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Preface

These are outline notes for lecture 16.

Introduction

This lecture is about the Taylor series and about the numerical solution of differential equations; two methods are considered, the Euler method and the Runge Kutta method.

The Taylor series

In the last lecture we used a series to describe a function, in that case the exponential function. In fact, it is often useful to use a series description of functions, it can help with numerical calculations of the values and it can be useful in studying properties of the function. Here, it will be used to work out how to efficiently calculate the solutions to differential equations accurately. The Taylor series is one commonly applicable approach to representing a function as a series.

Imagine you have a function $f(t)$ that can be represented as a series

$$f(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \dots \tag{1}$$

Now, putting $t = 0$ we get

$$f(0) = a_0 \tag{2}$$

Next, differentiate

$$\frac{df(t)}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + \dots = \sum_{n=1}^{\infty} a_n n t^{n-1} \tag{3}$$

so, putting $t = 0$ we get

$$\frac{df}{dt}\big|_{t=0} = a_1 \tag{4}$$

Differentiating again gives

$$\frac{d^2 f(t)}{dt^2} = 2a_2 + 6a_3 t + 12a_4 t^2 \dots = \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} \tag{5}$$

so

$$\frac{1}{2} \frac{d^2 f}{dt^2}\big|_{t=0} = a_2 \tag{6}$$

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and so on.

In fact, by this sort of calculation we see that

$$a_n = \frac{1}{n!} \left. \frac{d^n f}{dt^n} \right|_{t=0} \tag{7}$$

or, put another way,

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f}{dt^n} \right|_{t=0} t^n \tag{8}$$

This is the Taylor series. We haven't proven that $f(t)$ has a series of the form $\sum_{n=0}^{\infty} a_n t^n$ and not all functions do, in particular, if the function is badly behaved at $t = 0$ it may not. We also haven't proved that the series converges. If we write

$$f(t) = \sum_{n=0}^{N-1} \frac{1}{n!} \left. \frac{d^n f}{dt^n} \right|_{t=0} t^n + E_N(t) \tag{9}$$

where $E_N(t)$ represents the error from stopping at after N terms, we might expect $E_N(t)$ vanishes as N goes to infinity. In fact, this doesn't always happen and sometimes, even when the series does converge, it does so very slowly, so $E_N(t)$ remains large even for very large values of N. Frequently the series converges for some values of t but not for others, we will see an example of this in the worksheet. Nonetheless, the Taylor series is frequently useful.

The Taylor series - examples

We already know the series expansion of $\exp t$, but lets calculate it as a Taylor series. Since

$$\frac{d}{dt} \exp t = \exp t \tag{10}$$

we know

$$\frac{d^n}{dt^n} \exp t = \exp t \tag{11}$$

or

$$\left. \frac{d^n}{dt^n} \exp t \right|_{t=0} = 1 \tag{12}$$

for all n so

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \tag{13}$$

which is what we got before.

Next lets consider

$$f(t) = \sin(t) \tag{14}$$

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Now

$$\frac{d}{dt}f(t) = \cos(t) \tag{15}$$

and

$$\frac{d^2}{dt^2}f(t) = -\sin(t) \tag{16}$$

and so on. Putting $t = 0$ and using $\sin 0 = 0$ and $\cos 0 = 1$ we get

$$\sin(t) = \sum_{n\,\mathrm{odd}} \frac{(-1)^{(n-1)/2}t^n}{n!} \tag{17}$$

Finally, we have been expanding around $t = 0$, but you can expand around any point, here we expand around $t = t_0$

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dt^n}\right|_{t=t_0} (t - t_0)^n \tag{18}$$

or, writing $\epsilon = t - t_0$

$$f(t_0 + \epsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n f}{dt^n}\right|_{t=t_0} \epsilon^n \tag{19}$$

Numerical solutions of differential equations

Consider the differential equation

$$\frac{dy}{dt} = f(y) \tag{20}$$

This class of differential equations would include the growth equation we looked at last week:

$$\frac{dy}{dt} = ry \tag{21}$$

with $f(y) = ry$. Of course, the right hand side might also depend on t, but we'll worry about that later. Imagine we want to find numerical values for $y(t)$ where we know $y(0) = y_0$, some value, and

$$\frac{dy}{dt} = f(y) \tag{22}$$

Imagine further that we can't solve the equation analytically as we can for the growth equation, so we resort to solving it approximately on the computer. The normal approach would be to discretize time and to work out the solution approximately for each time step in turn, depending on the previous time step. In other words, say we choose δt, a small value, as the time step then we would work out $y(\delta t)$, then use that to work out $y(2\delta t)$ and so on.
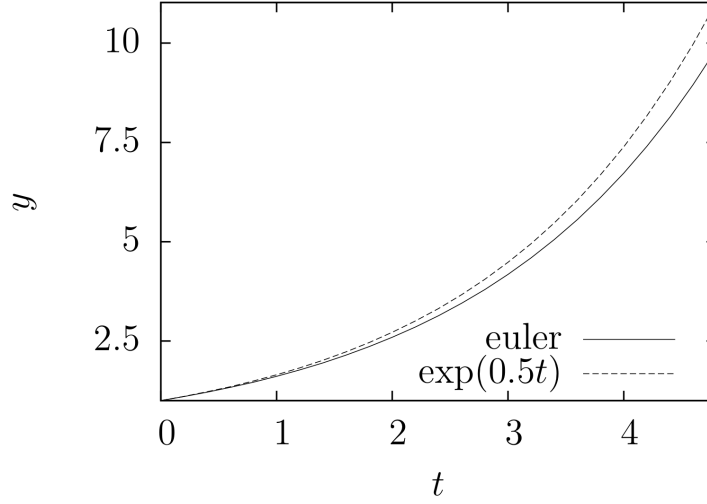
Figure 1: A comparison of the Euler method with the true solution for the growth equation. This equation in actually one where the Euler method works quite well for modest values of t, here a large time step of $\delta t = 0.2$ is used to emphasise the error, the growth rate is $r = 0.5$ and the actual solution is plotted for comparison.

Let us use the notation $y_n = y(n\delta t)$ and consider how we might get a computer to work out y_{n+1} approximately if y_n is already known. Now, by the Taylor expansion

$$y(n\delta t + \delta t) = y(n\delta t) + \left.\frac{dy}{dt}\right|_{t=n\delta t} \delta t + \frac{1}{2} \left.\frac{d^2y}{dt^2}\right|_{t=n\delta t} (\delta t)^2 + \ldots \tag{23}$$

so one simple approach is to ignore the $(\delta t)^2$ and smaller terms, since $dy/dt = f(y)$ this gives

$$y_{n+1} = y_n + f(y_n)\delta t \tag{24}$$

This approximation is known as the *Euler method*. A simple example is plotted in Fig. 1.

The Runge-Kutta method

The Runge-Kutta method uses the Taylor expansion in a clever way to find a better approximation than the euler method. It is a bit convoluted, so there is a lot of notation, but it does give a very useful numerical algorithm.

As before, we want to solve

$$\frac{dy}{dt} = f(y) \tag{25}$$

with a time discretization of δt, y_n is the approximate value the algorithm calculates for $y(n\delta t)$ and $y_0 = y(0)$, the initial condition. Now say we are at y_n and let

$$k_1 = f(y_n) \tag{26}$$

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so the Euler approximation would be $y_{n+1} = y_n + k_1\delta t$. Next, let

$$k_2 = f\left(y_n + \frac{\delta t k_1}{2}\right) \tag{27}$$

Now, using the Taylor expansion

$$k_2 = f(y_n + \delta t k_1/2) = f(y_n) + \frac{df}{dy}\bigg|_{y=y_n} \delta t \frac{k_1}{2} + \ldots \tag{28}$$

Substituting back for k_1 this gives

$$k_2 = f(y_n) + \frac{1}{2}\frac{df}{dy}\bigg|_{y=y_n} \frac{dy}{dt}\bigg|_{t=t_n} \delta t + \ldots \tag{29}$$

Using the chain rule

$$\frac{d^2y}{dt^2} = \frac{df}{dt} = \frac{df}{dy}\frac{dy}{dt} \tag{30}$$

so

$$k_2 = f(y_n) + \frac{1}{2}\frac{d^2y}{dt^2}\delta t + \ldots \tag{31}$$

Now, recall

$$y(n\delta t + \delta t) = y_n + \frac{dy}{dt}\bigg|_{t=t_n} \delta t + \frac{1}{2}\frac{d^2y}{dt^2}\bigg|_{t=t_n} \delta t^2 + \ldots \tag{32}$$

and from the formula for k_1 and k_2 we see that this can be written as

$$y_{n+1} = y_n + k_2\delta t \tag{33}$$

This is the *second order Runge Kutta method*, it is called second order because it include the first and second order terms in the Taylor expansion, the Euler method is like a first order Runge Kutta method.

The second order Runge Kutta method isn't usually used; it is the fourth order Runge Kutta that is considered the standard way of doing numerical integration. The idea is just the same as the one we saw above, by combining different terms more of the Taylor expansion is accounted for, in fact, as the name suggests, the fourth order Runge Kutta gets everything up to the fourth order, the errors are like δt^5.

Here I will give the fourth order Runge Kutta and will include the possibility that the right hand side of the differential equation also includes a dependence on t so, writing $t_n = n\delta t$

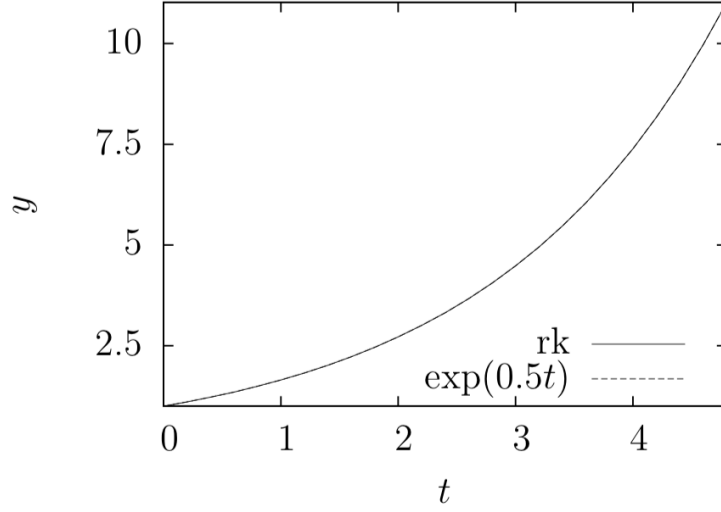$$\frac{dy}{dt} = f(t, y) \tag{34}$$

Now

$$k_1 = f(t_n, y_n)$$

Figure 2: A comparison of the fourth order Runge Kutta method with the true solution for the growth equation. This has the same values of r and δt as in Fig. 1, but the fourth order Runge-Kutta method is used instead of the Euler method. As you can see, the approximation and true solution are indistinguishable.

$$
\begin{aligned}
k_2 &= f\left(t_n + \frac{1}{2}\delta, y_n + \frac{1}{2}\delta t k_1\right) \\
k_3 &= f\left(t_n + \frac{1}{2}\delta, y_n + \frac{1}{2}\delta t k_2\right) \\
k_4 &= f\left(t_n + \delta, y_n + \delta t k_2\right)
\end{aligned}
\tag{35}
$$

and

$$
y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\tag{36}
$$