COMS10003 Work Sheet 14

Random Variables II

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1. Prove that the variance of a uniformly distributed random variable with density as given below is $(U-L)^2/12$.

$$p(x) = \begin{cases} 1/(U-L) & L \leq x \leq U \\ 0 & else \end{cases}$$

Answer:

$$Var(X) = \frac{1}{U-L}\int_L^U (x-E(X))^2 dx \qquad E(X) = (U+L)/2$$

Let $z = x - E(X)$

$$Var(X) = \frac{1}{U-L}\int_{(L-U)/2}^{(U-L)/2} z^2 dz = \frac{1}{U-L}\left[\frac{z^3}{3}\right]_{(L-U)/2}^{(U-L)/2} = \frac{(U-L)^2}{12}$$

2. Widgets are mass-produced. The target diameter is 45 mm but records show that the diameters are normally distributed with mean 45 mm and standard deviation 0.05 mm. An acceptable diameter is one within the range 44.95 mm to 45.05 mm. What proportion of the output is unacceptable?

Answer:

$$X \sim N(45, 0.05^2) \qquad Z = (X-45)/0.05$$

$$z_1 = (44.95-45)/0.05 = -1 \qquad z_2 = (45.05-45)/0.05 = 1$$

Hence number of acceptable widgets is within ± 1 *standard deviation of mean,* $\equiv$ *68.3%, so proportion of unacceptable widgets is approx 31.7%.*

3. The resistance of a strain gauge is normally distributed with a mean of 100 ohms and a standard deviation of 0.2 ohms. To meet the specification, the resistance must be within the range 100 ± 0.5 ohms. What percentage of gauges are unacceptable?

Answer:

$$X \sim N(100, 0.2^2) \qquad Z - (X-100)/0.2$$

$$z_1 = (99.5-100)/0.2 = -2.5 \qquad z_2 = (100.5-100)/0.2 = 2.5$$

$$Prob\{99.5 \leq X \leq 100.5\} = \frac{1}{\sqrt{2\pi}}\int_{-2.5}^{2.5} e^{-z^2/2} dz$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-z^2/2} dz$$

$$\Phi(2.5) \approx 0.49379 \qquad \rightarrow \qquad Prob\{99.5 \leq X \leq 100.5\} \approx= 2 \times 0.49379 = 0.98758$$

Hence percentage of unacceptable gauges is 100-98.758=1.242%

4. The Department estimates that 75% of the students taking undergraduate courses are in favour of studying of random variables as part of their studies. Use this estimate along with a normal distribution approximation to the binomial distribution to determine the probability that more than 780 students out of a random sample of 1000 will be in favour of studying random variables.

   Answer: *Let prob of student liking RVs be* $p = 0.75$. *Not liking is then* $q = 1 - p = 0.25$. *Hence approximate* $B(1000, 0.75)$ *by normal distribution with mean and variance*

   $$\mu = np = 1000 \times 0.75 = 750 \qquad \sigma^2 = npq \approx 13.7^2$$

   Hence let $X \sim N(750, 13.7^2)$ *represent number of students liking RVs. Now convert to standard form:*

   $$Z = (X - 750)/13.7 \qquad z_1 = (781 - 750)/13.7 = 2.26 \qquad z_2 = (1000 - 750)/13.7 = 18.25$$

   Hence

   $$Prob\{X > 780\} \approx 0.5 - \Phi(2.26) \approx 0.5 - 0.48809 \approx 0.012$$

   NB: We have not covered the 0.5 correction that is sometimes applied on the upper and lower values of the range

5. Overbooking of passengers on intercontinental flights is a common practice among airlines. Aircraft which are capable of carrying 300 passengers are booked to carry 320 passengers. If 10% of passengers who have a booking fail to turn up for their flights, what is the probability that at least one passenger who has a booking, will end up without a seat on a particular flight? Use the normal distribution approximation to the binomial distribution to obtain your answer.

   Answer: *Let* X *be RV denoted number of passengers who turn up with a booking. Let* $p = 0.9$ *be probability that people who have booking and turn up and let* $q = 0.1$ *be probability that those with booking do not turn up.* X *is from binomial distribution with* $n = 320$ *and* $p = 0.9$. *Hence approximate with* $Y \sim N(320 \times 0.9, 320 \times 0.9 \times 0.1) = N(288, 28.8)$. *We require Prob{ Y¿300}, hence using standard form:*

   $$Z = (Y - 288)/\sqrt{28.8} \approx (Y - 288)/5.37$$

   $$z_1 = (300 - 288)/5.37 = 2.23; \qquad z_2 = (320 - 288)/5.37 = 5.96$$

   Hence

   $$Prob\{Y > 300\} \approx 0.5 - \Phi(2.23) \approx 0.5 - 0.48713 = \approx 0.01$$

6. Over a period of 12 hours 180 requests are made to a web server at random. What is the probability that in a four-hour interval the number of requests is between 50 and 70? Use the normal distribution approximation to the binomial distribution to obtain your answer.

Answer: *Let* $p = 4/12 = 1/3$ *be probability that a request is received in a 4 hour interval. Let* X *be RV representing the number of requests received in 4 hour interval. We approximate* X *as being from* $N(180/3, 180/3 \times 2/3) = N(60, 40)$ *and then convert to standard form*

$$Z = (X - 60)/\sqrt{40} \quad z_1 = (50 - 60)/\sqrt{40} = -\sqrt{2.5} \quad z_2 = (70 - 60)/\sqrt{40} = \sqrt{2.5}$$

Hence $Prob\{50 < X < 70\} \approx 2\Phi(\sqrt{2.5}) \approx 2\Phi(1.58) = 2 \times 0.44295 = 0.8859.$