COMS10003: Workshop on Logic

Propositional Logic and Boolean Algebra

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Introduction

For this workshop you should read up on *Propositional Logic and on Boolean Algebra*, covering the following topics:

- Validity, satisfiability and contradictions;
- Logic equivalences and symbolic manipulation;
- Functional completeness of sets of connectives;
- Introduction to Boolean algebra, Boolean functions, normalisation and minimisation of circuits, and duality.

Reading will help you find solutions to the tasks in this worksheet.

Note, this worksheet contains tasks on several topics related to Propositional Logic and Boolean Algebra in no particular order. Review the worksheet. Schedule your work so that you find an answer to those parts of the worksheet that enable you to solve the rest of the questions alone.

For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Preparation

Extend your syntax/semantics reference card to include logical equivalences. This will help you remember the symbols used for the different connectives and the laws that you can use for symbolic manipulation. If necessary, add truth tables to this card for any new connectives if you find these hard to remember.

Before you start, compare the reference cards within your group. See whether you can improve your card based on what you have seen.

Task 1: Validity, Contradiction and Satisfiability

Task 1.1: For each of the following propositions, use truth tables to determine whether they are a Tautology, a Contradiction or a Contingency. Which are *satisfiable*? Explain and justify your answer.

1. $(p \Rightarrow (q \Rightarrow p))$
2. $(p \vee \neg q) \Rightarrow (r \wedge p)$
3. $\neg p \vee (\neg p \Rightarrow q)$
4. $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
5. $(\neg p \wedge \neg q) \Rightarrow (q \Leftrightarrow r)$
6. $(p \wedge q) \Rightarrow (p \vee q)$
7. $((p \wedge r) \vee (q \wedge \neg r)) \Leftrightarrow ((\neg p \wedge r) \vee (\neg q \wedge \neg r))$

Task 1.2: Which other techniques can you use to find the answers to the questions above? Where applicable, use a different technique and explain your approach.

Task 2: Logic Equivalences

Task 2.1: Demonstrate the correctness of the basic Logic Equivalences given in the table in this week's lecture slides. Practice the names of these equivalences.

Task 2.2: Can you find further logical equivalences in the literature? Bring these to the workshop, demonstrate their correctness and, if possible, give an intuitive explanation for each. Discuss these new logical equivalences in your group.

Task 3: Functional Completeness

Task 3.1: Determine whether the following sets of connectives are functionally complete:

Before you start, explain what you need to do. For each subtask, take advantage of earlier results and make sure you cover all connectives.

1. $\{\wedge, \vee, \neg\}$
2. $\{\wedge, \neg\}$
3. $\{\vee, \neg\}$
4. $\{\vee, \wedge\}$
5. $\{\uparrow\}$
6. $\{\downarrow\}$

Task 3.2: Express the following propositions using only NAND ($\uparrow$) as the logical connective:

1. $p \Rightarrow \neg q$
2. $\neg(p \oplus q)$
3. $p \wedge (q \wedge r)$

Task 3.3:

1. Is $\uparrow$ commutative?

   Explain what you need to show, then justify your answer.

2. Is $\uparrow$ associative?

   Explain what you need to show, then justify your answer.

Task 4: Boolean Algebra

Task 4.1:

1. Discuss the syntax that is used for Boolean algebra in the literature. Define the syntax you want to use as a group to answer the next question in this section.
2. List the following laws: associativity, commutativity, distributivity, identity, absorption, double negation. Where relevant, list for both conjunction and disjunction.
3. Draw truth tables for the following Boolean functions - these are deliberately given using a variety of symbols that are commonly found in textbooks on Boolean algebra:
   (a) $F = \neg A$
   (b) $F = \neg A \vee (B \wedge D)$
   (c) $F = \overline{A} \cdot \overline{B}$
   (d) $F = \overline{A + B}$
   (e) $F = (A \texttt{ || } C) \texttt{ \&\& } \sim B$
   (f) $F = \neg(D \vee B) \vee (B \vee D)$

Task 4.2:

1. Which Boolean function is represented by the following table? Write the expression in symbolic form.

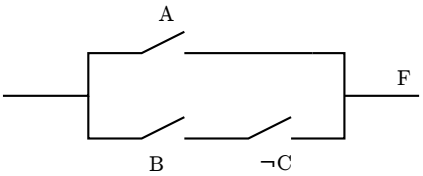
A	B	C
0	1	0
1	1	1
1	0	0
0	0	0

2. Which Boolean function is represented by the following table? Write the expression in symbolic form.

A	B	C
0	1	0
1	1	0
0	0	1
1	0	0

Task 4.3:

1. Draw the truth table for the following diagram.



2. Write the above circuit as a symbolic expression.

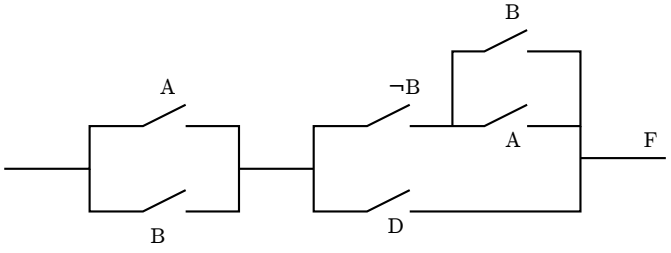
Task 4.4: Simplify the following expression: $XY + XYZ + XYZQ$. Which law(s) did you use?

Task 4.5:

1. Define Conjunctive Normal Form (CNF).
2. Can every expression be put into conjunctive normal form?
3. Identify if the following expressions are in CNF, and if they are not, normalise them.
   (a) $\neg C \wedge D$
   (b) $\neg D$
   (c) $\neg D \wedge \neg F$
   (d) $(\neg D) \wedge (\neg F)$
   (e) $\neg(D \wedge \neg F)$
   (f) $\neg(C \wedge D)$
   (g) $\neg(A \vee B)$
   (h) $(A \wedge B) \vee (C \wedge D) \vee (E \wedge F)$

Task 4.6

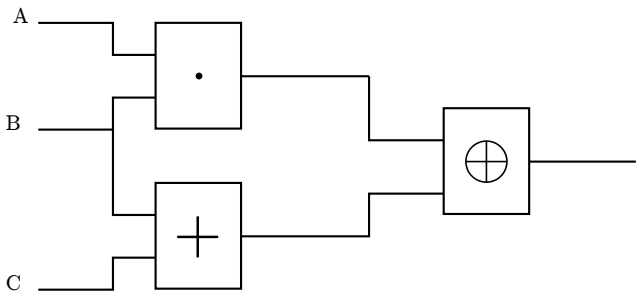
1. Write the exact symbolic equation for the circuit below.



2. Put the symbolic expression into a minimal disjunctive normal form.
3. Put the expression into a conjunctive normal form.

Task 4.7:

1. The circuit below has been designed to implement the expression
   $F = A + (A \cdot B) \oplus (B + C) \;=\; A + ((A \cdot B) \oplus (B + C))$,
   however, a mistake has been made.



   Give a combination of inputs that results in the incorrect output.
2. Give an expression for the inputs which result in incorrect output.
3. With as few modifications as possible, fix the circuit above, drawing the result.

Task 4.8: Draw as many different representations as you can find (in the literature) for the following logic gates:

1. AND
2. OR
3. NAND
4. NOT
5. XNOR

Task 4.9:

1. Define duality.
2. Give an example of a self-dual operation.
3. Give the dual of $F \vee G$.
4. Give the dual of $(A \wedge \neg B) \vee \neg C$.