COMS10003 Work Sheet 22

Linear Algebra: Solving Linear Equations and Inverting Matrices

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1. Use substitution to find solutions or otherwise to the following 2×2 linear systems. Sketch the geometric interpretation in each case.

$$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 2 \end{array} \qquad\qquad \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 3 \\ 3x_1 & + & 6x_2 & = & 3 \end{array}$$

   Answer:

   $x_1 = 1,\ x_2 = 0$

   no solution

2. Use Gaussian elimination (GE) to find a solution to the following 3×3 system

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & + & 3x_2 & + & 2x_3 & = & 3 \end{array}$$

   Answer:

   $x_1 = 1,\ x_2 = 0,\ x_3 = 1$

3. Use GE to show that the following system does not have a solution. Describe in geometric terms why this is so.

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & 3x_3 & = & 1 \\ 4x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 2x_1 & - & 3x_2 & + & 4x_3 & = & 2 \end{array}$$

   Answer:

   The 3 planes have no common intersection point or line

4. Use GE with matrix notation to solve the following linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 3 \\ 3x_1 & + & x_2 & - & x_3 & = & 2 \end{array}$$

   Answer:

   $x_1 = -0.5,\ x_2 = 2.5,\ x_3 = -1$

5. Use GE with matrix notation to find a general solution to the system

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & 3x_3 & - & x_4 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & + & 3x_4 & = & 1 \\ 4x_1 & + & x_2 & - & 2x_3 & + & 5x_4 & = & 2 \\ 6x_1 & - & 3x_2 & + & 4x_3 & + & 3x_4 & = & 4 \end{array}$$

Answer:

$x_1 = (2-a)/3,\ x_2 = (4+25a)/3,\ x_3 = 1+6a,\ x_4 = a$

6. Examine the GE algorithm and consider each division and each multiplication-subtraction as a single operation. Show that for an $n \times n$ system, GE is an $O(n^3)$ algorithm.

Answer:

To compute the pivot for all forward steps requires $\sum_{k=2}^{n} k$ *divisions*
To compute the reduced form coefficients for all forward steps requires $\sum_{k=2}^{n} k*(k-1)$ *multiplication-subtraction operations*
To solve for each unknown for all back substitution steps requires n *divisions and* $\sum_{k=2}^{n}(k-1)$ *multiplication-subtraction operations*
Thus, in total, approximately, requires $\sum_{k=2}^{n} k + \sum_{k=2}^{n} k*(k-1) + \sum_{k=2}^{n}(k-1) + n$ *which is of the order* $\sum_{k=1}^{n} k^2 = O(n^3)$

7. Use the Gauss-Jordan Method to find the inverses of the following matrices (if they exist). If you find one, confirm that it is correct using matrix multiplication.

$$\begin{bmatrix} 2 & 1 & -2 \\ 5 & -3 & 7 \\ 0 & -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -4 \\ -3 & 1 & -9 \\ 2 & -3 & 13 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 0.2881 & 0.0847 & 0.0169 \\ 0.0847 & -0.0339 & -0.4068 \\ -0.1695 & 0.0678 & -0.1864 \end{bmatrix}$$

No inverse