COMS21103: Data Structures and Algorithms

Problem Sheet - Week 12

1. **Polynomial Multiplication**

   For the two polynomials $f(x) = x^2 - 3$, $g(x) = -2x$, perform polynomial multiplication using

   (a) standard school-book multiplication method

   The standard schoolbook multiplication follows the equation

$$c_i = \sum_{j=0}^{i} a_j \cdot b_{i-j}$$

   Accordingly,

$$c_0 = \sum_{j=0}^{0} a_j \cdot b_{-j} = a_0 \cdot b_0 = -3 \cdot 0 = 0$$

$$c_1 = \sum_{j=0}^{1} a_j \cdot b_{1-j} = a_0 \cdot b_1 + a_1 \cdot b_0 = -3 \cdot -2 + 0 \cdot 0 = 6$$

$$c_2 = \sum_{j=0}^{2} a_j \cdot b_{2-j} = a_0 \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot b_0 = -3 \cdot 0 + 0 \cdot -2 + 1 \times 0 = 0$$

   notice that b_2 = 0 as it is only a polynomial of degree 2

$$c_3 = \sum_{j=0}^{3} a_j \cdot b_{3-j} = a_0 \cdot b_3 + a_1 \cdot b_2 + a_2 \cdot b_1 + a_3 \cdot b_0 = 0 \cdot 0 + 0 \cdot 0 + 1 \cdot -2 + 0 \cdot 0 = -2$$

   The resulting polynomial is $-2x^3 + 6x$

   (b) Discrete Fourier Transform

   First, evaluate A and B at ω_4^0, ω_4^1, ω_4^2, ω_4^3, that is $1, i, -1, -i$

$$\begin{aligned}
A(\omega_4^0) &= A(1) = (1)^2 - 3 = -2, & B(\omega_4^0) &= B(1) = -2(1) = -2 \quad (1)\\
A(\omega_4^1) &= A(i) = (i)^2 - 3 = -4, & B(\omega_4^1) &= B(i) = -2(i) = -2i \quad (2)\\
A(\omega_4^2) &= A(-1) = (-1)^2 - 3 = -2, & B(\omega_4^2) &= B(-1) = -2(-1) = 2 \quad (3)\\
A(\omega_4^3) &= A(-i) = (-i)^2 - 3 = -4, & B(\omega_4^3) &= B(-i) = -2(-i) = 2i \quad (4)
\end{aligned}$$

   Second, multiply point value representations
   $C(1) = A(1) \cdot B(1) = (-2) \cdot (-2) = 4$
   $C(i) = A(i) \cdot B(i) = (-4) \cdot (-2i) = 8i$
   $C(-1) = A(-1) \cdot B(-1) = (-2) \cdot (2) = -4$
   $C(-i) = A(-i) \cdot B(-i) = (-4) \cdot (2i) = -8i$
   Finally, interpolate... There are two ways to do this

   Method 1: Algebraic,
   For the function $C(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, solve the set of linear equations with four variables

Given the tuples $(1,4)$, $(i,8i)$, $(-1,-4)$, $(-i,-8i)$

$$c_0 + c_1 + c_2 + c_3 = 4 \quad (5)$$
$$c_0 + c_1 i - c_2 - c_3 i = 8i \quad (6)$$
$$c_0 - c_1 + c_2 - c_3 = -4 \quad (7)$$
$$c_0 - c_1 i - c_2 - c_3 i = -8i \quad (8)$$

When solving the equations, the answer would be $C = (0,6,0,-2)$

Method 2: from the lecture slides:
1. Switching roles of a and y
2. Replace ω_n *by* ω_n^{-1},
3. Divide the final result by n.
Thus,
evaluate $D(x) = 4 + 8ix - 4x^2 - 8ix^3$ *at* $(\omega_4^0, \omega_4^{-1}, \omega_4^{-2}, \omega_4^{-3})$
$D(\omega_4^0) = D(1) = 4 + 8i - 4 - 8i = 0$
$D(\omega_4^{-1}) = D(-i) = 4 + 8i(-i) - 4(-i)^2 - 8i(-i)^3 = 4 + 8 + 4 + 8 = 24$
$D(\omega_4^{-2}) = D(-1) = 4 + 8i(-1) - 4(-1)^2 - 8i(-1)^3 = 4 - 8i - 4 + 8i = 0$
$D(\omega_4^{-3}) = D(i) = 4 + 8i(i) - 4(i)^2 - 8i(i)^3 = 4 - 8 + 4 - 8 = -8$
Tuple is $(0,24,0,-8)$ *divided by* $N = 4$ *equals* $C = (0,6,0,-2)$

(c) Fast Fourier Transform

Livescribe document can be found on the unit's webpage