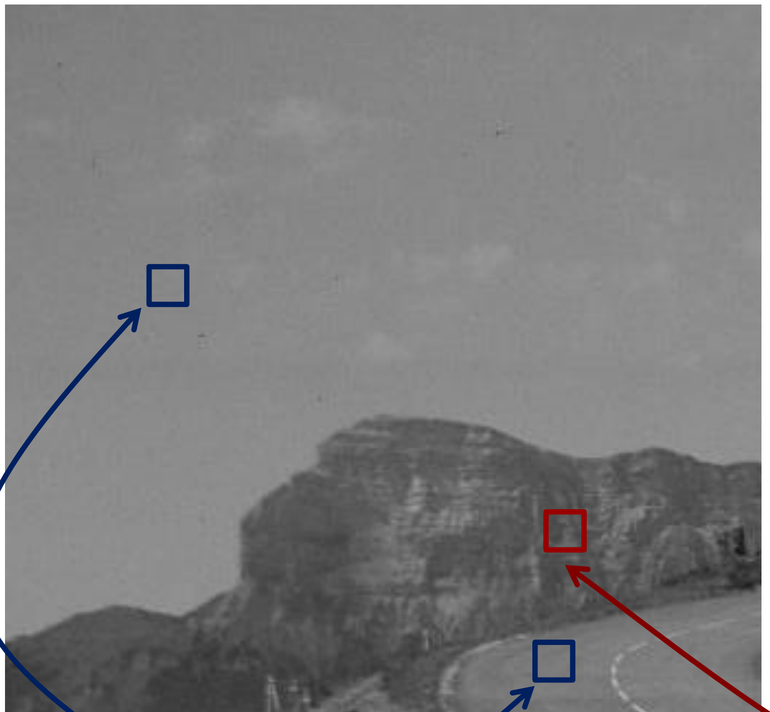
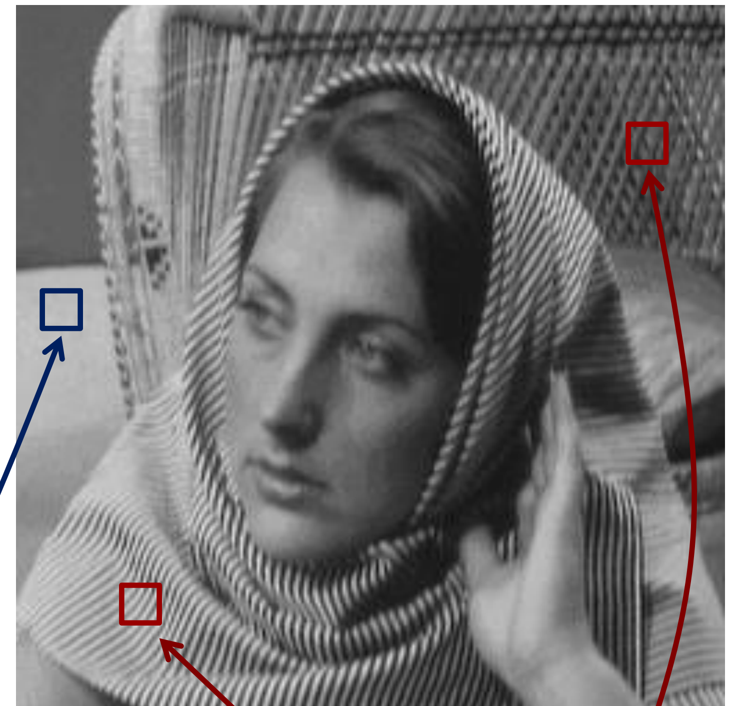
2D FT and Spatial Frequency

Fourier Transform → straightforward extension to 2D.

- Images are functions of two variables → e.g. $f(x,y)$
- Defined in terms of *spatial frequency* → 2D frequency.
- Fourier Transform is particularly useful for characterising this intensity variation across an image.
- *Rate of change of intensity* along each dimension.

Examples: Spatial Frequency





Slowly changing → low frequency

Rapidly changing →high frequency

2D Fourier Transform: Continuous Form

- The Fourier Transform of a continuous function of two variables $f(x,y)$ is:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)} dxdy$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse* Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)} dudv$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform: Discrete Form

- The FT of a discrete function of two variables, $f(x,y),\ x,y=0,1,2\ldots,N\text{-}1$, is:

$$F(u,v)=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux+vy}{N}\right)} \quad \text{for } u,v=0,1,2,\ldots,N-1.$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse FT:*

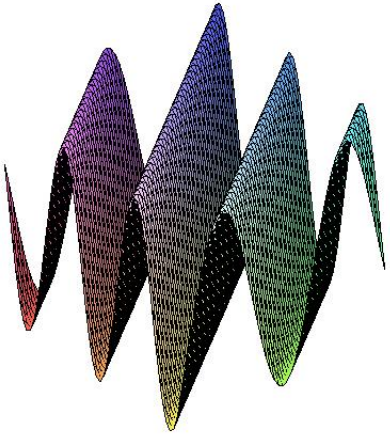
$$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi\left(\frac{ux+vy}{N}\right)} \quad \text{for } x,y=0,1,2,\ldots,N-1.$$

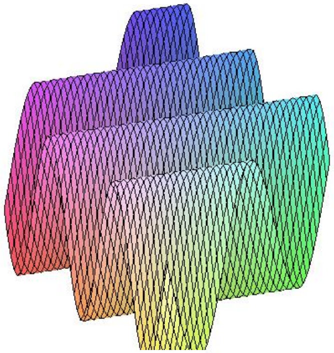
These two equations are also known as the Fourier Transform Pair.

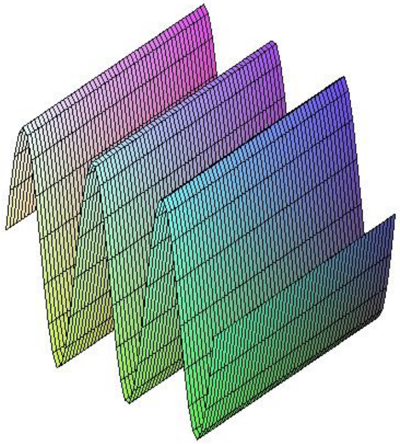
Note, they constitute a lossless representation of data.

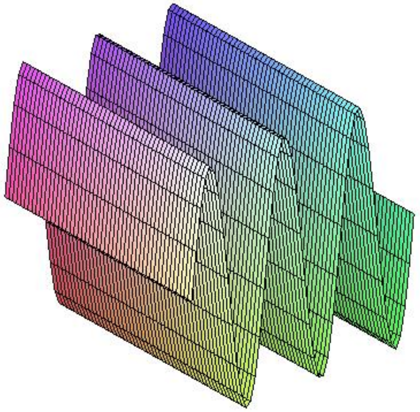
2D Fourier Decomposition

Weighted summation of 2D sines and cosines in all different directions…









2D Fourier Transform

- The concept of the frequency domain follows from Euler's Formula:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Thus each term of the Fourier Transform is composed of the sum of *all* values of the function $f(x,y)$ multiplied by sines and cosines of various frequencies:

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\cos\left(\frac{2\pi(ux+vy)}{N}\right) - j\sin\left(\frac{2\pi(ux+vy)}{N}\right)\right]$$

for $u,v = 0,1,2,\ldots, N-1.$

We have transformed from a time domain to a frequency domain representation.

2D Fourier Transform

- The concept of the frequency domain follows from Euler's Formula:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Thus each term of the Fourier Transform is composed of the sum of *all* values of the function $f(x,y)$ multiplied by sines and cosines of various frequencies:

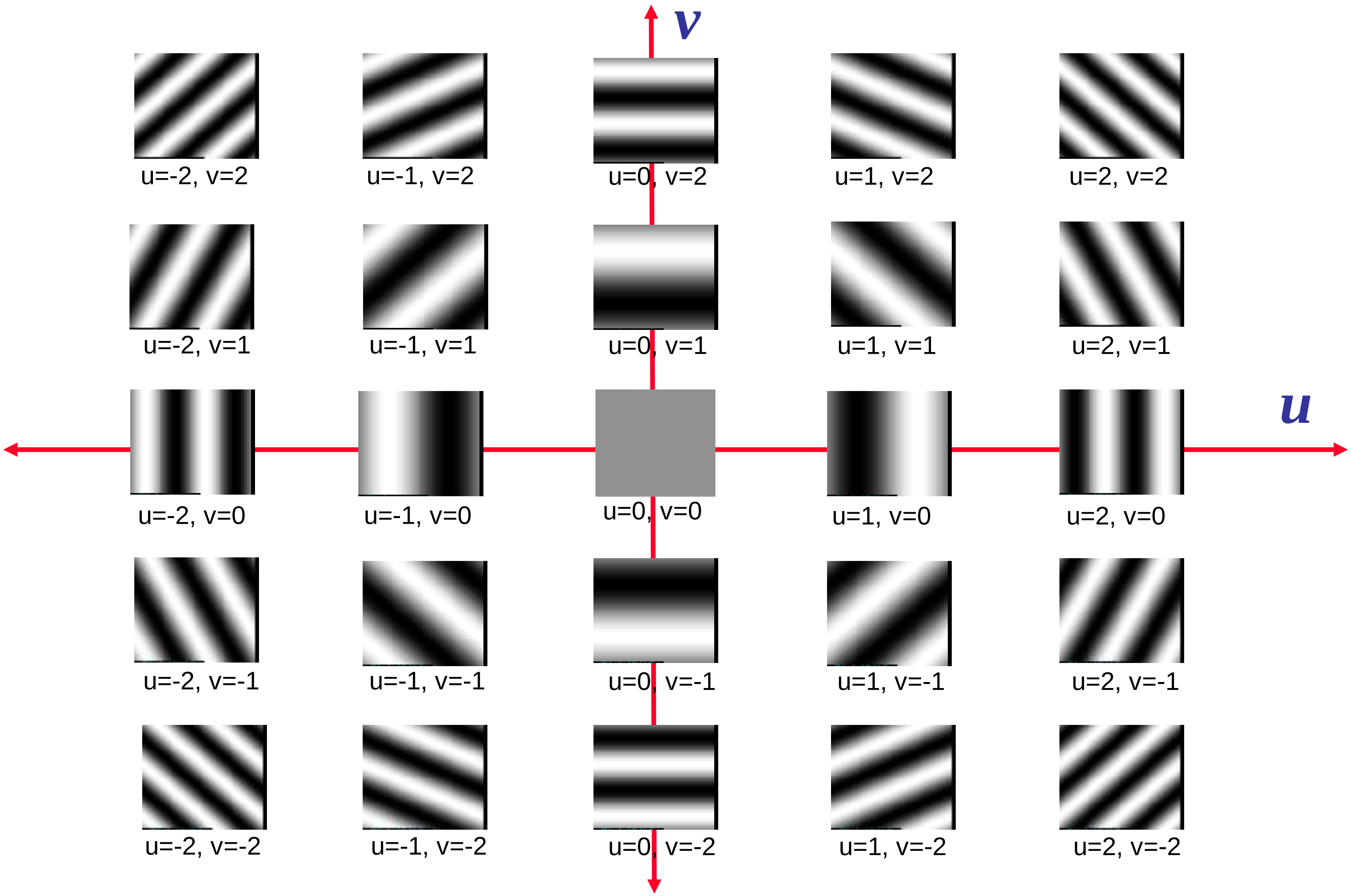
$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

The slowest varying frequency component, i.e. when $u=0,v=0$ → average image graylevel

for $u,v = 0,1,2,\ldots, N-1.$

We have transformed from a time domain to a frequency domain representation.

Another view: The 2D Basis Functions



2D Fourier Transform

- *F(u,v)* is a complex number & has real and imaginary parts:

$$F(u,v) = R(u,v) + jI(u,v)$$

- *Magnitude* or *spectrum* of the FT:

$$\left|F(u,v)\right| = \sqrt{R^2(u,v) + I^2(u,v)}$$

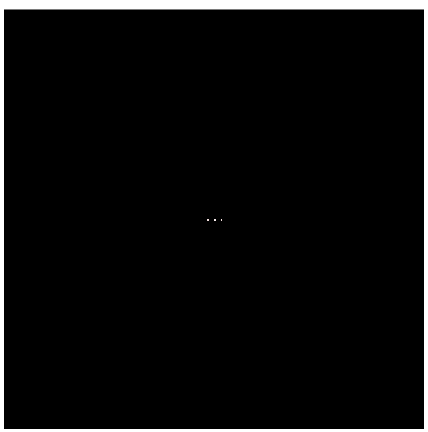
- Phase angle or phase spectrum:

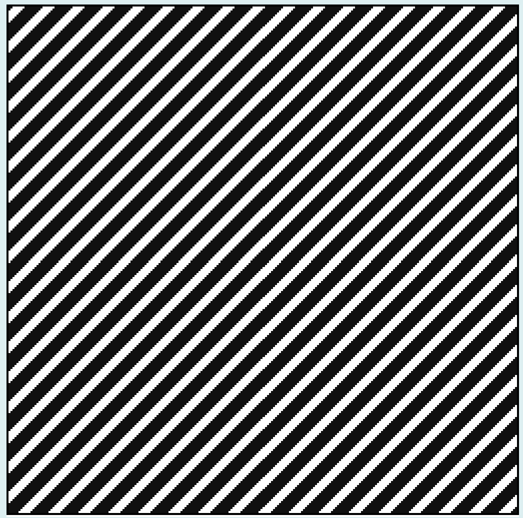
$$\phi(u,v) = \tan^{-1}\frac{I(u,v)}{R(u,v)}$$

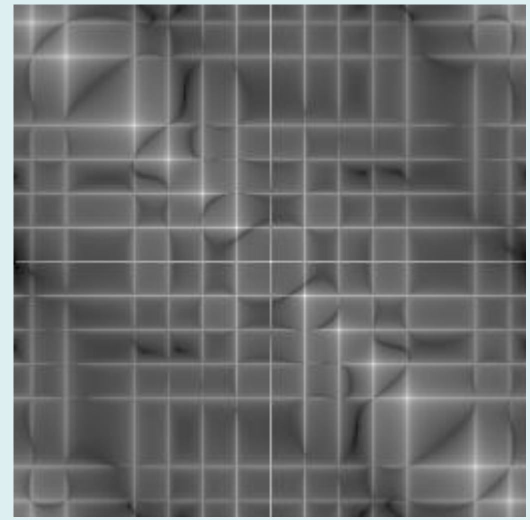
- Expressing *F(u,v)* in polar coordinates:

$$F(u,v) = \left|F(u,v)\right|e^{j\phi(u,v)}$$

Example I: Image Analysis

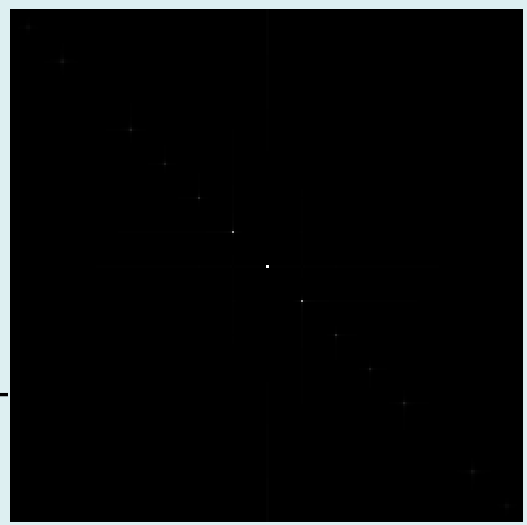


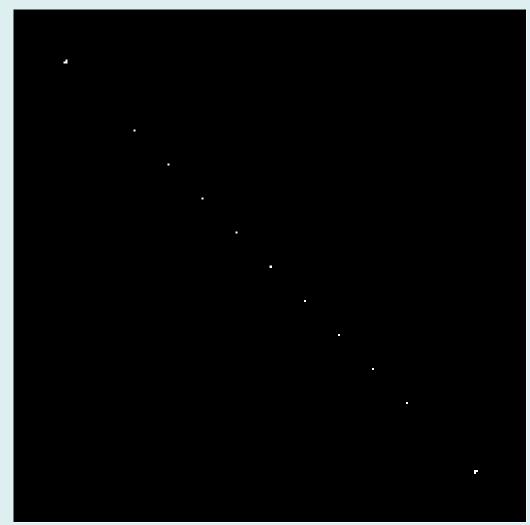




log
of FT

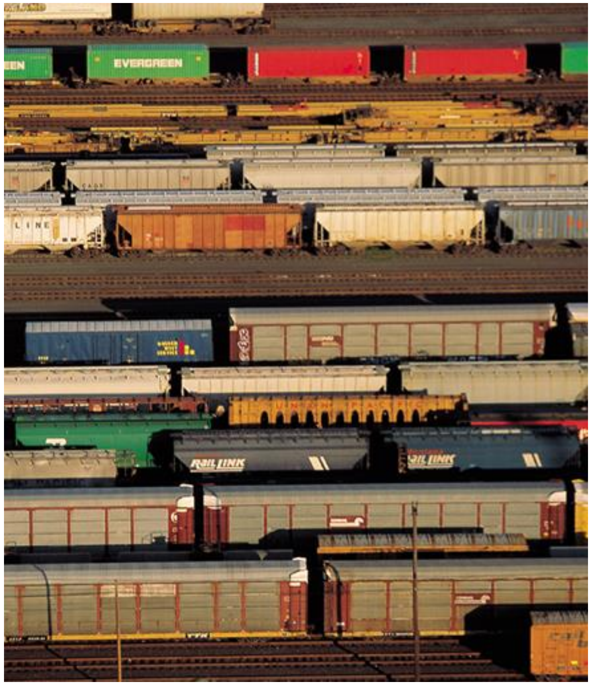
FT



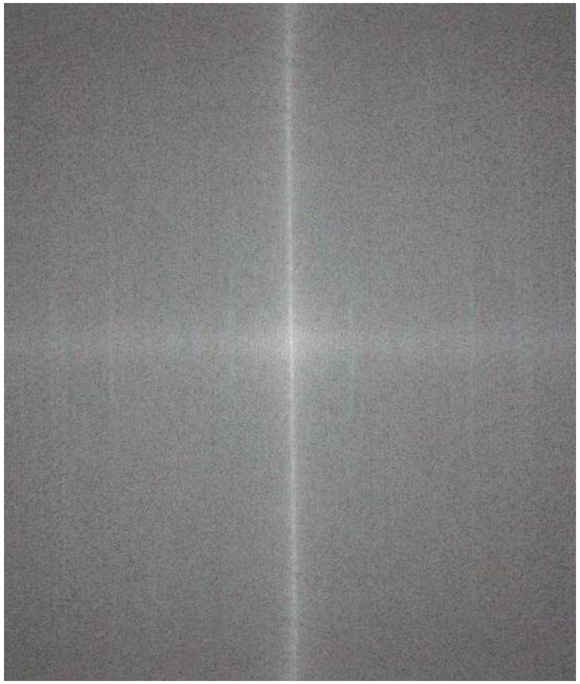


Thresholded
log of FT

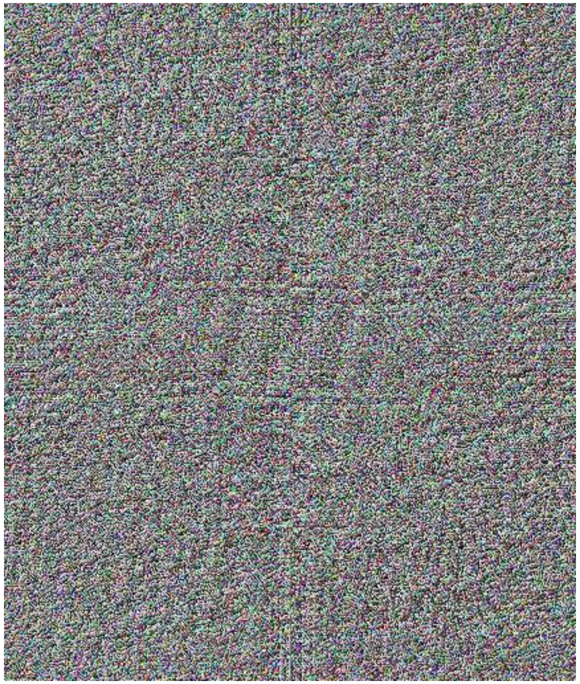
Example II: Magnitude + Phase



I

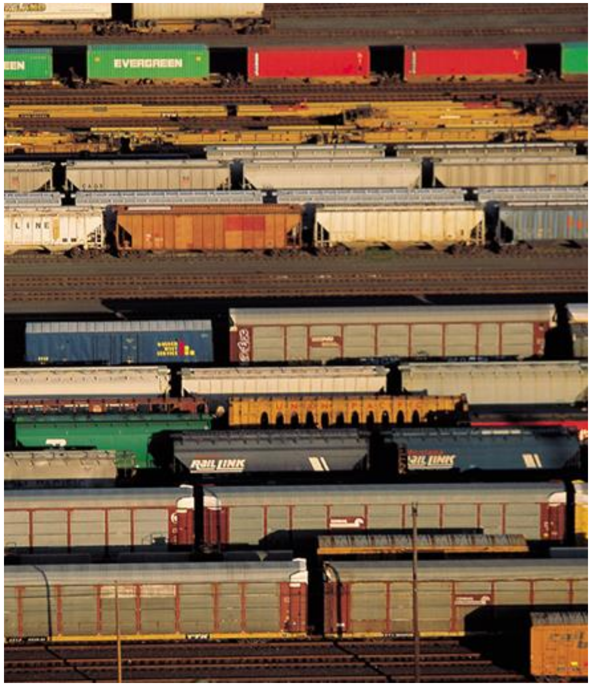


$\log(|F(I)|+1)$

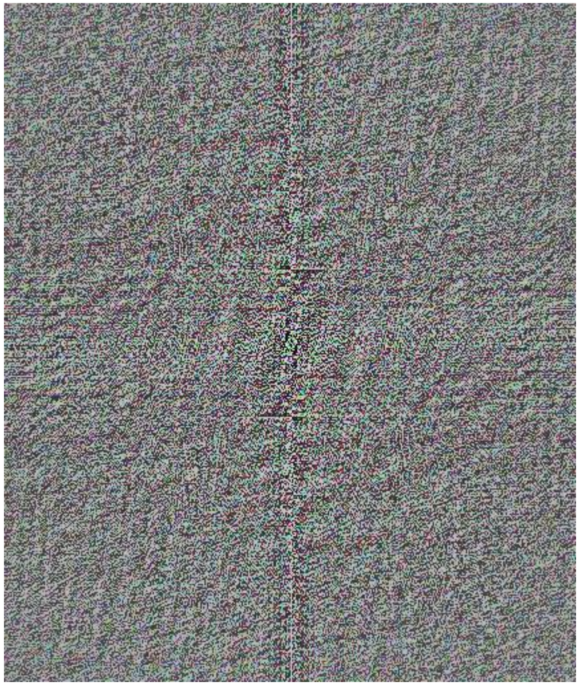


$\angle[F(I)]$

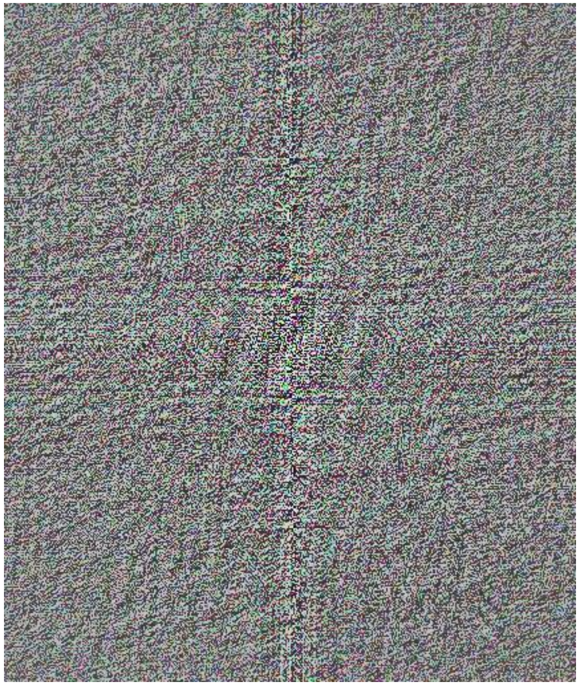
Example III: Real + Imaginary



I

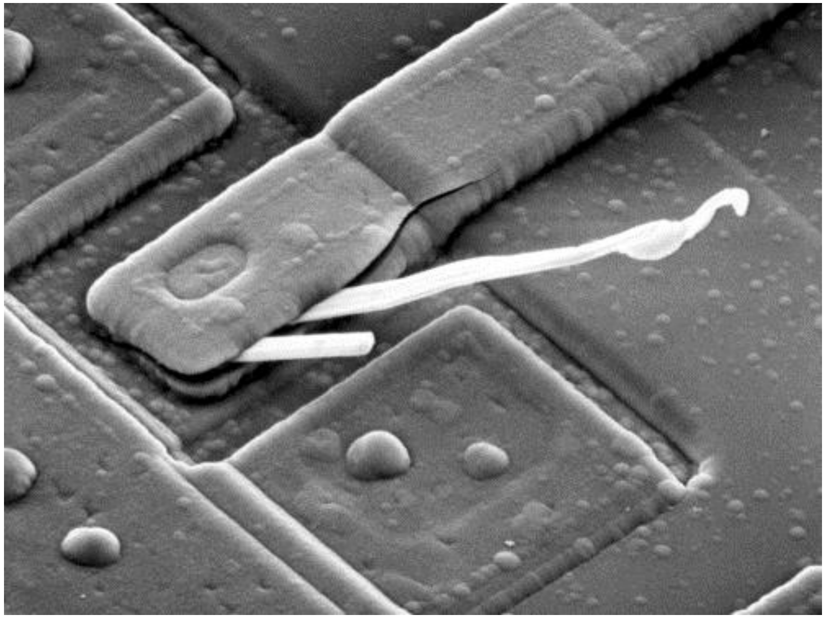


Re[F(I)]



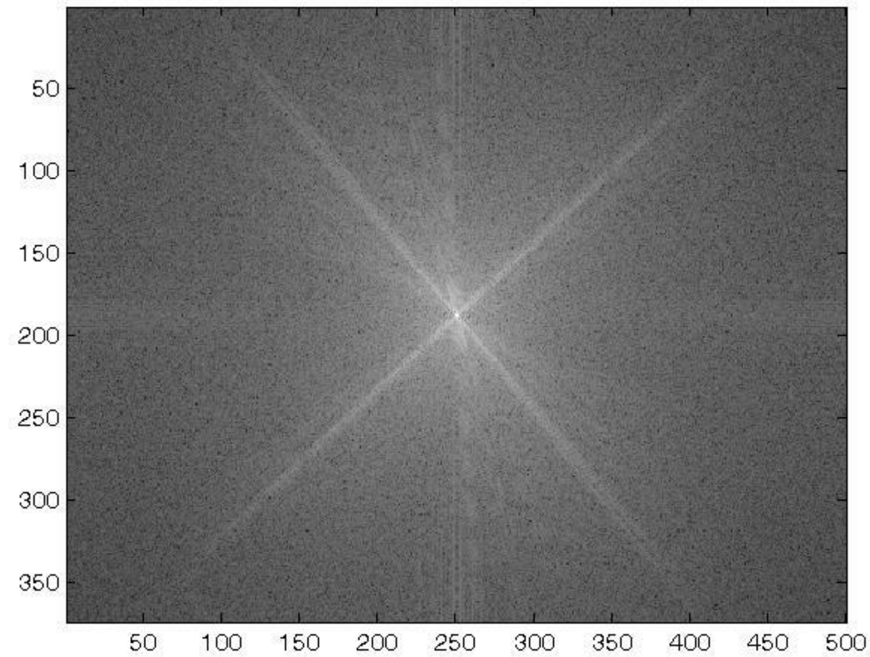
Im[F(I)]

Example IV: Interpreting the FS

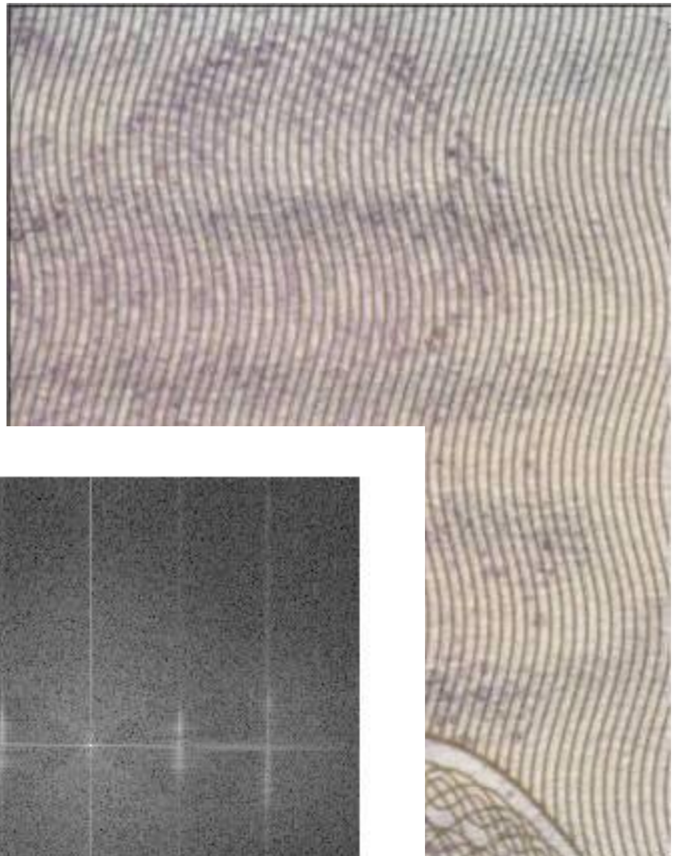


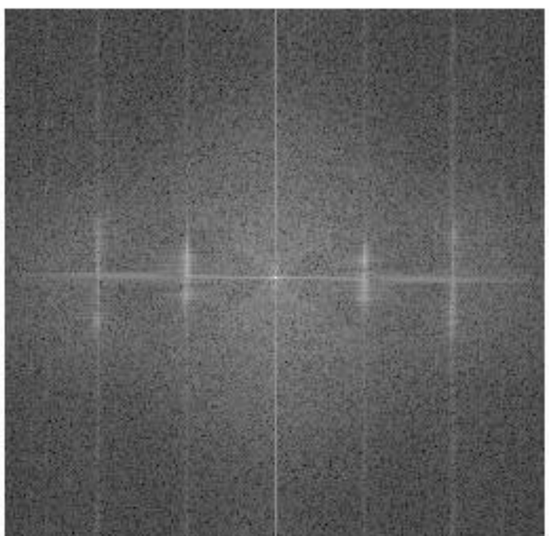
Scanning electron microscope image of an integrated circuit

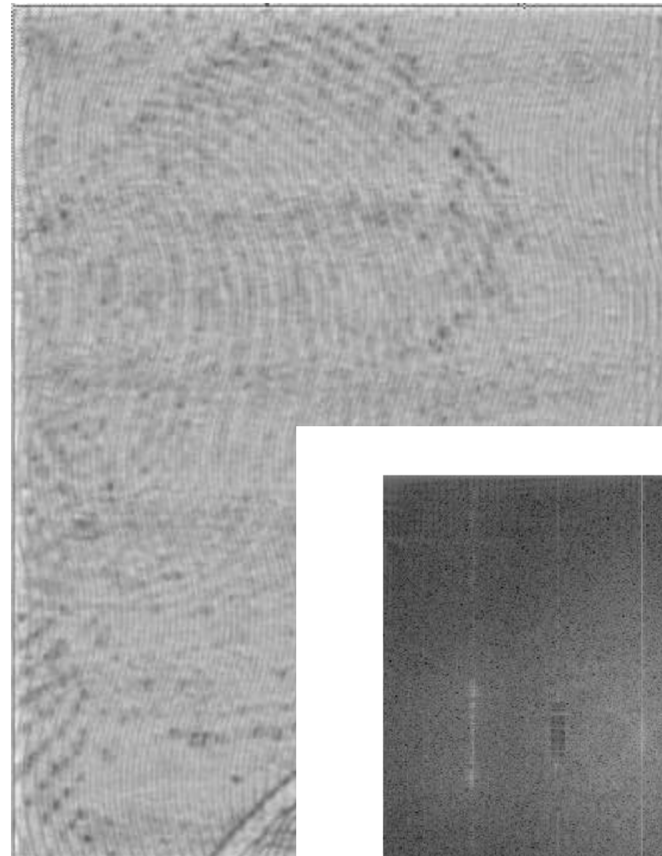
Can we interpret what the bright components mean?

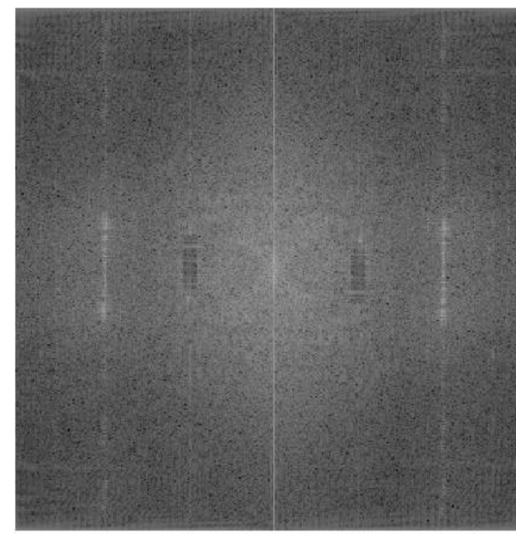


Example V: Image Analysis







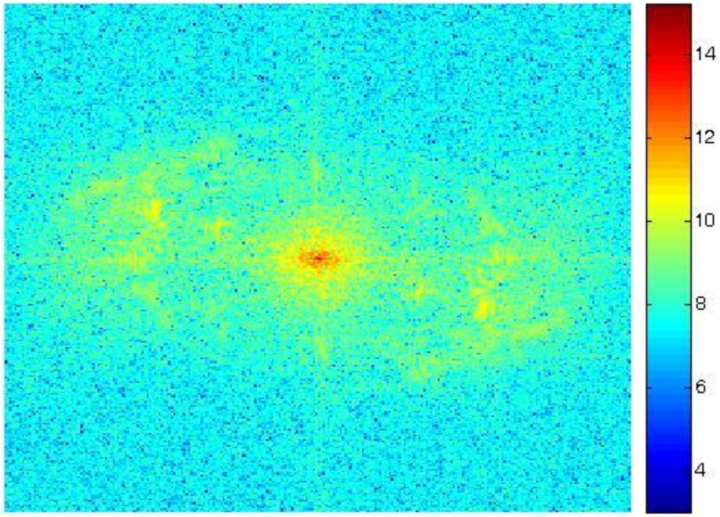


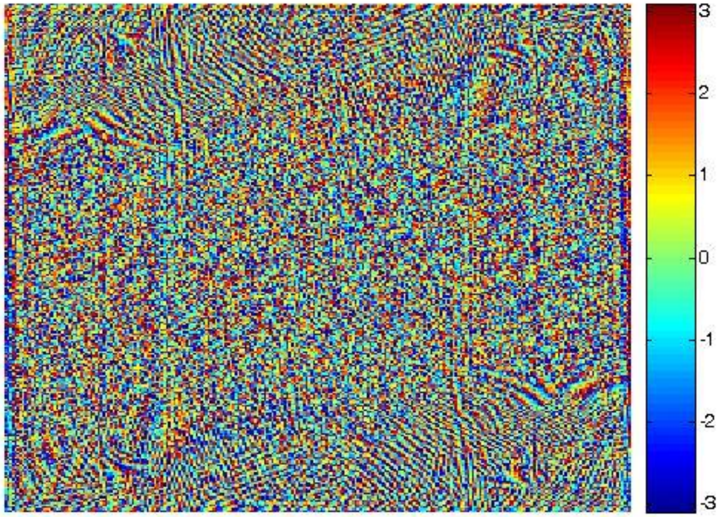
Matlab: 2D Fourier Transform

```
f = imread('barbara.gif');     %read in image
z = fft2(double(f));           % do fourier transform
q = fftshift(z);               % puts u=0,v=0 in the centre
Magq = abs(q);                 % magnitude spectrum
Phaseq=angle(q);               % phase spectrum
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usually for viewing purposes:
imagesc(log(abs(q)+1));
colorbar;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
w = ifft2(ifftshift(q));       % do inverse fourier transform
imagesc(w);
```

Matlab functions in RED

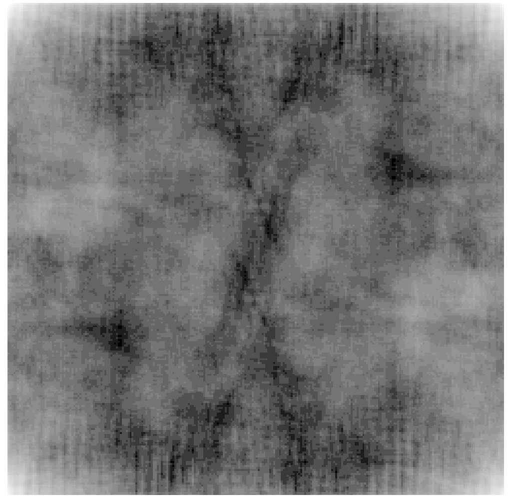
Viewing Magnitude and Phase



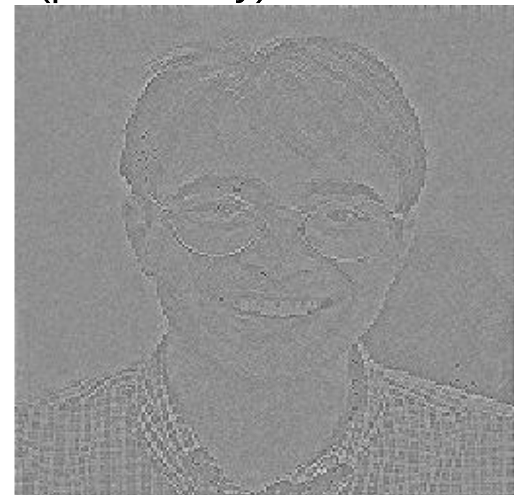


Importance of Phase

ifft(mag only)



ifft(phase only)



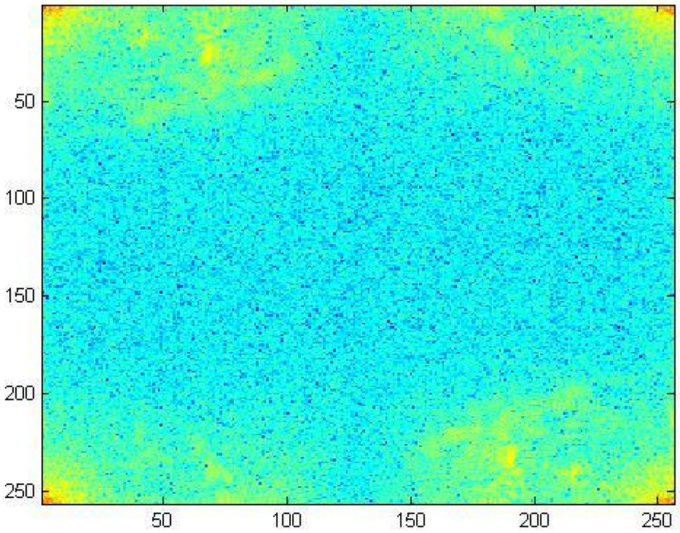
ifft(mag(Peter) and Phase(Andrew))

ifft(mag(Andrew) and Phase(Peter))

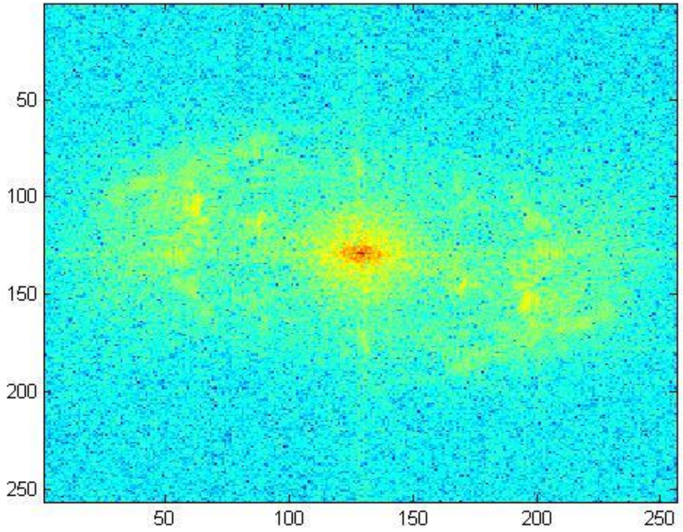
Symmetry

- Important property of the FT: *Conjugate Symmetry*
- The FT of a real function f(x,y) gives:

$$F(u,v) = F^*(-u,-v) \quad \longrightarrow \quad |F(u,v)| = |F^*(-u,-v)|$$



Before fftshift



After fftshift

Separability

- Important property of the FT: *Separability*
- If a 2D transform is separable, the result can be found by successive application of two 1D transforms.

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x,v)\, e^{\frac{-j2\pi ux}{N}} \quad \text{where} \quad F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-j2\pi vy}{N}}$$

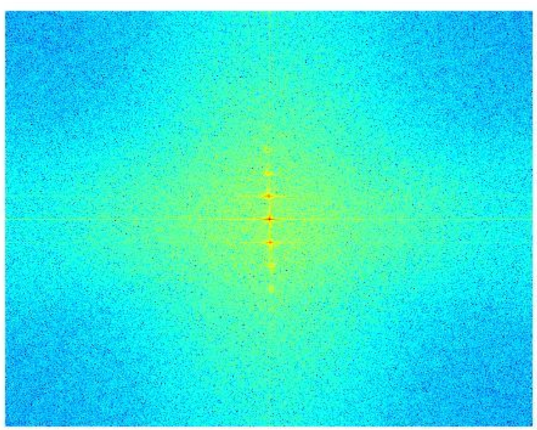
$$f(x,y) \xrightarrow[\text{1-D row transforms}]{} F(x,v) \xrightarrow[\text{1-D column transforms}]{} F(u,v)$$

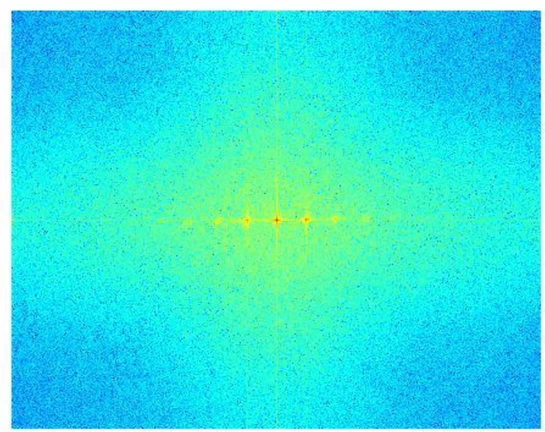
Rotation

- Important property of the FT: *Rotation*
- Rotate the image and the Fourier space rotates.

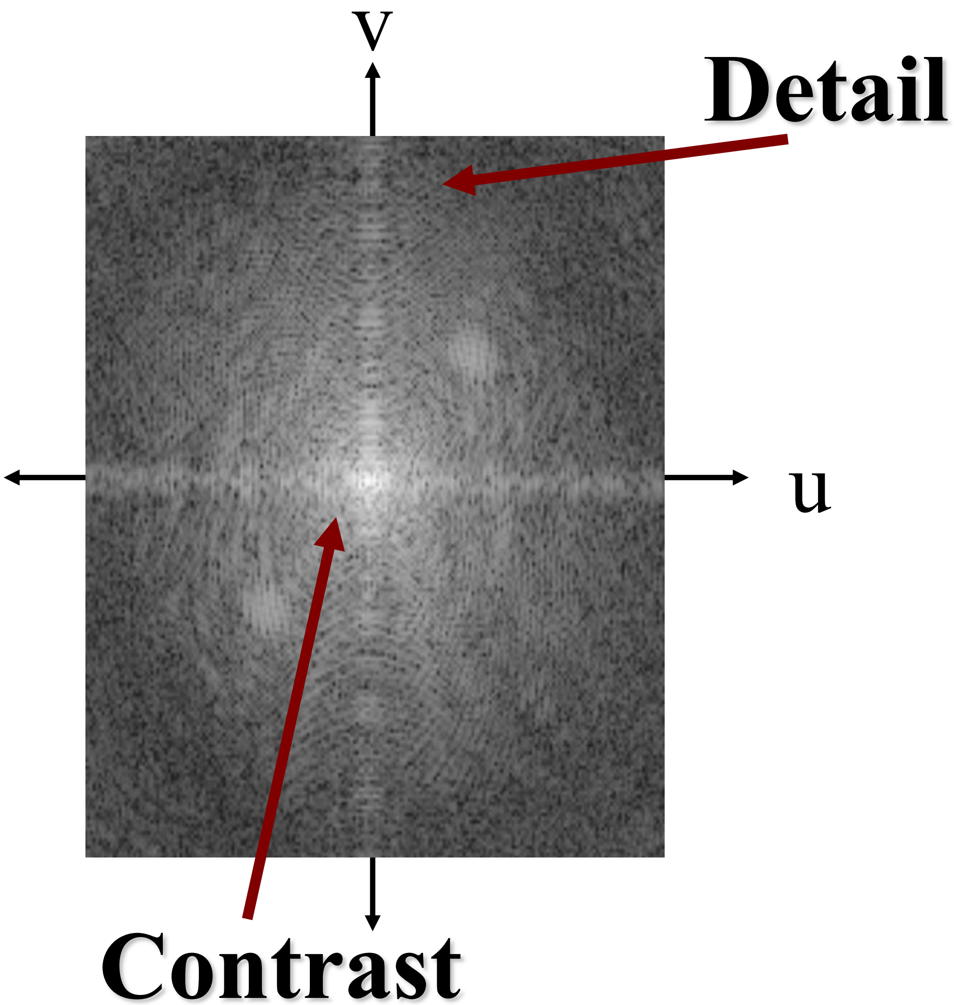
$$x = r\cos\theta \quad y = r\sin\theta \qquad u = \omega\cos\phi \quad v = \omega\sin\phi$$

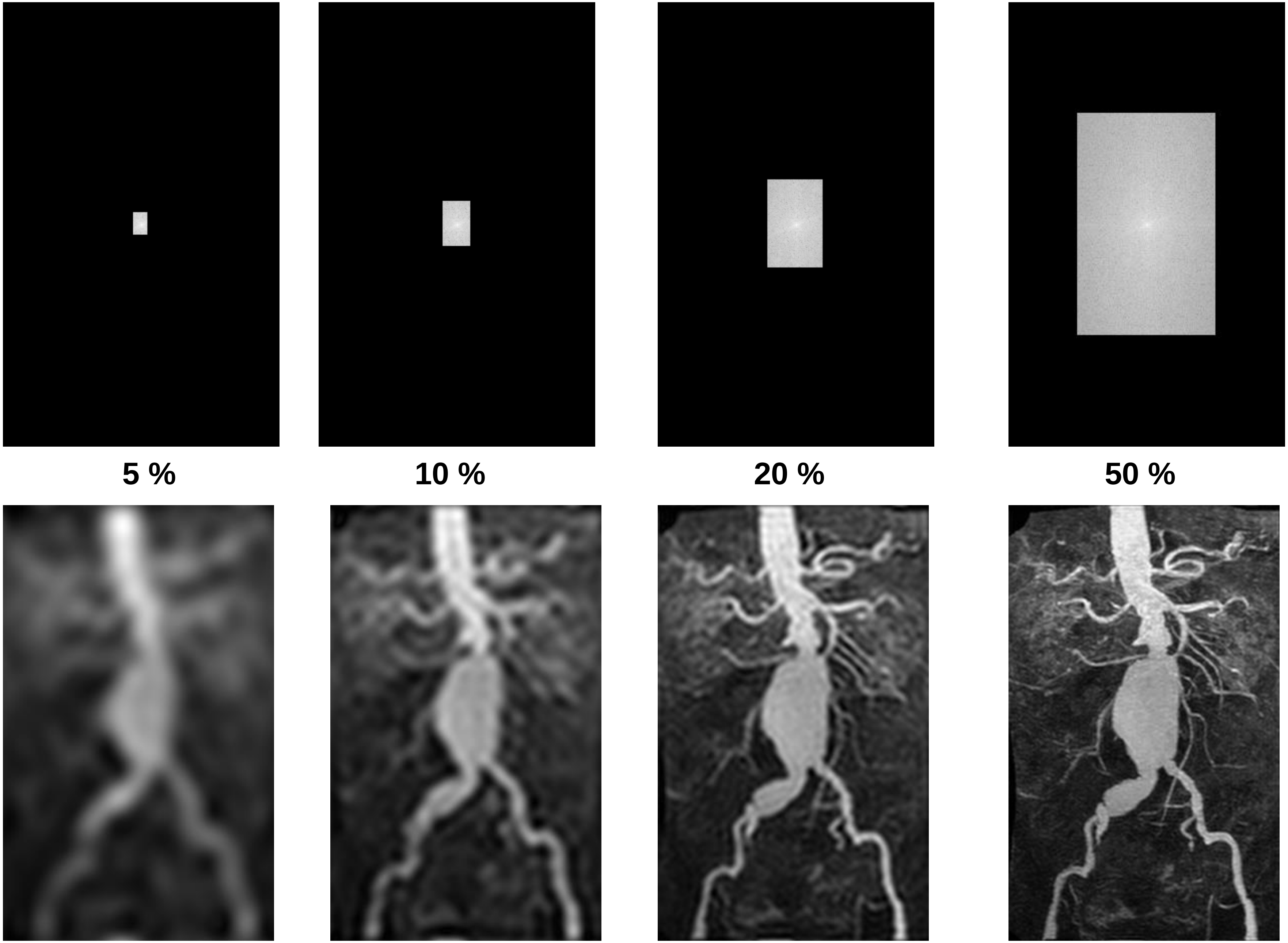
$$f(r, \theta + \theta_0) \Rightarrow F(\omega, \phi + \theta_0)$$





Manipulating the Fourier Frequencies

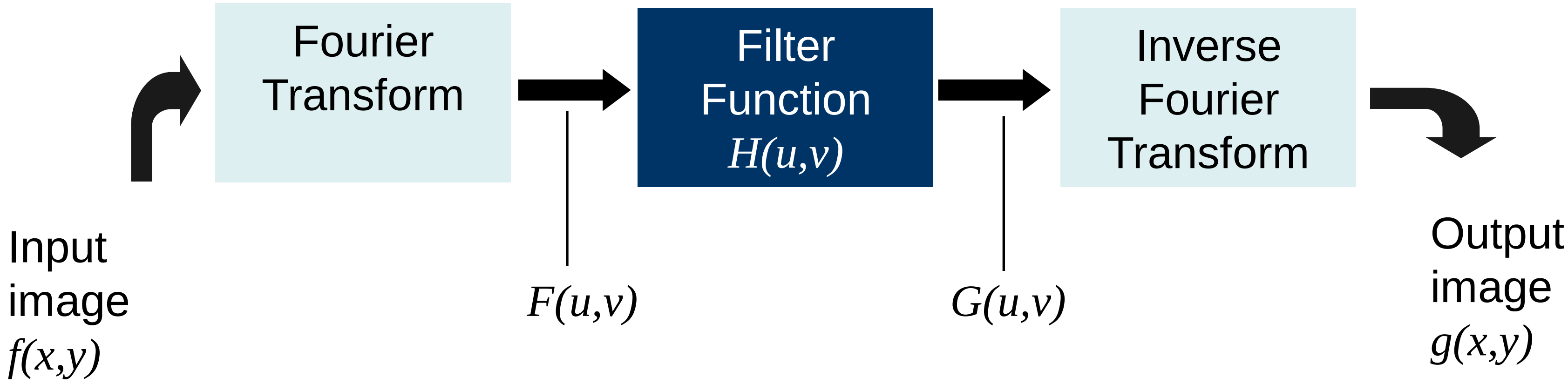



5 %
10 %
20 %
50 %

Filtering the Fourier Frequencies

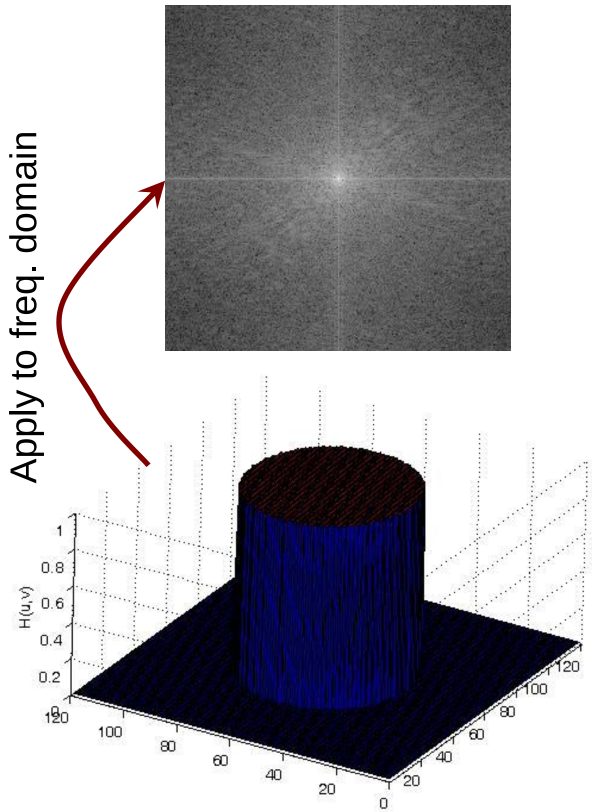
- Filtering → to manipulate the (signal/image/etc) data.

$$1\text{D}: \; G(u) = F(u)H(u) \qquad\qquad 2\text{D}: \; G(u,v) = F(u,v)H(u,v)$$



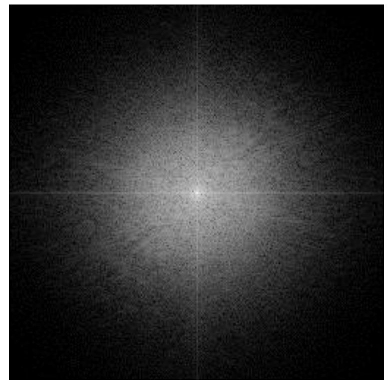
Low Pass Filtering

- 1D: turning the “treble” down on audio equipment!
- 2D: smooth image

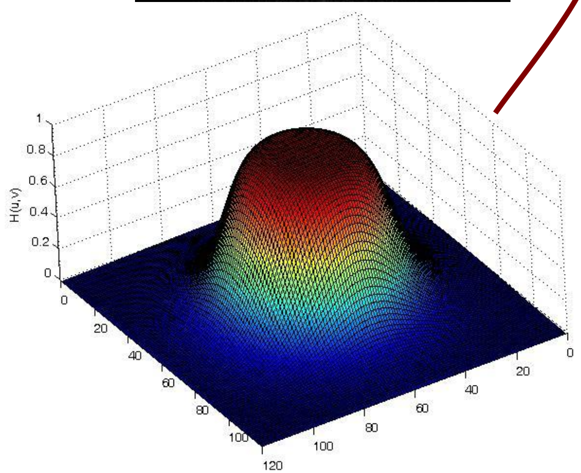


$$H(u,v)=\begin{cases} 1 & r(u,v) \le r_0 \\ 0 & r(u,v) > r_0 \end{cases} \qquad r(u,v)=\sqrt{u^2+v^2}\,,\ r_0 \text{ is the filter radius}$$

Butterworth's Low Pass Filter



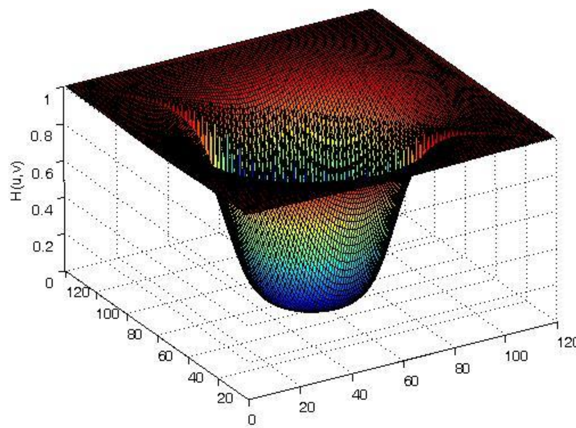
After applying to freq. domain



$$H(u,v) = \frac{1}{1 + [r(u,v)/r_0]^{2n}} \quad \text{of order } n$$

Butterworth's High Pass Filter

- 1D: turning the bass down on audio equipment!
- 2D: sharpen image



Order of $n=3$

$$H(u,v)=\frac{1}{1+[r_0/r(u,v)]^{2n}} \quad \text{of order } n$$

Filtering to Remove Periodic Noise

- This is a very common application of the FT.