

COMS10003 Workshop Sheet 16 outline solutions.

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Work sheet

1. Revise differentiating again!

(a) Find df/dx of $f(x) = \tan x$ using $\tan x = \sin x / \cos x$.

(b) We know $\sin^2 x + \cos^2 x = 1$, differentiate both sides of this equation.

Solutions: I guess you could use the quotient rule for the first one, but I can never remember it so I'll use the product rule. First let $y = 1/\cos x$ so, using the chain rule

$$\frac{dy}{dx} = \sin x / \cos^2 x \quad (1)$$

Thus

$$\frac{d}{dx} \tan x = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad (2)$$

As for the second,

$$\frac{d}{dx} \sin^2 x = 2 \sin x \cos x \quad (3)$$

whereas

$$\frac{d}{dx} \cos^2 x = -2 \sin x \cos x \quad (4)$$

which add to zero, which is good since $d/dx 1$ is obviously zero.

2. What is the Taylor expansion of $\tan t$ up to and include the t^3 term?

Solutions: Well we already saw that

$$\frac{d}{dt} \tan t = \sec^2 t \quad (5)$$

Differentiating that gives

$$\frac{d^2}{dt^2} \tan t = 2 \sin t \cos^{-3} t = 2 \tan t \sec^2 t \quad (6)$$

where I've used the notation $\sec t = \frac{1}{\cos t}$. Next

$$\frac{d^3}{dt^3} \tan t = 2 \sec^2 t \sec^2 t + 4 \tan^2 t \sec^4 t \quad (7)$$

Hence, using $\sec 0 = 1$ and $\tan 0 = 0$

$$\tan t = t + \frac{1}{3}t^3 + O(t^4) \quad (8)$$

3. If $f(t) = \arctan t$ then

$$\frac{df}{dt} = \frac{1}{1+t^2} \quad (9)$$

This is derived using the chain rule and a trick called implicit differentiation. Basically let $y = \arctan x$ so $x = \tan y$, differentiate both sides with respect to x

$$1 = \frac{dx}{dx} = \frac{d}{dx} \tan y = \frac{dy}{dx} \frac{d}{dy} \tan y \quad (10)$$

and then do some messing to get dy/dx in terms of x . Do that!

Solutions: Well it's mostly done

$$1 = \frac{dy}{dx} \sec^2 y \quad (11)$$

and we just need to get back to something involving $\tan y$, in fact

$$\sin^2 y + \cos^2 y = 1 \quad (12)$$

when you divide by $\cos^2 y$ gives

$$\tan^2 y + 1 = \sec^2 y \quad (13)$$

so our equation is

$$1 = \frac{dy}{dx} (1 + \tan^2 y) \quad (14)$$

which gives the answer since $\tan y = x$.

4. The Taylor expansion of $\arctan x$ is

$$\arctan x = \sum_{n \text{ odd}} (-1)^{(n-1)/2} \frac{x^n}{n} \quad (15)$$

Check this as far as the x^3 term. [There is a very elegant derivation of this Taylor series, basically $1/(1+x)$ has a known Taylor expansion, up to a sign this is asked in the exercise sheet; this allows you to calculate the expansion of $1/(1+x^2)$ and from this you can read off all the derivative you need for the expansion of $\arctan x$, if you are feeling ambitious you can try to do this.]

Solutions: So

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \quad (16)$$

and using the chain rule

$$\frac{d^2}{dx^2} \arctan x = -\frac{2x}{(1+x^2)^2} \quad (17)$$

and

$$\frac{d^3}{dx^3} \arctan x = -\frac{2(1+x^2)^2 - 8x^2(1+x^2)}{(1+x^2)^4} \quad (18)$$

and substituting in $x = 0$ gives one, zero and -2, as required.

5. What is $\arctan 1$?

Solutions:

$$\arctan 1 = \pi/4 \quad (19)$$

6. Use the Taylor expansion of $\arctan 1$ to write down a formula for calculating π and do the first six terms.

Solutions: Well we have

$$\pi = 4 \sum_{n \text{ odd}} (-1)^{(n-1)/2} \frac{1}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots \quad (20)$$

and so this give 3.3396 whici isn't very close, in fact, this series approximation of π is very slow to converge, there are other, similar, series that converge much faster.

7. The series for $\arctan 1$ is disasterously slow if you want to use it to calculate π ; in fact getting ten decimal places correct requires 5,000,000,000 terms. However, there are other approaches to using the $\arctan x$ series. According to Wikipedia, in 1699, English mathematician Abraham Sharp used this series with $x = \sqrt{1/3}$ to compute π to 71 digits, breaking the previous record of 39 digits, his record only stood for seven years when a new and faster converging series for π was found; this was

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \quad (21)$$

You can prove this with clever use of trigonometric identities, you shouldn't do that here, but you should use the formula to find an approximation of π , take three terms for example. By the way, there is a graph at

http://en.wikipedia.org/wiki/File:Record_pi_approximations.svg showing how the number of known digits of π has changed over history.

Solutions: Well

$$4 \arctan \frac{1}{5} = \frac{4}{5} - \frac{4}{3} \frac{1}{125} + \frac{4}{5} \frac{1}{3125} + \dots \approx 0.789589333 \quad (22)$$

and

$$\arctan \frac{1}{239} = \frac{1}{239} - \frac{1}{3} \frac{1}{57121} + \dots \approx 0.00417826 \quad (23)$$

8. The binomial theorem tells us that

$$(1+x)^n = \sum_{r=0}^n \binom{n}{r} x^r \quad (24)$$

Prove this using the Taylor expansion.

Solutions: Just a matter of differentiating:

$$f(x) = (1+x)^n \quad (25)$$

so

$$\frac{df}{dx} = n(1+x)^{n-1} \quad (26)$$

and so on, so

$$\frac{d^r f}{dx^r} = n(n-1)(n-2)\dots(n-r+1)(1+x)^{n-r} \quad (27)$$

and so on until $n = r$, in which case this is a constant and the next term is zero, hence the Taylor expansion is

$$(1+x)^n = \sum_{r=0}^n \frac{n(n-1)\dots(n-r+1)}{r!} x^r \quad (28)$$

and this is the binomial expansion since

$$\frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \quad (29)$$

9. Find the first three terms of the Taylor expansion of $\sqrt{1+x}$. Don't bother with this one if time is short since we've already done lots.

Solutions: Ok, so

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots \quad (30)$$

10. The general second order Runge Kutta method for $\dot{y} = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + \alpha \delta t k_1) \end{aligned} \quad (31)$$

and

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2 \right] \delta t \quad (32)$$

Show that this gives the Taylor series up to second order. Note that $\alpha = 1/2$ gives the midpoint method.

Solutions: Same craic as in the lectures, expand out k_2 using the Taylor series

$$k_2 = f(y_n + \alpha \delta t k_1) = f(y_n) + \left. \frac{df}{dy} \right|_{y=y_n} \alpha \delta t k_1 + \dots \quad (33)$$

then use the chain rule

$$\frac{df}{dy} k_1 = \frac{df}{dy} \frac{dy}{dt} = \frac{df}{dt} = \frac{d^2 y}{dt^2} \quad (34)$$

so

$$k_2 = \dot{y}(t_n) + \alpha \delta t \ddot{y}(t_n) + \dots \quad (35)$$

so

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2 \right] \delta t = y(t_n) + \delta t \dot{y}(t_n) + \frac{\delta t^2}{2} \ddot{y}(t_n) + \dots \quad (36)$$

which is the first three terms of the Taylor expansion, as required.