COMS10003 Work Sheet 21

Linear Algebra: Matrices

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1. For the following matrices A, B and C, and vector $\mathbf{v}$

$$A = \begin{bmatrix} 3 & 1 & 5 \\ -4 & 1 & 0 \\ 2 & 4 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 8 & -3 \\ 5 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 \\ 6 & 3 \\ -4 & -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Find:

(a) $A\mathbf{v}$ (c) BA (e) A^TB^T (g) AC (i) A^2
(b) AB (d) $\mathbf{v}\mathbf{v}^T$ (f) $(A+B)\mathbf{v}$ (h) $\mathbf{v}^TC$

Answer:

$$A\mathbf{v} = \begin{bmatrix} 6 \\ -6 \\ -12 \end{bmatrix} \quad AB = \begin{bmatrix} -2 & 30 & -5 \\ 1 & -31 & 11 \\ 34 & 14 & -16 \end{bmatrix} \quad BA = \begin{bmatrix} -35 & -3 & 23 \\ 9 & 2 & 31 \\ -8 & 3 & -16 \end{bmatrix}$$

$$\mathbf{v}\mathbf{v}^T = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \quad A^TB^T = (BA)^T = \begin{bmatrix} -35 & 9 & -8 \\ -3 & 2 & 3 \\ 23 & 31 & -16 \end{bmatrix} \quad (A+B)\mathbf{v} = \begin{bmatrix} -12 \\ -4 \\ -15 \end{bmatrix}$$

$$AC = \begin{bmatrix} -20 & 1 \\ 14 & -1 \\ 44 & 20 \end{bmatrix} \quad \mathbf{v}^TC = [-18 \quad -6] \quad A^2 = \begin{bmatrix} 15 & 24 & 45 \\ -16 & -3 & -20 \\ 2 & 30 & 46 \end{bmatrix}$$

2. Determine the rank of the following matrices (by observation):

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 2 & 6 & -4 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Answer:

(a) rank is 2 since eg col3=col1-col2 so linearly dependent. col 1 and col2 are independent

(b) rank is 1 since all 3 cols are dependent

(c) rank is 2 as cols are independent

Hence for the following linear systems determine whether a solutions exists or not, and if so, how many (again, by observation)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 2 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Answer:

(a) no solutions - vector is outside col space

(b) infinite number of solutions - vector is multiple of all 3 dependent cols

(c) 1 solution $= (1, 2)$

3. Find the matrices which corresponds to the following linear transformations in $\mathbf{R}^2$:
   (a) A projection onto the vector (1,0).
   (b) A counterclockwise rotation through an angle θ followed by a projection onto the vector (1,0).
   (c) Multiplication by a scalar k followed by a counterclockwise rotation through 90^o

   For (b) and (c), are the matrices the same if the order of the two transformations in each case are reversed?

   Answer:

$$(a) \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (b) \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix} \qquad (c) \quad \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}$$

   Order matters for (b) but not for (c).

4. Find the matrix representing the linear transformation which projects a vector in $\mathbf{R}^2$ onto the vector $(\cos\theta, \sin\theta)$.

   Answer:

   Determine projection of (1,0) and (0,1) onto vector $(\cos\theta, \sin\theta)$*, giving*

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

5. Find the 3×3 matrices which correspond to the following linear transformations
   (a) projection of a vector onto the x-y plane

(b) reflection of a vector through the x-y plane

(d) counterclockwise rotation of a vector by an angle θ around the y-axis.

Answer:

$$(a) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (b) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (c) \quad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

6. Prove that $(AB)^T = B^T A^T$. Hint: Let A and B be of size $m \times n$ and $n \times p$, respectively, and represent them by column vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_m$ and $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_p$, i.e.

$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} \qquad B = [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \ldots \;\; \mathbf{b}_p]$$

Answer:

$$C = AB \quad \rightarrow \quad C_{ij} = \mathbf{a}_i^T \mathbf{b}_j \quad \rightarrow \quad C_{ij}^T = \mathbf{a}_j^T \mathbf{b}_i$$

$$D = B^T A^T \quad \rightarrow \quad D_{ij} = \mathbf{b}_i^T \mathbf{a}_j = C_{ij}^T$$

7. Determine the inverse (if it exists) of the following matrices

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$(a) \quad \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad (b) \;\; \textit{determinant zero - no inverse} \qquad (c) \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Let A and B be invertible matrices of the same size. Show that the product AB is also invertible with inverse $B^{-1}A^{-1}$. Hence by induction show that $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$.

Answer: $ABB^{-1}A^{-1} = AIA^{-1} = I$, *which also generalises to products of multiple matrices.*

9. Show that: (i) if A has a row consisting of all zeros (a zero row), then the product AB also has a zero row; (ii) if B has a zero column, then AB has a zero column; and (iii) any matrix with a zero row or a zero column is not invertible.

   Answer: *(i) Let* $\mathbf{r}_i$ *be the zero row of* A *and* $\mathbf{b}_j$, $1 \leq j \leq n$, *the columns of* B, *then the* i*th row of* AB *is* $(\mathbf{r}_i.\mathbf{b}_1, \mathbf{r}_i.\mathbf{b}_2, \ldots, \mathbf{r}_i.\mathbf{b}_n) = (0, 0, \ldots, 0)$
   (ii) Let $\mathbf{c}_i$ *be the zero column of* B *and* $\mathbf{a}_j$, $1 \leq j \leq n$, *the rows of* A, *then the* i*th column of* AB *is* $(\mathbf{a}_1.\mathbf{c}_i, \mathbf{a}_2.\mathbf{c}_i, \ldots, \mathbf{a}_n.\mathbf{c}_i)^T = (0, 0, \ldots, 0)^T$
   (iii) If A *is invertible, then there exists* A^{-1} *st* $AA^{-1} = I$, *but* I *has no zero rows or columns, hence from above then neither can* A.

10. For the following matrix A, determine $B = A^n$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

   Answer:

$$B = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$$