COMS21202: Symbols, Patterns and Signals

Problem Sheet 3: Probabilistic Models

1. You were consulted by a Physics student who is trying to estimate the voltage (V) given current (I) and resistance (R) information. The student informs you that the physical model is

$$I = \frac{V}{R} + \epsilon$$

subject to a measurement error $\epsilon \sim \mathcal{N}(0, \sigma^2)$. Assuming *i.i.d* observations, the likelihood of $p(D|V)$ where V is the model's only parameter and D is the observed data is equal to:

$$p(D|V) = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(I_i - \frac{V}{R_i})^2}{\sigma^2}}$$

where I_i is the current value for observation i and R_i is the resistance value for observation i. Prove, using the Maximum Likelihood Estimation (MLE) recipe, that $V_{ML} = \sum \frac{I_i}{R_i} / \sum \frac{1}{R_i^2}$

Answer:

Take the natural logarithm $\ln p(D|V) = N \ln \frac{1}{\sqrt{2\pi}\sigma} + \sum_i \frac{(I_i - \frac{V}{R_i})^2}{-2\sigma^2}$

Take the derivative $\frac{d}{dV} \ln p(D|V) = -\frac{1}{\sigma^2} \sum_i \frac{1}{R_i}(I_i - \frac{V}{R_i})$

Equate the derivative to 0 $-\frac{1}{\sigma^2} \sum_i \frac{1}{R_i}(I_i - \frac{V}{R_i}) = 0$

$\sum_i \frac{I_i}{R_i} - V_{ML} \sum_i \frac{1}{R_i^2} = 0$

$V_{ML} = \sum \frac{I_i}{R_i} / \sum \frac{1}{R_i^2}$

2. Suppose that X is a discrete random variable with the following probability mass function, where $0 \leq \theta \leq 1$ is a parameter.

X	0	1	2	3
$P(X)$	$\frac{2\theta}{3}$	$\frac{\theta}{3}$	$\frac{2(1-\theta)}{3}$	$\frac{(1-\theta)}{3}$

The following 10 independent observations were taken from this distrubution:

3	0	2	1	3	2	1	0	2	1

(a) What is the Maximum Likelihood estimate of θ

(b) Assume you have prior knowledge that $p(\theta) = b\,\theta\,(1-\theta)$, what would the Maximum a Posteriori (MAP) be?

Answer:

(a) Following the MLE recipe,

$$\begin{aligned} p(D|\theta) &= \prod_i P(d_i|\theta) \\ &= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2 \\ &= c\,\theta^5\,(1-\theta)^5 \text{(where c is a constant)} \end{aligned}$$

Take the natural log, so

$$lnp(D|\theta) = \ln c + 5\ln\theta + 5\ln(1-\theta)$$

Take the derivative

$$\frac{d}{d\theta}\ln p(D|\theta) = \frac{5}{\theta} - \frac{5}{1-\theta}$$

Set the derivative to 0

$$\frac{5}{\theta_{ML}} - \frac{5}{1-\theta_{ML}} = 0$$

$$5(1-\theta_{ML}) - 5\theta_{ML} = 0$$

$$\theta_{ML} = \frac{1}{2}$$

(b) *When introducing prior,*

$$p(D|\theta)p(\theta) = \left(\frac{2\theta}{3}\right)^2\left(\frac{\theta}{3}\right)^3\left(\frac{2(1-\theta)}{3}\right)^3\left(\frac{1-\theta}{3}\right)^2 b\theta(1-\theta)$$

$$= c\,\theta^6\,(1-\theta)^6 \textit{(where c is a constant)}$$

Following the same formula as in (a), θ_{MAP} *= 0.5 (same as* θ_{ML} *for this particular case)*