COMS10003 Work Sheet 19

Linear Algebra: Vectors and the Dot Product

Andrew Calway March 11, 2015

This worksheet is about vectors and the dot product. Much of it is straightforward if you are careful and you have probably seen some of it before. The problems deal with some of the basics about vectors and it is important that you get a good understanding of this before we move on. Have fun and don't forget to make good use of all the resources online. As before, some questions have been taken, or adapted, from my bookshelf books:

Theory and problems of linear algebra by Seymour Lipschutz, McGraw-Hill,1981.

Linear Algebra and Probability for CS Applications by Ernest Davis, CRC Press, 2012.

Coding the Matrix by Philip N Klein, Newtonian Press, 2013.

1. For the vectors $\mathbf{u} = (2, -1, 3, -2)$ and $\mathbf{v} = (4, -1, 0, 1)$, determine the following, where '.' denotes the dot product:

   (a) 2**u**-3**v**

   (b) **u**.**v**

   (c) **u**.**u**

   (d) **v**.**v**

   (e) **u**.**v**/|**u**|

   Answer:

   (a) $2\mathbf{u} - 3\mathbf{v} = (-8, 1, 6, -7)$

   (b) $\mathbf{u}.\mathbf{v} = (2 * 4 + 1 * 1 + 3 * 0 - 2 * 1) = 7$

   (c) $\mathbf{u}.\mathbf{u} = (2^2 + 1 + 3^2 + 2^2) = 18 = |\mathbf{u}|^2$

   (d) $\mathbf{v}.\mathbf{v} = (4^2 + 1 + 0 + 1) = 18 = |\mathbf{v}|^2$

   (e) $\mathbf{u}.\mathbf{v}/|\mathbf{u}| = 7/\sqrt{18}$

2. Prove that $\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|cos\theta$ where θ is the angle between the vectors **u** and **v**. Hint: geometry and cosine rule.

   Answer:

   Using cosine rule:

   $|\mathbf{v} - \mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|cos\theta$

$|\mathbf{v}-\mathbf{u}|^2 = \sum_i (v_i - u_i)^2 = \sum_i v_i^2 + u_i^2 - 2u_i v_i = |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2\mathbf{u}.\mathbf{v}$
Hence $\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|cos\theta$

3. Use the dot product to find the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ in Q1. **Answer:** $\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|cos\theta \rightarrow \theta = acos(7/18) = 67.4^o$

4. Determine the value of the scalar a such that the magnitude of the difference vector $\mathbf{d} = a\mathbf{u} - \mathbf{v}$ is minimised, where $\mathbf{u} = (1, 1, 3)$ and $\mathbf{v} = (1, 4, 5)$.

   Answer: $a = \mathbf{u}.\mathbf{v}/|\mathbf{u}|^2 = (1 + 4 + 15)/11 = 20/11$

5. For the 3-D vectors $\mathbf{x} = (3, 1, 2)$ and $\mathbf{y} = (2, 2, 1)$, determine another 3-D vector which is orthogonal to $\mathbf{x}$ and in the same 2-D plane as $\mathbf{x}$ and $\mathbf{y}$. Use the dot product to show that the vector is orthogonal to $\mathbf{x}$.

   Answer: *Project* $\mathbf{y}$ *onto direction of* $\mathbf{x}$ *to find difference vector*

   Difference vector $= \mathbf{y} - (\mathbf{y}.\mathbf{x})\mathbf{x}/|\mathbf{x}|^2 = \mathbf{y} - 10\mathbf{x}/14 = (2 - 30/14, 2 - 10/14, 1 - 20/14)$

   Dot product is zero: 6-90/14+2-10/142+2-40/14=0

6. Now do the same for the vectors in Q1, i.e. determine a vector which is orthogonal to $\mathbf{u}$ and in the same 2-D plane as $\mathbf{u}$ and $\mathbf{v}$. Again, now use the dot product to show that the vector is orthogonal to $\mathbf{x}$.

   Answer: *Project* $\mathbf{v}$ *onto direction of* $\mathbf{u}$ *to find difference vector*

   Difference vector $= \mathbf{v} - (\mathbf{v}.\mathbf{u})\mathbf{u}/|\mathbf{u}|^2 = \mathbf{v} - 7\mathbf{u}/18 \approx (3.22, -0.61, -1.17, 1.78)$

   Dot product is zero: $2 * 3.22 + 0.61 - 3 * 1.17 - 2 * 1.78 \approx 0$

7. Determine, if possible, the values of the scalars a_1 and a_2 such that the sum $a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ is equal to the following vectors, where $\mathbf{v}_1 = (2, -1, 1)$ and $\mathbf{v}_2 = (-3, 1, 2)$

   (a) $(13, -5, -4)$,

   (b) $(3, -1.5, 1.5)$ and

   (c) $(6, -2, -3)$.

   Answer:

   $(13, -5, -4) = a(2, -1, 1) + b(-3, 1, 2)$ *–¿ 3 eqns in 2 unknowns –¿* $a = 2$, $b = -3$

   $(3, -1.5, 1.5) = 1.5(2, -1, 1)$ *by observation*

   $(6, -2, -3) = a(2, -1, 1) + b(-3, 1, 2)$

   $2a - 3b = 6$

   $-a + b = -2$

   $a + 2b = -3$

   inconsistent eqns - no linear combination (out of space)

8. Prove the following properties of the dot product.

   (a) $\mathbf{v}.\mathbf{u} = \mathbf{u}.\mathbf{v}$ - commutative

   (b) $(\mathbf{v} + \mathbf{u}).\mathbf{w} = \mathbf{v}.\mathbf{w} + \mathbf{u}.\mathbf{w}$ - distributive

   Answer: *do the algebra*

9. The n components of the n-D vector $\mathbf{p}$ correspond to the population of each county. The n components of the n-D vector $\mathbf{q}$ correspond to the average income in each county. Determine expressions involving **only** the dot product and division operators for (a) the total income of everyone in the country and (b) the average income across the whole country. You may wish to make use of the vector $\mathbf{1}$ whose components are all equal to 1.

   Answer: $\mathbf{p}.\mathbf{q}$ *and* $\mathbf{p}.\mathbf{q}/\mathbf{p}.\mathbf{1}$

10. In a simple recognition problem, 'objects' are represented by a 2-D feature vector whose components represent some characteristic of the object. Consider the case in which there are two different object classes (A and B) and we want to design a simple classifier so that we can recognize to which class an unknown test object belongs based on its 2-D feature vector $\mathbf{f}$. Based on a labelled set of feature vectors from both classes, i.e. we know to which class each feature vector belongs, we find that the average feature vector for class A is $\mathbf{u} = (1, 2)$ and for class B is $\mathbf{v} = (3, 1)$. One of the simplest forms of classifer is a linear classifier in which the projection of the feature vector of the test object onto a 'weight' vector $\mathbf{w}$ is compared to a threshold, t. For example, if $\mathbf{f}.\mathbf{w} > t$ then it belongs to class A, otherwise it belongs to class B (or vice versa). For this type of classifier and given the two class problem described, determine suitable values for t given the following two possible weight vectors: (a) $(1, 1)$ and (b) $(-1, 1)$. Compare the two classifiers and indicate which one you think would perform best, giving reasons. *Hint: you may benefit from sketching the vectors in the 2-D plane.*

    Answer: *Project means onto each weight vector: (a) A:3 and B:4, hence* $t = 3.5$ *(b) A:1 and B:-2, hence* $t = -0.5$. *Second classifier is better as separates the means better.*