S1. a The truth table for this function is as follows

a	b	c	$(a \wedge b \wedge \neg c)$	$(a \wedge \neg b \wedge c)$	$(\neg a \wedge \neg b \wedge c)$	$f(a,b,c)$
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	1	0	1	0	1
1	1	0	1	0	0	1
1	1	1	0	0	0	0

Since there are $n = 3$ input variables, there are clearly $2^n = 2^3 = 8$ input combinations; three of these produce 1 as an the output from the function.

b The truth table for this function is as follows

a	b	c	d	$f(a,b,c,d)$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

so since there is only one case where $f(a,b,c,d) = 1$, the only assignment given which matches the criteria is $a = 0$, $b = 1$, $c = 0$ and $d = 1$.

This hints at a general principle: when we have an expression like this, a term such as $\neg x$ can be read as "x should be 0" and x as "x should be 1". So the expression as a whole is read as "a should be 0 and b should be 1 and c should be 0 and d should be 1". Since we have basically fixed all four inputs, only *one* entry of the truth table matches. On the other hand, if we instead had

$$f(a,b,c,d) = \neg a \wedge b \wedge \neg c$$

for example, we would be saying "a should be 0 and b should be 1 and c should be 0, and d can be anything" which gives *two* possible assignments (i.e., $a = 0$, $b = 1$, $c = 0$ and either $d = 0$ or $d = 1$).

c Informally, SoP form means there are say n terms in the expression: each term is the conjunction of some variables (or their complement), and the expression is the disjunction of the terms. As conjunction and disjunction basically means the AND and OR operators, and AND and OR act sort of like multiplication and addition, the SoP name should make some sense: the expression is sort of like the sum of terms which are themselves each a product of variables. The second option is correct as a result; the first and last violate the form described above somehow (e.g., the first case is in the opposite, PoS form).

d One can easily make a comparison using a truth table such as

a	b	$a \vee 1$	$a \oplus 1$	$\neg a$	$a \wedge 1$	$\neg(a \wedge b)$	$\neg a \vee \neg b$
0	0	1	1	1	0	1	1
0	1	1	1	1	0	1	1
1	0	1	0	0	1	1	1
1	1	1	0	0	1	0	0

from which it should be clear that all the equations are correct except for the first one. That is, $a \vee 1 \neq a$ but rather $a \vee 1 = 1$.

e i Inspecting the following truth table

a	$\neg a$	$\neg\neg a$
0	1	0
0	1	0
1	0	1
1	0	1

shows this equivalence is correct (this is the involution axiom).

ii Inspecting the following truth table

a	b	$\neg a$	$\neg b$	$a \wedge b$	$\neg(a \wedge b)$	$\neg a \vee \neg b$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

shows this equivalence is correct (this is the de Morgan axiom).

iii Inspecting the following truth table

a	b	$\neg a$	$\neg b$	$\neg a \wedge b$	$a \wedge \neg b$
0	0	1	1	0	0
0	1	1	0	1	0
1	0	0	1	0	1
1	1	0	0	0	0

shows this equivalence is incorrect.

iv Inspecting the following truth table

a	$\neg a$	$a \oplus a$
0	1	0
1	0	0

shows this equivalence is incorrect.

S2. a The dual of any expression is constructed by using the principle of duality, which informally means swapping each AND with OR (and vice versa) and each 0 with 1 (and vice versa); this means, for example, we can take the OR form of each axiom and produce the AND form (and vice versa).

So in this case, we start with an OR form: this means the dual will the corresponding AND form. Making the swaps required means we end up with

$$x \wedge 0 \equiv 0$$

so the second option is correct.

b This question is basically asking for the complement of f, since the options each have $\neg f$ on the left-hand side: this means using the principle of complements, a generalisation of the de Morgan axiom, by swapping each variable with the complement (and vice versa), each AND with OR (and vice versa), and each 0 with 1 (and vice versa). If we apply these rules (taking care with the parenthesis) to

$$f = \neg a \wedge \neg b \vee \neg c \vee \neg d \vee \neg e,$$

we end up with

$$\neg f = (a \vee b) \wedge c \wedge d \wedge e$$

which matches the last option.

c The de Morgan axiom, which can be generalised using by the principle of complements, says that

$$\neg(x \wedge y) \quad \equiv \quad \neg x \vee \neg y$$

or conversely that

$$\neg(x \vee y) \quad \equiv \quad \neg x \wedge \neg y$$

You can think of either form as "pushing" the NOT operator on the left-hand side into the parentheses: this acts to complement each variable, and swap the AND to an OR (or vice versa). We know that

$$\begin{array}{rcl} x \overline{\wedge} y & \equiv & \neg(x \wedge y) \\ x \overline{\vee} y & \equiv & \neg(x \vee y) \end{array}$$

so pattern matching against the options, it is clear the first one is correct because

$$x \overline{\vee} y \equiv \neg(x \vee y) \equiv \neg x \wedge \neg y$$

where the right-hand side matches the description of an AND whose two inputs are complemented.

S3. a The third option, i.e., $\neg a \wedge \neg b$ is the correct one; the three simplification steps, via two axioms, are as follows:

$$\begin{array}{llllll} & \neg(a \vee b) & \wedge\ \neg(c \vee d \vee e) & \vee\ \neg(a \vee b) & \\ = & (\neg a \wedge \neg b) & \wedge\ \neg(c \vee d \vee e) & \vee\ (\neg a \wedge \neg b) & \text{(de Morgan)} \\ = & (\neg a \wedge \neg b) & \wedge\ (\neg c \wedge \neg d \wedge \neg e) & \vee\ (\neg a \wedge \neg b) & \text{(de Morgan)} \\ = & \neg a \wedge \neg b & & & \text{(absorption)} \end{array}$$

b We can clearly see that

$$\begin{array}{lll} & (a \vee b \vee c) \wedge \neg(d \vee e) \vee (a \vee b \vee c) \wedge (d \vee e) & \\ = & (a \vee b \vee c) \wedge (\neg(d \vee e) \vee (d \vee e)) & \text{(distribution)} \\ = & (a \vee b \vee c) \wedge ((d \vee e) \vee \neg(d \vee e)) & \text{(commutativity)} \\ = & (a \vee b \vee c) \wedge 1 & \text{(inverse)} \\ = & a \vee b \vee c & \text{(identity)} \end{array}$$

meaning the first option is the correct one.

c We can clearly see that

$$\begin{array}{lll} & a \wedge c \vee c \wedge (\neg a \vee a \wedge b) & \\ = & c \wedge (a \vee \neg a \vee a \wedge b) & \text{(distribution)} \\ = & c \wedge (1 \vee a \wedge b) & \text{(commutativity)} \\ = & c \wedge (a \wedge b \vee 1) & \text{(inverse)} \\ = & c \wedge 1 & \text{(null)} \\ = & c & \text{(identity)} \end{array}$$

meaning the last option is the correct one: *none* of the above is correct, since the correct simplification is actually just c.

d The fourth option, i.e., $a \wedge b$ is correct. This basically stems from repeated application of the absorption axiom, the AND form of which states

$$x \vee (x \wedge y) \equiv x.$$

Applying it from left-to-right, we find that

$$\begin{array}{lll} & a \wedge b \vee a \wedge b \wedge c \vee a \wedge b \wedge c \wedge d \vee a \wedge b \wedge c \wedge d \wedge e \vee a \wedge b \wedge c \wedge d \wedge e \wedge f & \\ = & (a \wedge b) \vee (a \wedge b) \wedge (c) \vee (a \wedge b) \wedge (c \wedge d) \vee (a \wedge b) \wedge (c \wedge d \wedge e) \vee (a \wedge b) \wedge (c \wedge d \wedge e \wedge f) & \text{(precedence)} \\ = & (a \wedge b) \vee (a \wedge b) \wedge (c \wedge d) \vee (a \wedge b) \wedge (c \wedge d \wedge e) \vee (a \wedge b) \wedge (c \wedge d \wedge e \wedge f) & \text{(absorption)} \\ = & (a \wedge b) \vee (a \wedge b) \wedge (c \wedge d \wedge e) \vee (a \wedge b) \wedge (c \wedge d \wedge e \wedge f) & \text{(absorption)} \\ = & (a \wedge b) \vee (a \wedge b) \wedge (c \wedge d \wedge e \wedge f) & \text{(absorption)} \\ = & (a \wedge b) & \text{(absorption)} \end{array}$$

e We can simplify this function as follows

$$\begin{array}{rcll} f(a,b,c) & = & (a \wedge b) \vee a \wedge (a \vee c) \vee b \wedge (a \vee c) & \\ & = & (a \wedge b) \vee a \vee b \wedge (a \vee c) & \text{(absorbtion)} \\ & = & a \vee (a \wedge b) \vee b \wedge (a \vee c) & \text{(commutitivity)} \\ & = & a \vee b \wedge (a \vee c) & \text{(absorbtion)} \\ & = & a \vee (b \wedge a) \vee (b \wedge c) & \text{(distribution)} \\ & = & a \vee (a \wedge b) \vee (b \wedge c) & \text{(commutitivity)} \\ & = & a \vee (b \wedge c) & \text{(commutitivity)} \end{array}$$

at which point there is nothing else that can be done: we end up with 2 operators (and AND and an OR), so the second option is correct.

f Since $z \vee \neg z = 1$ and $z \vee z = z$, we have that

$$\begin{array}{rcl} \neg x \vee \neg y & = & (\neg x \wedge (y \vee \neg y)) \vee (\neg y \wedge (x \vee \neg x)) \\ & = & (\neg x \wedge y) \vee (\neg x \wedge \neg y) \vee (\neg y \wedge x) \vee (\neg y \wedge \neg x) \\ & = & (\neg x \wedge y) \vee (\neg y \wedge x) \vee (\neg x \wedge \neg y). \end{array}$$

g By writing

$$\begin{array}{lclcl} t_0 & = & x & \wedge & y \\ t_1 & = & y & \wedge & z \\ t_2 & = & y & \vee & z \\ t_3 & = & x & \vee & z \\ t_4 & = & t_1 & \wedge & t_2 \end{array}$$

we can shorten the LHS and RHS to

$$\begin{array}{lclcl} f & = & t_0 & \vee & t_4 \\ g & = & y & \wedge & t_3 \end{array}$$

and then perform a brute-force enumeration

x	y	z	t_0	t_1	t_2	t_3	t_4	f	g
0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	1	0	0	0
0	1	0	0	0	1	0	0	0	0
0	1	1	0	1	1	1	1	1	1
1	0	0	0	0	0	1	0	0	0
1	0	1	0	0	1	1	0	0	0
1	1	0	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1	1	1

to demonstrate that $f = g$, i.e., the equivalence holds. Note that this approach is not as robust if the intermediate steps are not shown; simply including f and g in the truth table does not give much more confidence that simply writing the equivalence!

To prove the equivalence using an axiomatic approach, the following steps can be applied:

$$\begin{array}{cll} & (x \wedge y) \vee (y \wedge z \wedge (y \vee z)) & \\ = & (x \wedge y) \vee (y \wedge z \wedge y) \vee (y \wedge z \wedge z) & \text{(distribution)} \\ = & (x \wedge y) \vee (y \wedge y \wedge z) \vee (y \wedge z \wedge z) & \text{(commutativity)} \\ = & (x \wedge y) \vee (y \wedge z) \vee (y \wedge z) & \text{(idempotency)} \\ = & (x \wedge y) \vee (y \wedge z) & \text{(idempotency)} \\ = & (y \wedge x) \vee (y \wedge z) & \text{(commutativity)} \\ = & y \wedge (x \vee z) & \text{(distribution)} \end{array}$$