COMS22202: 2015/16

Language Engineering

Dr Oliver Ray
(csxor@Bristol.ac.uk)

Department of Computer Science
University of Bristol

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5. Prove the so-called *substitution lemma* for arithmetic expressions, which states that $\mathcal{A}[\![\, a[y \mapsto a'] \,]\!]s = \mathcal{A}[\![a]\!](s[y \mapsto \mathcal{A}[\![a']\!]s])$ for all a, a', y, s.

By structural induction on the arithmetic a.

Basis Cases:

If a is a numeral n representing the integer i then $a[y \mapsto a'] = n$ and $\mathcal{A}[\![n]\!]s = i$. Thus $\mathcal{A}[\![\, a[y \mapsto a'] \,]\!]s = \mathcal{A}[\![n]\!]s = i = \mathcal{A}[\![n]\!]s' = \mathcal{A}[\![a]\!](s[y \mapsto \mathcal{A}[\![a']\!]s])$.

If a is a variable x then there are two sub-cases: either it holds that $x = y$ - in which case $a[y \mapsto a'] = a'$ and $\mathcal{A}[\![a]\!]s' = s'y$ so that $\mathcal{A}[\![\, a[y \mapsto a'] \,]\!]s = \mathcal{A}[\![a']\!]s = \mathcal{A}[\![a]\!](s[y \mapsto \mathcal{A}[\![a']\!]s])$ - or it does not - in which case $a[y \mapsto a'] = x$ and $s\,x = (s[y \mapsto i])\,x$ so that $\mathcal{A}[\![\, a[y \mapsto a'] \,]\!]s = \mathcal{A}[\![x]\!]s = s\,x = (s[y \mapsto \mathcal{A}[\![a']\!]s])\,x = \mathcal{A}[\![a]\!](s[y \mapsto \mathcal{A}[\![a']\!]s])$.

Composite Cases:

If a is a sum $a_1 + a_2$ then $\mathcal{A}[\![\, a[y \mapsto a'] \,]\!]s =$ (by defn of $[]$) $\mathcal{A}[\![\, a_1[y \mapsto a'] + a_2[y \mapsto a'] \,]\!]s =$ (by defn of $\mathcal{A}[\![]\!]$) $\mathcal{A}[\![\, a_1[y \mapsto a'] \,]\!]s + \mathcal{A}[\![\, a_2[y \mapsto a'] \,]\!]s =$ (by ind hyp) $\mathcal{A}[\![a_1]\!](s[y \mapsto \mathcal{A}[\![a']\!]s]) + \mathcal{A}[\![a_2]\!](s[y \mapsto \mathcal{A}[\![a']\!]s]) =$ (by defn of $\mathcal{A}[\![]\!]$) $\mathcal{A}[\![a_1 + a_2]\!](s[y \mapsto \mathcal{A}[\![a']\!]s]) = \mathcal{A}[\![a]\!](s[y \mapsto \mathcal{A}[\![a']\!]s])$

If a is a product $a_1 * a_2$ then an analogous argument holds.

If a is a subtraction $a_1 - a_2$ then an analogous argument also holds.