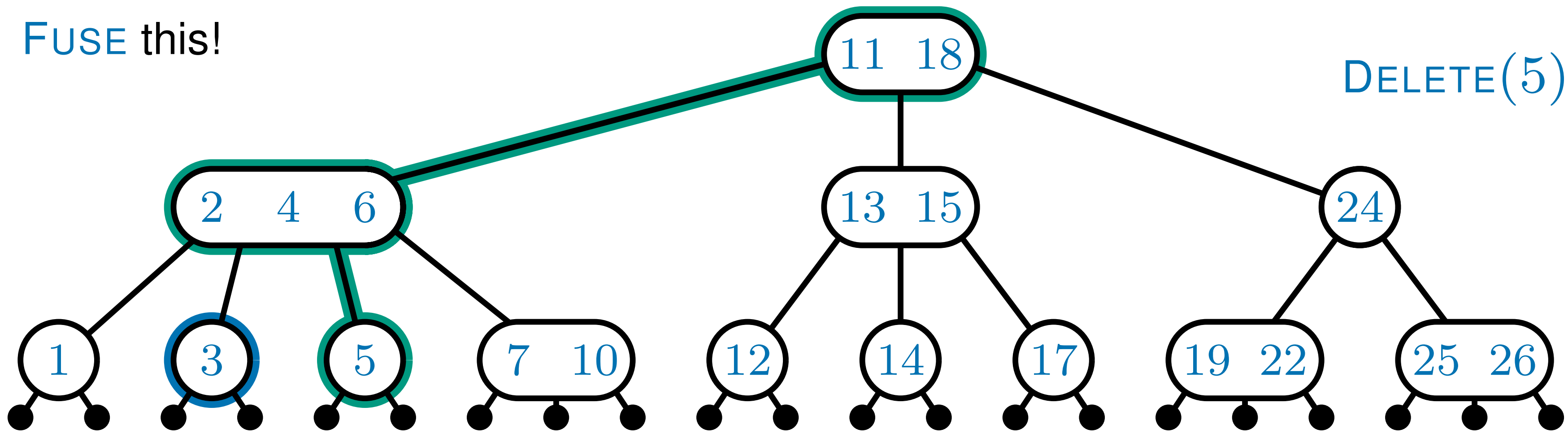

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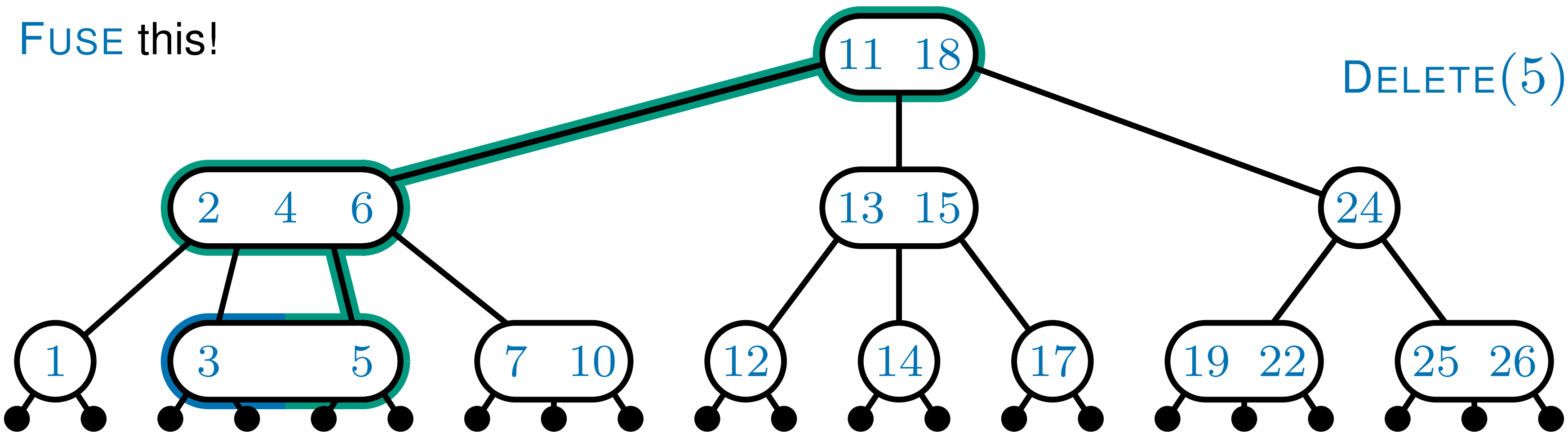

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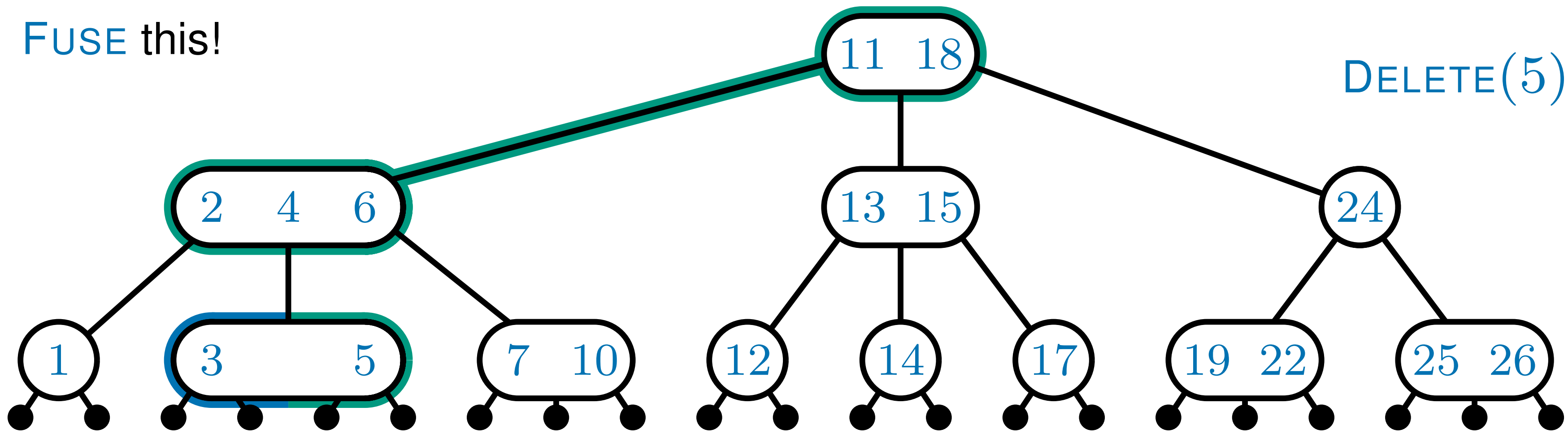

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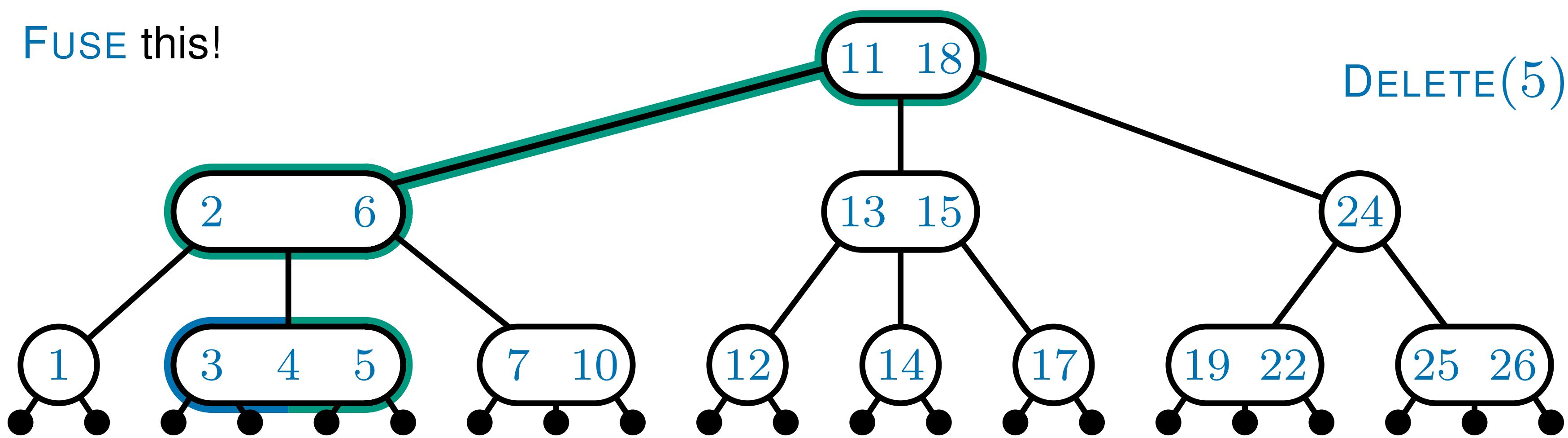

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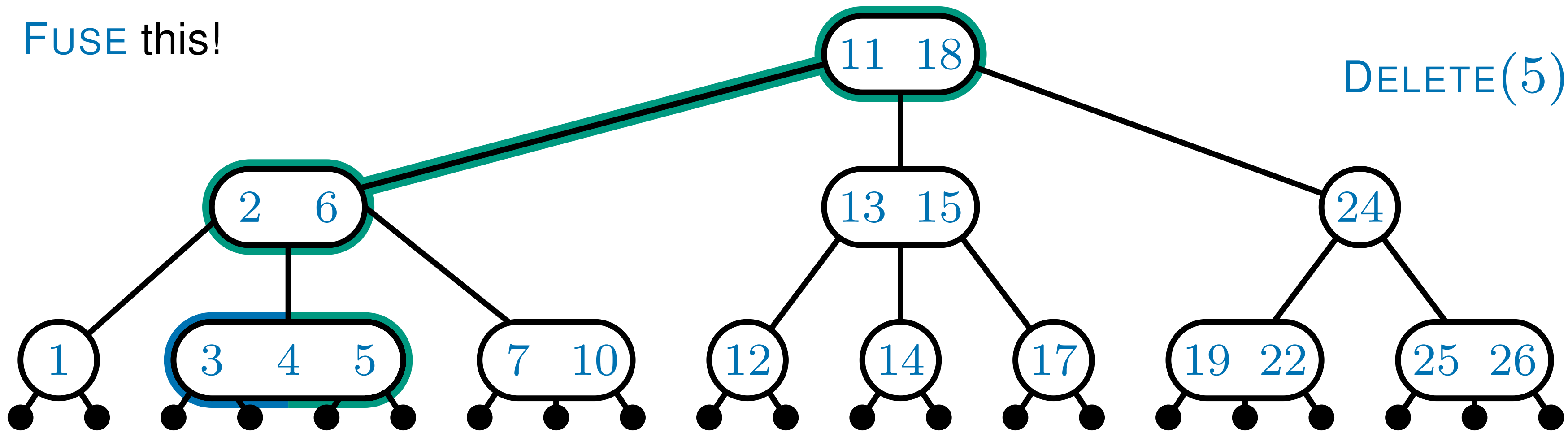

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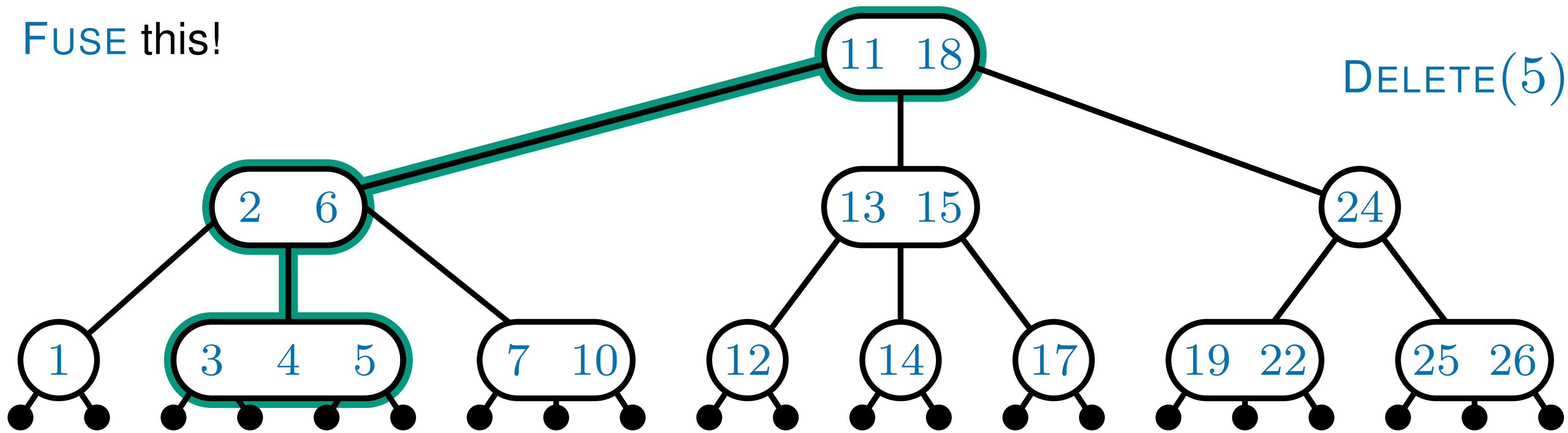
FUSE this!



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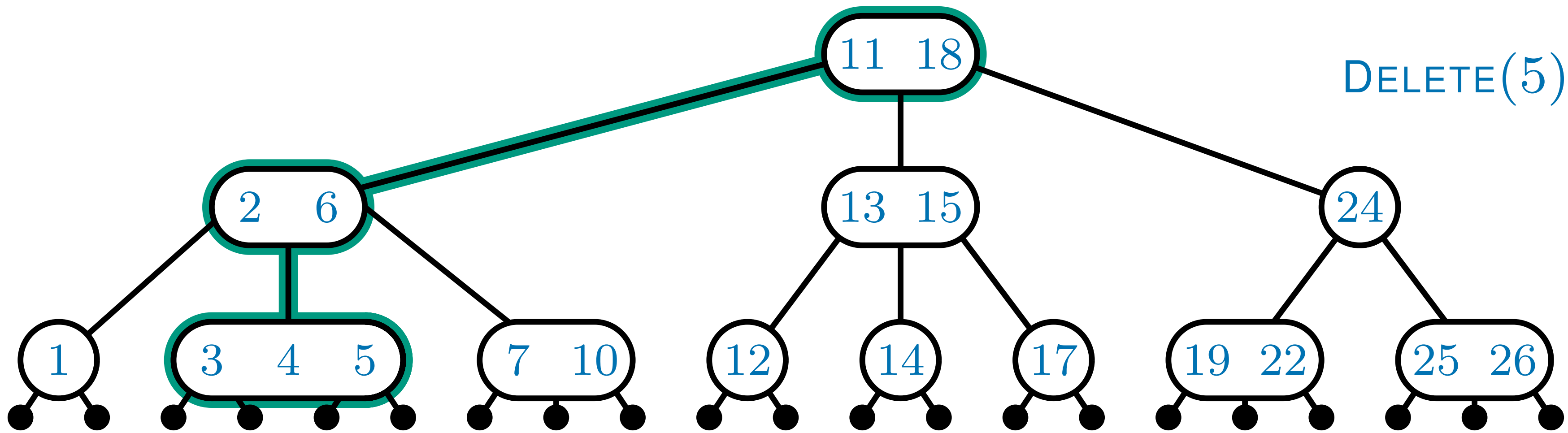

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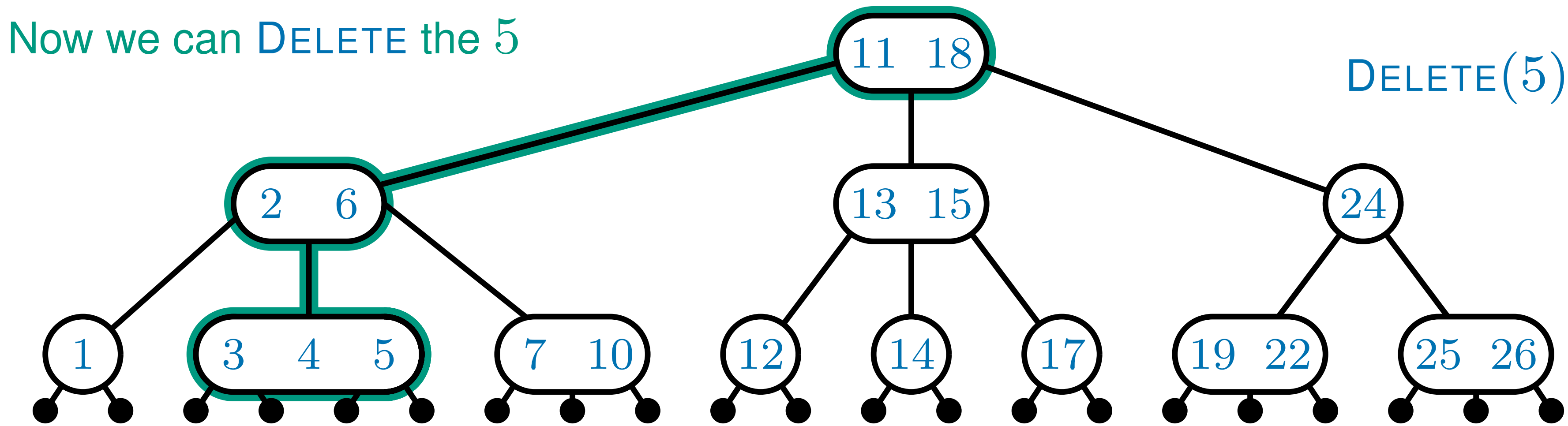

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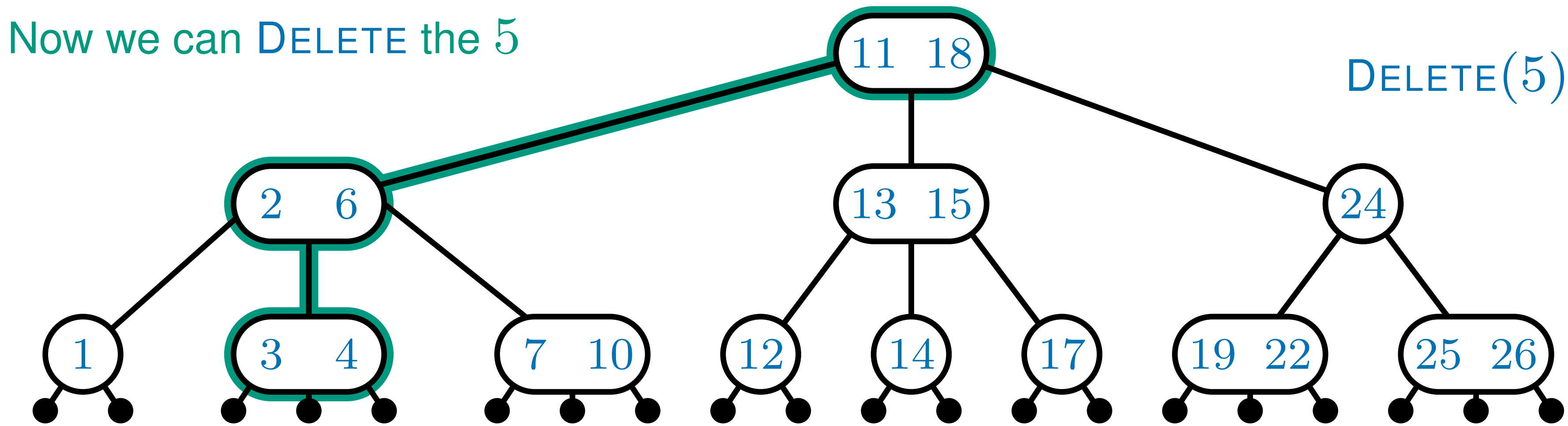

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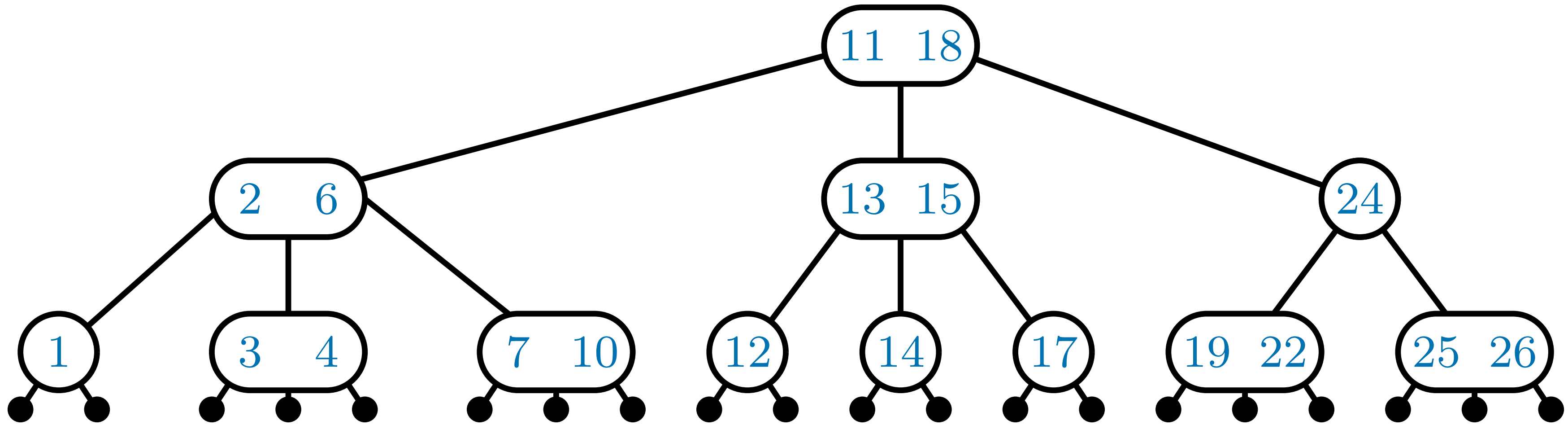

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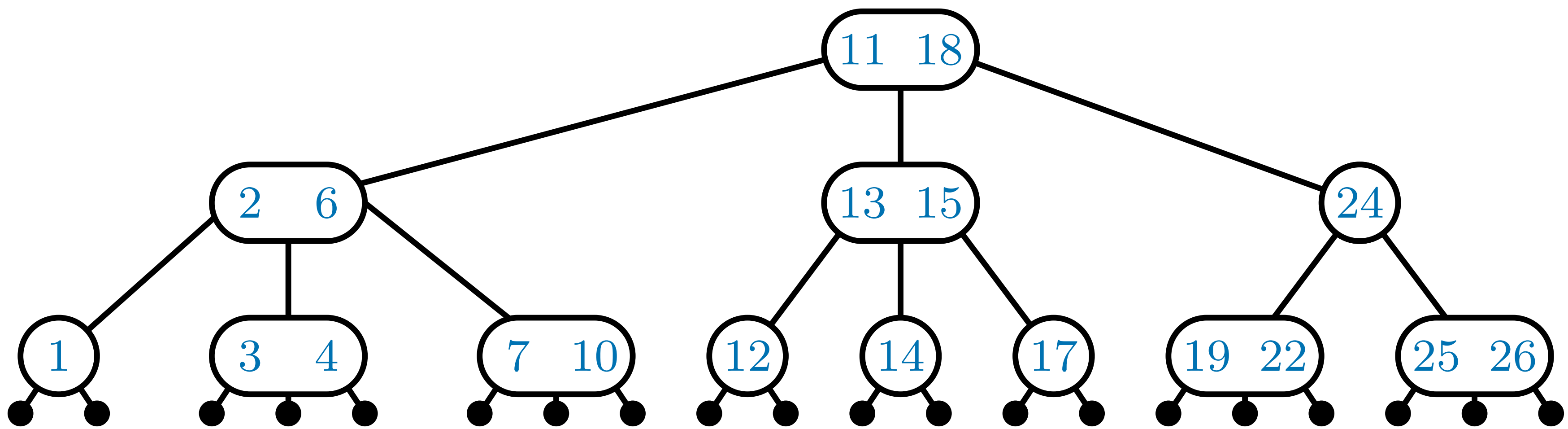

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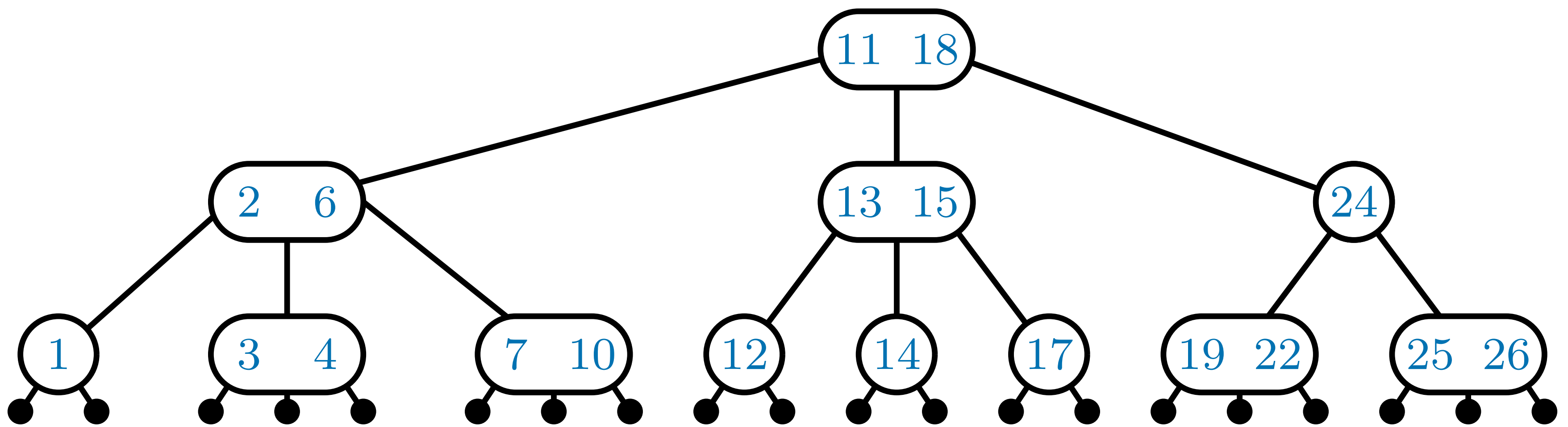
OK, one more thing...

The DELETE operation



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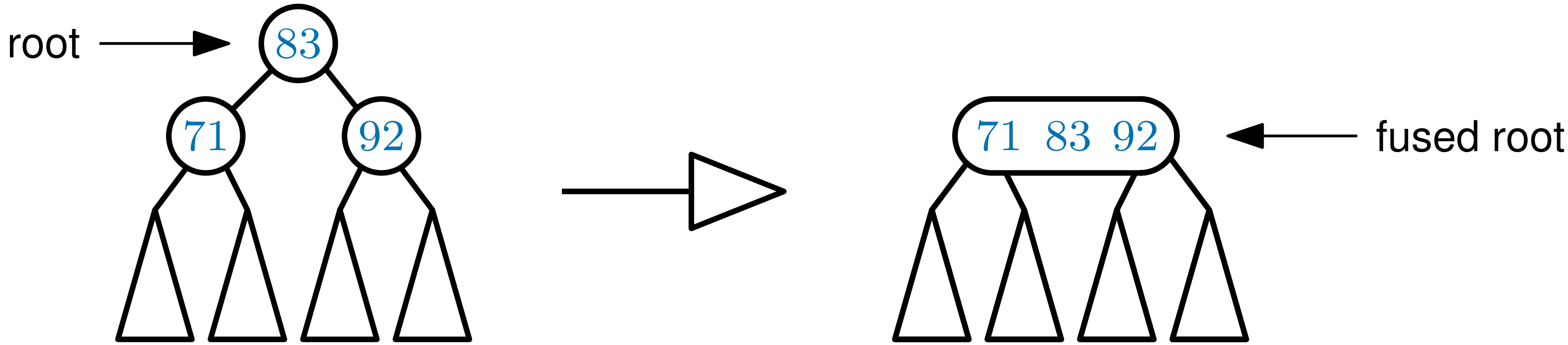
OK, one more thing... what happens when we FUSE the root?

FUSING the root

We said that we could only FUSE two 2-nodes if the parent was not a 2-node...

we make an exception for the root



FUSING the root can decrease the height of the tree

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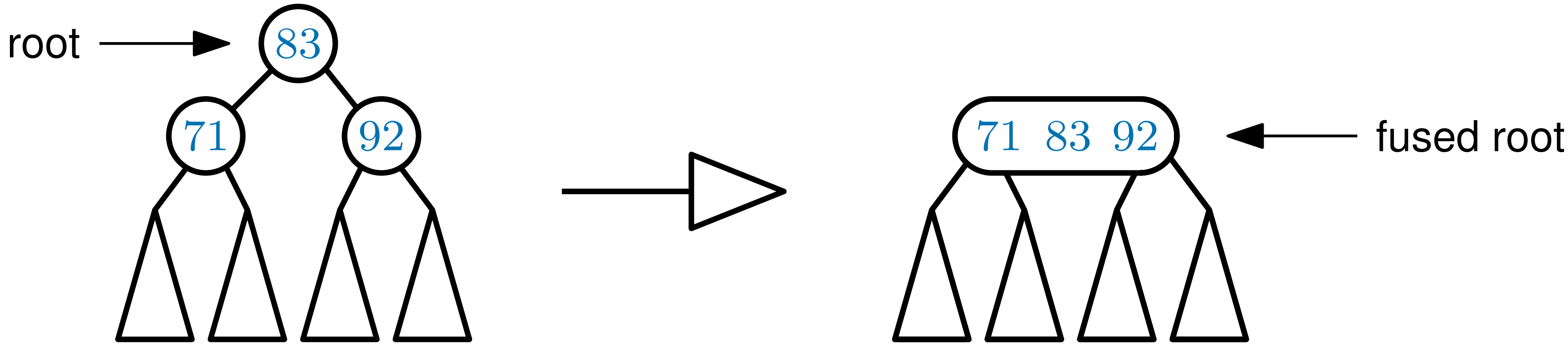

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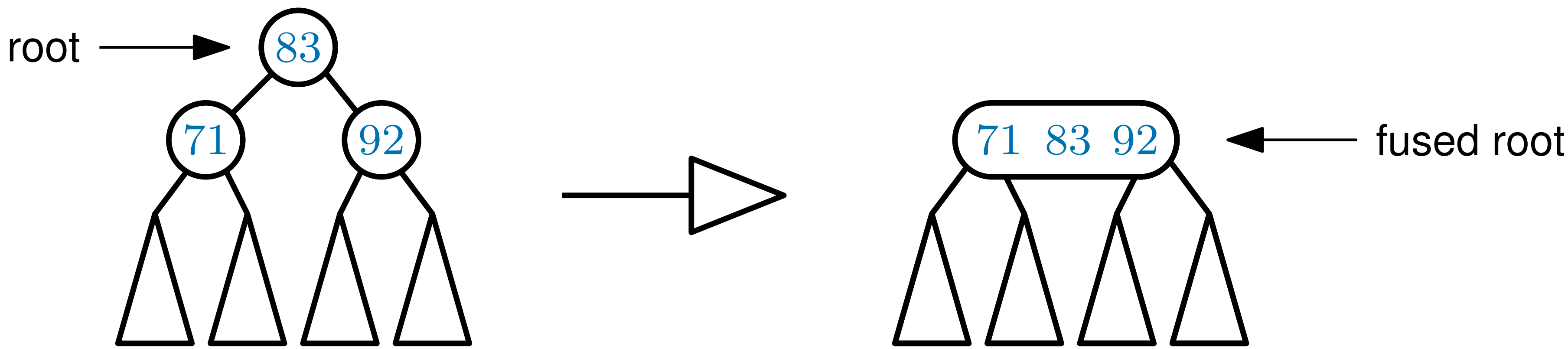

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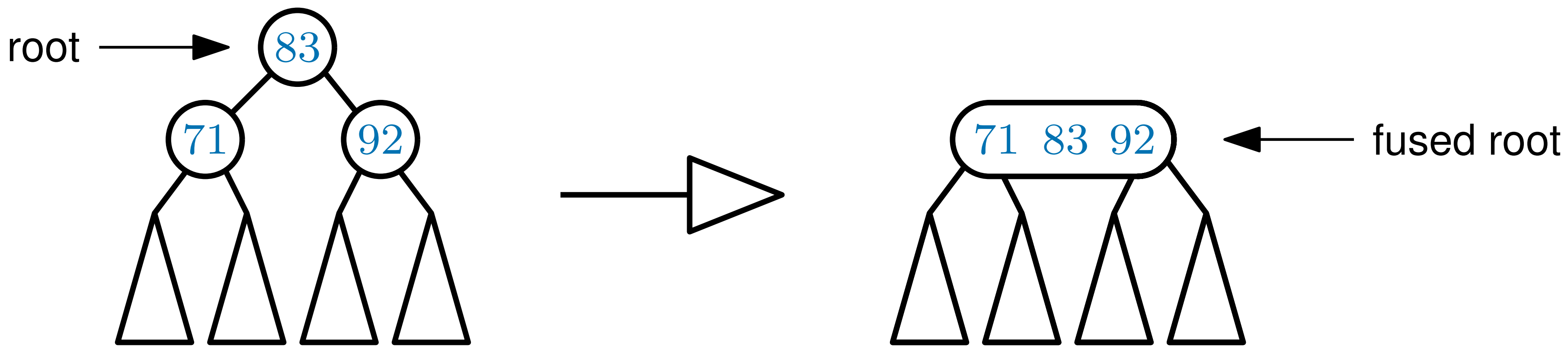

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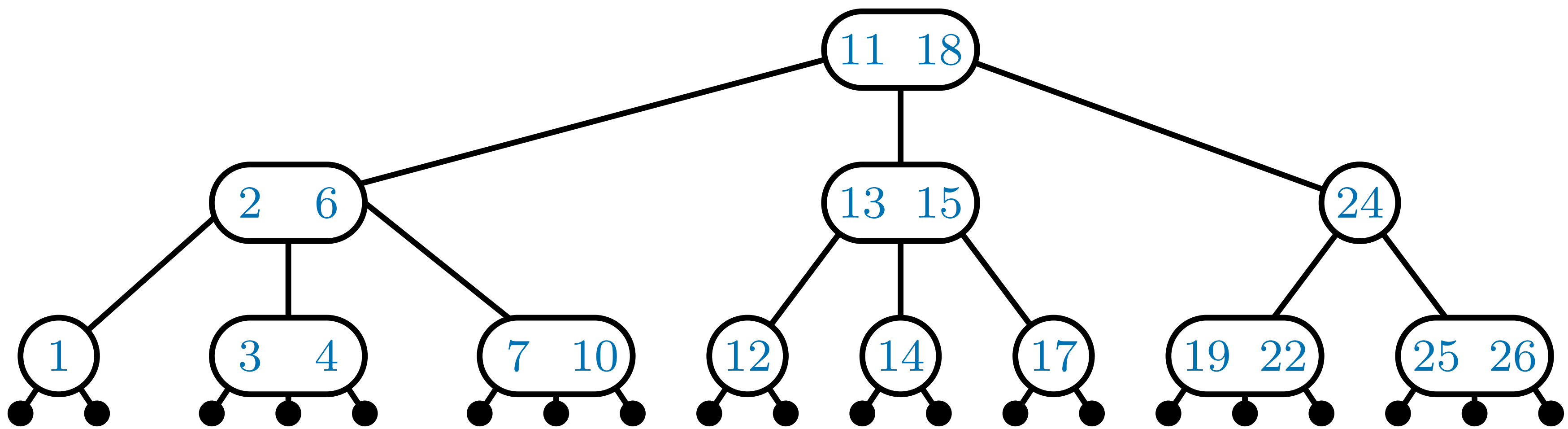
As each FUSE or TRANSFER takes $O(1)$ time, overall DELETE takes $O(\log n)$ time

The DELETE operation



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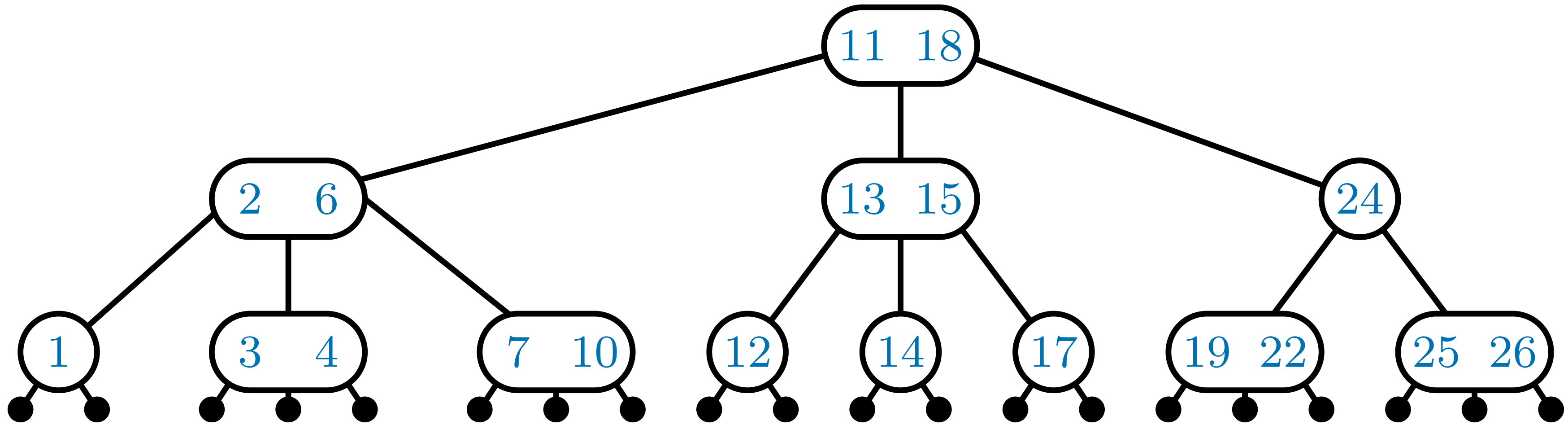
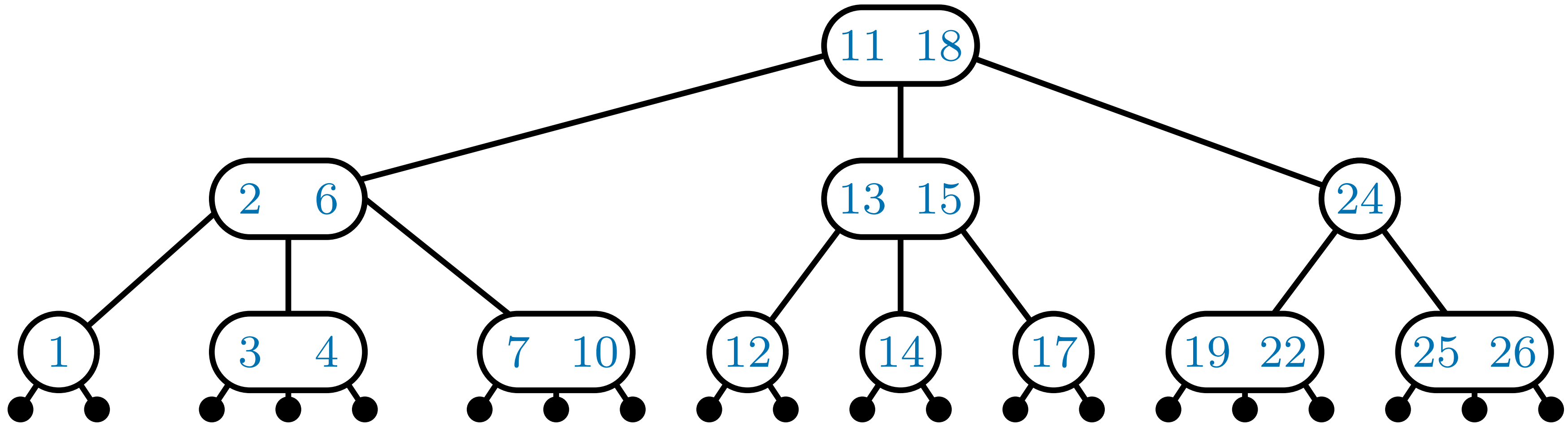


What if we want to DELETE something other than a leaf?

The DELETE operation



What if we want to DELETE something other than a leaf?

Step 1: Find the PREDECESSOR of k (this is essentially the same as FIND)

- that's the element with the largest key k' such that $k' < k$

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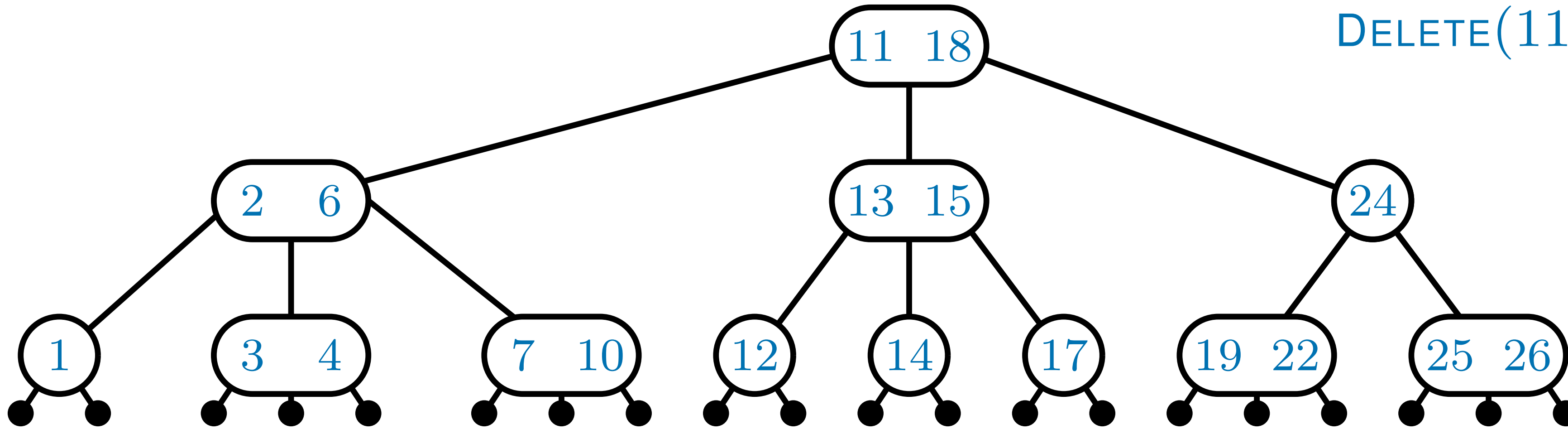

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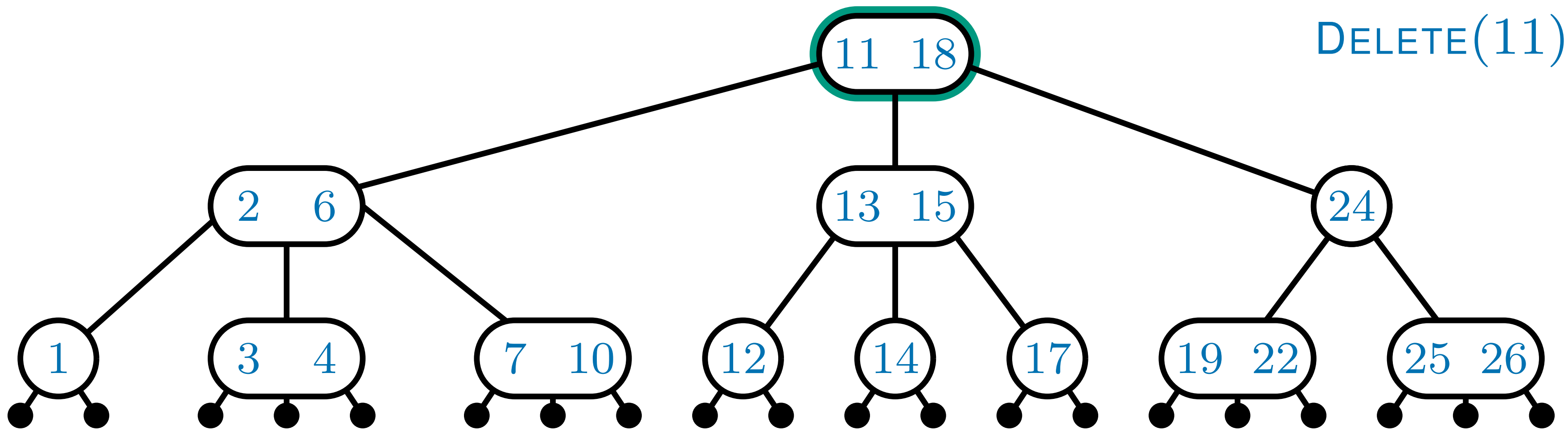

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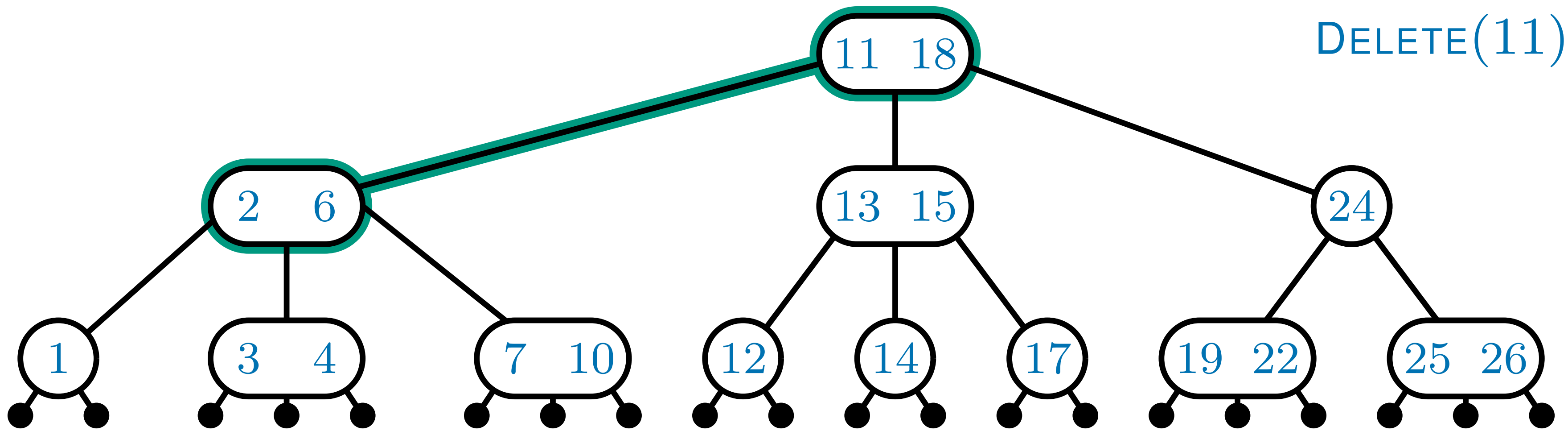

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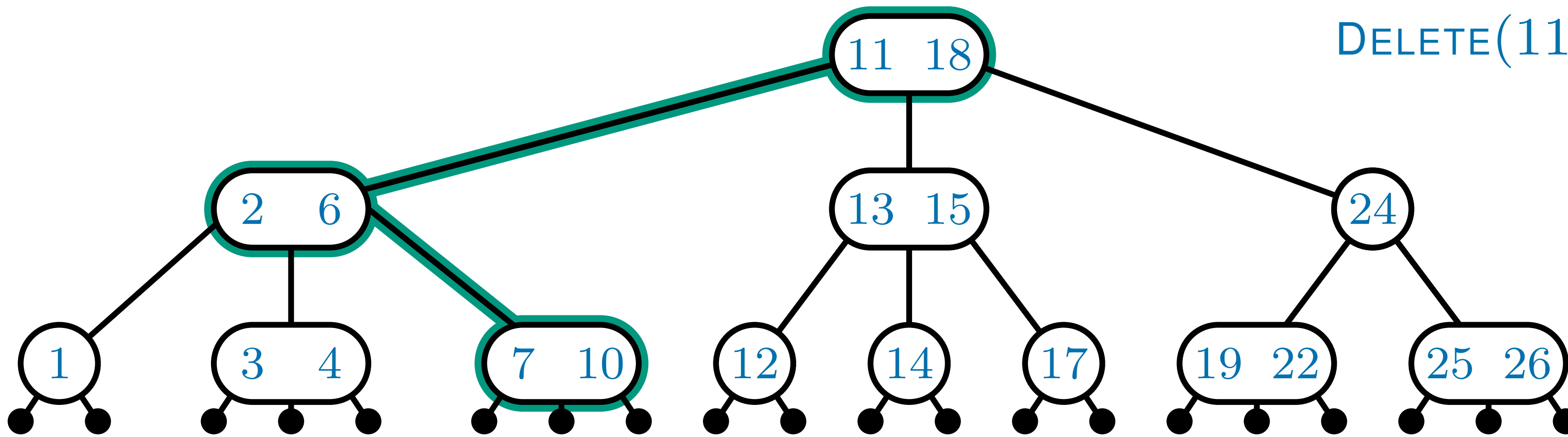

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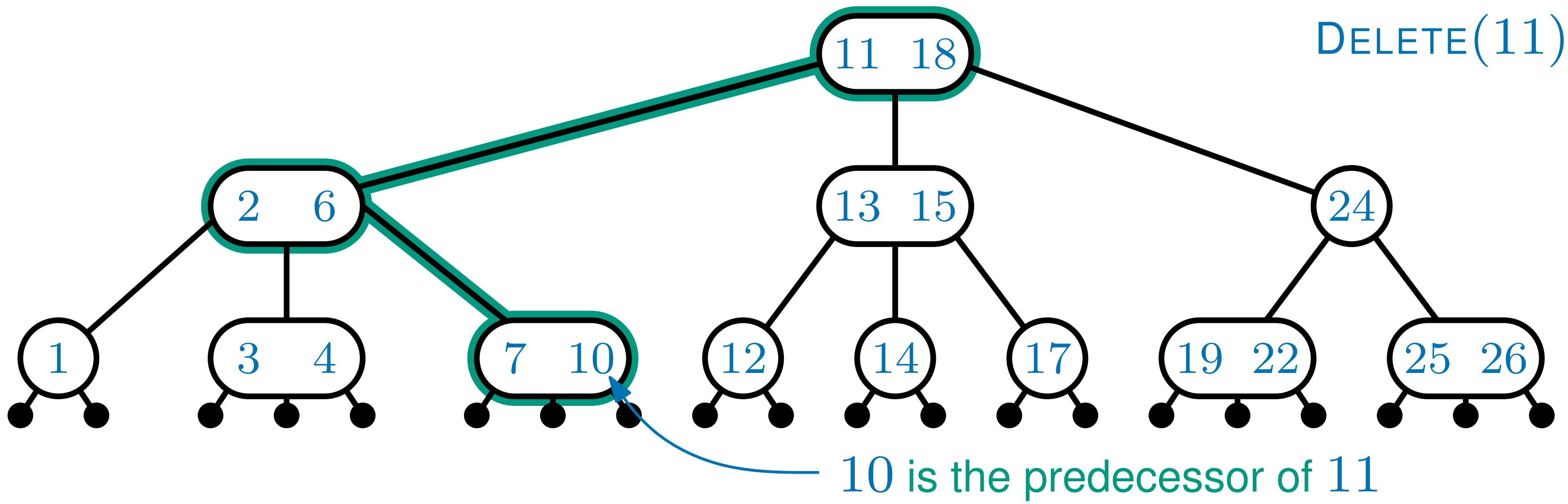

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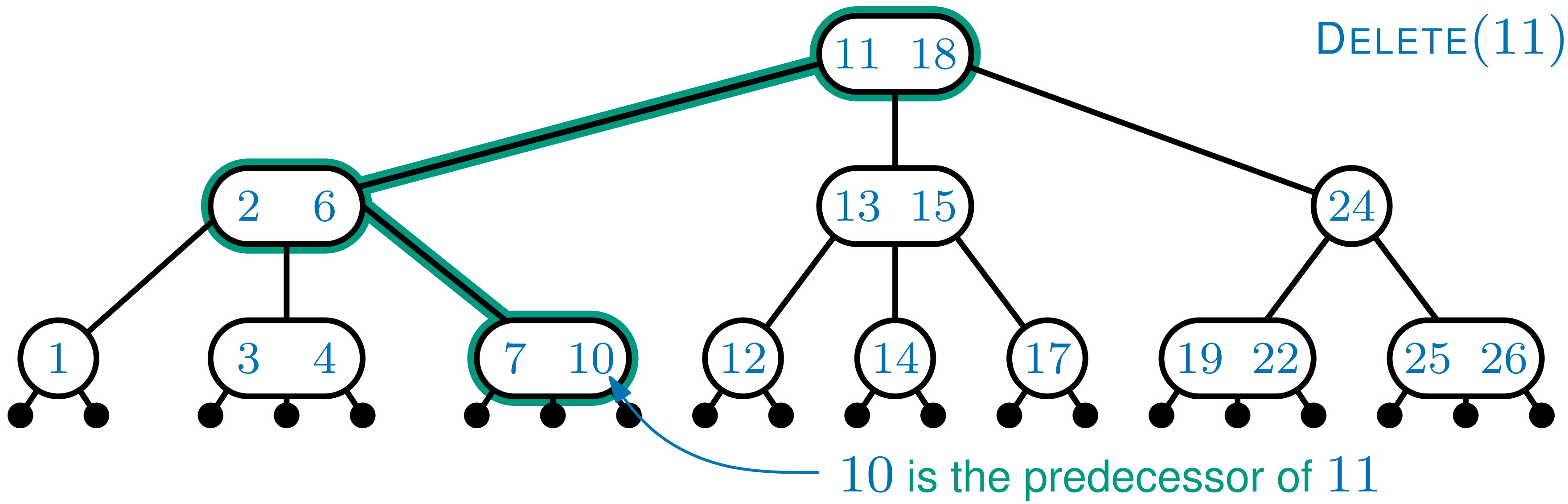

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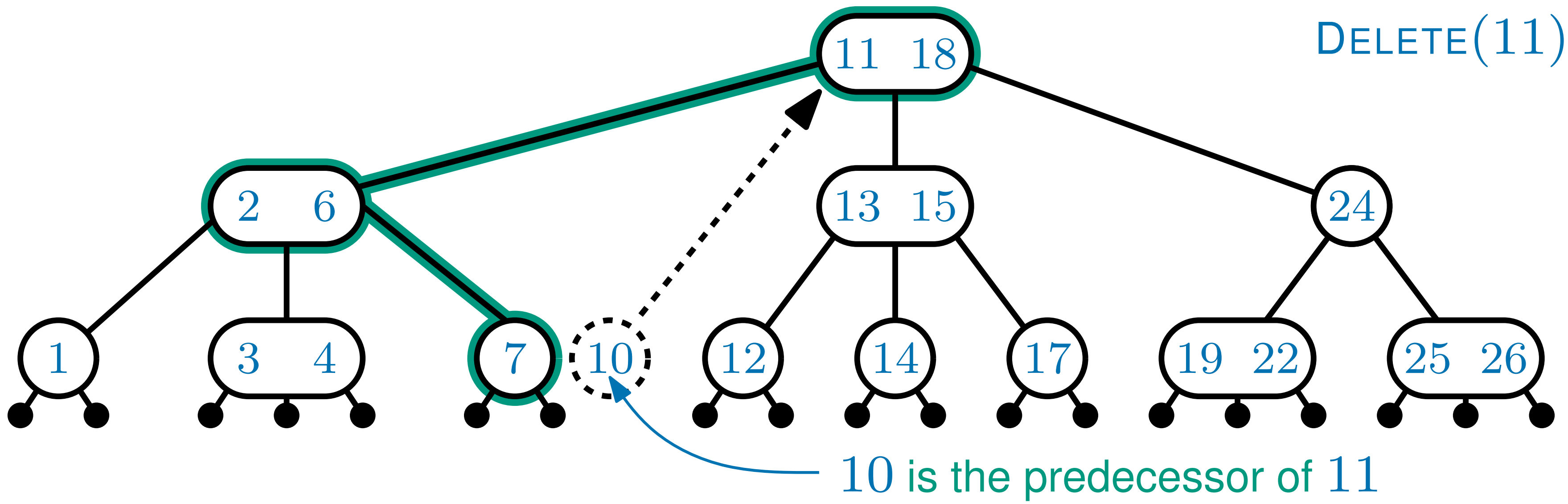

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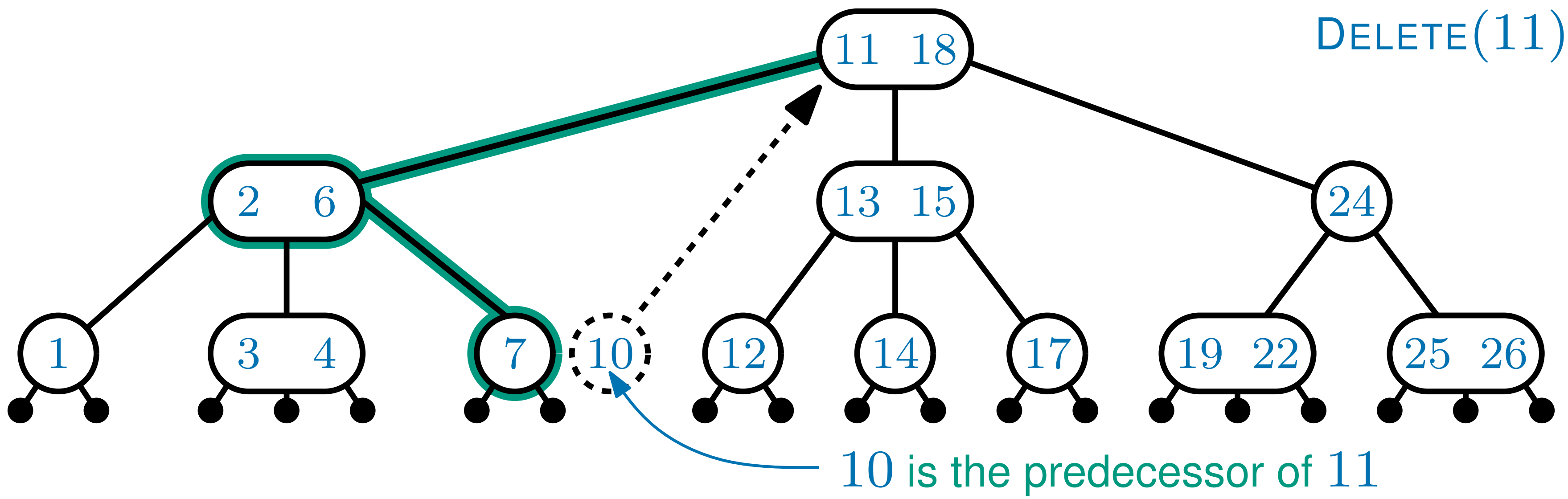

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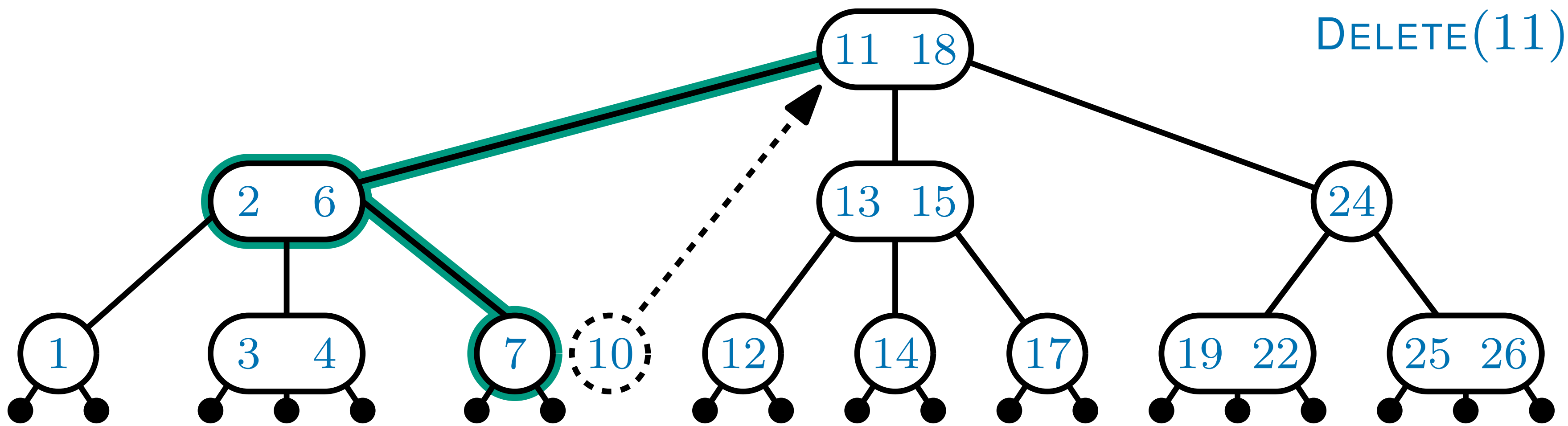

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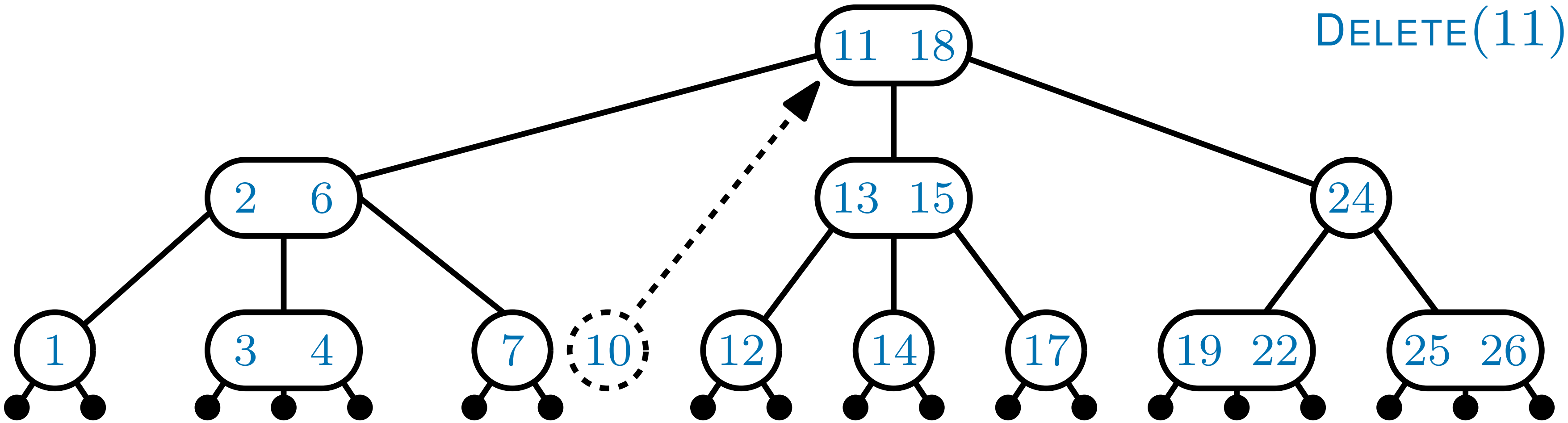

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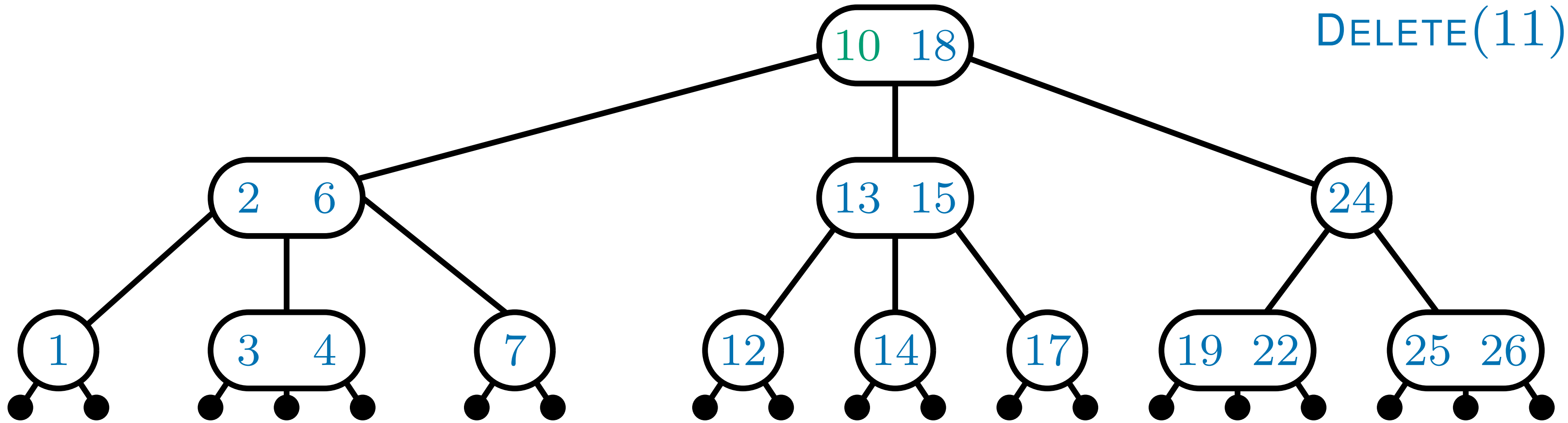

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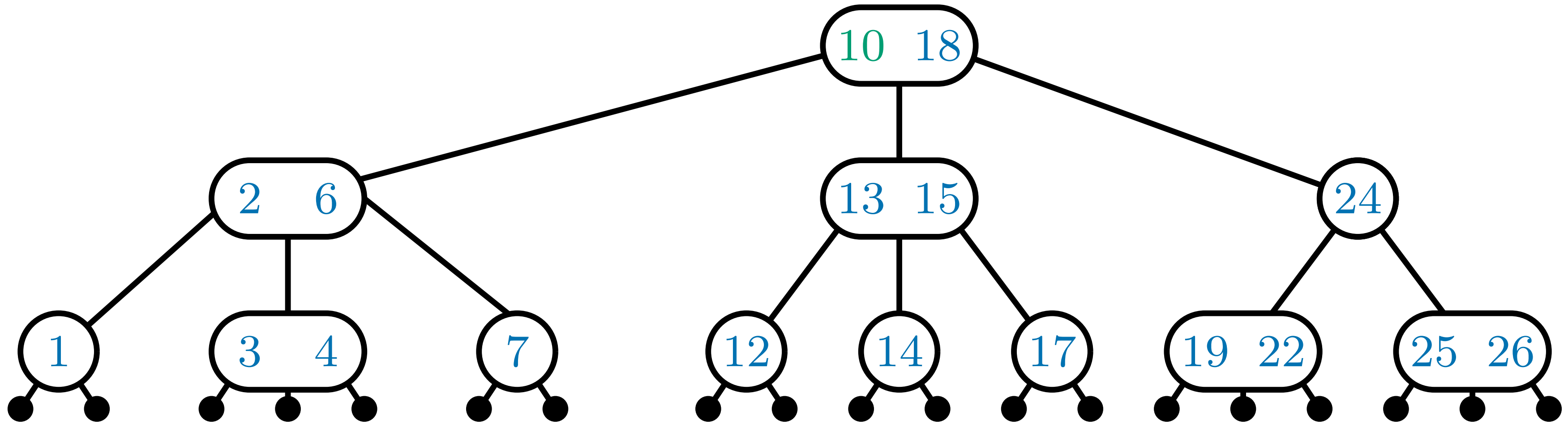

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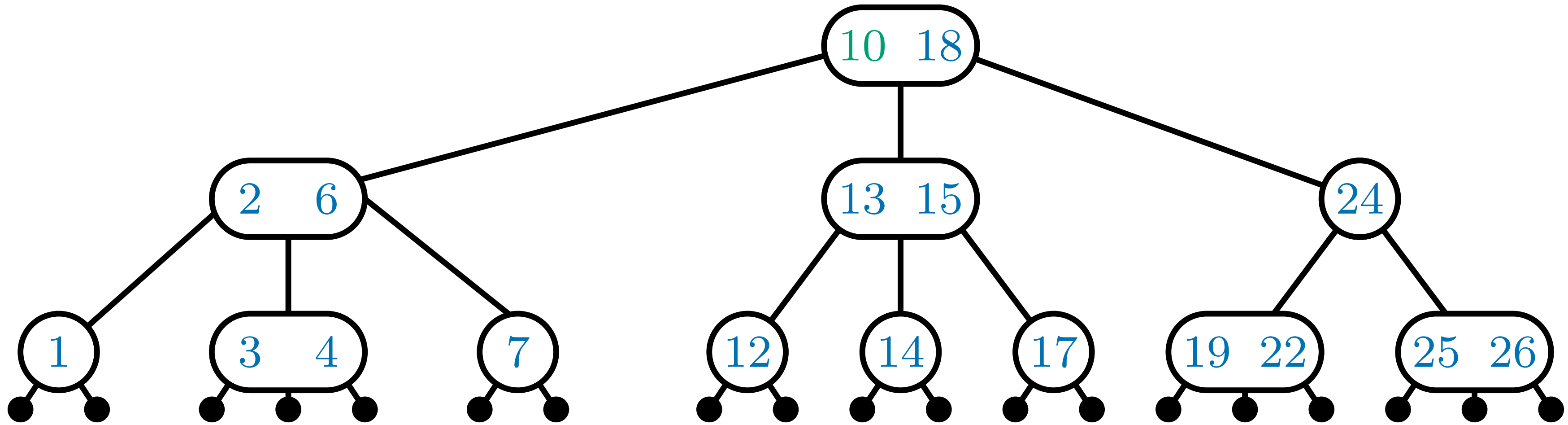

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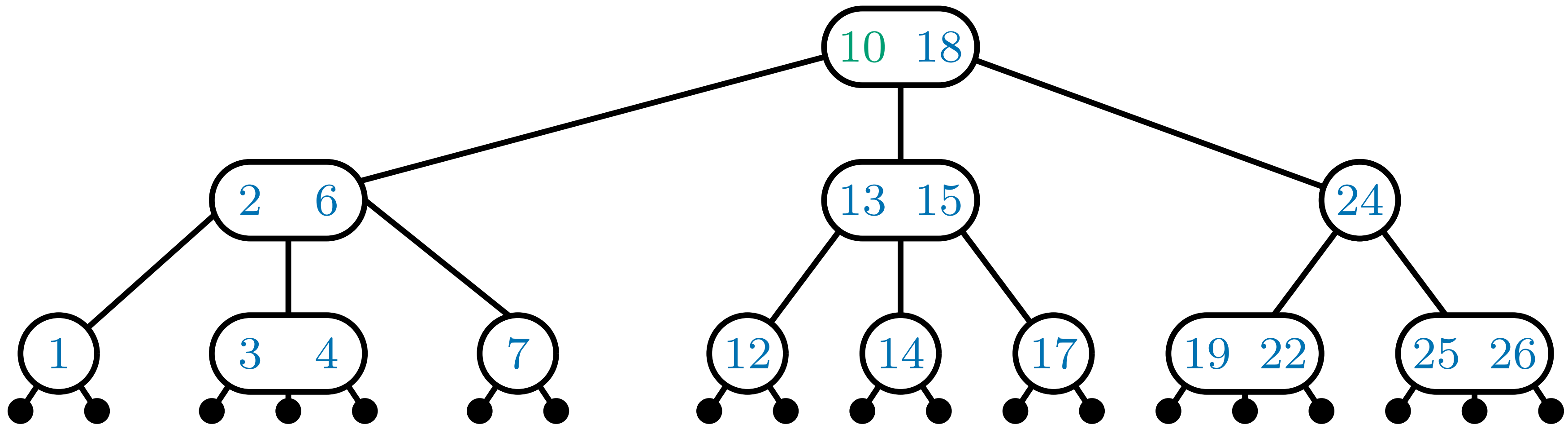
This also takes $O(\log n)$ time

2-3-4 tree summary

A 2-3-4 is a data structure based on a tree structure

which supports $\textsc{Insert}(x, k)$, $\textsc{Find}(k)$ and $\textsc{Delete}(k)$



each of these operations takes *worst case* $O(\log n)$ time

2-3-4 tree summary

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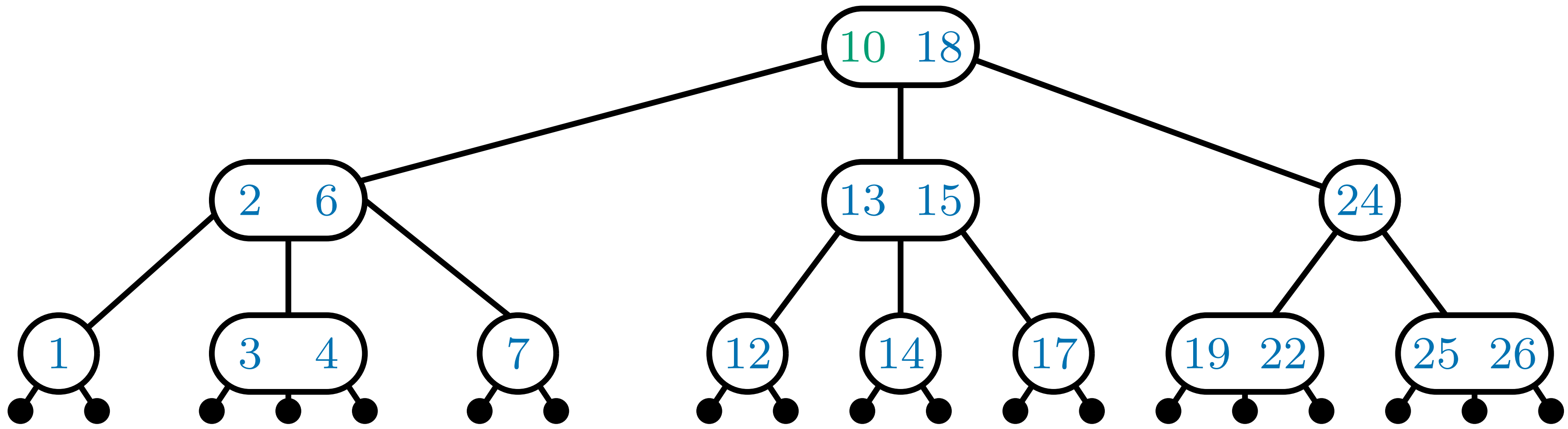

each of these operations takes *worst case* $O(\log n)$ time

Unfortunately, 2-3-4 trees are awkward to implement

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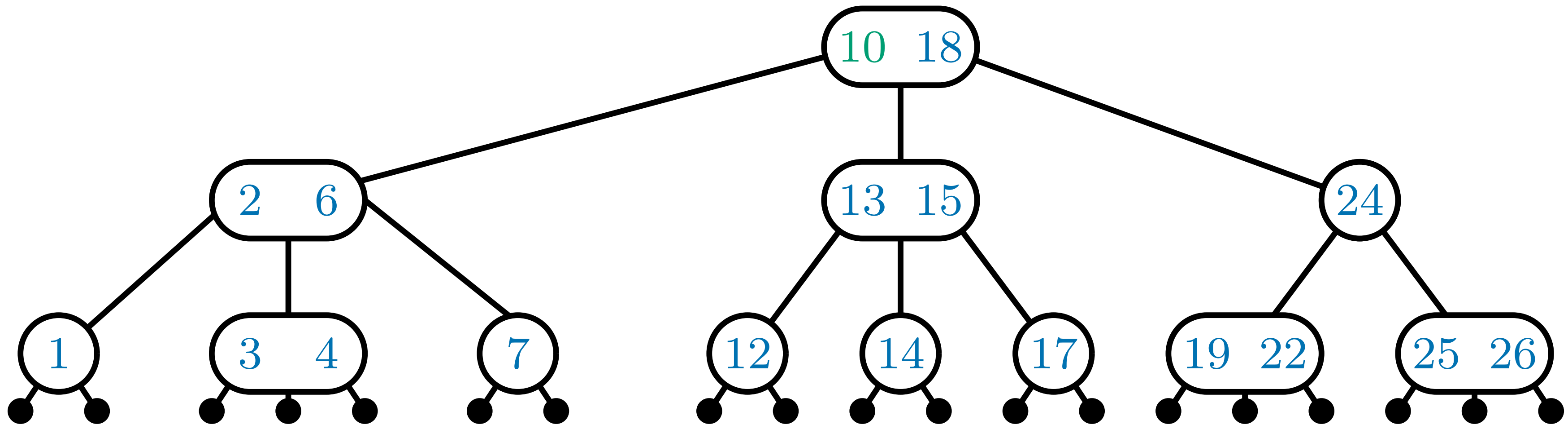

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So, what is used in practice?

Red-Black tree summary

A Red-Black tree is a data structure based on a **binary** tree structure

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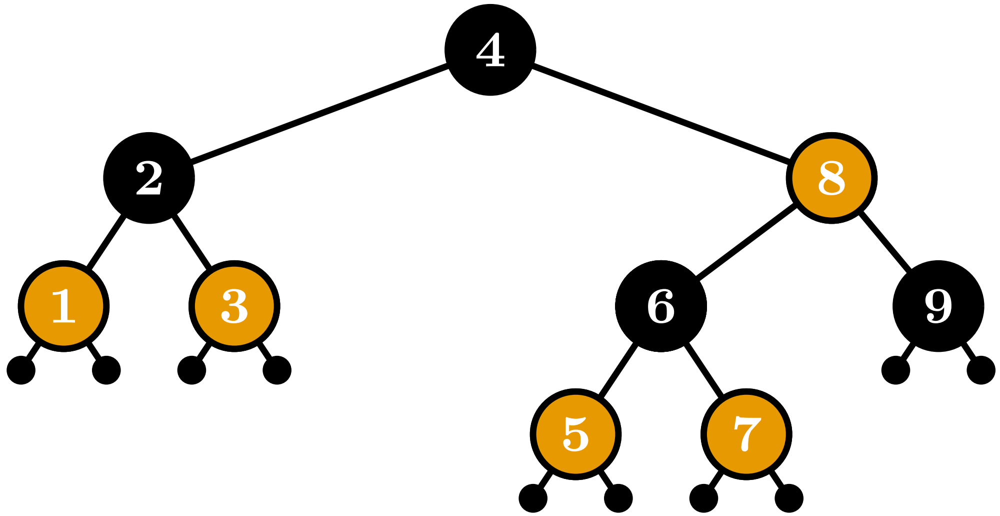

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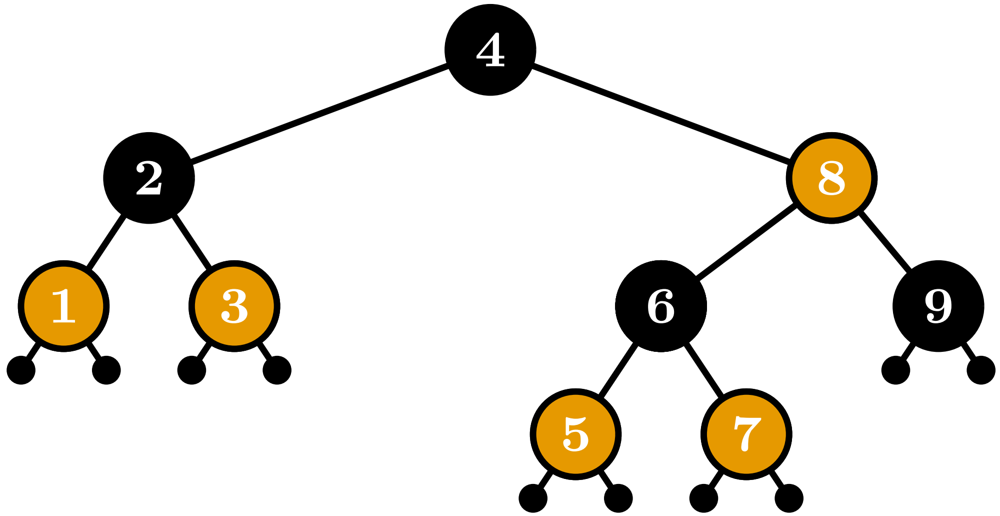

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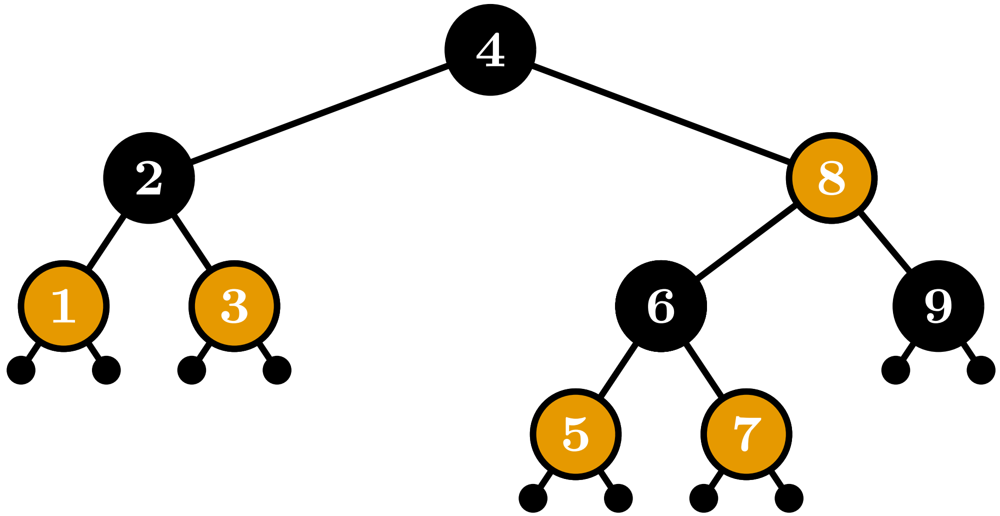

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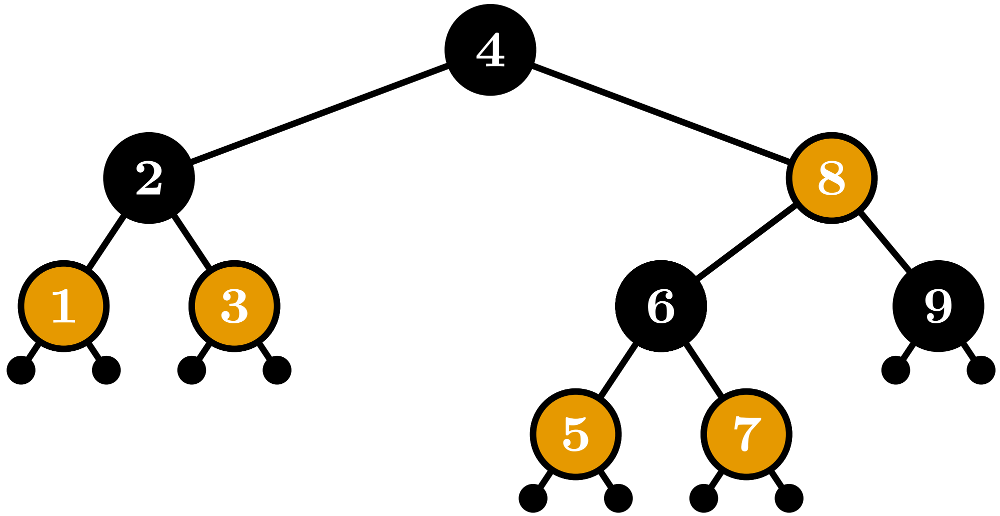

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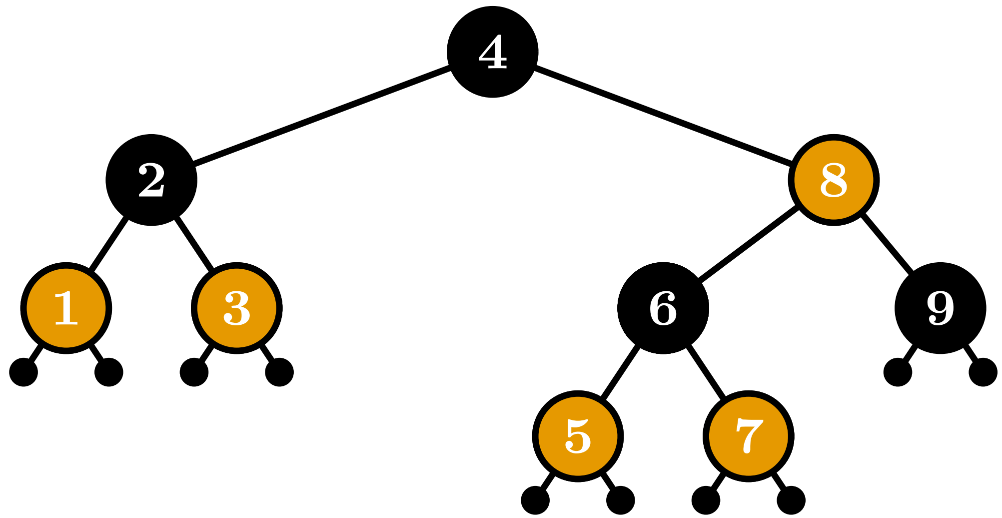

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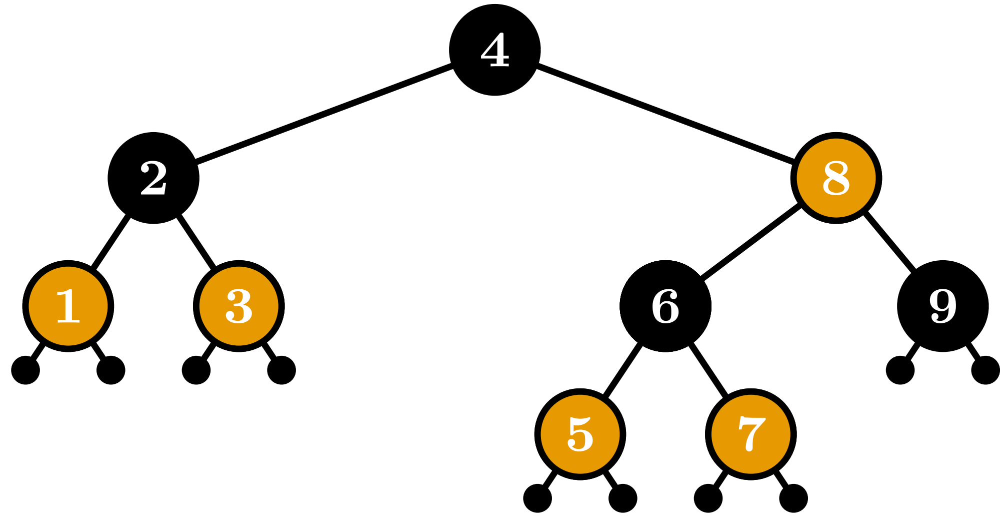

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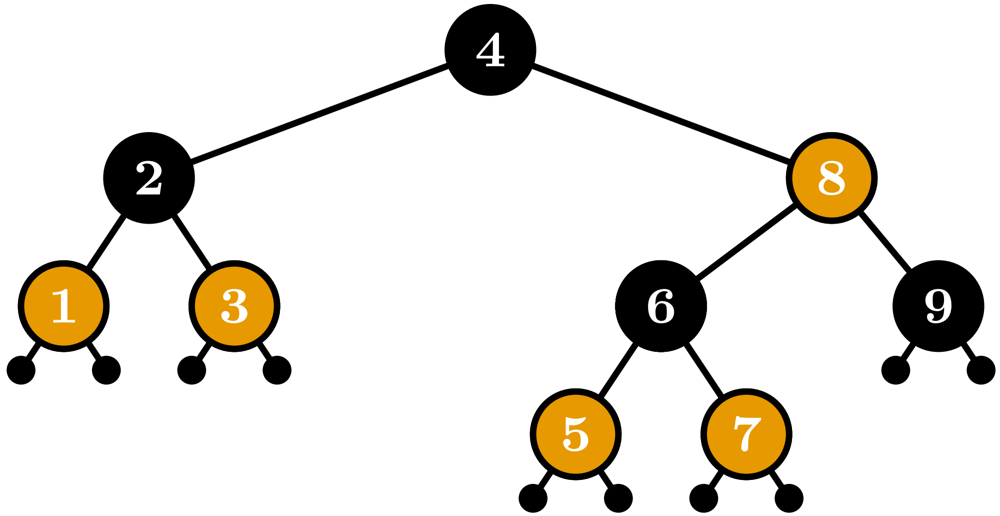

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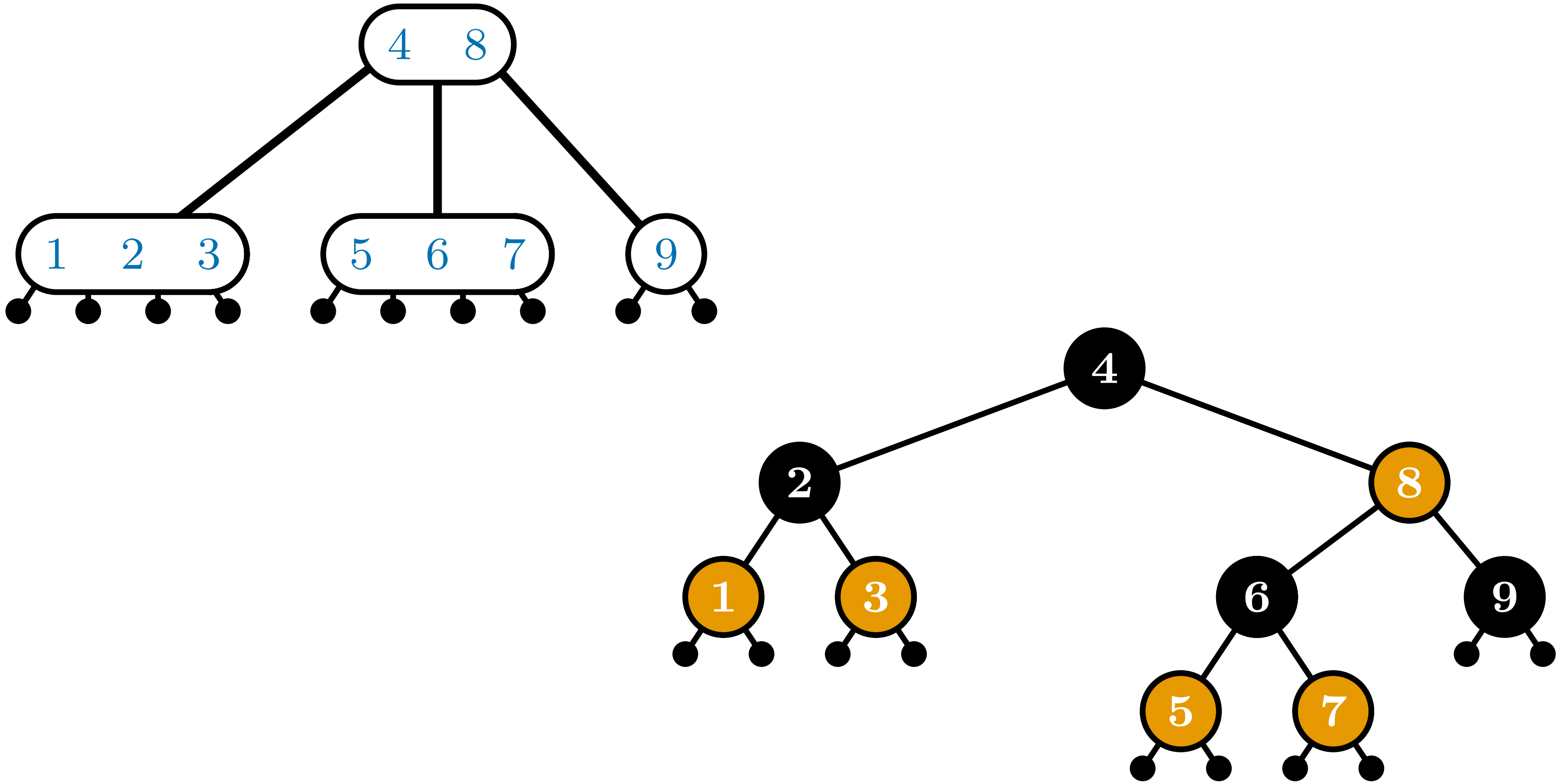
2-3-4 trees vs. Red-Black trees

Any 2-3-4 tree can be converted into a Red-Black tree (and visa-versa)



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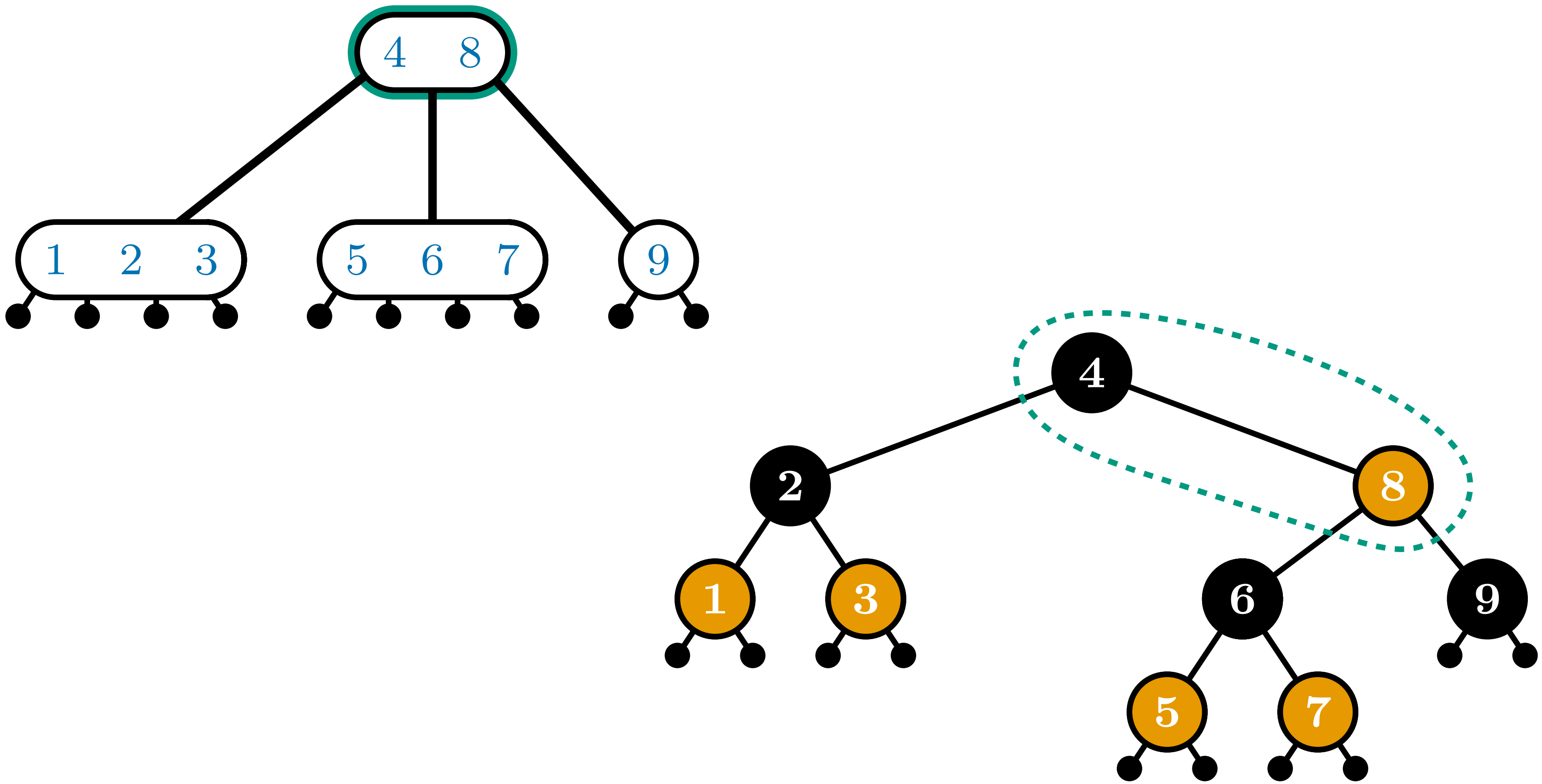

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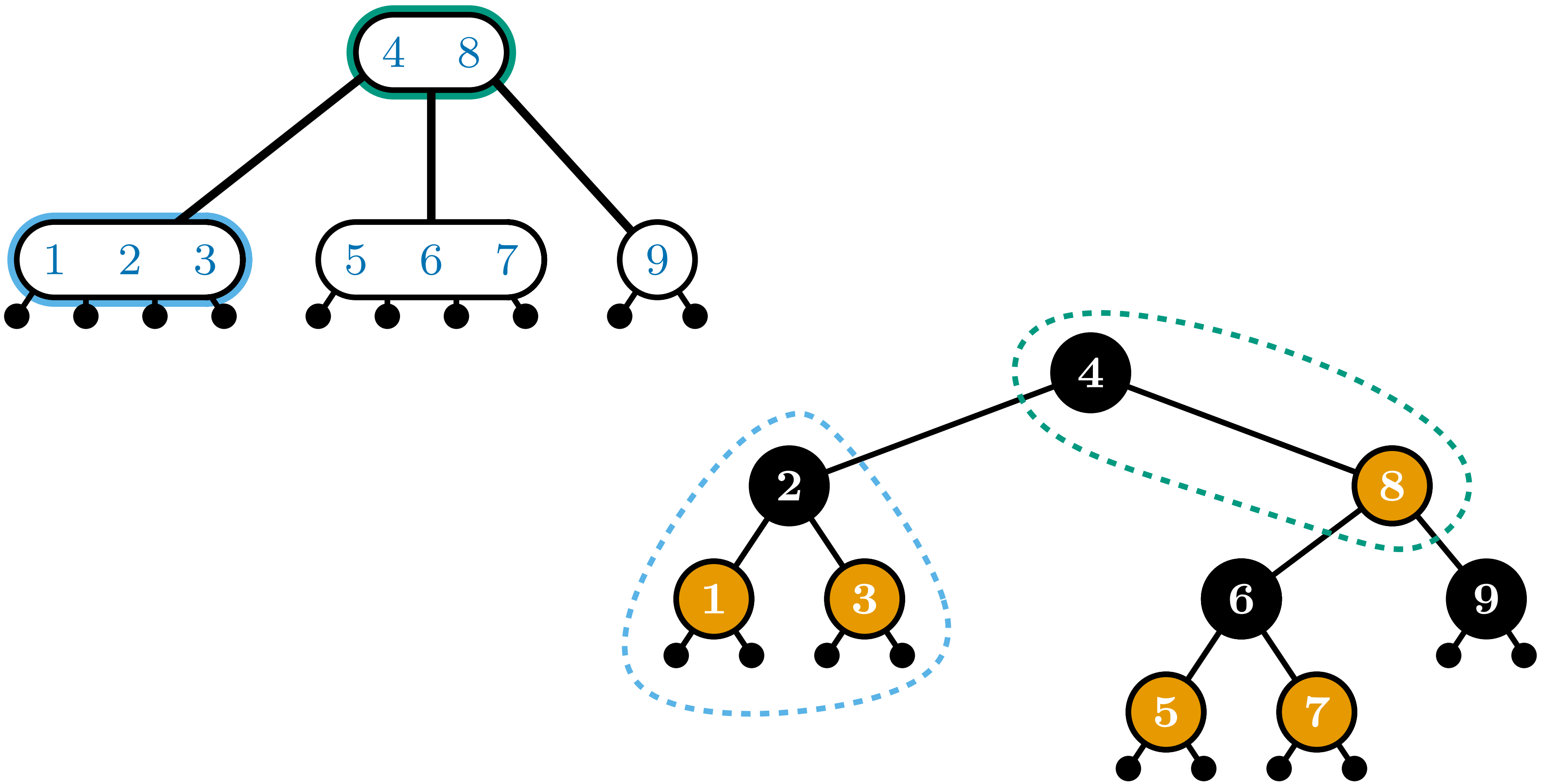

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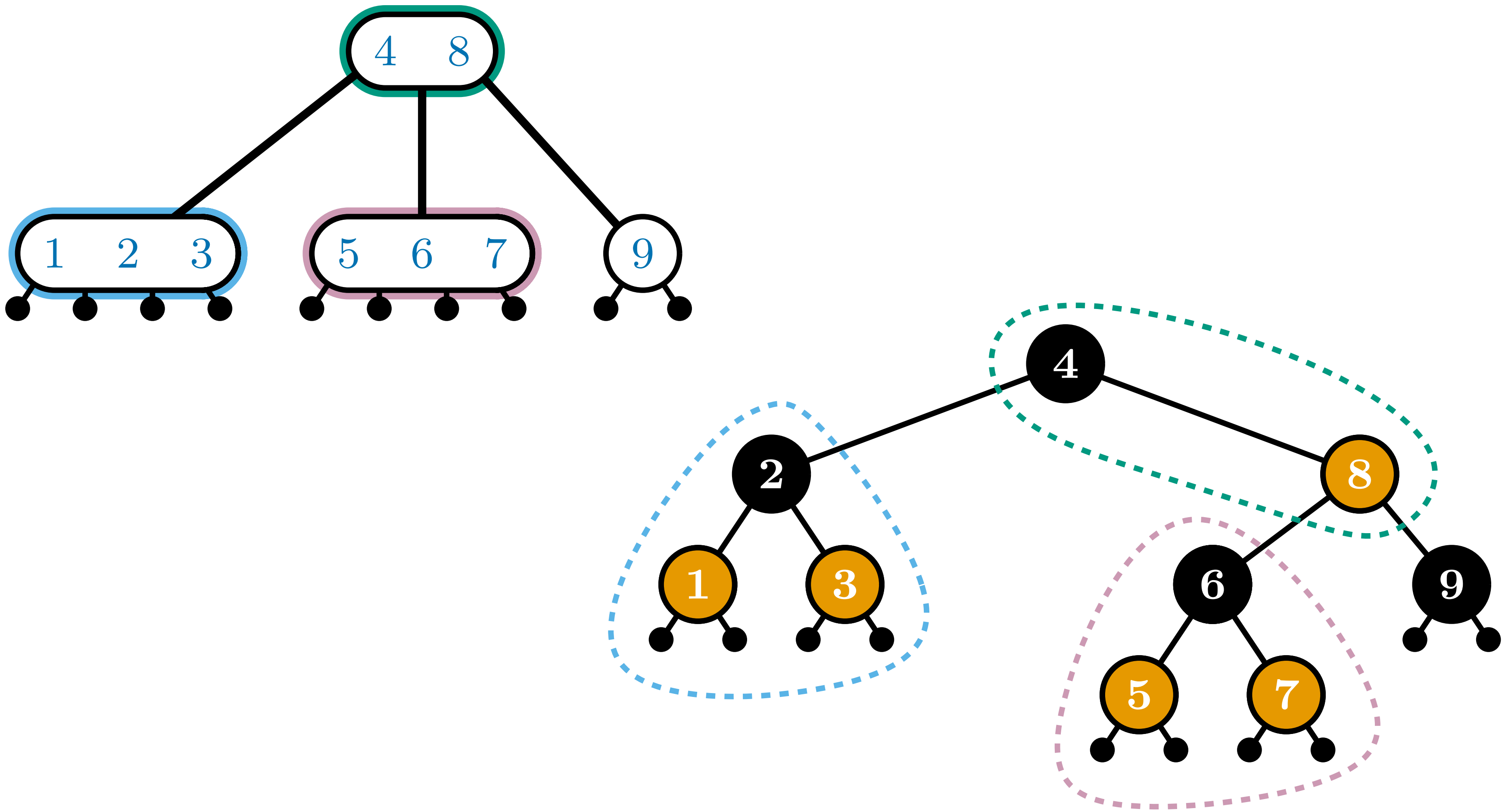

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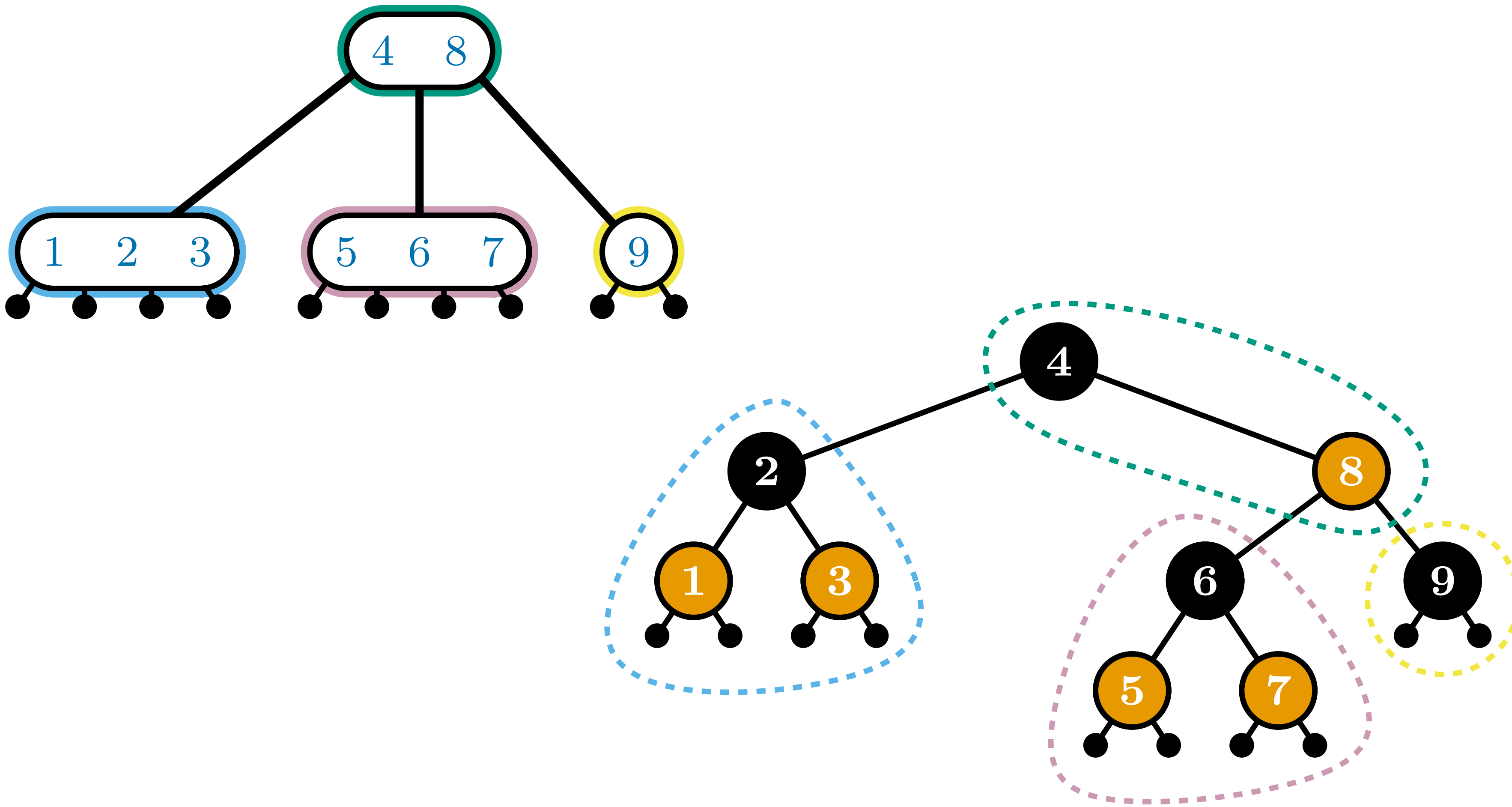

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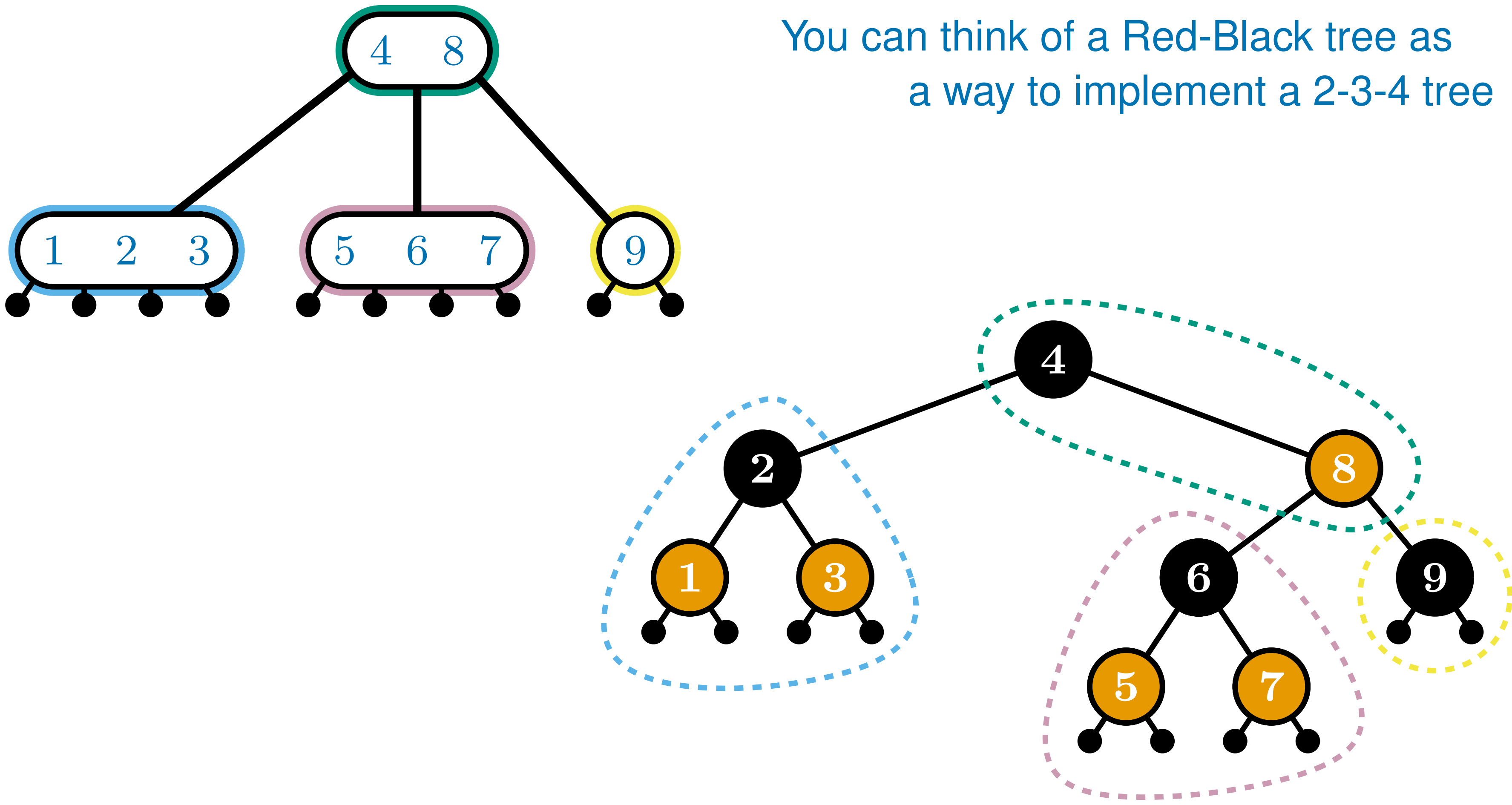
You can think of a Red-Black tree as a way to implement a 2-3-4 tree



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Dynamic Search Structure Summary

A **dynamic search structure** supports (at least) the following three operations

DELETE(k) - deletes the (unique) element x with x.key $= k$

INSERT(x, k) - inserts x with key $k = x$.key

FIND(k) - returns the (unique) element x with x.key $= k$

Here are the *worst case* time complexities of the structures we have seen...

	INSERT	DELETE	FIND
Unsorted Linked List	$O(1)$	$O(n)$	$O(n)$
Binary Search Tree	$O(n)$	$O(n)$	$O(n)$
2-3-4 Tree	$O(\log n)$	$O(\log n)$	$O(\log n)$
Red-Black Tree	$O(\log n)$	$O(\log n)$	$O(\log n)$

End of part one

Part two

Skip lists

inspired by slides by Ashley Montanaro

Dynamic Search Structures

A **dynamic search structure**,

stores a set of elements

Each element x must have a unique key - x.key

The following operations are supported:

INSERT(x, k) - inserts x with key $k = x$.key

FIND(k) - returns the (unique) element x with x.key $= k$

(or reports that it doesn't exist)

DELETE(k) - deletes the (unique) element x with x.key $= k$

(or reports that it doesn't exist)

We would also like it to support:

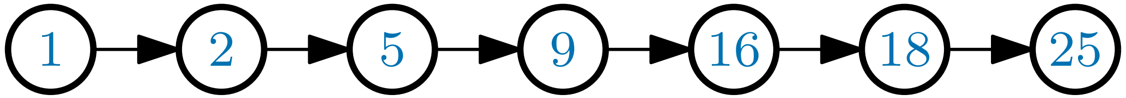
PREDECESSOR(k) - returns the (unique) element x
with the largest key such that x.key $< k$

RANGEFIND(k_1, k_2) - returns every element x with $k_1 \leqslant x$.key $\leqslant k_2$

Using a Linked List as a Dynamic Search Structure (again)

Earlier we briefly considered using an unsorted Linked List as a dynamic search structure

What about using a *sorted* Linked List?



The bottleneck is FIND, which is *very inefficient*,

- we have to look through the entire linked list to find an item *(in the worst case)*

INSERT and DELETE also take $O(n)$ time *but only because they rely on* FIND

How can we speed up the FIND operation?

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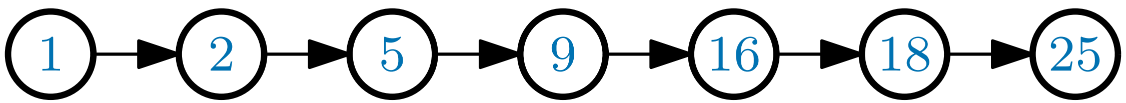

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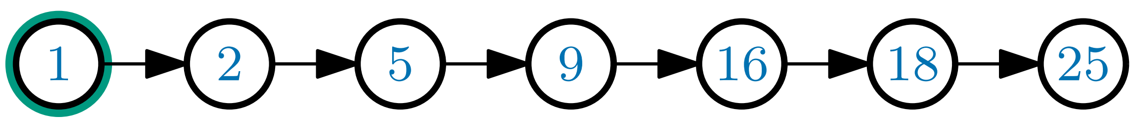

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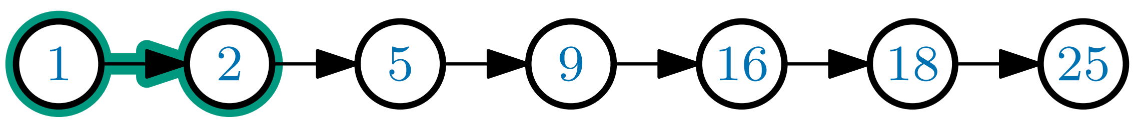

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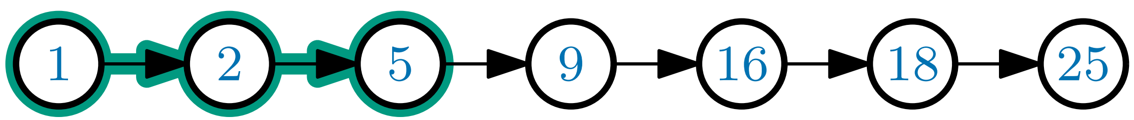

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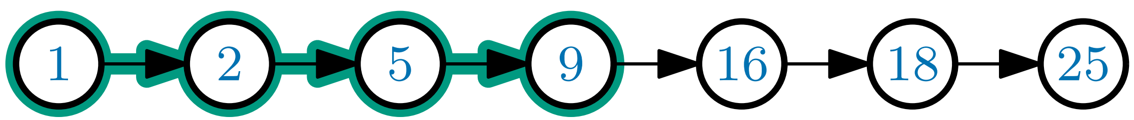

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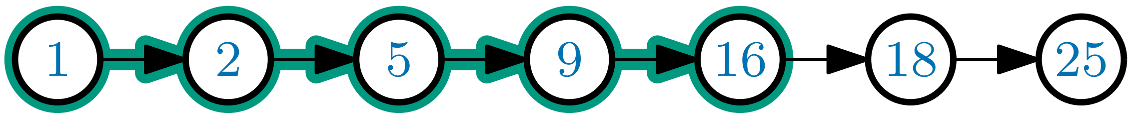

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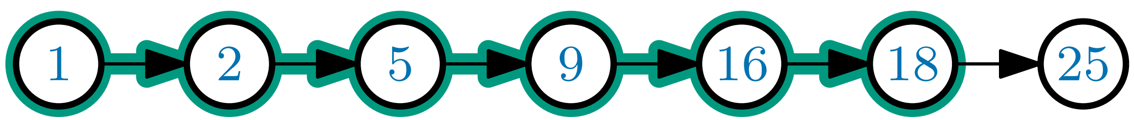

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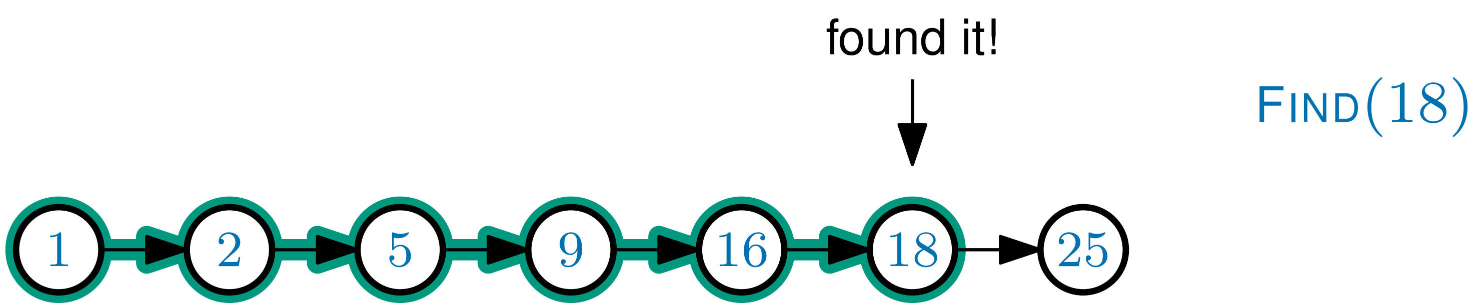

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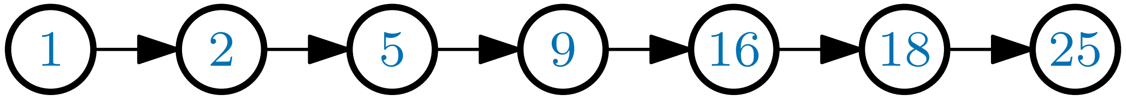

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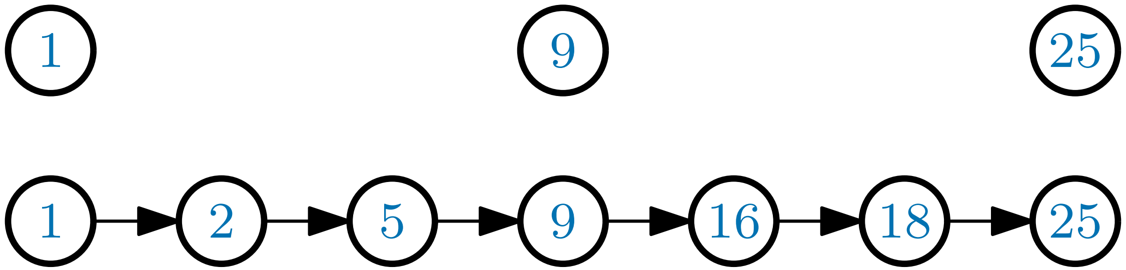

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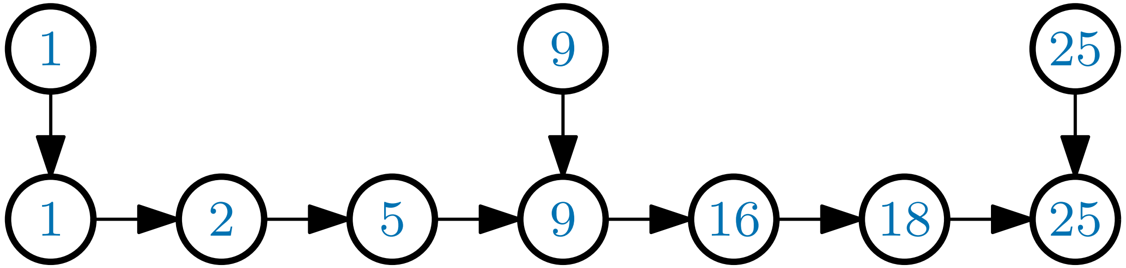

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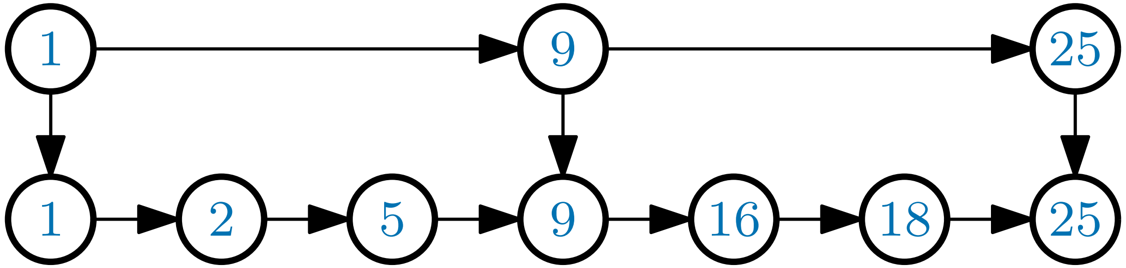

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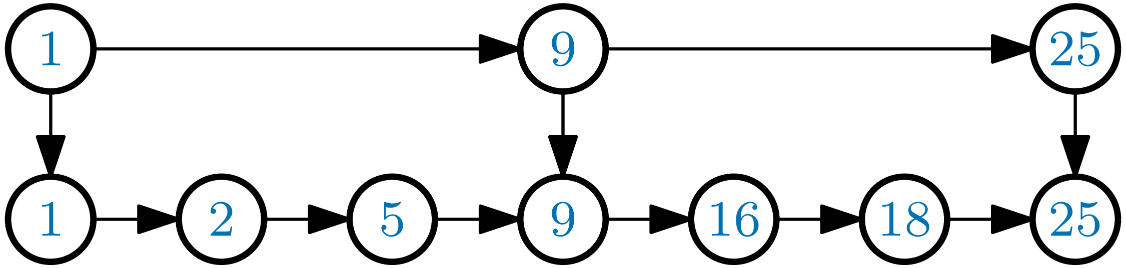

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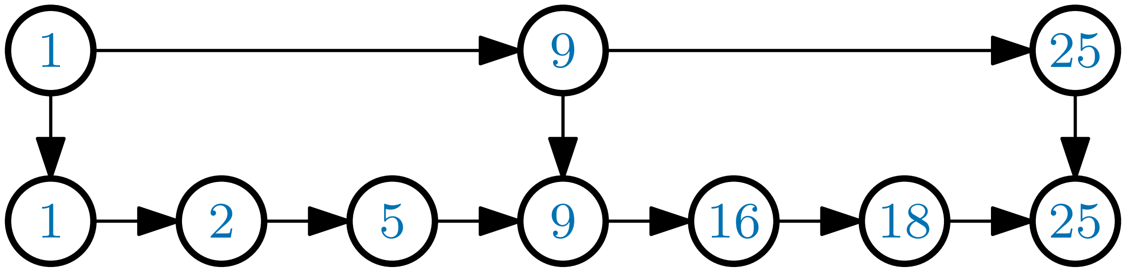

To perform $\text{FIND}(k)$ we start in the top list
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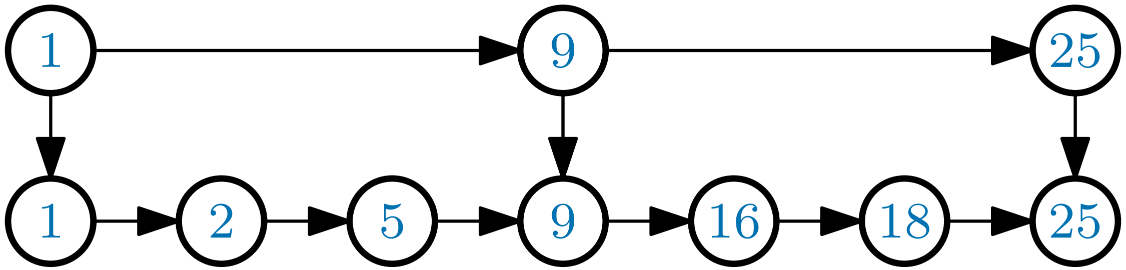

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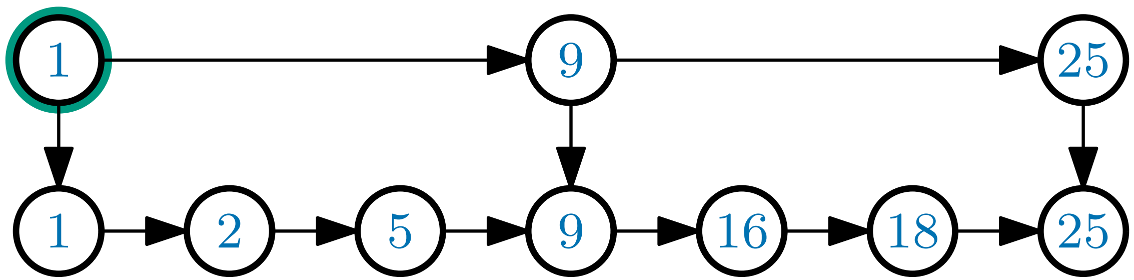

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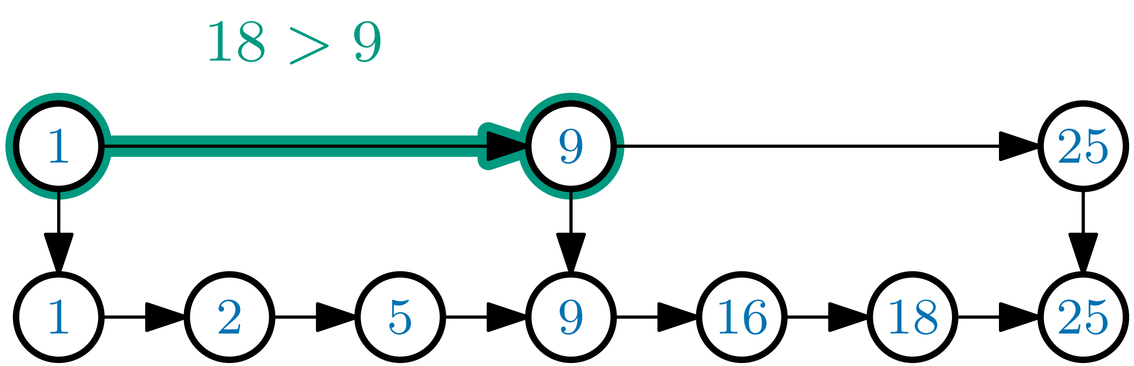

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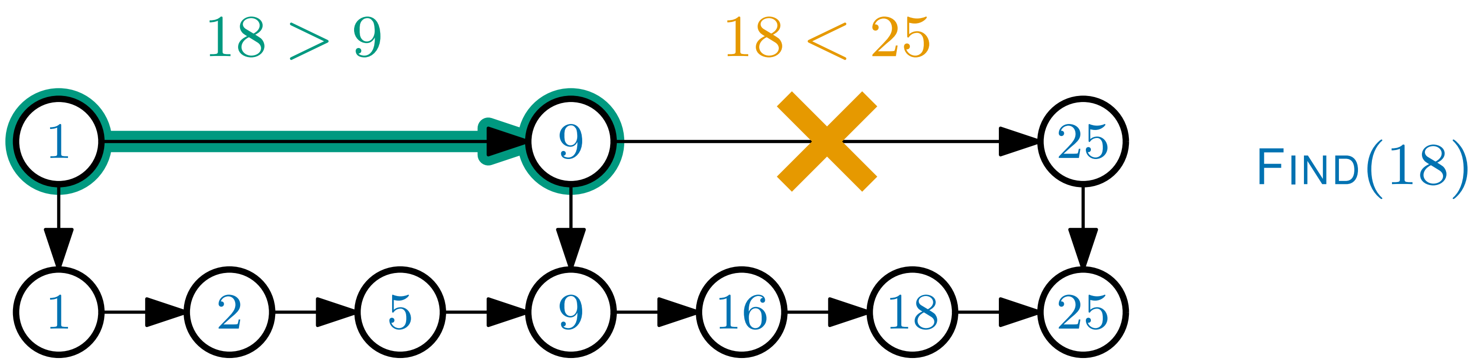

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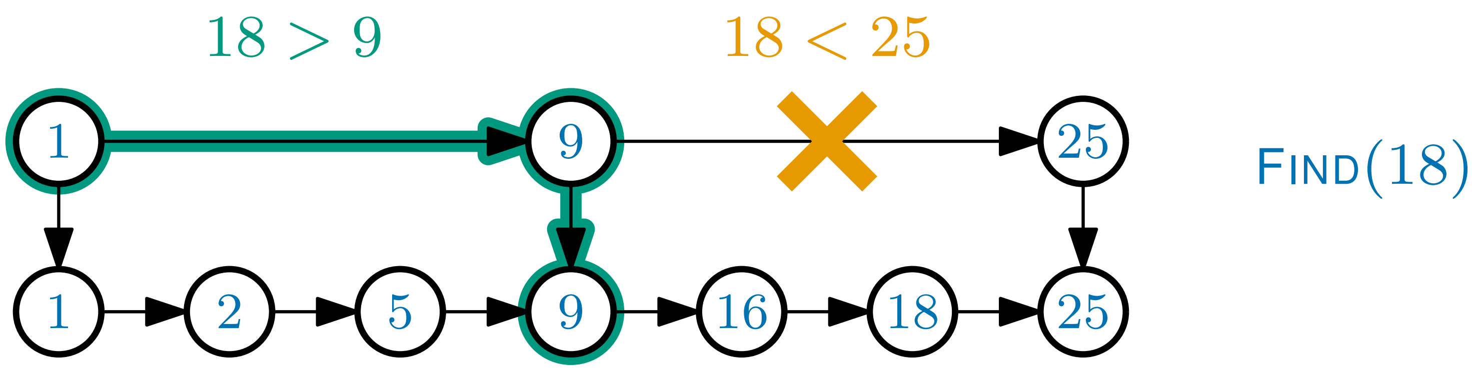

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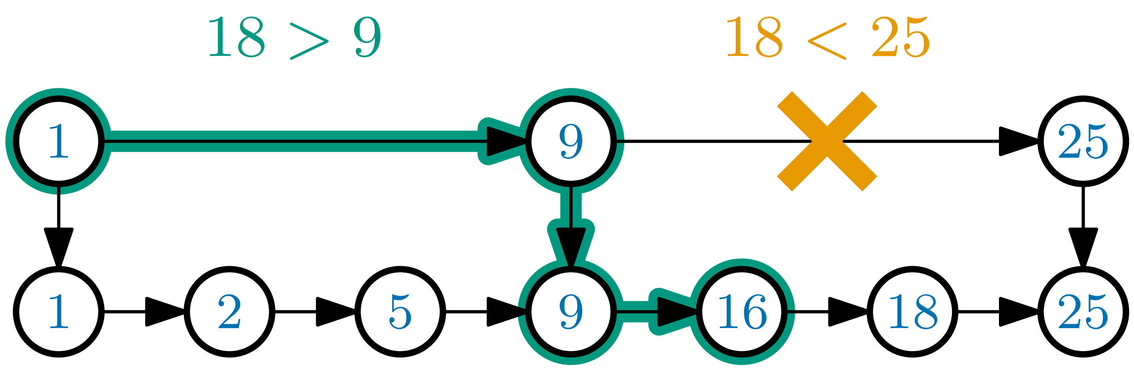

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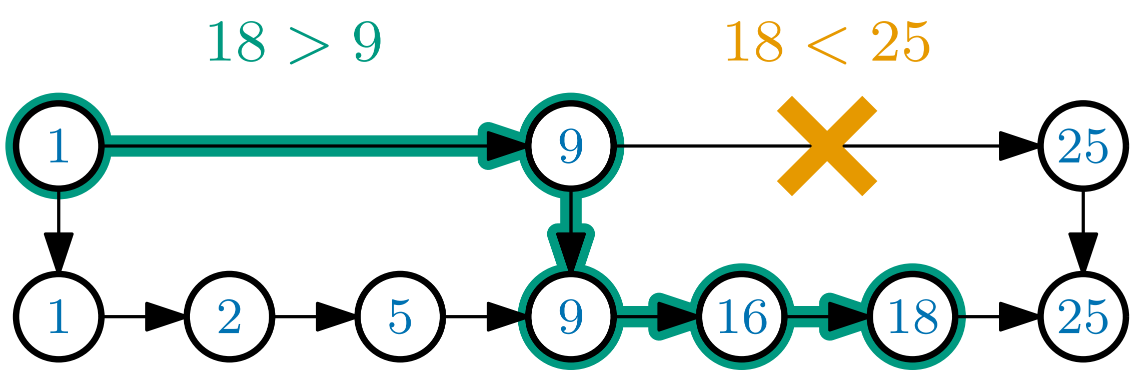

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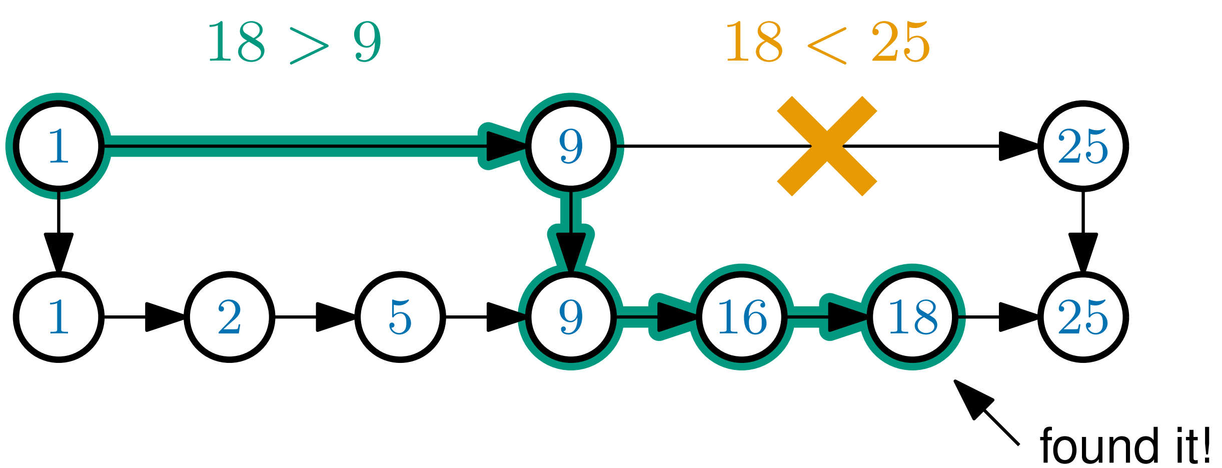

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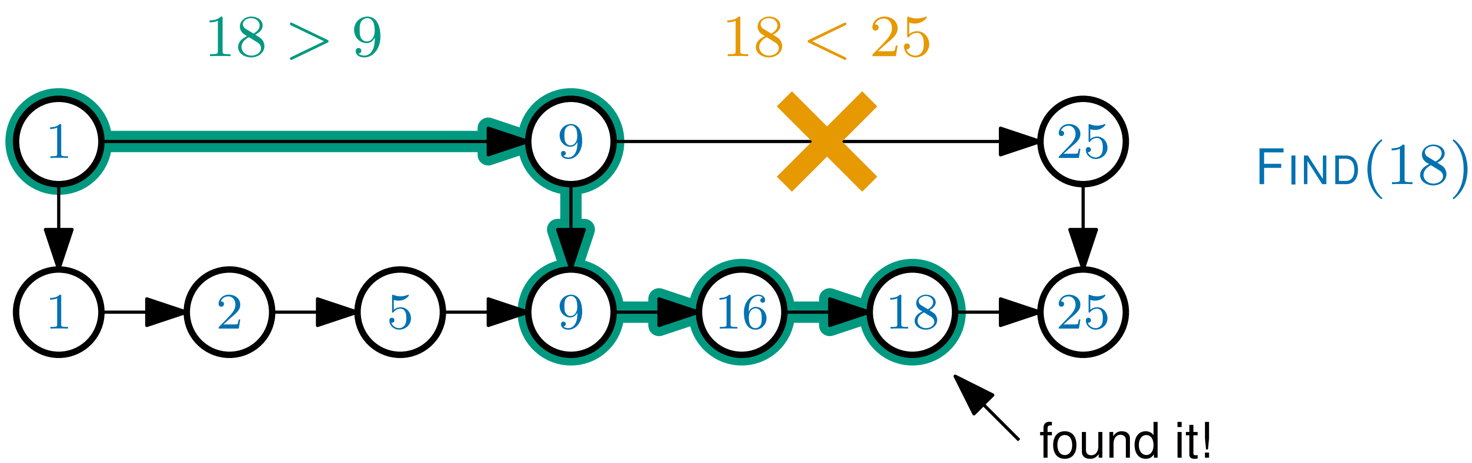

To perform $\textsc{Find}(k)$ we start in the top list

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How long does this take?

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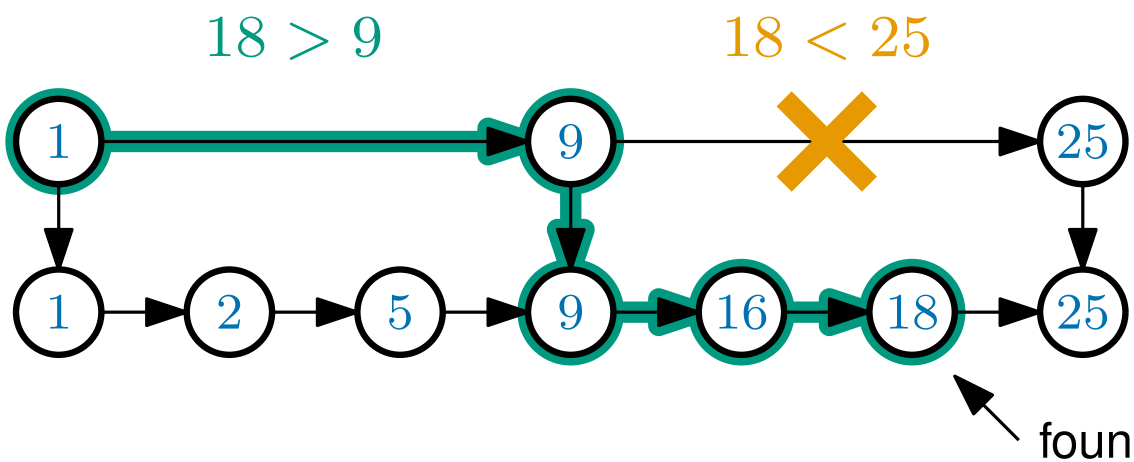
How about adding some shortcuts?

We've attached a second linked list containing only some of the keys...



$\text{FIND}(18)$

To perform $\text{FIND}(k)$ we start in the top list
and go right until we come to a key $k' > k$

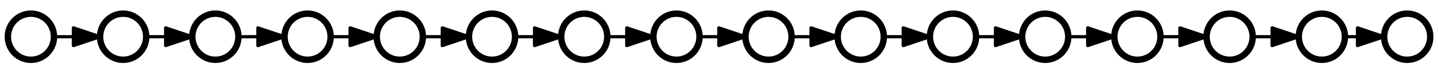
then we move down to the bottom list
and go right until we find k

How long does this take?
That depends on where we place the shortcuts

Linked Lists with two levels

Imagine that we decide to place m keys in the top list...

(the bottom list always contains all n keys)



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where should we put them to minimise

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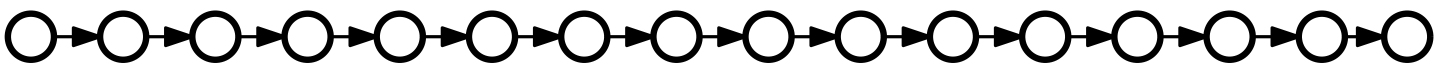

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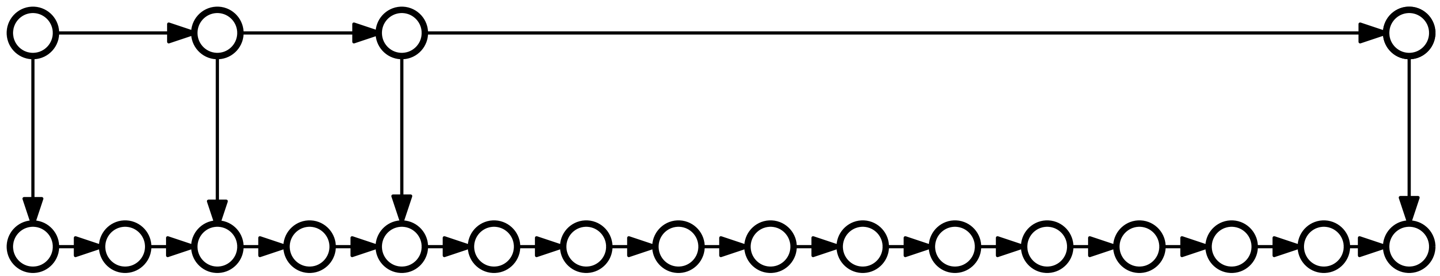

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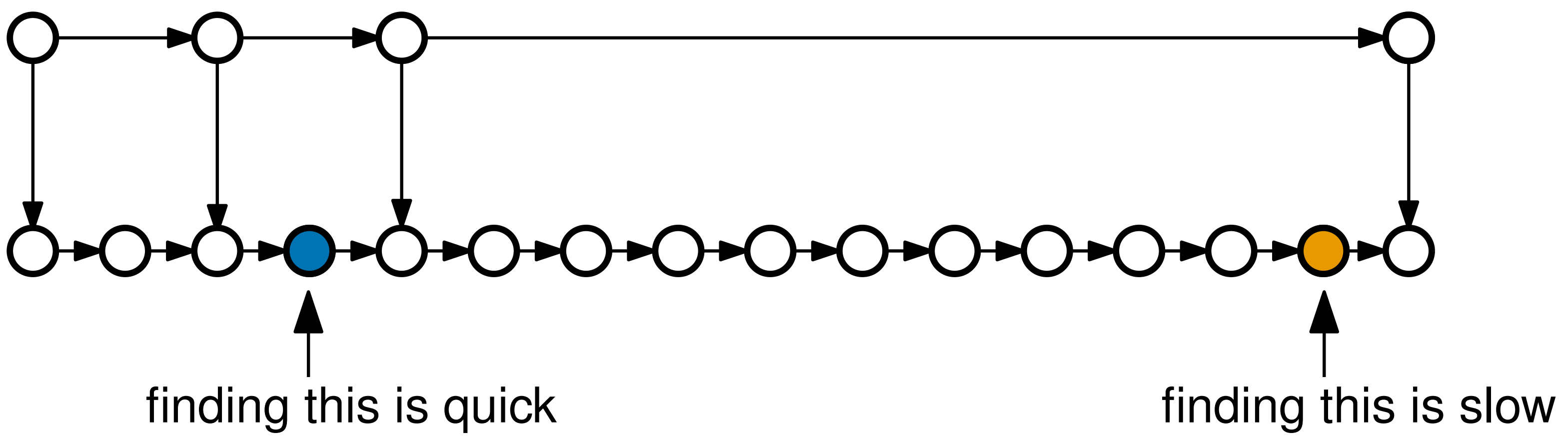

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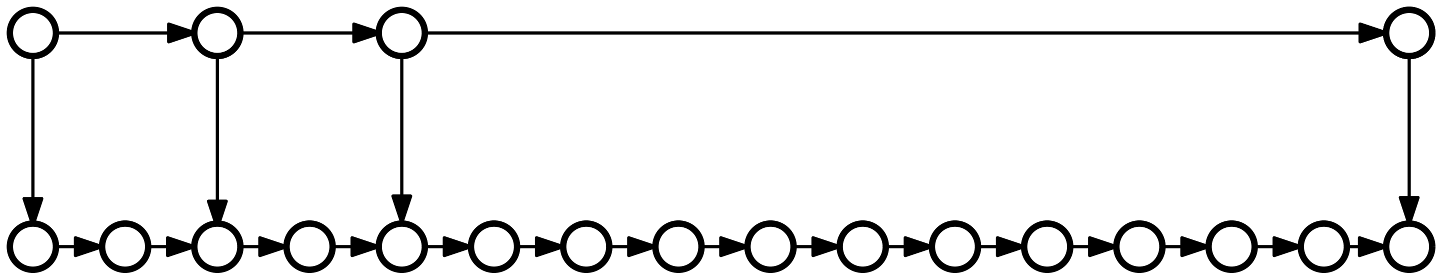

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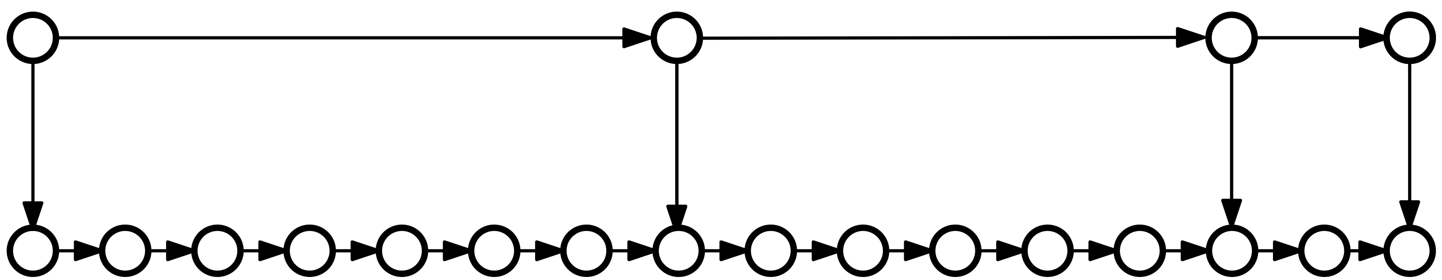

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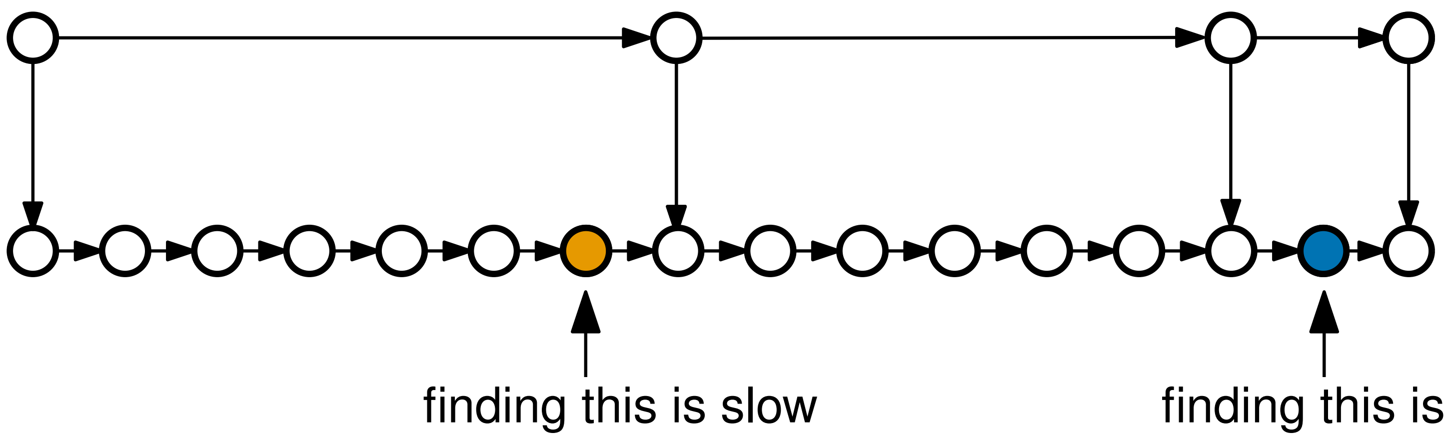

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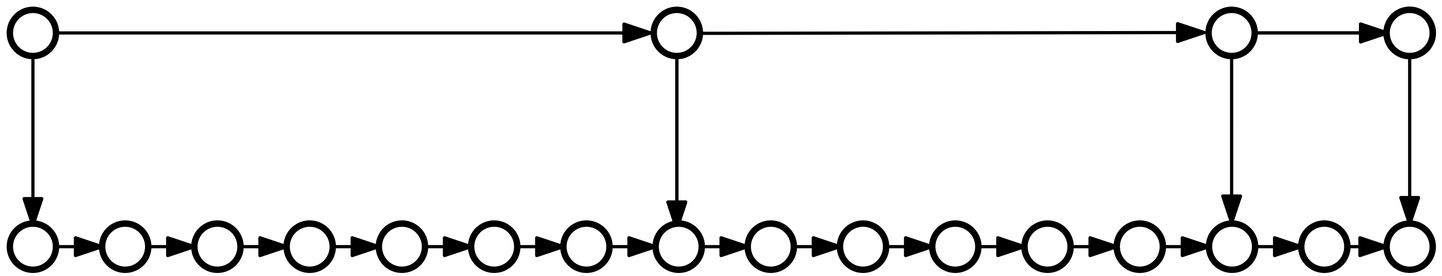

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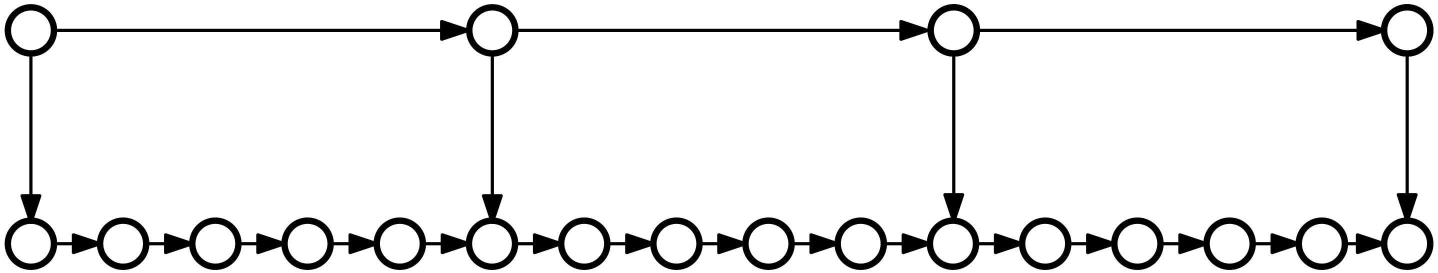

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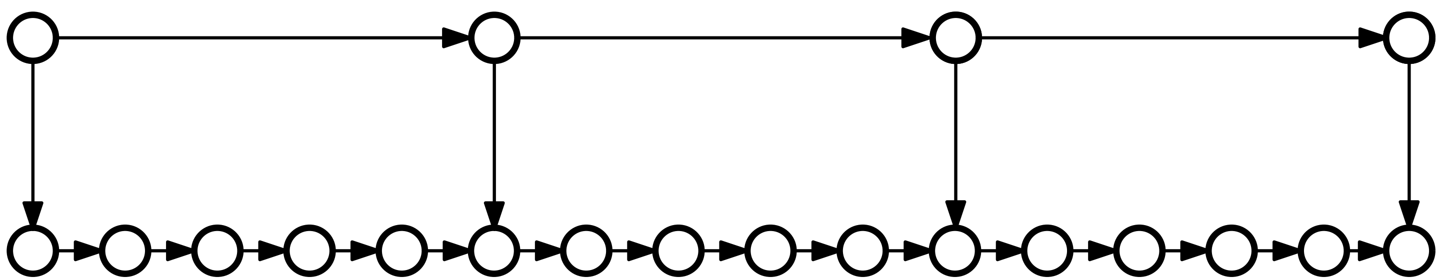

If we spread out the m keys in the top list evenly ...

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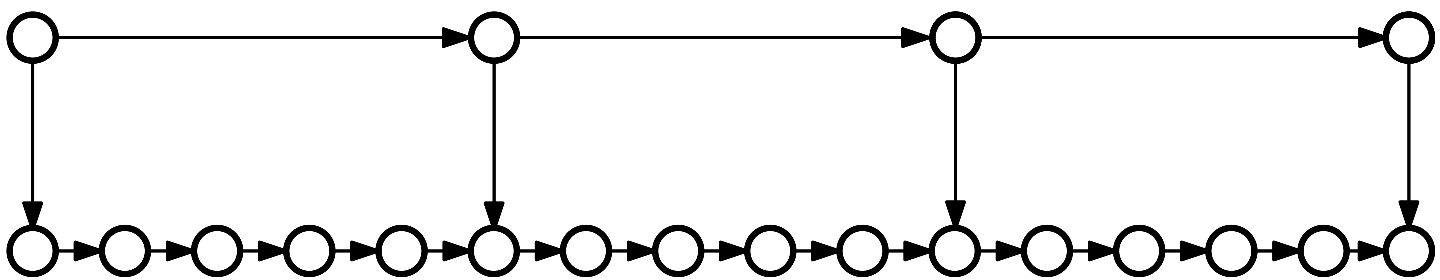

If we spread out the m keys in the top list evenly ...

the *worst case* time for a FIND operation becomes $O(m + n/m)$

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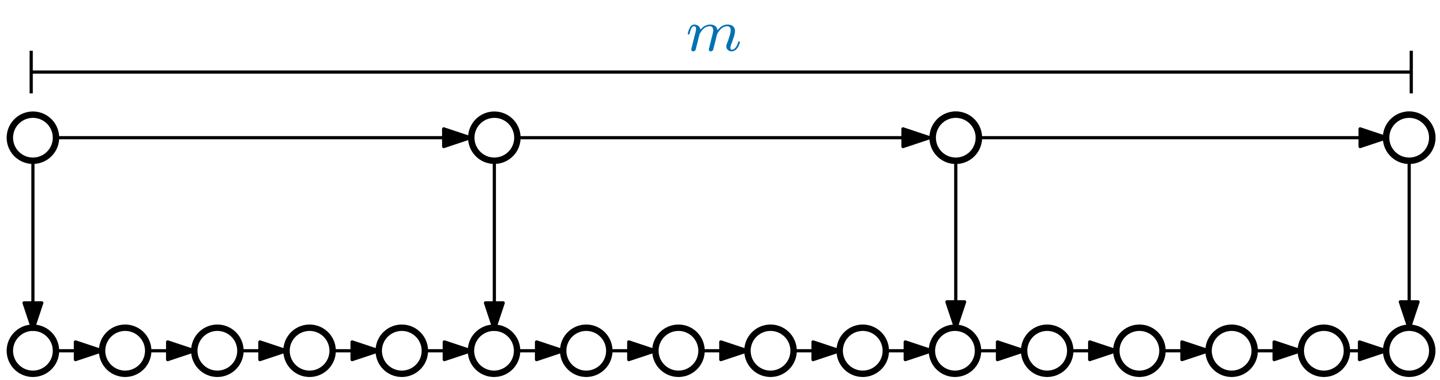

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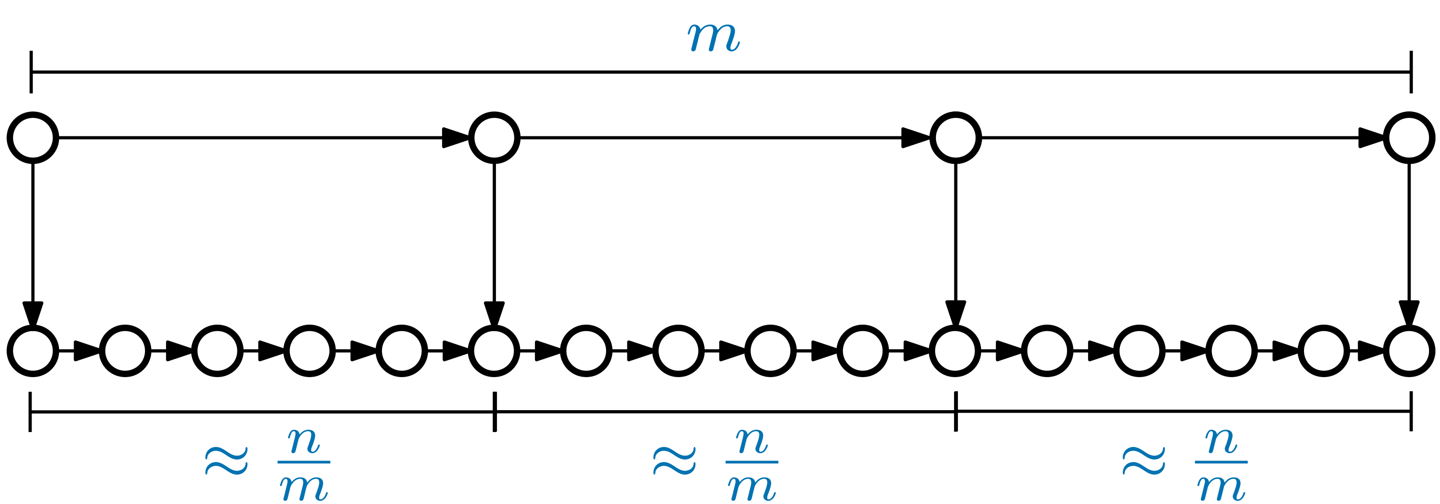

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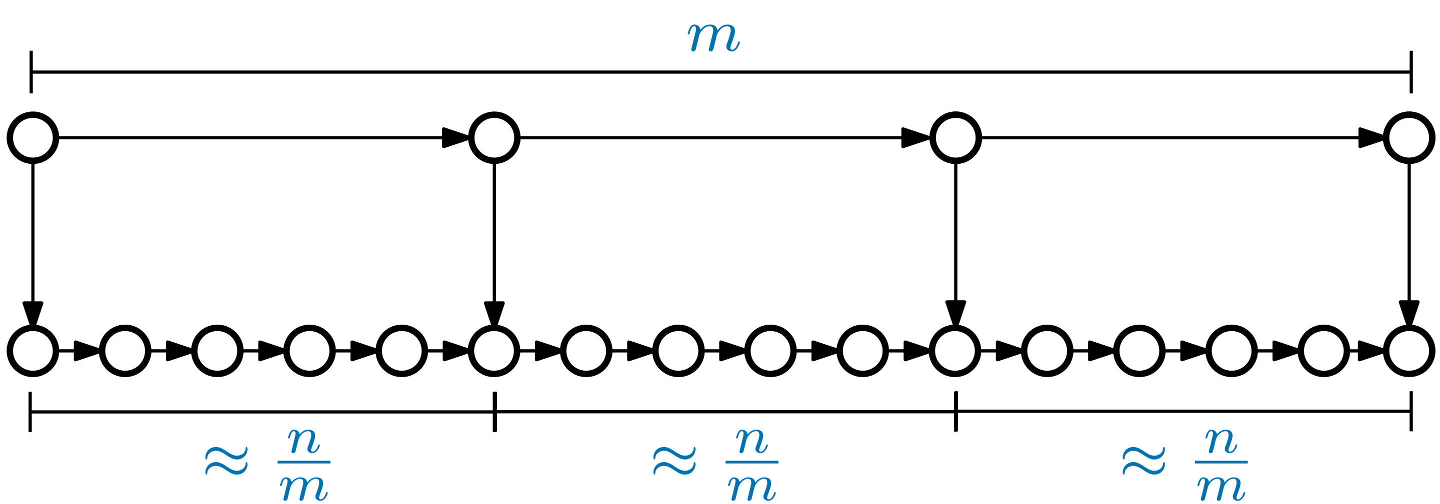

If we spread out the m keys in the top list evenly ...

the *worst case* time for a FIND operation becomes $O(m + n/m)$

By setting $m = \sqrt{n}$, we get

the *worst case* time for a FIND operation is $O(\sqrt{n})$

Linked Lists with many levels

How about adding even more lists? *(each list is called a level)*

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Each level will now contain **half** of the keys *(rounding up)* from the level below

They are chosen to be as evenly spread as possible

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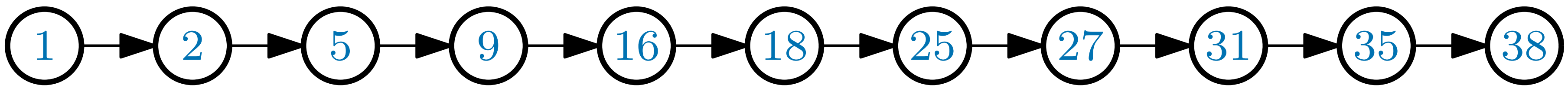

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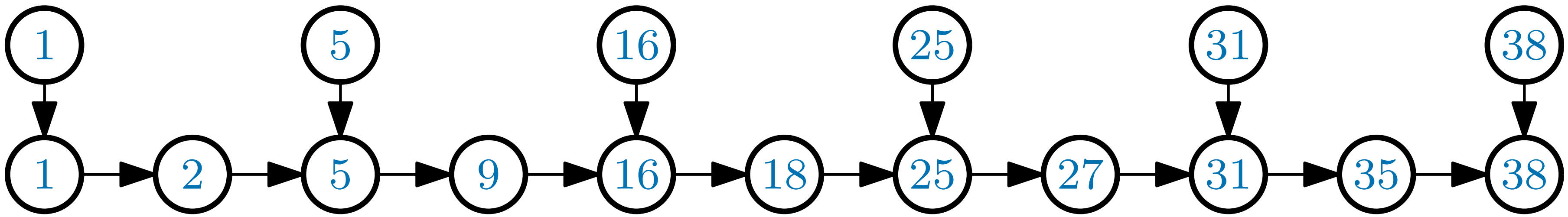

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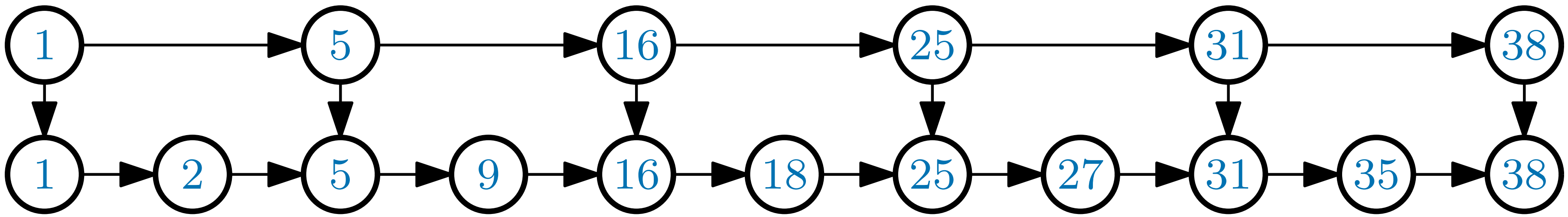

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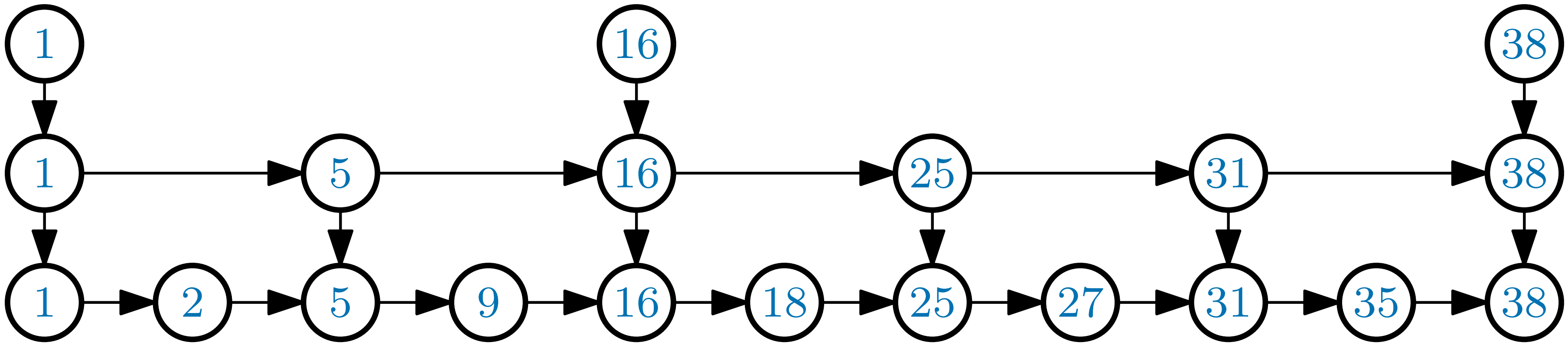

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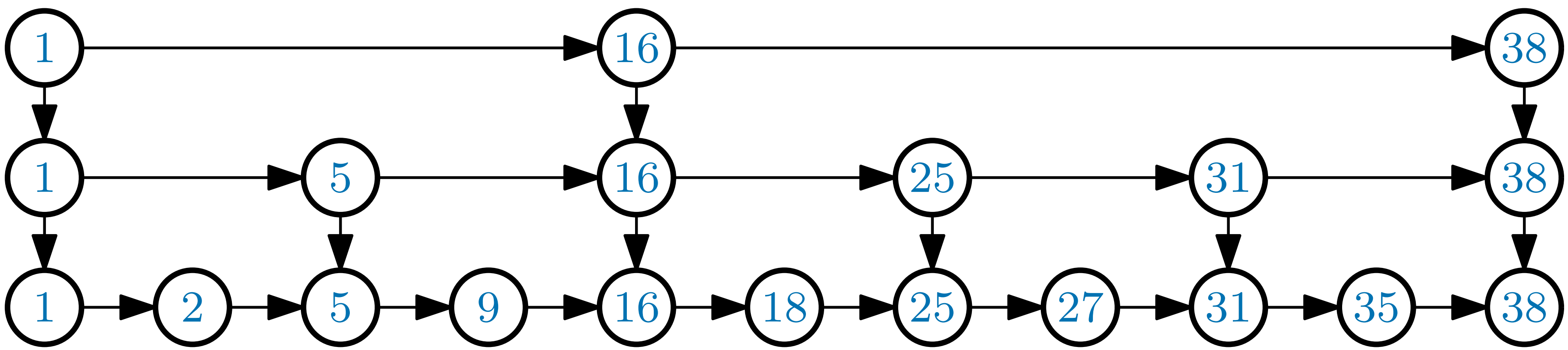

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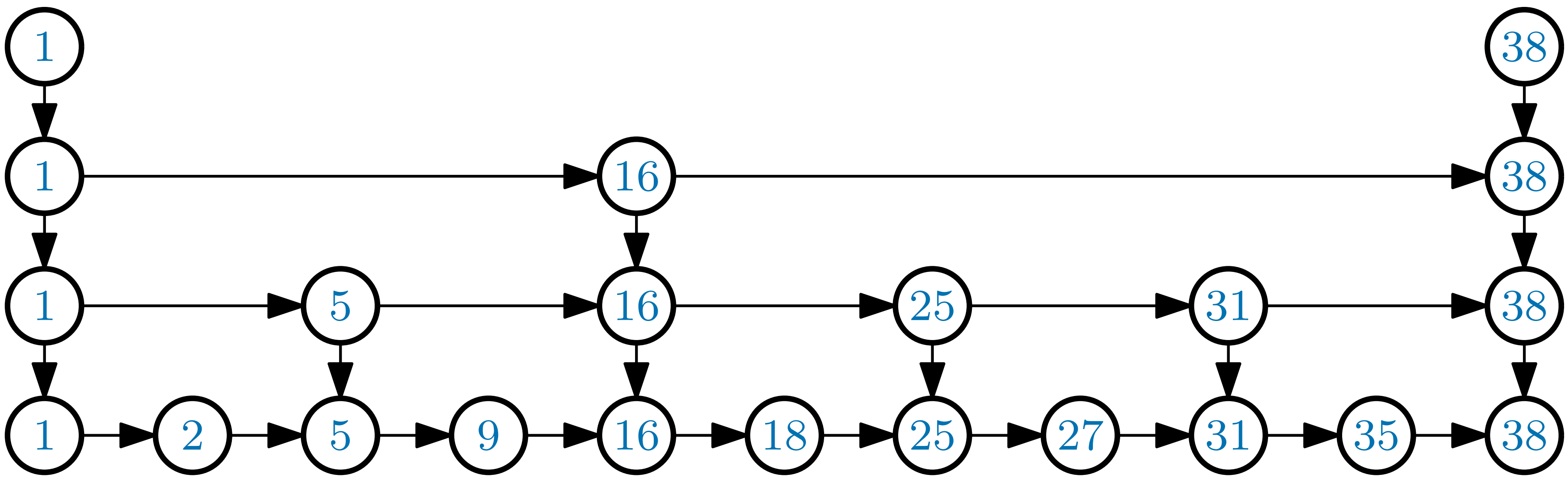

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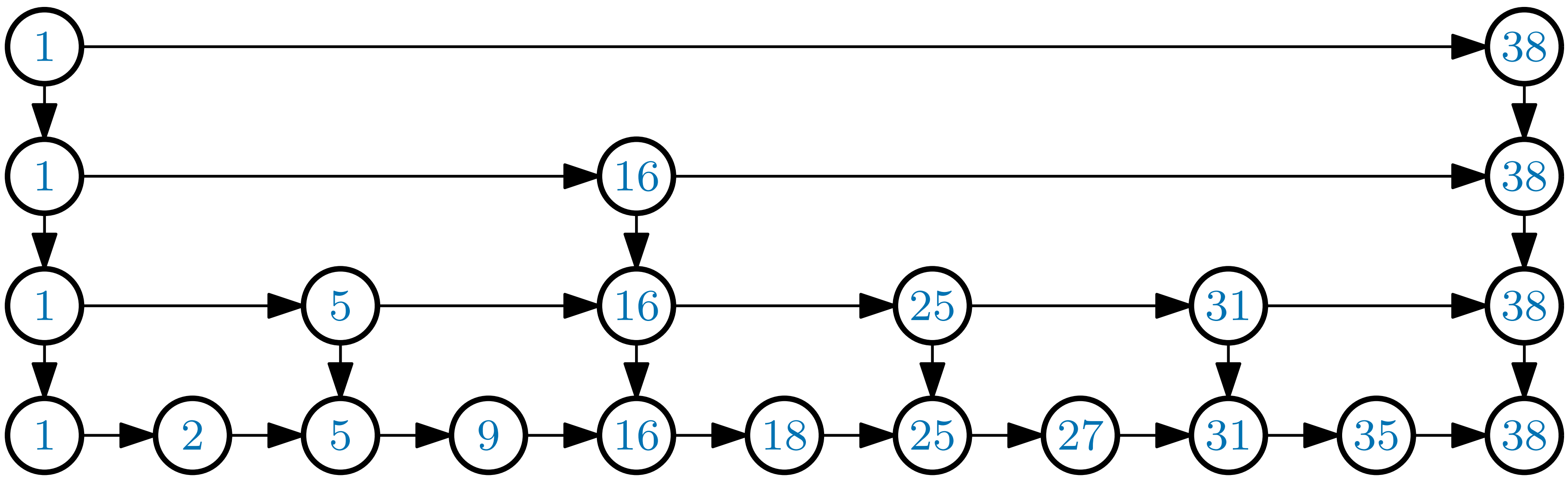

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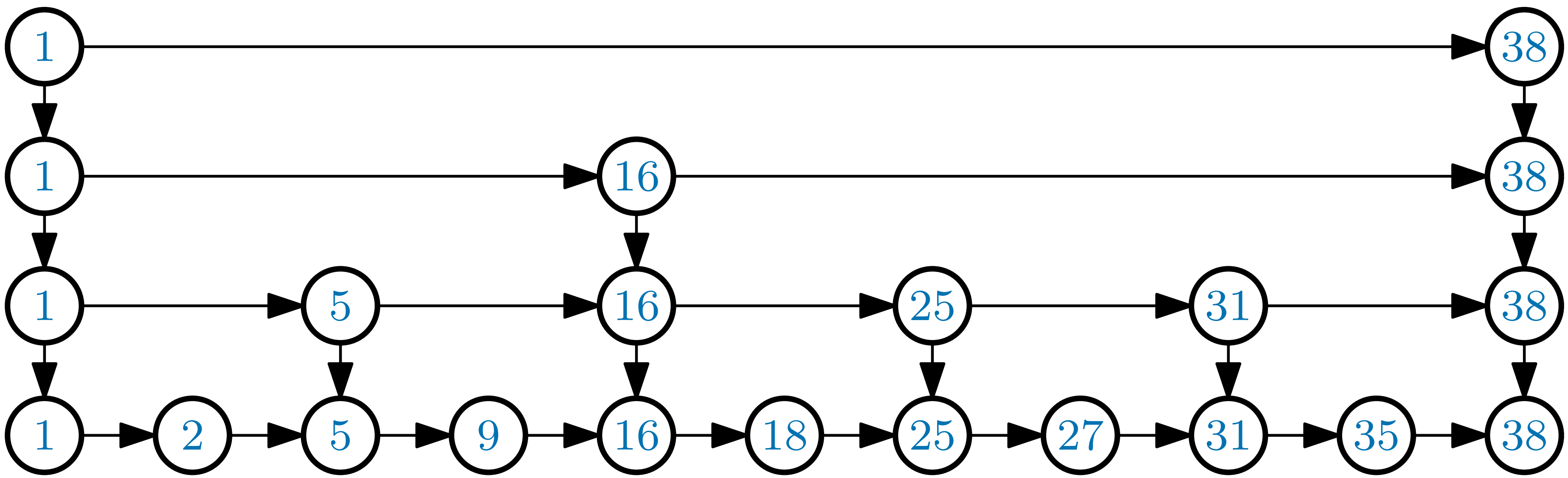

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and every level contains the leftmost and rightmost keys

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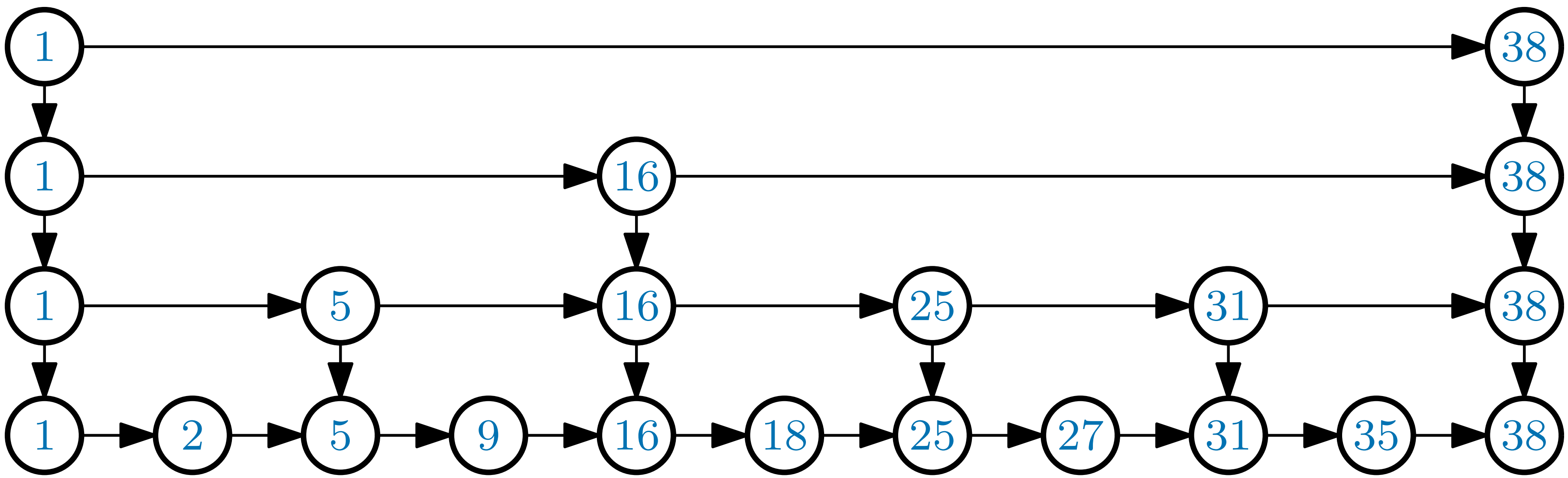


The bottom level contains every key

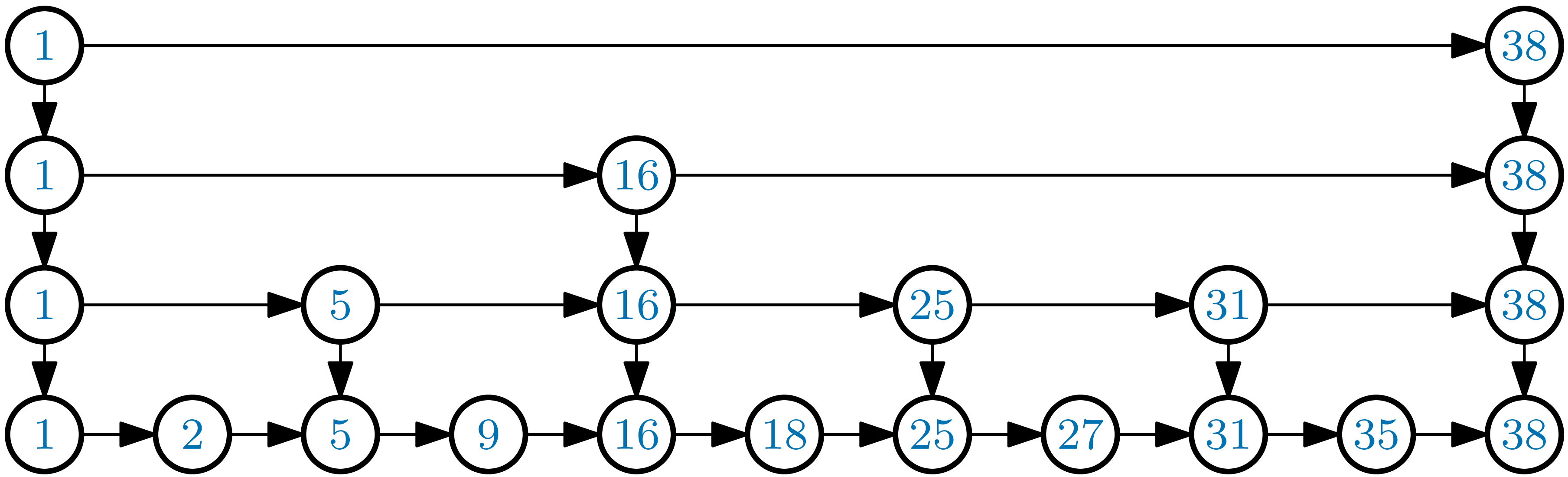
and every level contains the leftmost and rightmost keys

As each level contains half of the keys from the level below, there are $O(\log n)$ levels

FIND in multi-level linked lists

How do we perform FIND(k) in multi-level linked list?

(essentially just like before)



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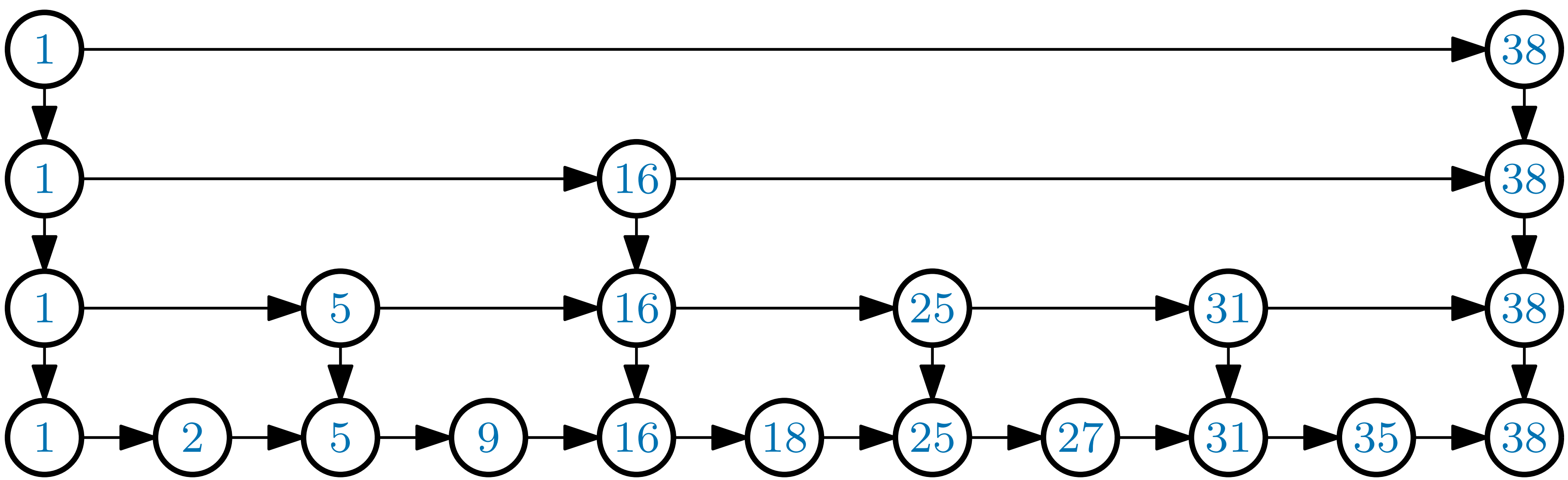

To perform FIND(k),

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Start at the top-left (the head of the top level)
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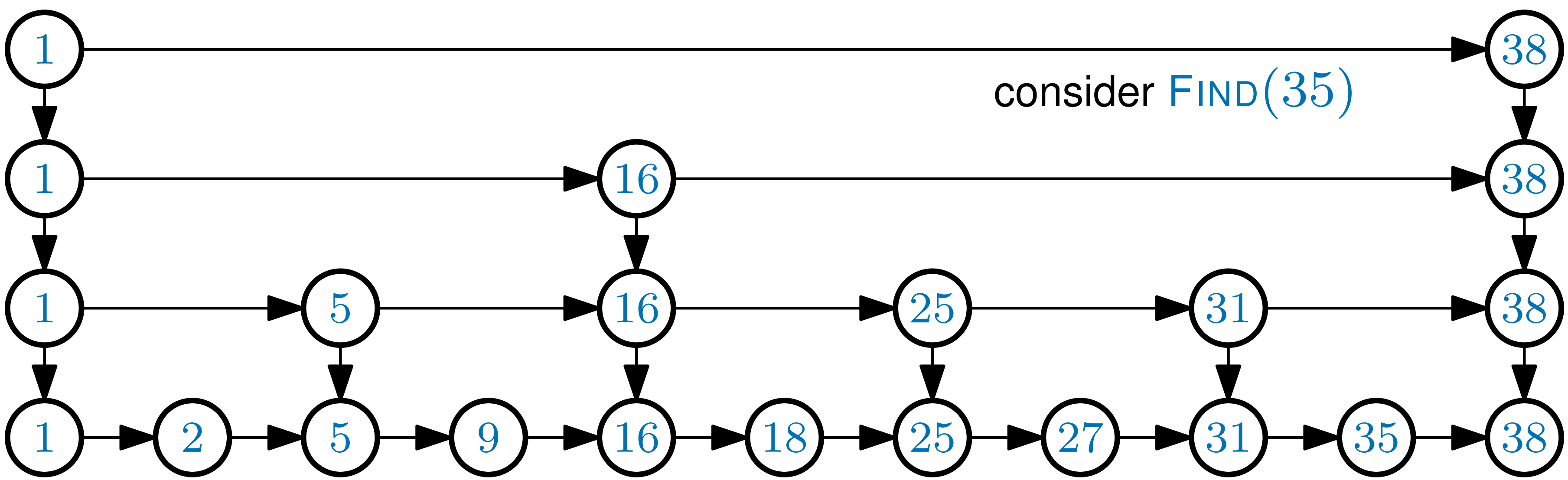

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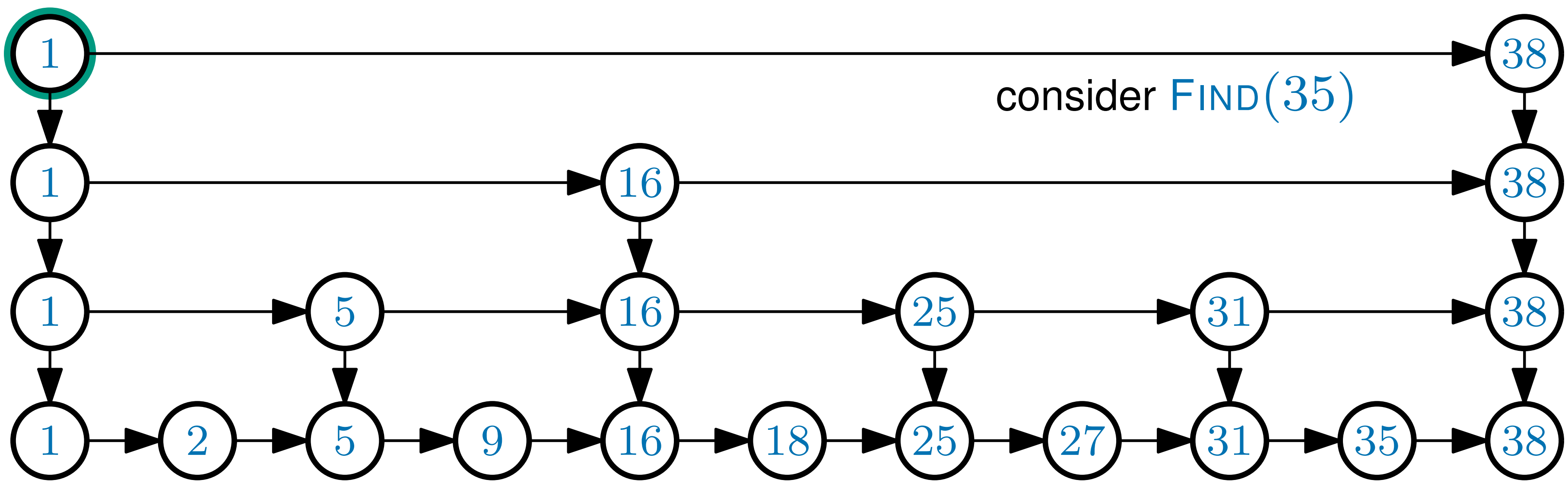

consider FIND(35)

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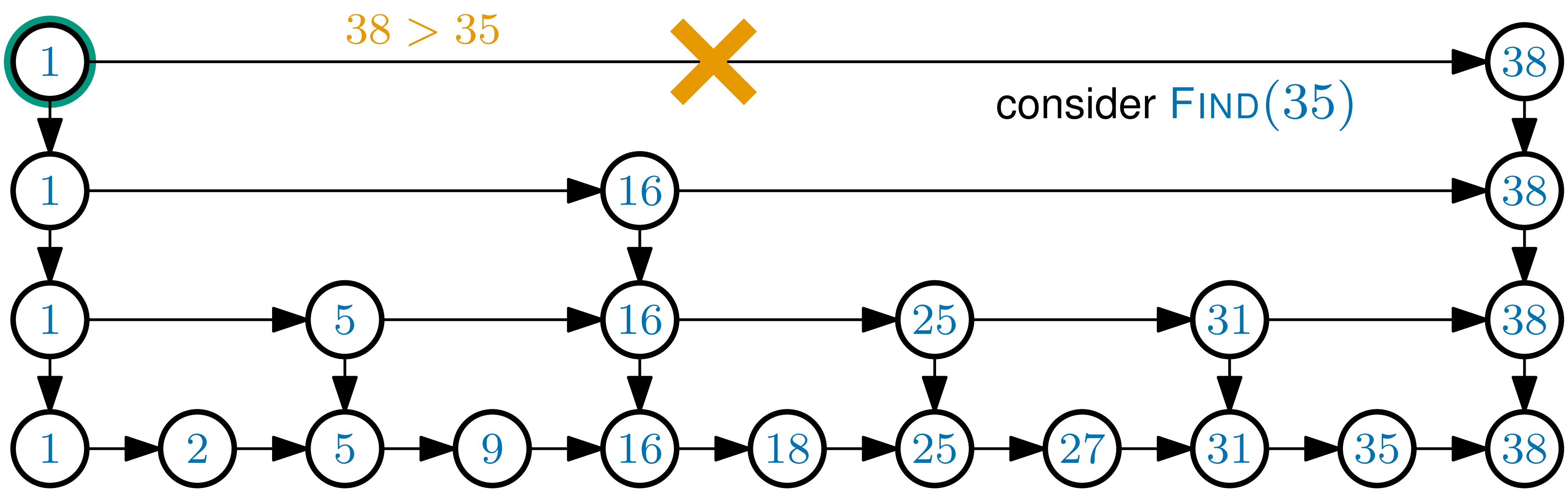

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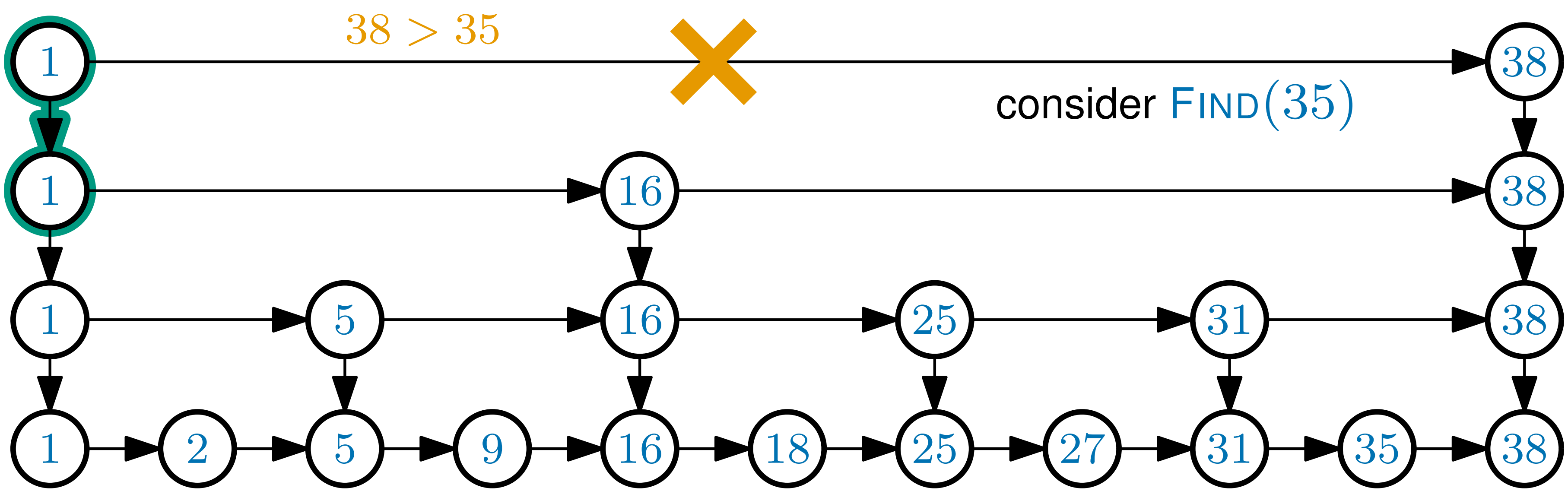

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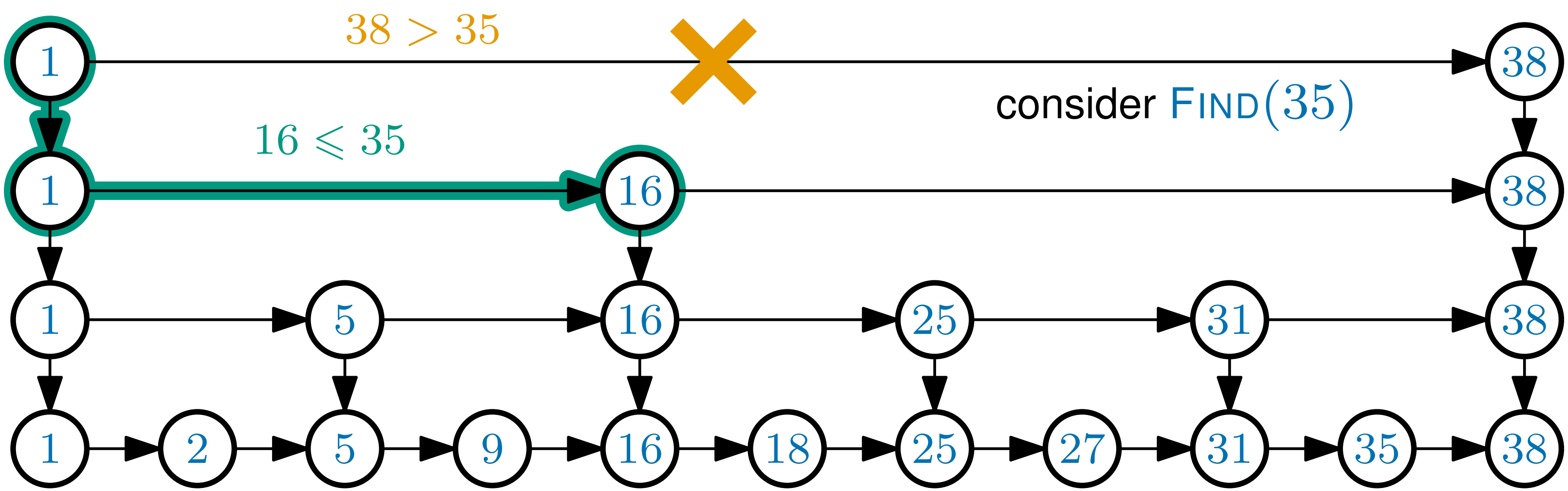

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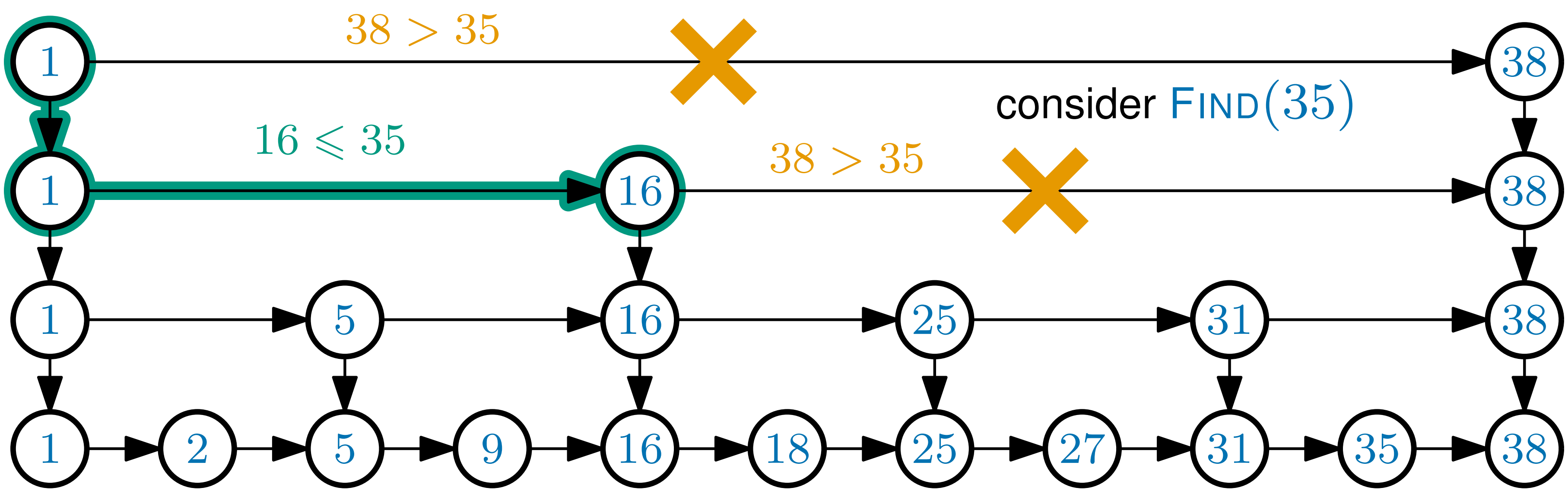

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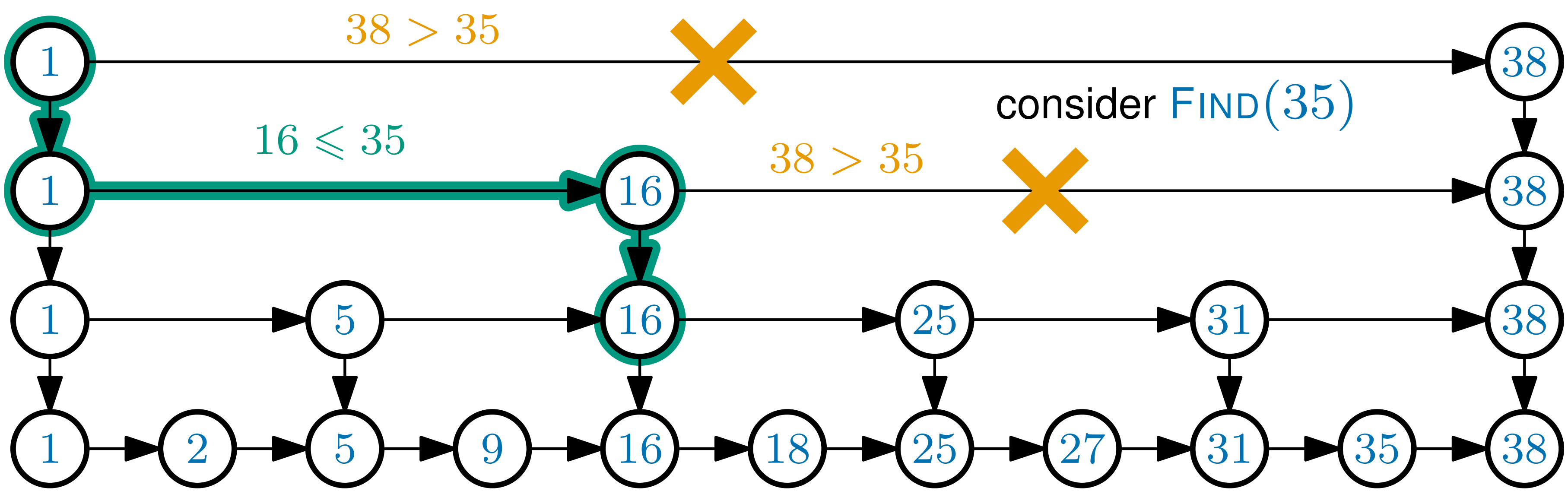

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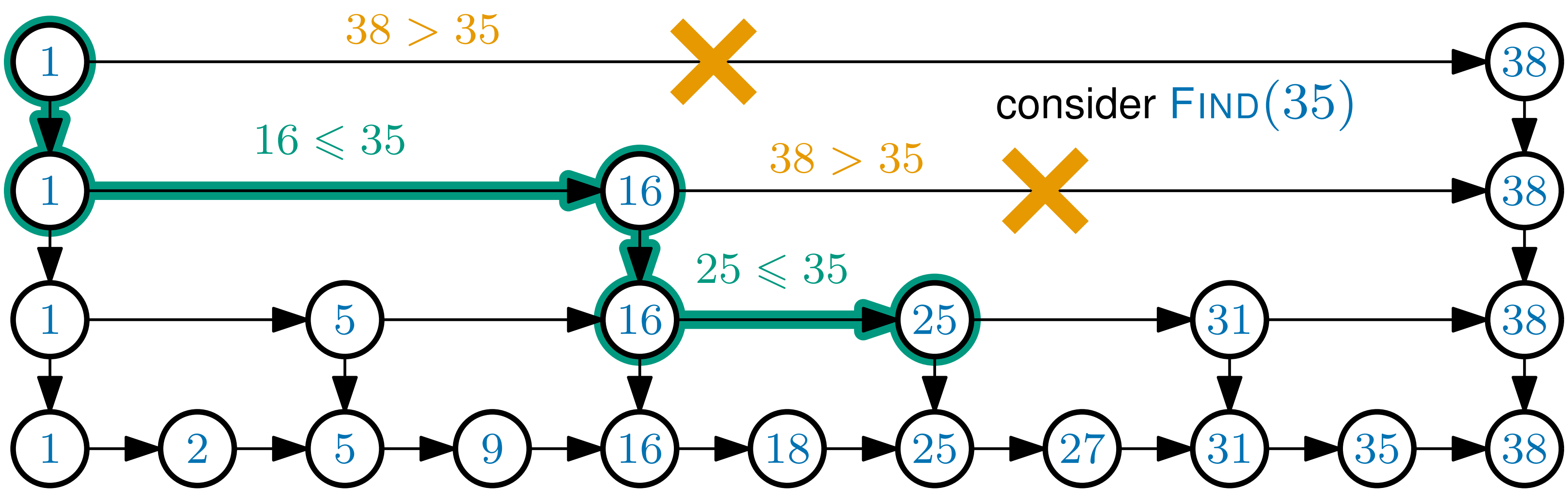

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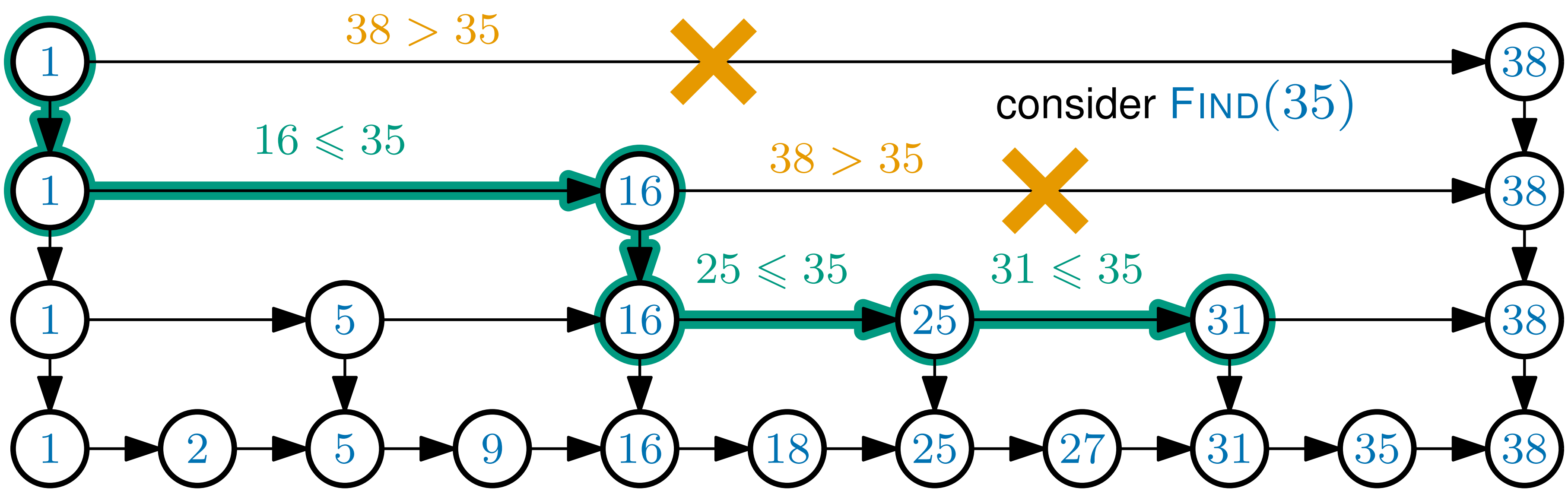

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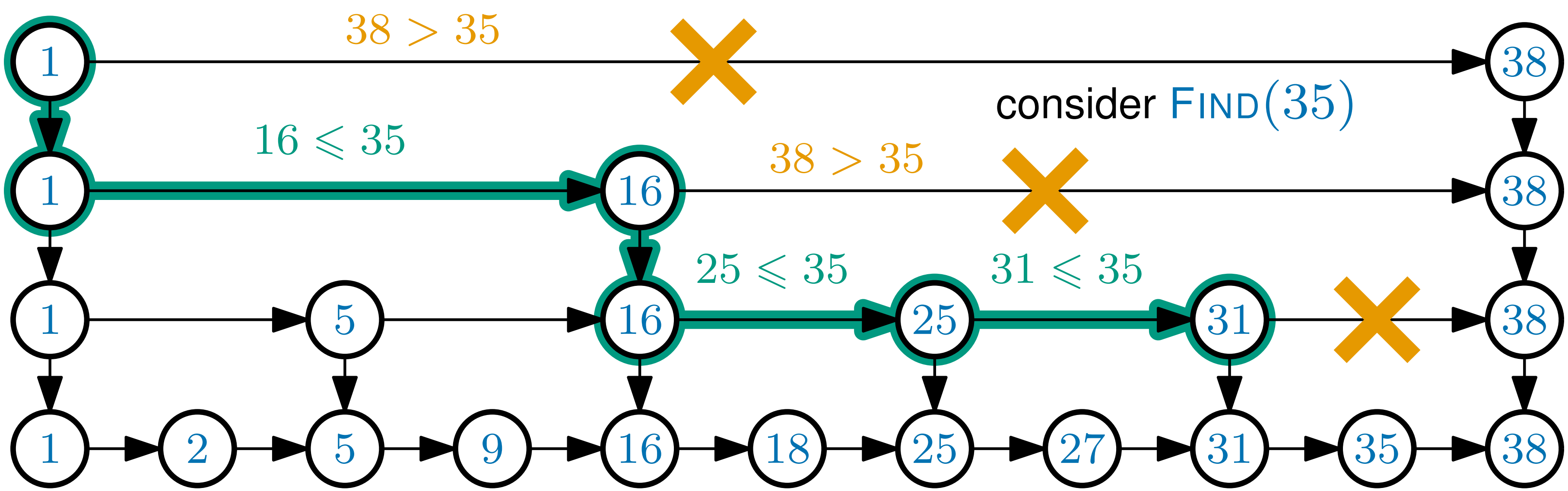

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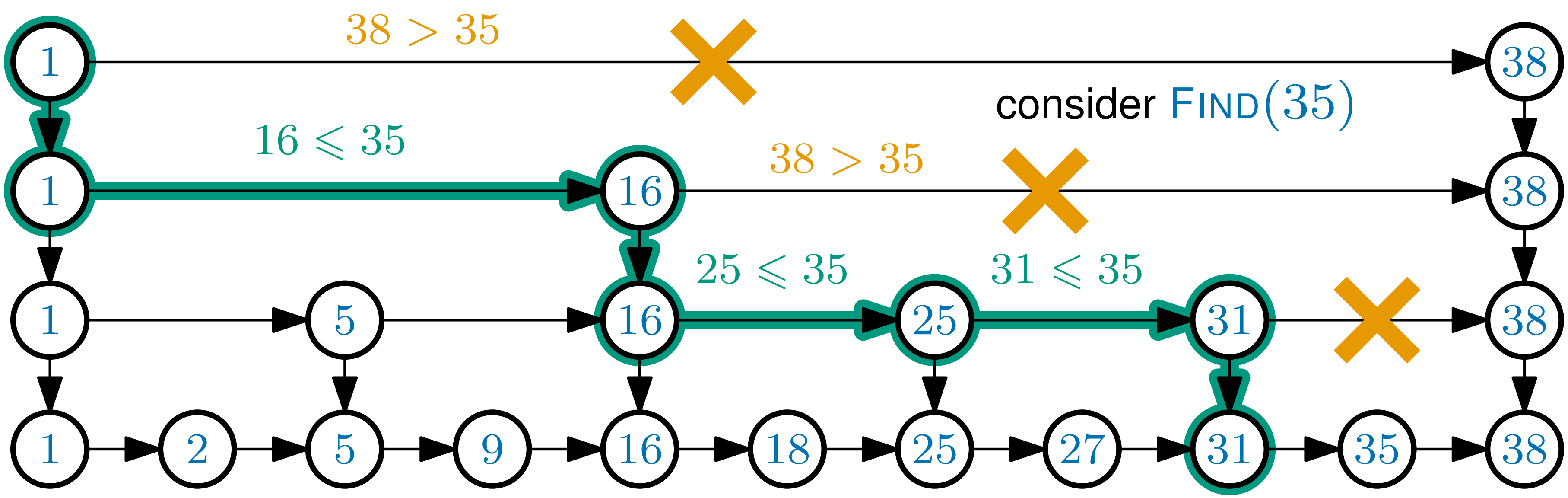

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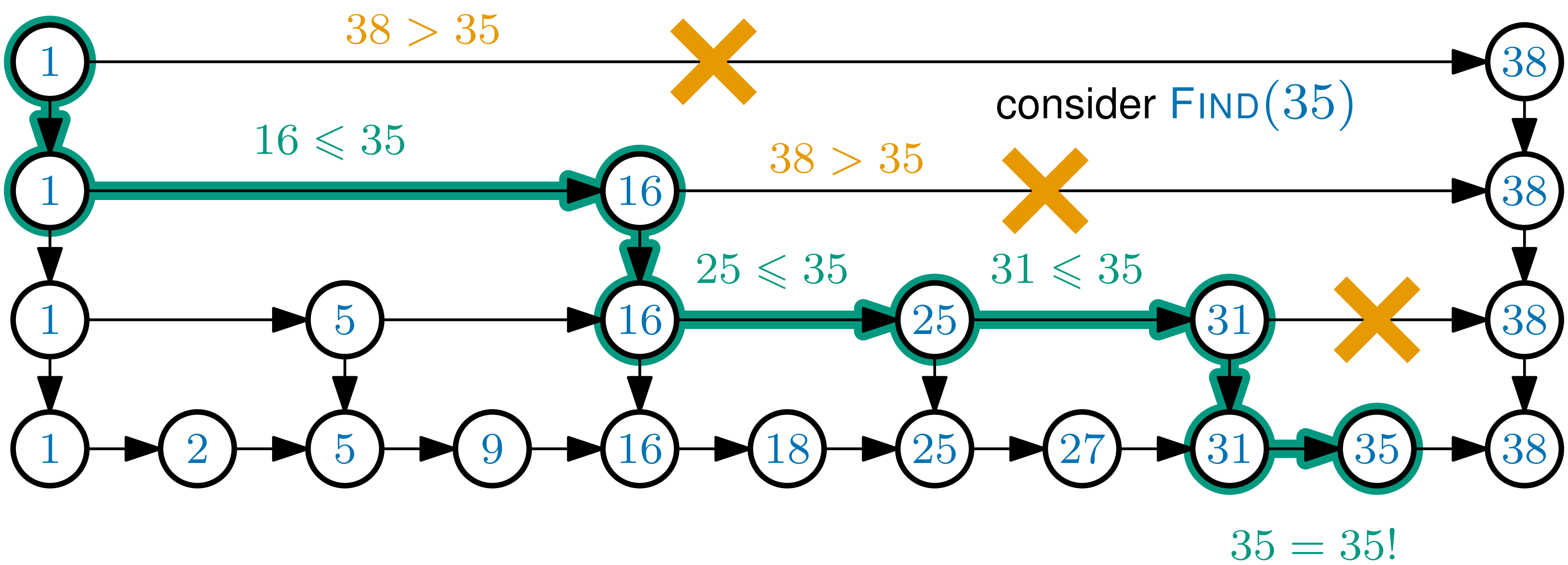

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How long does FIND(k) take in a multi-level linked list?

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Observation 1 We only **move down** at most $O(\log n)$ times

because there are only $O(\log n)$ *levels*

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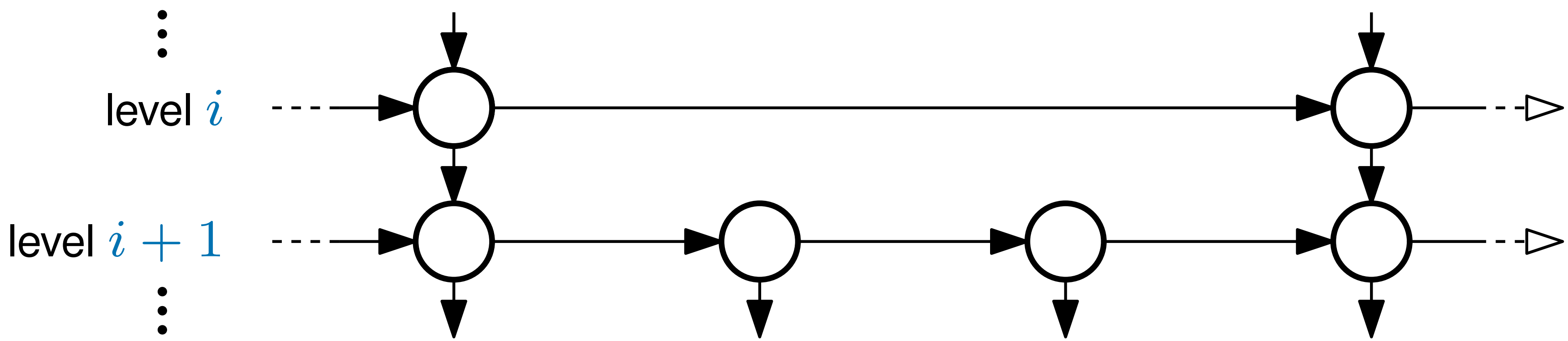

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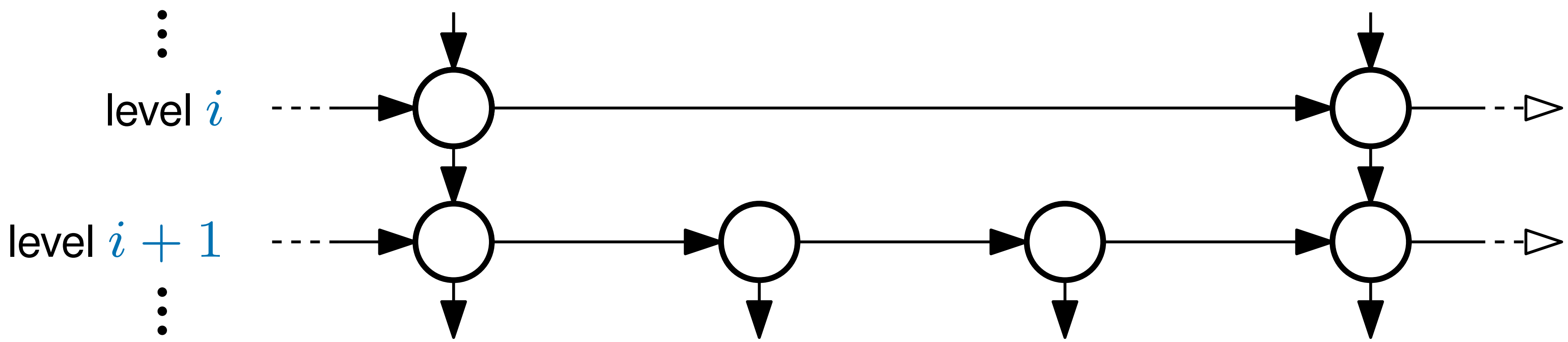

Observation 3 We only **move right** at most 2 times on any level $i+1$

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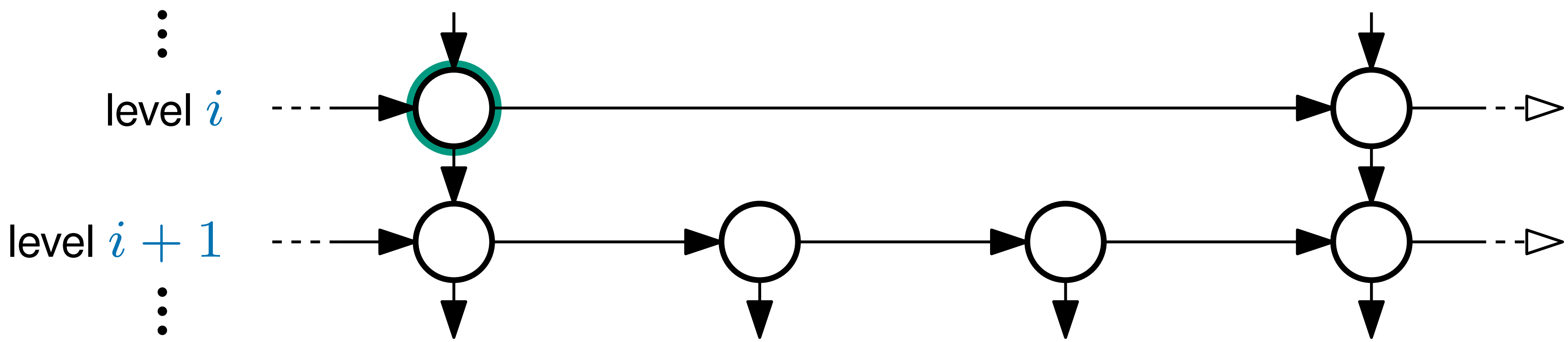

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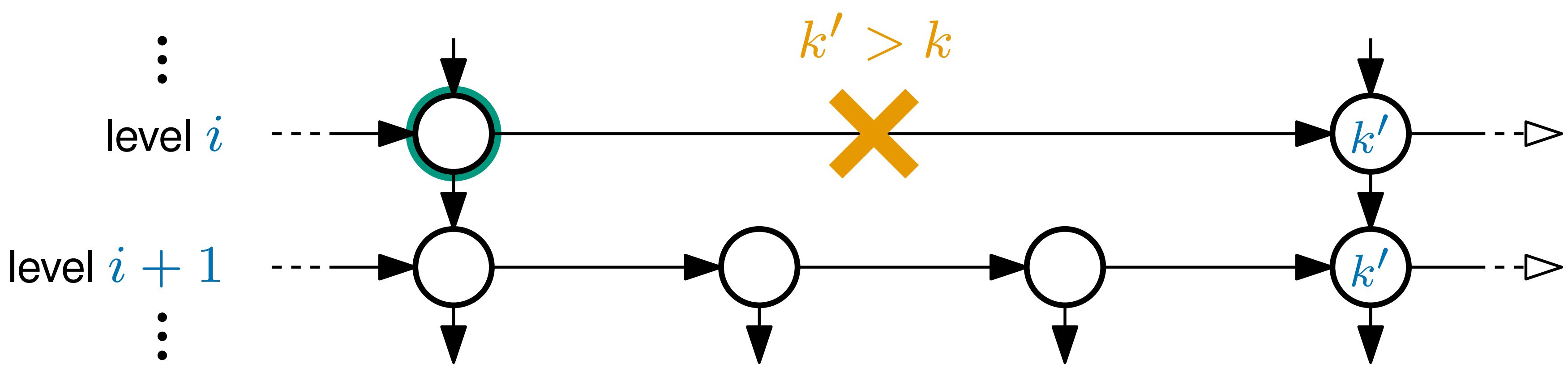

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The complexity of FIND

How long does FIND(k) take in a multi-level linked list?

Observation 1 We only **move down** at most $O(\log n)$ times

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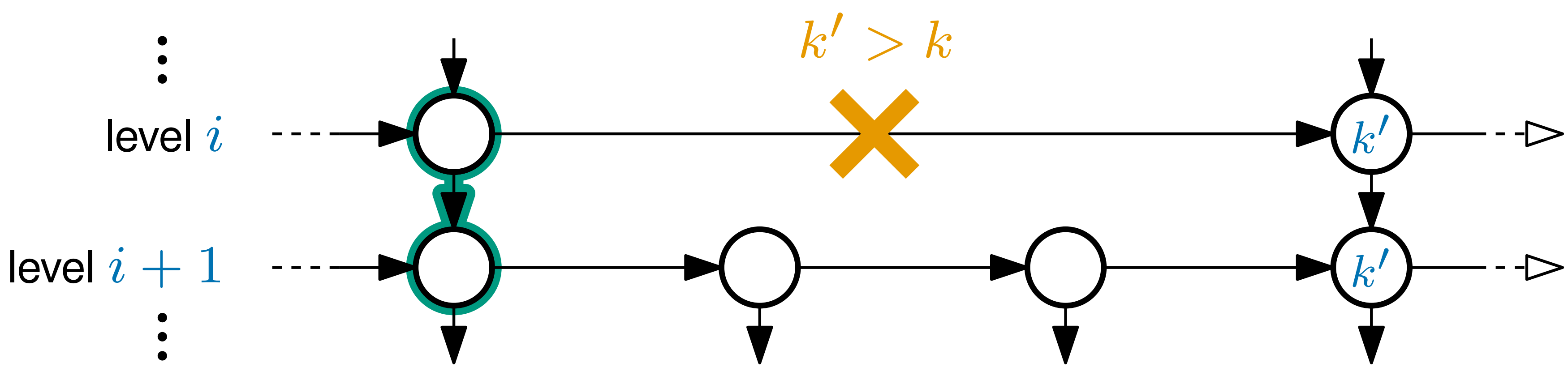

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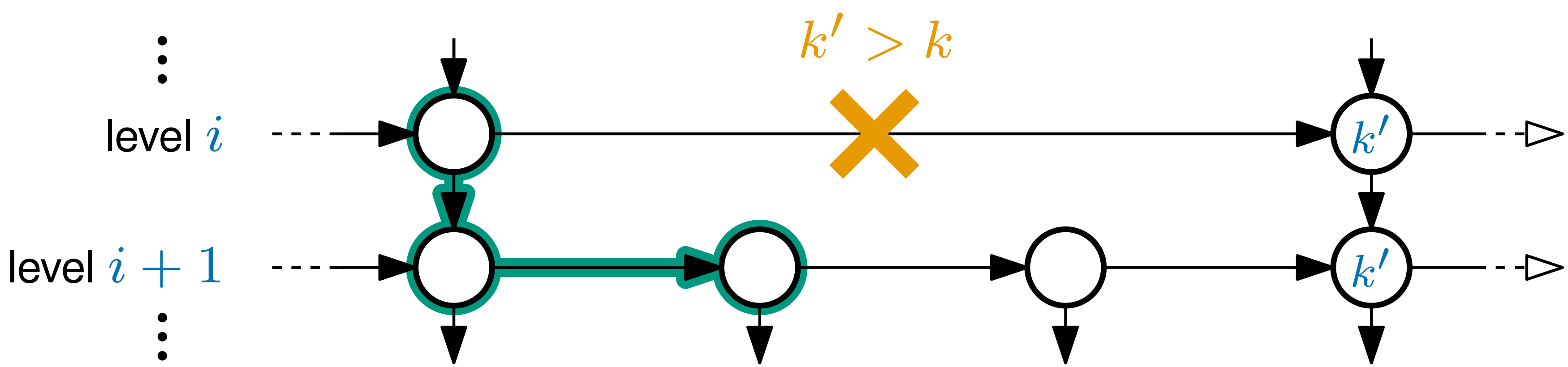

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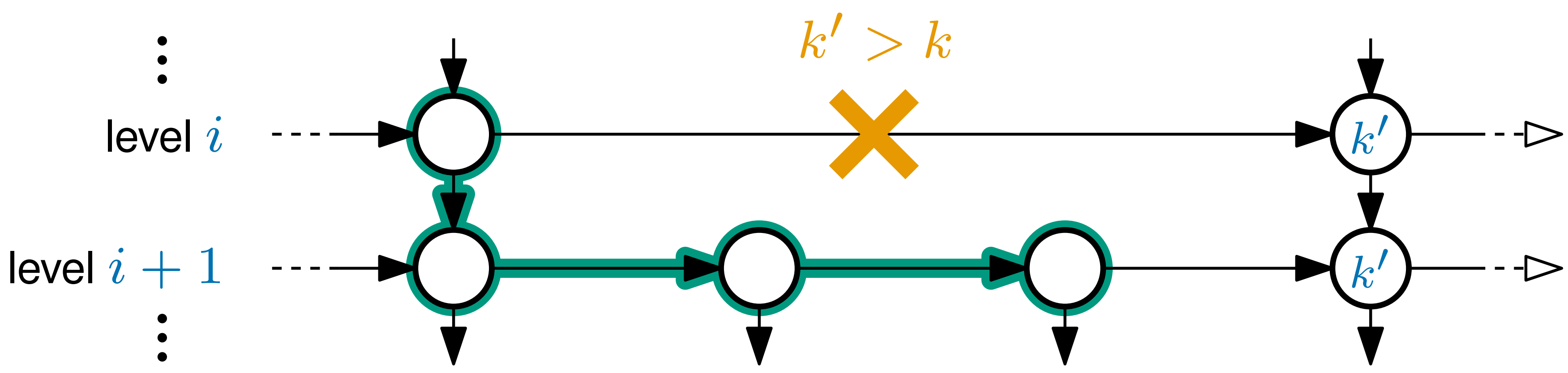

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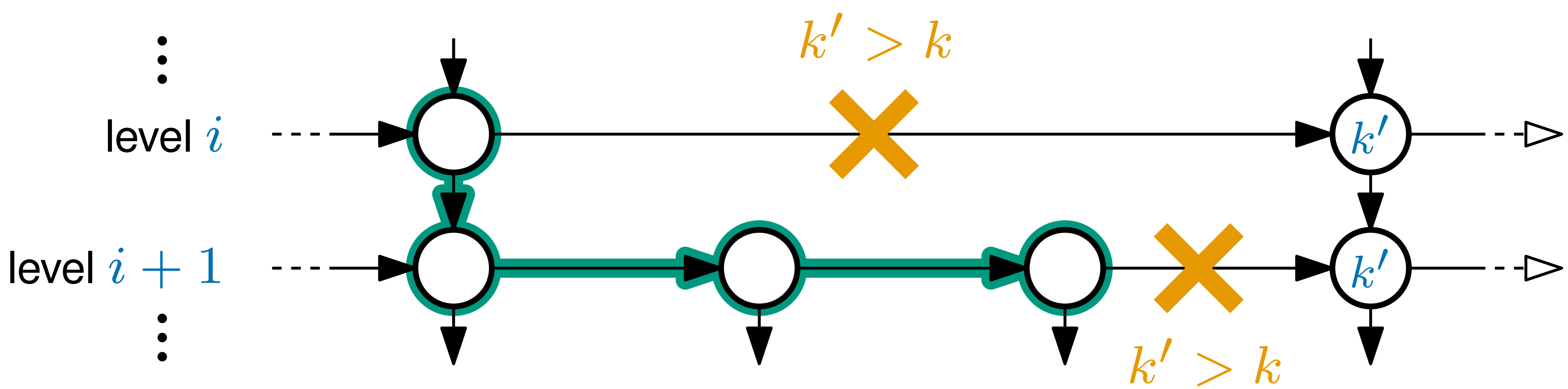

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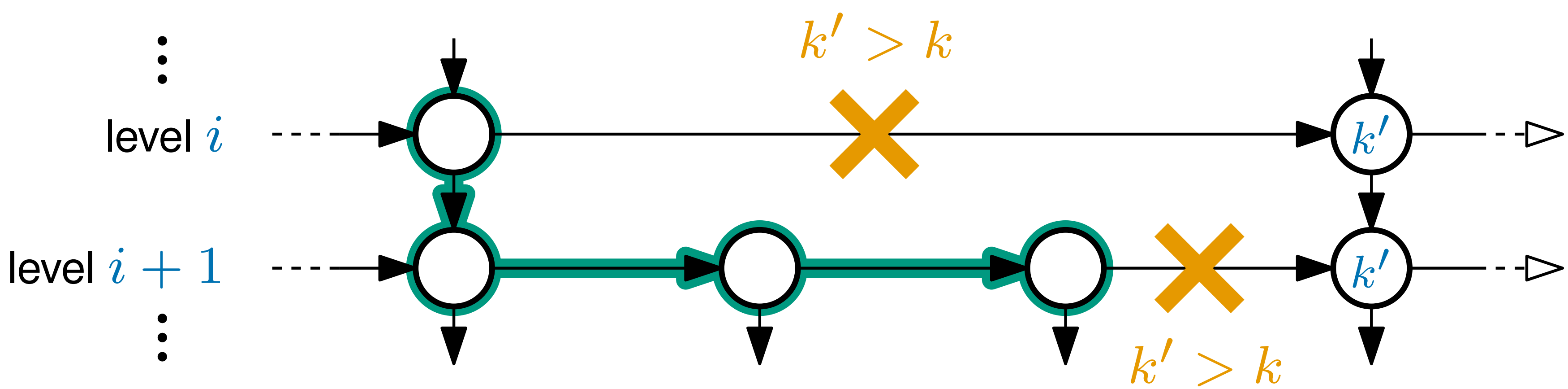

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Fact We only **move** at most $O(\log n)$ times while performing a FIND

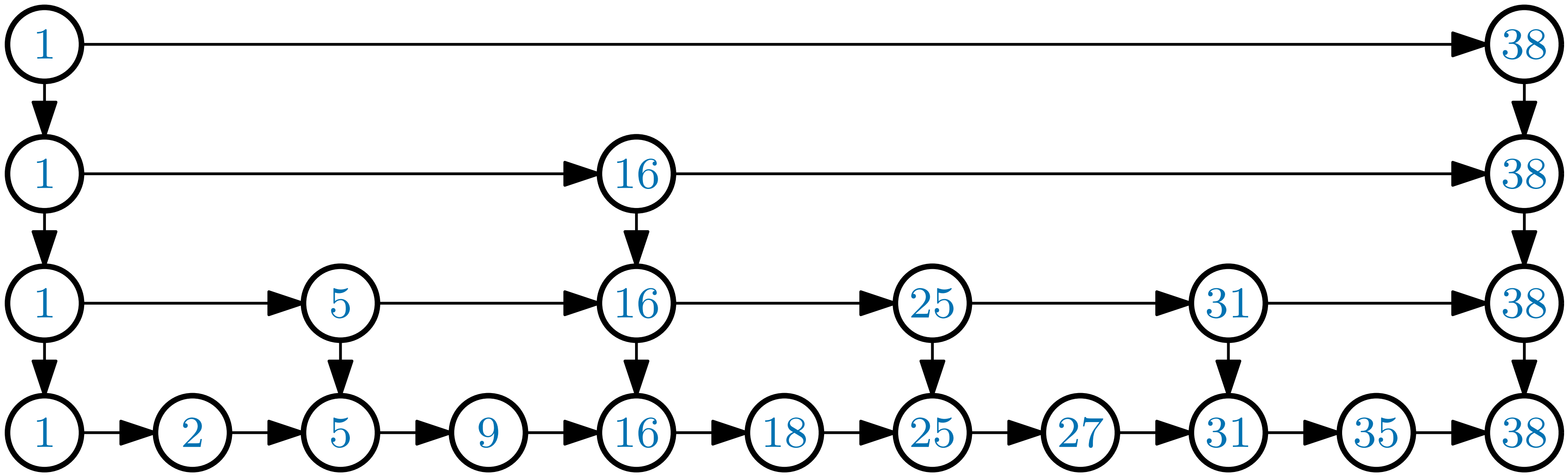
Multi-level Linked Lists

If we had a multi-level linked list with $O(\log n)$ levels

where each level contained half of the keys from the level below

and the keys were evenly spread as possible

then we could perform FIND in $O(\log n)$ time



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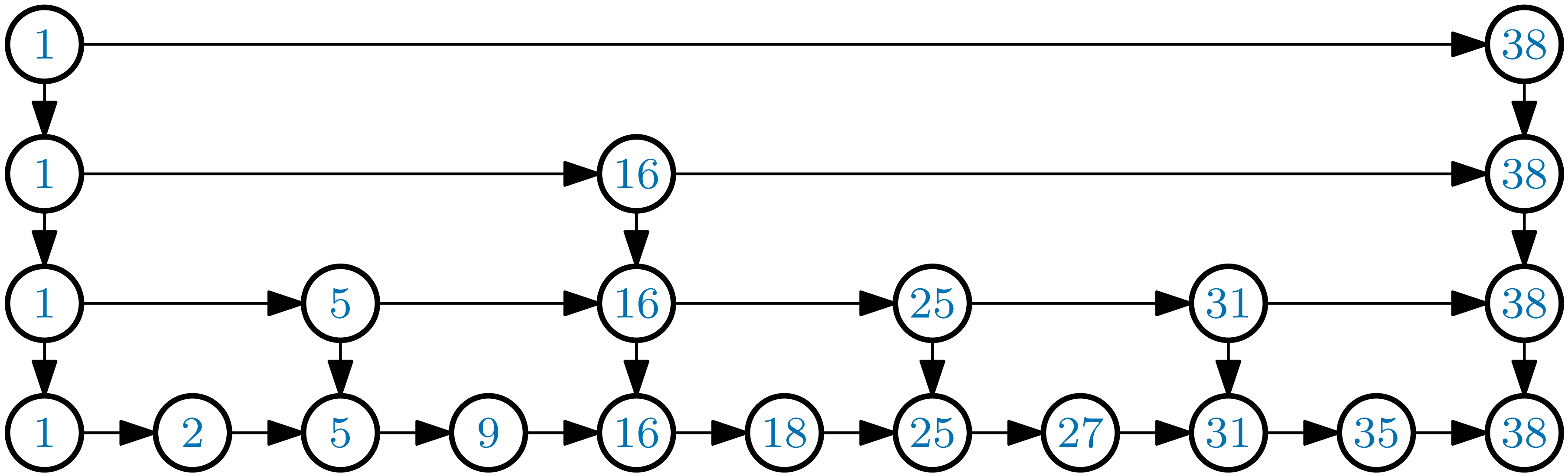

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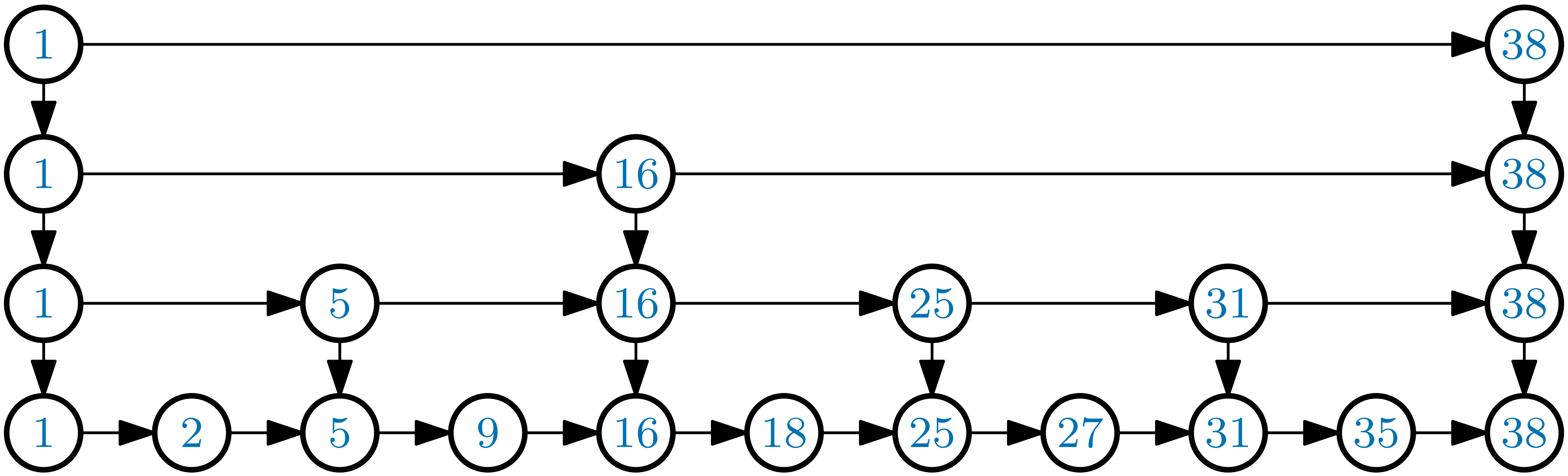

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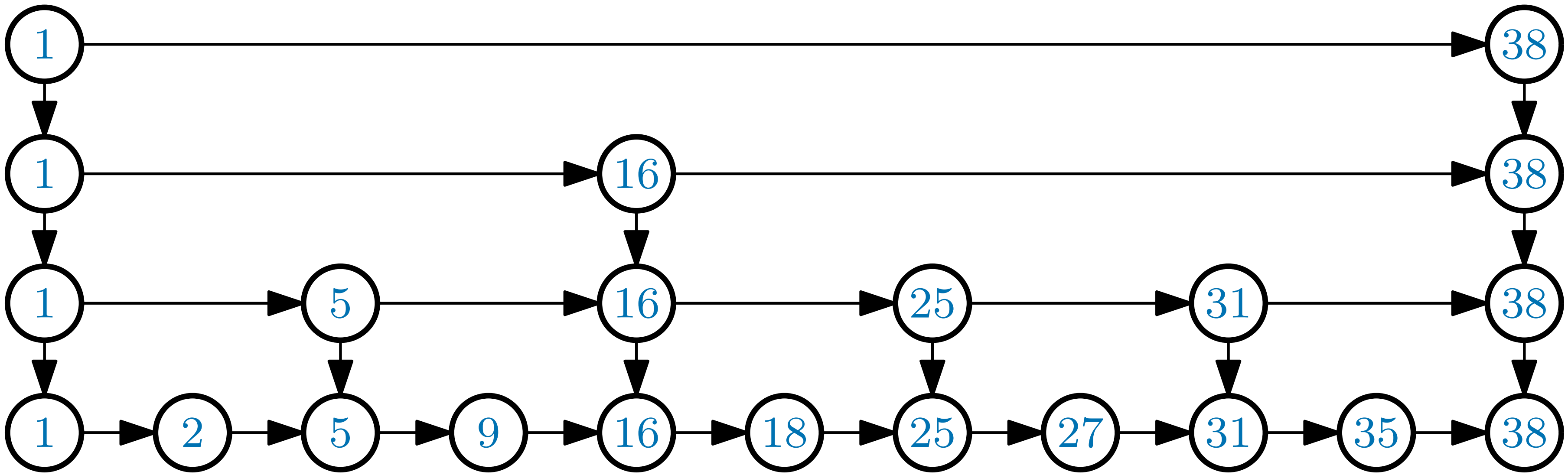

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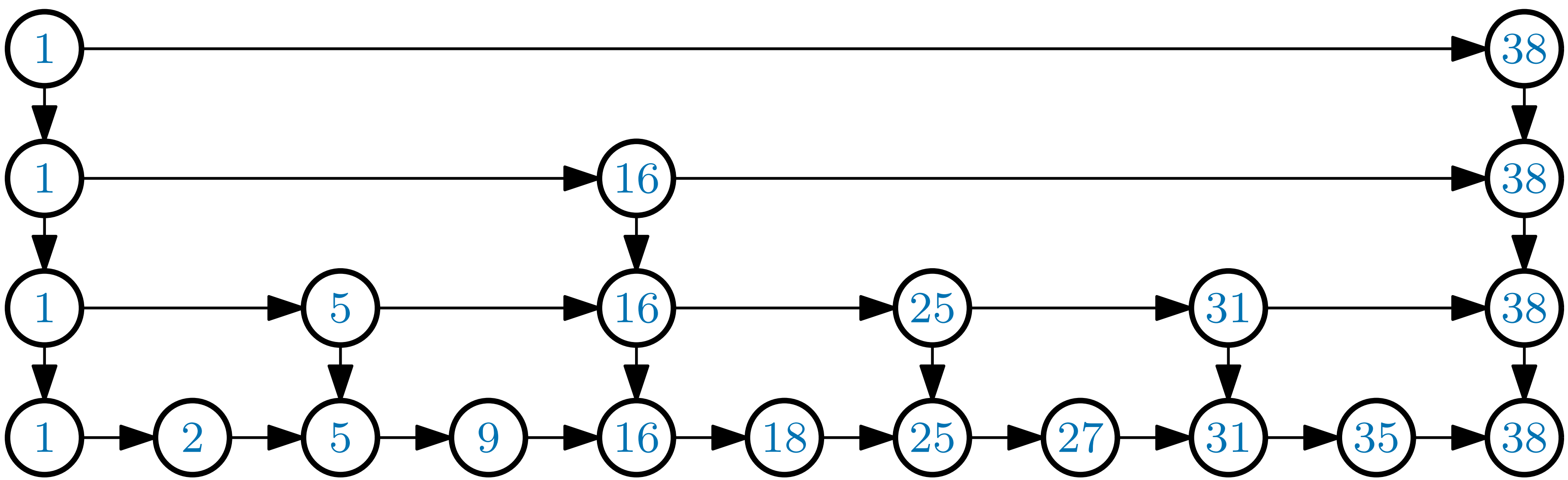

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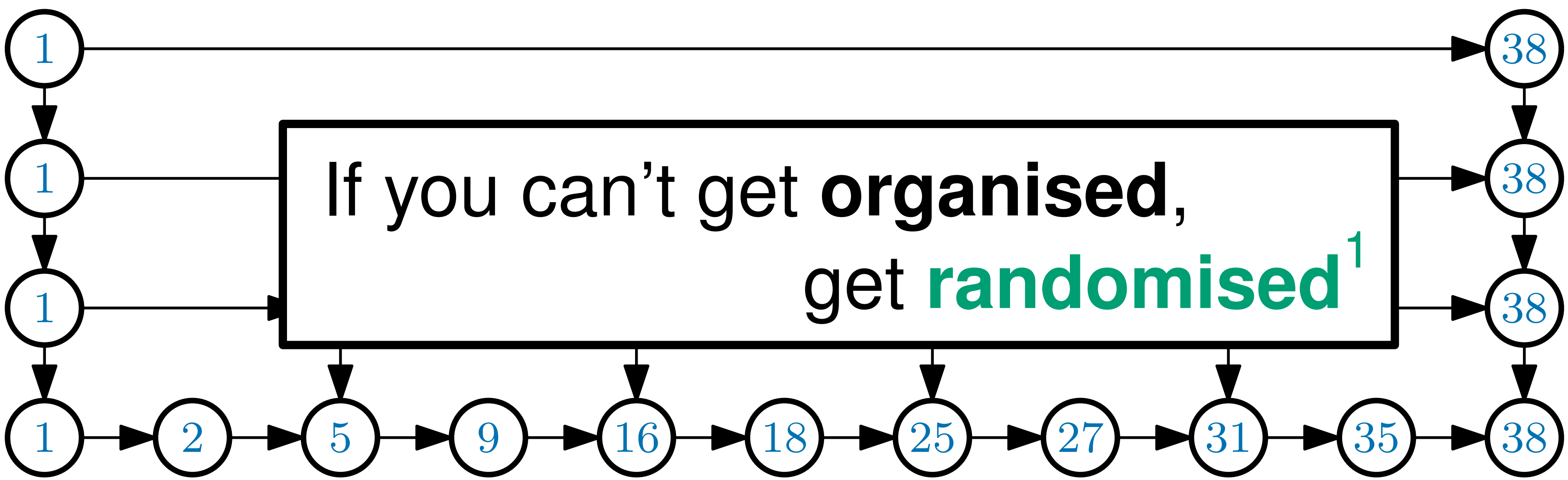

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Before we formally introduce Skip Lists, we let's rewind

and try building another Multi-level Linked List...

by flipping coins

$$1 \rightarrow 2 \rightarrow 5 \rightarrow 9 \rightarrow 16 \rightarrow 18 \rightarrow 25 \rightarrow 27 \rightarrow 31 \rightarrow 35 \rightarrow 38$$

(we still always include the smallest and largest keys in every level)

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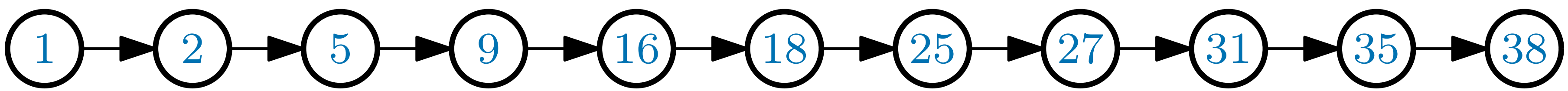

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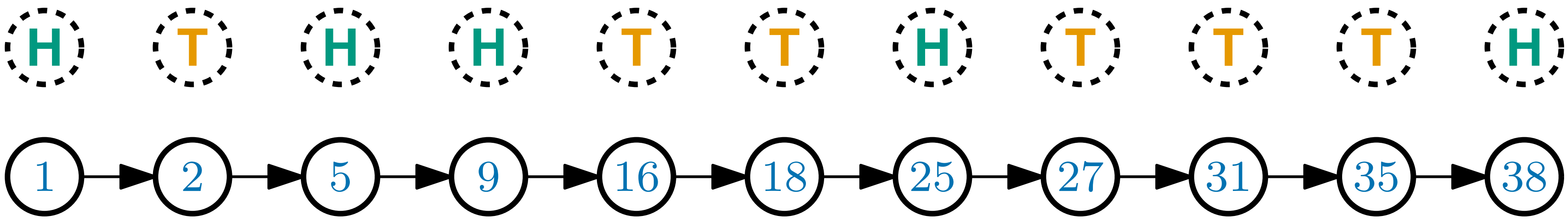

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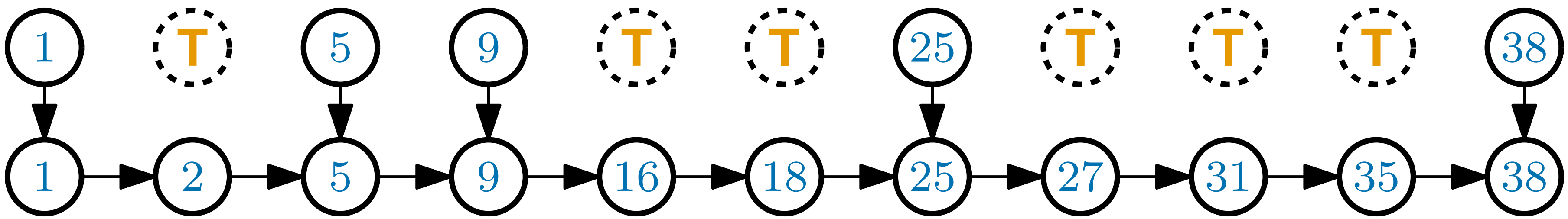

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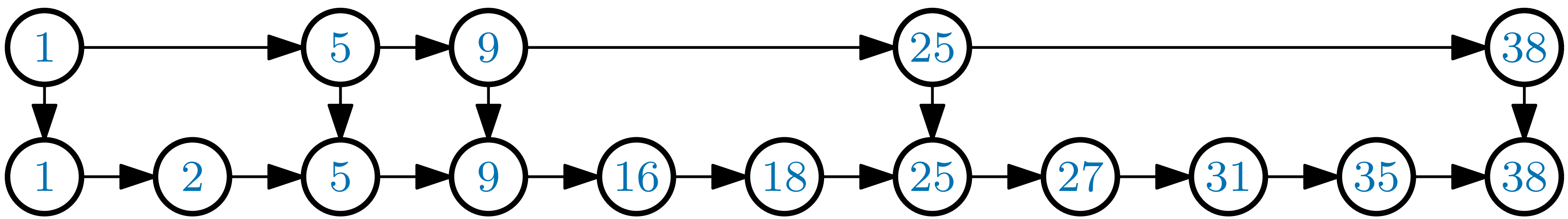

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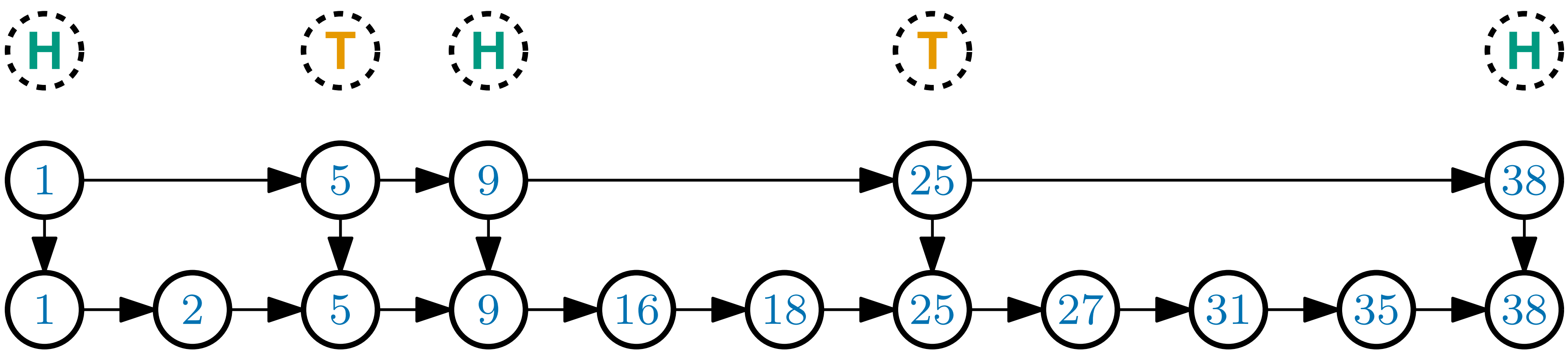

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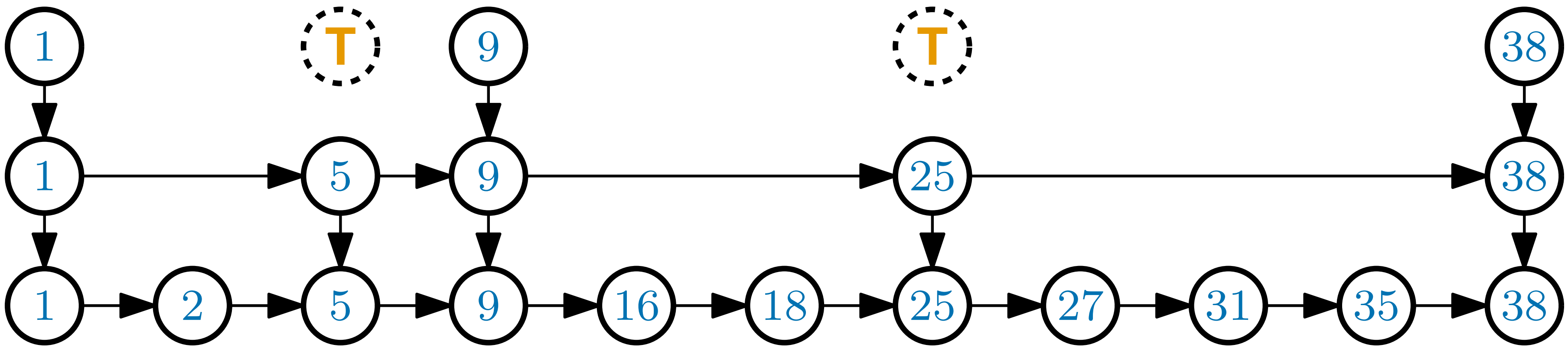

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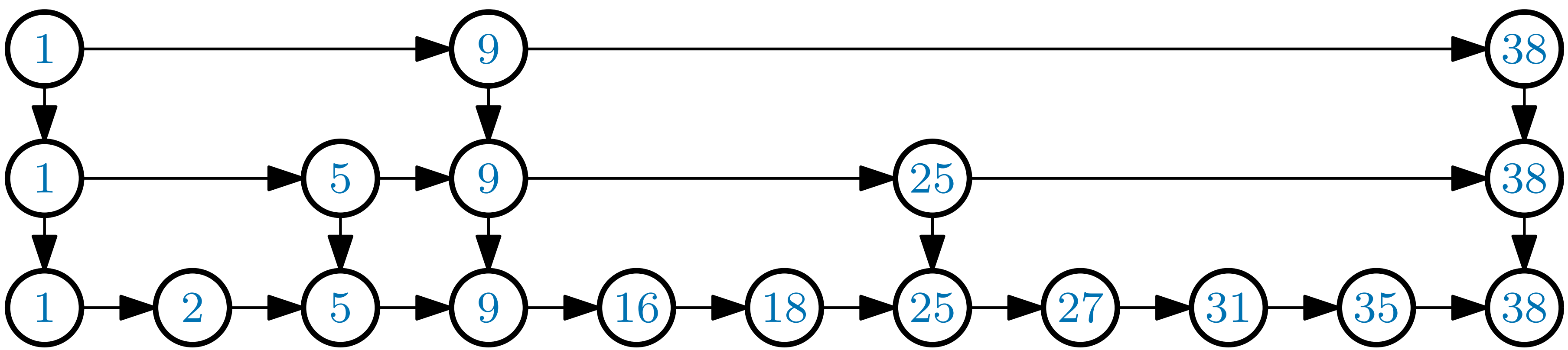

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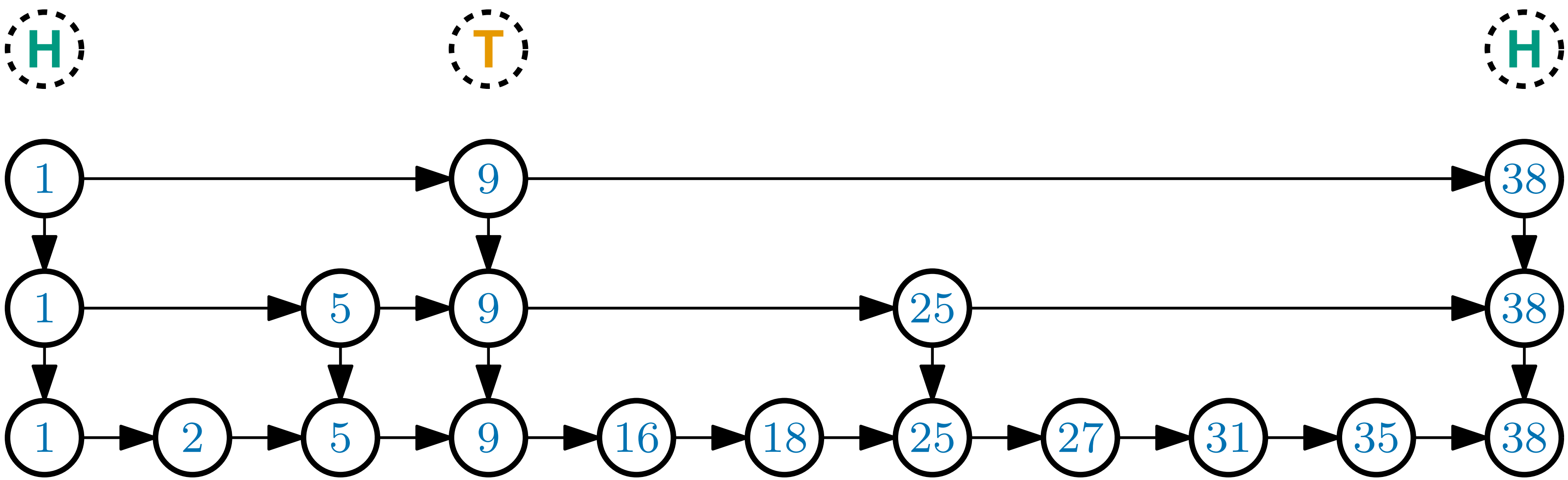

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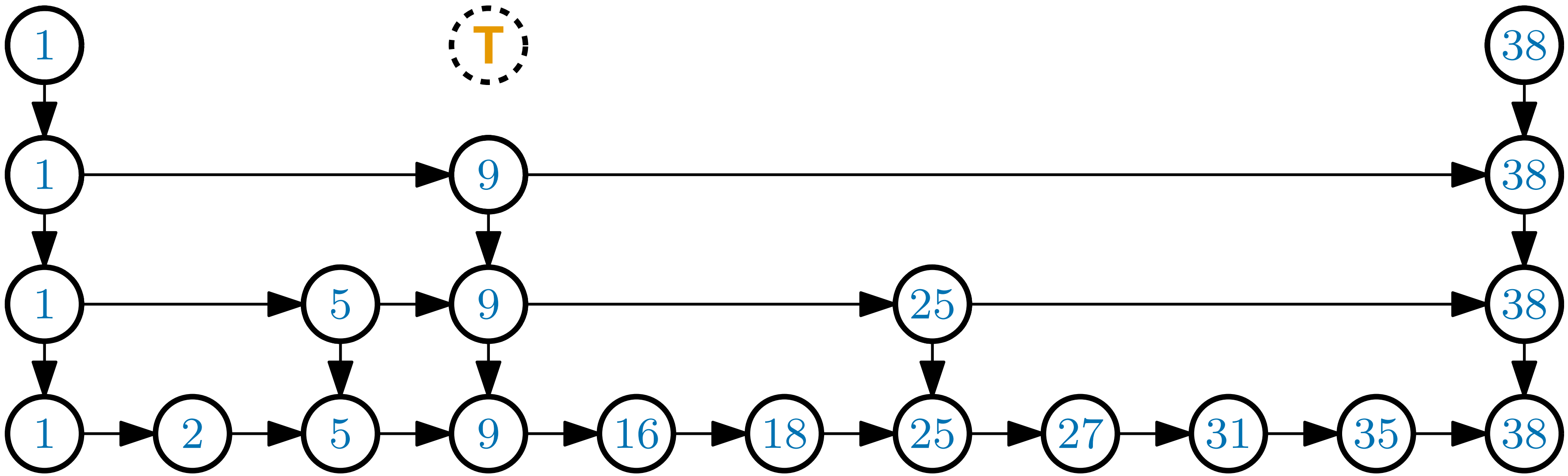

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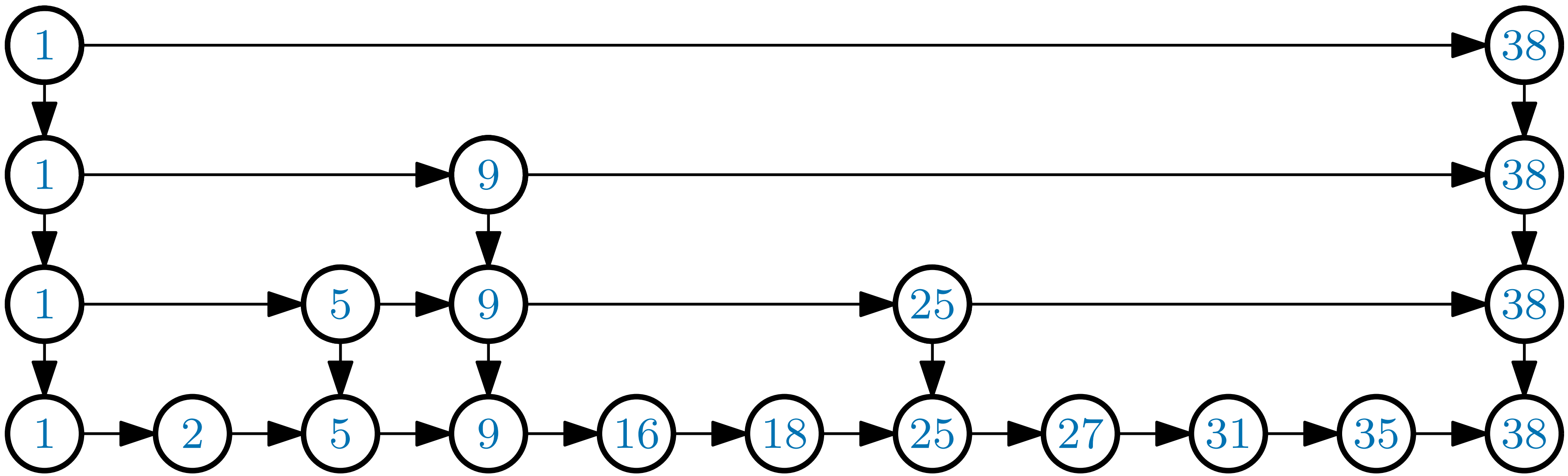

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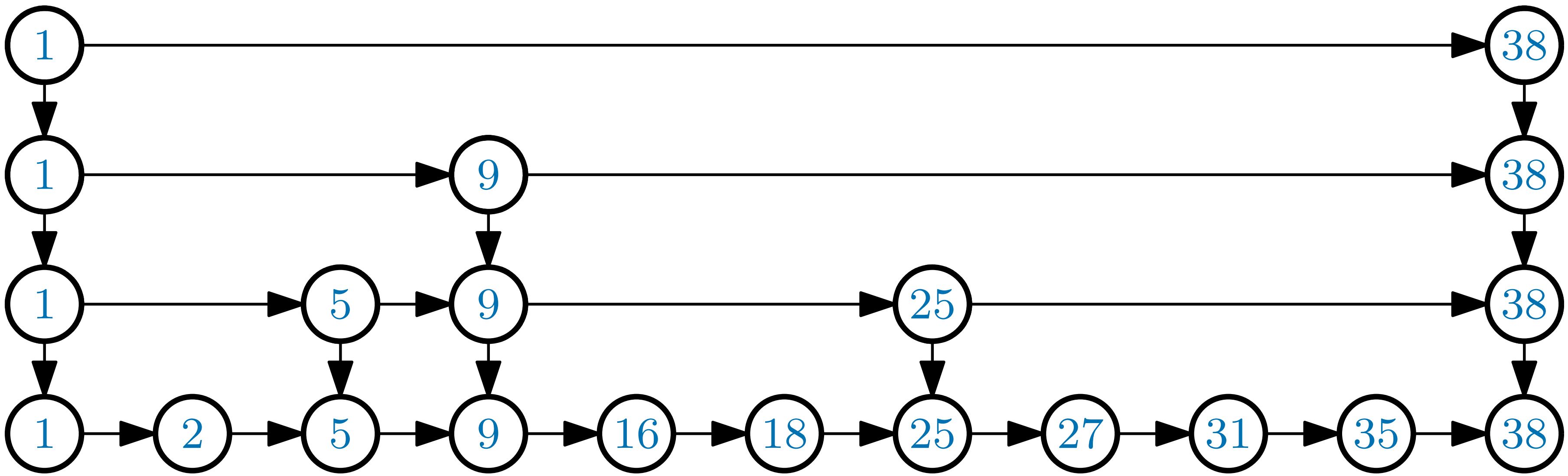

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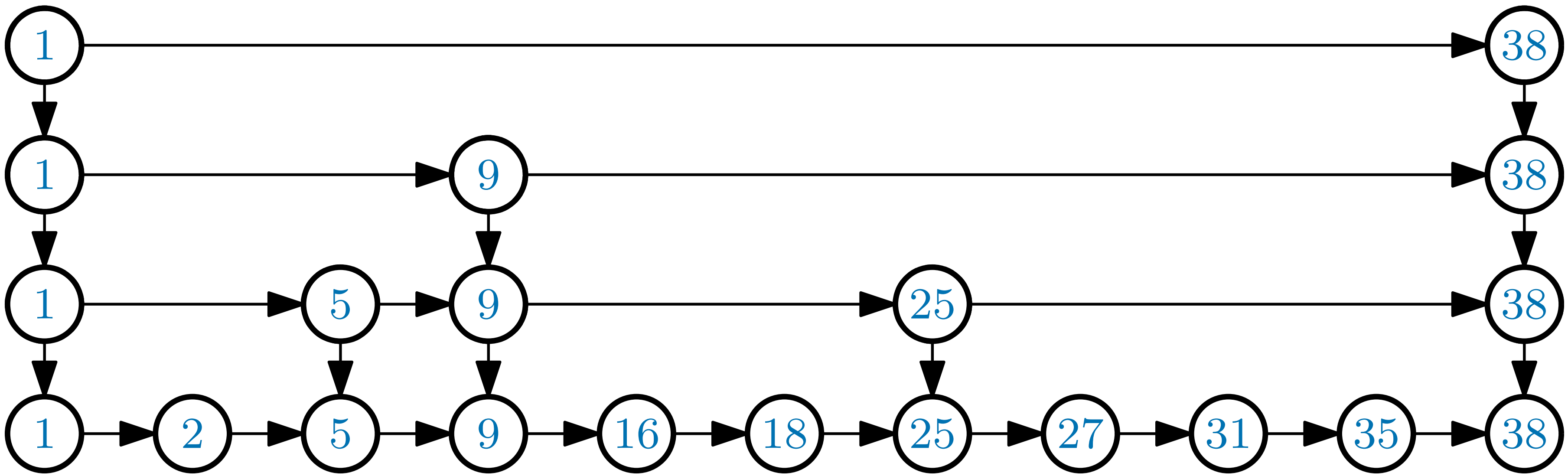

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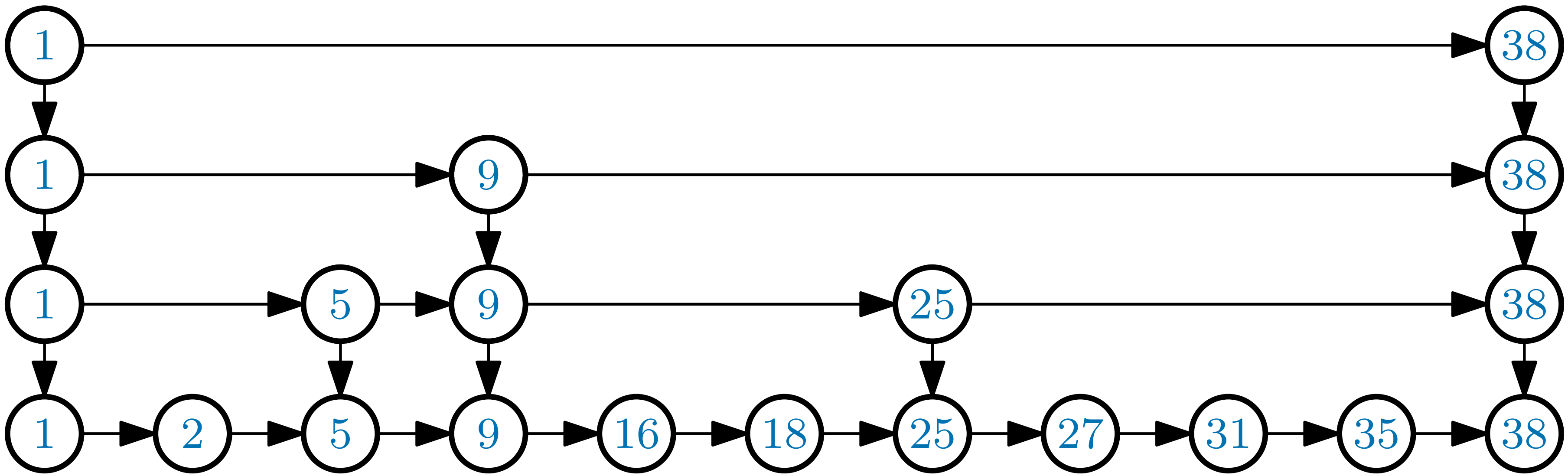

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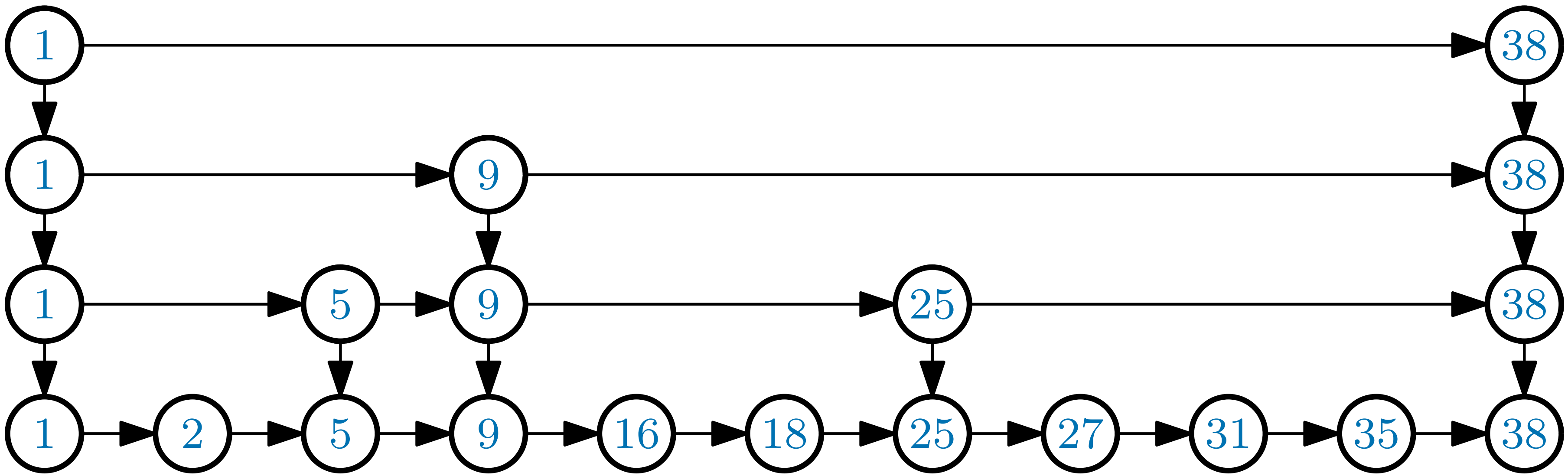

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The intuition is that n coin flips contain about $\frac{n}{2}$ heads and about $\frac{n}{2}$ tails

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Skip Lists

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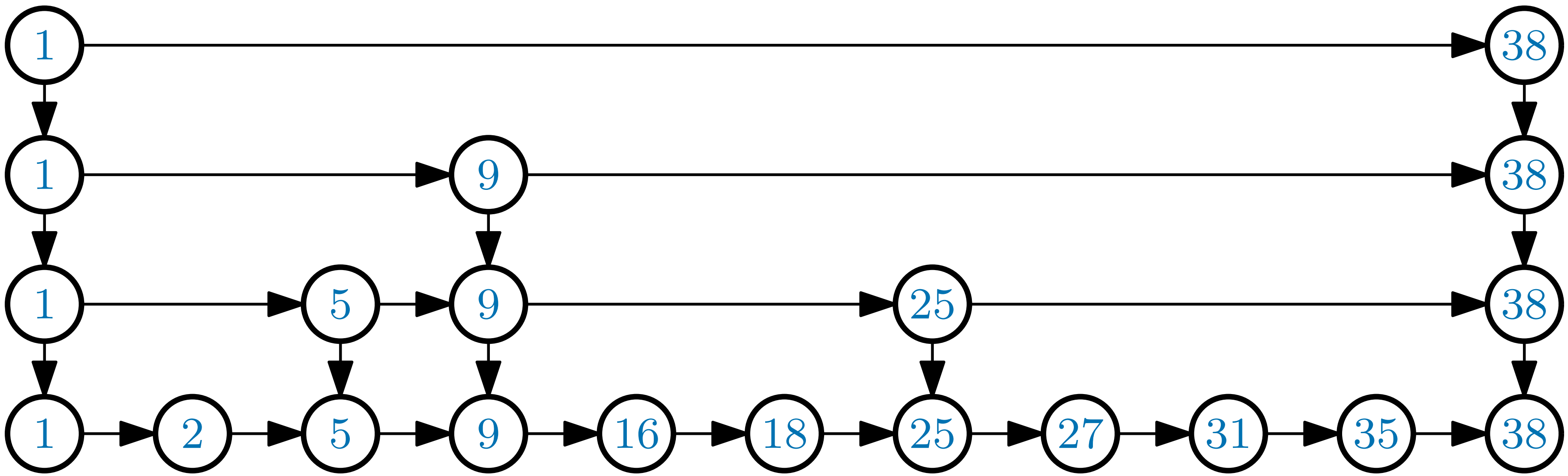
i.e. this is *a skip list...*



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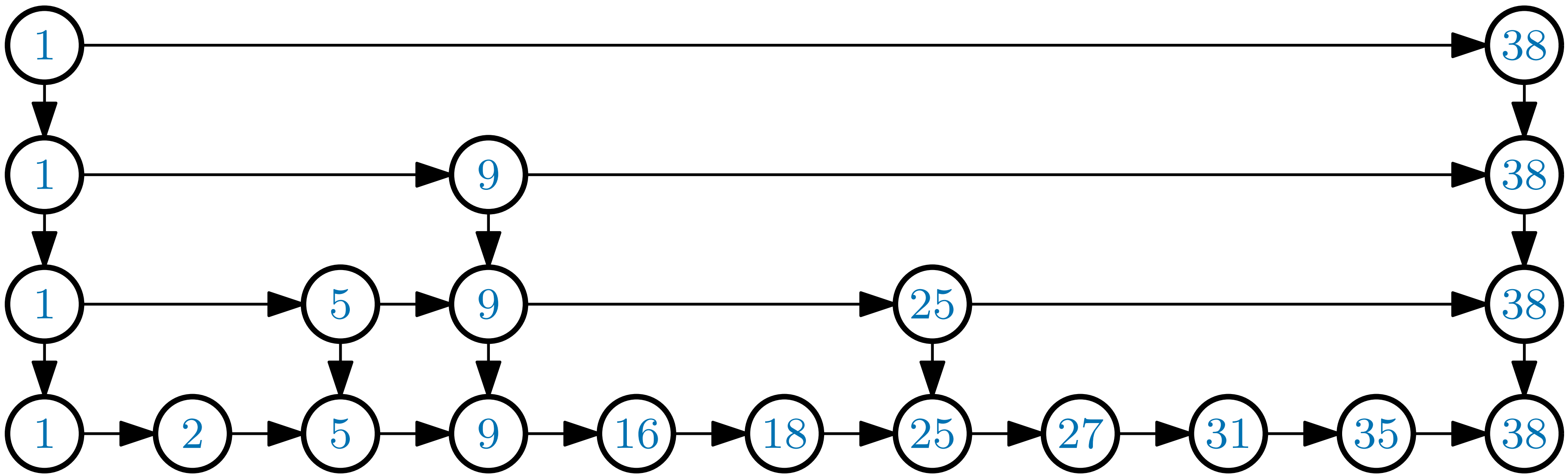

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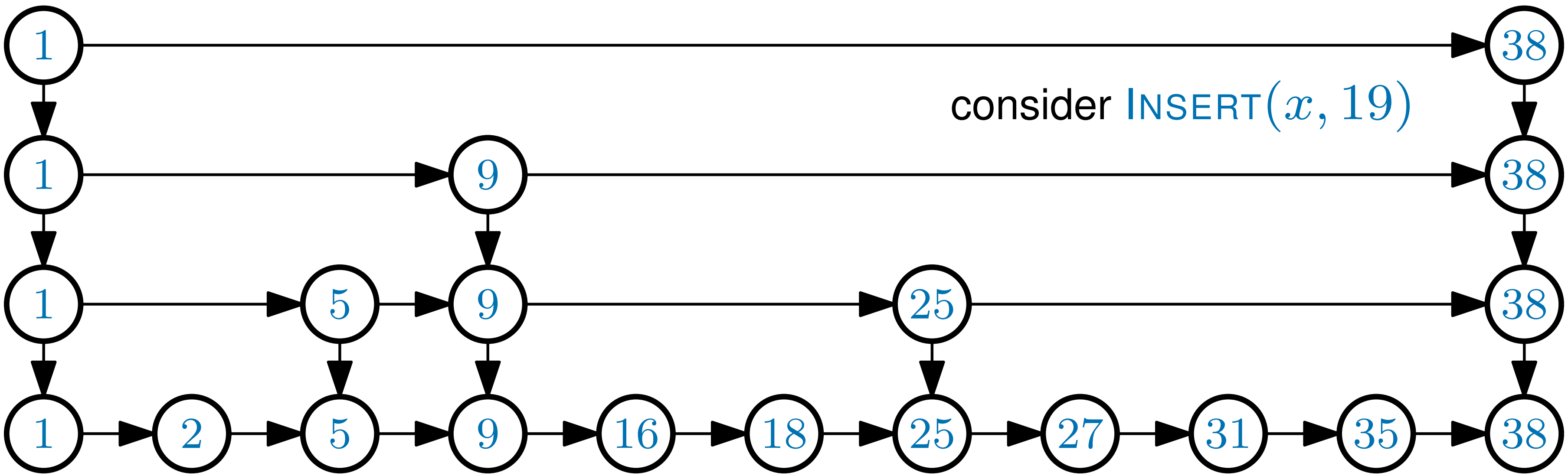

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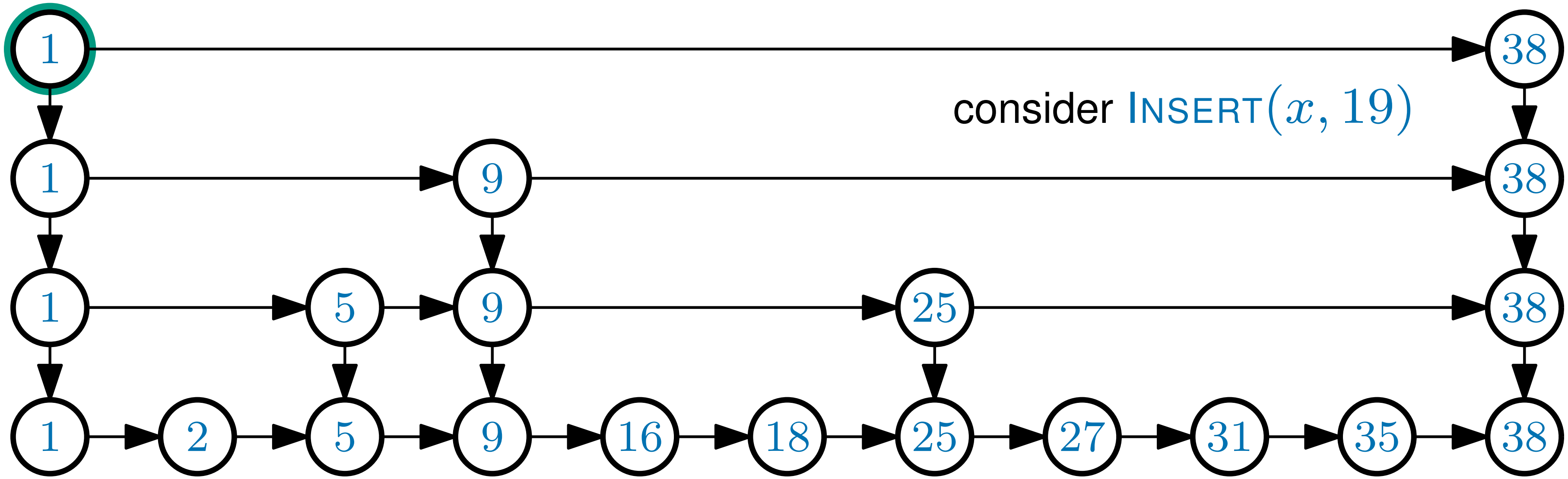

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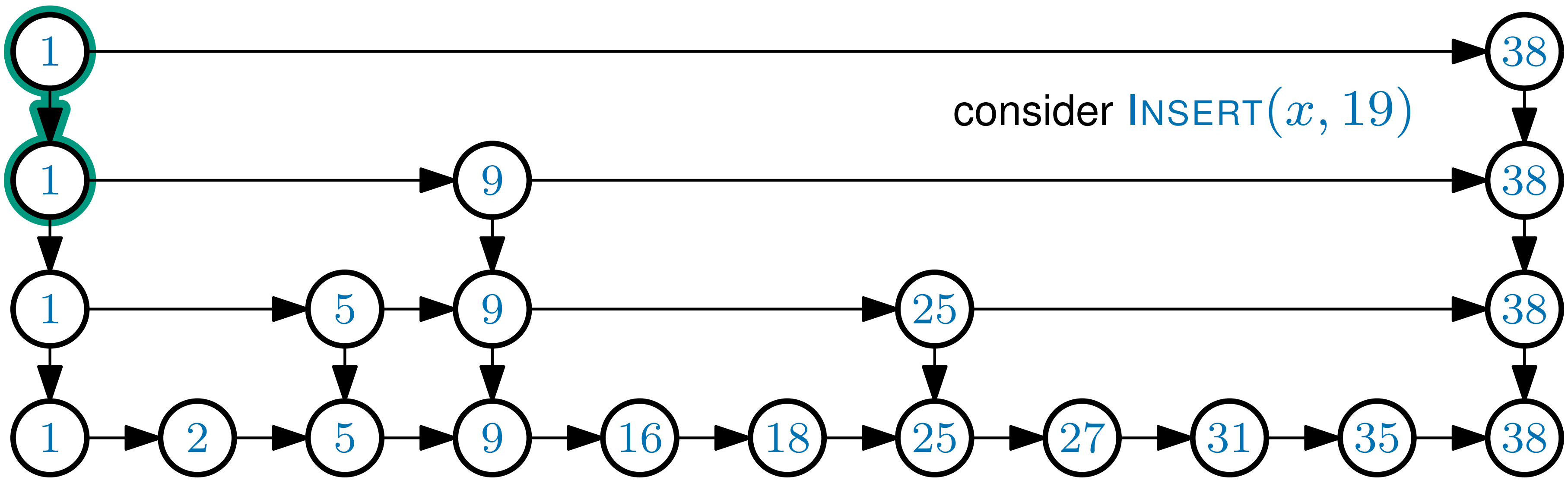

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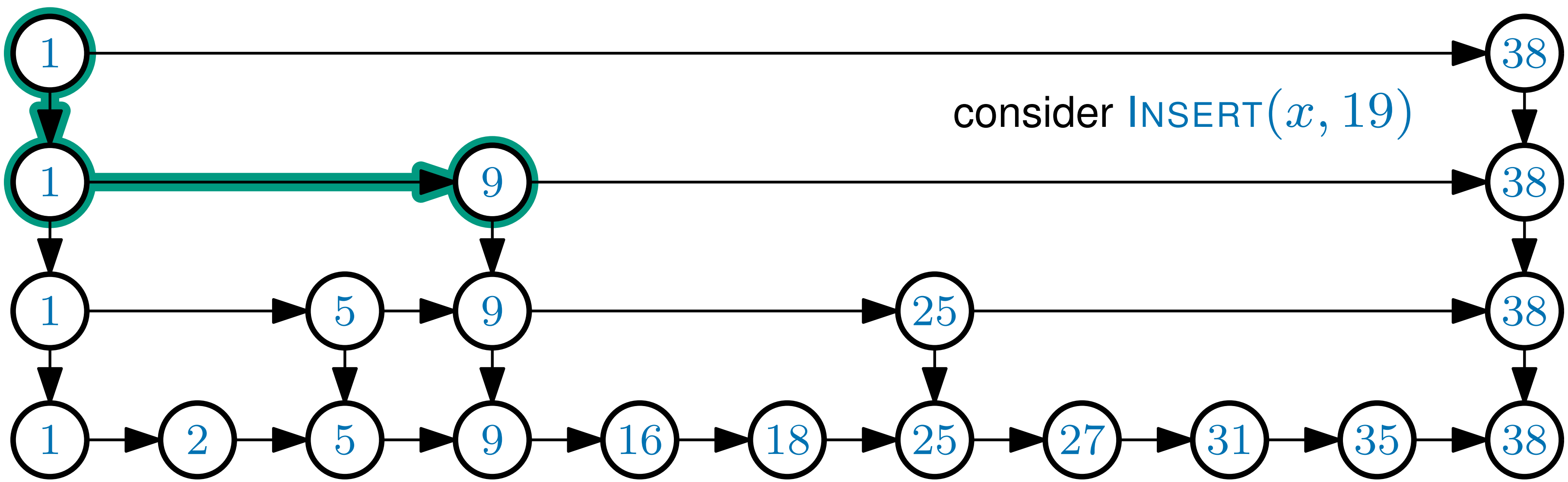

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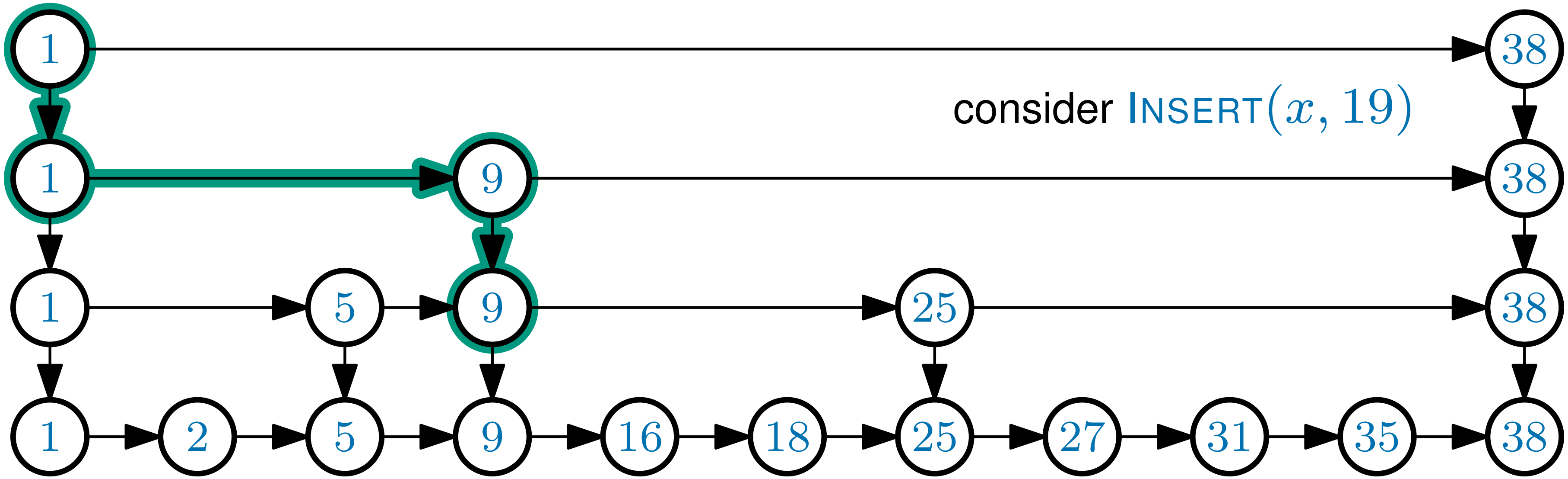

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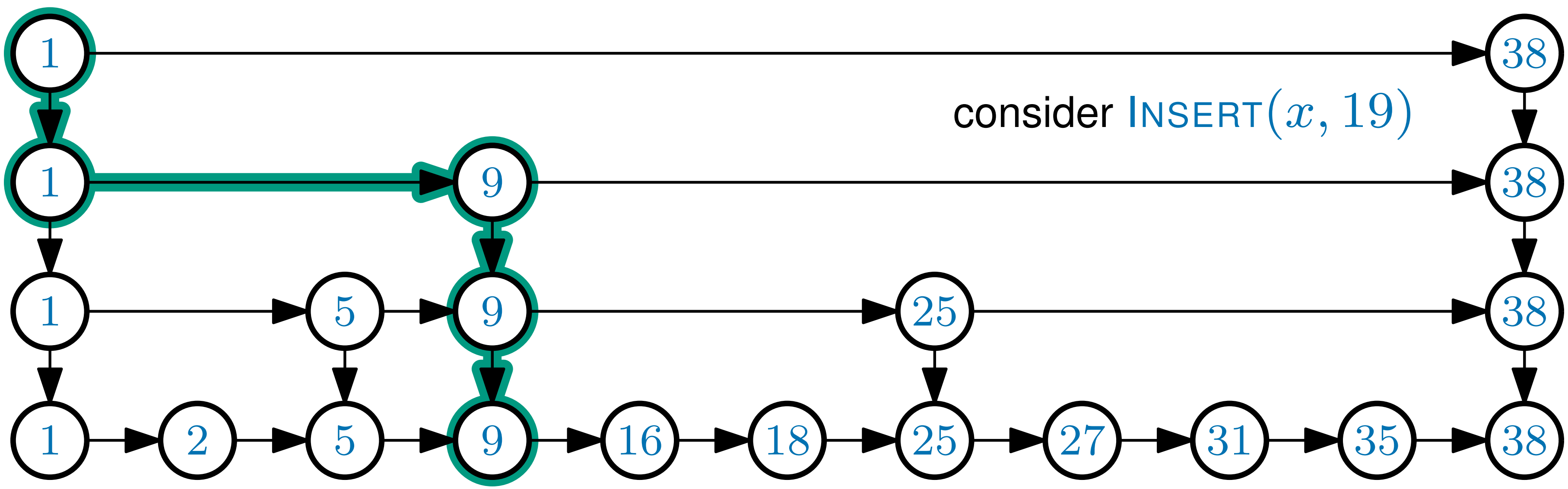

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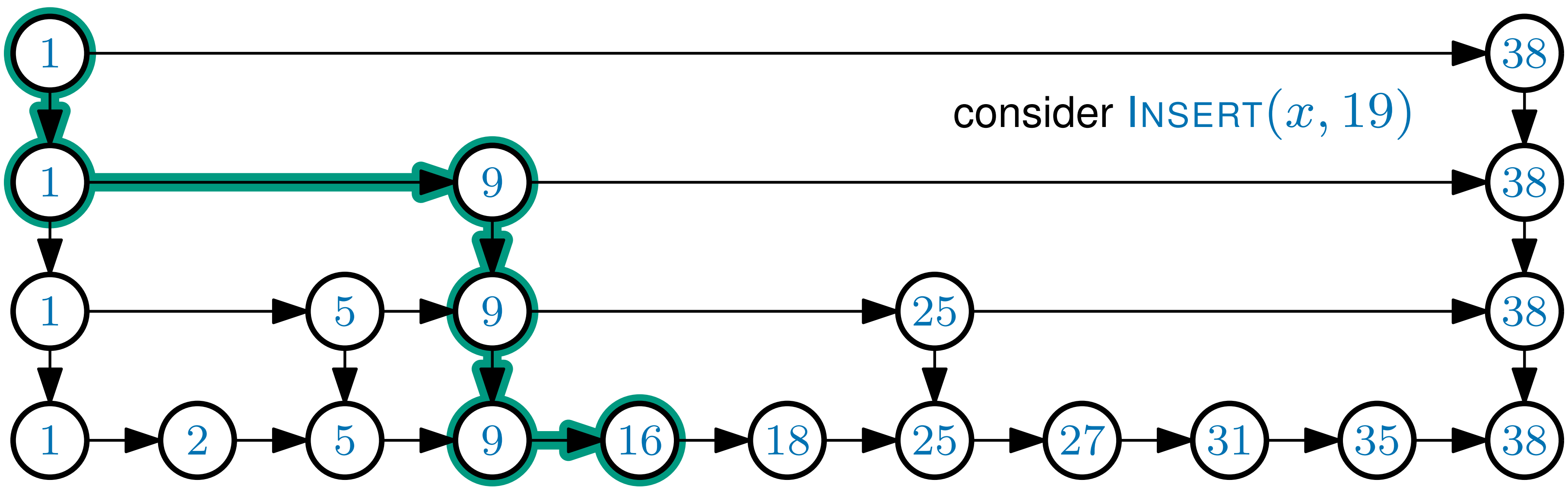

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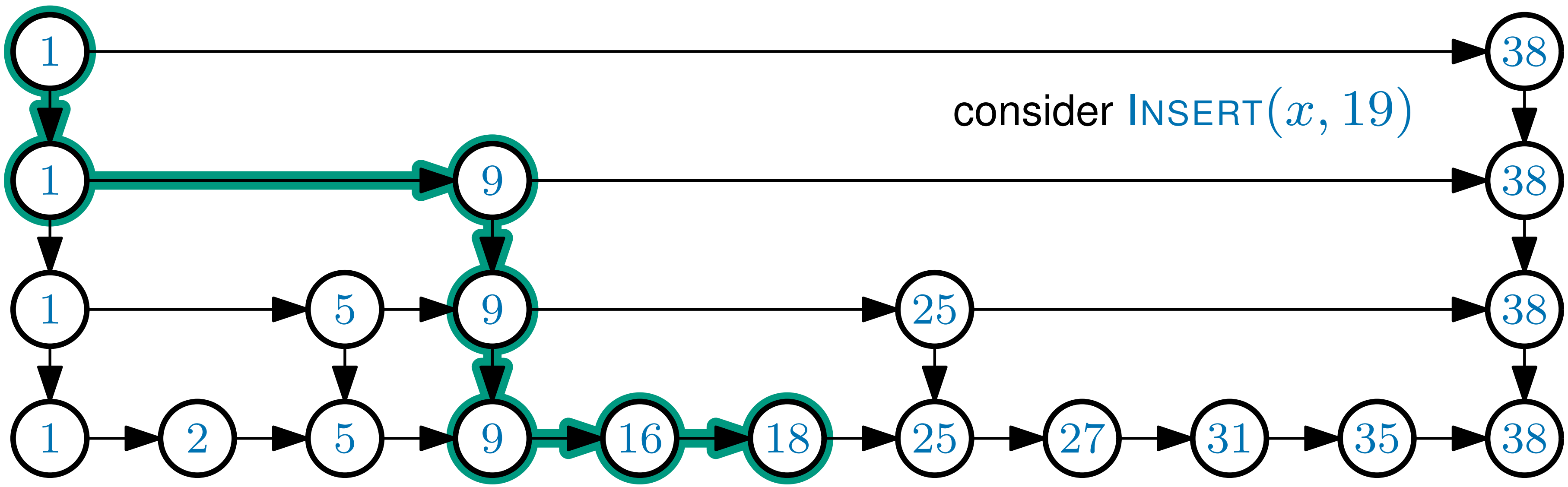

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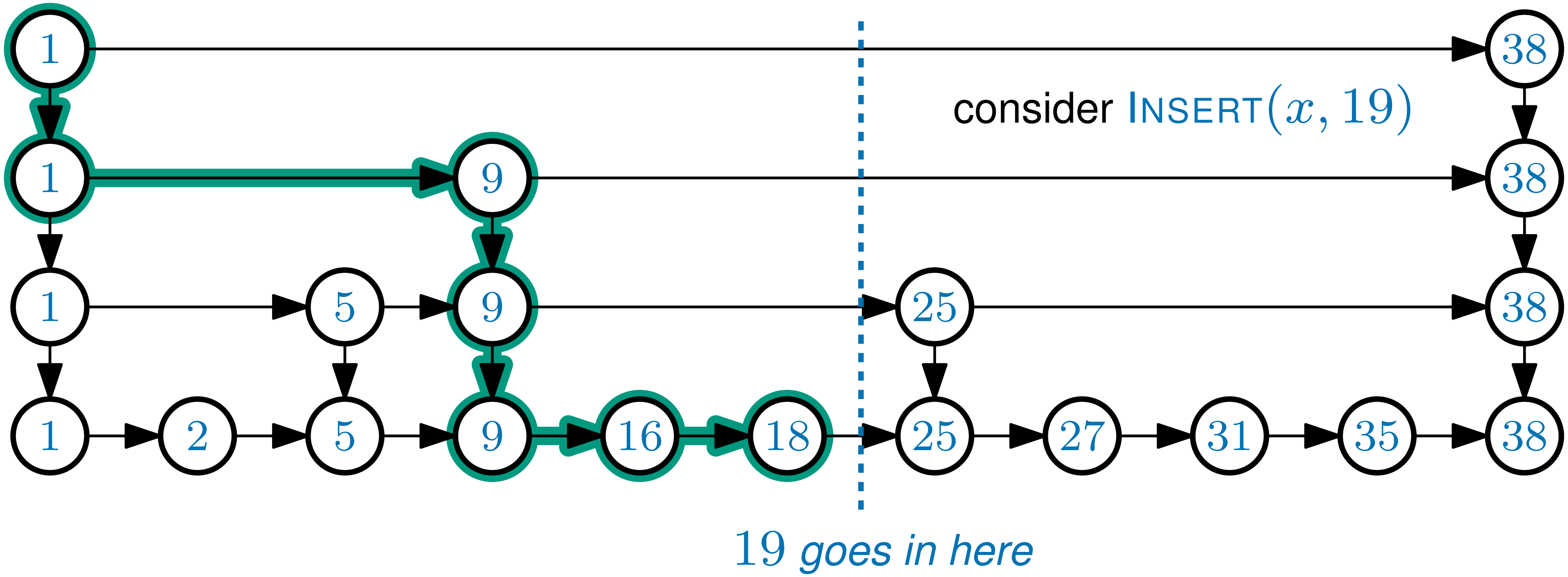

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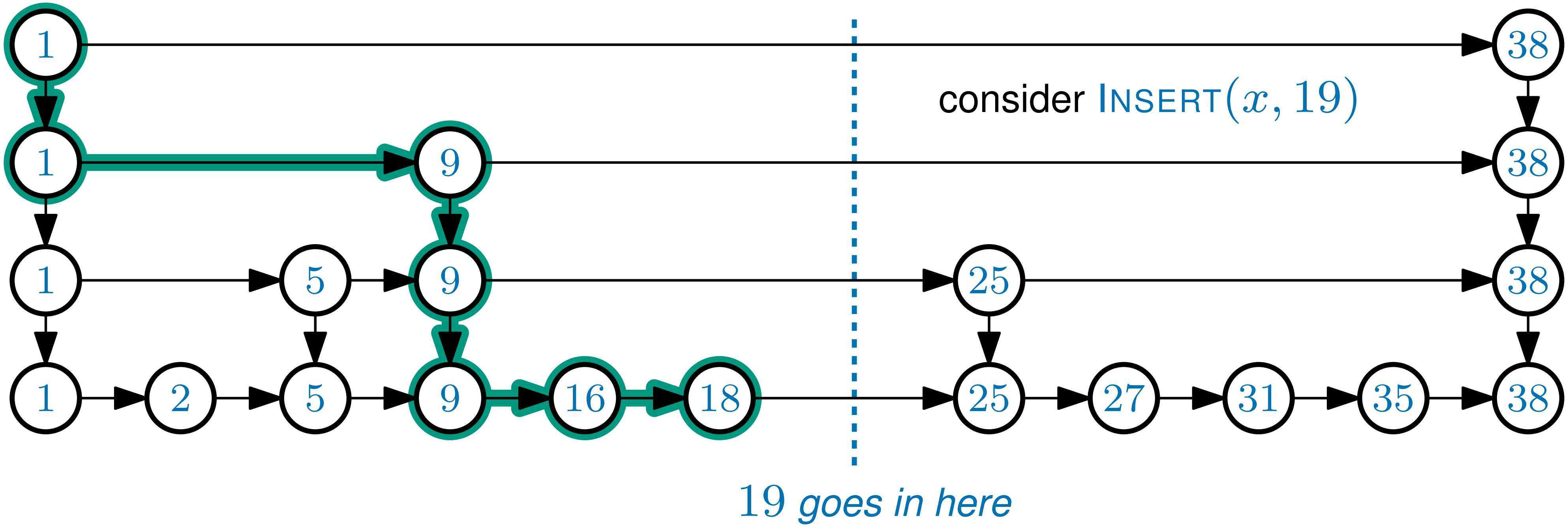

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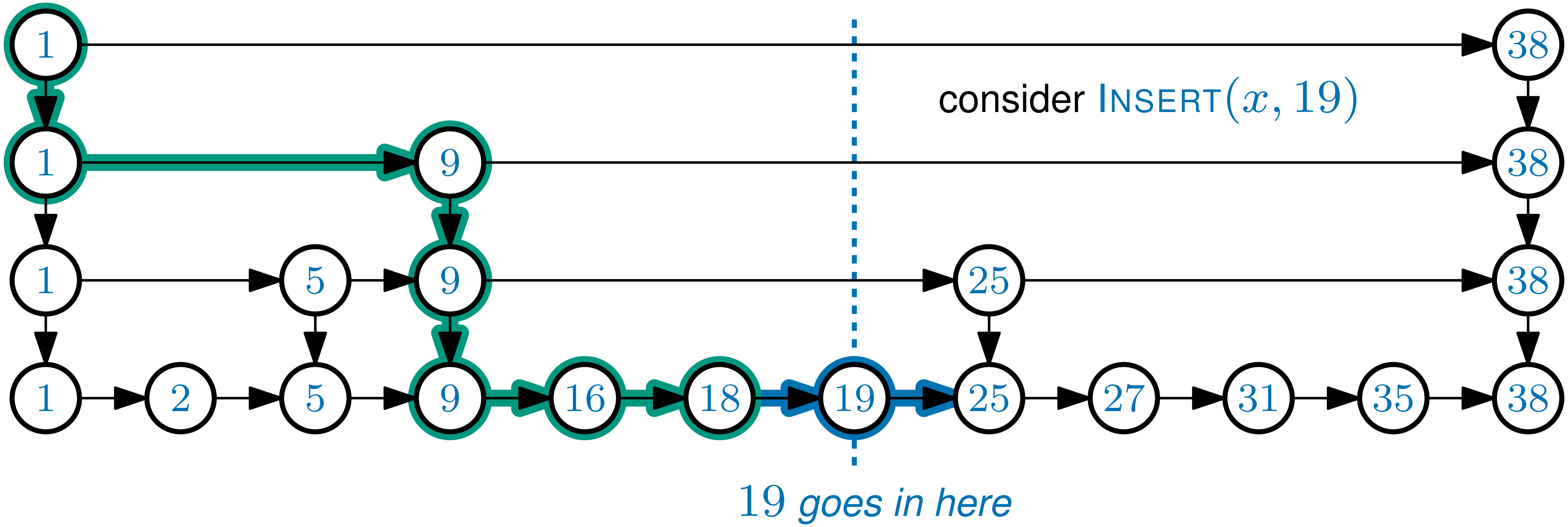

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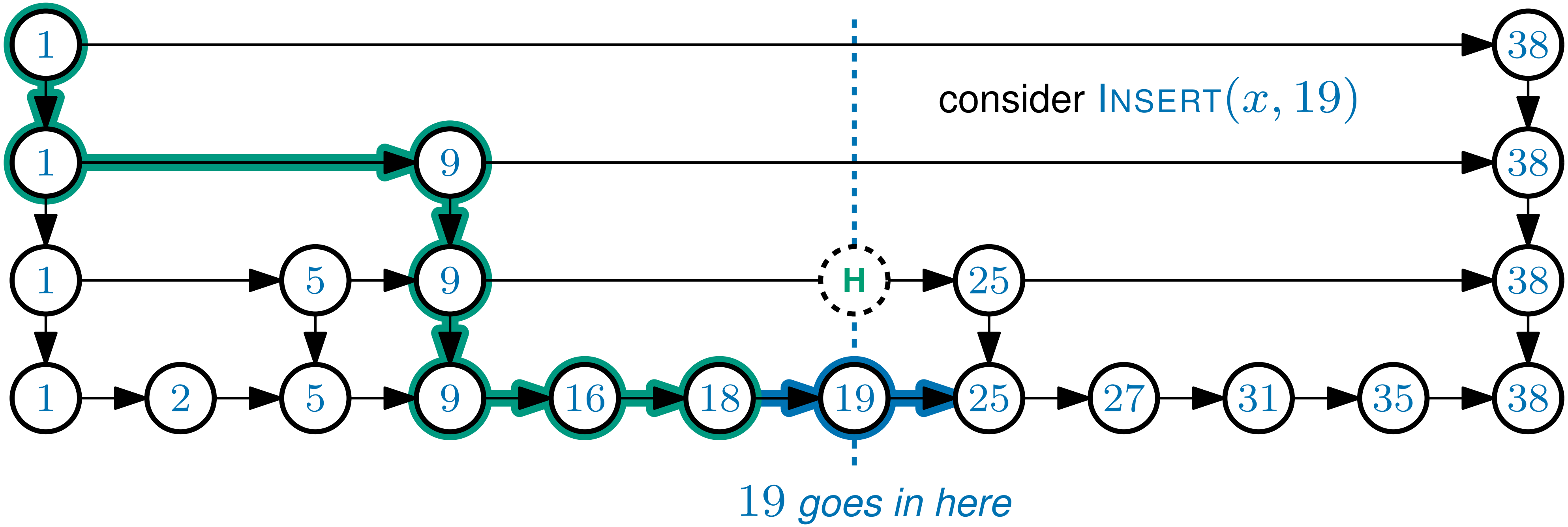

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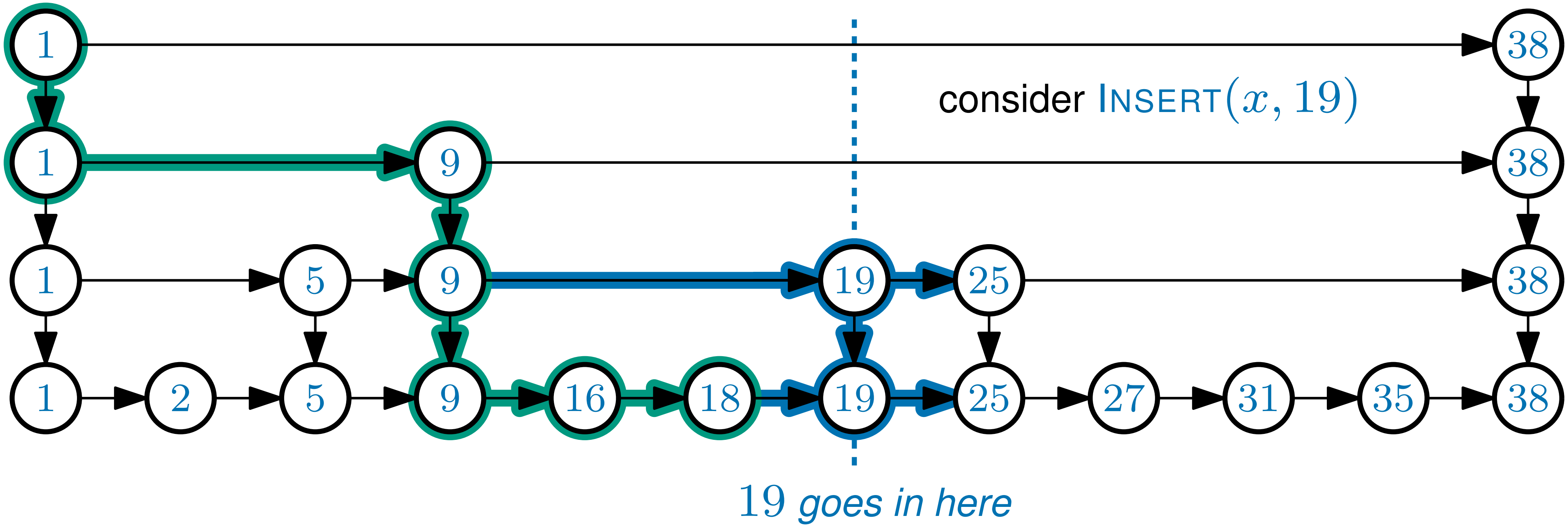

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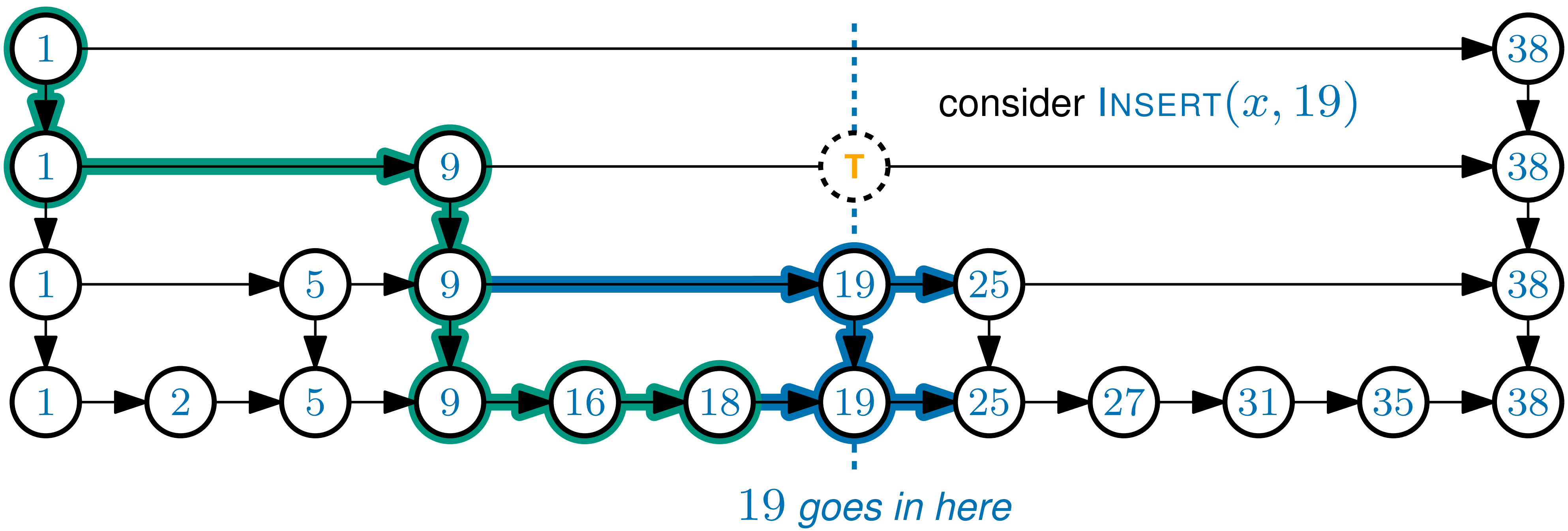

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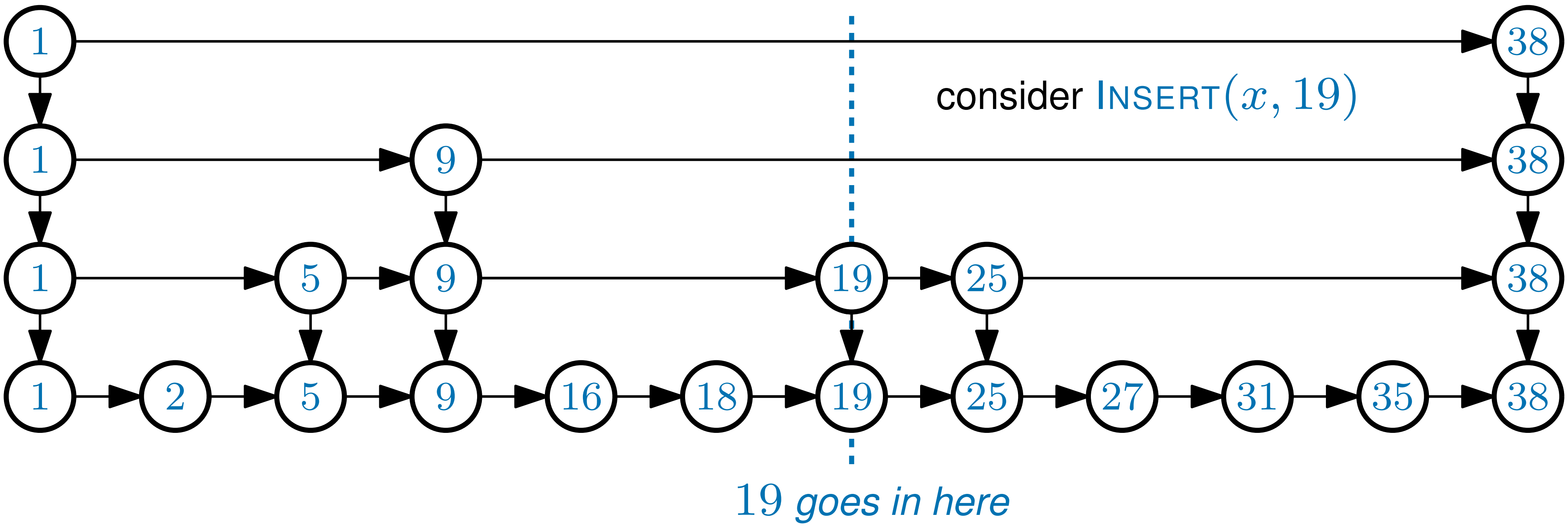

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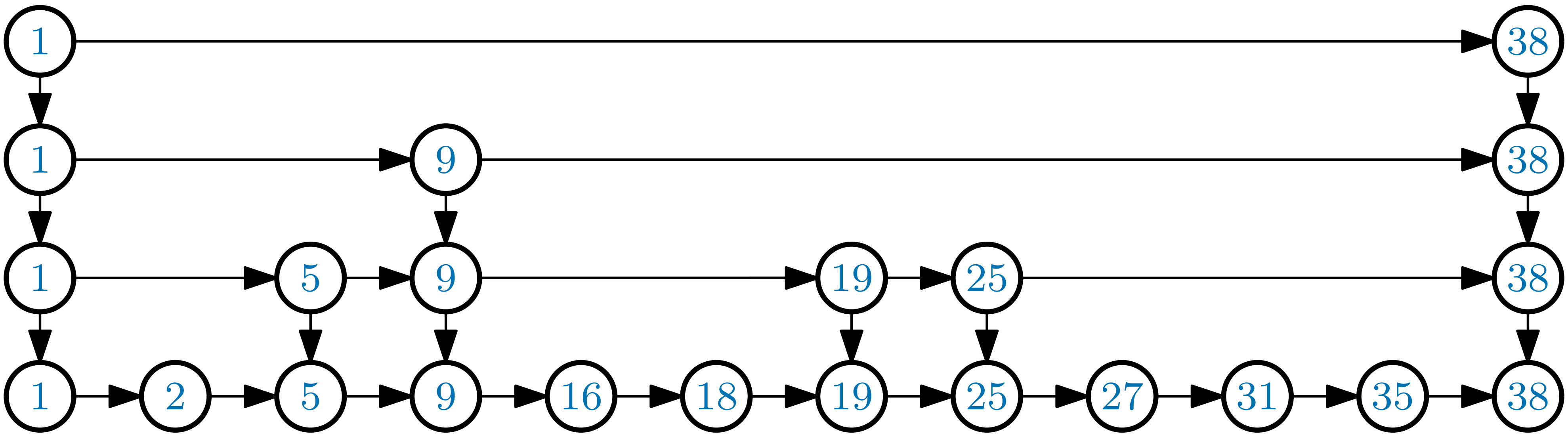

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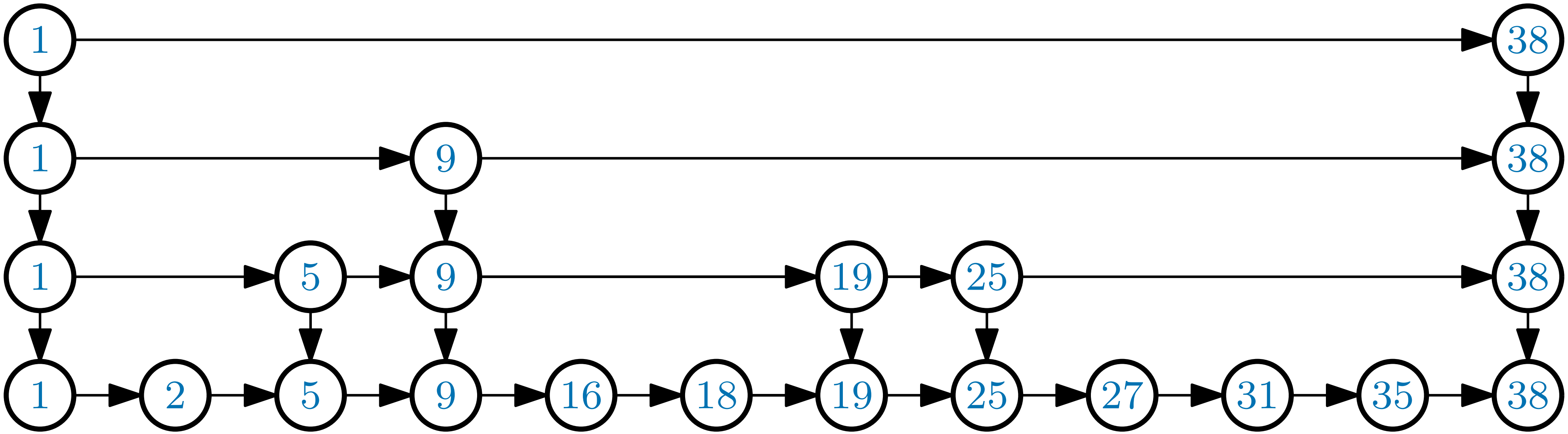

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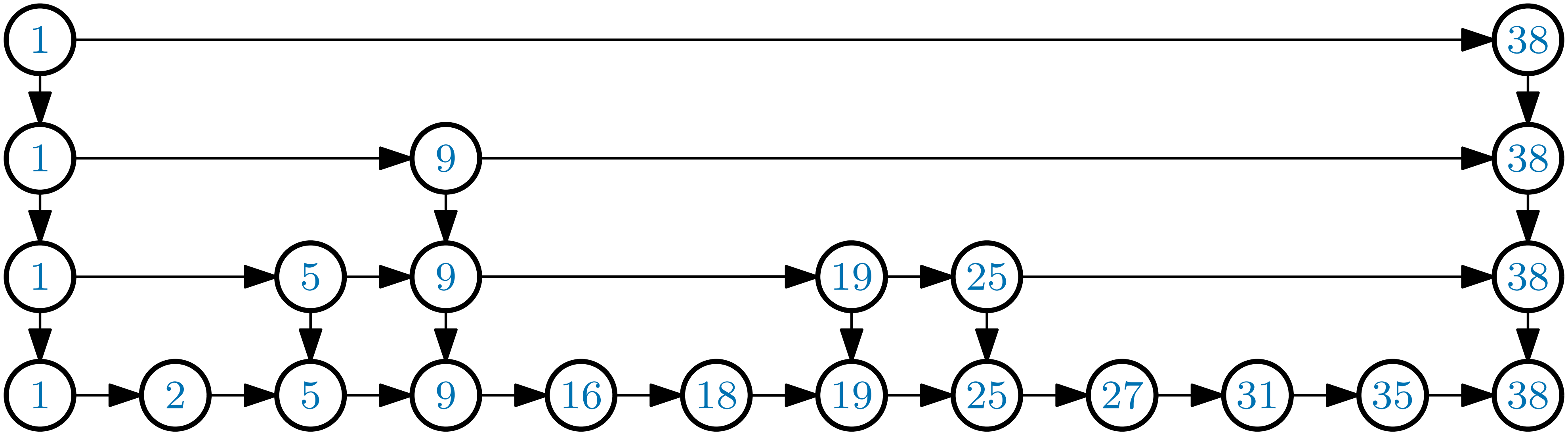
DELETING is straightforward, just FIND the key and DELETE it from all levels



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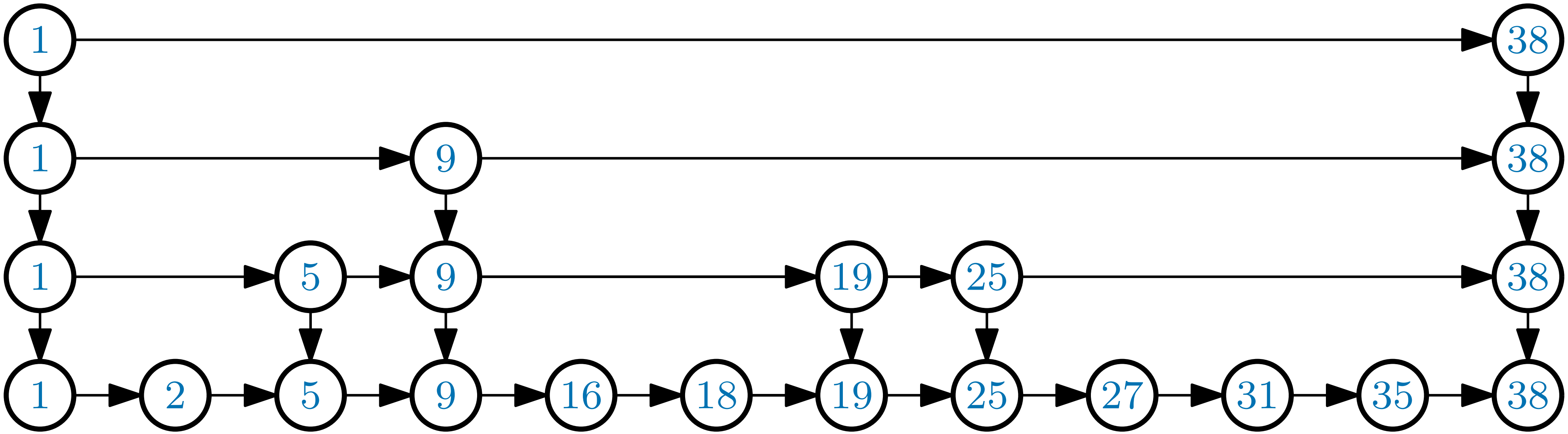

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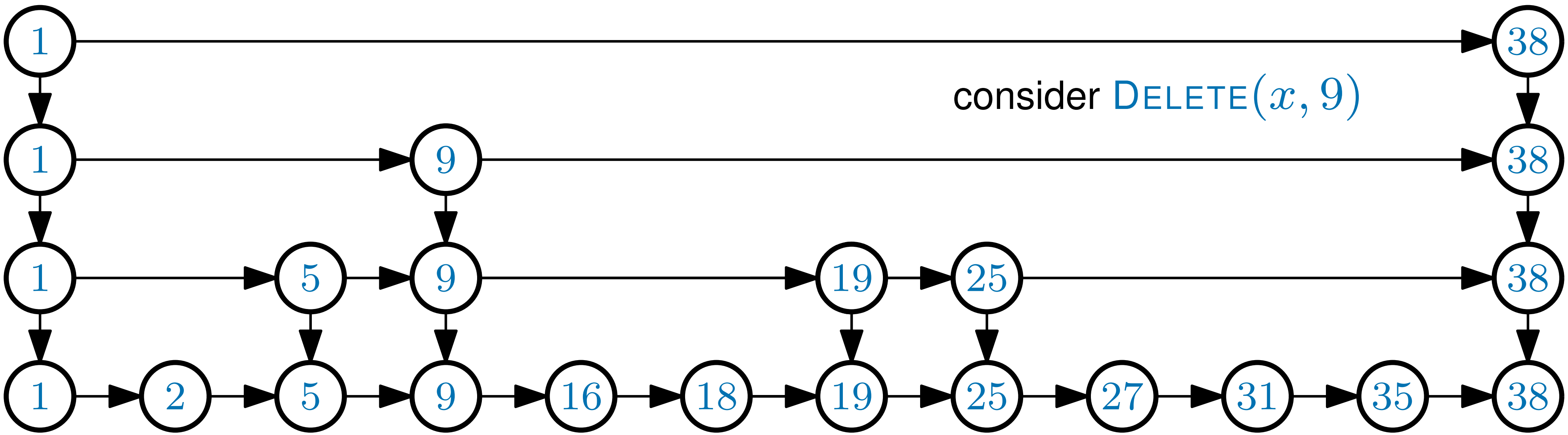

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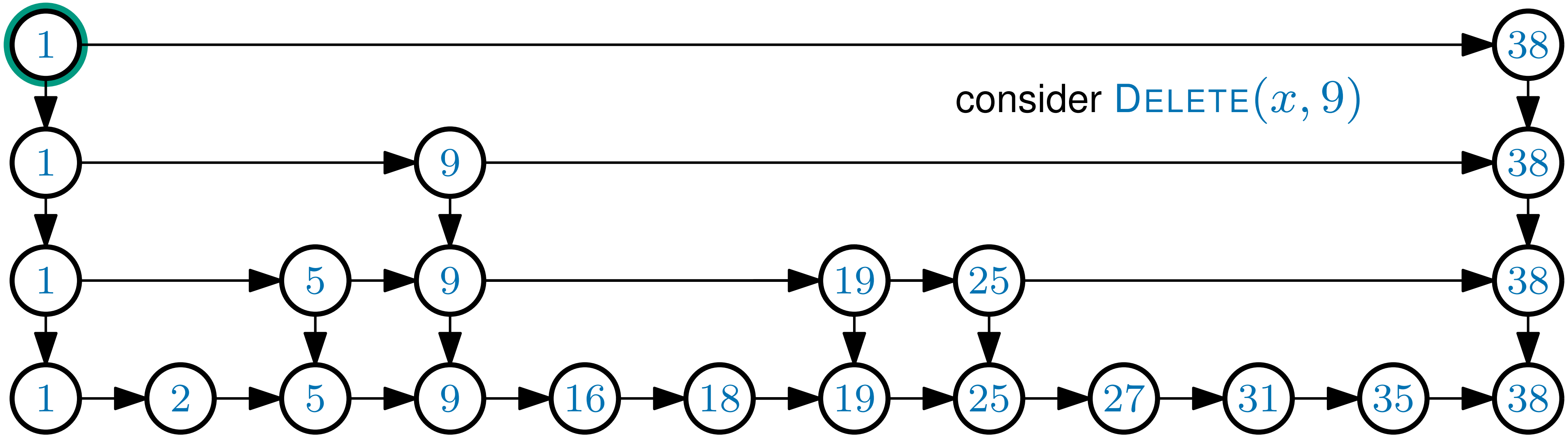

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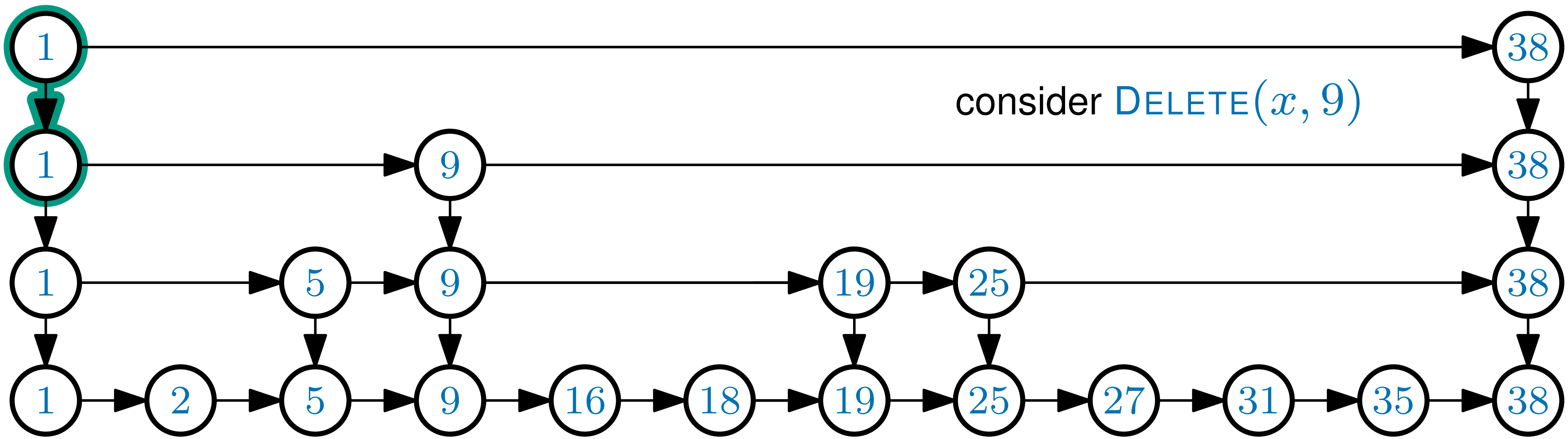

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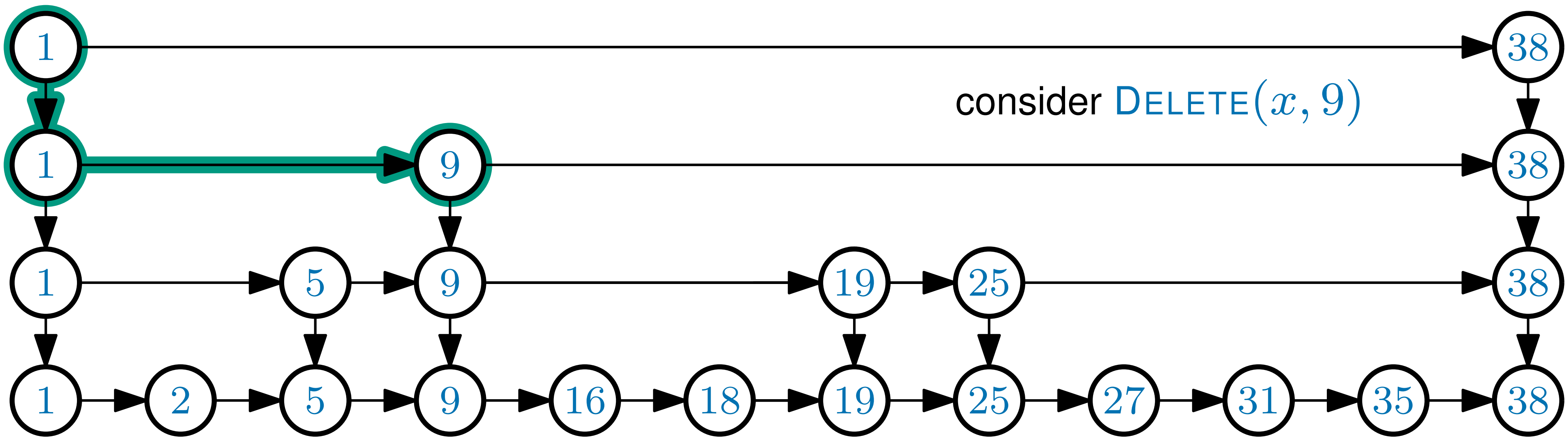

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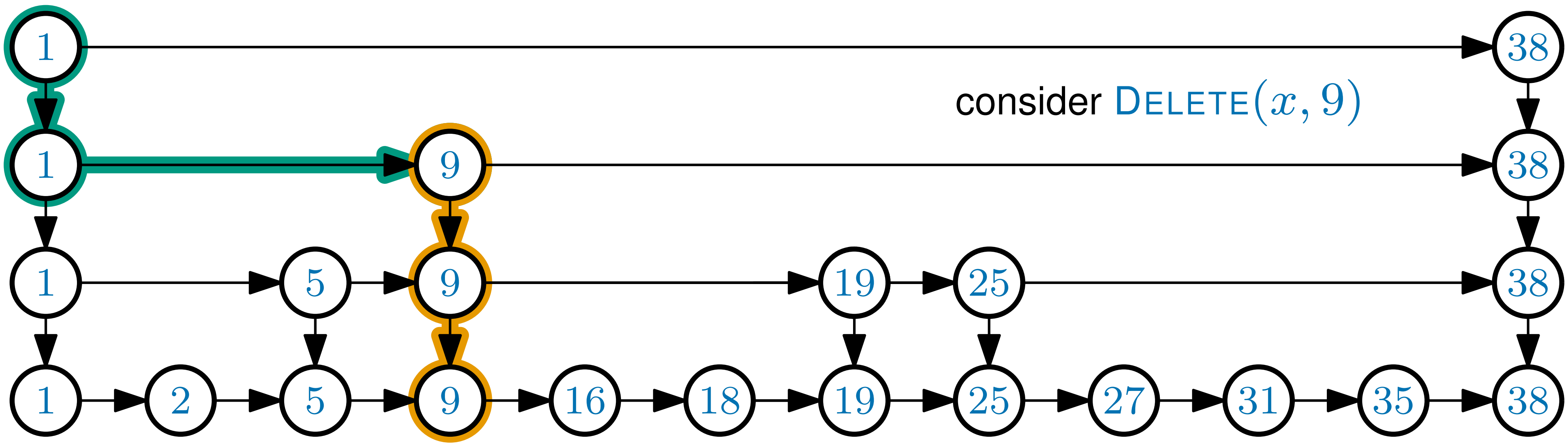

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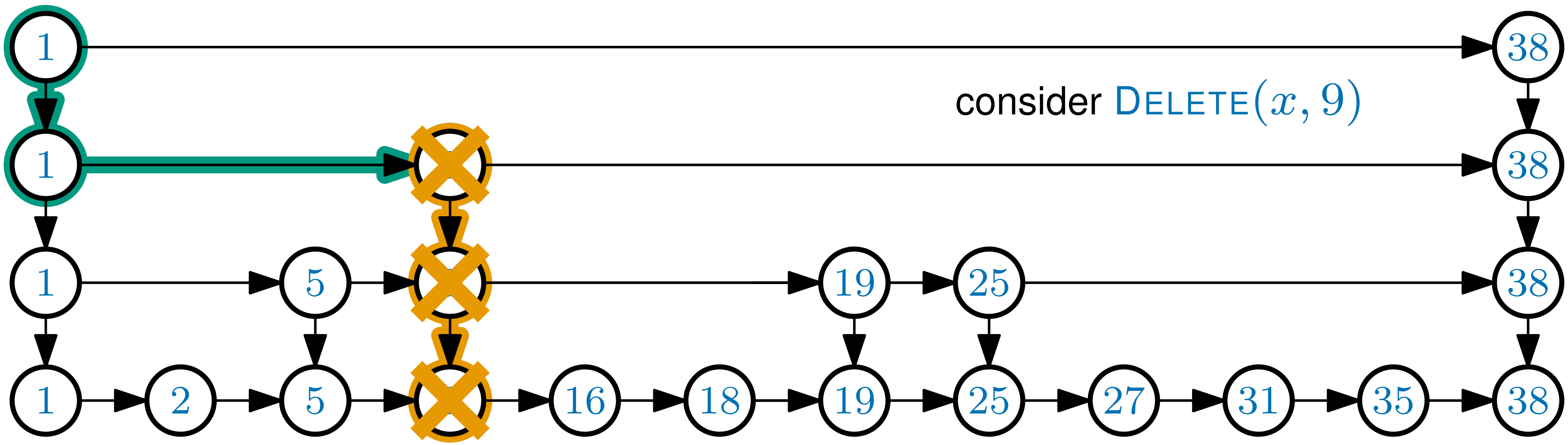

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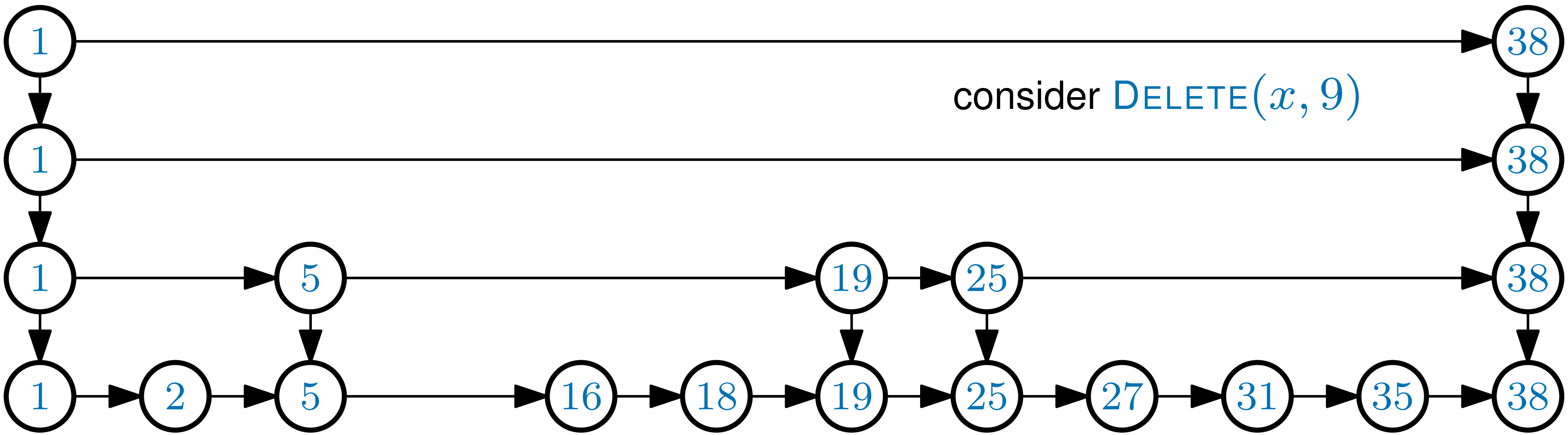

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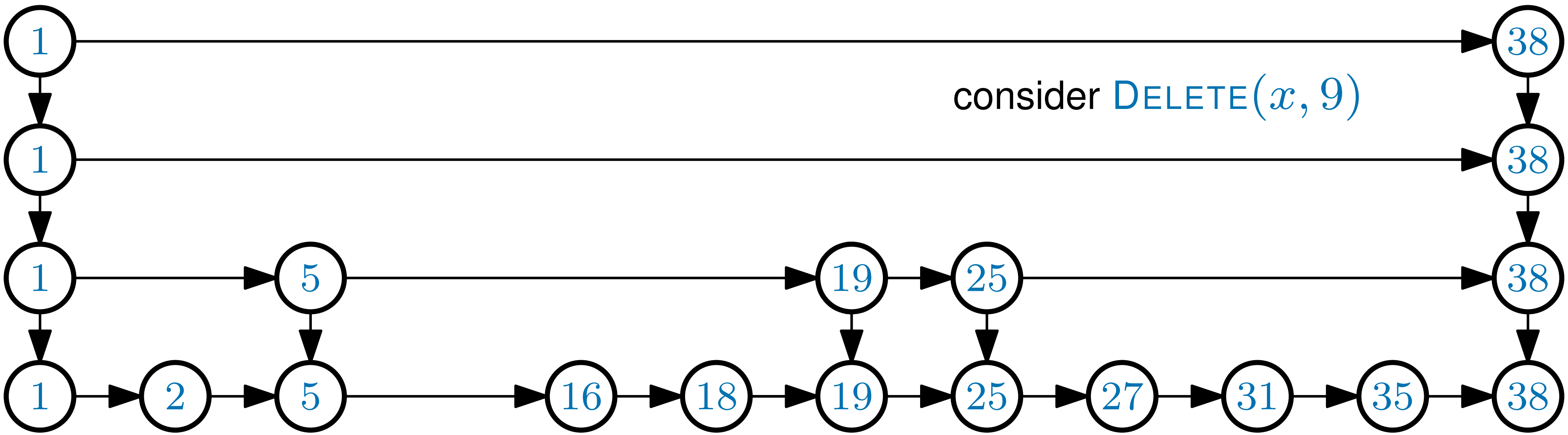

consider DELETE$(x, 9)$

To perform DELETE(k),

Step 1: Use FIND(k) to find (x, k)

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Step 3: Remove any empty levels

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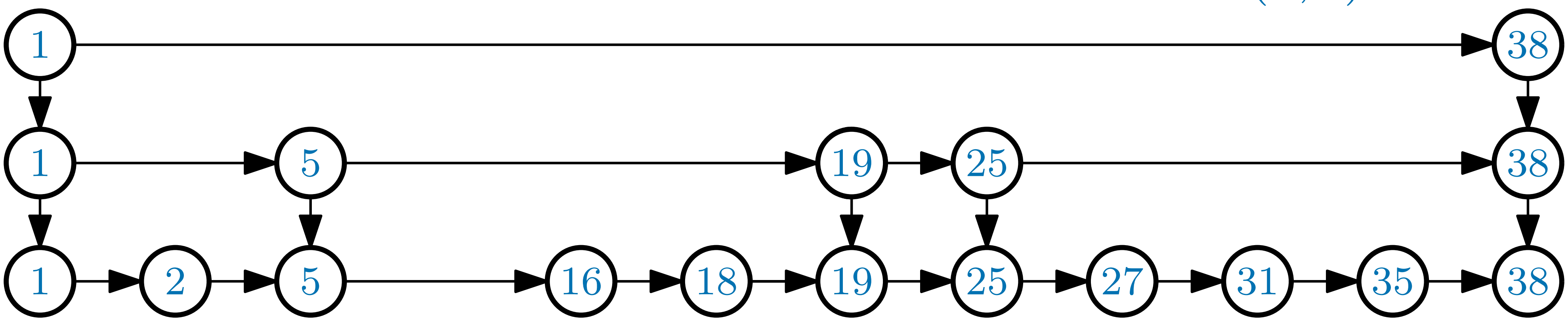

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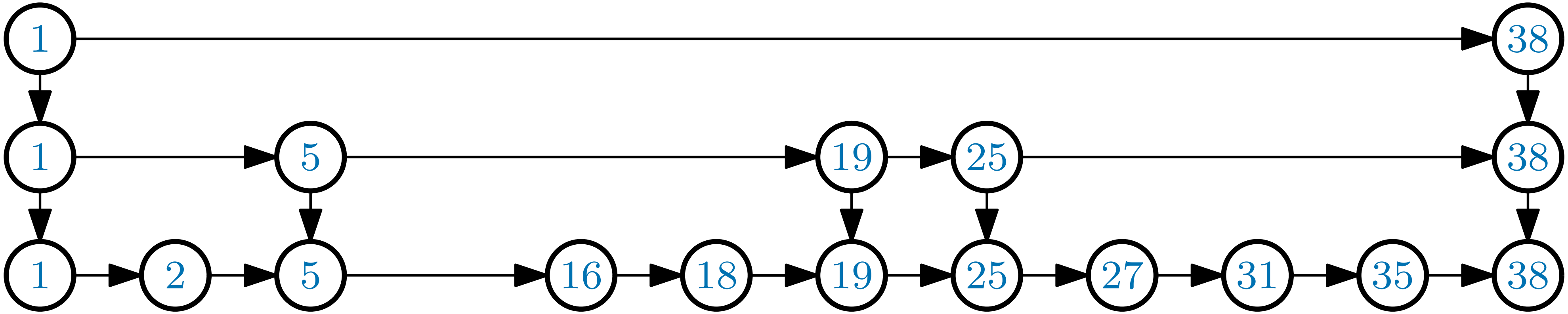

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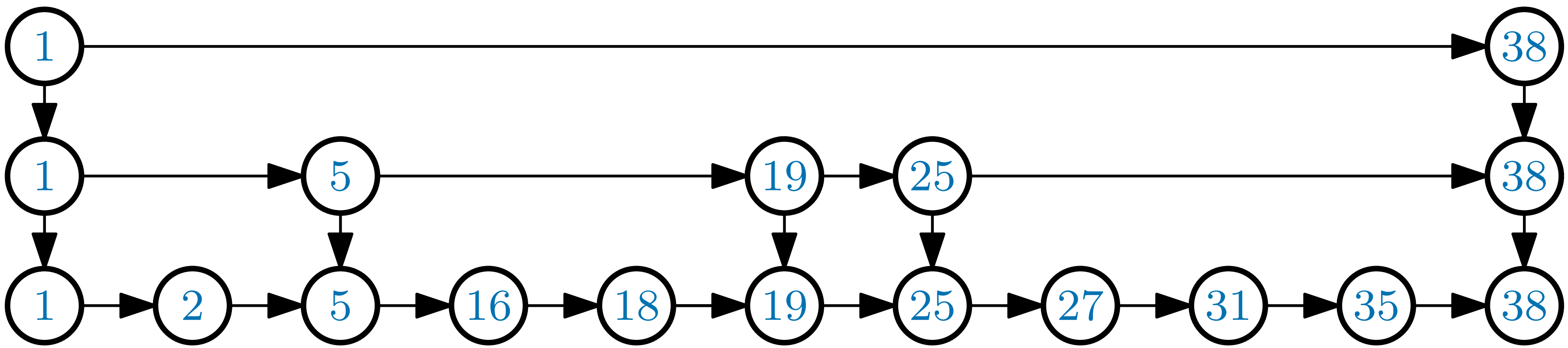
Skip Lists (pre-proof) summary

A skip list is a randomised data structure, based on link lists with shortcuts

which supports $\textsc{Insert}(x, k)$, $\textsc{Find}(k)$ and $\textsc{Delete}(k)$



We will show that each of these operations takes *expected* $O(\log n)$ time

That is, they take $O(\log n)$ time 'on average'

Important There is *no randomness in the data*,
the only randomness is in the coin flips

On *the worst case* input sequence, the expected time is $O(\log n)$

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We begin by proving that after n INSERT operations, a skip list is very unlikely to have more than $2\log n$ levels. . .

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Let $E_1, E_2 \ldots E_n$ be events where E_j occurs with probability p_j

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It gets better as n increases!

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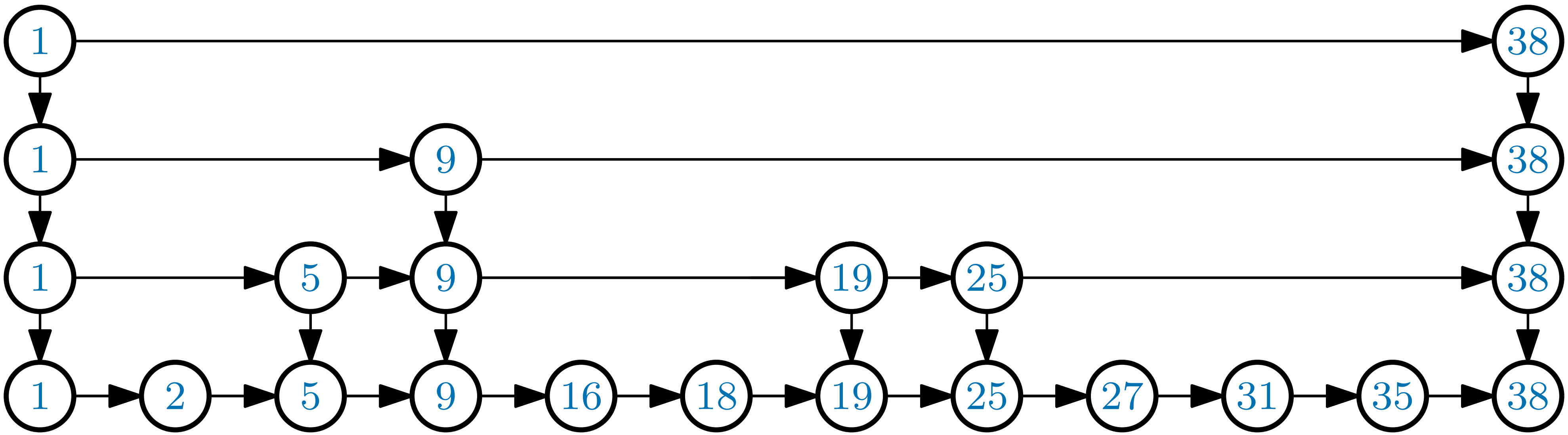
As the number of levels is $O(\log n)$ (with high probability),

we can conclude that the number of times we **move down** is very likely to be $O(\log n)$



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`Start at the top-left` *(the head of the top level)*

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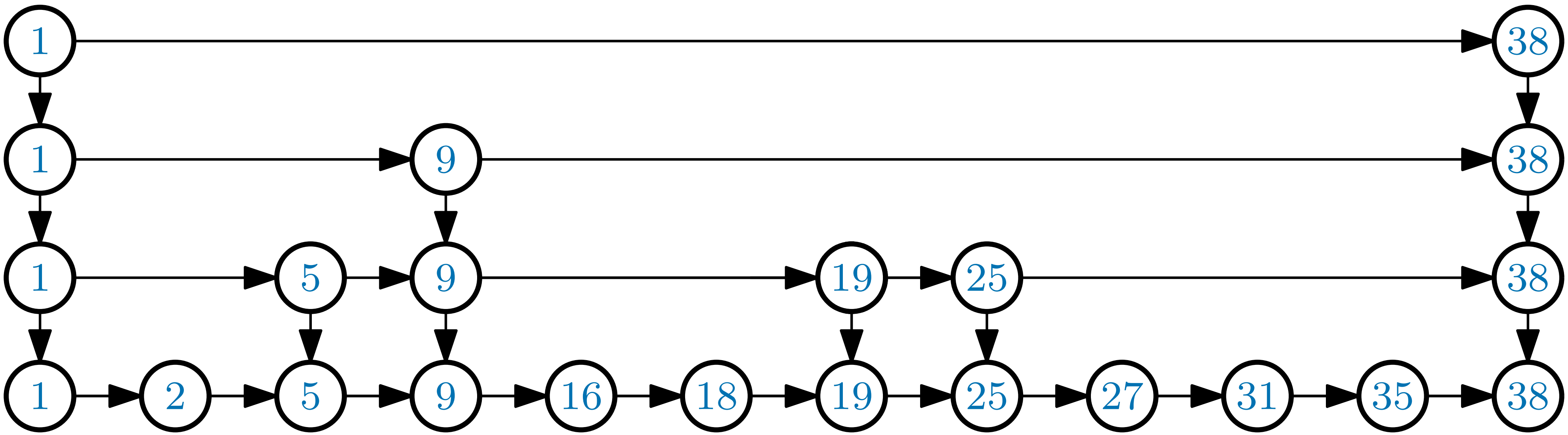

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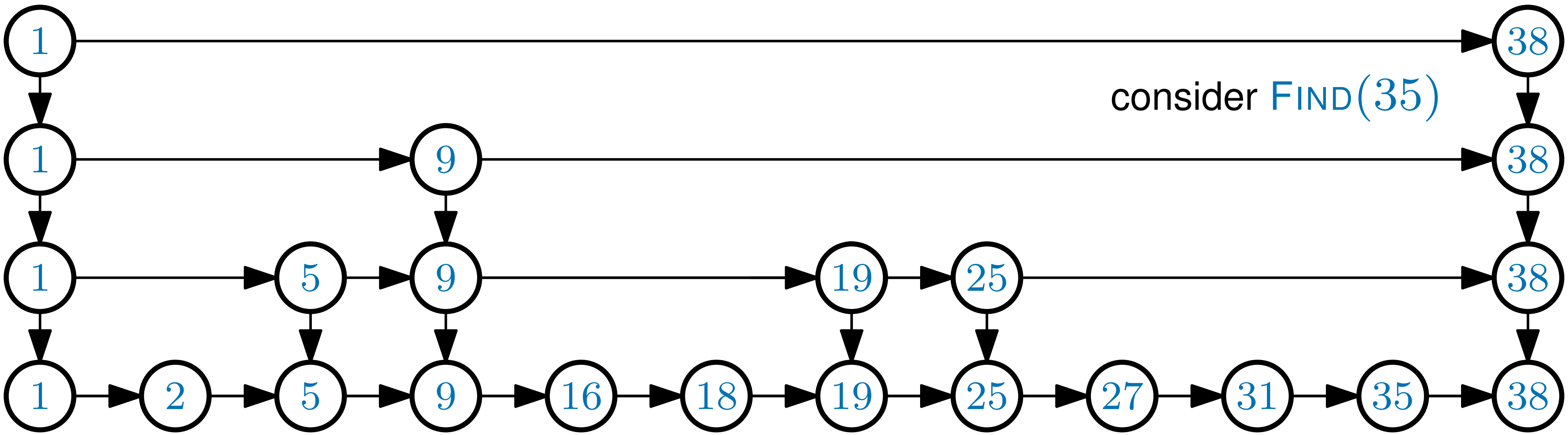

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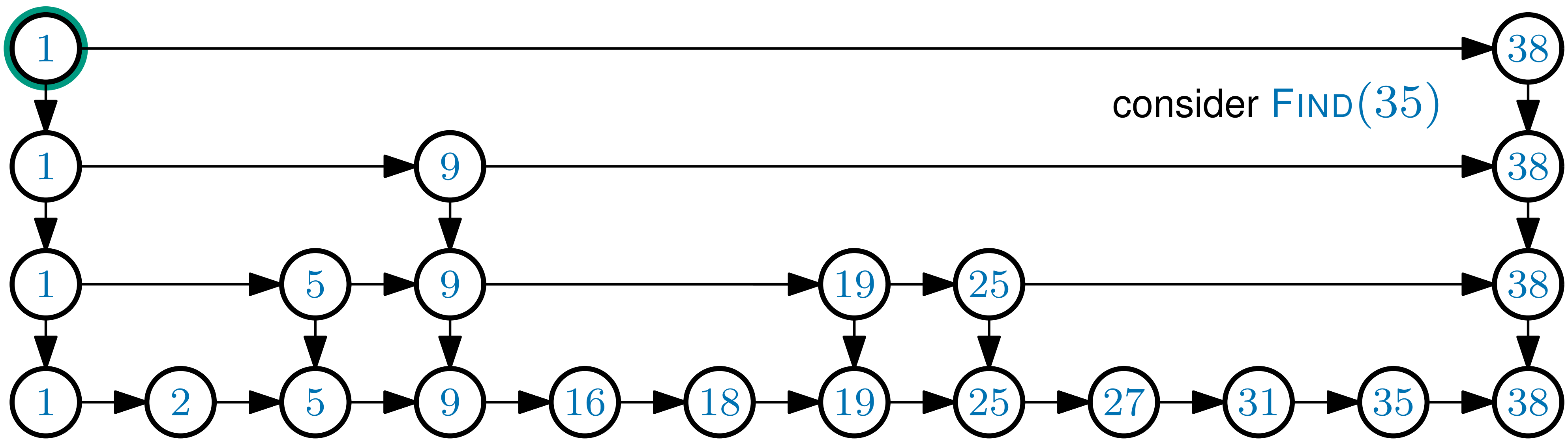

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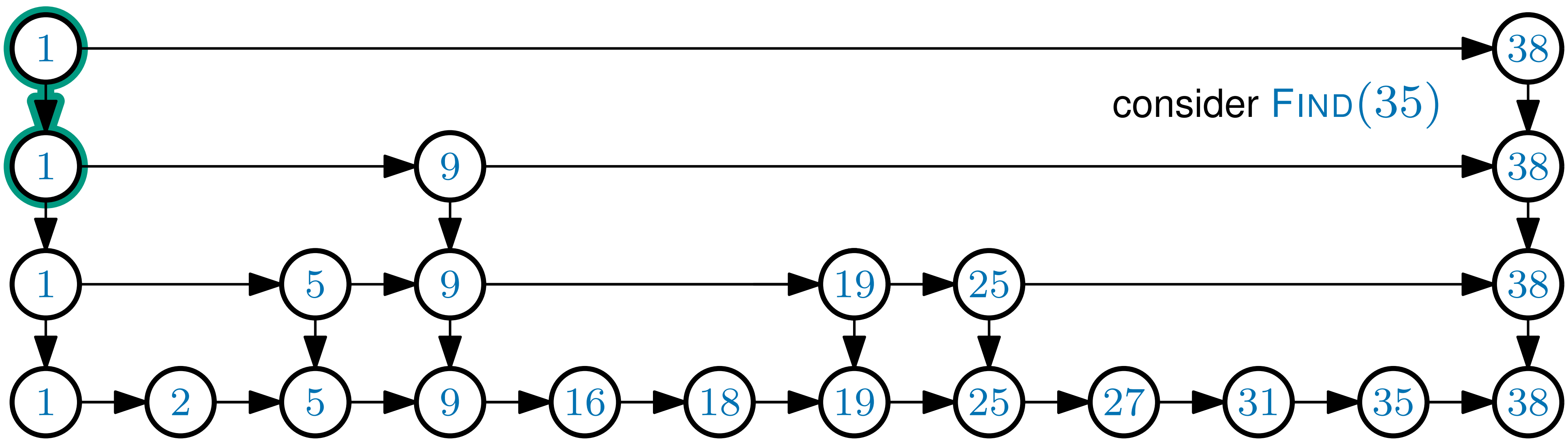

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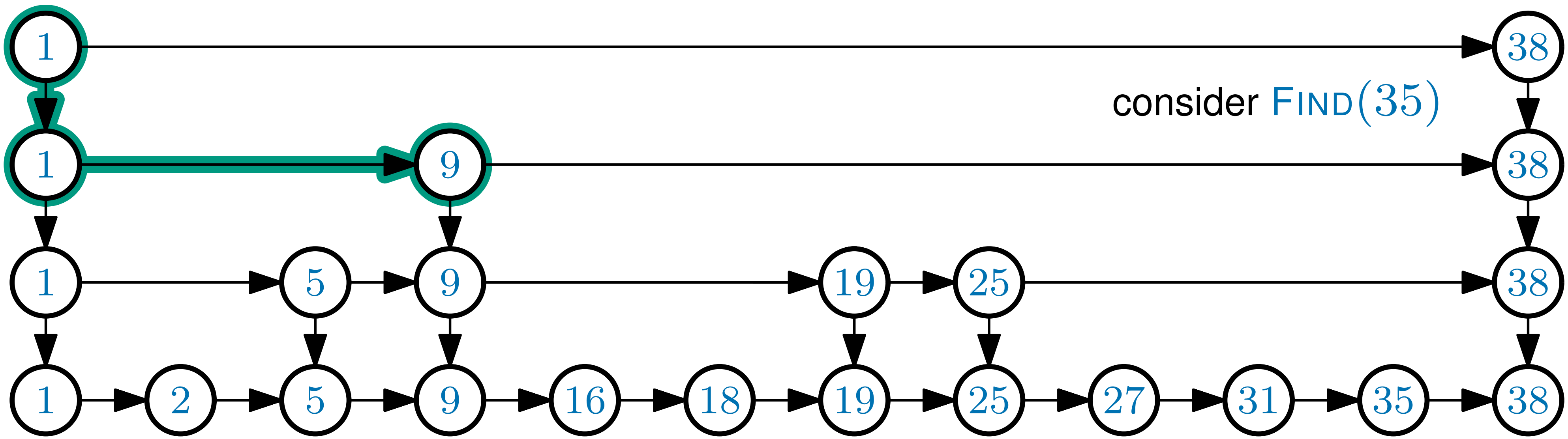

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        If the node to the right's key, k' ≤ k
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So how long does a FIND take? (sketch proof)

As the number of levels is $O(\log n)$ (with high probability),

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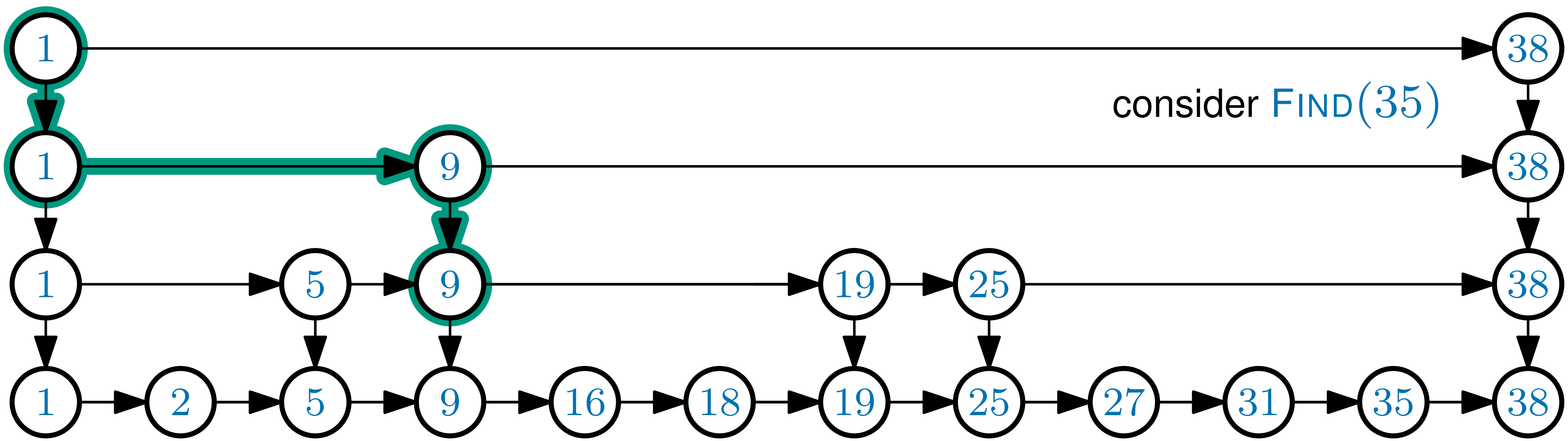

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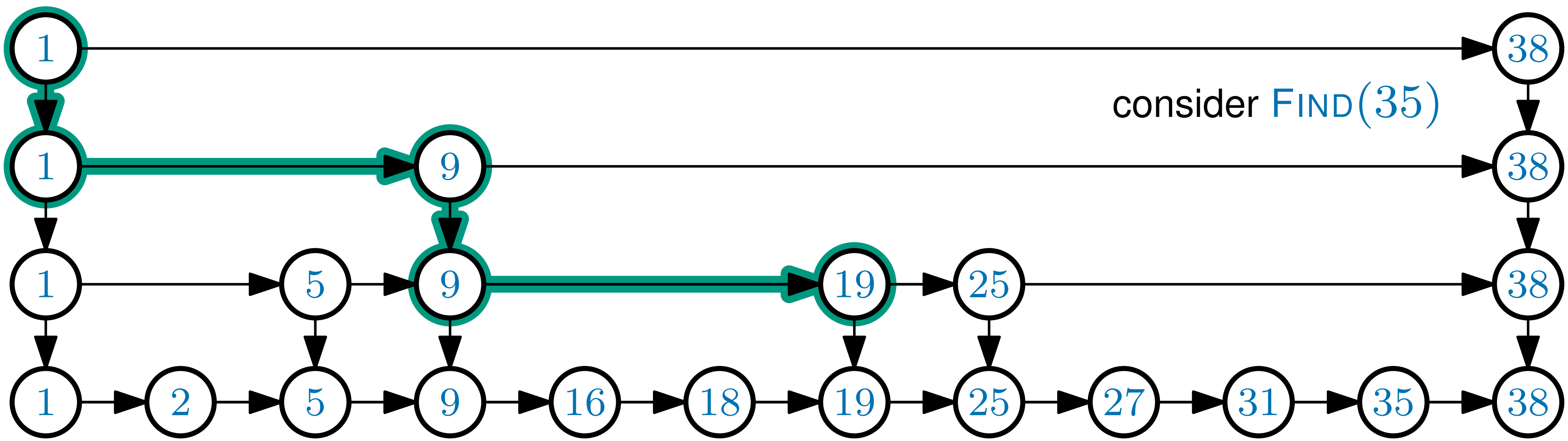

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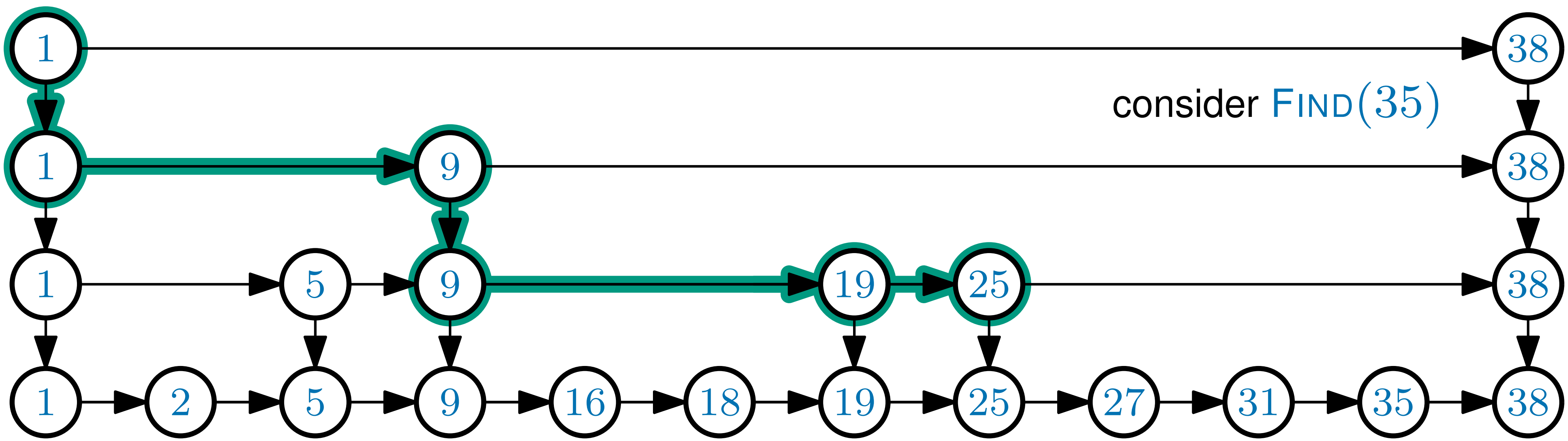

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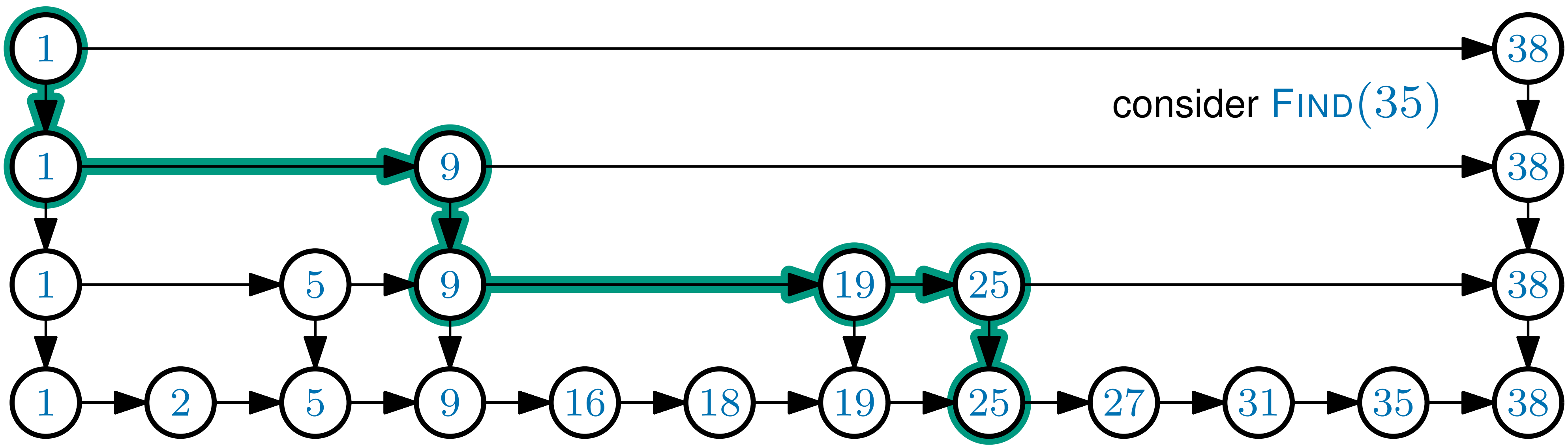

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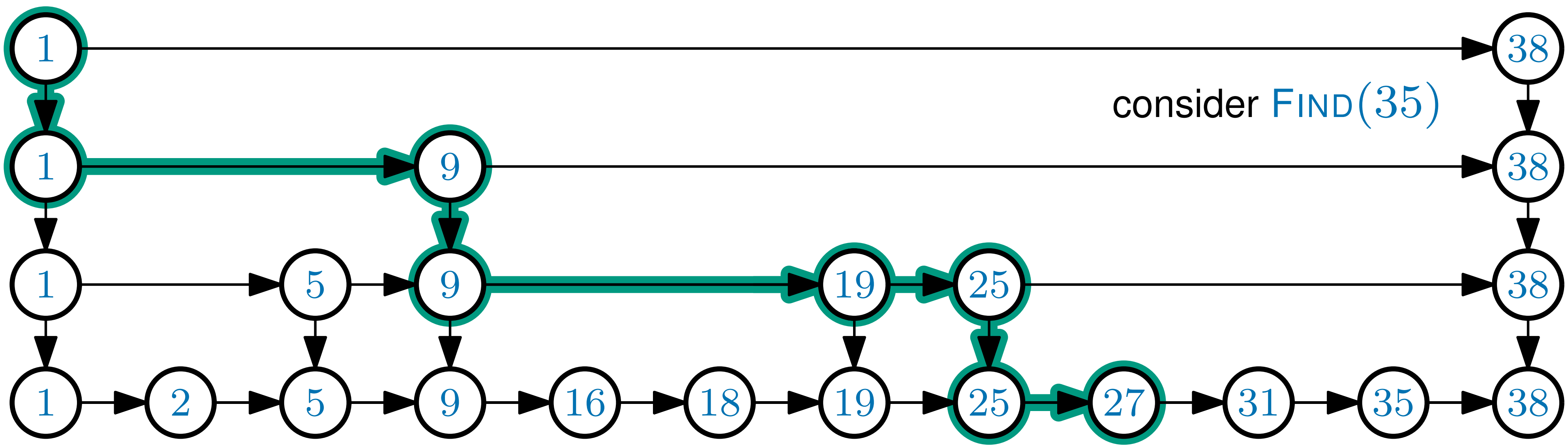

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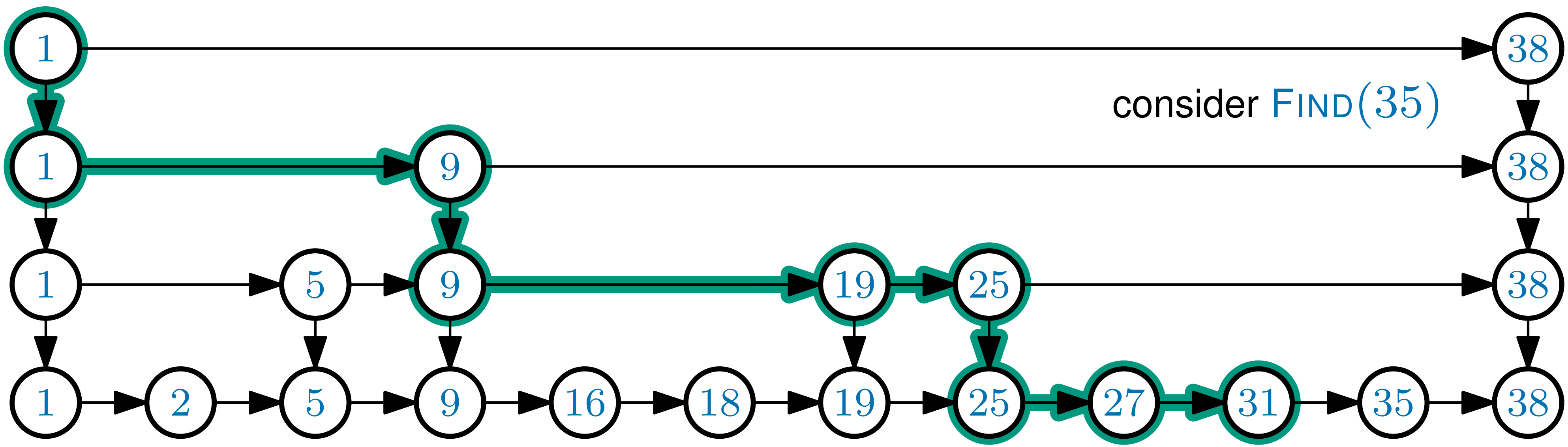

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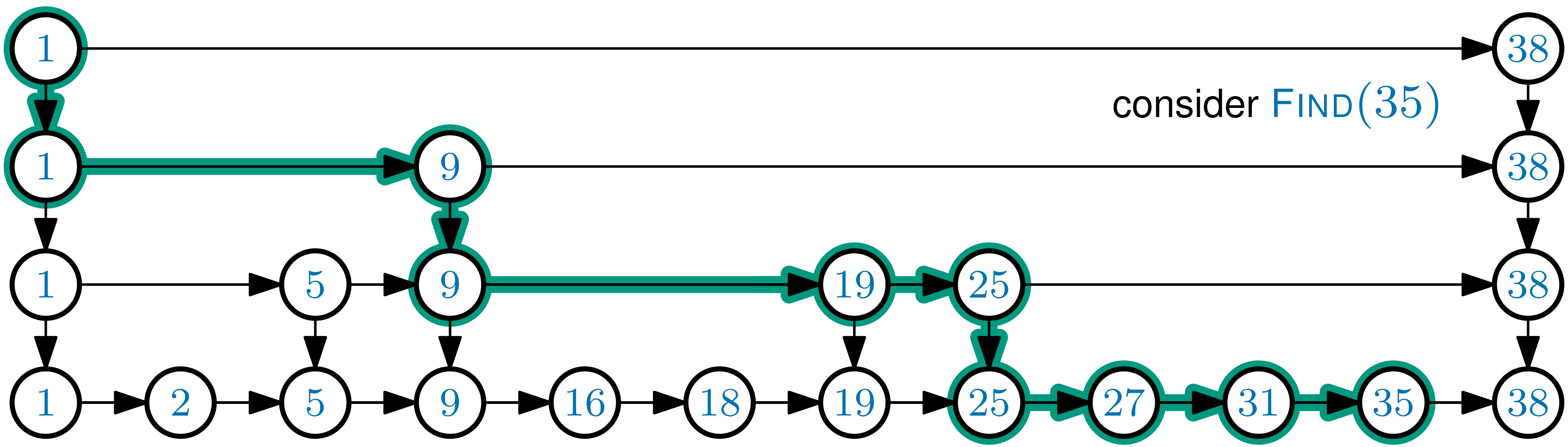

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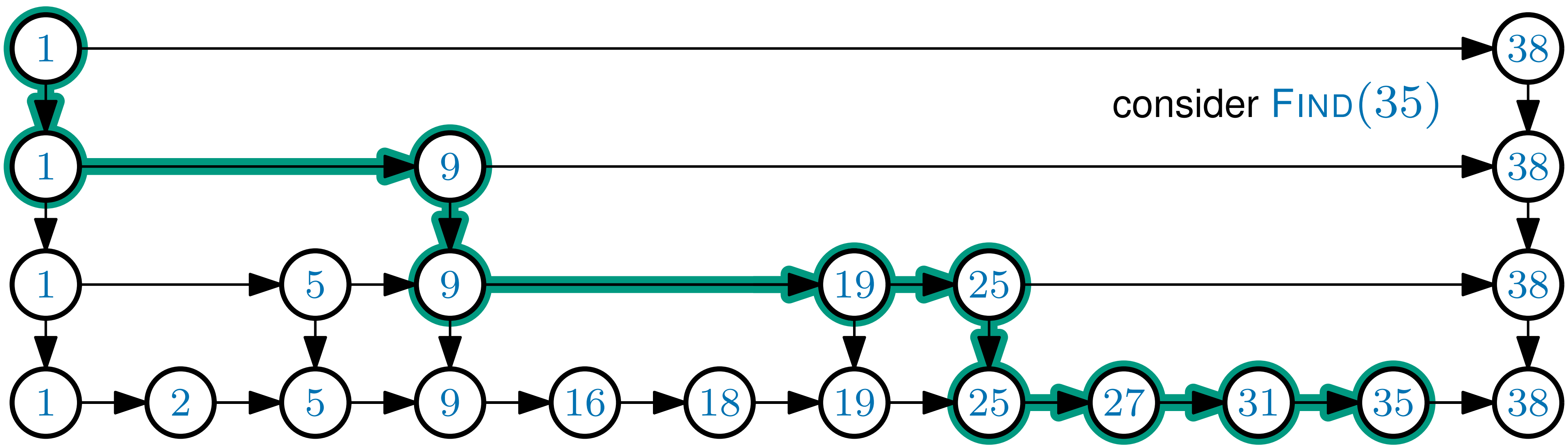

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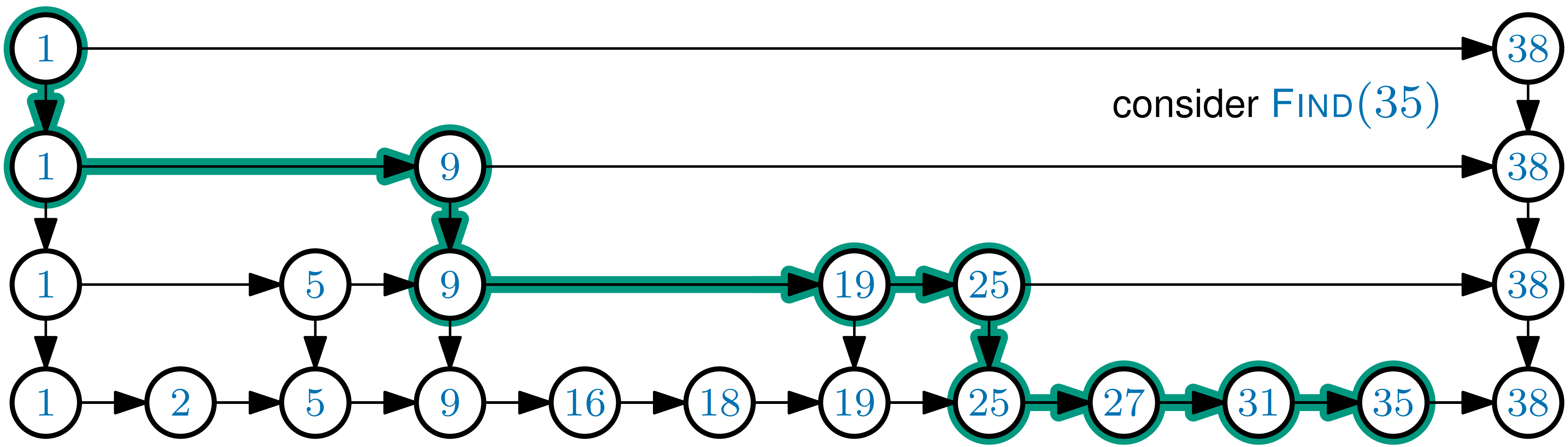

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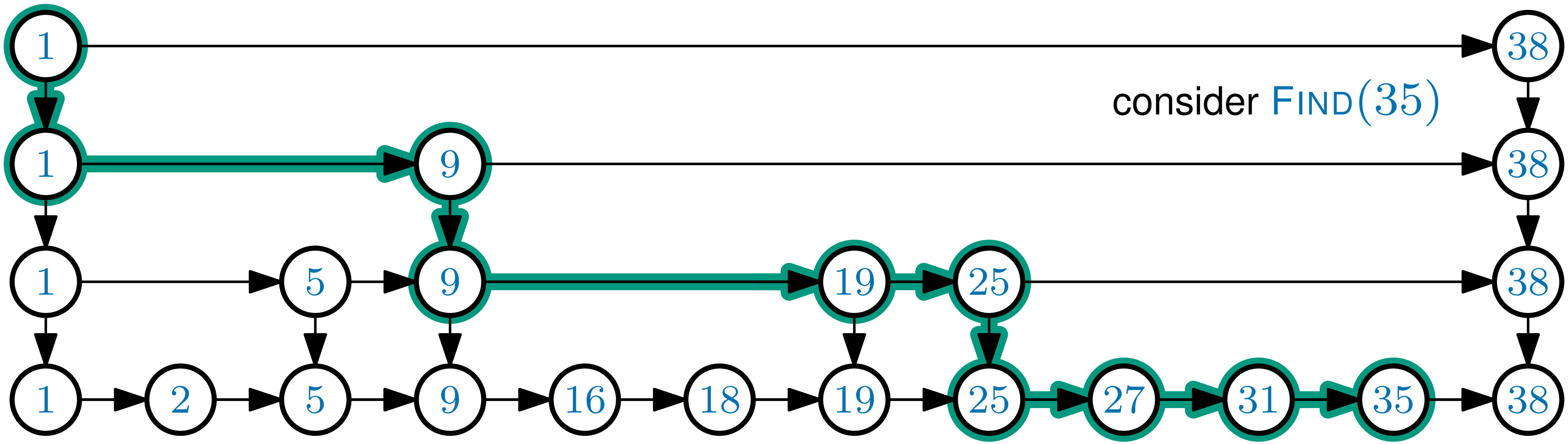

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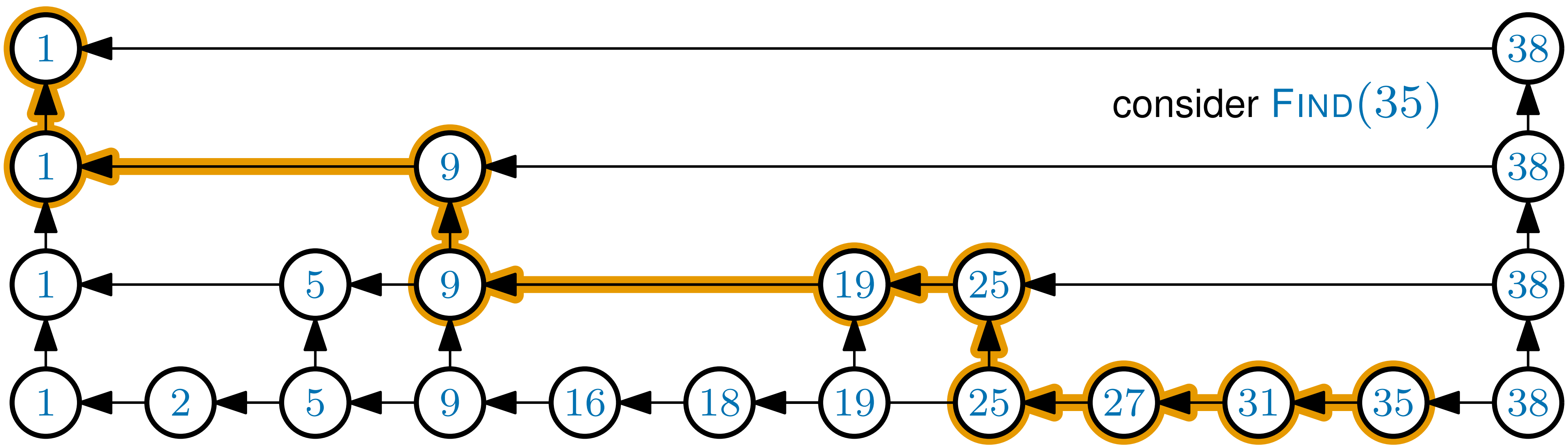

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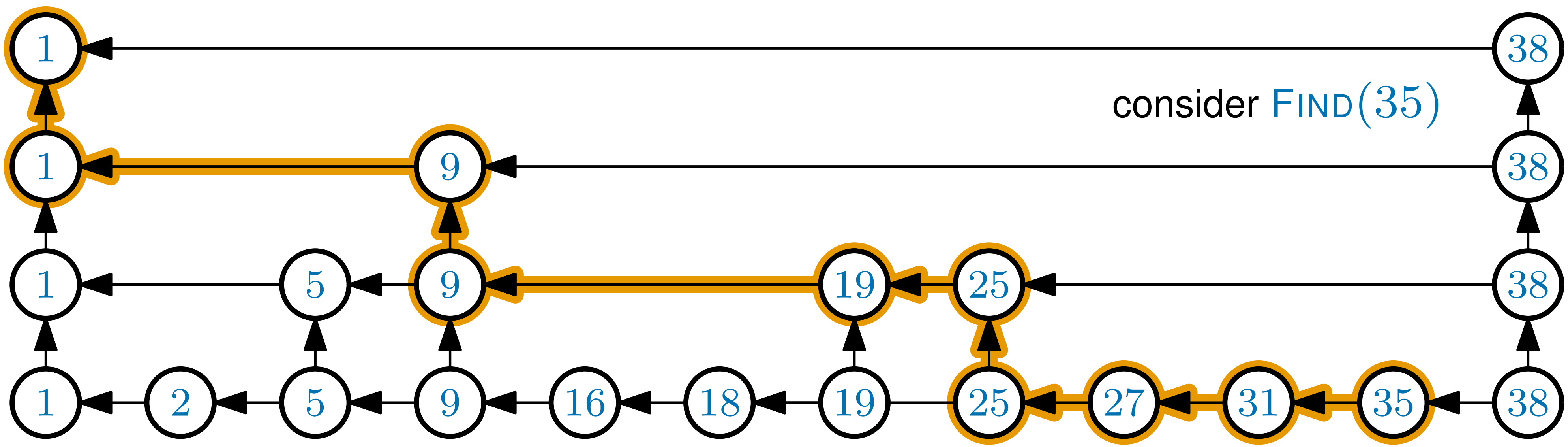

How long is this path?

1. Reverse it
2. Convince yourself this is the same path:

```
Start at k
While not at the top-left:
      If you can,
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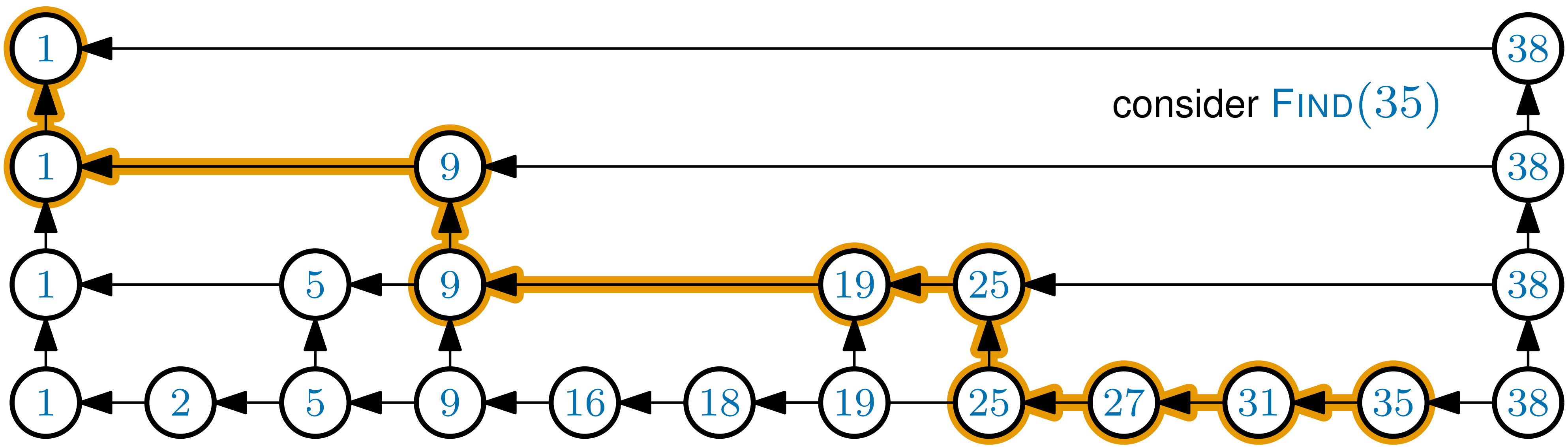

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3. Now convince yourself

   it takes the same time as this:

   (in expectation)

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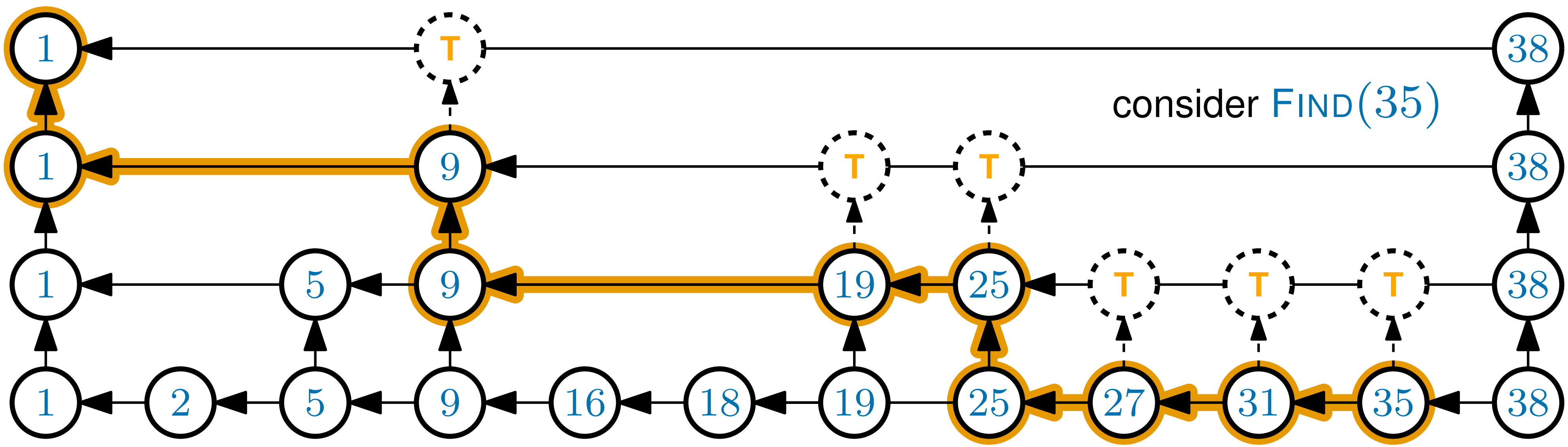

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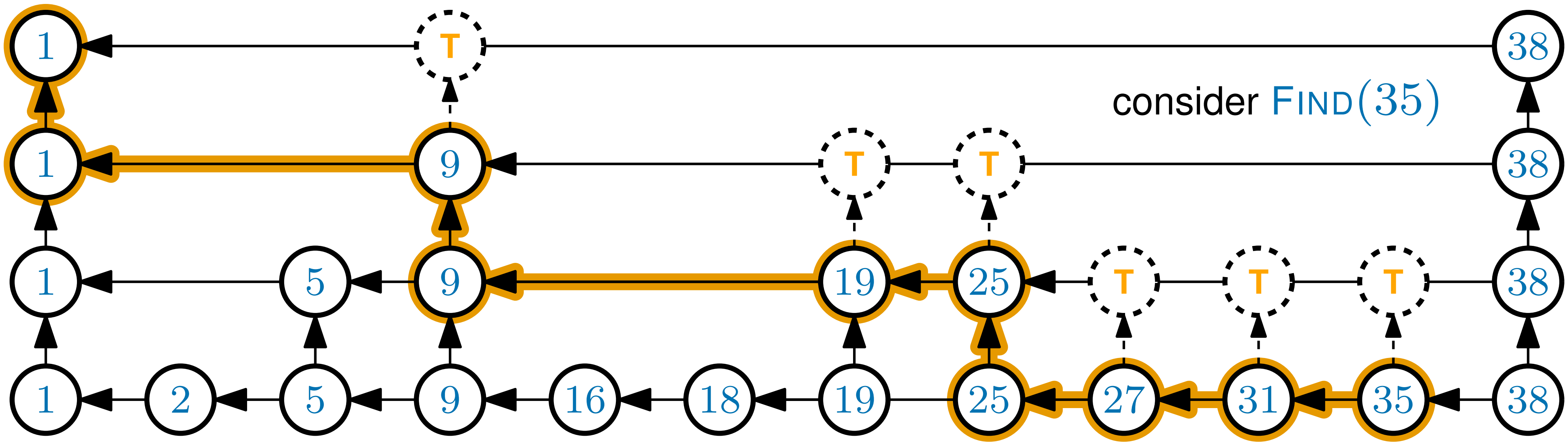

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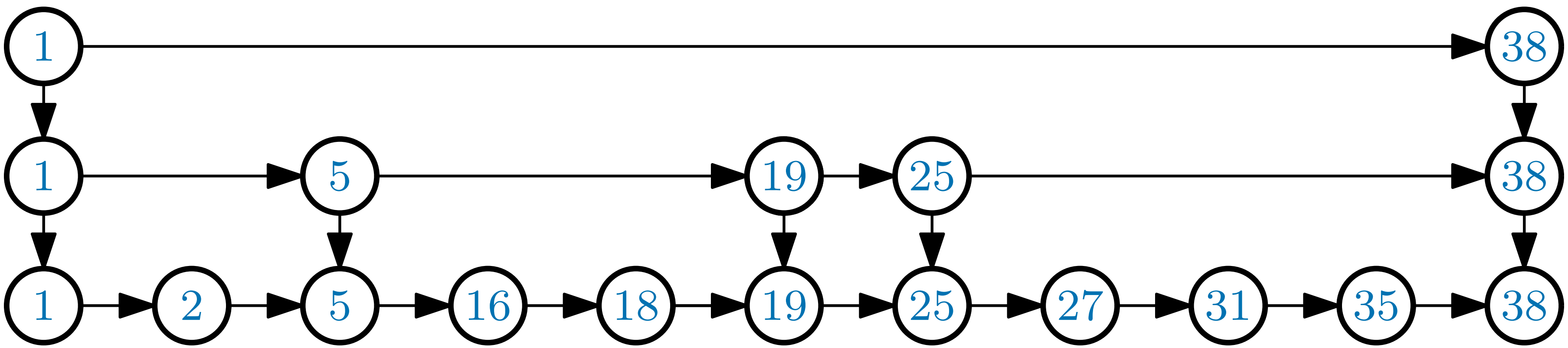
(this is a stronger claim but proving it is harder)

Skip Lists (post-proof) summary

A skip list is a randomised data structure, based on link lists with shortcuts

which supports $\textsc{Insert}(x, k)$, $\textsc{Find}(k)$ and $\textsc{Delete}(k)$



each of these operations takes *expected* $O(\log n)$ time

That is, they take $O(\log n)$ time 'on average'

Important There is *no randomness in the data*,
the only randomness is in the coin flips

On *the worst case* input sequence, the expected time is $O(\log n)$

Dynamic Search Structure Summary

A **dynamic search structure** supports (at least) the following three operations

DELETE(k) - deletes the (unique) element x with x.key $= k$

INSERT(x, k) - inserts x with key $k = x$.key

FIND(k) - returns the (unique) element x with x.key $= k$

Here are the time complexities of the structures we have seen...

	INSERT	DELETE	FIND
Unsorted Linked List	$O(1)$	$O(n)$	$O(n)$
Binary Search Tree	$O(n)$	$O(n)$	$O(n)$
2-3-4 Tree	$O(\log n)$	$O(\log n)$	$O(\log n)$
Red-Black Tree	$O(\log n)$	$O(\log n)$	$O(\log n)$
Skip list	$O(\log n)$	$O(\log n)$	$O(\log n)$

The time complexities for the Skip list are *expected*, for the others, they are *worst case*