COMS10003: Workshop on Proof

Propositional Logic and Inductive Proof

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November 2, 2014

Introduction

In preparation for this workshop you've been asked to read up on *proof techniques in general and, in particular, on inductive proof.*

Note, this worksheet contains tasks on several topics related to Proof in Propositional Logic and Inductive Proof. Review the worksheet. Schedule your work so that you find an answer to those parts of the worksheet that enable you to solve the rest of the questions alone.

For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Task 1: Proof in Propositional Logic

Task 1.1: Valid Arguments Determine whether the following arguments are valid.

Hint: First identify all propositions, then formalise the argument. Follow the approach demonstrated in the first lecture on proof.

Answer: *An argument is valid if the conjunction of the premises P_i imply the conclusion C, i.e.*

$$P_1 \land P_2 \land \ldots \land P_n \Rightarrow C$$

is a tautology.

a) To be great is to be misunderstood.
She is great.
She is misunderstood.

Answer:

P_1 : *To be great is to be misunderstood.*
P_2 : *She is great.*
C : *She is misunderstood.*

Let A stand for "being great" and let B stand for "being misunderstood". Use these to build up P_1 etc. Show that $((A \Rightarrow B) \land A) \Rightarrow B$ is a tautology. You can use a truth table to do this, or, alternatively, you can use symbolic manipulation and show that the expression is logically equivalent to True. To gain more practice you should try both!

b) If Frank does not go running regularly he will not be able to complete the race.
Frank did not complete the race.
Frank did not go running regularly.

Answer:

P_1 : *If Frank does not go running regularly he will not be able to complete the race.*
P_2 : *Frank did not complete the race.*
C : *Frank did not go running regularly.*

With appropriate formalisation (similar to what was discussed in the lectures, you will find that $((\neg A \Rightarrow B) \land B) \Rightarrow \neg A$ is not a tautology. Note that a single counter example (i.e. an assignment of truth values to the propositions that demonstrates that the expression is not a tautology) is sufficient to show this.

c) n is odd or n is even.
If n is odd, then $2n$ is even.
If n is even, then $2n$ is even.
$2n$ is even.

Answer:

P_1 : *n is odd or n is even.*
P_2 : *If n is odd, then $2n$ is even.*
P_3 : *If n is even, then $2n$ is even.*
C $\quad$ *$2n$ is even.*

A valid argument; follow the approach from above. Again, practice the use of a truth table and try symbolic manipulation.

Task 1.2 Rules of Inference We have already used **Modus Ponens** as an inference rule. Find other inference rules and explain their intuitive meaning. In particular, make sure you include the following in your discussion:

- Modus Tollens

$$\begin{array}{c} \neg q \\ p \Rightarrow q \\ \hline \therefore \neg p \end{array}$$

- Conjunction introduction and conjunction elimination

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

and

$$\begin{array}{c} p \wedge q \\ \hline \therefore p \end{array}$$

- Disjunction introduction

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$$

- Hypothetical and disjunctive syllogism

$$\begin{array}{c} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore p \Rightarrow r \end{array}$$

and

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Are the following arguments valid? If so, which inference rules have been used, if not, why not? Discuss. Use this opportunity to remember the names of the inference rules.

1. The taxi is either red or green. The taxi is not red, therefore the taxi is green.
   Answer: *Disjunctive syllogism*
2. If I can't make it tomorrow, then I will call you this evening. I have not called you this evening. Therefore I can make it tomorrow.
   Answer: *Modus tollens*
3. Speeding is illegal. Therefore it is prohibited by law.
   Answer: *INVALID! Circular reasoning!*
4. If I can't make it tomorrow, then I will call you this evening. I can make it tomorrow. Therefore, I won't call you this evening.
   Answer: *INVALID! Fallacy of denying the hypothesis*
5. Logic is fun and it is sunny today. Therefore, logic is fun.
   Answer: *Conjunction elimination (simplification)*
6. If I go to London tomorrow, then I can't pick you up tomorrow. If I do not pick you up tomorrow, then grandma will pick you up. Therefore, if I go to London tomorrow, then grandma will pick you up tomorrow.
   Answer: *Hypothetical syllogism*
7. It is sunny today. Therefore, it is either sunny or raining today.
   Answer: *Disjunction intro (addition)*
8. Whenever it rains, I go by car. I went by by car yesterday. Therefore, it rained yesterday.
   Answer: *INVALID! Fallacy of affirming the conclusion*
9. If it rains, then I'll go by car. It rains today, therefore I go by car today.
   Answer: *Modus ponens*

Task 1.3 Logical Reasoning Which conclusions can you draw from the following statements? Clearly explain each step in each argument. Justify your reasoning. State the rules of inference used.

In particular, state whether or not you can get help with unloading the car?

"It is not snowing this morning. We will go skiing only if it is snowing. If we do not go skiing, then, if it is cold we will stay in. If we are in, then we can help you unload the car when you arrive. It is very cold today."

Answer: *Yes, you get help with unloading the car. The arguments should be clearly laid out and each step justified.*

Task 2: Inductive Proof

Task 2.1: Use inductive proof to prove the following properties $P(n)$ for all $n \in \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, ...\}$:

1.

$$P(n) = \sum_{i=1}^{n} (2i - 1) = n^2 \tag{1}$$

Hint: First try for a few examples to convince yourself that it works. Once you've gained confidence, proceed to drafting the proof.

Answer:

Base case for $n = 1$*:* $P(1) = \sum_{i=1}^{1} (2i - 1) = 1 = 1^2$

Assume for some k, $n = k$*:* $P(k) = \sum_{i=1}^{k} (2i - 1) = 1 + 3 + ... + (2k - 1) = k^2$

Show for $n = k + 1$*:*

$$P(k+1) \quad = \quad \sum_{i=1}^{k+1} (2i - 1) = (k+1)^2 \quad | \textit{ substitute into Equation 1}$$

It is important to note that $P(k+1)$ *can also be written as* $1 + 3 + ... + (2k - 1) + (2(k+1) - 1)$*. This is the expression we start from in our proof of* $P(k+1)$ *following from* $P(k)$*.*

$$\begin{aligned} P(k+1) &= \sum_{i=1}^{k+1} (2i - 1) \\ &= \underline{1 + 3 + ... + (2k - 1)} + (2(k+1) - 1) \quad | \textit{ replace with assumption } P(k) \\ &= k^2 + (2(k+1) - 1) \\ &= k^2 + (2k + 2 - 1) \\ &= k^2 + (2k + 1) \\ &= (k+1)^2 \end{aligned}$$

Thus, we have shown that if $P(k)$ *then* $P(k+1)$ *for some* k*, and that* $P(1)$ *holds. Therefore, by the principle of mathematical induction,* $P(n)$ *holds for all* $n \in \mathbb{N}^+$*.*

2.

$$P(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \tag{2}$$

Answer:

Base case for $n = 1$*:* $P(1) = \sum_{i=0}^{1} 2^i = 2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$

Assume for some k, n = k: $P(k) = \sum_{i=0}^{k} 2^i = 1 + 2 + 4 + ... + 2^k = 2^{k+1} - 1$

Show for $n = k+1$*:*

$$P(k+1) \quad = \quad \sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1 = 2^{k+2} - 1 \quad | \textit{ substitute into Equation 2}$$

Note that $P(k+1)$ *can also be written as* $1 + 2 + 4 + ... + 2^k + 2^{(k+1)}$.

$$\begin{aligned} P(k+1) &= \sum_{i=0}^{k+1} 2^i \\ &= \underline{1 + 2 + 4 + ... + 2^k} + 2^{k+1} \ | \textit{ replace with assumption } P(k) \\ &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \times 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

Thus, we have shown that if $P(k)$ *then* $P(k+1)$ *for some k, and that* $P(1)$ *holds. Therefore, by the principle of mathematical induction,* $P(n)$ *holds for all* $n \in \mathbb{N}^+$.

3.

$$P(n) = \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} \qquad (r \neq 1) \tag{3}$$

Answer:

Base case for $n = 1$*:*

$$\begin{aligned} P(1) &= r^0 + r^1 = 1 + r^1 = 1 + r \\ &= \tfrac{1-r^{1+1}}{1-r} = \tfrac{1-r^2}{1-r} = \tfrac{(1-r)(1+r)}{1-r} = 1 + r \end{aligned}$$

Assume for some k, n = k: $P(k) = \sum_{i=0}^{k} r^i = 1 + r + r^2 + ... + r^k = \frac{1-r^{k+1}}{1-r}$

Show for $n = k+1$*:*

$$\begin{aligned} P(k+1) &= \sum_{i=0}^{k+1} r^i = \tfrac{1-r^{(k+1)+1}}{1-r} \quad | \textit{ substitute into Equation 3} \\ &= \tfrac{1-r^{k+2}}{1-r} \end{aligned}$$

Note that $P(k+1)$ *can also be written as* $1 + r + r^2 + ... + r^k + r^{k+1}$.

$$
\begin{aligned}
P(k+1) &= \textstyle\sum_{i=0}^{k+1} r^i \\
&= \underline{1 + r + r^2 + ... + r^k} + r^{k+1} \mid \textit{replace with assumption } P(k) \\
&= \tfrac{1-r^{k+1}}{1-r} + r^{k+1} \\
&= \tfrac{1-r^{k+1}}{1-r} + \tfrac{1-r}{1-r} \times r^{k+1} \\
&= \tfrac{1-r^{k+1}}{1-r} + \tfrac{(1-r)\times r^{k+1}}{1-r} \\
&= \tfrac{1-r^{k+1}}{1-r} + \tfrac{r^{k+1}-r\times r^{k+1}}{1-r} \\
&= \tfrac{1-r^{k+1}}{1-r} + \tfrac{r^{k+1}-r^{k+2}}{1-r} \\
&= \tfrac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} \\
&= \tfrac{1-r^{k+2}}{1-r}
\end{aligned}
$$

Thus, we have shown that if $P(k)$ then $P(k+1)$ for some k, and that $P(1)$ holds. Therefore, by the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}^+$.

Task 2.2:

1. Prove that $n! > 2^n$ for $n \in \mathbb{N}$ and $n \geq 4$.

   Answer:
   Base case $n = 4$: $4! > 2^4 \Rightarrow 20 > 16$.

   Note: *Check for $n = 3$: $3! = 6$ and $2^3 = 8$, so the above statement indeed does not hold below 4.*

   Assume for some k, $n = k$: $k! > 2^k$

   Show that for $n = k+1$ it is true that: $(k+1)! > 2^{(k+1)}$

   $$
   \begin{array}{rcl}
   (k+1)! & > & 2^{(k+1)} \\
   k! \times (k+1) & > & 2^k \times 2
   \end{array}
   $$

   The above statement holds because $k! > 2^k$ based on our assumption, and $k+1 > 2$.

   We have shown that if the statement holds for any k, then it holds for $k+1$, and have established that it holds for $n = 4$. Therefore, by the principle of mathematical induction the statement holds for all $n \geq 4$.

2. Prove that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

   Answer:
   Base case for $n = 0$: $n^3 - n = 0^3 - 0 = 0$ is trivially divisible by 3.

   Assume for some k, $n = k$: $(k^3 - k)$ is divisible by 3.

Show for $n = k+1$: $(k+1)^3 - (k+1)$ is divisible by 3.

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \\ &= \underline{(k^3 - k)} + 3(k^2 + k) \end{aligned}$$

The above term is divisible by 3 because $k^3 - k$ is divisible by 3 based on the assumption, and $3(k^2 + k)$ is a multiple of 3, so also divisible by 3.

Thus, we have shown that if the statement holds for any $n = k$, then it holds for $n = k+1$, and that it holds for $n = 0$. Therefore, by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$.

3. Let $S_n = \{1, 2, ..., n\}$. Prove that $|\mathcal{P}(S_n)| = 2^n$ for $n \in \mathbb{N}^+$.

   ($\mathcal{P}(S)$ is the power set of S — every possible subset, including the empty set and S itself.)

   Answer:

   Base case for $n = 1$, $S_1 = \{1\}$ and $|\mathcal{P}(\{1\})| = |\{\{\}, \{1\}\}| = 2 = 2^1$

   Assume that for some k, $n = k$: $S_k = \{1, 2, ..., k\}$ and $|\mathcal{P}(S_k)| = 2^k$.

   For $n = k+1$, $S_{k+1} = \{1, 2, ..., k+1\}$, show that $|\mathcal{P}(S_{k+1})| = 2^{k+1}$.

   Note *that*

$$\begin{aligned} S_{k+1} &= \{1, 2, ..., k, k+1\} = S_k \cup \{k+1\}, \text{ and that} \\ \mathcal{P}(S_{k+1}) &= \mathcal{P}(S_k) \cup \{S \cup \{k+1\} \mid S \in \mathcal{P}(S_k)\}. \end{aligned}$$

$$\begin{aligned} |\mathcal{P}(S_{k+1})| &= |\mathcal{P}(S_k) \cup \{S \cup \{k+1\} \mid S \in \mathcal{P}(S_k)\}| \\ &= |\mathcal{P}(S_k)| + |\{S \cup \{k+1\} \mid S \in \mathcal{P}(S_k)\}| \\ &= |\mathcal{P}(S_k)| + |\mathcal{P}(S_k)| && \mid \textit{see}^* \textit{ below} \\ &= 2^k + 2^k && \mid \textit{based on assumption} \\ &= 2 \times 2^k \\ &= 2^{k+1} \end{aligned}$$

   * $\{S \cup \{k+1\} \mid S \in \mathcal{P}(S_k)\}$ *contains the same number of elements (which are sets) as $\mathcal{P}(S_k)$.*

   Thus, we have shown that if the statement holds for any k, then it holds for $k+1$, and that it holds for the base case $n = 1$. Therefore, by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}^+$.