COMS21202: Symbols, Patterns and Signals
Data Acquisition and Data Characteristics

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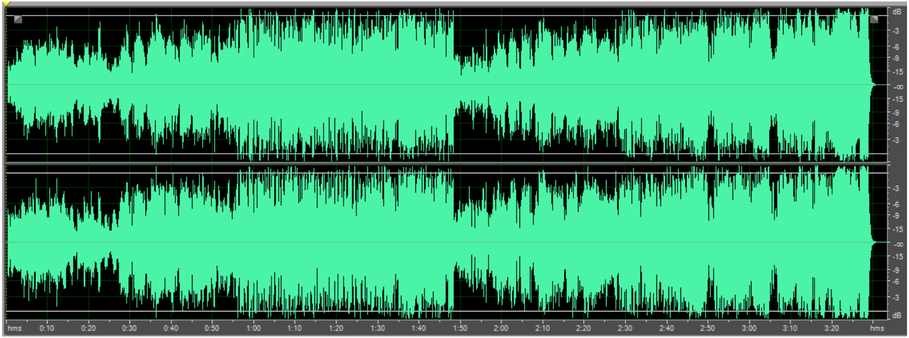
Data Acquisition - Analogue to Digital Conversion

Analogue to Digital conversion involves

1. Sampling
2. Quantisation

e.g. Audio Signal - 1D



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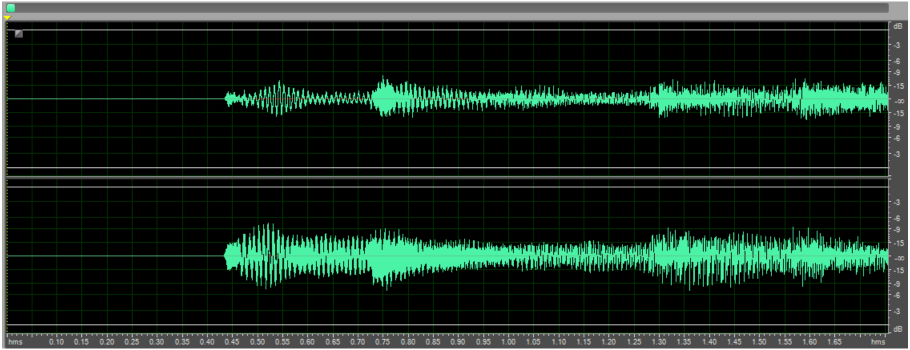

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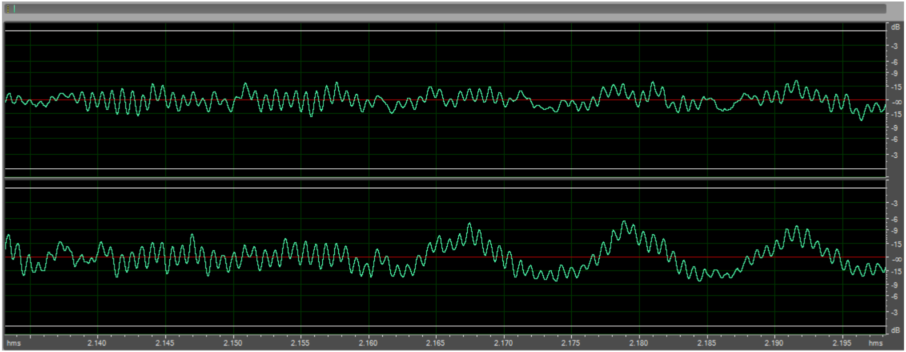

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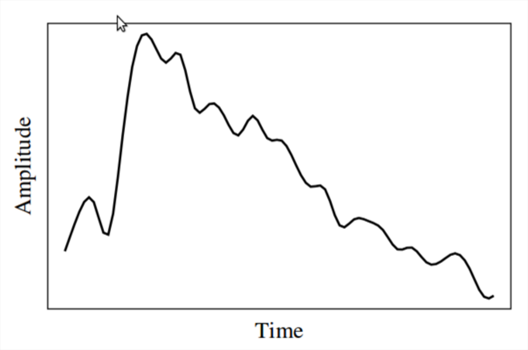

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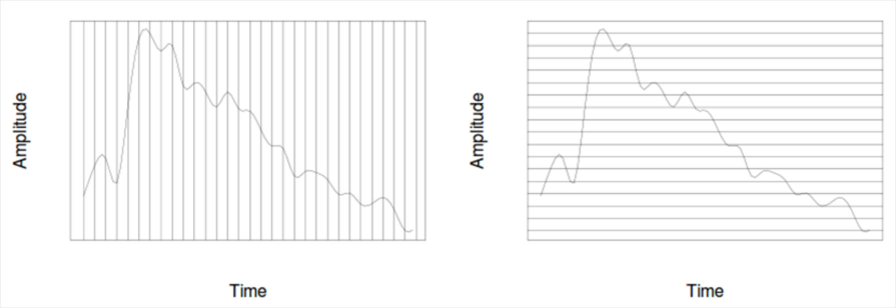

Data Acquisition - Analogue to Digital Conversion

Theorem

Nyquist Shannon sampling theorem:
If a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2B}$ seconds apart.

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Accordingly,

- Suppose the highest frequency for a given analog signal is f_{max},
- According to the Theorem, the sampling rate must be at least $2f_{max}$

Data Acquisition - Analogue to Digital Conversion

Standard audio formats

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- Audio CD
  - Sampling: 44 KHz samples
  - Quantisation: 16 bits / sample
  - Stereo (2 channels)

Data Acquisition - Analogue to Digital Conversion

Images - Multi-Dimensional

- Sampling: Resolution in digital photography
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- Binary Images: Black/White 1 bit per pixel

Data Characteristics

- Distance
- Mean and Variance
- Covariance and Correlation

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- Distance is important as it:
  - enables data to be ordered
  - allows numeric calculations
  - enables calculating similarity and dissimilarity
- Without defining a distance measure, almost all statistical and machine learning algorithms will not be able to function.

Distance

A valid distance measure $D(a, b)$ between two components a and b has properties

- non-negative: $D(a, b) > 0$
- reflexive: $D(a, b) = 0 \iff a = b$
- symmetric: $D(a, b) = D(b, a)$
- satisfies triangular inequality: $D(a, b) + D(b, c) \geq D(a, c)$

Distance (Numerical)

Distances between numerical data points in Euclidean space $\mathbb{R}^n$, for a point $x = (x_1, x_2, .., x_n)$ and a point $y = (y_1, y_2, .., y_n)$, the Minkowski distance of order p (p-norm distance) is defined as:

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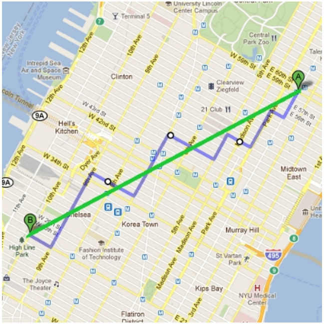

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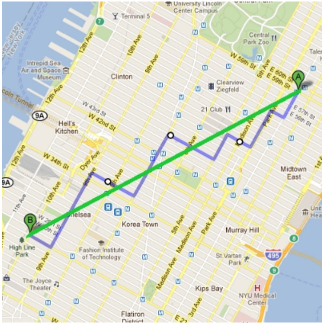

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$$\begin{aligned} D(x, y) &= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \\ &= \|\mathbf{x} - \mathbf{y}\| \\ &= \sqrt{(\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y})} \end{aligned} \tag{1}$$

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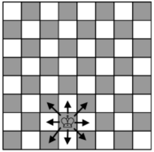

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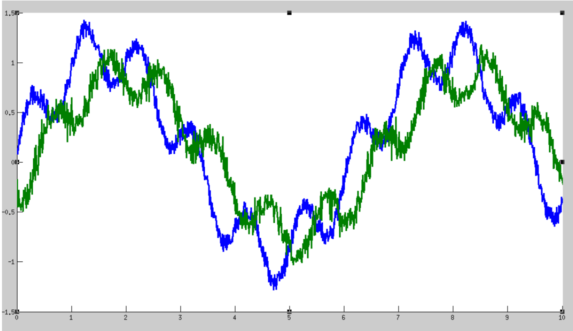

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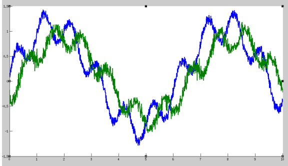

P-Norm distances can only

- Compare time series of the same length
- very sensitive respect to signal transformations:
  - shifting
  - uniform amplitude scaling
  - non-uniform amplitude scaling
  - uniform time scaling

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e.g. Dynamic Time Warping (Berndt and Clifford, 1994)

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- Dynamic Time Warping (DTW) can be defined recursively as
  For two time series $\mathbf{X} = (x_0, .., x_n)$ and $\mathbf{Y} = (y_0, .., y_m)$

  $$DTW(\mathbf{X}, \mathbf{Y}) = D(x_0, y_0) + min\{DTW(\mathbf{X}, REST(\mathbf{Y})), DTW(REST(\mathbf{X}), \mathbf{Y}), DTW(REST(\mathbf{X}), REST(\mathbf{Y}))\}$$

  where $REST(\mathbf{X}) = (x_1, .., x_n)$

Distance (Numerical Time Series)

e.g. Dynamic Time Warping

- Solved efficiently using dynamic programming by building an $n \times m$ distance matrix

$$distMatirx = \begin{bmatrix} D(x_0, y_0) & D(x_0, y_1) & \cdots & D(x_0, y_m) \\ D(x_1, y_0) & D(x_1, y_1) & \cdots & D(x_1, y_m) \\ \vdots & \ddots & & \vdots \\ D(x_n, y_0) & D(x_n, y_1) & \cdots & D(x_n, y_m) \end{bmatrix}$$

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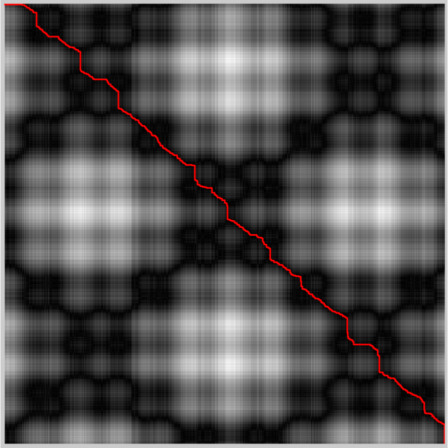

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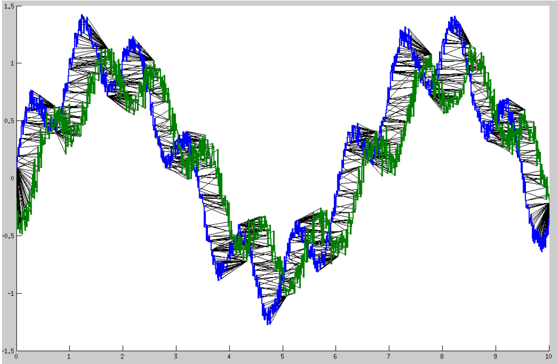
- Also used for aligning sequences



Distance (Symbolic)

- Distance is not always between numerical data
- Distance between symbolic data is less well-defined, but gaining interest (e.g. text data)
- Distance in text could be:
  - syntactic
  - semantic

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Syntactic - e.g. Hamming Distance

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- For binary strings, hamming distance equals L_1

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- Measures the *minimum* number of ‘operations’ required to transform one sequence of characters into another
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- used in spelling correction, DNA string comparisons

Distance (Symbolic)

Semantic - e.g. WUP Distance

- Built on top of a hierarchy of word semantics
- Most commonly used is WordNet (Princeton)
  `http://wordnet.princeton.edu/`
- WordNet contains more than 117,000 synsets (synset: set of one or more synonyms that are interchangeable in some context)

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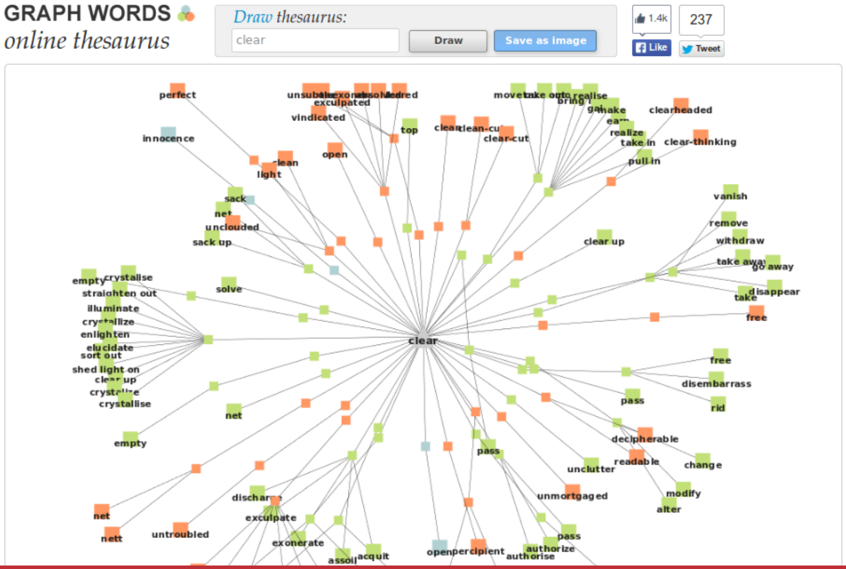

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  - antonymy (strong contract) e.g. wet $\leftrightarrow$ dry

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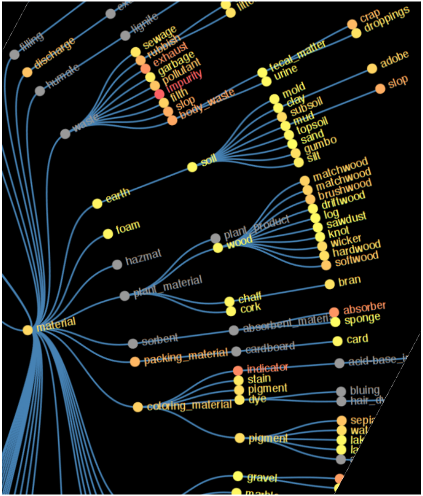
Semantic - e.g. hyponymy



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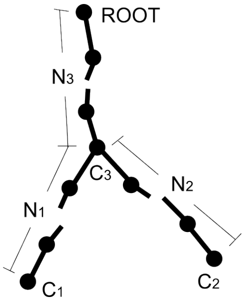

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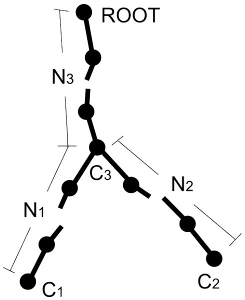

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  `http://lyle.smu.edu/~tspell/jaws/`

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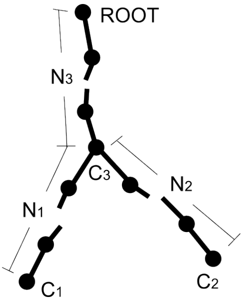

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- or online: `http://ws4jdemo.appspot.com/`

Distance - Conclusion

- Once you define a distance measure on your data, you can perform numeric operations
- Different distance measures will enable you to use the same data for various goals

Mean and Variance (Reminder)

For one-dimensional data $\{x_1, .., x_n\}$,
Mean: [average]

$$\mu = \frac{1}{N} \sum_i x_i$$

Variance: [spread]

$$\sigma^2 = \frac{1}{N-1} \sum_i (x_i - \mu)^2$$

Standard Deviation:

$$\sigma = \sqrt{\frac{1}{N-1} \sum_i (x_i - \mu)^2}$$

Mean and Covariance

For multi-dimensional data $\{\mathbf{x}_1, .., \mathbf{x}_n\}$ where $\mathbf{x}_i$ is an m-dimensional vector,
Mean: calculated independently for each dimension

$$\boldsymbol{\mu} = \frac{1}{N} \sum_i \mathbf{x}_i$$

Variance can still be computed along each dimension

Mean and Covariance

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Variance can still be computed along each dimension

Covariance Matrix: spread and correlation

$$\begin{aligned} \Sigma &= \frac{1}{N-1} \sum_i (\mathbf{x}_i - \boldsymbol{\mu})^2 \\ &= \frac{1}{N-1} \sum_i (\mathbf{x}_i - \boldsymbol{\mu})^T (\mathbf{x}_i - \boldsymbol{\mu}) \end{aligned}$$

WARNING: Σ is the capital letter of σ, not the summation sign!

Covariance Matrix

In two dimensions,

$$\Sigma = \frac{1}{N-1} \sum_i \begin{bmatrix} (v_{i1} - \mu_1)^2 & (v_{i1} - \mu_1)(v_{i2} - \mu_2) \\ (v_{i1} - \mu_1)(v_{i2} - \mu_2) & (v_{i2} - \mu_2)^2 \end{bmatrix}$$

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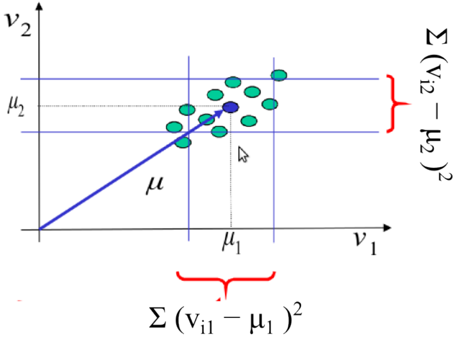

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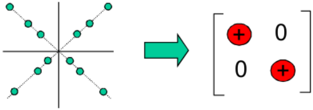
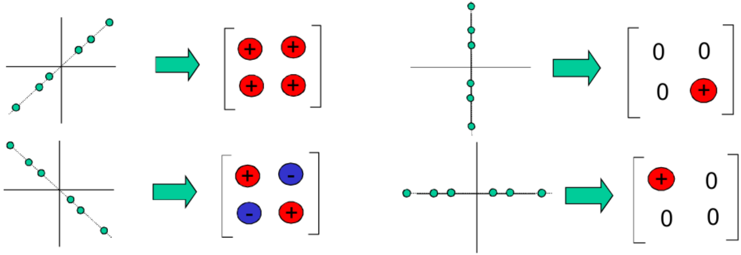
- In addition to the variances along each dimension, the covariance matrix measures the correlation between components
- A positive covariance between two components means a proportional relationship between the variables.
- A negative covariance value indicates and inverse proportional relationship.

Covariance Matrix

$$C = \frac{1}{N-1} \sum_i \begin{bmatrix} (v_{i1}-\mu_1)^2 & (v_{i1}-\mu_1)(v_{i2}-\mu_2) \\ (v_{i1}-\mu_1)(v_{i2}-\mu_2) & (v_{i2}-\mu_2)^2 \end{bmatrix}$$





Covariance Matrix

In three dimensions,

$$\Sigma = \frac{1}{N-1}\sum_i \begin{bmatrix} (v_{i1}-\mu_1)^2 & (v_{i1}-\mu_1)(v_{i2}-\mu_2) & (v_{i1}-\mu_1)(v_{i3}-\mu_3) \\ (v_{i1}-\mu_1)(v_{i2}-\mu_2) & (v_{i2}-\mu_2)^2 & (v_{i2}-\mu_2)(v_{i3}-\mu_3) \\ (v_{i1}-\mu_1)(v_{i3}-\mu_3) & (v_{i2}-\mu_2)(v_{i3}-\mu_3) & (v_{i3}-\mu_3)^2 \end{bmatrix}$$

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Covariance matrix is always

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Covariance matrix is always

- square and symmetric

Covariance Matrix

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Covariance matrix is always

- square and symmetric
- variances on the diagonal

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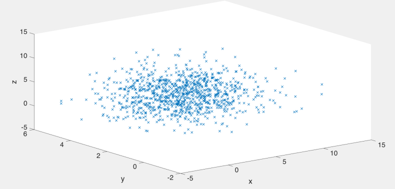
Covariance matrix is always

- square and symmetric
- variances on the diagonal
- covariance between each pair of dimensions is included in non-diagonal elements

Covariance Matrix - e.g.

For the covariance matrix,

$$\Sigma = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & 7 \end{bmatrix}$$



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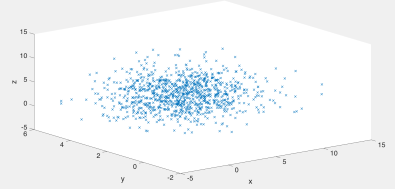

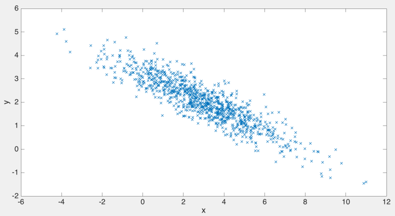

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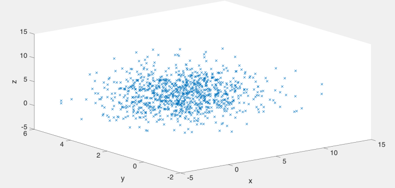

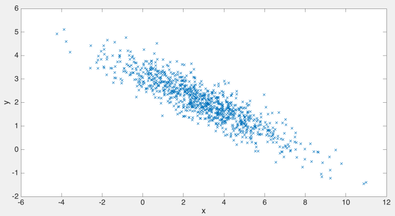

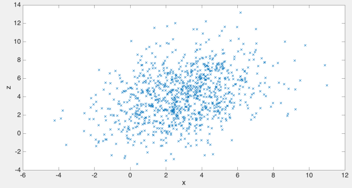

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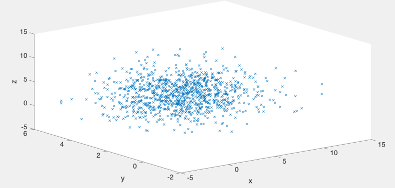

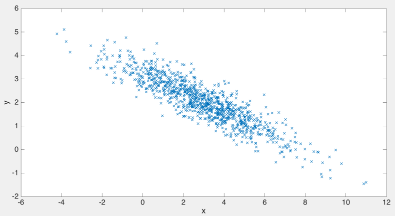

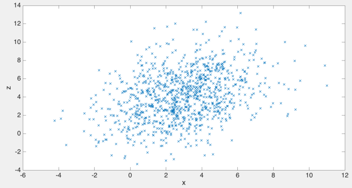

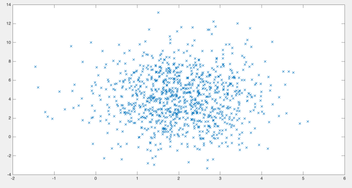

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Covariance Matrix

Definition

For a square matrix A,
if there exists a non-zero column vector v where

$$Av = \lambda v$$

then,
$v \rightarrow$ eigenvector of matrix A
$\lambda \rightarrow$ is eigenvalue of matrix A

e.g.

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}, v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Covariance Matrix

- To calculate eigenvectors of a square matrix, solve $|A - \lambda\mathbf{I}| = 0$ where
  - $\mathbf{I}$ is the identity matrix
  - |A| is the determinant of the matrix
- For 2×2 matrices, two eigenvalues are found λ_1, λ_2

e.g.

$$A - \lambda\mathbf{I} = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 2 & 3 - \lambda \end{bmatrix}$$

$$|A - \lambda\mathbf{I}| = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

$$\lambda_1 = 1, \lambda_2 = 2$$

Covariance Matrix

- After the eigenvalues are found, the eigenvectors can be calculated

For $\lambda_1 = 1$

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \tag{2}$$

$v_{11} = -v_{12}$

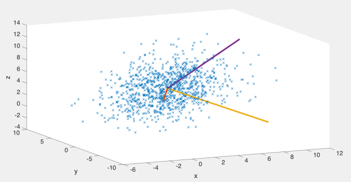
$\|v_1\| = 1$ (Normalising vector)

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Covariance Matrix

- Eigenvectors and eigenvalues define **principal axes** and spread of points along directions
- Major axis - eigenvector corresponding to larger eigenvalue
- Minor axis - eigenvector corresponding to smaller eigenvalue
- Represented using major and minor axes of ellipses

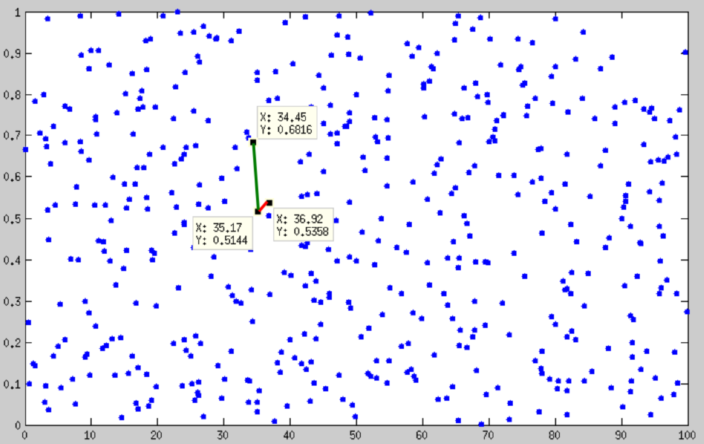
Covariance Matrix-ex



- $\lambda_1 = 0.08 \qquad \lambda_2 = 4.52 \qquad \lambda_3 = 8.40$
- $v_1 = \begin{bmatrix} -0.42 \\ -0.90 \\ 0.12 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.71 \\ -0.40 \\ -0.57 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0.57 \\ -0.15 \\ 0.81 \end{bmatrix}$
- Principal/Major axis is v_3 (corresponding to largest eigenvalue)

Data Characteristic - Data Normalisation

- Multi-dimensional data may need to be normalised before distance is calculated.



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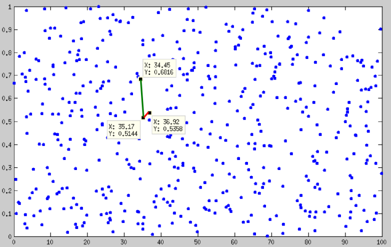

Data Characteristic - Data Normalisation

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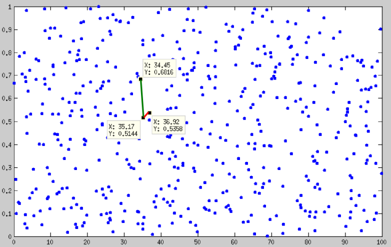
  1. Rescaling

$$x' = \frac{x - min(x)}{max(x) - min(x)}$$



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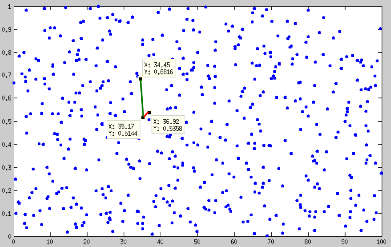
  2. Standardisation (also known as *z*-score)

$$x' = \frac{x - \mu}{\sigma}$$



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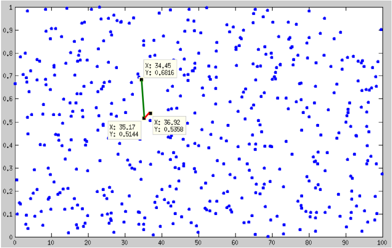
  3. Scaling to unit length

$$x' = \frac{x}{\|x\|}$$



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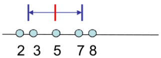
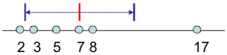


$\mu=5 \quad \sigma=2.28$



$\mu=7 \quad \sigma=4.93$

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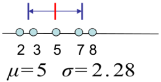
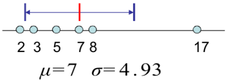
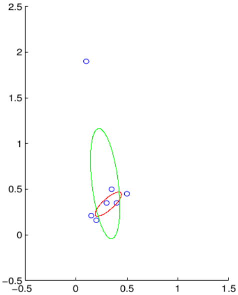
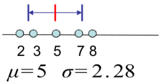


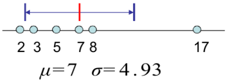


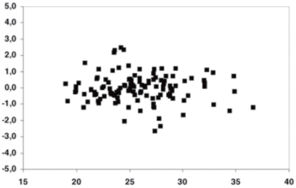


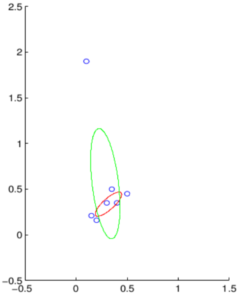
Data Characteristic - Outliers

- Mean, variance and covariance can provide concise description of ‘average’ and ‘spread’
  - but not when outliers are present in the data
  - **outliers**: small number of points with values significantly different from that other points
  - usually due to fault in measurement
  - not always easy to remove









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Mean vs. Median

- An alternative to arithmetic mean is the **median value**
- But median is difficult to work with
- e.g. median of two sets cannot be defined in terms of the individual medians

Note - Sample Variance vs. Variance

Given sample $\{x_1, x_2, .., x_N\}$

$$\mu \approx \bar{x} = \frac{1}{N}\sum_i x_i \tag{3}$$

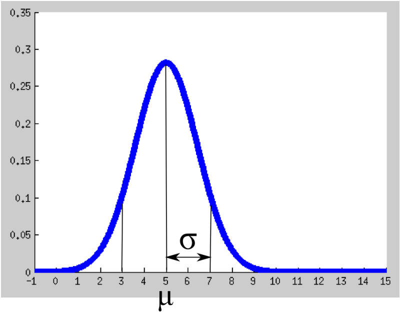
$$\sigma^2 \approx s^2 = \frac{1}{N-1}\sum_i (x_i - \bar{x})^2 \tag{4}$$

- These are only estimates of the ‘true’ mean and variance
- $N - 1$ gives unbiased estimate of the variance
- As $N \to \infty$
  - $\bar{x} \to \mu$
  - $s^2 \to \sigma^2$

Normal Distribution (Reminder)

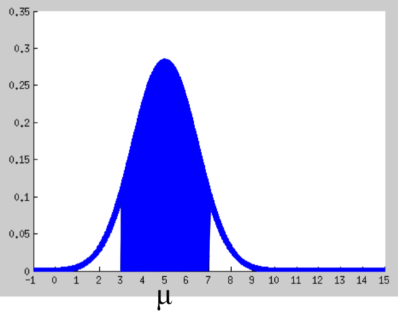
For a normal distribution $\mathcal{N}(\mu, \sigma^2)$ in one dimension, the probability density function (pdf) can be calculated as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{5}$$



Normal Distribution (Reminder)

- 68% of the sample *should* lies within one standard deviation of the mean
- 95% of that area lies within two standard deviations of the mean
- 99.9% of that area lies within three standard deviations of the mean



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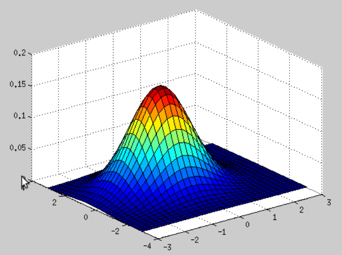
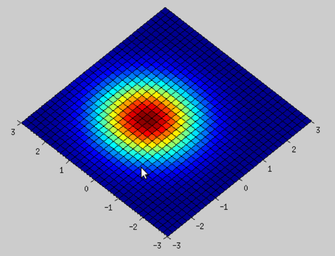
Normal Distribution - Multi-dimensional

For multi-dimensional normal distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ in M dimensions, the probability density function (pdf) can be calculated as

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^M |\Sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})} \tag{6}$$





WARNING: **Σ is the capital letter of σ, not the summation sign!**

Further Reading

- Fundamentals of Multimedia
  Li and Drew (2004)
  - Section 6.1 Digitization of Sound
- Applied Multivariate Statistical Analysis
  Hardle and Simar (2003)
  - Section 1.2
  - Section 1.4
  - Section 3.1
  - Section 3.2
- Linear Algebra and its applications
  Lay (2012)
  - Section 6.5
  - Section 6.6
- Advances in Data Mining Knowledge Discovery and applications
  Karahoca (Ed.) (2012)
  - Chapter 3. Similarity Measures and Dimensionality Reduction Techniques for Time Series Data Mining