Assignment - Portfolio 2
Mathematical Methods for CS

Andreas Georgiou - ag14774

May 16, 2015

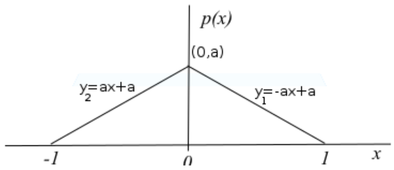
Question 1
Part a

A: Number of requests per hour to server A. $E(A) = 60 \quad Var(A) = 10^2 = 100$
B: Number of requests per hour to server B. $E(B) = 80 \quad Var(B) = 5^2 = 25$

$$E(A+B) = E(A) + E(B) = 60 + 80 = 140$$
$$^*Var(A+B) = Var(A) + Var(B) = 100 + 25 = 125$$
$$\sigma_{A+B} = \sqrt{125} = 5\sqrt{5} \approx 11.18$$

*The assumption was made that the number of requests per hour to server A do not affect the number of requests per hour to server B or vice-versa. (Assume that A and B are independent).

Part b



$$\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1 \Rightarrow \int_0^1 (-ax+a)\,\mathrm{d}x + \int_{-1}^{0} (ax+a)\,\mathrm{d}x = 1$$
The area of the triangle is equal to 1.
$\Rightarrow \frac{1-(-1)\times a}{2} = 1 \Rightarrow a = 1$

$$p(x) = \begin{cases} y_1 = -x+1 & x \geq 0 \\ y_2 = x+1 & x < 0 \end{cases}$$

(i)
$$E(x) = \int_{-\infty}^{\infty} xp(x)\,\mathrm{d}x = \int_0^1 x(-x+1)\,\mathrm{d}x + \int_{-1}^{0} x(x+1)\,\mathrm{d}x = \int_0^1 (-x^2+x)\,\mathrm{d}x + \int_{-1}^{0} (x^2+x)\,\mathrm{d}x$$
$$= \left[\frac{-x^3}{3} + \frac{x^2}{2} + c\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^2}{2} + c\right]_{-1}^{0} = -\frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = 0$$
$\boxed{E(x) = 0}$

(ii)
$$Var(x) = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\,\mathrm{d}x = \int_0^1 x^2(-x+1)\,\mathrm{d}x + \int_{-1}^{0} x^2(x+1)\,\mathrm{d}x = \int_0^1 (-x^3+x^2)\,\mathrm{d}x + \int_{-1}^{0} (x^3+x^2)\,\mathrm{d}x$$
$$= \left[\frac{-x^4}{4} + \frac{x^3}{3} + c\right]_0^1 + \left[\frac{x^4}{4} + \frac{x^3}{3} + c\right]_{-1}^{0} = -\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3} = \frac{1}{6}$$
$\boxed{Var(x) = \frac{1}{6}}$

(iii)
$$P(0.2 \leq x \leq 0.8) = \int_{0.2}^{0.8} p(x)\,\mathrm{d}x = \int_{0.2}^{0.8} (-x+1)\,\mathrm{d}x = \left[\frac{-x^2}{2} + x + c\right]_{0.2}^{0.8} = -\frac{0.8^2}{2} + 0.8 + \frac{0.2^2}{2} - 0.2 = \frac{3}{10}$$
$\boxed{P(0.2 \leq x \leq 0.8) = \frac{3}{10}}$

Part c

X = discrete variable that represents the number of requests per hour
$A \sim N(60, 10^2)$
$Z = \frac{(A-\mu)}{\sigma}$
We apply continuity correction:
$P(X > 70) \approx P(A \geq 70.5) = P\left(\frac{A-\mu}{\sigma} \geq \frac{70.5-60}{10}\right) = P(Z \geq 1.05) = 0.5 - \Phi(1.05)^* = 0.5 - 0.3531 = 0.1469$

$\boxed{P(A \geq 70) = 0.1469}$
14.69% of the time

*From standard normal distribution tables

Question 2

a) $\dfrac{\partial f}{\partial x} = 3x^2 \qquad \dfrac{\partial f}{\partial y} = 3y^2 \qquad \dfrac{\partial f}{\partial z} = 3z^2$

b) $f(g) = e^g \quad g(x) = x + y$

$$\frac{\mathrm{d}f}{\mathrm{d}g} = e^g \qquad \frac{\mathrm{d}g}{\mathrm{d}x} = 1 \qquad \frac{\mathrm{d}g}{\mathrm{d}y} = 1 \qquad \frac{\mathrm{d}g}{\mathrm{d}z} = 0$$

Using chain rule:

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}g} \times \frac{\mathrm{d}g}{\mathrm{d}x} = e^{x+y} \qquad \frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}g} \times \frac{\mathrm{d}g}{\mathrm{d}y} = e^{x+y} \qquad \frac{\partial f}{\partial z} = \frac{\mathrm{d}f}{\mathrm{d}g} \times \frac{\mathrm{d}g}{\mathrm{d}z} = 0$$

c) $u = xy \qquad v = \sin xy$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = y \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = x \qquad \frac{\mathrm{d}u}{\mathrm{d}z} = 0$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (xy)'(\sin xy)' = y \cos xy \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = (xy)'(\sin xy)' = x \cos xy \qquad \frac{\mathrm{d}v}{\mathrm{d}z} = 0$$

Product rule: $(uv)' = vu' + uv'$

$$\frac{\partial f}{\partial x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x} = y \sin xy + xy^2 \cos xy \qquad \frac{\partial f}{\partial y} = v\frac{\mathrm{d}u}{\mathrm{d}y} + u\frac{\mathrm{d}v}{\mathrm{d}y} = x \sin xy + x^2 y \cos xy \qquad \frac{\partial f}{\partial z} = 0$$

d) $u = xy \qquad v = \ln z$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = y \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = x \qquad \frac{\mathrm{d}u}{\mathrm{d}z} = 0$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = 0 \qquad \frac{\mathrm{d}v}{\mathrm{d}z} = \frac{1}{z}$$

Product rule: $(uv)' = vu' + uv'$

$$\frac{\partial f}{\partial x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x} = y \ln z \qquad \frac{\partial f}{\partial y} = v\frac{\mathrm{d}u}{\mathrm{d}y} + u\frac{\mathrm{d}v}{\mathrm{d}y} = x \ln z \qquad \frac{\partial f}{\partial z} = v\frac{\mathrm{d}u}{\mathrm{d}z} + u\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{xy}{z}$$

e) $u = x \qquad v = \sin xyz$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = 0 \qquad \frac{\mathrm{d}u}{\mathrm{d}z} = 0$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (xyz)'(\sin xyz)' = yz \cos xyz \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = (xyz)'(\sin xyz)' = xz \cos xyz \qquad \frac{\mathrm{d}v}{\mathrm{d}z} = (xyz)'(\sin xyz)' = xy \cos xyz$$

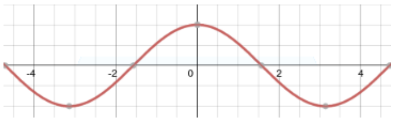
Product rule: $(uv)' = vu' + uv'$

$$\frac{\partial f}{\partial x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x} = \sin xyz + xyz \cos xyz \qquad \frac{\partial f}{\partial y} = v\frac{\mathrm{d}u}{\mathrm{d}y} + u\frac{\mathrm{d}v}{\mathrm{d}y} = x^2 z \cos xyz \qquad \frac{\partial f}{\partial z} = v\frac{\mathrm{d}u}{\mathrm{d}z} + u\frac{\mathrm{d}v}{\mathrm{d}z} = x^2 y \cos xyz$$

Question 3

If $f(t) = f(-t) \Rightarrow even$ else $f(t) = -f(-t) \Rightarrow odd$

a)



From the graph we can clearly see that $\cos t = \cos(-t) \Rightarrow f(t) = f(-t) \Rightarrow even$

b)

$f(t) = t^3 + t$
$-f(-t) = -[(-t)^3 - t] = -(-t^3 - t) = t^3 + t = f(t)$
$f(t) = -f(-t) \Rightarrow odd$

c)

$f(t) = 3t^2 \sin t$

Since $\sin t$ is an odd function $\sin(-t) = -\sin t$
$-f(-t) = -[3(-t)^2 \sin(-t)] = -[3t^2(-\sin t)] = 3t^2 \sin t = f(t)$
$f(t) = -f(-t) \Rightarrow odd$

d)

$f(t) = t^3 + 2t^2 + t$
$-f(-t) = -[(-t)^3 + 2(-t)^2 - t] = -[-t^3 + 2t^2 - t] = t^3 - 2t + t$
$f(t) \neq -f(-t) \qquad f(t) \neq f(-t)$
$f(t)$ is neither even nor odd

e)

$$f(t) = |t| = \begin{cases} t & t \geq 0 \\ -t & t < 0 \end{cases}$$

$$f(-t) = |-t| = \begin{cases} t & t \geq 0 \\ -t & t < 0 \end{cases} = f(t)$$

$f(t) = f(-t) \Rightarrow even$

Question 4
Part a

$\vec{v}_1 = (-1, 0, 1) \qquad \vec{v}_2 = (1, -1, 2)$

$$\Omega = a\begin{pmatrix}-1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}-a\\0\\a\end{pmatrix} + \begin{pmatrix}b\\-b\\2b\end{pmatrix} = \begin{pmatrix}-a+b\\-b\\a+2b\end{pmatrix}$$

Since we want $\vec{v}_1$ to be orthogonal to $\vec{v}_3 \implies \vec{v}_1.\vec{v}_3 = 0$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} . \begin{pmatrix}-1\\0\\1\end{pmatrix} = -x + z$$

$-x + z = 0 \Rightarrow x = z$

We need a vector $\vec{v}_3$ such that its x-coordinate is equal to its z-coordinate and of the form $\begin{pmatrix}-a+b\\-b\\a+2b\end{pmatrix}$

$-a + b = a + 2b \Rightarrow -2a = b$

Hence, by substituing $b = -2a$ back to the general form of the subspace we get:

$$\vec{v}_3 = \begin{pmatrix}-3a\\2a\\-3a\end{pmatrix}$$

Any vector of that form is orthogonal to $\vec{v}_1$ and is also within the subspace Ω.

$$\boxed{\text{We choose an arbitrary value for } a = 1 \Rightarrow \vec{v}_3 = \begin{pmatrix}-3\\2\\-3\end{pmatrix}}$$

Part b

Convert $\vec{v}_3$ and $\vec{v}_1$ to unit vectors:

$\|\vec{v}_3\| = \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$

Unit vector $\vec{v}_3' = \frac{1}{\sqrt{22}}(-3, 2, -3) = \left(\frac{-3}{\sqrt{22}}, \frac{2}{\sqrt{22}}, \frac{-3}{\sqrt{22}}\right)$

$\|\vec{v}_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

Unit vector $\vec{v}_1' = \frac{1}{\sqrt{2}}(-1, 0, 1) = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

The vectors $\{\vec{v}_1', \vec{v}_3'\}$ form an orthonormal basis(orthogonal unit vectors) which spans the subspace Ω since $\vec{v}_2$ can be written as a linear combination of the first two.

Now we are looking for projections of $\vec{w}$ and $\vec{z}$ ($\vec{w}'$ and $\vec{z}'$ respectively) of the form $a\vec{v}_1' + b\vec{v}_3'$ where a, b are scalars. Because $\{\vec{v}_1', \vec{v}_3'\}$ is an orthonormal basis, we can easily calculate the scalars a, b using the dot product, i.e. the projection of the vector onto each of the basis vectors:

$$a_1 = \vec{w}.\vec{v}_1' \Rightarrow a = \begin{pmatrix}1\\-2\\5\end{pmatrix} . \begin{pmatrix}\frac{-1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = 1 \times \frac{-1}{\sqrt{2}} + 5 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$b_1 = \vec{w}.\vec{v}_3' \Rightarrow a = \begin{pmatrix}1\\-2\\5\end{pmatrix} . \begin{pmatrix}\frac{-3}{\sqrt{22}}\\\frac{2}{\sqrt{22}}\\\frac{-3}{\sqrt{22}}\end{pmatrix} = 1 \times \frac{-3}{\sqrt{22}} - 2 \times \frac{2}{\sqrt{22}} + 5 \times \frac{-3}{\sqrt{22}} = -\sqrt{22}$$

$$\vec{w}' = 2\sqrt{2} \times \begin{pmatrix}\frac{-1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} - \sqrt{22} \times \begin{pmatrix}\frac{-3}{\sqrt{22}}\\\frac{2}{\sqrt{22}}\\\frac{-3}{\sqrt{22}}\end{pmatrix} = \begin{pmatrix}-2\\0\\2\end{pmatrix} + \begin{pmatrix}3\\-2\\3\end{pmatrix} = \begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$\boxed{\vec{w}' = \begin{pmatrix}1\\-2\\5\end{pmatrix}}$$

$$a_2 = \vec{z}.\vec{v}_1' \Rightarrow a = \begin{pmatrix}-2\\-2\\3\end{pmatrix} . \begin{pmatrix}\frac{-1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = -2 \times \frac{-1}{\sqrt{2}} + 3 \times \frac{1}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

$$b_2 = \vec{z}.\vec{v}_3' \Rightarrow a = \begin{pmatrix}-2\\-2\\3\end{pmatrix} . \begin{pmatrix}\frac{-3}{\sqrt{22}}\\\frac{2}{\sqrt{22}}\\\frac{-3}{\sqrt{22}}\end{pmatrix} = -2 \times \frac{-3}{\sqrt{22}} - 2 \times \frac{2}{\sqrt{22}} + 3 \times \frac{-3}{\sqrt{22}} = -\frac{7\sqrt{22}}{22}$$

$$\vec{z}' = \frac{5\sqrt{2}}{2} \times \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} - \frac{7\sqrt{22}}{22} \times \begin{pmatrix} \frac{-3}{\sqrt{22}} \\ \frac{2}{\sqrt{22}} \\ \frac{-3}{\sqrt{22}} \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 0 \\ \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{21}{22} \\ -\frac{14}{22} \\ \frac{21}{22} \end{pmatrix} = \begin{pmatrix} -\frac{34}{22} \\ -\frac{14}{22} \\ \frac{76}{22} \end{pmatrix}$$

$$\boxed{\vec{z}' = \begin{pmatrix} -\frac{34}{22} \\ -\frac{14}{22} \\ \frac{76}{22} \end{pmatrix}}$$

Part c

$\vec{w}$ is the same vector as its projection $\vec{w}'$. Since the projection of a vector onto a subspace is how much of that vector is within that subspace, it is clear that vector $\underline{\vec{w} \text{ is within the subspace } \Omega}$.

Because $\vec{z}'$ is the closest vector to $\vec{z}$, vector $\underline{\vec{z} - \vec{z}' \text{ must be orthogonal to the subspace}}$, that is, orthogonal to all the vectors of the spanning set.

$$\vec{z} - \vec{z}' = \begin{pmatrix} -2 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{34}{22} \\ -\frac{14}{22} \\ \frac{76}{22} \end{pmatrix} = \begin{pmatrix} -\frac{5}{11} \\ -\frac{15}{11} \\ -\frac{5}{11} \end{pmatrix}$$

Check orthogonality using the dot product. If two vectors are orthogonal to each other, their dot product is equal to 0:

$$(\vec{z} - \vec{z}') \,.\vec{v}_1 = \begin{pmatrix} -\frac{5}{11} \\ -\frac{15}{11} \\ -\frac{5}{11} \end{pmatrix} . \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = (-1) \times \frac{-5}{11} - \frac{5}{11} = 0$$

$$(\vec{z} - \vec{z}') \,.\vec{v}_2 = \begin{pmatrix} -\frac{5}{11} \\ -\frac{15}{11} \\ -\frac{5}{11} \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = -\frac{5}{11} - 1 \times \frac{-15}{11} + 2 \times \frac{-5}{11} = 0$$

$$\boxed{\text{Hence } \vec{z} - \vec{z}' = \begin{pmatrix} -\frac{5}{11} \\ -\frac{15}{11} \\ -\frac{5}{11} \end{pmatrix} \text{ is orthogonal to the subspace } \Omega.}$$

Question 5
Part a

$A\vec{v} = \lambda I \vec{v}$

Since we are looking for non-zero vectors $\vec{v} \Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 7-\lambda & -2 & 3 \\ 8 & -1-\lambda & 6 \\ 4 & -2 & 6-\lambda \end{vmatrix} = 0$$

$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 0$
$A_{ij} = (-1)^{i+j}|M_{ij}|$

$$A_{11} = \begin{vmatrix} -1-\lambda & 6 \\ -2 & 6-\lambda \end{vmatrix} = (-1-\lambda)(6-\lambda) - 6\times(-2) = -6 + \lambda - 6\lambda + \lambda^2 + 12 = \lambda^2 - 5\lambda + 6$$

$$A_{12} = -\begin{vmatrix} 8 & 6 \\ 4 & 6-\lambda \end{vmatrix} = -[8\times(6-\lambda) - 4\times 6] = -48 + 8\lambda + 24 = 8\lambda - 24$$

$$A_{13} = \begin{vmatrix} 8 & -1-\lambda \\ 4 & -2 \end{vmatrix} = -2\times 8 - 4\times(-1-\lambda) = -16 + 4 + 4\lambda = 4\lambda - 12$$

$(7-\lambda)\times(\lambda^2 - 5\lambda + 6) - 2\times(8\lambda - 24) + 3\times(4\lambda - 12) = (7-\lambda)\times(\lambda-3)\times(\lambda-2) - 16\times(\lambda-3) + 12\times(\lambda-3)$
$\Rightarrow (7-\lambda)(\lambda-3)(\lambda-2) - 4(\lambda-3) = (\lambda-3)\times[(7-\lambda)(\lambda-2) - 4] = (\lambda-3)\times(9\lambda - 18 - \lambda^2)$

$(\lambda - 3) = 0 \Rightarrow \lambda_1 = 3$
$-\lambda^2 + 9\lambda - 18 = 0 \Rightarrow (\lambda-6)(\lambda-3) = 0$
$\lambda_2 = 6 \qquad \lambda_3 = 3$

$\boxed{\text{Eigenvalues are } \lambda = 3 \text{ and } \lambda_2 = 6}$.

For $\lambda = 3$:

$$\begin{bmatrix} 4 & -2 & 3 \\ 8 & -4 & 6 \\ 4 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix(Reduce to row echelon form):

$$\begin{bmatrix} 4 & -2 & 3 & 0 \\ 8 & -4 & 6 & 0 \\ 4 & -2 & 3 & 0 \end{bmatrix} \overset{1}{\Rightarrow} \begin{bmatrix} 4 & -2 & 3 & 0 \\ 8 & -4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{2}{\Rightarrow} \begin{bmatrix} 4 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow 4v_1 - 2v_2 + 3v_3 = 0$$

1)Subtract first row from third row.
2)Multiply first row by two and subtract from second.

Set $v_3 = a$ and $v_2 = b \Rightarrow 4v_1 - 2b + 3a = 0 \Rightarrow v_1 = \frac{1}{2}b - \frac{3}{4}a \Rightarrow (\frac{1}{2}b - \frac{3}{4}a, b, a)$

We can choose an arbitrary basis for the plane by selecting two independent vectors which satisfy $4v_1 - 2v_2 + 3v_3 = 0$

$$\boxed{\begin{array}{ll} \vec{u} = (-0.75, 0, 1) & \\ \vec{w} = (0.5, 1, 0) & \qquad \vec{v} = a\begin{bmatrix} -0.75 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} \end{array}}$$

For $\lambda_2 = 6$:

$$\begin{bmatrix} 1 & -2 & 3 \\ 8 & -7 & 6 \\ 4 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix(Reduce to row echelon form):

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 8 & -7 & 6 & 0 \\ 4 & -2 & 0 & 0 \end{bmatrix} \overset{1}{\Rightarrow} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 6 & -12 & 0 \end{bmatrix} \overset{2}{\Rightarrow} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} v_1 - 2v_2 + 3v_3 = 0 \\ 9v_2 - 18v_3 = 0 \end{array}$$

1)Multiply first row with (-8) and add it to second row. Multiply first row by (-4) and add it to third row
2)Divide second row by 3 and third row by 2 and add them together.

Set $v_3 = a \Rightarrow 9v_2 - 18a = 0 \Rightarrow v_2 = 2a$
$v_1 - 2(2a) + 3a = 0 \Rightarrow v_1 = a \Longrightarrow (a, 2a, a)$

$$\boxed{\vec{v} = a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

Part b

Let P be the matrix whose collumns are the eigenvectors calculated in part a.

$$P = \begin{bmatrix} -0.75 & 0.5 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the inverse of P using the Gauss-Jordan elimination method:

$$\left[\begin{array}{cccccc} -0.75 & 0.5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{1}{\Rightarrow} \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -0.75 & 0.5 & 1 & 1 & 0 & 0 \end{array}\right] \overset{2}{\Rightarrow} \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -3 & 2 & 4 & 4 & 0 & 0 \end{array}\right] \overset{3}{\Rightarrow} \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 7 & 4 & 0 & 3 \end{array}\right] \overset{4}{\Rightarrow}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 4 & -2 & 3 \end{array}\right] \overset{5}{\Rightarrow} \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{8}{3} & \frac{7}{3} & -2 \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{2}{3} & 1 \end{array}\right] \Longrightarrow P^{-1} = \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ -\frac{8}{3} & \frac{7}{3} & -2 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

1)Swap first and third rows.
2)Multiply third row by 4
3)Multiply first row by 3 and add to third row.
4)Multiply second row by 2 and subtract from third row.
5)Divide third row by 3. Subtract it two times from the second row and once from the first.

$$D = P^{-1}AP = \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ -\frac{8}{3} & \frac{7}{3} & -2 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 3 \\ 8 & -1 & 6 \\ 4 & -2 & 6 \end{bmatrix} \begin{bmatrix} -0.75 & 0.5 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ -8 & 7 & -6 \\ 8 & -4 & 6 \end{bmatrix} \begin{bmatrix} -0.75 & 0.5 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

As expected, the values of the main diagonal correspond to the eigenvalues calculated in part a.

$$B = A^5 = PD^5P^{-1} = \begin{bmatrix} -0.75 & 0.5 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 6^5 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ -\frac{8}{3} & \frac{7}{3} & -2 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -0.75 & 0.5 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 243 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & 7776 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ -\frac{8}{3} & \frac{7}{3} & -2 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix} = \begin{bmatrix} -182.25 & 121.5 & 7776 \\ 0 & 243 & 15552 \\ 243 & 0 & 7776 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ -\frac{8}{3} & \frac{7}{3} & -2 \\ \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix} =$$

$$\boxed{\begin{bmatrix} 10287 & -5022 & 7533 \\ 20088 & -9801 & 15066 \\ 10044 & -5022 & 7776 \end{bmatrix}}$$