Continued from last lecture ...

Notes:

Mechanical Derivation (2): Method #2 (Karnaugh map)

Example

Consider an example 3-input, 1-output function:

x	y	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Notes:

-
  - The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
  - The green group spans column 3 and rows 0 and 1; provided $x = 1$ and $y = 0$ we specify *just* those cells, so the expression is $x \wedge \neg y$. That is, $x = 1$ and $y = 0$ restricts us to column 3 (columns 0, 1 and 2 have at least one of $x = 0$ or $y = 1$) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
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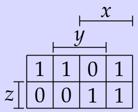
1	0	1	1
1	1	0	0
1	1	1	1



			x	x
		y	y	
	1	1	0	1
z	0	0	1	1

Each group translates into one term of the SoP form expression

$$r =$$

Notes:

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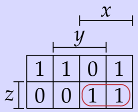
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



			x	x
		y	y	
	1	1	0	1
z	0	0	1	1

Each group translates into one term of the SoP form expression

$$r = (\quad x \quad \wedge \quad z \quad)$$

Notes:

- – The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
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1	0	1	1
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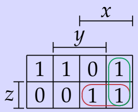

1	1	0	1
0	0	1	1

Each group translates into one term of the SoP form expression

$$\begin{array}{rcl} r & = & (x \wedge z) \vee \\ & & (x \wedge \neg y) \end{array}$$

Notes:

- – The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
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Mechanical Derivation (2): Method #2 (Karnaugh map)

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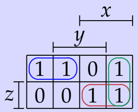
x	y	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



1	1	0	1
0	0	1	1

Each group translates into one term of the SoP form expression

$$\begin{array}{rcl} r & = & (x \wedge z) \vee \\ & & (x \wedge \neg y) \vee \\ & & (\neg x \wedge \neg z) \end{array}$$

Notes:

- – The red group spans columns 2 and 3, in row 1; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. $x = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $x = 0$) and $z = 1$ restricts us to row 1 (row 0 has $z = 0$). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
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Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	**?**	**?**
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	**?**	**?**
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	**?**	**?**
1	1	0	0	**?**	**?**
1	1	0	1	**?**	**?**
1	1	1	0	**?**	**?**
1	1	1	1	**?**	**?**

Notes:

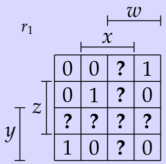
- – The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
  - – The green group spans column 1 and rows 2 and 3; provided $w = 0$, $x = 0$ and $y = 1$ we specify *just* those cells, so the expression is $\neg w \wedge \neg x \wedge y$. That is, $w = 0$ and $x = 0$ restricts us to column 0 (columns 1, 2 and 3 have at least one of $w = 1$ or $x = 1$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
  - – The blue group spans columns 1 and 2 and rows 1 and 2; provided $x = 1$ and $z = 1$ we specify *just* those cells, so the expression is $x \wedge z$. That is, $x = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $x = 0$) and $z = 1$ restricts us to rows 1 and 2 (rows 0 and 3 have $z = 0$). Note that the value of w and y don't matter: cells in the group hold the value 1 regardless of w and y.
  - – The magenta group spans columns 1 and 2 in row 0; provided $x = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $x \wedge \neg y \wedge \neg z$. That is, $x = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $x = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
  - – The yellow group spans column 0 and rows 1 and 2; provided $w = 0$, $x = 0$ and $z = 1$ we specify *just* those cells, so the expression is $\neg w \wedge \neg x \wedge z$. That is, $w = 0$ and $x = 0$ restricts us to column 0 (columns 1, 2 and 3 have at least one of $w = 1$ or $x = 1$) and $y = 1$ restricts us to rows 1 and 2 (rows 0 and 1 have $y = 0$) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
  - – The orange group spans columns 2 and 3 and rows 2 and 3; provided $w = 1$ and $y = 1$ we specify *just* those cells, so the expression is $w \wedge y$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

Mechanical Derivation (3): Method #2 (Karnaugh map)

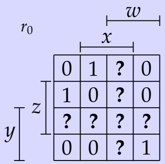
Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	**?**	**?**
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	**?**	**?**
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	**?**	**?**
1	1	0	0	**?**	**?**
1	1	0	1	**?**	**?**
1	1	1	0	**?**	**?**
1	1	1	1	**?**	**?**



0	0	**?**	1
0	1	**?**	0
?	**?**	**?**	**?**
1	0	**?**	0



0	1	**?**	0
1	0	**?**	0
?	**?**	**?**	**?**
0	0	**?**	1

Each group translates into one term of the SoP form expressions

$r_1 =$

$r_0 =$

Notes:

- – The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
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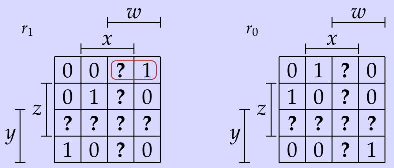
Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



r_1

0	0	?	1
0	1	?	0
?	?	?	?
1	0	?	0

r_0

0	1	?	0
1	0	?	0
?	?	?	?
0	0	?	1

Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \qquad r_0 =$$

Notes:

- - The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
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Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
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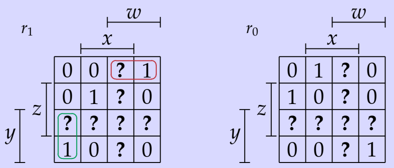
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?



r_1

0	0	?	1
0	1	?	0
?	?	?	?
1	0	?	0

r_0

0	1	?	0
1	0	?	0
?	?	?	?
0	0	?	1

Each group translates into one term of the SoP form expressions

$$\begin{array}{lll} r_1 = & (w \wedge \neg y \wedge \neg z) \vee & \qquad r_0 = \\ & (y \wedge \neg w \wedge \neg x) & \end{array}$$

Notes:

- - The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
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  - The magenta group spans columns 1 and 2 in row 0; provided $x = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $x \wedge \neg y \wedge \neg z$. That is, $x = 1$ restricts us to columns 1 and 2 (columns 0 and 3 have $x = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
  - The yellow group spans column 0 and rows 1 and 2; provided $w = 0$, $x = 0$ and $z = 1$ we specify *just* those cells, so the expression is $\neg w \wedge \neg x \wedge z$. That is, $w = 0$ and $x = 0$ restricts us to column 0 (columns 1, 2 and 3 have at least one of $w = 1$ or $x = 1$) and $y = 1$ restricts us to rows 1 and 2 (rows 0 and 1 have $y = 0$) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
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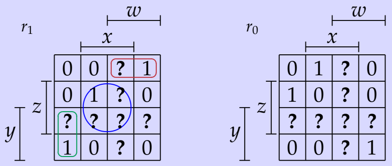
Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	**?**	**?**
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	**?**	**?**
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	**?**	**?**
1	1	0	0	**?**	**?**
1	1	0	1	**?**	**?**
1	1	1	0	**?**	**?**
1	1	1	1	**?**	**?**



r_1

0	0	**?**	1
0	1	**?**	0
?	**?**	**?**	**?**
1	0	**?**	0

r_0

0	1	**?**	0
1	0	**?**	0
?	**?**	**?**	**?**
0	0	**?**	1

Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \vee (y \wedge \neg w \wedge \neg x) \vee (x \wedge z) \qquad r_0 =$$

Notes:

- – The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
  - – The green group spans column 1 and rows 2 and 3; provided $w = 0$, $x = 0$ and $y = 1$ we specify *just* those cells, so the expression is $\neg w \wedge \neg x \wedge y$. That is, $w = 0$ and $x = 0$ restricts us to column 0 (columns 1, 2 and 3 have at least one of $w = 1$ or $x = 1$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
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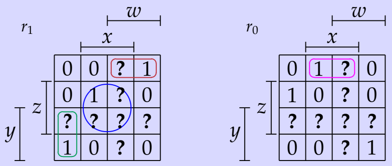
  - – The orange group spans columns 2 and 3 and rows 2 and 3; provided $w = 1$ and $y = 1$ we specify *just* those cells, so the expression is $w \wedge y$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 1$ restricts us to rows 2 and 3 (rows 0 and 1 have $y = 0$). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	**?**	**?**
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	**?**	**?**
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	**?**	**?**
1	1	0	0	**?**	**?**
1	1	0	1	**?**	**?**
1	1	1	0	**?**	**?**
1	1	1	1	**?**	**?**



r_1

0	0	**?**	1
0	1	**?**	0
?	**?**	**?**	**?**
1	0	**?**	0

r_0

0	1	**?**	0
1	0	**?**	0
?	**?**	**?**	**?**
0	0	**?**	1

Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \vee (y \wedge \neg w \wedge \neg x) \vee (x \wedge z) \qquad r_0 = (x \wedge \neg y \wedge \neg z)$$

Notes:

- – The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
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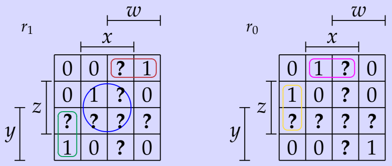
Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	**?**	**?**
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	**?**	**?**
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	**?**	**?**
1	1	0	0	**?**	**?**
1	1	0	1	**?**	**?**
1	1	1	0	**?**	**?**
1	1	1	1	**?**	**?**



r_1

0	0	**?**	1
0	1	**?**	0
?	**?**	**?**	**?**
1	0	**?**	0

r_0

0	1	**?**	0
1	0	**?**	0
?	**?**	**?**	**?**
0	0	**?**	1

Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \vee (y \wedge \neg w \wedge \neg x) \vee (x \wedge z)$$

$$r_0 = (x \wedge \neg y \wedge \neg z) \vee (z \wedge \neg w \wedge \neg x)$$

Notes:

- – The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
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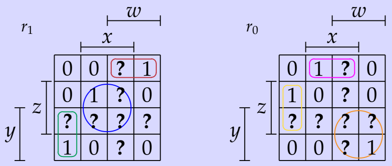
Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	**?**	**?**
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	**?**	**?**
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	**?**	**?**
1	1	0	0	**?**	**?**
1	1	0	1	**?**	**?**
1	1	1	0	**?**	**?**
1	1	1	1	**?**	**?**



r_1

0	0	**?**	1
0	1	**?**	0
?	**?**	**?**	**?**
1	0	**?**	0

r_0

0	1	**?**	0
1	0	**?**	0
?	**?**	**?**	**?**
0	0	**?**	1

Each group translates into one term of the SoP form expressions

$$r_1 = (w \wedge \neg y \wedge \neg z) \vee (y \wedge \neg w \wedge \neg x) \vee (x \wedge z)$$

$$r_0 = (x \wedge \neg y \wedge \neg z) \vee (z \wedge \neg w \wedge \neg x) \vee (w \wedge y)$$

Notes:

- – The red group spans columns 2 and 3 in row 0; provided $w = 1$, $y = 0$ and $z = 0$ we specify *just* those cells, so the expression is $w \wedge \neg y \wedge \neg z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 0$ and $z = 0$ restricts us to row 0 (rows 1, 2 and 3 have at least one of $y = 1$ or $z = 1$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
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"Building Block" Components (1) – Choice

- The idea of choice is crucial in constructing larger components:
  1. a **multiplexer** continuously drives one of many inputs onto a single output depending on a control signal, and
  2. a **demultiplexer** continuously drives a single input onto one of many outputs depending on a control signal.
- An m-input (resp. m-output), n-bit multiplexer (resp. demultiplexer) has
  1. m inputs (resp. outputs), each having n bits, and
  2. a ($\lceil \log_2(m) \rceil$)-bit control signal that selects between the inputs (resp. outputs).

Notes:

"Building Block" Components (2) – Choice

- As an analogy, the C `switch` statement

  Listing (C)

  ```
  switch( c ) {
    case 0 : r = w; break;
    case 1 : r = x; break;
    case 2 : r = y; break;
    case 3 : r = z; break;
  }
  ```

  acts similarly to a 4-input multiplexer.
- Likewise,

  Listing (C)

  ```
  switch( c ) {
    case 0 : r0 = x; break;
    case 1 : r1 = x; break;
    case 2 : r2 = x; break;
    case 3 : r3 = x; break;
  }
  ```

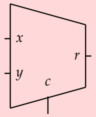
  acts similarly to a 4-output demultiplexer.

Notes:

"Building Block" Components (3) – Choice

Definition (2-**input**, 1-**bit multiplexer**)

The behaviour of the component



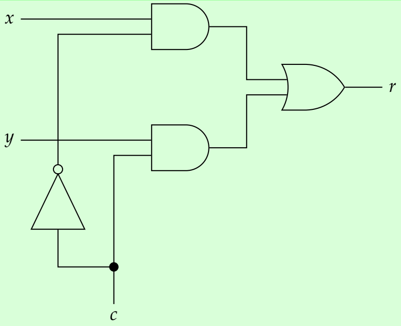
is described by the truth table

MUX2			
c	x	y	r
0	0	**?**	0
0	1	**?**	1
1	**?**	0	0
1	**?**	1	1

which can be used to derive the following implementation:

$$r = (\neg c \wedge x) \vee (c \wedge y)$$

Circuit (2-input, 1-bit multiplexer)

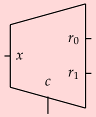


Notes:

"Building Block" Components (4) – Choice

Definition (2-**output**, 1-**bit demultiplexer**)

The behaviour of the component



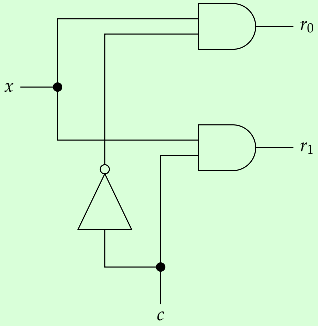
is described by the truth table

DEMUX2			
c	x	r_1	r_0
0	0	**?**	0
0	1	**?**	1
1	0	0	**?**
1	1	1	**?**

which can be used to derive the following implementation:

$$\begin{array}{rcl} r_0 & = & \neg c \wedge x \\ r_1 & = & c \wedge x \end{array}$$

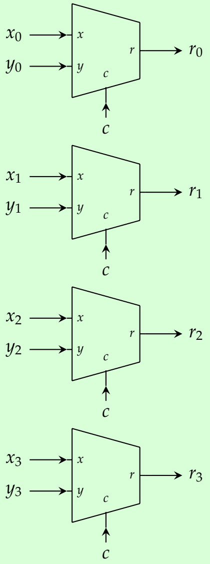
Circuit (2-output, 1-bit demultiplexer)



Notes:

An Aside: Applying our design patterns

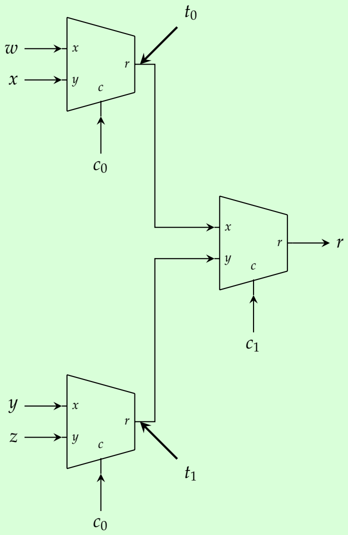
Circuit (2-input, 4-bit multiplexer via replication)



Notes:

An Aside: Applying our design patterns

Circuit (4-input, 1-bit multiplexer via a cascading)



Notes:

“Building Block” Components (7) – Arithmetic

- Given two 1-bit operands x and y,
  1. a **half-adder** computes $x + y$ to produce a sum s and a carry-out co,
  2. a **full-adder** computes $x + y + ci$, where ci is a 1-bit carry-in, to produce a sum s and a carry-out co,
  3. an **equality comparator** computes

$$r = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

     and
  4. a **less than comparator** computes

$$r = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

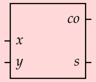
  all of which are used to create larger, more general components.

Notes:

“Building Block” Components (8) – Arithmetic

Definition (half-adder)

The behaviour of the component



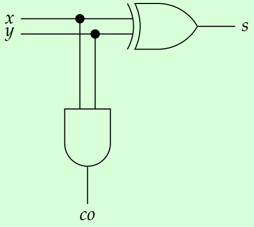
is described by the truth table

Half-Adder			
x	y	co	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

which can be used to derive the following implementation:

$$\begin{array}{lcl} co & = & x \wedge y \\ s & = & x \oplus y \end{array}$$

Circuit (half-adder)

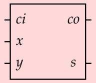


Notes:

"Building Block" Components (9) – Arithmetic

Definition (**full-adder**)

The behaviour of the component



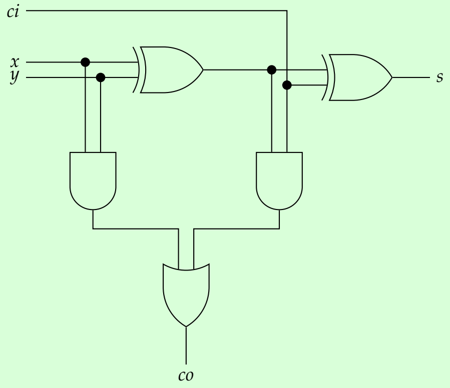
is described by the truth table

Full-Adder				
ci	x	y	co	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

which can be used to derive the following implementation:

$$\begin{array}{rcl} co & = & (x \wedge y) \vee ((x \oplus y) \wedge ci) \\ s & = & x \oplus y \oplus ci \end{array}$$

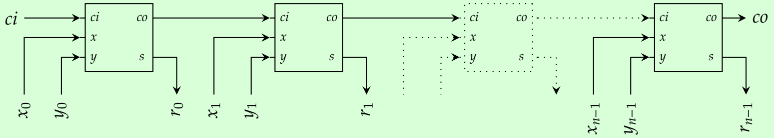
Circuit (full-adder)



Notes:

"Building Block" Components (10) – Arithmetic

Circuit (n-bit addition)

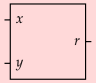


Notes:

“Building Block” Components (11) – Arithmetic

Definition (**equality comparator**)

The behaviour of the component



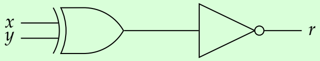
is described by the truth table

EQUAL		
x	y	r
0	0	1
0	1	0
1	0	0
1	1	1

which can be used to derive the following implementation:

$$r = \neg(x \oplus y)$$

Circuit (equality comparator)

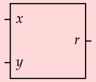


Notes:

“Building Block” Components (12) – Arithmetic

Definition (**less than comparator**)

The behaviour of the component



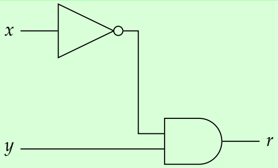
is described by the truth table

LESS-THAN		
x	y	r
0	0	0
0	1	1
1	0	0
1	1	0

which can be used to derive the following implementation:

$$r = \neg x \wedge y$$

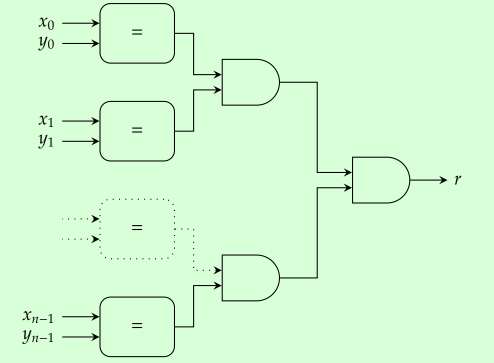
Circuit (less than comparator)



Notes:

“Building Block” Components (13) – Arithmetic

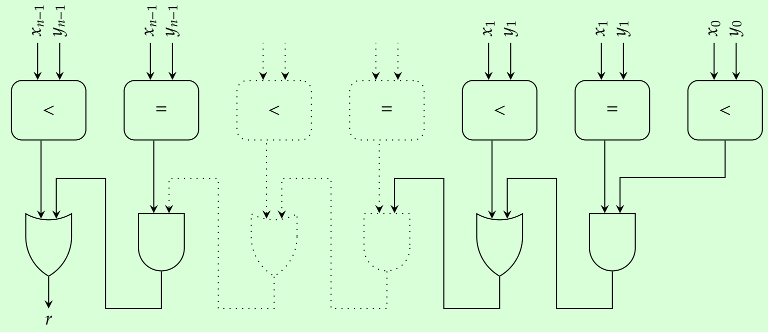
Circuit (n-bit equality comparison)



Notes:

“Building Block” Components (14) – Arithmetic

Circuit (n-bit less than comparison)



Notes:

Conclusions

- Take away points:
  1. There are a *huge* number of issues to consider when designing even simple components, e.g.,
     - how do we describe what the circuit should do?
     - what sort of standard library do we use?
     - do we aim for the fewest gates?
     - do we aim for shortest critical path?
     - how do we cope with propagation delay and fan-out?
  2. The design patterns, and mechanical techniques are the key concepts to focus on: these allow you do produce an effective implementation most easily.
  3. In many cases, use of appropriate **Electronic Design Automation (EDA)** tools can provide (semi-)automatic solutions.

Notes:

References and Further Reading

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Notes:

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Notes: