COMS10003: Workshop on Logic

Propositional Logic and Boolean Algebra

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Introduction

For this workshop you should read up on *Propositional Logic and on Boolean Algebra*, covering the following topics:

- Validity, satisfiability and contradictions;
- Logic equivalences and symbolic manipulation;
- Functional completeness of sets of connectives;
- Introduction to Boolean algebra, Boolean functions, normalisation and minimisation of circuits, including Karnaugh maps, and duality.

Reading will help you find solutions to the tasks in this worksheet.

Note, this worksheet contains tasks on several topics related to Propositional Logic and Boolean Algebra in no particular order. Review the worksheet. Schedule your work so that you find an answer to those parts of the worksheet that enable you to solve the rest of the questions alone.

For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Preparation

Extend your syntax/semantics reference card to include logical equivalences. This will help you remember the symbols used for the different connectives and the laws that you can use for symbolic manipulation. If necessary, add truth tables to this card for any new connectives if you find these hard to remember.

Before you start, compare the reference cards within your group. See whether you can improve your card based on what you have seen.

Task 1: Validity, Contradiction and Satisfiability

Task 1.1: For each of the following propositions, use truth tables to determine whether they are a Tautology, a Contradiction or a Contingency. Which are *satisfiable*? Explain and justify your answer.

1. $(p \Rightarrow (q \Rightarrow p))$

   Answer: *Tautology (true under every assignment), satisfiable (true under at least one value assignment)*

p	q	$(p \Rightarrow (q \Rightarrow p))$
T	T	T
T	F	T
F	T	T
F	F	T

2. $(p \vee \neg q) \Rightarrow (r \wedge p)$

   Answer: *Contingency (neither tautology nor contradiction), satisfiable*

p	q	r	$(p \vee \neg q) \Rightarrow (r \wedge p)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

3. $\neg p \vee (\neg p \Rightarrow q)$

   Answer: *Tautology, satisfiable*

p	q	$\neg p \vee (\neg p \Rightarrow q)$
T	T	T
T	F	T
F	T	T
F	F	T

4. $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$

   Answer: *Contradiction, not satisfiable*

p	q	$(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
T	T	F
T	F	F
F	T	F
F	F	F

5. $(\neg p \land \neg q) \Rightarrow (q \Leftrightarrow r)$
6. $(p \land q) \Rightarrow (p \lor q)$
7. $((p \land r) \lor (q \land \neg r)) \Leftrightarrow ((\neg p \land r) \lor (\neg q \land \neg r))$

Task 1.2: Which other techniques can you use to find the answers to the questions above? Where applicable, use a different technique and explain your approach.

Answer: *We could have used logical equivalences instead of truth tables.*

1. $(p \Rightarrow (q \Rightarrow p)) = \neg p \lor \neg q \lor p = \top \lor \neg q = \top$ *(Tautology, domination laws)*
2. *N/A*
3. $\neg p \lor (\neg p \Rightarrow q) = \neg p \lor p \lor q = \top \lor q = \top$ *(Tautology, domination laws)*
4. $$\begin{aligned} & (p \lor q) \land (p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q) && (1)\\ = & (((p \lor (q \land p)) \lor ((p \land \neg q) \lor \bot))) \land (((\neg p \lor (q \land \neg p) \lor ((\neg p \land \neg q) \lor \bot))) && (2)\\ = & ((p \lor (p \land \neg q)) \land ((\neg p \lor (\neg p \land \neg q)) && (3)\\ = & p \land \neg p && (4)\\ = & \bot && (5) \end{aligned}$$

   (Distributive laws, idempotent laws, identity laws, absorption)
5. *N/A*
6. $(p \land q) \Rightarrow (p \lor q) = (\neg p \lor \neg q) \lor (p \lor q) = \top$ *(De Morgan's law)*
7. *No solution given. Hint: Reduce to* $\bot$.

Task 2: Logic Equivalences

Task 2.1: Demonstrate the correctness of the basic Logic Equivalences given in the table in this week's lecture slides. Practice the names of these equivalences.

Answer:

1. *Identity laws*

p	$p \wedge \top$	$p \vee \bot$
T	T	T
F	F	F

2. *Domination laws*

p	$p \vee \top$	$p \wedge \bot$
T	T	F
F	T	F

3. *Double negation law*

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

4. *Commutative laws*

p	q	$p \vee q$	$q \vee p$	$p \wedge q$	$q \wedge p$
T	T	T	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	F	F

5. *Associative laws*

p	q	r	$p \vee (q \vee r)$	$(p \vee q) \vee r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	T	T	F	F
F	F	F	F	F	F	F

6. *Distributive laws*

p	q	r	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

7. *De Morgan's laws*

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	F	F
T	F	T	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

Task 2.2: Can you find further logical equivalences in the literature? Bring these to the workshop, demonstrate their correctness and, if possible, give an intuitive explanation for each. Discuss these new logical equivalences in your group.

Answer: *No answer given.*

Task 3: Functional Completeness

Task 3.1: Determine whether the following sets of connectives are functionally complete:

Before you start, explain what you need to do. For each subtask, take advantage of earlier results and make sure you cover all connectives.

Answer: *To show that a set is functionally complete, we need to show that we can express all truth functions using just this set by either giving a truth table or using the equivalences we saw in lectures.*

1. $\{\wedge, \vee, \neg\}$
   Answer: *(Note that negated functions have been omitted in the answer below. You may want to complete this as self study exercise.)*
   (a) $p \Rightarrow q = \neg p \vee q$
   (b) $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p) = (\neg p \vee q) \wedge (\neg q \vee p)$
   (c) $\top = p \Leftrightarrow p = (p \Rightarrow p) \wedge (p \Rightarrow p) = \neg p \vee p$
   (d) $p \oplus q = \neg(p \Leftrightarrow q) = \neg((p \Rightarrow q) \wedge (q \Rightarrow p)) = \neg((\neg p \vee q) \wedge (\neg q \vee p)) = (\neg(\neg p \vee q) \vee \neg(\neg q \vee p)) = (p \wedge \neg q) \vee (q \wedge \neg p)$
   (e) $\bot = p \oplus p = (p \wedge \neg p) \vee (p \wedge \neg p) = (p \wedge \neg p)$
   (f) $p = p \wedge p = p \vee p$ *(Projection)*
   (g) $p \uparrow q = \neg(p \wedge q)$
   (h) $p \downarrow q = \neg(p \vee q)$
2. $\{\wedge, \neg\}$
   Answer: *We already know that $\{\wedge, \vee, \neg\}$ is functionally complete, and that any formula can be transformed into DNF/CNF. (This is VERY important to state. Alternatively, the steps in the above solution would need to be repeated here.) So, now we just have to express $\vee$ using just $\{\wedge, \neg\}$.*
   (a) $p \vee q = \neg(\neg p \wedge \neg q)$
3. $\{\vee, \neg\}$
   Answer: *We already know that $\{\wedge, \vee, \neg\}$ and $\{\wedge, \neg\}$ are functionally complete, so now we just have to express $\wedge$ using just $\{\vee, \neg\}$*
   (a) $p \wedge q = \neg(\neg p \vee \neg q)$
4. $\{\vee, \wedge\}$
   Answer: *This set is not functionally complete; for example, $p \uparrow q = \neg(p \wedge q) = \neg p \vee \neg q$ cannot be expressed without $\neg$.*
5. $\{\uparrow\}$
   Answer: *We know that the set $\{\wedge, \neg\}$ is functionally complete, so we just have to express the functions in this set using NAND:*

(a) $\neg p = p \uparrow p$

(b) $p \wedge q = (p \uparrow q) \uparrow (p \uparrow q)$

6. $\{\downarrow\}$
Answer: *We know that the set $\{\vee, \neg\}$ is functionally complete, so we just have to express the functions in this set using NOR:*

(a) $\neg p = p \downarrow p$

(b) $p \vee q = (p \downarrow q) \downarrow (p \downarrow q)$

Task 3.2: Express the following propositions using only NAND ($\uparrow$) as the logical connective:

Answer: *Hint: You may want to transfer into a normal form first, e.g. DNF. Double negation and De Morgan's laws are very useful from there.*

1. $p \Rightarrow \neg q$
Answer: $p \Rightarrow \neg q = p \uparrow q$
2. $\neg(p \oplus q)$
Answer: $\neg(p \oplus q) = (p \uparrow q) \uparrow (\neg p \uparrow \neg q) = (p \uparrow q) \uparrow ((p \uparrow p) \uparrow (q \uparrow q))$
3. $p \wedge (q \wedge r)$
Answer: $p \wedge (q \wedge r) = (p \wedge ((q \uparrow r) \uparrow (q \uparrow r))) = (p \uparrow ((q \uparrow r) \uparrow (q \uparrow r))) \uparrow (p \uparrow ((q \uparrow r) \uparrow (q \uparrow r)))$

Task 3.3:

1. Is $\uparrow$ commutative?

Explain what you need to show, then justify your answer.

Answer: *Need to show that $p \uparrow q = q \uparrow p$ (using truth tables or logical equivalences).*

p	q	$p \uparrow q$	$q \uparrow p$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Since the last two columns show the same truth values, we can conclude that $\uparrow$ is commutative.

2. Is $\uparrow$ associative?

Explain what you need to show, then justify your answer.

Answer: *Need to show that $(p \uparrow q) \uparrow r = p \uparrow (q \uparrow r)$, which turns out to be false, so NAND is not associative.*

p	q	r	$(p \uparrow q) \uparrow r$	$p \uparrow (q \uparrow r)$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	T	F
F	T	T	F	T
F	T	F	T	T
F	F	T	F	T
F	F	F	T	T

The truth table clearly shows that the last two columns differ. This demonstrates that $\uparrow$ is not associative.

Note: It would be sufficient to find one counterexample, i.e. one assignment of truth values to the propositional variables (i.e. one interpretation) that demonstrates that the two sides evaluate to different truth values. It is important that you clearly state your reasoning in an answer sentence.

Task 4: Boolean Algebra

Task 4.1:

1. Discuss the syntax that is used for Boolean algebra in the literature. Define the syntax you want to use as a group to answer the next question in this section.

   Answer: *No answer given. Students typically find a variety of notations and should discuss these. What has been taught in Computer Architecture would be a suitable example.*

2. List the following laws: associativity, commutativity, distributivity, identity, absorption, double negation. Where relevant, list for both conjunction and disjunction.

   Answer: *Most of these have been shown in lectures, so we are just going to focus on absorption.*

   (a) $p \vee (p \wedge q) = p$

   (b) $p \wedge (p \vee q) = p$

   If these two laws are not intuitive, draw truth tables for them.

3. Draw truth tables for the following Boolean functions - these are deliberately given using a variety of symbols that are commonly found in textbooks on Boolean algebra:

   (a) $F = \neg A$

   Answer:

A	$F(A)$
1	0
0	1

   (b) $F = \neg A \vee (B \wedge D)$

   Answer:

A	B	D	$F(A, B, D)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

   (c) $F = \overline{A} \cdot \overline{B}$

Answer:

A	B	$F(A, B)$
1	1	0
1	0	0
0	1	0
0	0	1

(d) $F = \overline{A + B}$

Answer:

A	B	$F(A, B)$
1	1	0
1	0	0
0	1	0
0	0	1

(e) $F = (A \texttt{ || } C) \texttt{ \&\& } {\sim}B$

Answer:

A	B	C	$F(A, B, C)$
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

(f) $F = \neg(D \vee B) \vee (B \vee D)$

Answer:

B	D	$F(B, D)$
1	1	1
1	0	1
0	1	1
0	0	1

Task 4.2:

1. Which Boolean function is represented by the following table? Write the expression in symbolic form.

A	B	C
0	1	0
1	1	1
1	0	0
0	0	0

Answer: *The Boolean function represented by the above table is* $F = A \wedge B$.

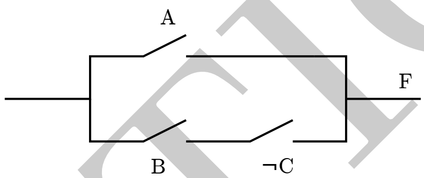
2. Which Boolean function is represented by the following table? Write the expression in symbolic form.

A	B	C
0	1	0
1	1	0
0	0	1
1	0	0

Answer: *The Boolean function represented by the above table is* $F = \neg A \wedge \neg B$.

Task 4.3:

1. Draw the truth table for the following diagram.



Answer:

A	B	C	$F(A, B, C)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

2. Write the above circuit as a symbolic expression.

Answer:
$F = A \vee (B \wedge \neg C)$

Task 4.4: Simplify the following expression: $XY + XYZ + XYZQ$. Which law(s) did you use?

Answer: $XY + XYZ + XYZQ = XY$ *(distributivity, domination)*

Task 4.5:

1. Define Conjunctive Normal Form (CNF).

   Answer: *A formula in Conjunctive Normal Form is a conjunction of disjunctive clauses, it is an AND of ORs. Example:* $\neg A \wedge (B \vee C)$

2. Can every expression be put into conjunctive normal form?

   Answer: *Yes, as, as we have seen above, the set* $\{\wedge, \vee, \neg\}$ *is functionally complete.*

3. Identify if the following expressions are in CNF, and if they are not, normalise them.

   (a) $\neg C \wedge D$
   Answer: *(This is already in CNF, discuss why.)*

   (b) $\neg D$
   Answer: *(This is already in CNF, discuss why.)*

   (c) $\neg D \wedge \neg F$
   Answer: *(This is already in CNF, discuss why.)*

   (d) $(\neg D) \wedge (\neg F)$
   Answer: $(\neg D) \wedge (\neg F) = \neg D \wedge \neg F$

   (e) $\neg(D \wedge \neg F)$
   Answer: $\neg(D \wedge \neg F) = \neg D \vee F$

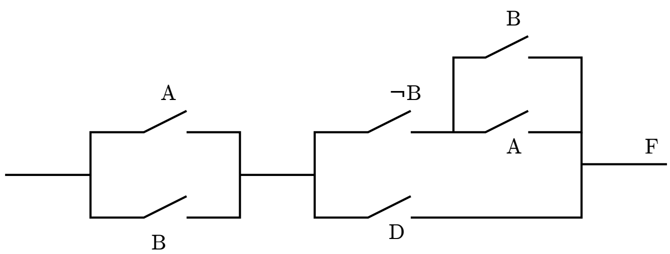
   (f) $\neg(C \wedge D)$
   Answer: $\neg(C \wedge D) = \neg C \vee \neg D$

   (g) $\neg(A \vee B)$
   Answer: $\neg(A \vee B) = \neg A \wedge \neg B$

   (h) $(A \wedge B) \vee (C \wedge D) \vee (E \wedge F)$
   Answer: $(A \wedge B) \vee (C \wedge D) \vee (E \wedge F) = (A \vee C \vee E) \wedge (A \vee C \vee F) \wedge (A \vee D \vee E) \wedge (A \vee D \vee F) \wedge (B \vee C \vee E) \wedge (B \vee C \vee F) \wedge (B \vee D \vee E) \wedge (B \vee D \vee F)$

Task 4.6

1. Write the exact symbolic equation for the circuit below.



Answer: $(A \vee B) \wedge ((\neg B \wedge (A \vee B)) \vee D)$

2. Put the symbolic expression into a minimal disjunctive normal form.

   Hint: Use a Karnaugh map to simplify this expression.

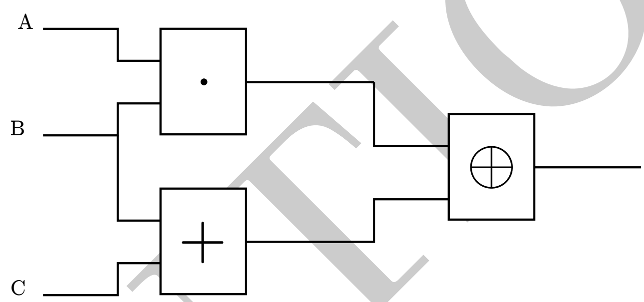
   Answer: $(A \wedge \neg B) \vee (B \wedge D)$

3. Put the expression into a conjunctive normal form.

   Answer: $(A \vee B) \wedge (\neg B \vee D)$

Task 4.7: **Answer:** *Assumption:* $\oplus$ *takes precedence over* $+$*, i.e.* $F = A + (A \cdot B) \oplus (B + C) = A + ((A \cdot B) \oplus (B + C))$.

1. The circuit below has been designed to implement the expression
   $F = A + (A \cdot B) \oplus (B + C) \; = \; A + ((A \cdot B) \oplus (B + C))$,
   however, a mistake has been made.



   Give a combination of inputs that results in the incorrect output.

   Answer: $(A = 1, B = 1, C = 1)$ *or* $(A = 1, B = 0, C = 0)$ *or* $(A = 1, B = 0, C = 1)$

2. Give an expression for the inputs which result in incorrect output.

   Answer: $A \cdot \overline{B} + A \cdot B \cdot C$

3. With as few modifications as possible, fix the circuit above, drawing the result.

   Answer: *Put an OR-Gate behind the XOR-Gate, with A as the other input.*

Task 4.8: Draw as many different representations as you can find (in the literature) for the following logic gates:

1. AND
2. OR
3. NAND
4. NOT

5. XNOR

Answer: *No answer is given. There is a standard that defines the symbols for logic gates. You should familiarise yourself with the symbols used in that standard.*

Task 4.9:

1. Define duality.

   Answer: *Given an Boolean algebra consisting of expressions formed by using the connectives* $\{\wedge, \vee, \neg\}$*, duality says that we can exchange* $\vee$ *and* $\wedge$*, and* $\perp$ *and* $\top$ *to obtain another valid expression.*

2. Give an example of a self-dual operation.

   Answer: *Forming the complement is a self-dual operation.*

3. Give the dual of $F \vee G$.

   Answer: *The dual of* $F \vee G$ *is* $F \wedge G$.

4. Give the dual of $(A \wedge \neg B) \vee \neg C$.

   Answer: *The dual of the above formula is* $(A \vee \neg B) \wedge \neg C$.