COMS10003: Workshop on Proof

Proof Strategies

Kerstin Eder

November 9, 2014

SOLUTIONS

Introduction

This worksheet contains problems that require you to use different proof strategies, including direct proof, indirect proof, proof by contradiction, existence proofs, proof by exhaustion and proof by induction.

For some problems, more than one proof strategy can be applied. In these cases, try all of those applicable and see which leads to the most elegant, the most natural, the simplest or the shortest proof. **Always state the method of proof you are using.**

It is also important that you practice how to write up proofs. This requires you to clearly lay the proof out on paper, use proper English sentences and to justify each step explicitly.

Proposed method of working:

1. **Draft:** Outline your argument based on a proof strategy, indicate how the steps follow from each other. You don't need to pay attention to the notation at this stage. Instead, focus on the logic of the argument.
2. **Write up:** Based on the draft, write it up as a proper proof with all steps itemized. Formalize the argument using formal notation. Formalize your assumptions. Justify each step, clearly indicating how it follows from which previous steps. Use full sentences where appropriate. Do not forget to state the conclusion of your proof.
3. For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Get your write up checked over by a Teaching Assistant before you move to the next proof.

Task 1: Basic Proof Strategies

Prove or disprove the following statements. For each proof, state clearly which proof strategy you are using and justify your choice of strategy. Which other strategies work? Attempt different strategies and discuss the differences and similarities in the resulting proofs.

1. The sum of an even integer and an odd integer is an odd integer.
   Answer: *direct proof*
2. Two integers are said to have the same parity if they are both odd or both even. Show that if x and y are two integers for which $x + y$ is even, then x and y have the same parity.
   Answer: *valid, e.g. proof by contrapositive*
3. All primes are odd.
   Answer: *not valid, find a counter example*
4. For all integers n, $n^3 - n$ is even.
   Answer: *Prove for both cases, n odd and n even.*
5. The square of any odd integer has the form $8m + 1$ for some integer m.
   Answer: *Observe that any integer can be written as either $4k, 4k + 1, 4k + 2$ or $4k + 3$ for some integer k. If an integer n is odd, then $n = 4k + 1$ or $n = 4k + 3$. Then prove each case individually.*
6. For all real numbers a and b, if $a^2 = b^2$, then $a = b$.
   Answer: *not valid, counter examples exist, e.g. $a = 1$ and $b = -1$*

Task 2:

Prove that the square of an even integer is an even integer, using:

1. a direct proof,
2. an indirect proof,
3. a proof by contradiction.

Answer: *Formalization:*

P: n is even
Q: n^2 is even

Statement: $P \rightarrow Q$, i.e. for any integer n, if n is even, then n^2 is even.

1. *The direct proof is simple and elegant: n can be written as $2k$ for some integer k. Then n^2 can be written as $(2k)^2$. Now, $(2k)^2 = 4k^2 = 2(2k^2)$, which is a multiple of 2, so it is even.*

SOLUTIONS

2. *The indirect proof relies on the contrapositive, i.e.* $\neg Q \rightarrow \neg P$.
   The contrapositive would be: if n^2 *is odd, then* n *is odd.*

   First attempt: *We could rephrase the contrapositive to: The square root of an odd square number is an odd number.*
   Now, n^2, *an odd square number, can be written as* $n^2 = 2k + 1$ *for some* k. *So,*

$$\begin{aligned} n^2 &= 2k+1 \\ \sqrt{n^2} &= \sqrt{2k+1} \end{aligned}$$

   and we need to show that $\sqrt{n^2} = \sqrt{2k+1}$ *is odd. Hmm. It seems that the contrapositive is a lot harder to prove than its equivalent statement above.*

   Second attempt: *We know that* n^2 *can be written as* $2k + 1$ *for some* k. *Now,*

$$\begin{array}{rcll} n^2 & = & 2k+1 & \mid \; -1 \\ n^2 - 1 & = & 2k & \mid \; \textit{Factorize LHS} \\ (n+1)(n-1) & = & 2k & \end{array}$$

   and therefore either $(n-1)$ *or* $(n+1)$ *is divisible by 2, i.e. is even. If* $(n-1)$ *is even then* n *is odd. If* $(n+1)$ *is even then* n *is odd. Therefore* n *must be odd.*

   Note *that in some cases the contrapositive may be much easier to prove than the original statement. It is therefore often worth formulating the contrapositive to see whether it is more intuitive and hence easier to prove.*

3. *The proof by contradiction means you aim to derive a contradiction from the negation of the statement you want to prove.*

   (Note that the negation of $P \Rightarrow Q$ *is* $P \wedge \neg Q$*!)*

   The negation of our statement is: "n *is even and* n^2 *is odd", which is* $P \wedge \neg Q$. *This should lead to a contradiction.*
   An even n *can be written as* $n = 2k$ *for some* k.
   n^2, *with* $n = 2k$, *can then be written as* $(2k)^2$. *Now,* $(2k)^2 = 4k^2 = 2(2k^2)$, *which is even, i.e. we have inferred* Q *(from* $P \wedge \neg Q$*). This contradicts* $\neg Q$, *i.e that* n^2 *is odd. Therefore, our original statement holds, i.e. "if* n *is even, then* n^2 *is even".*

Task 3:

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even using:

1. a direct proof,
   Answer: $n^3 + 5 = 2k + 1$ *for some* k, *and develop to show that* n *must be even.*
2. an indirect proof,
   Answer: *Contrapositive: If* n *is odd, then* $n^3 + 5$ *is even.*
   Assume n *is odd, and show that* $n^3 + 5$ *is even.*

SOLUTIONS

$n = 2k + 1$ for some integer k (definition of odd numbers)
$n^3 + 5 = (2k + 1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$
As $2(4k^3 + 6k^2 + 3k + 3)$ is 2 times an integer, it is even.

3. a proof by contradiction.
   Answer: *We formalize as follows:*
   p stands for $n^3 + 5$ is odd, q stands for n is even.
   Assume p and $\neg q$, i.e. $n^3 + 5$ is odd, and n is odd.
   $n = 2k + 1$ for some integer k (definition of odd numbers).
   $n^3 + 5 = (2k + 1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$.
   Since $2(4k^3 + 6k^2 + 3k + 3)$ is 2 times an integer, n must be even, which is our q.
   Thus we derived a contradiction.
   Intuition: We assumed q was false, and showed that this assumption implies that q must be true. But, since q can't be both true and false, we have reached our contradiction.

Task 4:

Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Hint: Formalize this statement using a biconditional. Then prove each implication separately. Note that there are also different cases.

Answer:
$\Leftarrow$ direction:
$[(m = n) \vee (m = -n)] \Rightarrow (m^2 = n^2)$
Case 1: $(m = n) \Rightarrow (m^2 = n^2)$
$(m)^2 = m^2, and (n)^2 = n^2$, which proves this case
Case 2: $(m = -n) \Rightarrow (m^2 = n^2)$
$(m)^2 = m^2, and (-n)^2 = n^2$, which proves the other case

$\Rightarrow$ direction:
$(m^2 = n^2) \Rightarrow [(m = n) \vee (m = -n)]$
Subtract n^2 from both sides to get $m^2 - n^2 = 0$
Factor to get $(m + n)(m - n) = 0$
Since that equals zero, one of the factors must be zero
Thus, either $m + n = 0$ (which means $m = -n$) or $m - n = 0$ (which means $m = n$).

Task 5:

Prove that $n^2 = (n - 1)^2 + 2n - 1$ using:

1. direct proof,
2. induction.

SOLUTIONS

Hint: Make sure that you use the assumption in the inductive proof. If the assumpiotn is not used, the proof structure is incorrect, even though the proof result may still be correct (due to the direct proof you've used.)

Answer: *No solution is given.*

Task 6:

Prove that every positive integer greater than or equal to 2 has a prime decomposition, i.e. can be written as a product of prime numbers.

Clearly state the proof strategy you use. You may need to experiment with different proof strategies. If you get stuck, ask a Teaching Assistant to help you find a proof strategy that works.

Answer: *It may not be obvious that this proof can be done using induction. Well done if you spotted this yourself.*

Statement: $P(n)$*: Every positive integer* $n \geq 2$ *has a prime decomposition.*

Base case for $n = 2$*:* $P(2)$ *holds because* 2 *is one of the primes, it is already decomposed into primes.*

Assumption for $n = k$*:* $P(k)$*: We assume that up to some* $k \geq 2$ *all integers* $2, 3, ..., k$ *can be decomposed into primes.*

For $n = k + 1$*: To show* $P(k + 1)$ *we use a case split as follows:*

- *Either* $k + 1$ *is already a prime, in which case it is already decomposed into primes, or*
- $k + 1$ *is not a prime. In this case,* $k + 1$ *has a divisor* d *other than* 1 *and* $k + 1$*. We can therefore write* $k + 1$ *as* $k + 1 = cd$*, where* c *and* d *are divisors between* 2 *and* k*. Using our assumption, we know that* c *and* d *have prime decompositions, i.e.* $c = c_1c_2...c_i$ *and* $d = d_1d_2...d_j$*. We can use this to write* $k + 1$ *as a prime decomposition as follows:* $k + 1 = c_1c_2...c_id_1d_2...d_j$*.*

We have now shown that, for some $k \geq 2$*,* $P(k) \rightarrow P(k + 1)$*, and that* $P(2)$ *holds. Therefore, by the principle of mathematical induction,* $P(n)$ *holds for all* $n \geq 2$*.*

Task 7:

Prove that $\sqrt{2}$ is irrational. Which proof strategy did you use?

SOLUTIONS

Answer: *Proof by contradiction. Suppose $\sqrt{2}$ is rational, then $\sqrt{2} = a/b$, where a and b are whole numbers and b is not zero.*

Assume a and b have no common factors; thus at least one of a or b must be odd.

$$\begin{aligned} \sqrt{2} &= a/b \\ 2 &= a^2/b^2 \\ 2b^2 &= a^2 \end{aligned}$$

This means that a^2 is an even number, so a is also even. (Because the square of an even number is even.)

Now, $a = 2k$ for some k. Therefore $2b^2 = 4k^2$, and $b^2 = 2k^2$. This means that b is also even.

We have now shown that both a and b must be even, i.e. have 2 as common factor. This contradicts our assumption. Therefore $\sqrt{2}$ is irrational.

Task 8: Resolution

Research the foundations of resolution as a proof method. State the inference rule on which resolution is based.

Answer: $((p \vee q) \wedge (\neg p \vee r) \Rightarrow (q \vee r))$

Using resolution, prove the statement: "Sam will drive to the meeting place or Sam will get a lift." given the following facts:

- Sam will not take the bus or Sam will get unwell.
- Sam will take the bus or Sam will drive to the meeting place.
- Sam will not get unwell or Sam will get a lift.

Answer: *No solution is given.*

Task 9:

Prove that $n^4 - 1$ is divisible by 5 when n is not divisible by 5. Which proof strategy did you use?

Answer: *Proof by exhaustion. Prove for the four nonzero remainders that an integer not divisible by 5 can have.*

SOLUTIONS