COMS10003 Workshop Sheet 18 outline solutions.

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Work sheet

1. Establish that
$$\int_{-\pi}^{\pi} dt \ \sin mt \ \cos nt = 0 \tag{1}$$
for integers n and m.

   Solution: This is an exercise in trigonometry and uses first the trignometric identity
$$\int_{-\pi}^{\pi} dt \ \sin mt \ \cos nt = \frac{1}{2}\int_{-\pi}^{\pi} dt \ [\sin(n-m)t - \sin(n+m)t)] = 0 \tag{2}$$
and then the fact that the integral of a trignometric function over its whole period is zero. This fact is used a lot; take sine for example, and let N be an integer. Consider
$$I = \int_{-\pi}^{\pi} \sin Ntdt \tag{3}$$
Let $s = Nt$ so $ds = Ndt$ and when $t = \pi$ then $s = N\pi$ and $t = -\pi$ then $s = -N\pi$. Hence
$$I = \frac{1}{N}\int_{-N\pi}^{N\pi} \sin tdt \tag{4}$$
and then, if N is an integer, periodicity gives
$$I = \int_{-\pi}^{\pi} \sin tdt = -\cos\pi + \cos\pi = 0 \tag{5}$$
The exception is $n = m$ in which case the first term is just zero.

2. What is $\sum_{n=0}^{\infty} a_n\delta_{n3}$?

   Solution: The δ_{n3} is zero unless $n = 3$, we are summing over all n, including $n = 3$ so the answer is a_3.

3. Show by checking whether $f(t) = -f(-t)$ for odd, $f(t) = f(-t)$ for even and neither for neither which of the following are odd, even or neither: $\sin t$, $t^3 + t$, $t^3 + 2t^2$ and $|t|$.

   Solutions: : odd, odd, neither, even.

4. Consider an odd function $f(t)$. By doing a change of variable $t' = -t$ show

$$\int_{-1}^{1} f(t)dt = 0 \tag{6}$$

Solution:

$$I = \int_{1}^{-1} f(-t')(-dt') = -\int_{1}^{-1} [-f(t')](-dt') = \int_{-1}^{1} f(t')dt' = -I \tag{7}$$

therefore $I = -I$ so $I = 0$.

5. If $f(t)$ is even and $g(t)$ is odd, what is $f(t)g(t)$?

Solution: Let $h(t) = f(t)g(t)$ so $h(-t) = f(-t)g(-t) = -f(t)g(t) = -h(t)$, so odd.

6. What is the Fourier series for

$$f(t) = \begin{cases} -1 & t \in (-\pi, -\pi/2) \\ 1 & t \in (-\pi/2, \pi/2) \\ -1 & t \in (\pi/2, \pi) \end{cases} \tag{8}$$

with $f(t+2\pi) = f(t)$. Try to use odd and even arguments to avoid doing the integral for b_n.

Solution: So, since $f(t)$ is even and $sinnt$ is odd, all the integrals involve sine are zero by the arguments above, the a_0 is also zero because the function is one and -1 an equal amount, making its integral zero. So

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos ntdt \tag{9}$$

where I have used the evenness of the integrand, what happens from $-\pi$ to zero is the same as what happens from zero to π. Now, putting in the value of $f(t)$

$$\frac{\pi}{2}a_n = \int_{0}^{\pi/2} \cos ntdt - \int_{\pi/2}^{\pi} \cos ntdt \tag{10}$$

and integrating

$$\frac{\pi}{2}a_n = \frac{1}{n}\sin nt]_{0}^{\pi/2} - \frac{1}{n}\sin nt]_{\pi/2}^{\pi} \tag{11}$$

or

$$\frac{\pi}{2}a_n = \frac{2}{n}\sin n\pi/2 - \sin n\pi \tag{12}$$

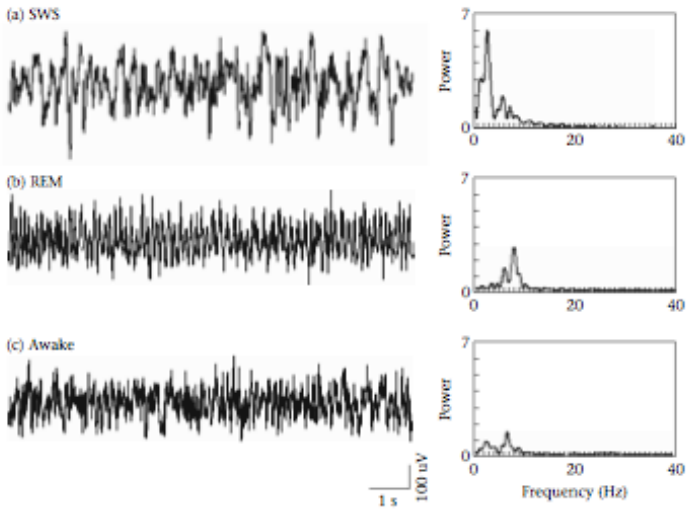
and $\sin n\pi = 0$ for all n, as for $n\pi/2$, if $n = 1$ this is the sine of $\pi/2$ which is one, for $n = 2$ it is zero, for $n = 3$ it is -1, for $n = 4$ it is zero again and then it repeats, so it is zero for n even and for n odd it is $(-1)^{(n-1)/2}$, so

$$a_n = \frac{4}{n\pi}(-1)^{(n-1)/2} \tag{13}$$

for n odd and zero for n even, giving

$$f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi}(-1)^{(n-1)/2} \cos nt \tag{14}$$

7. There are, roughly speaking, two different types of sleep, slow wave sleep (SWS) when the brain is largely in active and the body is relaxed, and rapid eye movement (REM) sleep when we dream, the brain has a similar activity pattern to waking and the body is paralysed. This was discovered using electroencephalogram (EEG), the recording of the electrical activity in the brain using electrodes placed on the scalp. Here we see, on the left, EEG traces for a mouse during REM, SWS and waking, on the right we see a measure of the size of the Fourier components, roughly the a_n and b_n for different frequencies: $\sin(nt)$ has period $2\pi/n$ and so has frequency $n/2\pi$. Although the brain is largely inactive during SWS what little activity there is is synchronized and is characterized by delta waves and theta waves. Can you guess what frequency delta waves and theta waves are at? [Picture taken from Lima SL, Rattenborg NC, Lesku JA, Amlaner CJ. (2005) *Sleeping under the risk of predation.* Animal Behaviour 70:723-736.]



Solution: Long question, short answer, you can see from the graph that delta is 1-3 Hz and theta is 7-8 Hz.