Lecture I

Introduction to coding theory

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CoCoNuT, 2015

Lecture outline

References

- van Lint, J.H., Introduction to Coding Theory, *Graduate Texts in Mathematics*, 86, (Third Edition) 1999, Springer-Verlag.
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- Hamming, R. W., *Error detecting and Error Correcting Codes*, Bell System Technical Journal 29, 1950, pp. 147-160.
- MacKay, D., Information Theory, Inference, and Learning Algorithms, 2003, *Cambridge University Press*.
- Huffman, W. C. and Brualdi, Richard A., Handbook of Coding Theory, 1998, *Elsevier Science Inc.*, New York, NY, USA.
- Steven Roman, Introduction to coding and information theory, Springer, *Undergraduate texts in mathematics*, 1997

- Official page COMS20002:

  Communication, Complexity and Number Theory

- My personal homepage for news, slides and (maybe) exercises

  `http://www.cs.bris.ac.uk/home/cseao/`

Lecture outline

1. Introduction
   - Motivation
2. Error correcting codes
   - Error detection and correction
   - Hamming code
3. Block codes
   - First definitions
   - Hamming distance and minimum distance of a code
   - Maximum Likelihood Decoding (MLD)
   - Error correction and error detection capability
4. Modelling noisy channel: a simple example

What does she say?

“Wel*ome to t*is c*ass!” $\longrightarrow$

What does she say?

“Wel*ome to t*is c*ass!” $\longrightarrow$ “Welcome to this class!”

What does she say?

"Wel*ome to t*is c*ass!" $\longrightarrow$ "Welcome to this class!"

Why is this example working?

- English has in built redundancy, so that it can tolerate *errors*.

Motivation

More in general, consider the following applications of *data storage* or *transmission*:

- CDs and DVDs
- Satellite/Digital Television
- Deep space probes
- Internet communications
- Mobile phones
- Computer hard disks/memory/floppy etc

Motivation

More in general, consider the following applications of *data storage* or *transmission*:

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In all of these the data can become corrupted.

- It is prone to errors

However they still work

- How?

Motivation

Goal: reliable systems of communication

Motivation

Goal: reliable systems of communication

HOW?

Motivation

Goal: reliable systems of communication

HOW?

1. **Physical solution**: channels with no noise

Motivation

Goal: reliable systems of communication

HOW?

1. **Physical solution**: channels with no noise
2. **System solution**: accept channels as they are, but manipulate the data in order to obtain reliable systems

Coding theory - Applications

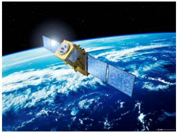
- Internet

Coding theory - Applications

- Internet
- Mobile phones

Coding theory - Applications

- Internet
- Mobile phones
- Satellite broadcast
    - TV



Coding theory - Applications

- Internet
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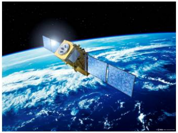

Coding theory - Applications

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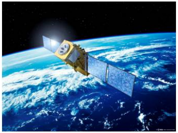

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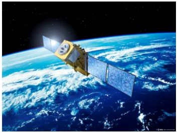
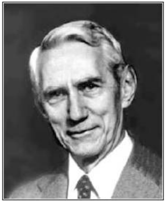


Codes are all around us!

Coding theory - The birth

"The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point"
(Claude Shannon, 1948)

- In 1948, Claude E. Shannon wrote *"A Mathematical Theory of Communication"*, which marked the beginning of both Information and Coding Theory



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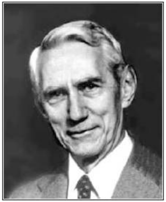


- In 1950, Richard W. Hamming wrote *"Error Detecting and Error Correcting Codes"*, which was the first paper explicitly introducing error-correcting codes

Coding and Information Theory

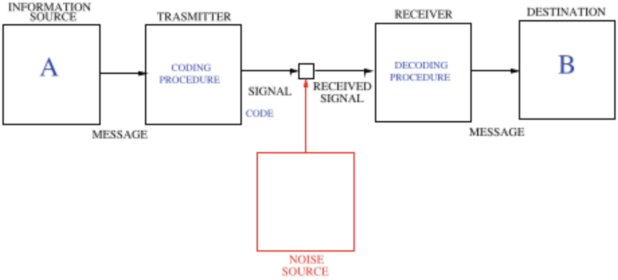
- **Coding Theory**: mainly concerned on explicit methods for efficient and reliable data transmission and storage

Coding and Information Theory

- **Coding Theory**: mainly concerned on explicit methods for efficient and reliable data transmission and storage
- **Information Theory**: it provides the performance limits on what can be done by suitable encoding of information
  - How should information be measured?
  - How much additional information is gained by some reduction in uncertainty?
  - What is the information content of a random variable?
  - How does the noise level in a communication channel limit its capacity to transmit information?
  - How does the bandwidth (in cycles/second) of a communication channel limit its capacity to transmit information?

Coding theory

The general idea is that of adding some kind of redundancy to the message that we want to send over a communication channel



Transmitted signal + noise = received signal

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Digital Data

Digital data is sent as a series of ones and zeros.

- 11110101111101010100011010101011

Sometimes an error occurs:

- 1111011**1**1111101010100011010101011

We would like to be able to either detect or correct such errors.

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Detection

- Good if we can request a resend of the data

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Detection

- Good if we can request a resend of the data

Correction

- Needed if data cannot be resent (e.g. CD/DVD) or too costly to resend (e.g. deep space probe)

Simple Error Detection

Most data is first bundled up into a group of bits before sending

- e.g. 4, 8, 32 or 64 bits at a time

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A simple detection trick is to add a parity bit

Suppose we wish to transmit 4 bits

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We add in an extra bit which signals whether the original data

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\begin{aligned}
0110 &\longrightarrow 0110\mathbf{0} \\
1111 &\longrightarrow 1111\mathbf{0} \\
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Simple Error Detection

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To do this we add a fifth bit equal to

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where

$$x \oplus y = x + y \pmod 2$$

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The resulting five bits is called a codeword

$$C = \{c = m_1 m_2 m_3 m_4 m_5 \mid \sum_{i=1}^{5} m_i = 0\} \subseteq \{0, 1\}^5$$

Simple Error Detection

We can now detect whether a single error has occurred.

Suppose you receive the following data using the previous example:

- 10101
- 01110
- 11101
- 11111
- 00000
- 00001

Are there any errors ?

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We can now detect whether a single error has occurred.

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Trouble is we do not know where the errors occurred

Lecture outline

Detecting errors - Hamming code I

Again sticking to four bits of message

$$m_1 m_2 m_3 m_4$$

The idea is to use multiple parity-check bits.

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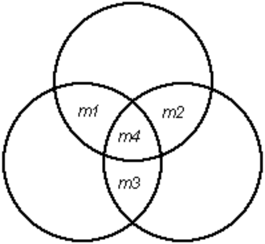
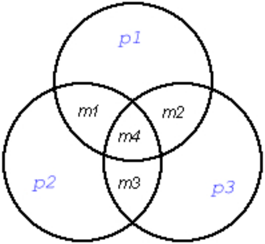
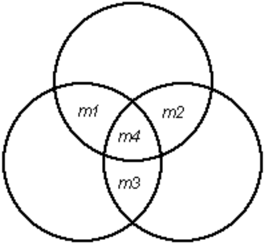
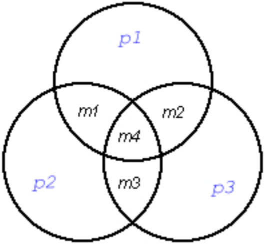




Detecting errors - Hamming code II





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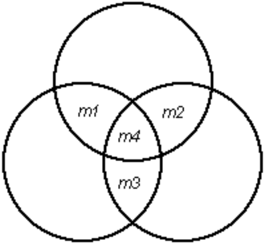
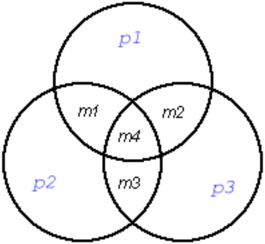
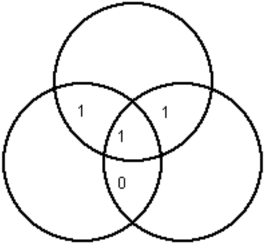




Suppose $\mathbf{m} = 1101$ is the message



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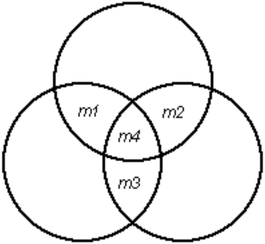

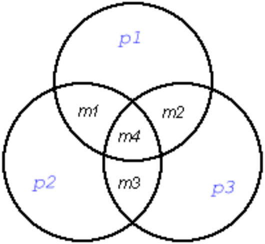

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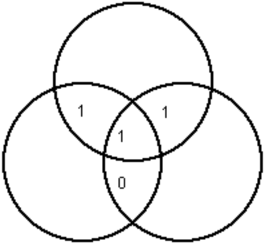

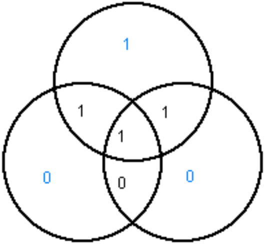

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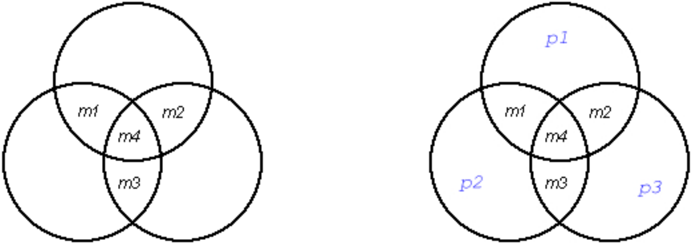
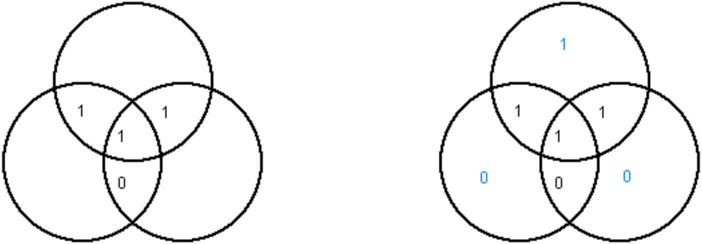


Suppose $\mathbf{m} = 1101$ is the message $\rightarrow 1101100$ is the codeword



Detecting errors - Hamming code III

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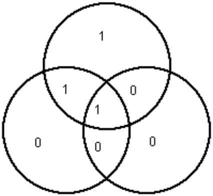

Detecting errors - Hamming code III

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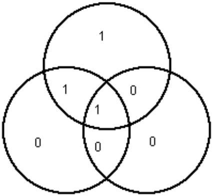

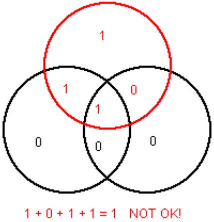

Detecting errors - Hamming code III

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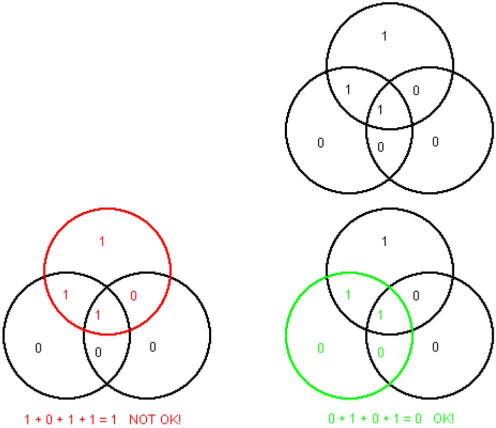

Detecting errors - Hamming code III

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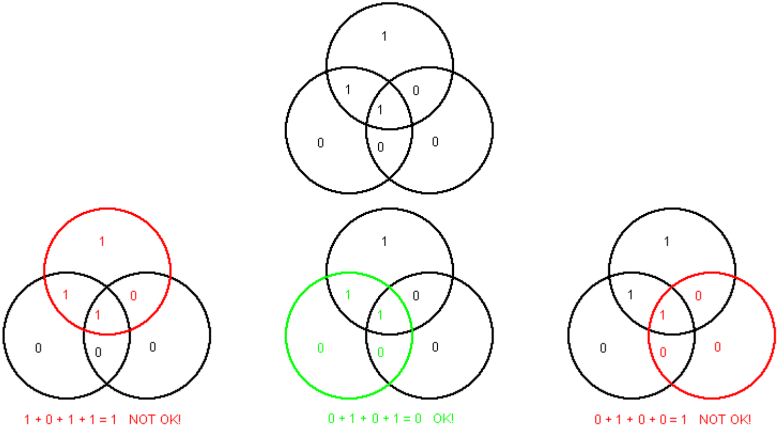
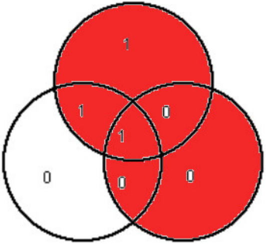


Correcting errors - Hamming code IV

$\mathbf{c} = 1101100$ and $\mathbf{r} = 1001100$



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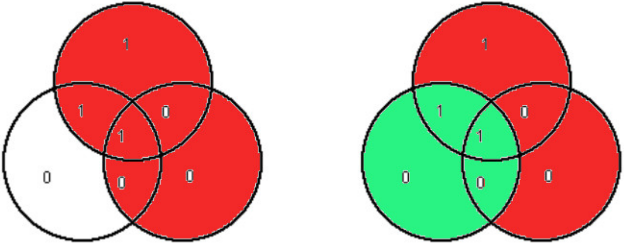

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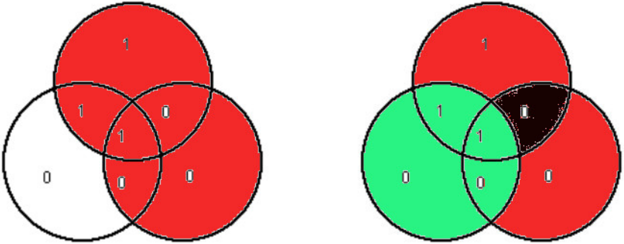

$\rightarrow$ the error is at m_2

Hamming code - Basic idea

- Use multiple parity bits, each covering a subset of the message bits

$$
\begin{aligned}
m_1 + m_4 + m_2 + p_1 = 0 \quad &\longrightarrow \quad \{m_1, m_4, m_2, p_1\} \\
m_1 + m_3 + m_4 + p_2 = 0 \quad &\longrightarrow \quad \{m_1, m_3, m_4, p_2\} \\
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- the subsets overlap, i.e. each message-bit belongs to multiple subsets

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- the subsets overlap, i.e. each message-bit belongs to multiple subsets
- No two message bits belong to exactly the same subsets. In this way it is possible to correct one error and to detect two errors.

Hamming code - Basic idea

- Use multiple parity bits, each covering a subset of the message bits

$$m_1 + m_4 + m_2 + p_1 = 0 \quad \longrightarrow \quad \{m_1, m_4, m_2, p_1\} \; OK$$
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Example

$\mathbf{c} = 1101100$ and $\mathbf{r} = 1001000$, so two errors occurred, at m_2 and p_1. Then:

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Example

$\mathbf{c} = 1101100$ and $\mathbf{r} = 1001000$, so two errors occurred, at m_2 and p_1. Then:

- The code detects that some errors occurred

Hamming code - Basic idea

- Use multiple parity bits, each covering a subset of the message bits

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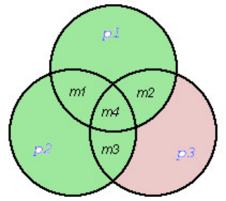

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- C contains 16 codewords of length 7

Lecture outline

Block codes - Notation I

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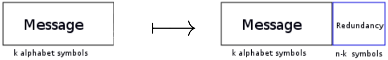

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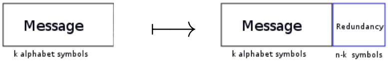

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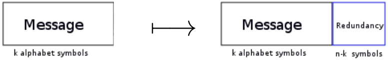

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Hamming

Example

The Hamming code we have seen before is a binary $(7, 16)$ block code with information rate $4/7$.

Lecture outline

Hamming distance

Definition (Hamming distance)

Given two strings $\mathbf{x}$ and $\mathbf{y} \in \mathcal{A}^n$, the **Hamming distance** between $\mathbf{x}$ and $\mathbf{y}$ is

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Definition (Code distance)

The (Hamming) **minimum distance of a code** C is given by

$$d(C) = min\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}.$$

Definition (Hamming weight)

The **Hamming weight** of a string $\mathbf{x}$, $\mathrm{wt}(\mathbf{x})$, is defined as the number of non-zero symbols in the string.

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1. if $\mathbf{r} \in C$ then no correction is needed 
2. if $\mathbf{r} \notin C$, then some errors occurred 

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We will see that for some types of channel MLD is equivalent to finding the coderword $\mathbf{c}$ closest to $\mathbf{r}$ in the Hamming distance (Nearest neighbour decoding):

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From now on we will assume a type of channel such that we can use the minimum distance decoding to perform MLD

Decoding problem - Why is $d(C)$ important?

Let $\mathbf{x} \in \mathcal{A}^n$ and $t \in \mathbb{N}$, define

$$\mathcal{B}_t(\mathbf{x}) = \{\mathbf{y} \in \mathcal{A}^n \mid d(\mathbf{x}, \mathbf{y}) \leq t\}$$

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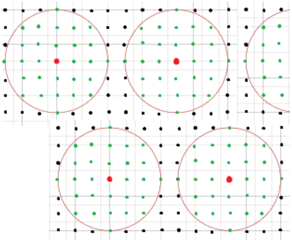

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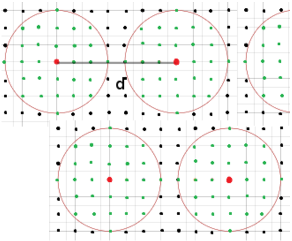

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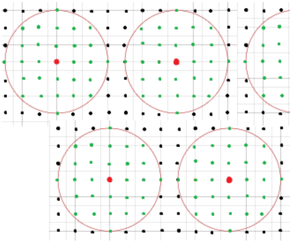

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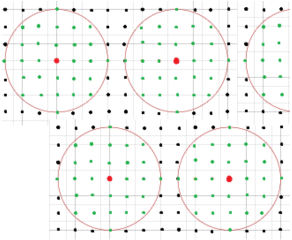

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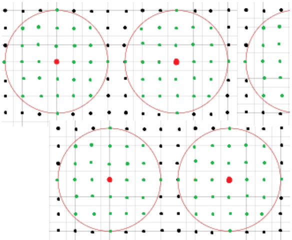

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- if $\mathbf{r}$ = green word, we should correct it to the red coderword that is the center of the ball it lies in
- if $\mathbf{r}$ = black word, then we are not able to correct because, if we increase the radii, balls would overlap

Lecture outline

More formally, we have the following definition:

- The error detection capability of a code C is the number e of errors that the code can detect. A *e-error detecting* code has minimum distance $d = e + 1$.

Error correction and error detection capability

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In a similar way we can define the erasure correction capability of a code. An **erasure** occurs when a transmitted symbol is unreadable and at its place an extra symbol ϵ is introduced.

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- The condition for simultaneous correction of t errors and s erasures is

$$d \geq 2t + s + 1.$$

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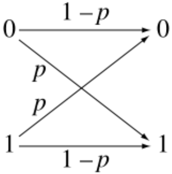
$\mathcal{X} = \{0, 1\}$ input alphabet and $\mathcal{Y} = \{0, 1\}$ output alphabet.
A BSC is parametrized by the probability p, $0 \leq p < 1/2$, that an input bit is flipped. p depends on the noise level and is called *crossover probability*.



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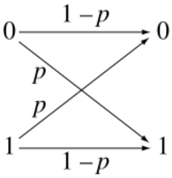

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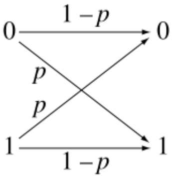

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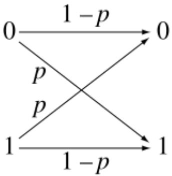

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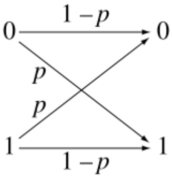

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\Pr(\mathbf{r}|\mathbf{c}_1) \leq \Pr(\mathbf{r}|\mathbf{c}_2) &\iff d(\mathbf{r}, \mathbf{c}_1) \geq d(\mathbf{r}, \mathbf{c}_2) \\
&\iff \mathrm{wt}(\mathbf{r} + \mathbf{c}_1) \geq \mathrm{wt}(\mathbf{r} + \mathbf{c}_2) \\
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The most likely codeword sent is the one corresponding to the error of smallest weight