Data Structures and Algorithms – COMS21103

2015/2016

Shortest Paths Revisited

Negative Weights and All-Pairs

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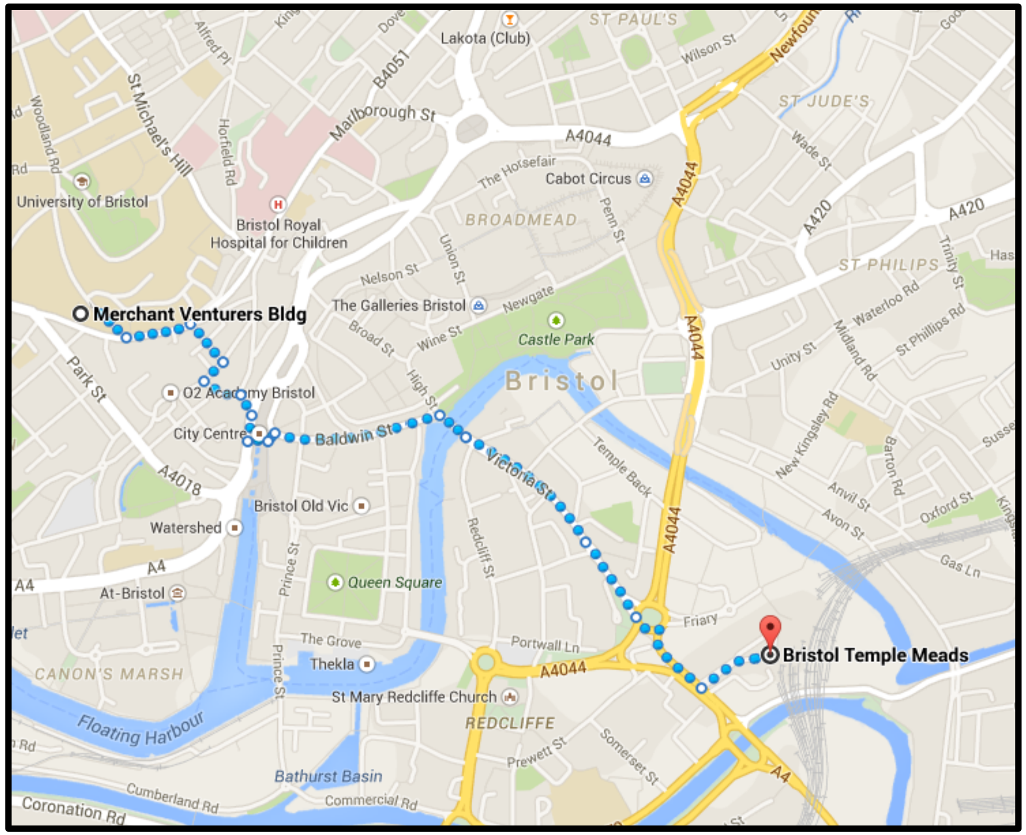
inspired by slides by Ashley Montanaro

University of BRISTOL

In today's lectures we'll be revisiting the **shortest paths** problem

in a weighted, directed graph. . .



The shortest path from MVB to Temple Meads

(according to Google Maps)

In particular we'll be interested in algorithms which allow **negative edge weights**

and algorithms which compute the shortest path between **all pairs** of vertices

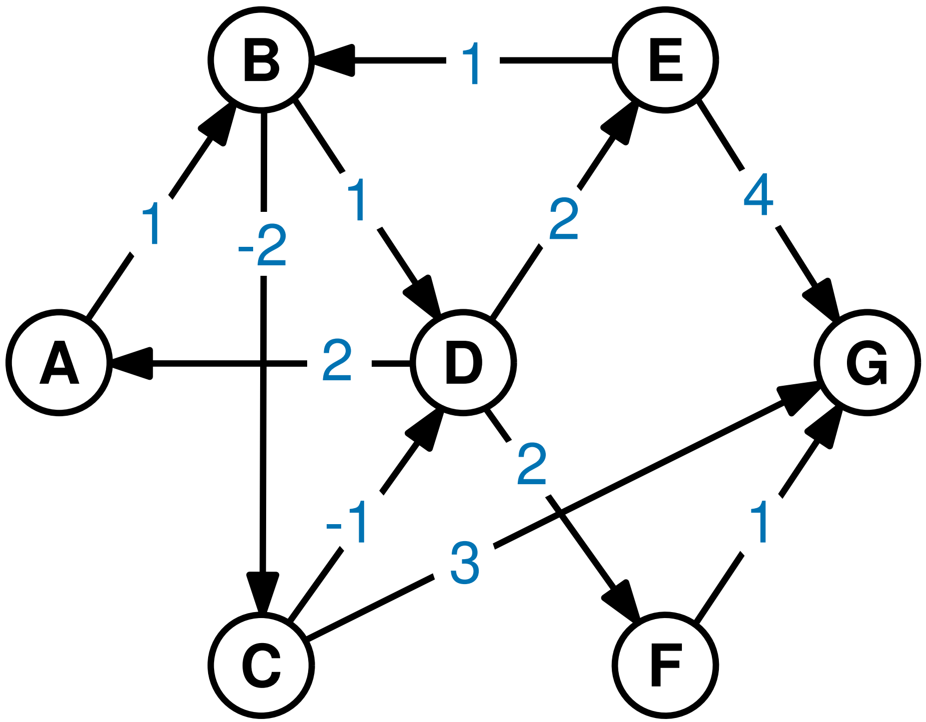
Part one

Single Source Shortest Paths with negative weights

Single source shortest paths with negative weights

Bellman-Ford's algorithm solves the **single source shortest paths** problem

in a **weighted**, directed graph...



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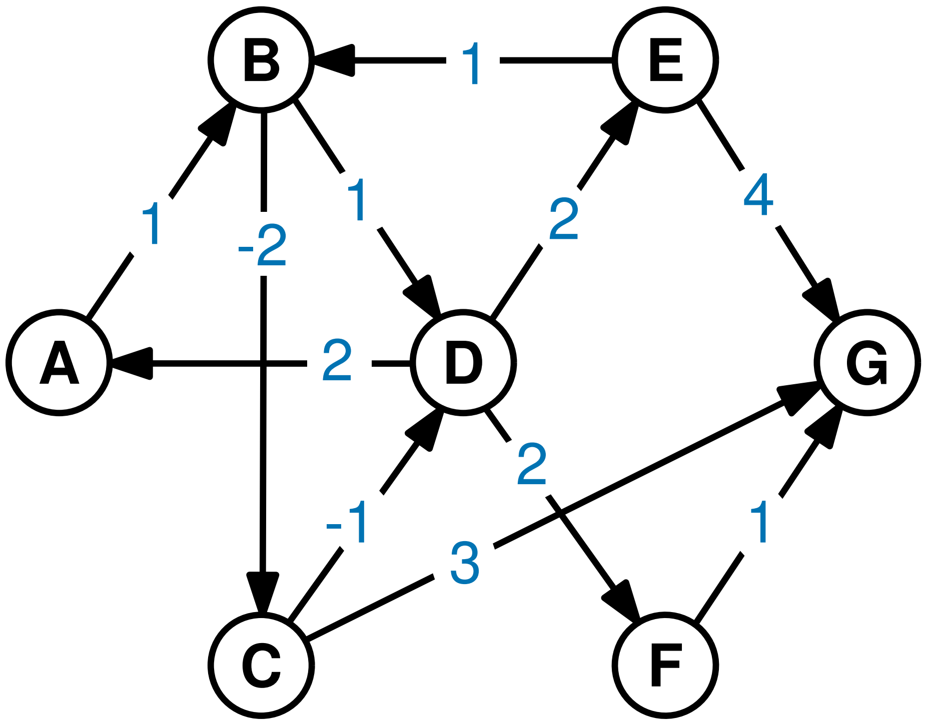

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to every other vertex

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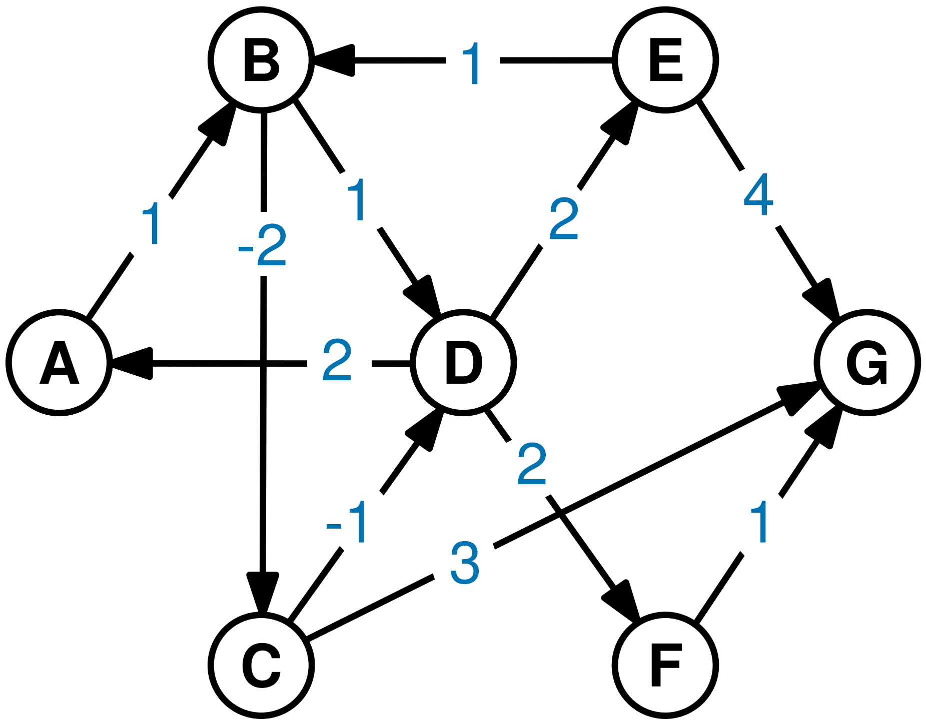

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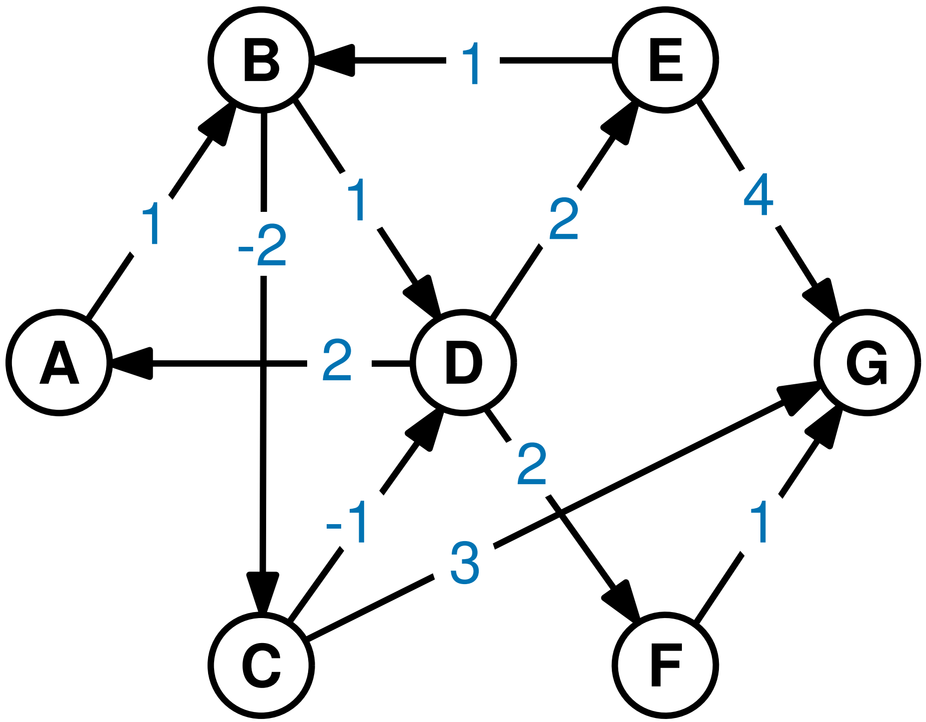

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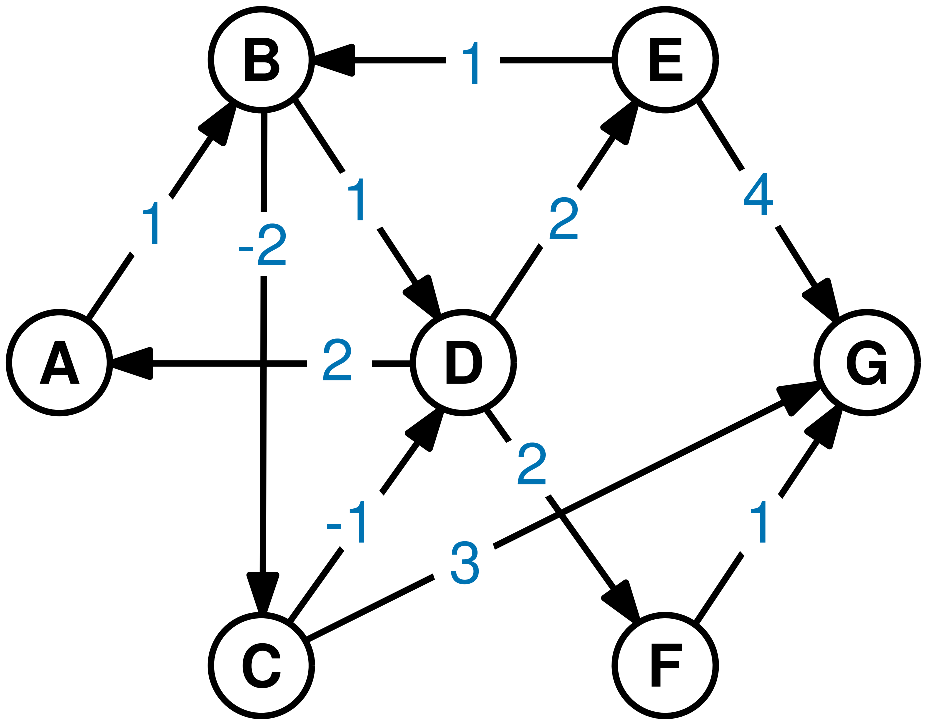

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We will not need any non-elementary data structures

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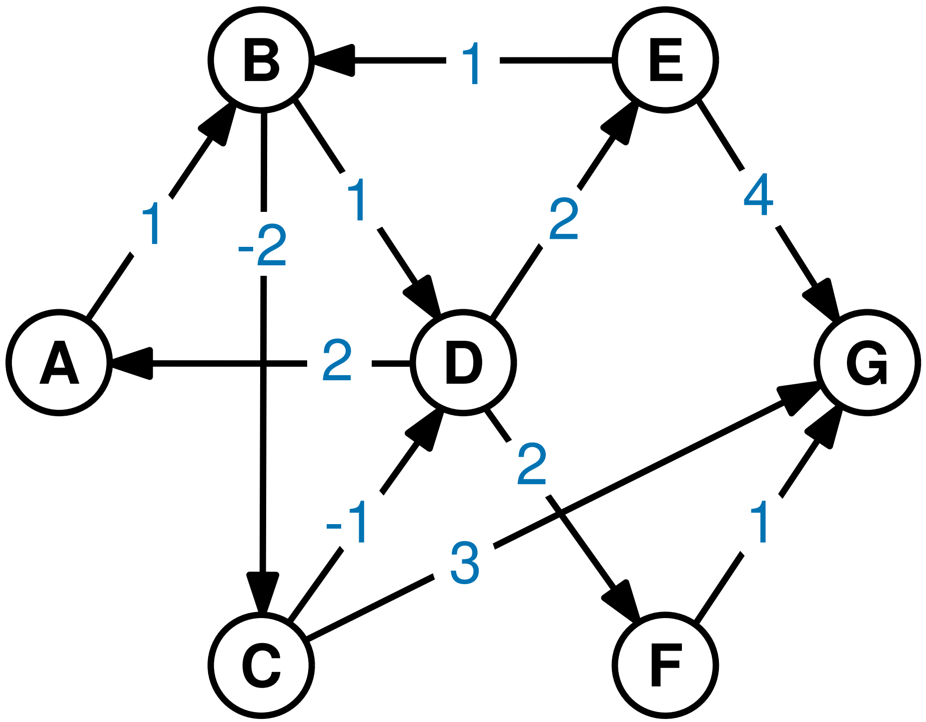

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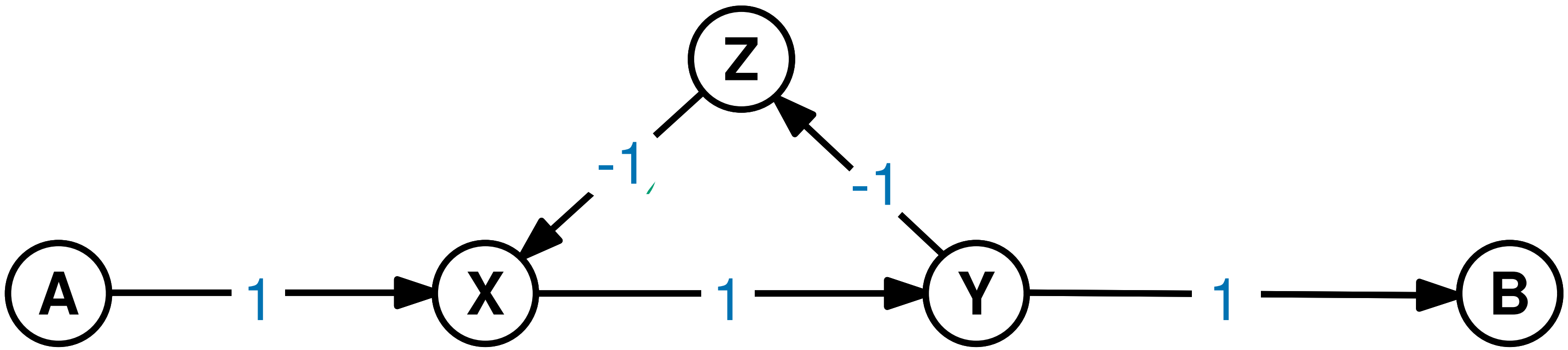
Previously we saw that Dijkstra's algorithm *(implemented with a binary heap)* solves this problem in $O((|V| + |E|) \log |V|)$ time when the edges have **non-negative weights**

$|V|$ is the number of vertices and $|E|$ is the number of edges

Negative weight cycles

If some of the edges in the graph have negative weights
the idea of a shortest path might not make sense:



What is the shortest path from **A** to **B**?

A negative weight cycle is a path from a vertex v back to v such that
the sum of the edge weights is **negative**

If there is a path from s to t which includes a negative weight cycle,
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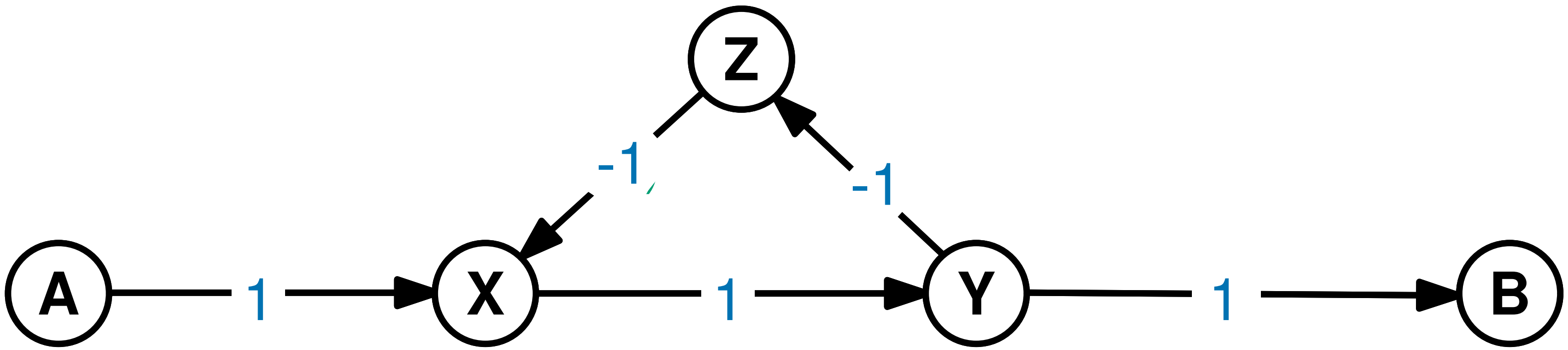

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We will first discuss a (slightly) simpler version of Bellman-Ford
that assumes there are no such cycles

Most of the Bellman-Ford algorithm

MOSTOFBELLMAN-FORD(s)

```
For all v, set dist(v) = ∞
set dist(s) = 0
For i = 1, 2, ..., |V|,
  For every edge (u, v) ∈ E
    If dist(v) > dist(u) + weight(u, v)
      dist(v) = dist(u) + weight(u, v)
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$(u, v) \in E$ iff there is an edge *from* u *to* v

$\texttt{weight}(u, v)$ is the weight of the edge from u to v

$\texttt{dist}(v)$ is the length of the shortest path between s and v, found so far

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The algorithm repeatedly asks, for each edge (u, v)

"can I find a shorter route to v if I go via u?"

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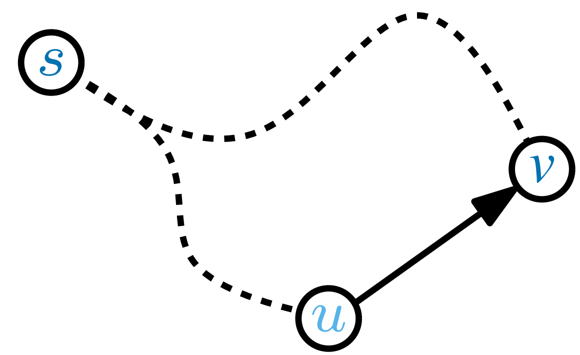

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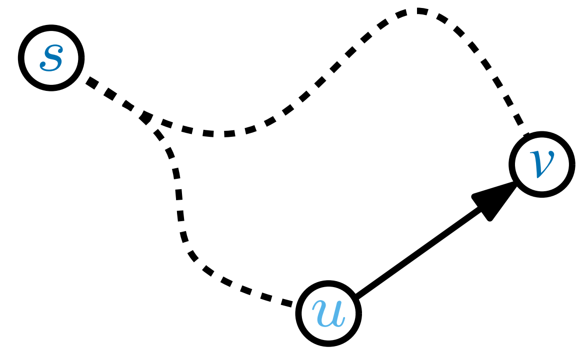
This is called Relaxing edge (u, v)



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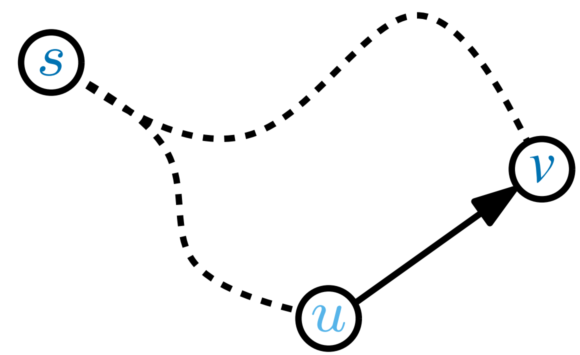

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Claim When the MOSTOFBELLMAN-FORD algorithm terminates,

for each vertex v, $\texttt{dist}(v)$ is the length of the shortest path between s and v

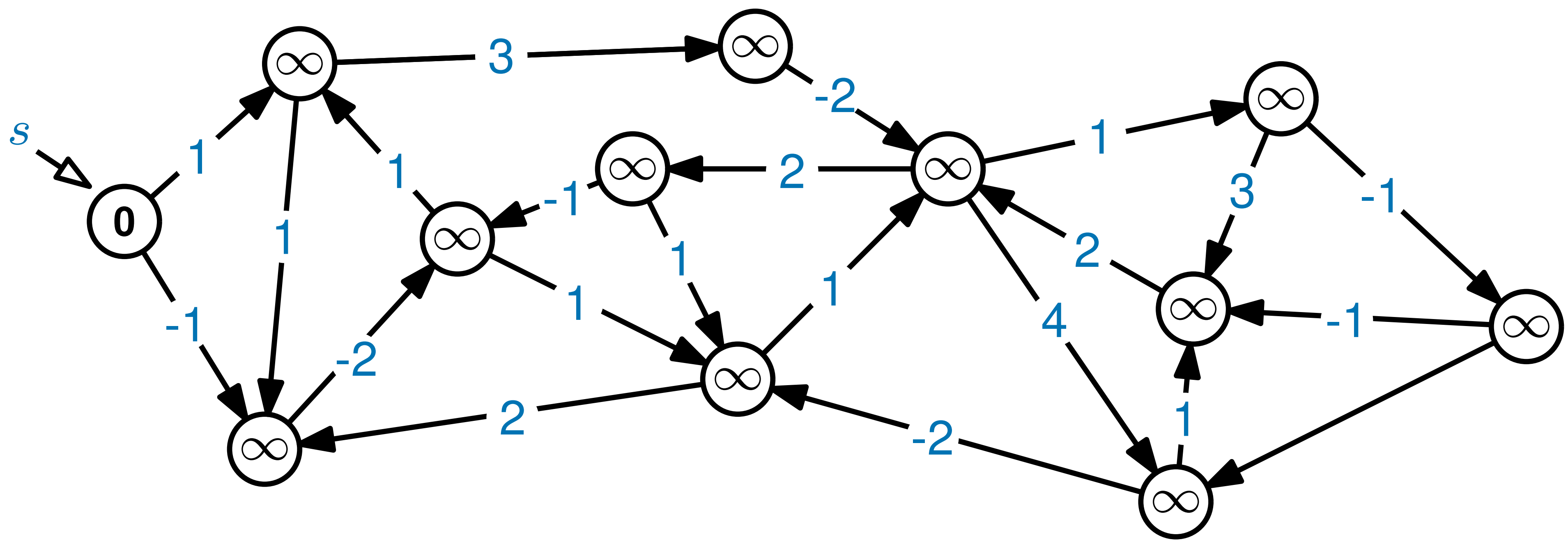
(assuming there are no-negative weight cycles)

37 ← vertex v

$\texttt{dist}(v)$ - *the length of the best path between s and v, found so far*

We're going to simulate

MOSTOFBELLMAN-FORD(s)



MOSTOFBELLMAN-FORD runs $|V|$ iterations,

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RELAX(u, v)

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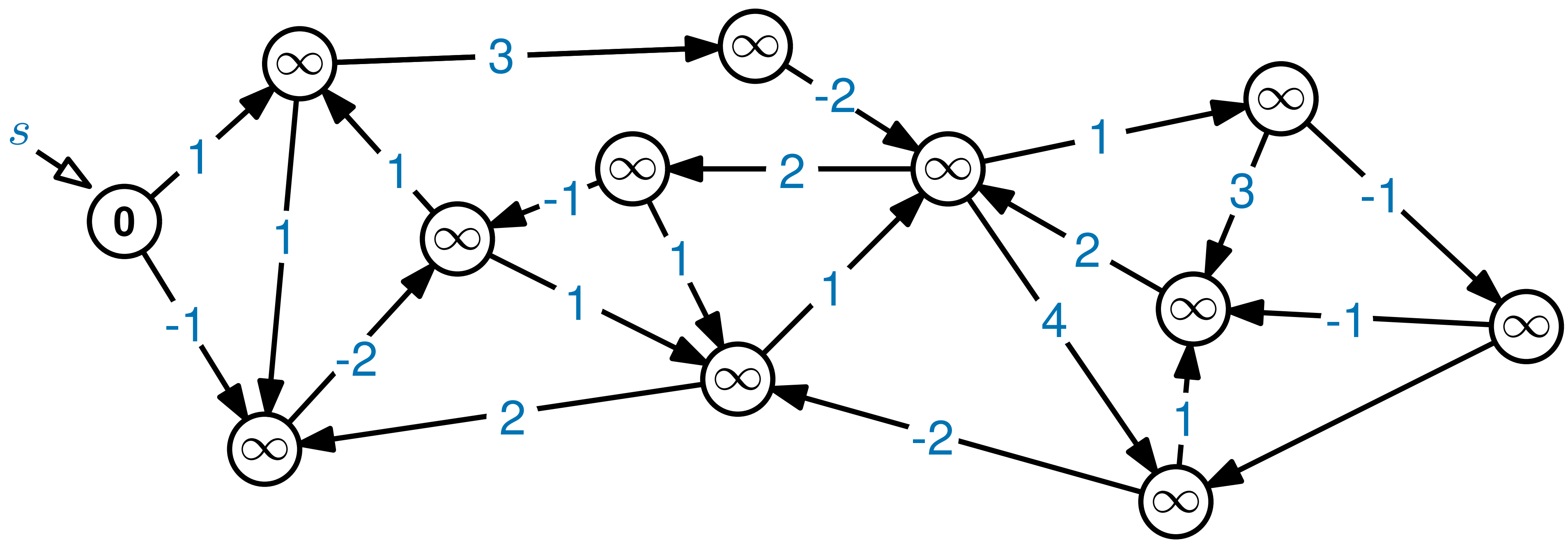
We're going to simulate MostOfBellman-Ford(s)

We start by setting $\texttt{dist}(s) = 0$ and every other $\texttt{dist}(v) = \infty\ldots$



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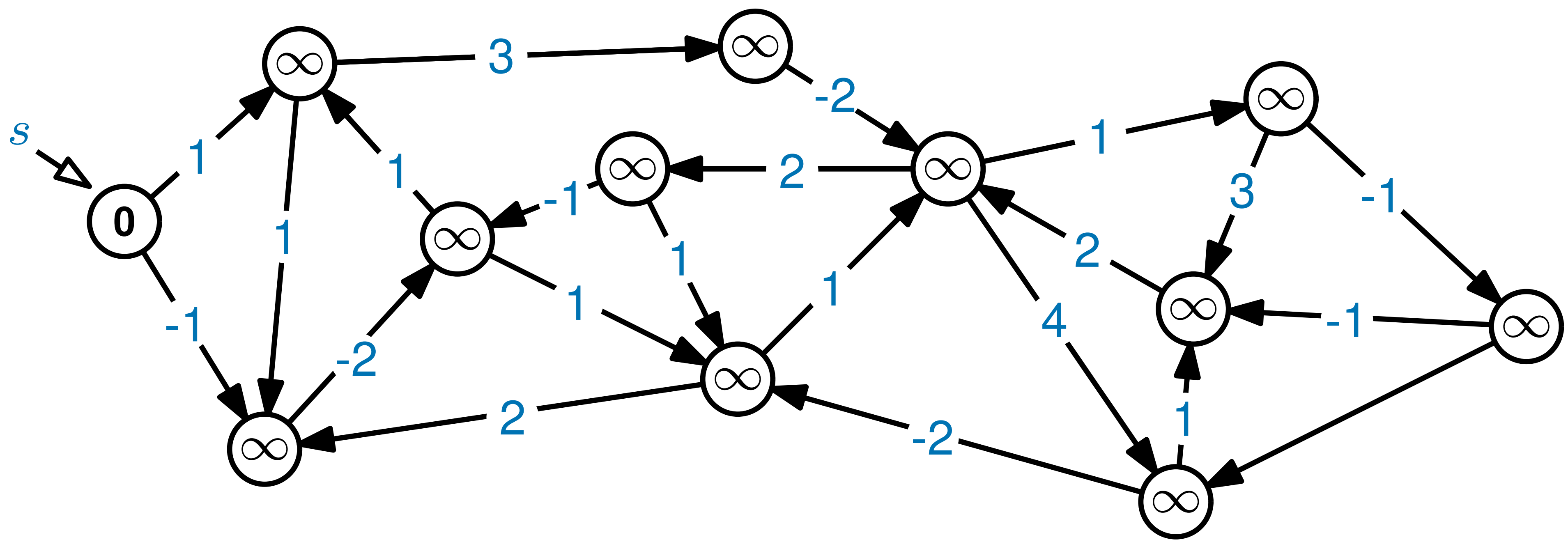

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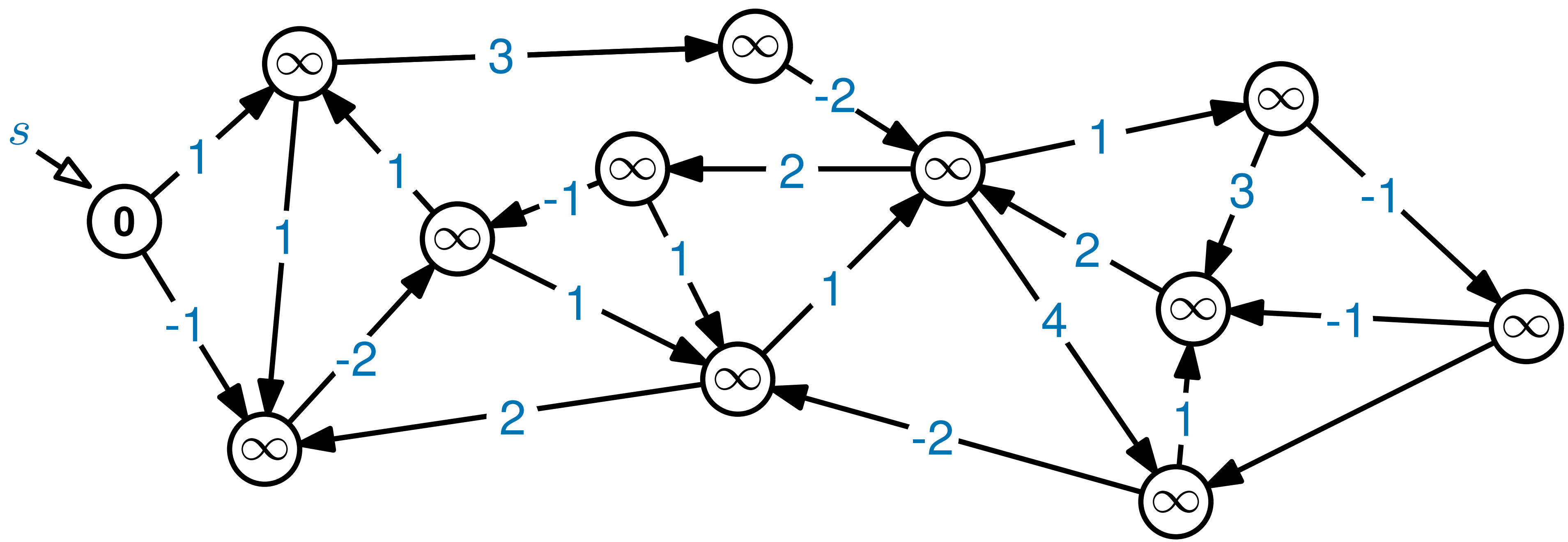
We now start iteration $1\ldots$



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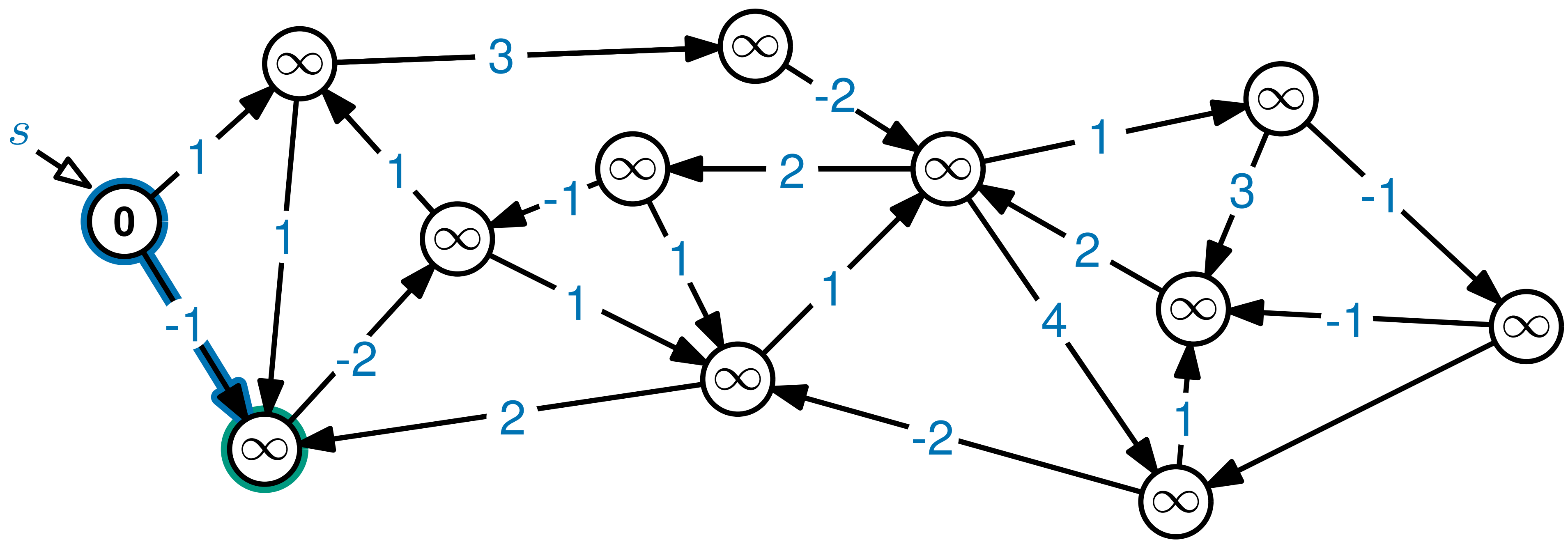

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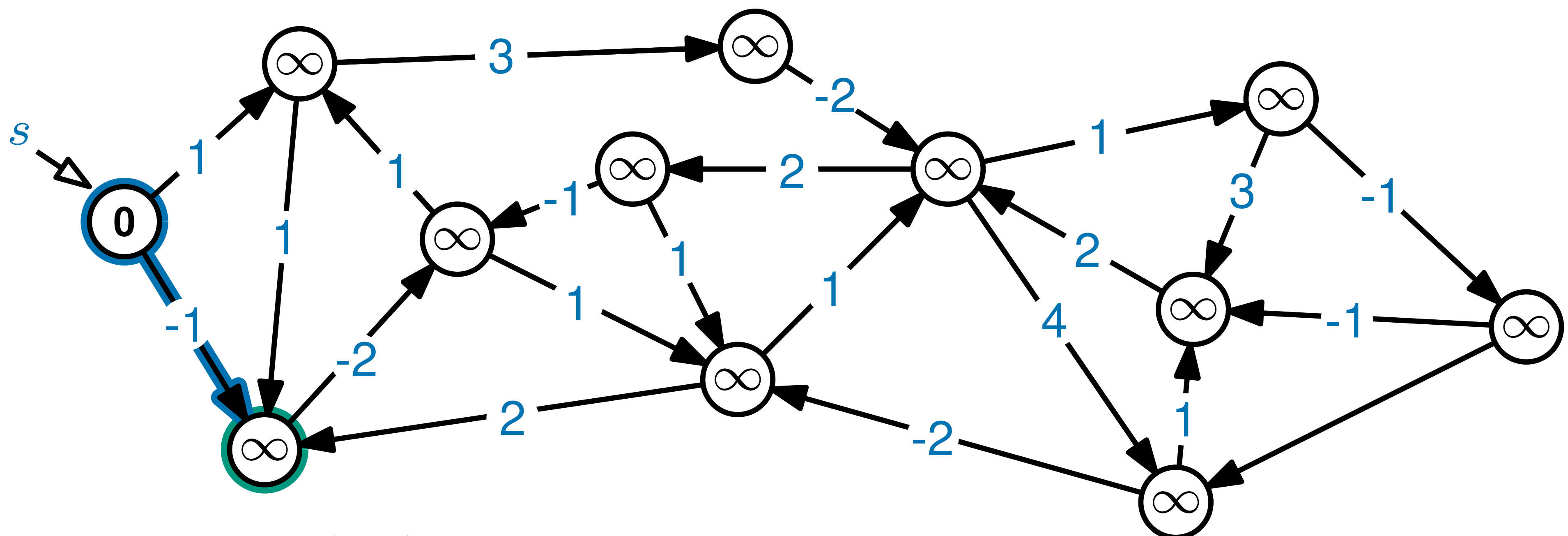

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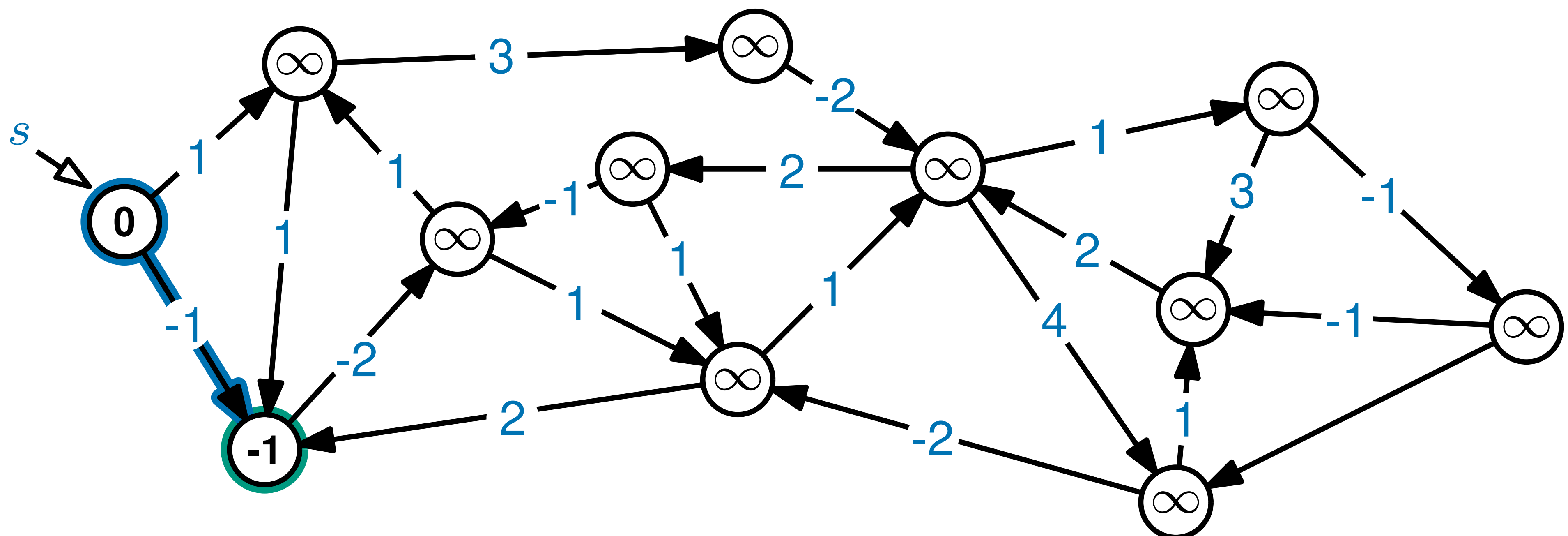

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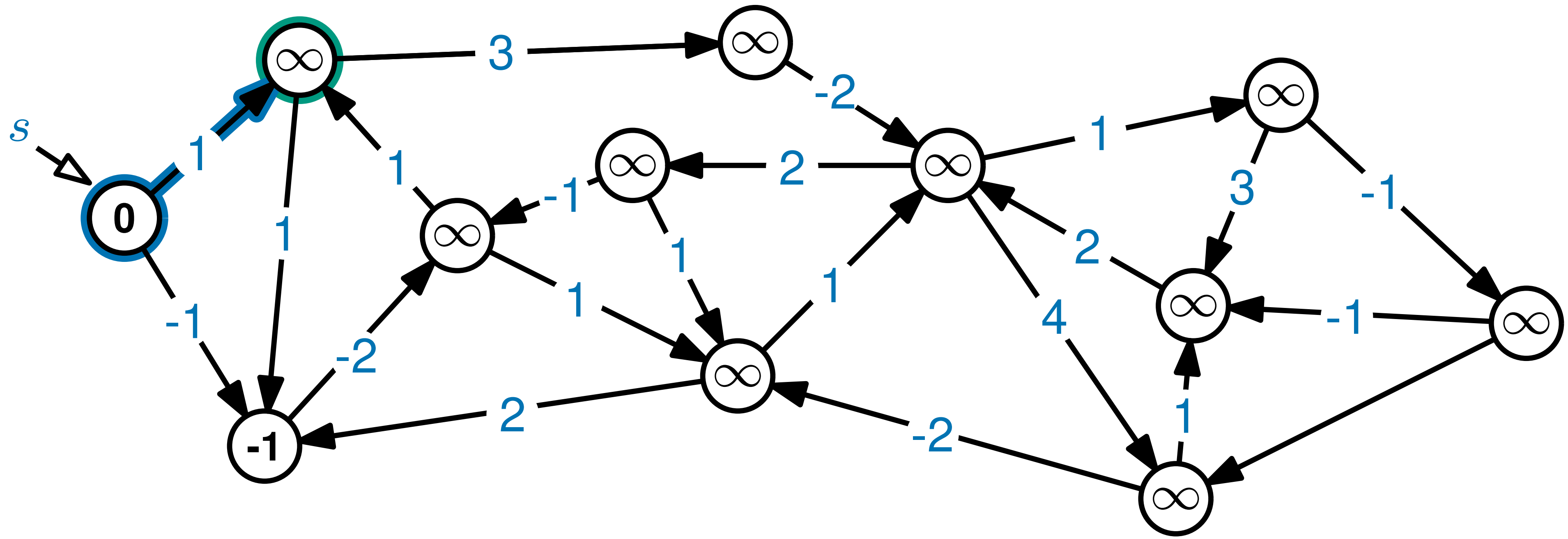

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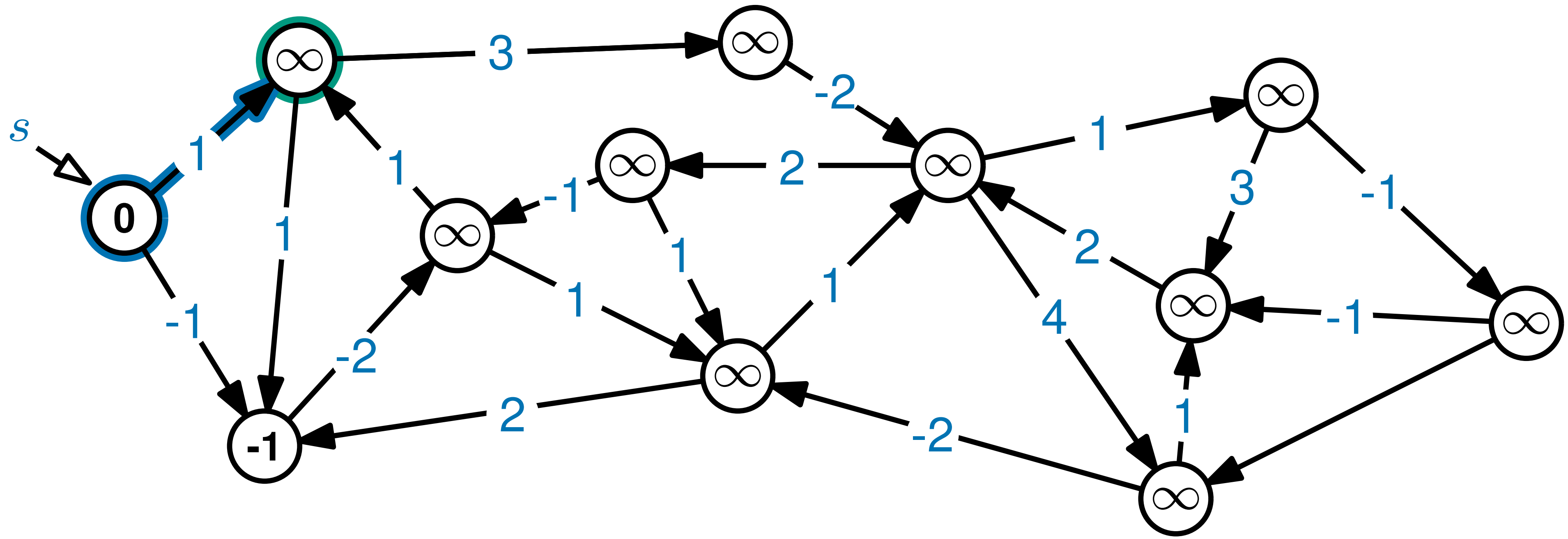

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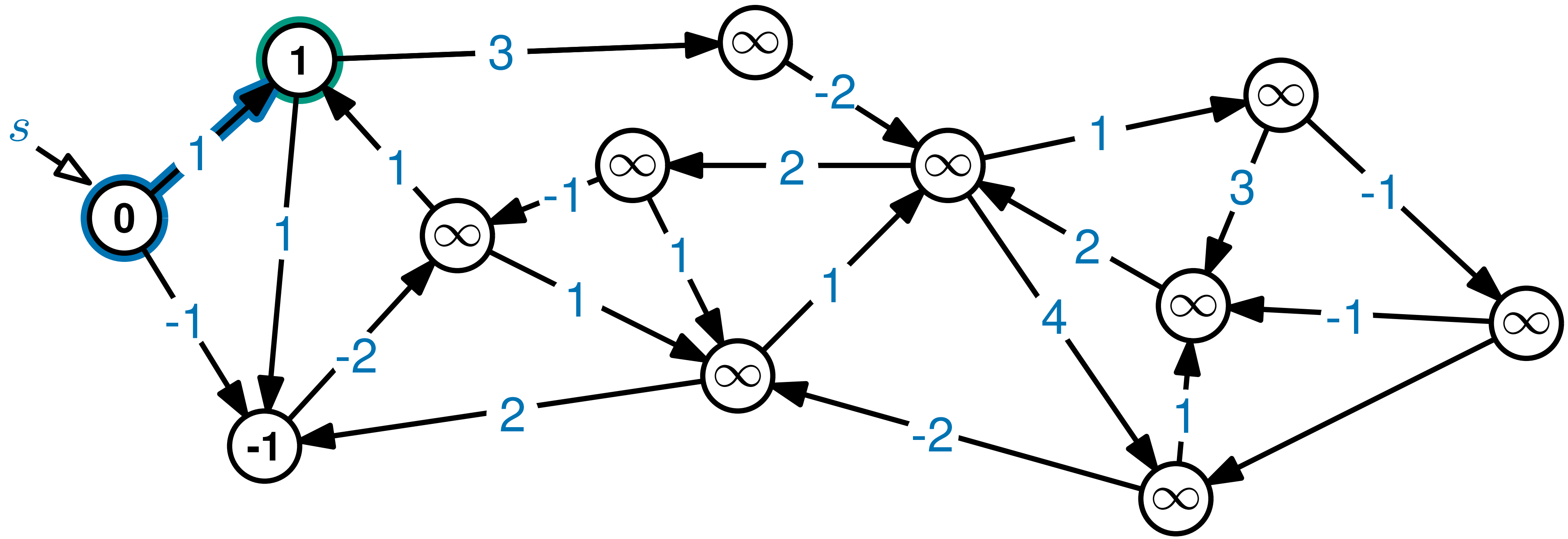

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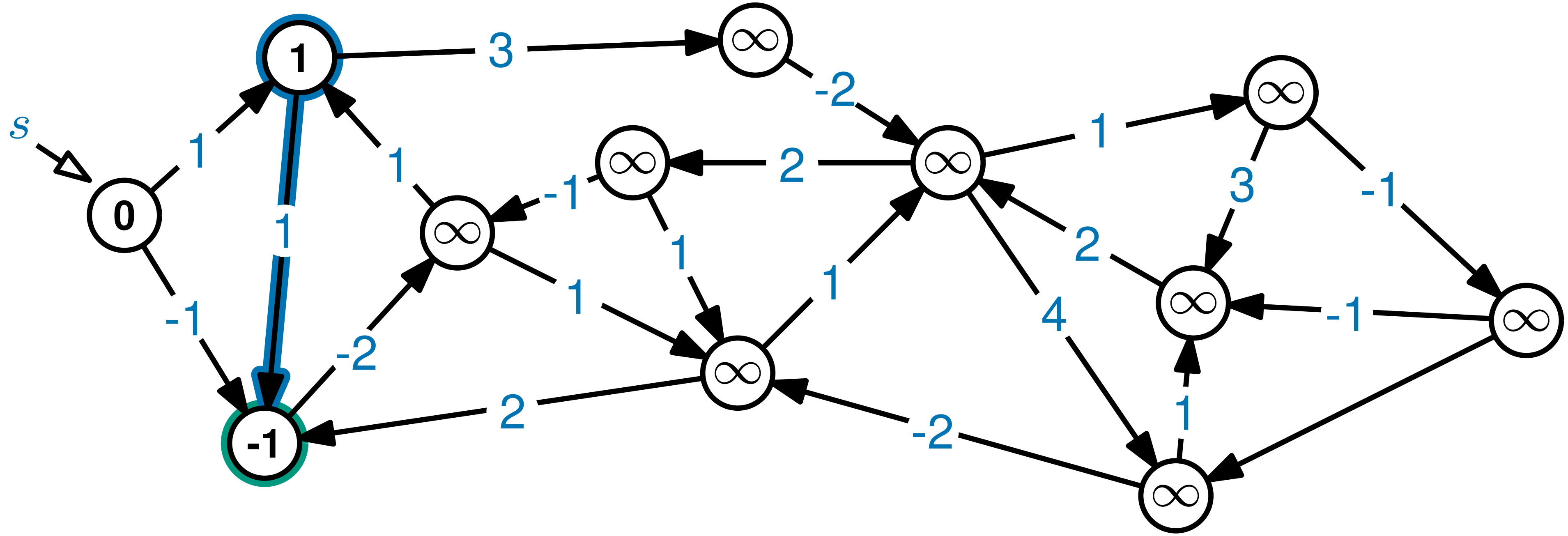

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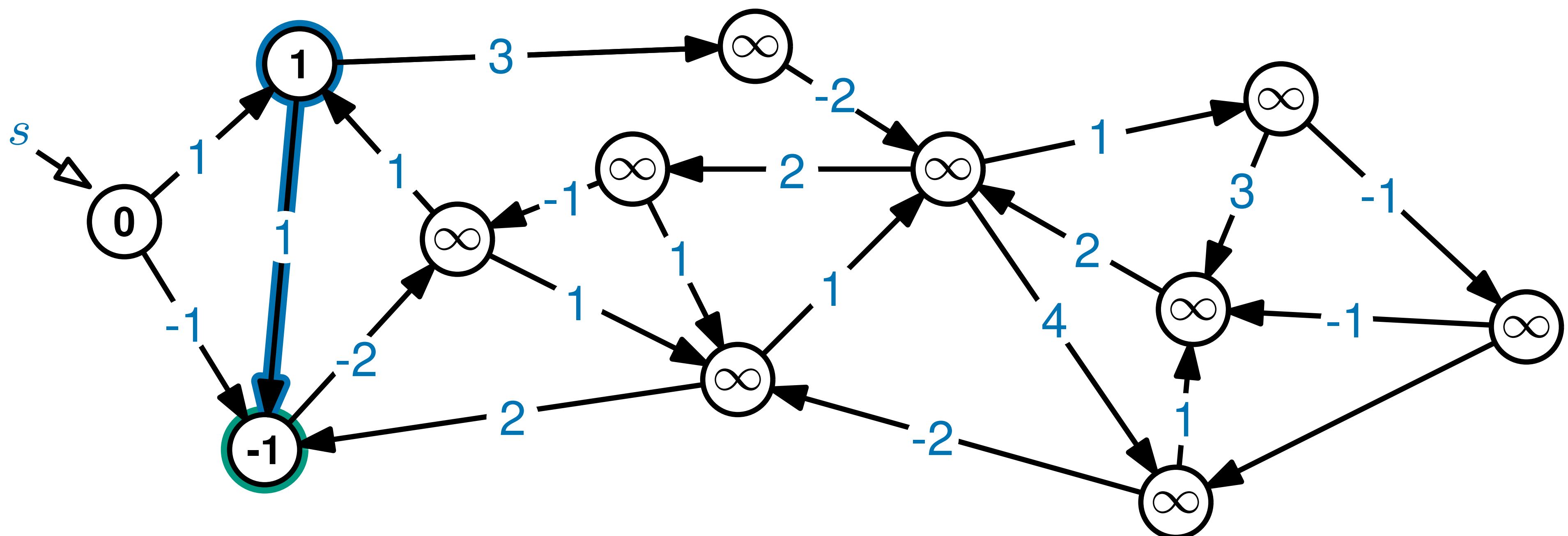

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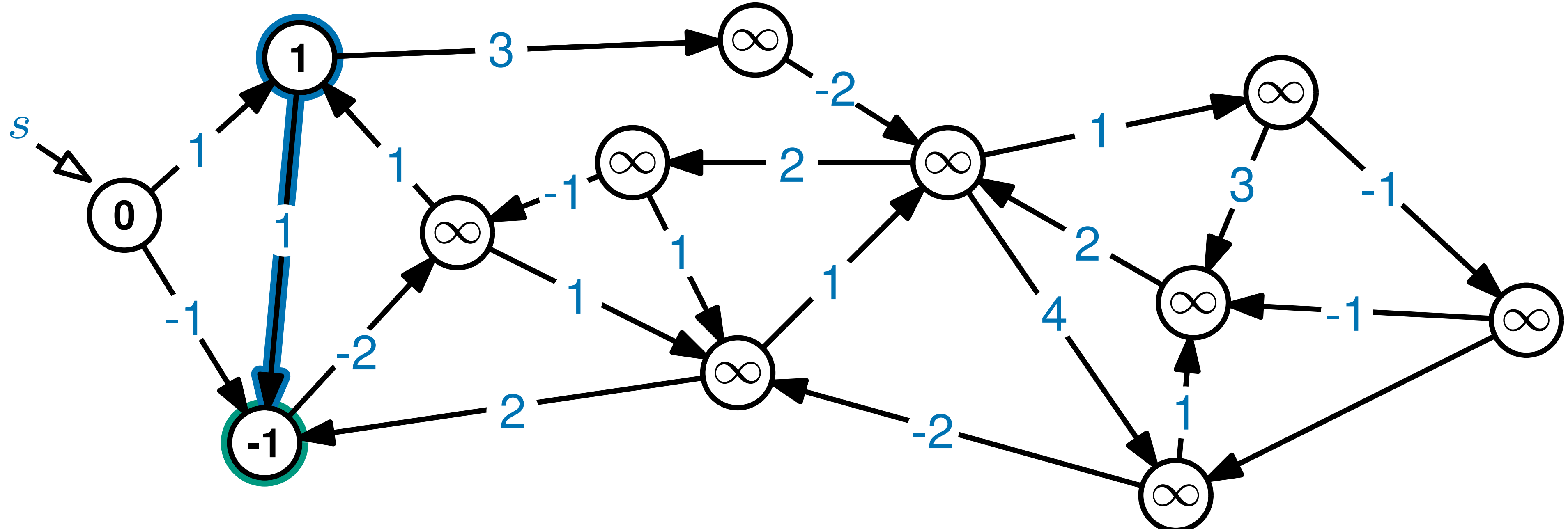

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(so we don't change $\texttt{dist}$*)*

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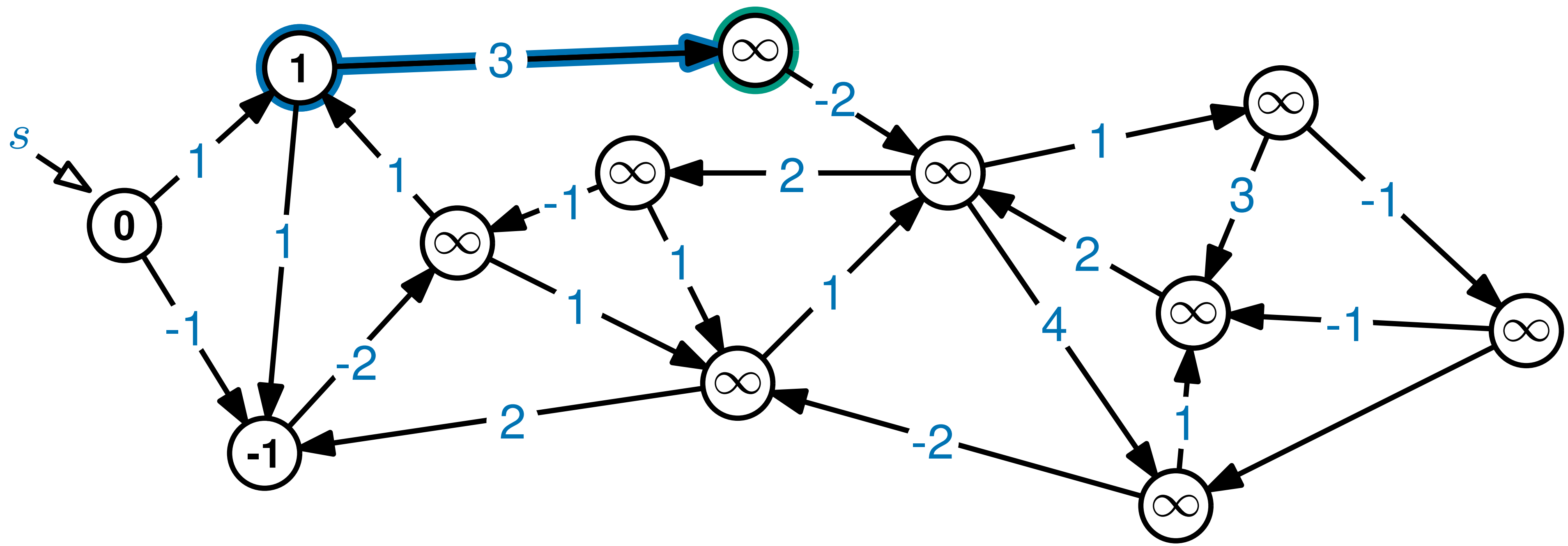

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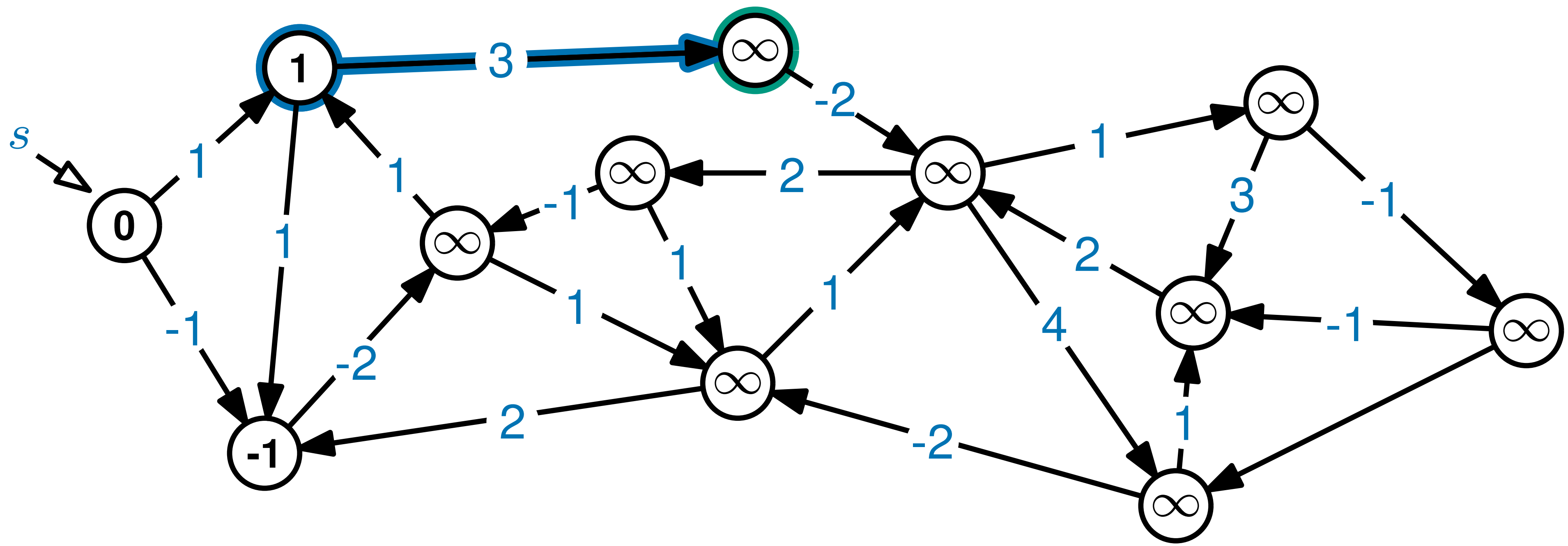
We're going to simulate MostOfBellman-Ford(s)

We now start iteration $1\ldots$

$\infty > 1 + 3$



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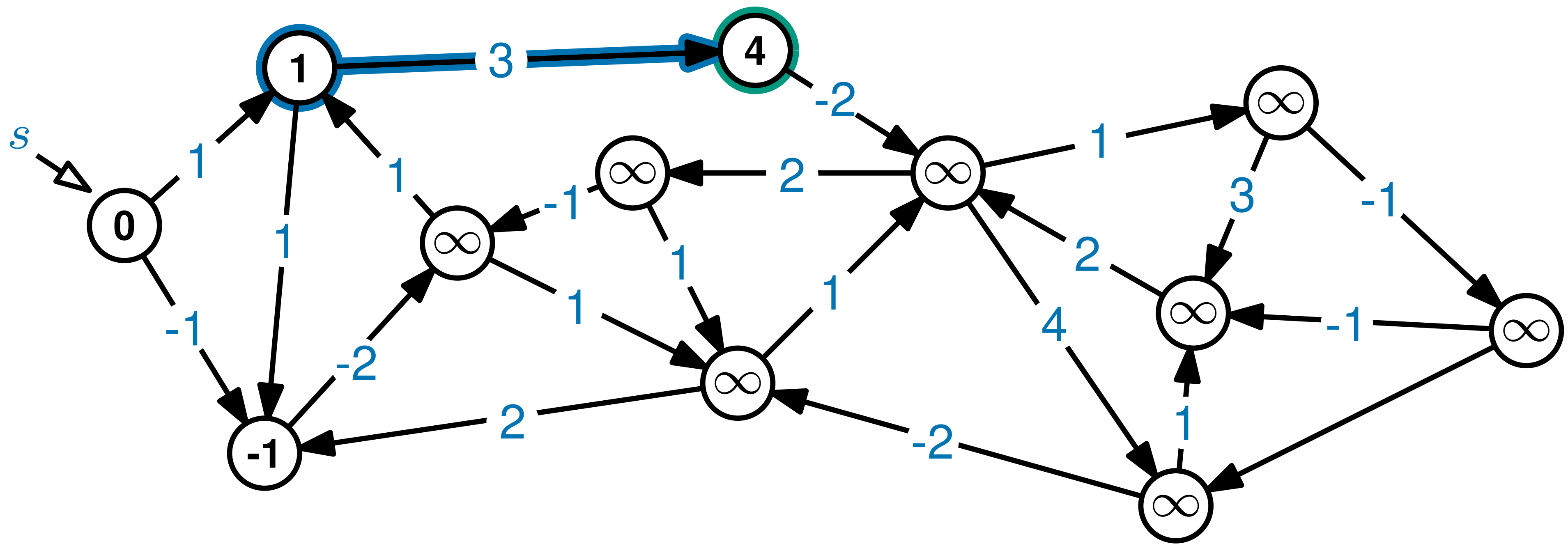

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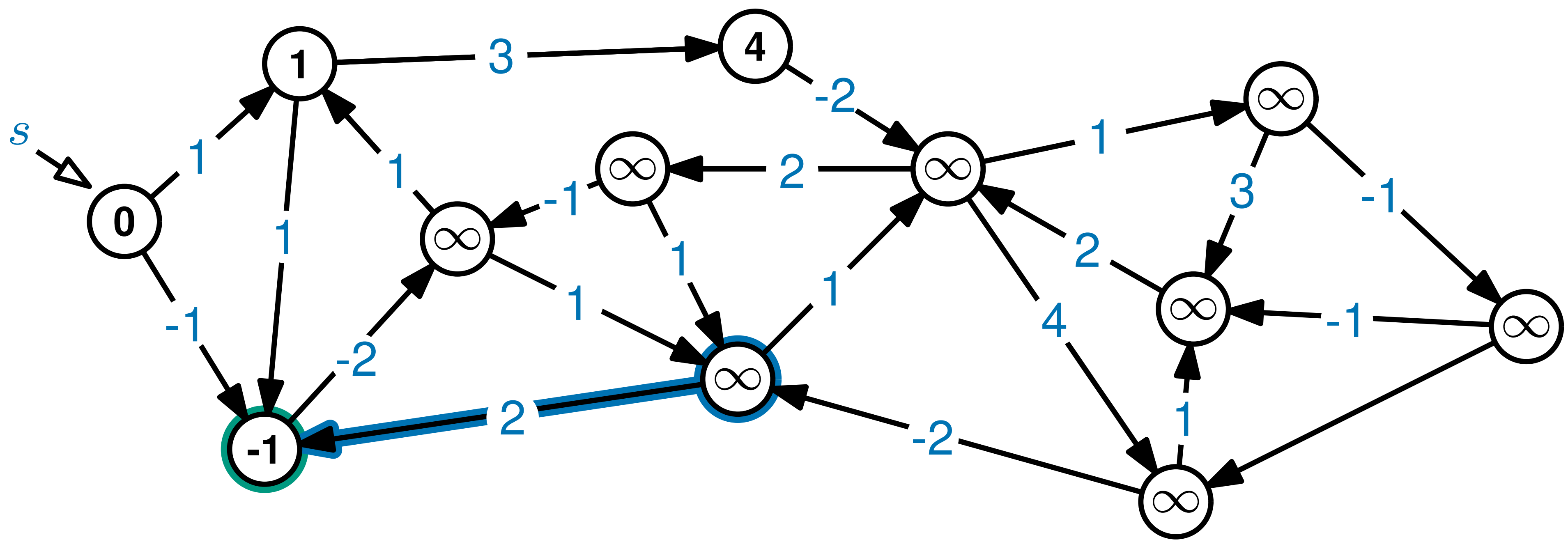

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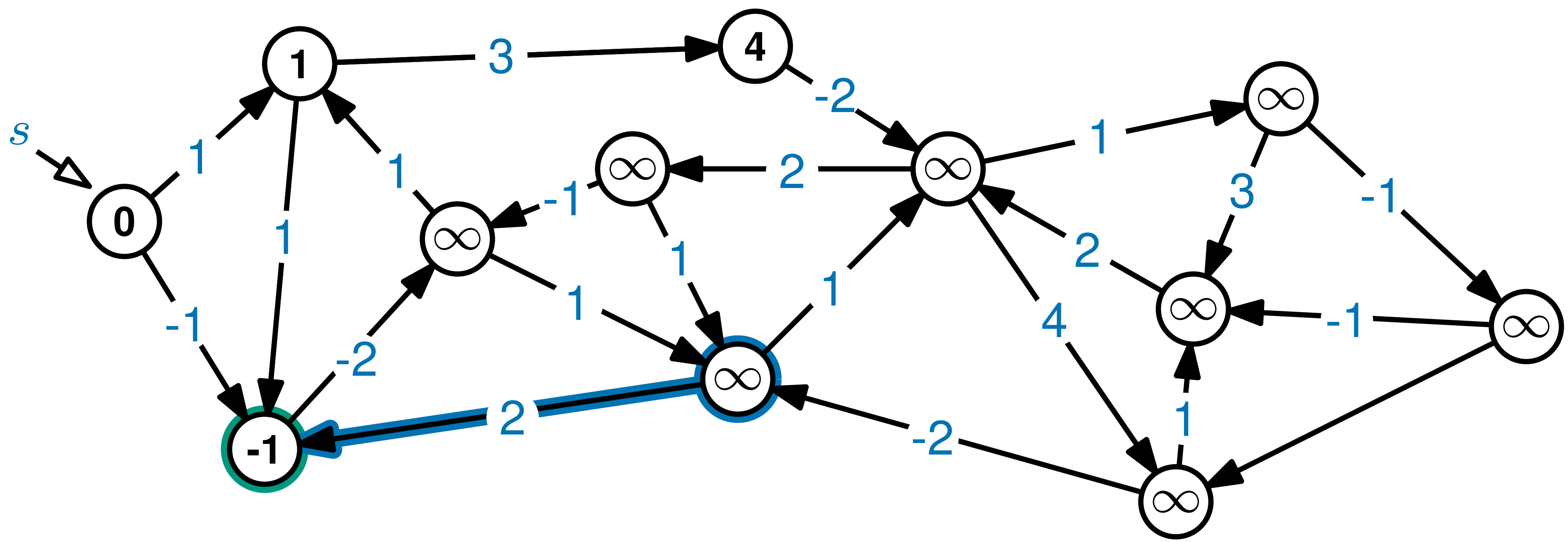

MOSTOFBELLMAN-FORD runs $|V|$ iterations,

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(in the order they occur in the adjacency list)

RELAX(u, v)

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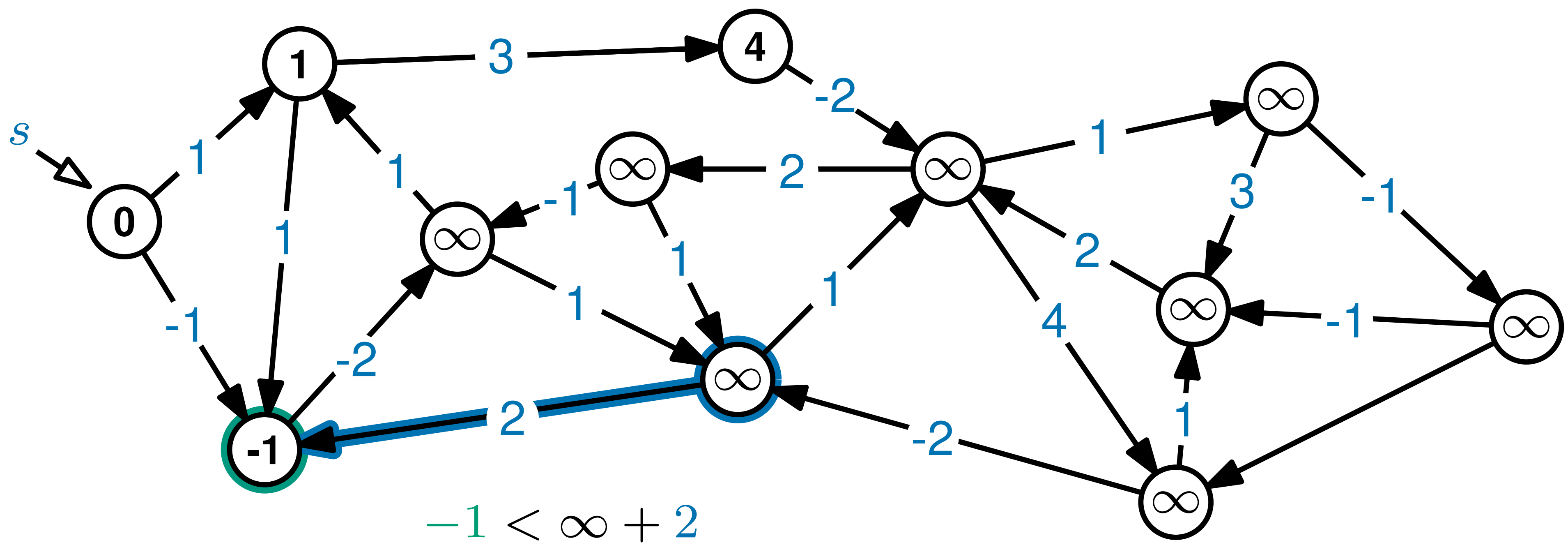
We're going to simulate MOSTOFBELLMAN-FORD(s)

We now start iteration $1\ldots$



$-1 < \infty + 2$

MOSTOFBELLMAN-FORD runs $|V|$ iterations,

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RELAX(u, v)

```
if dist(v) > dist(u) + weight(u,v)
   dist(v) = dist(u) + weight(u,v)
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vertex v

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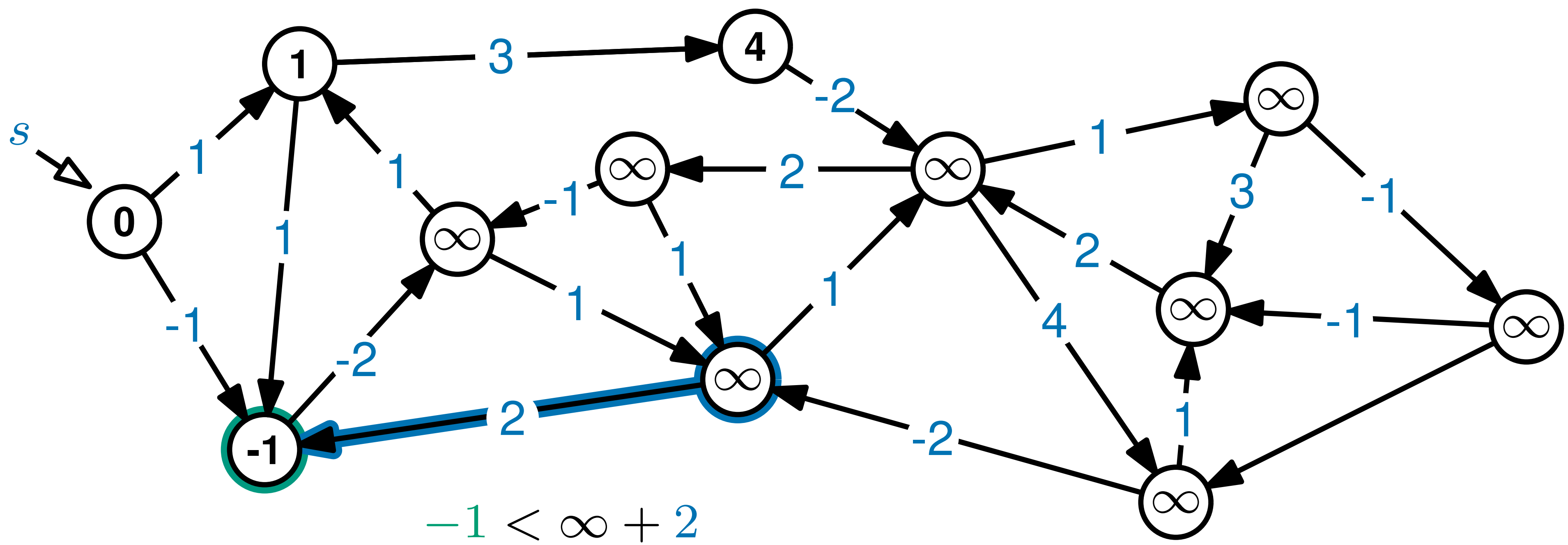

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(so we don't change $\texttt{dist}$*)*

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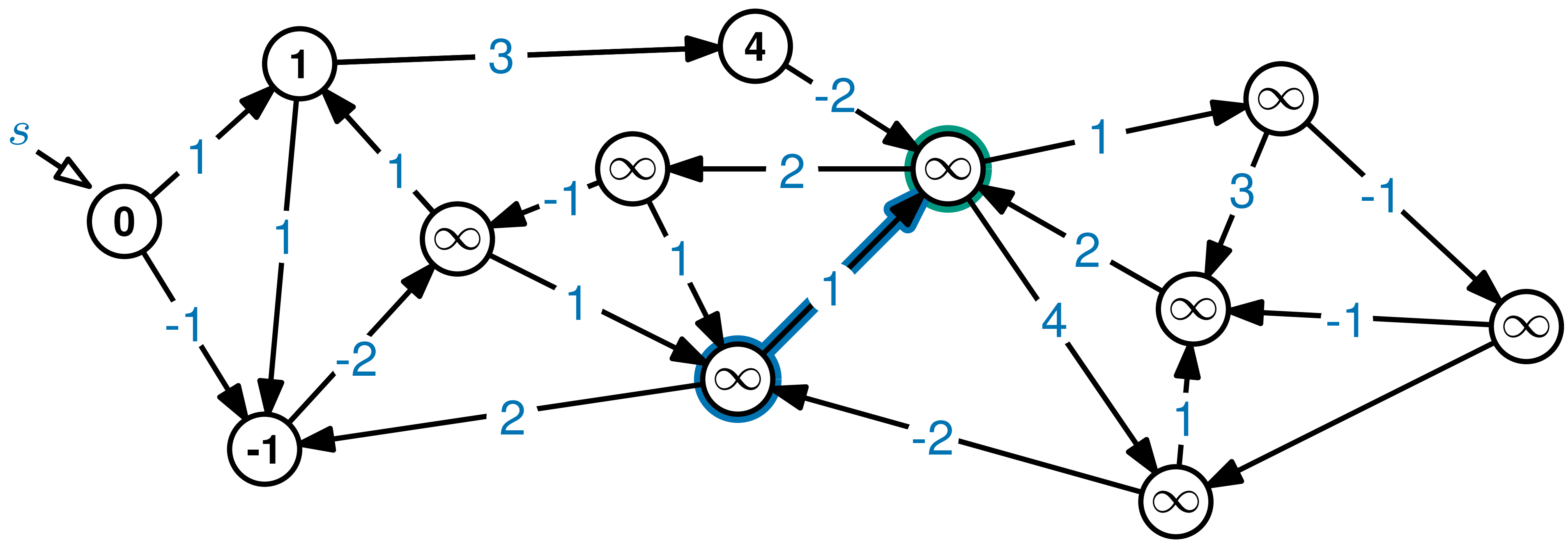
vertex v

We're going to simulate

MOSTOFBELLMAN-FORD(s)

We now start iteration $1\dots$



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$\quad \texttt{dist}(v) = \texttt{dist}(u) + \texttt{weight}(u, v)$

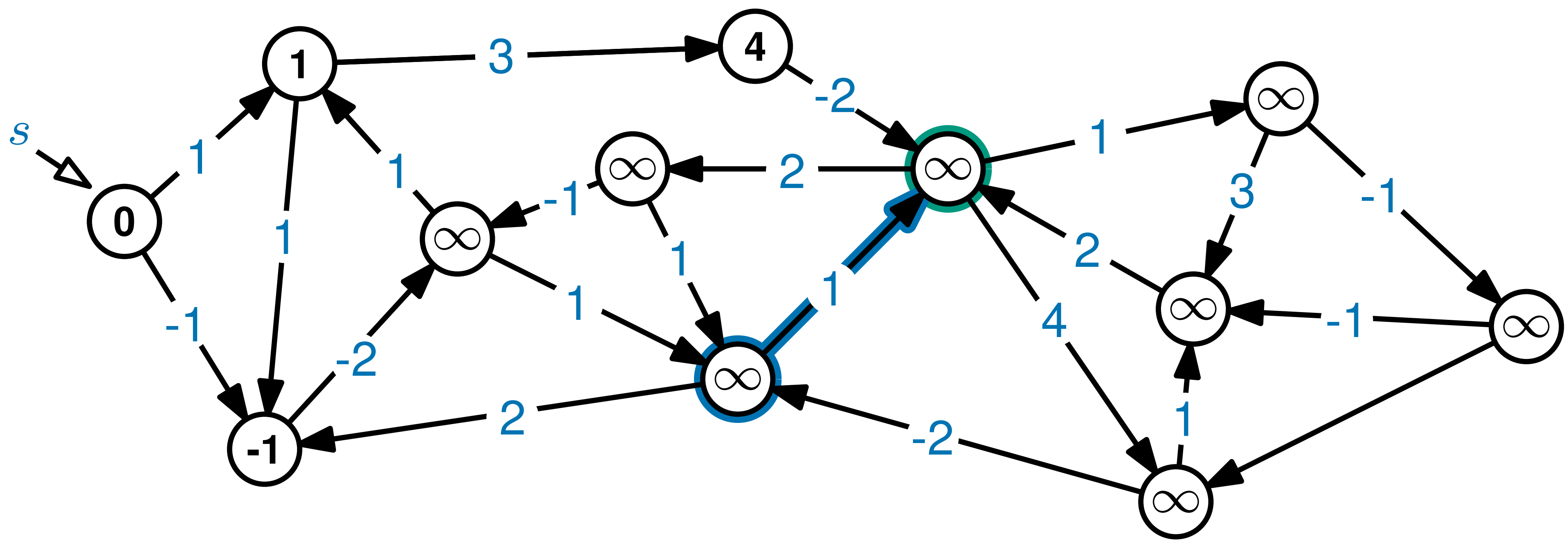
37 vertex v

$\texttt{dist}(v)$ - *the length of the best path between s and v, found so far*

We're going to simulate MostOfBellman-Ford(s)

We now start iteration $1 \ldots$

$\infty < \infty + 2$



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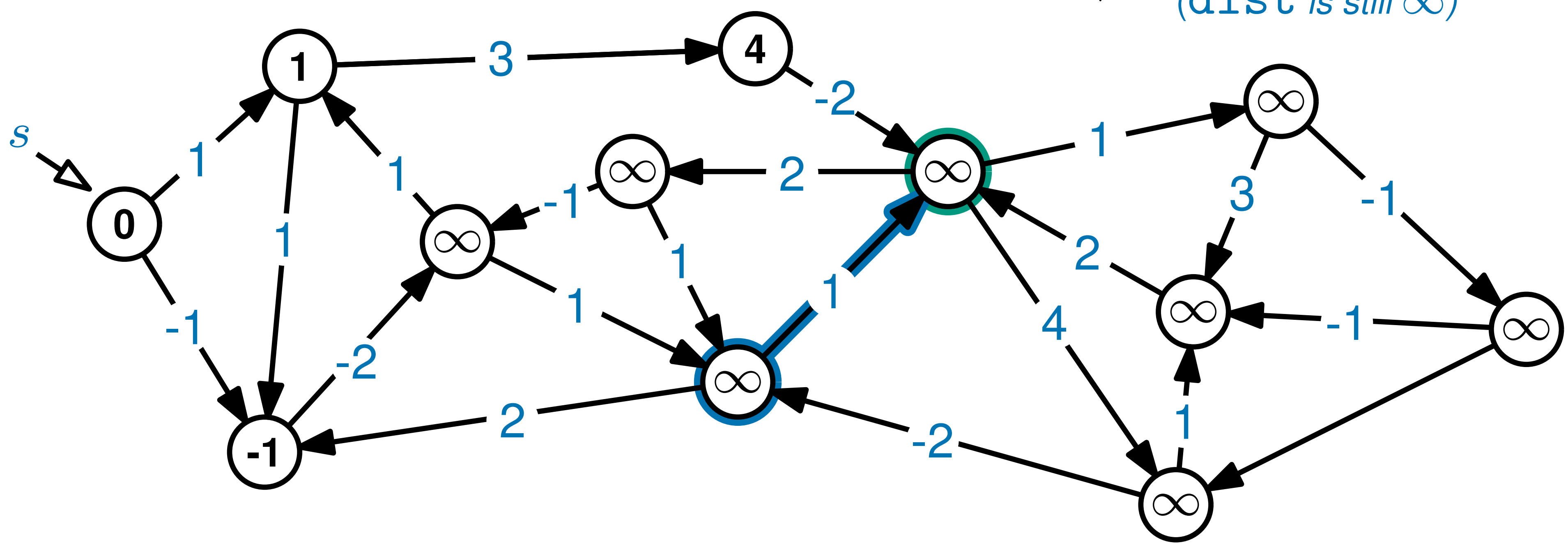
$\infty < \infty + 2$ *($\texttt{dist}$ is still ∞)*



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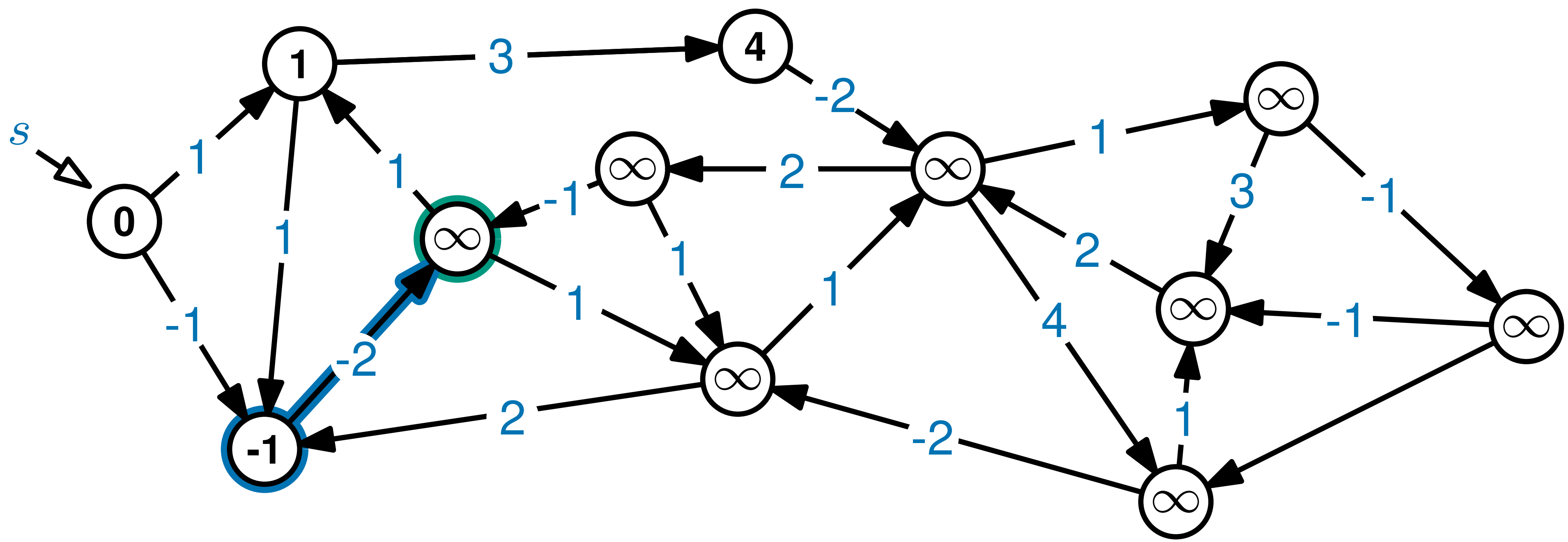

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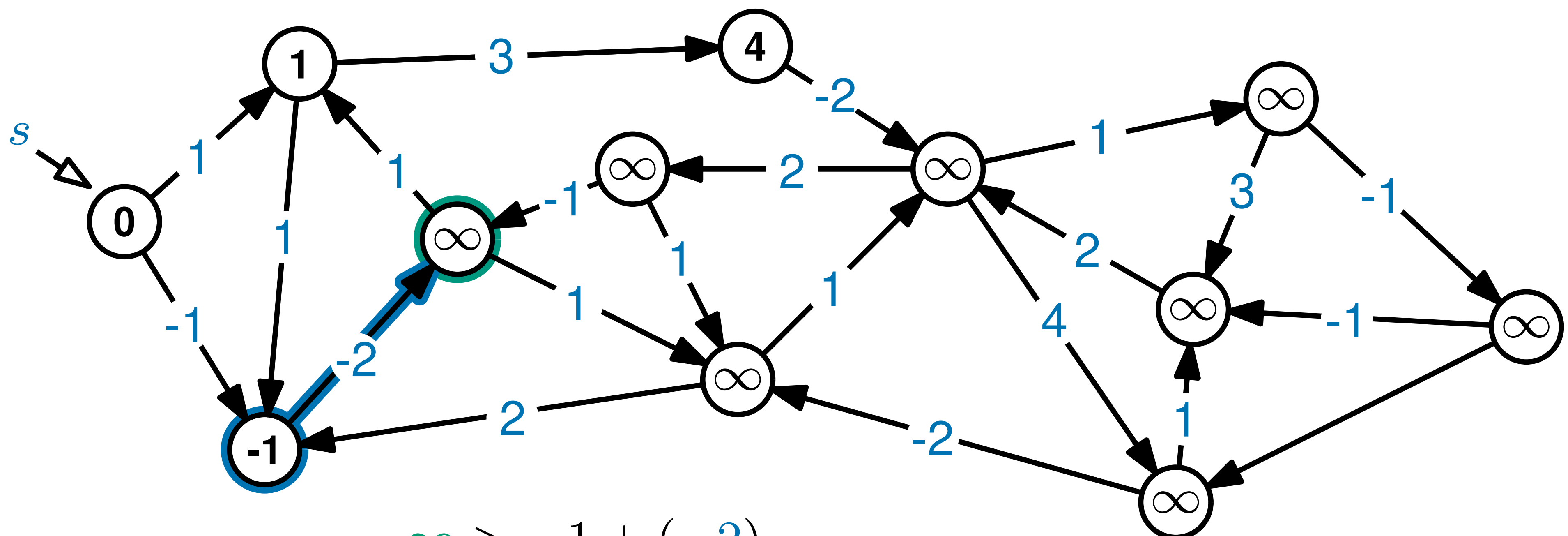

$\infty > -1 + (-2)$

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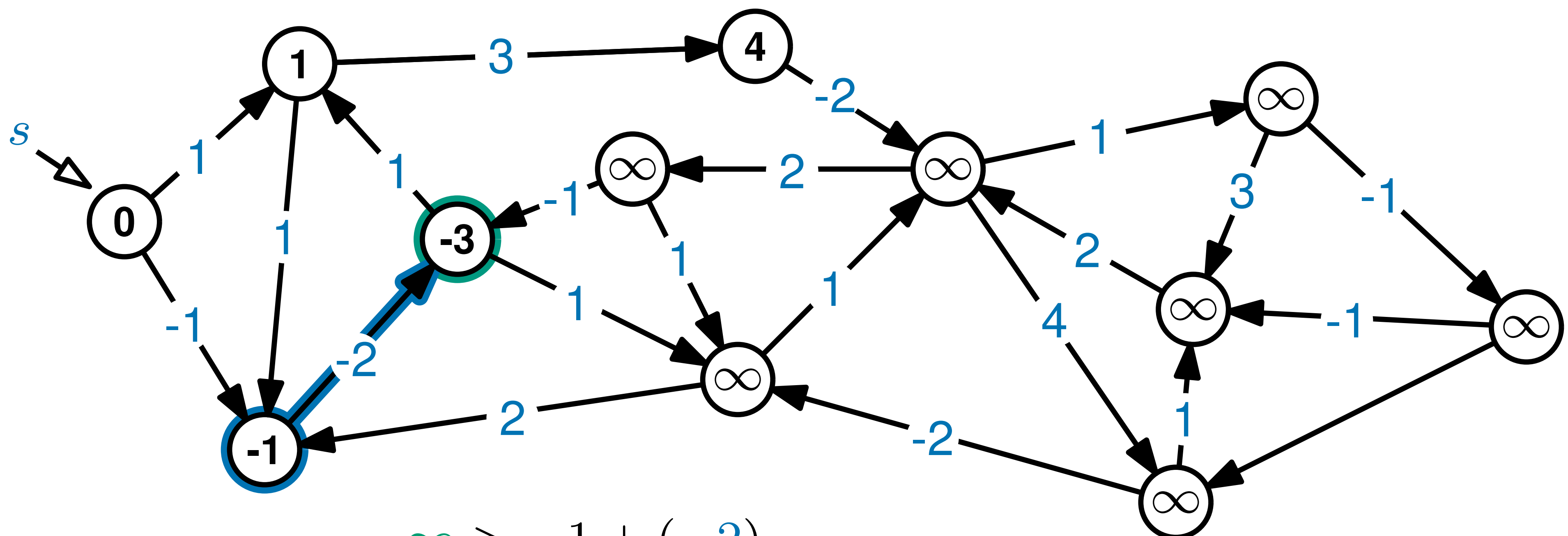

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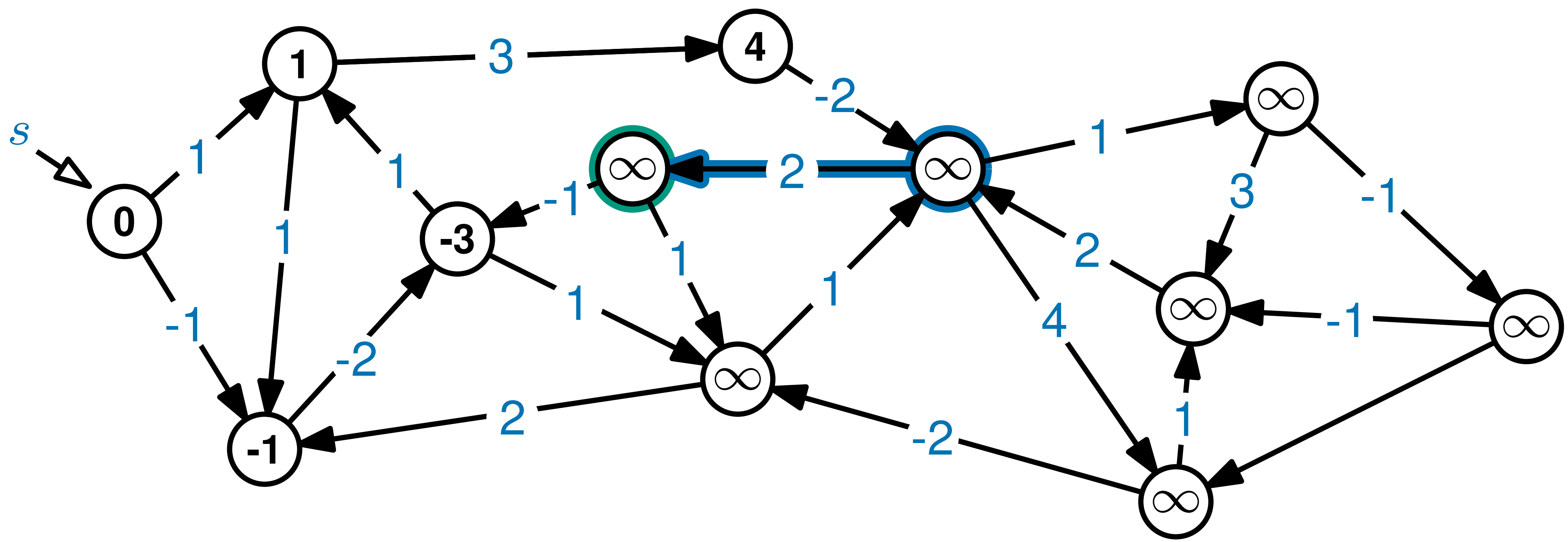

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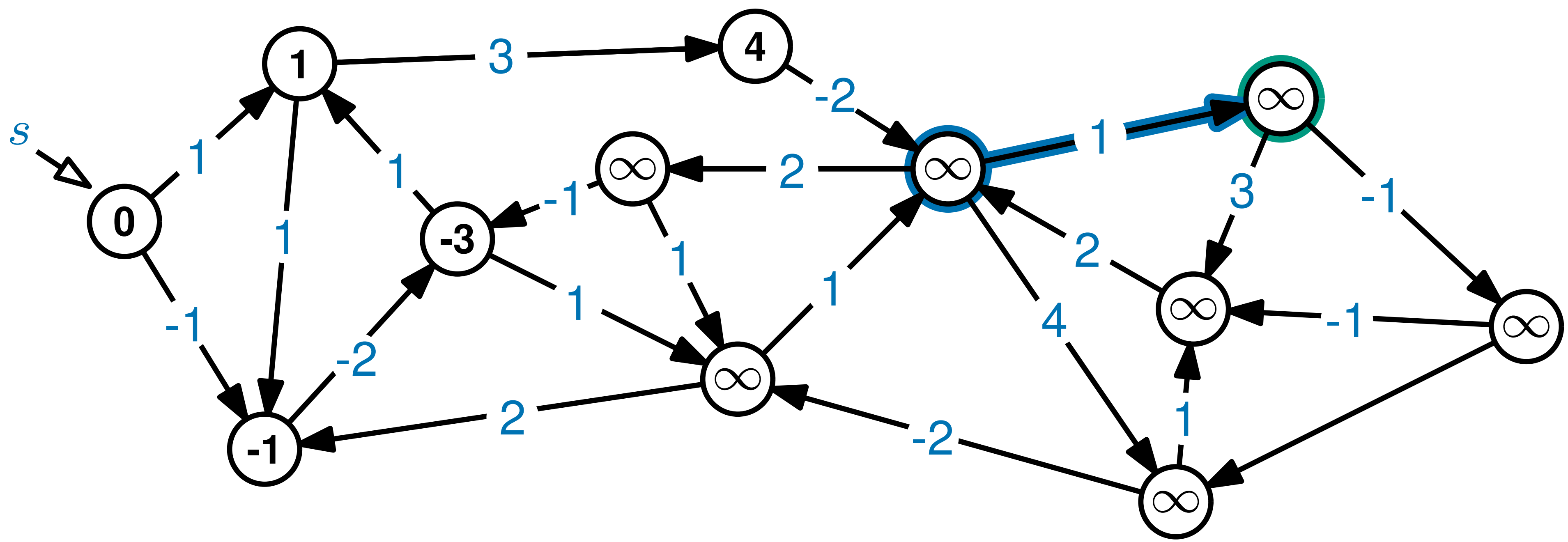

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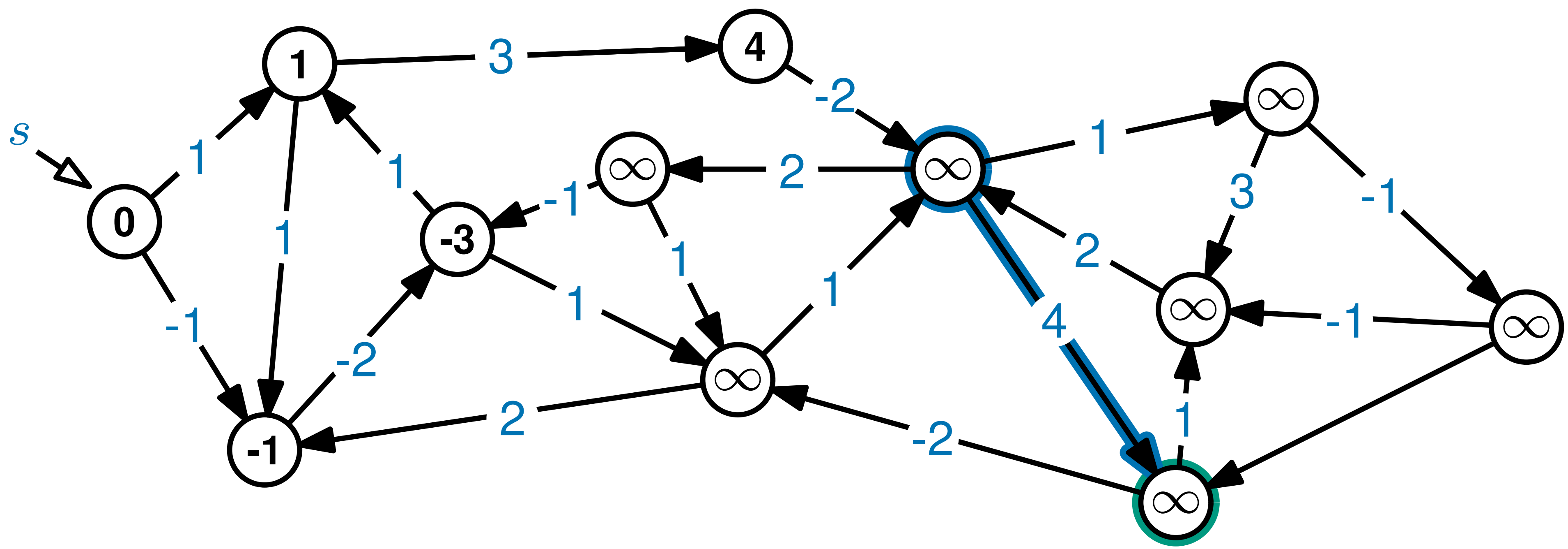

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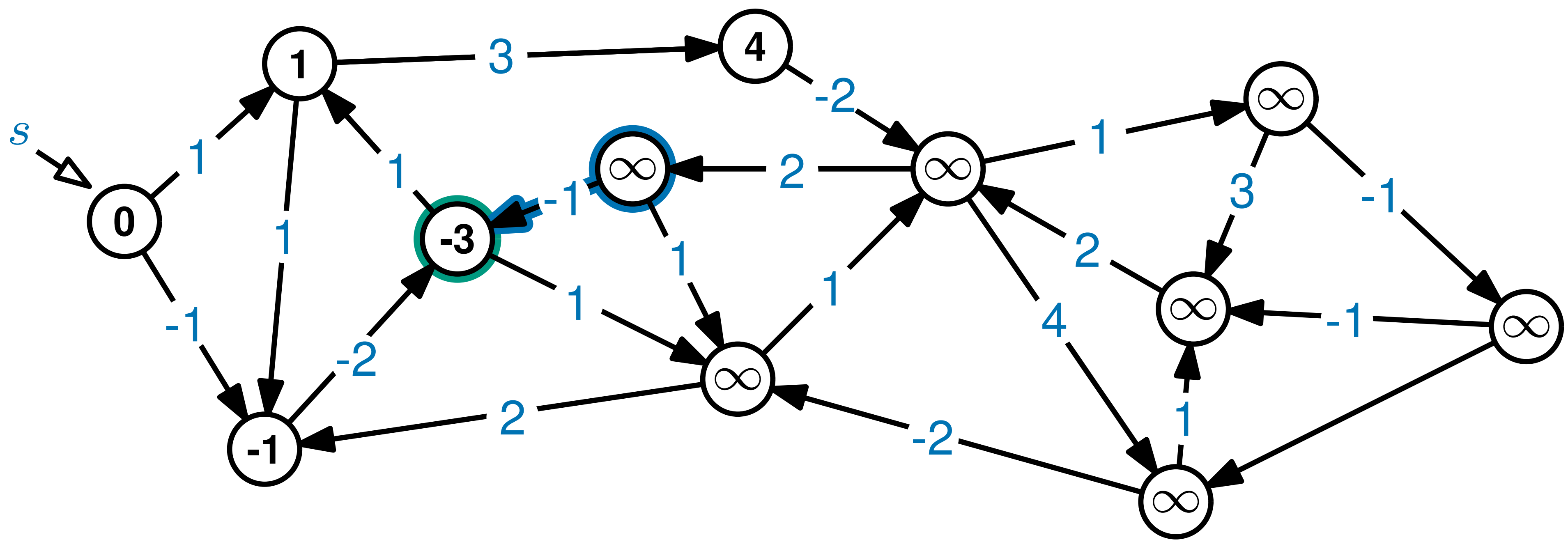

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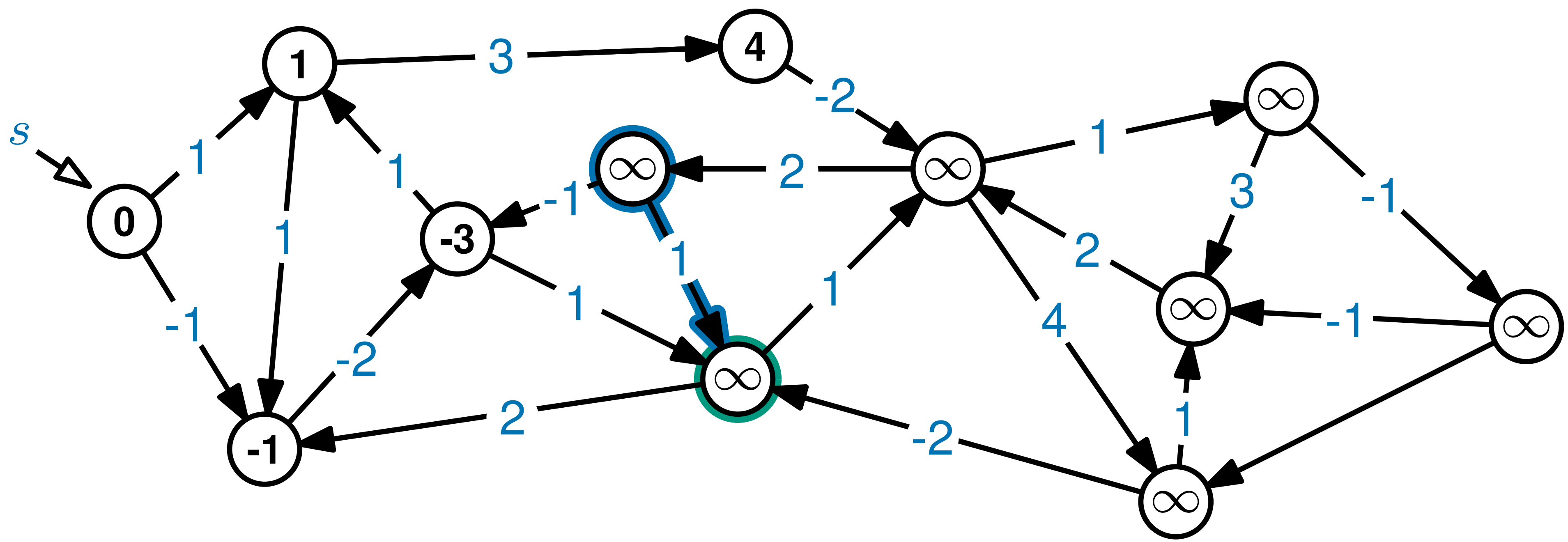

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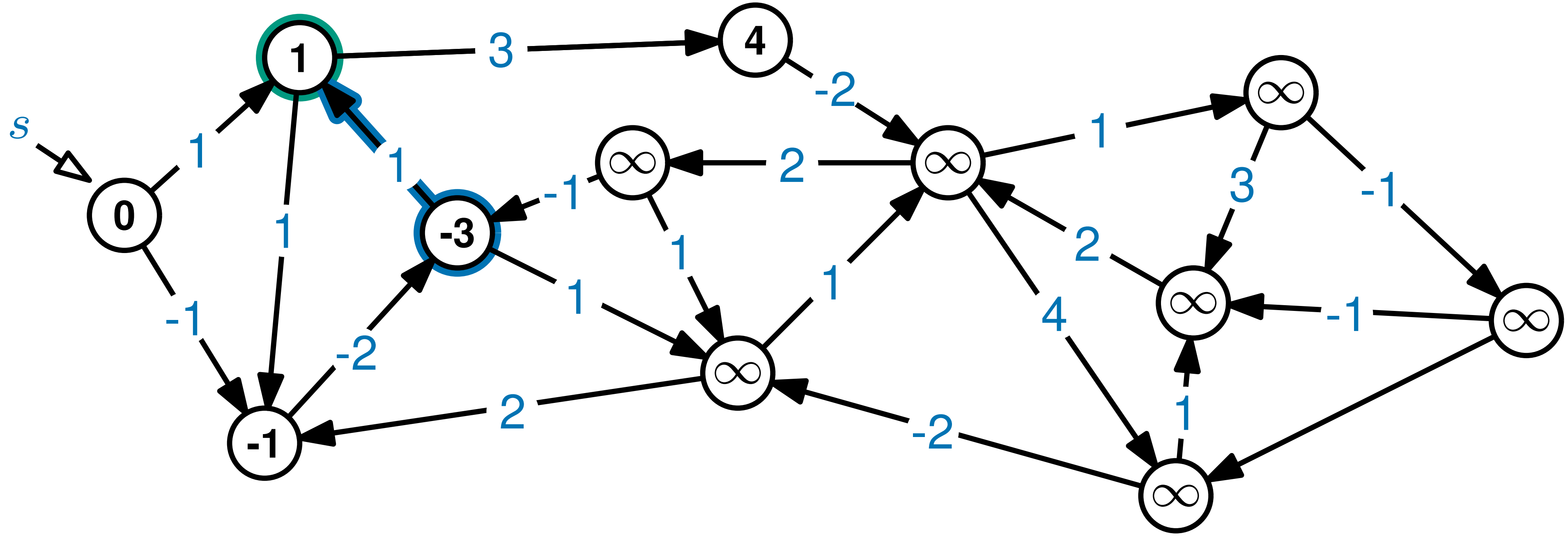

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MostOfBellman-Ford(s)

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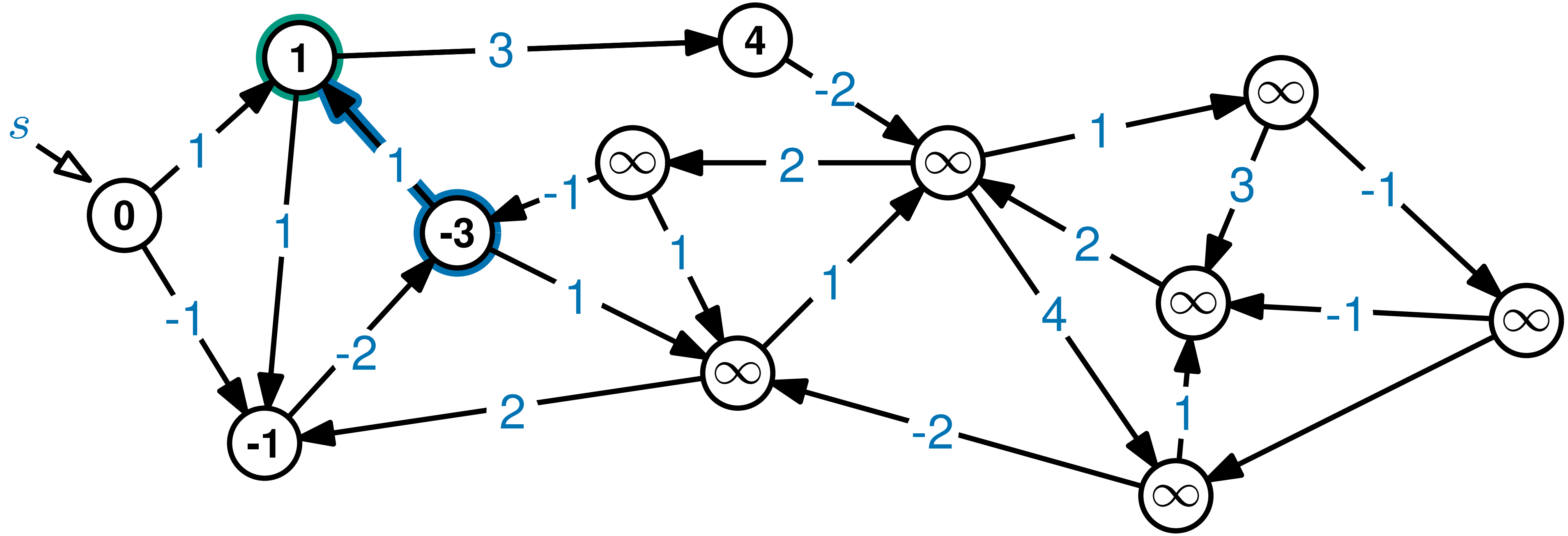
$$1 > -3 + 1$$



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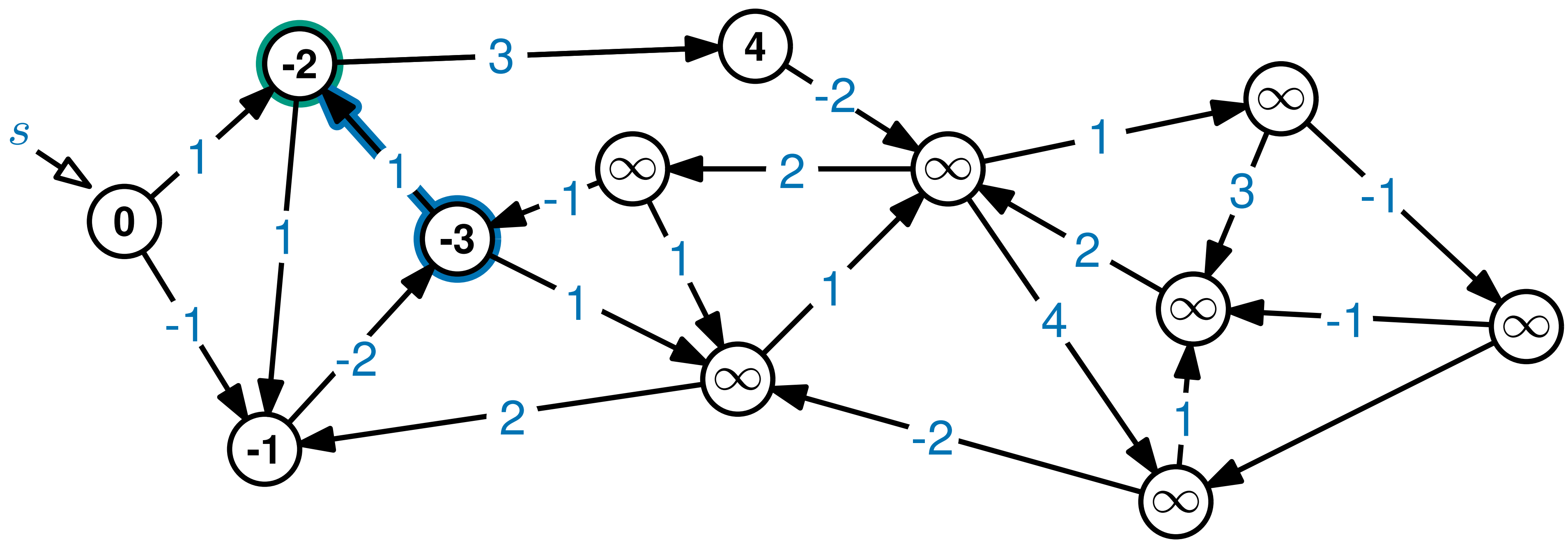

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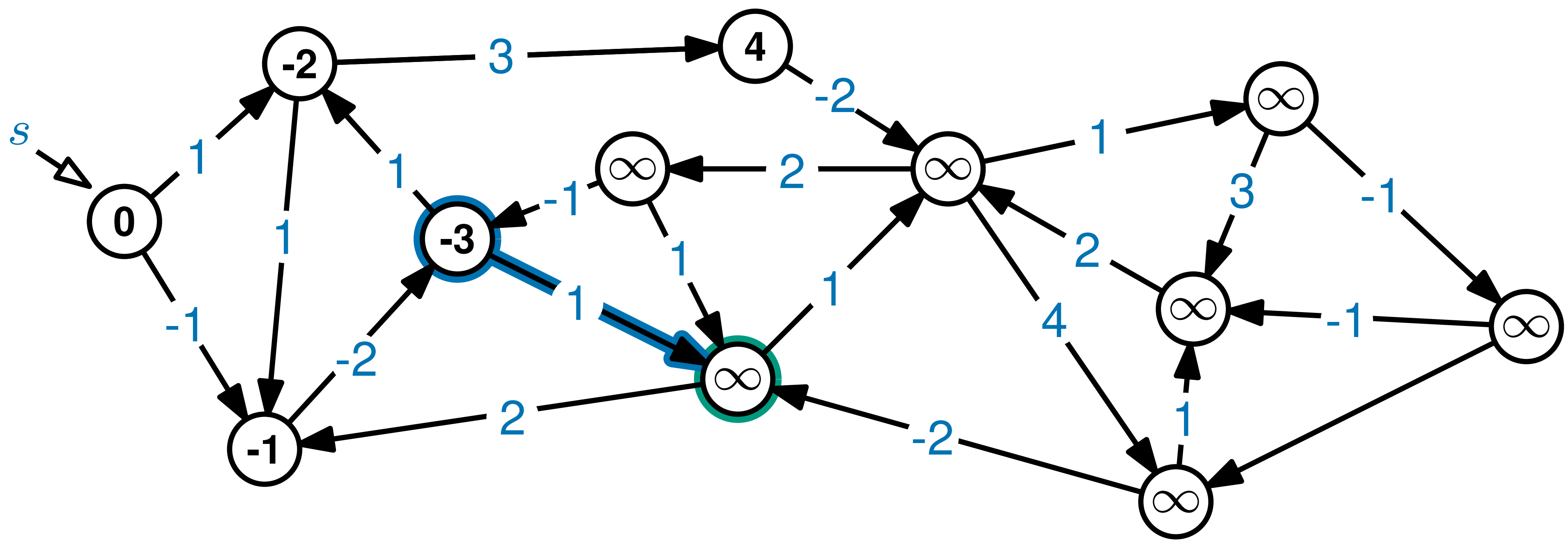

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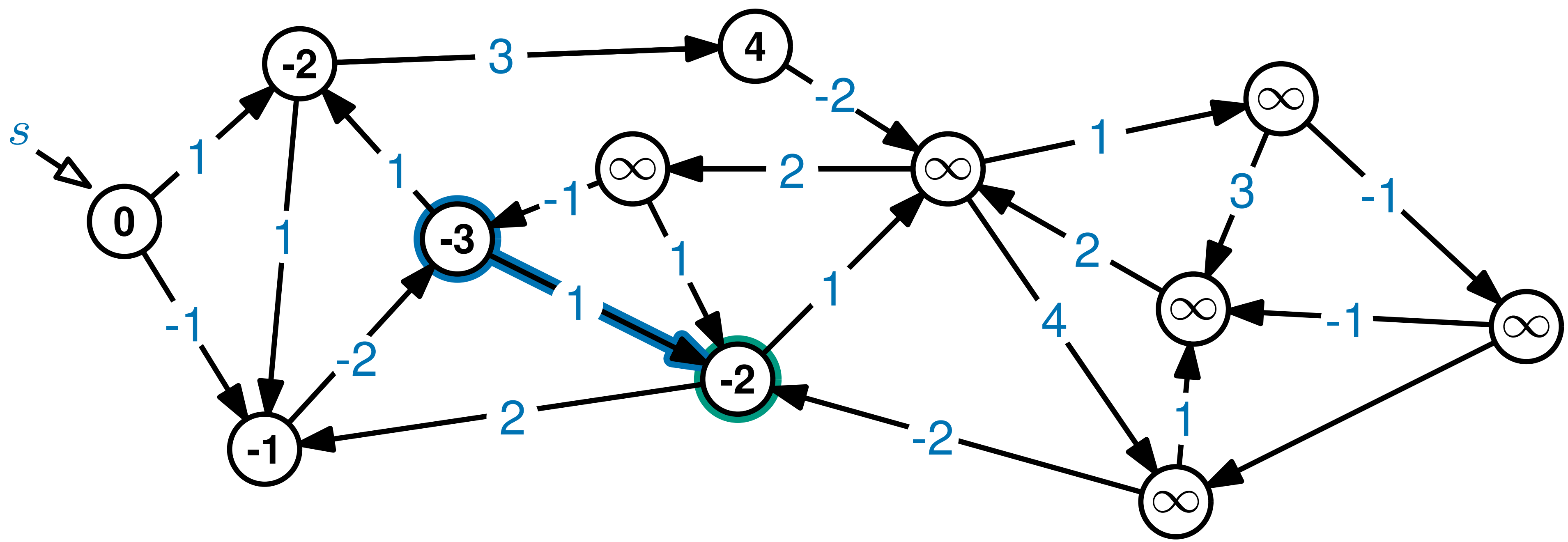

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University of BRISTOL

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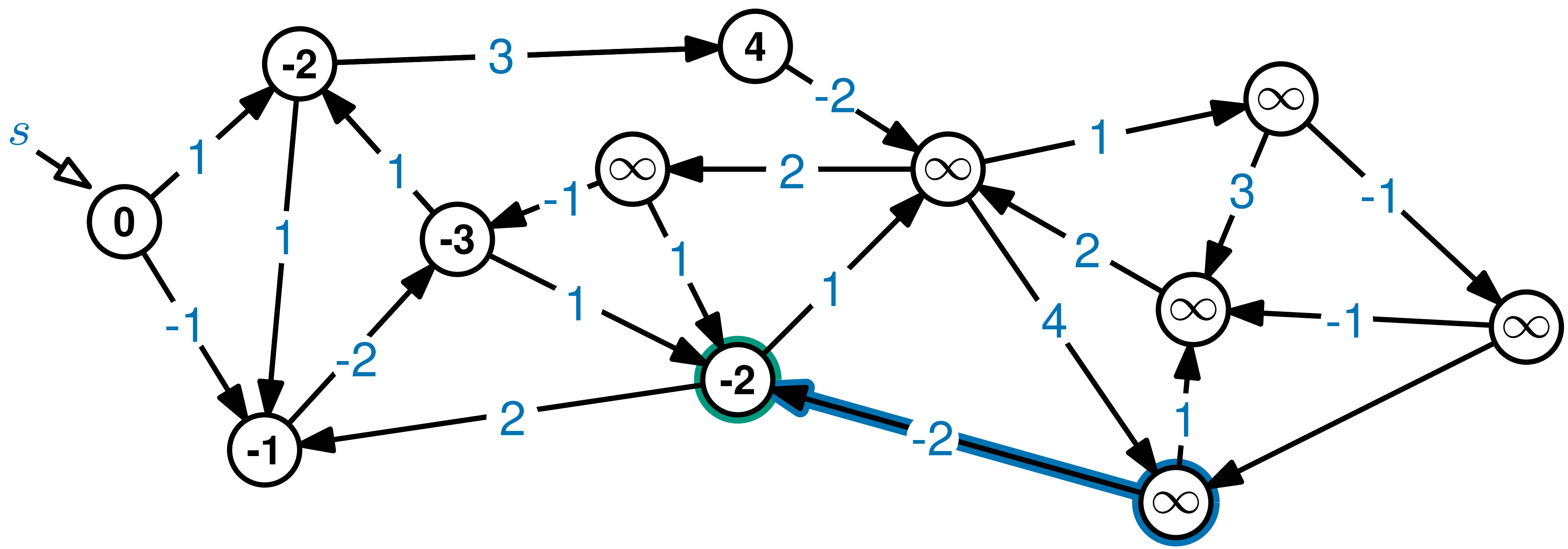

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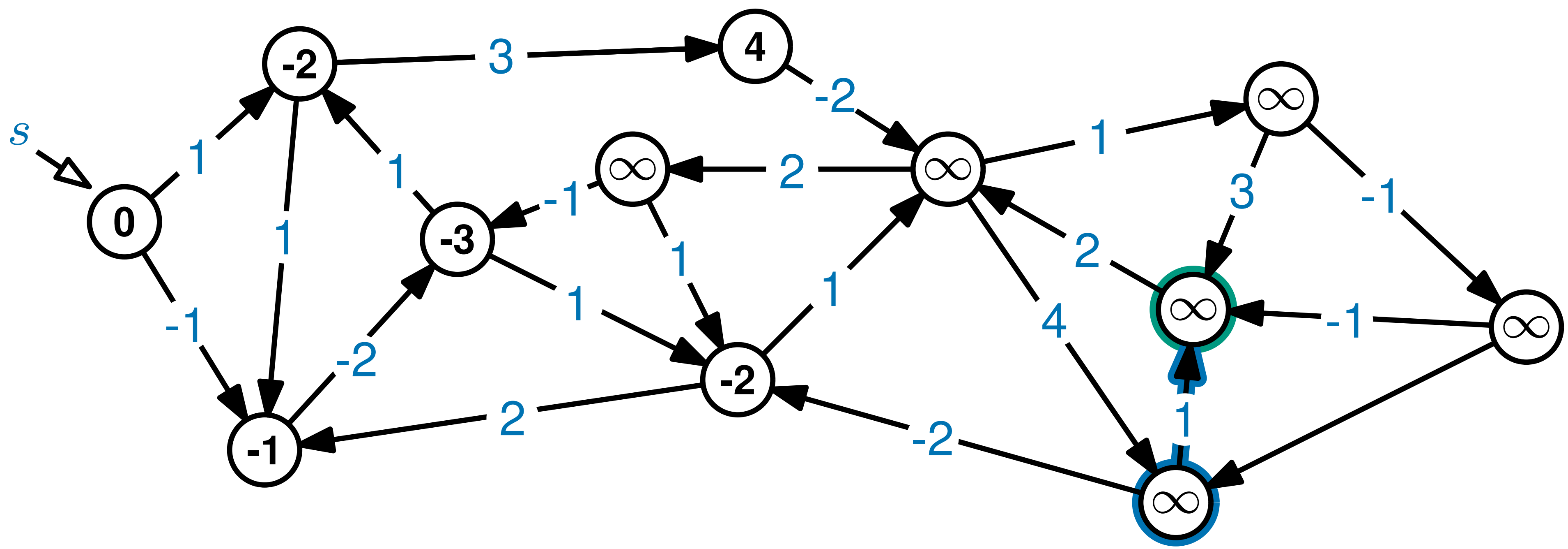

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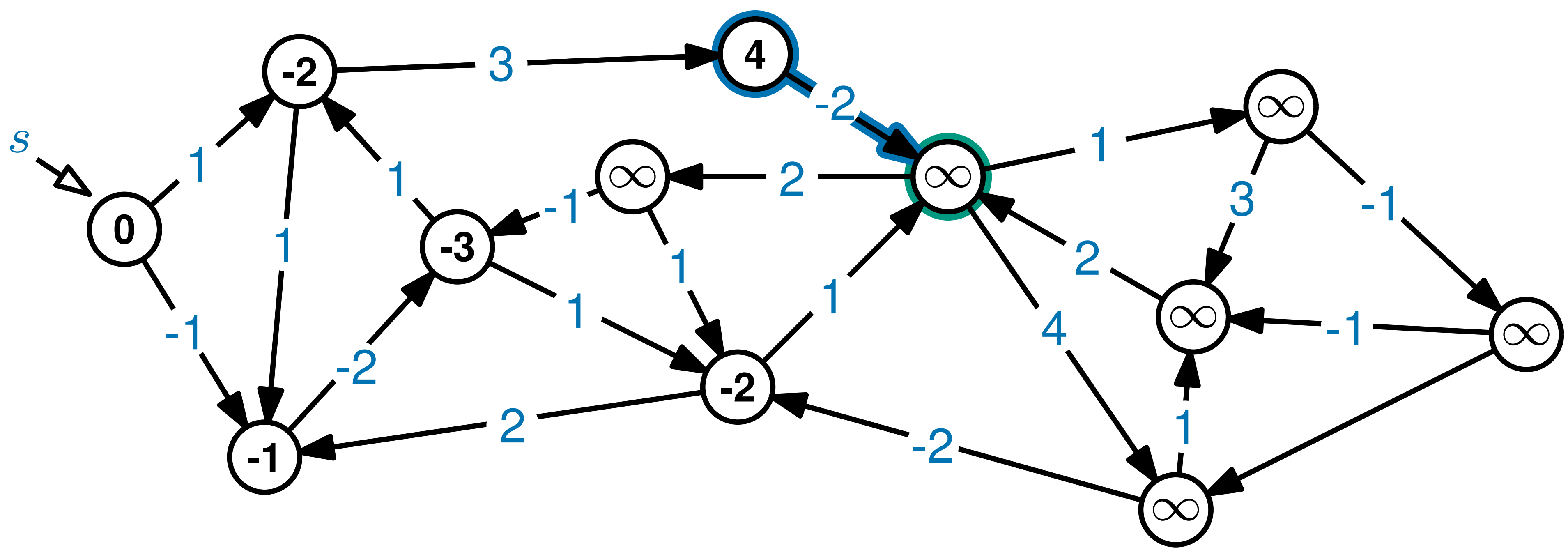

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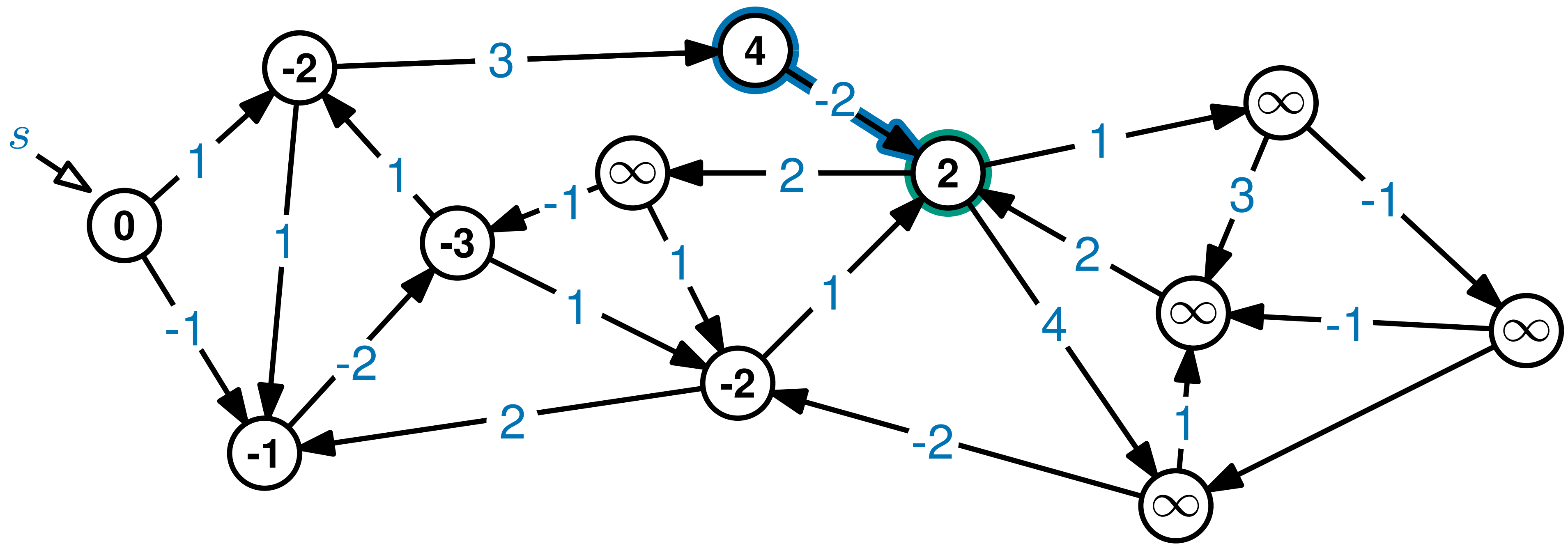

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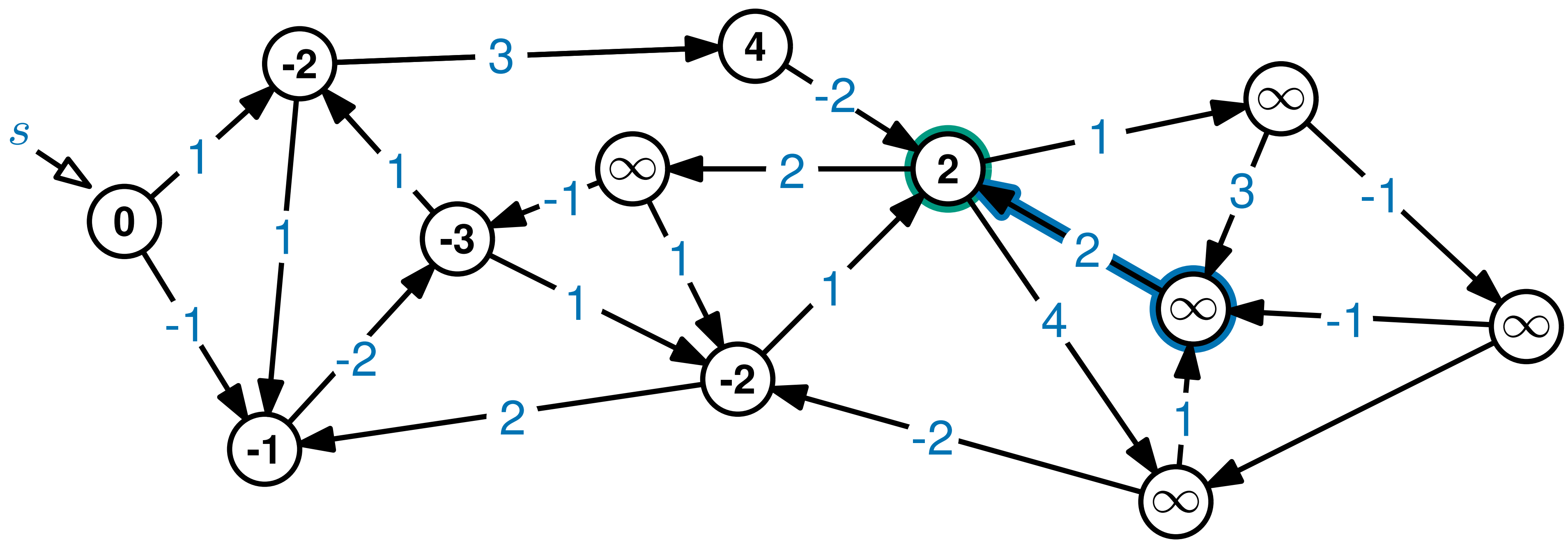

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University of BRISTOL

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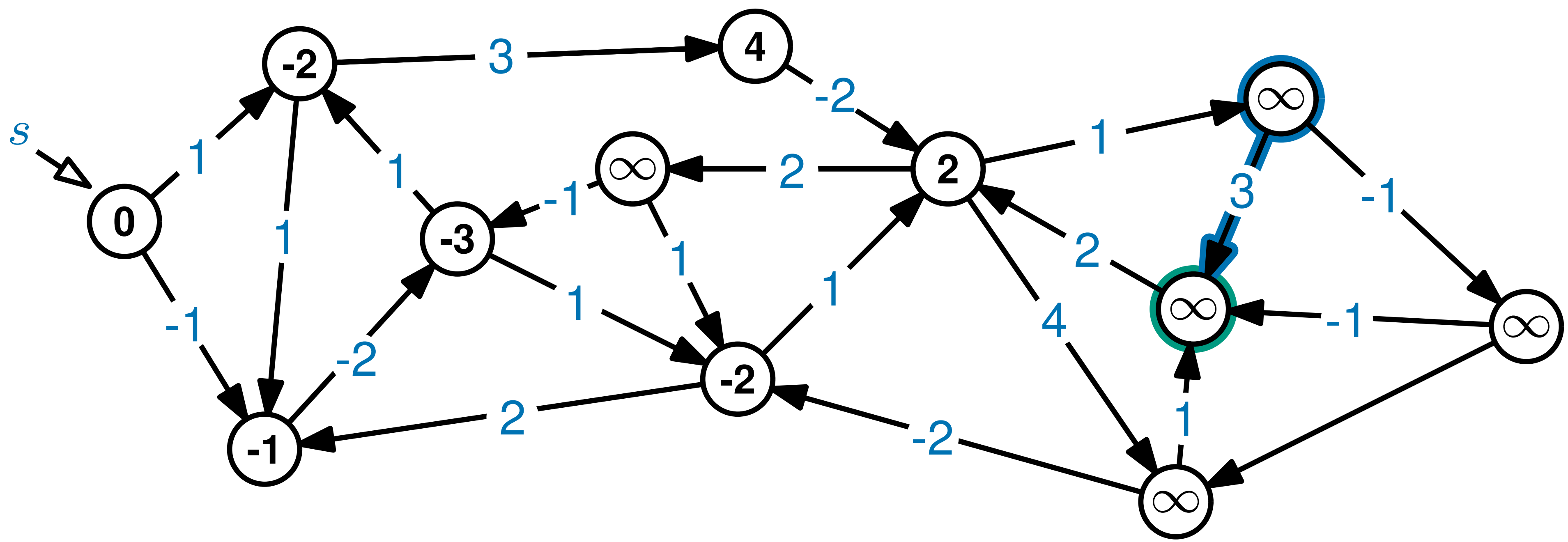

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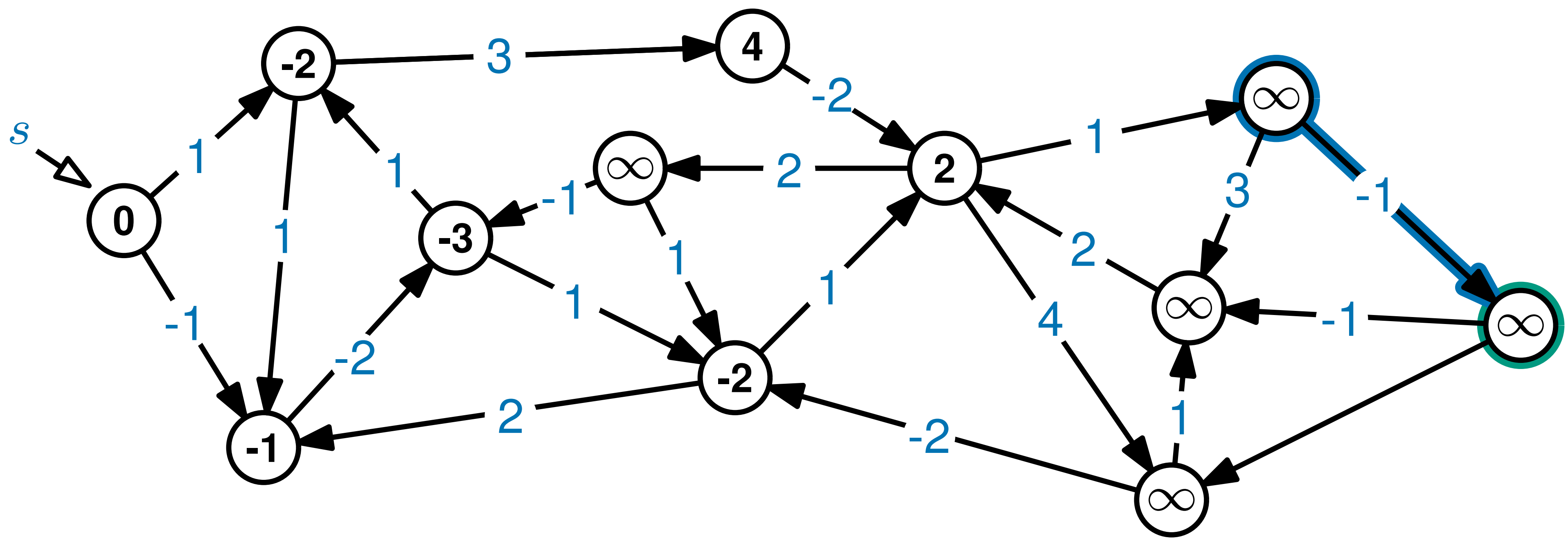

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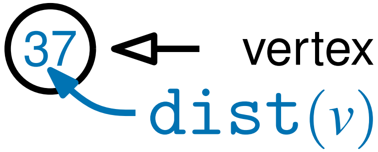

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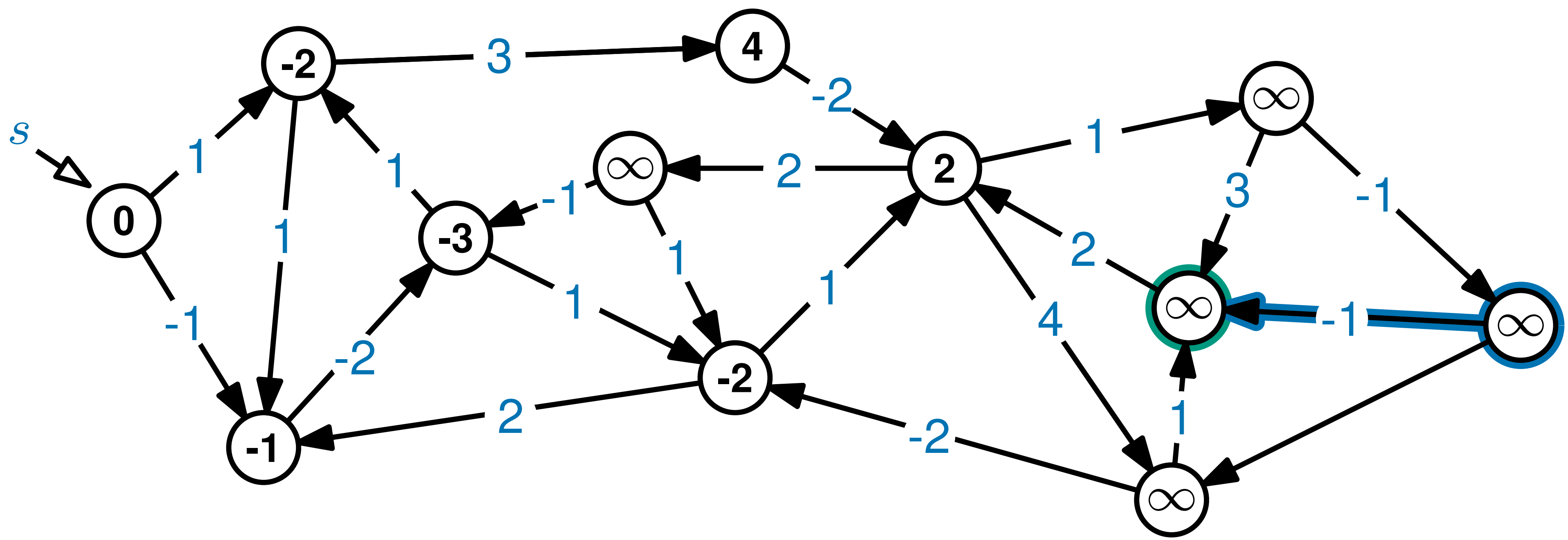

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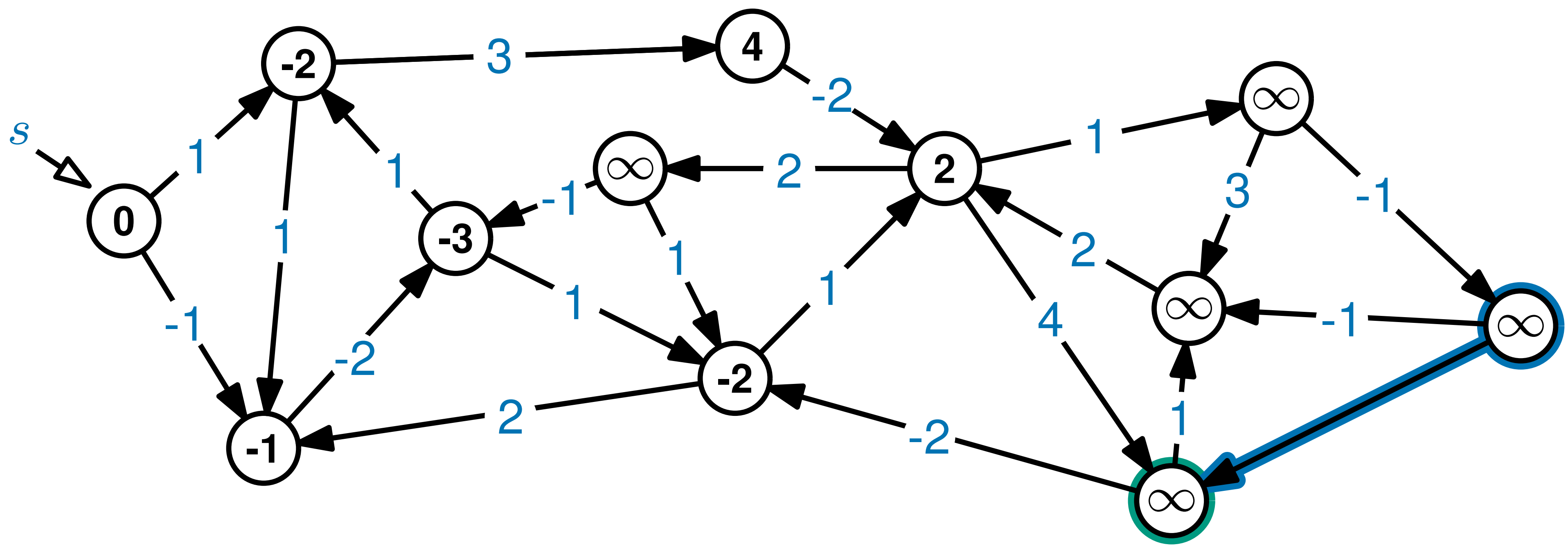

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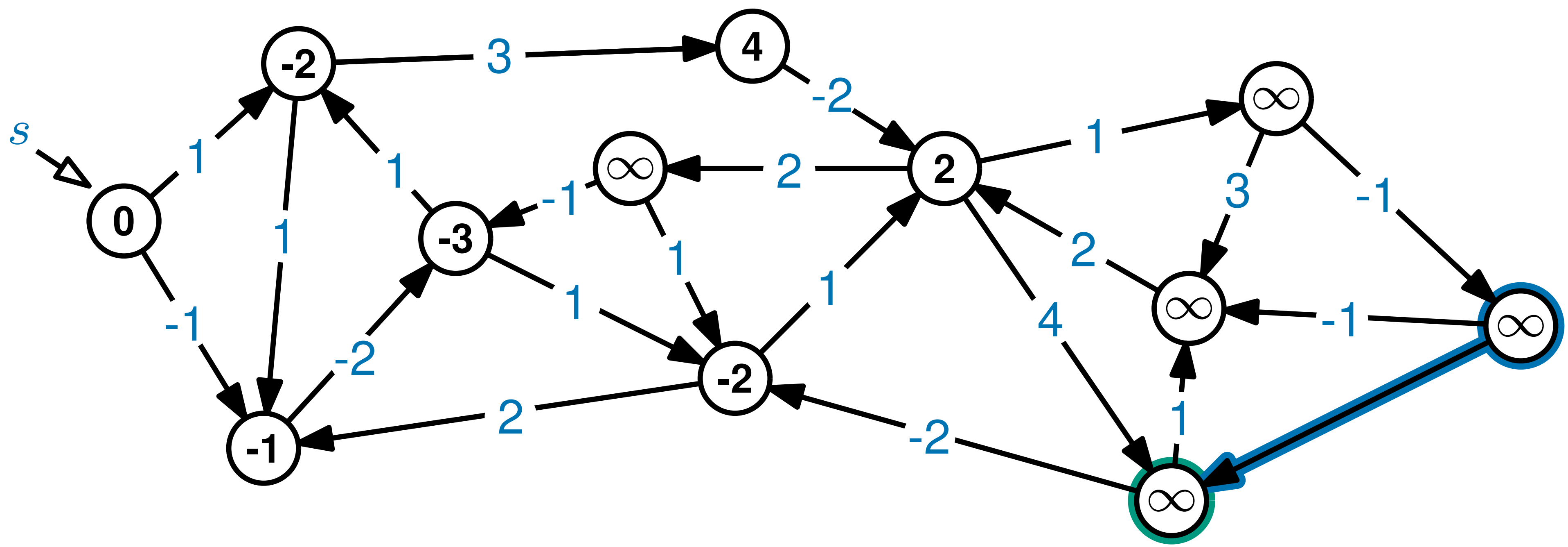
We're going to simulate MOSTOFBELLMAN-FORD(s)

This is the end of iteration 1



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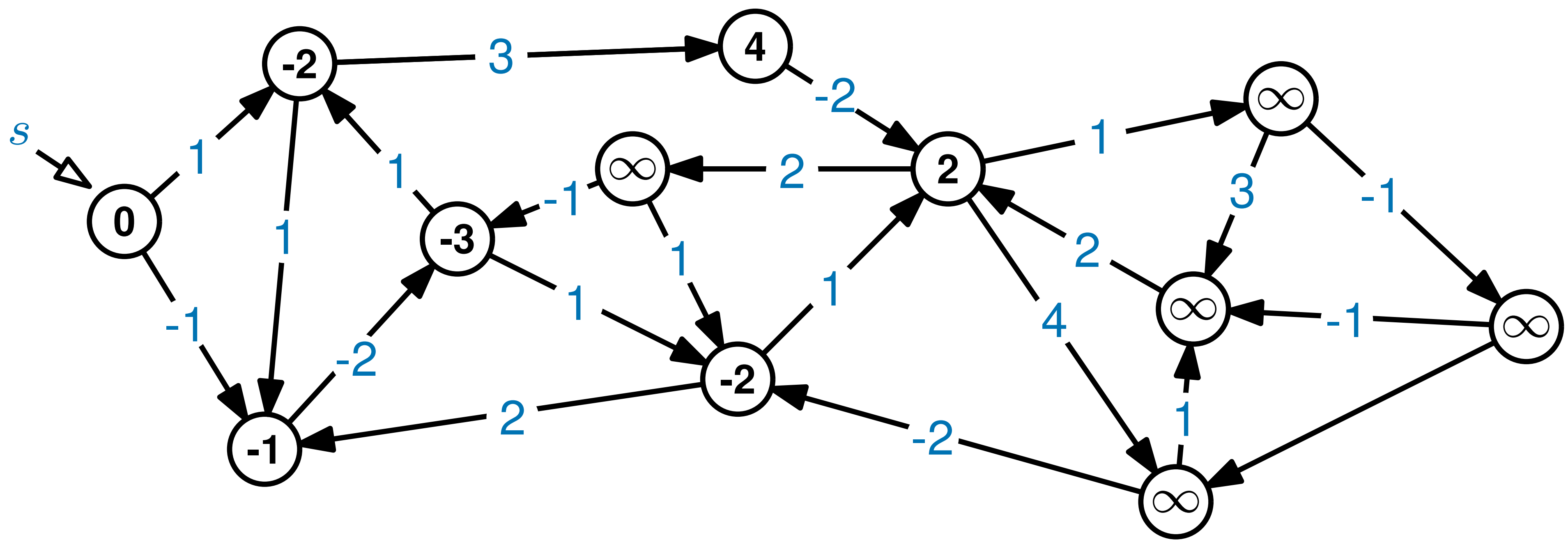

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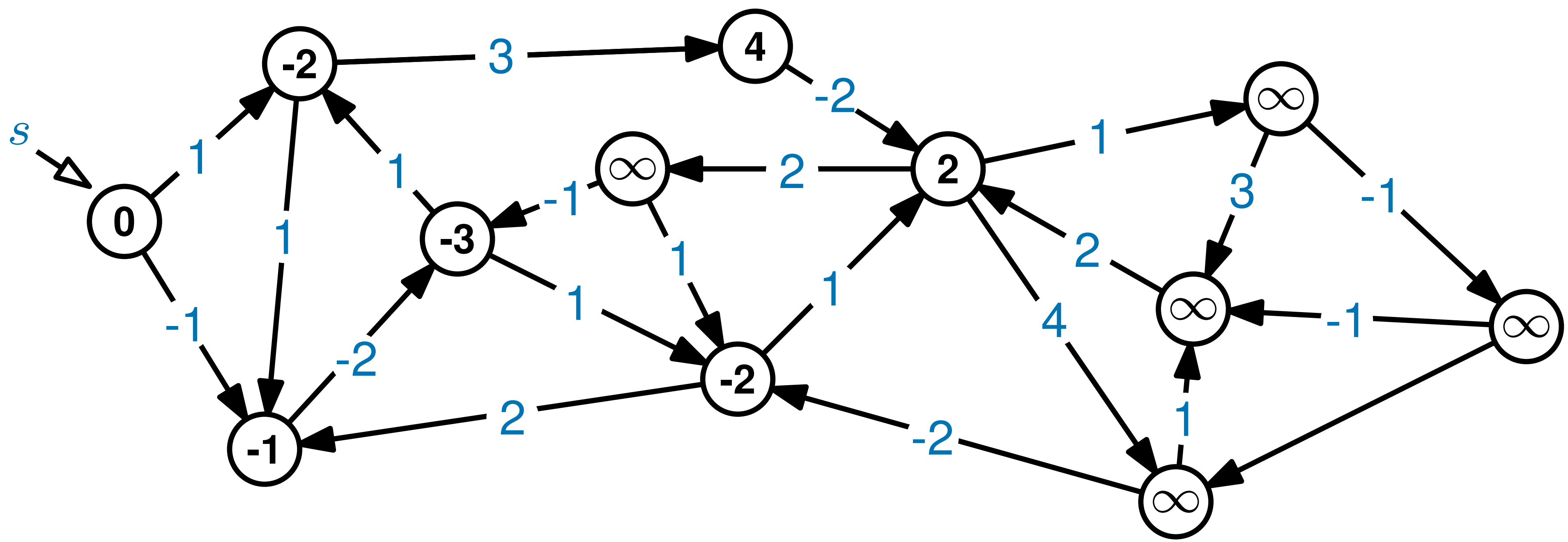
We're going to simulate MOSTOFBELLMAN-FORD(s)

How are things looking?



MOSTOFBELLMAN-FORD runs $|V|$ iterations,

In each iteration we RELAX every edge (u, v)

(in the order they occur in the adjacency list)

RELAX(u, v)

$\texttt{if}\ \ \texttt{dist}(v) > \texttt{dist}(u) + \texttt{weight}(u, v)$
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37 vertex v

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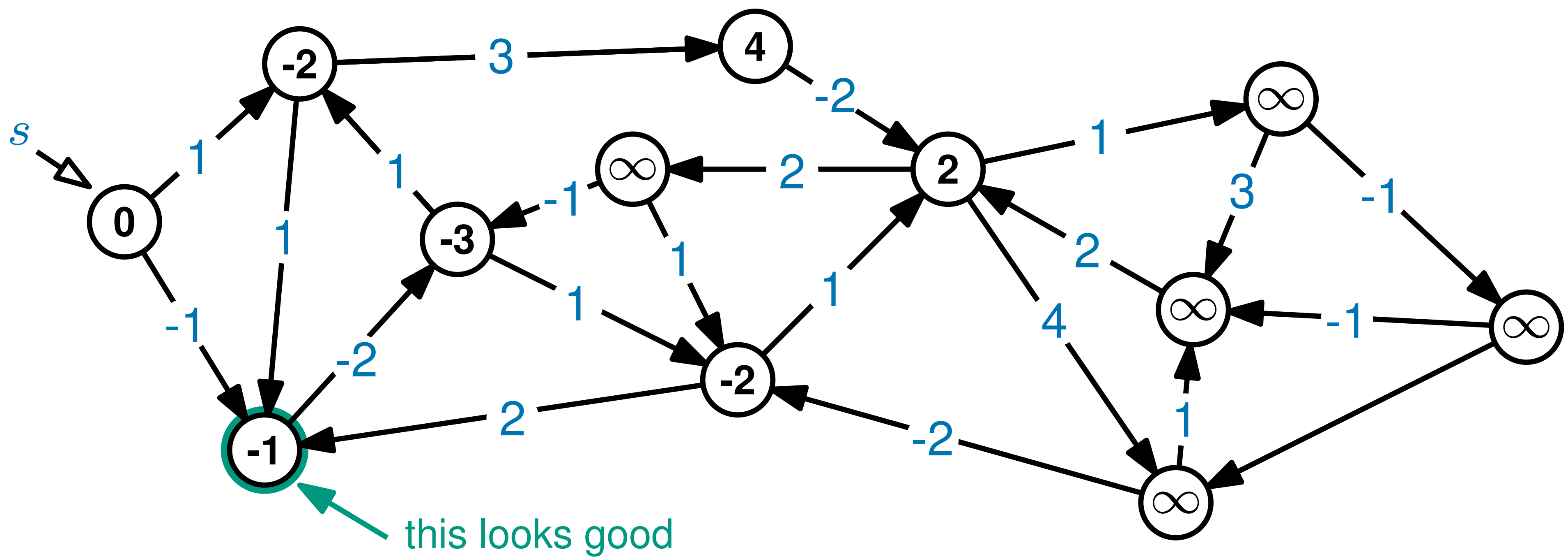

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MOSTOFBELLMAN-FORD(s)

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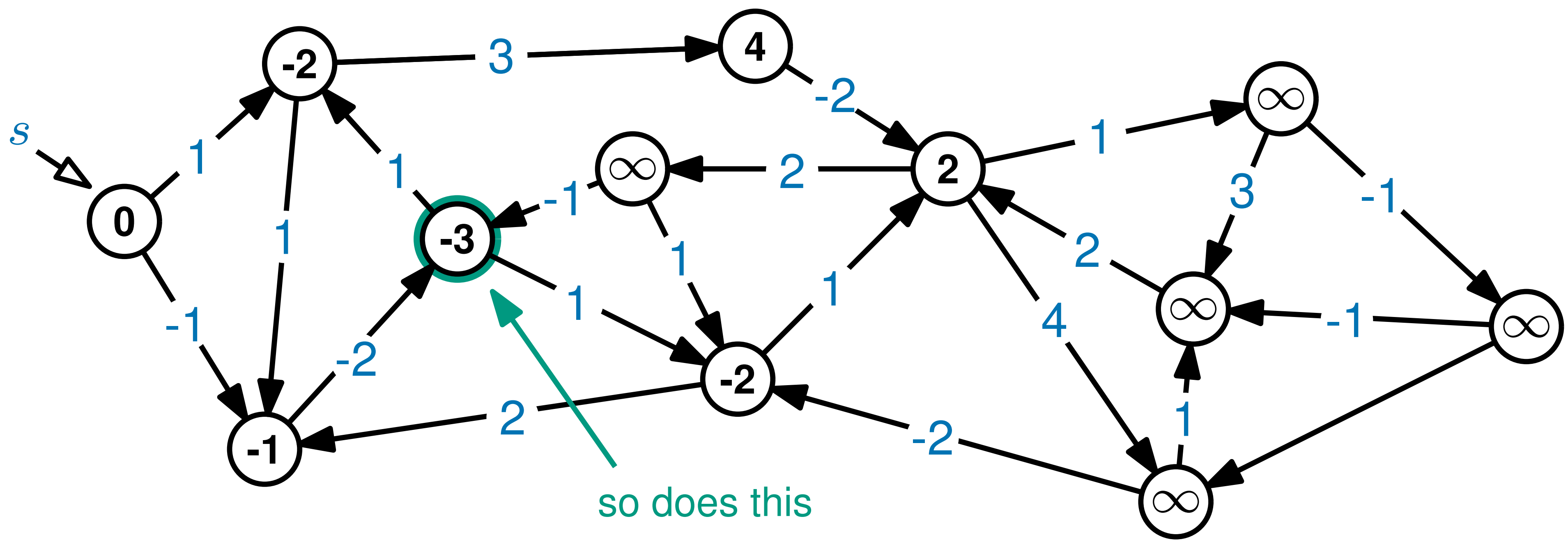

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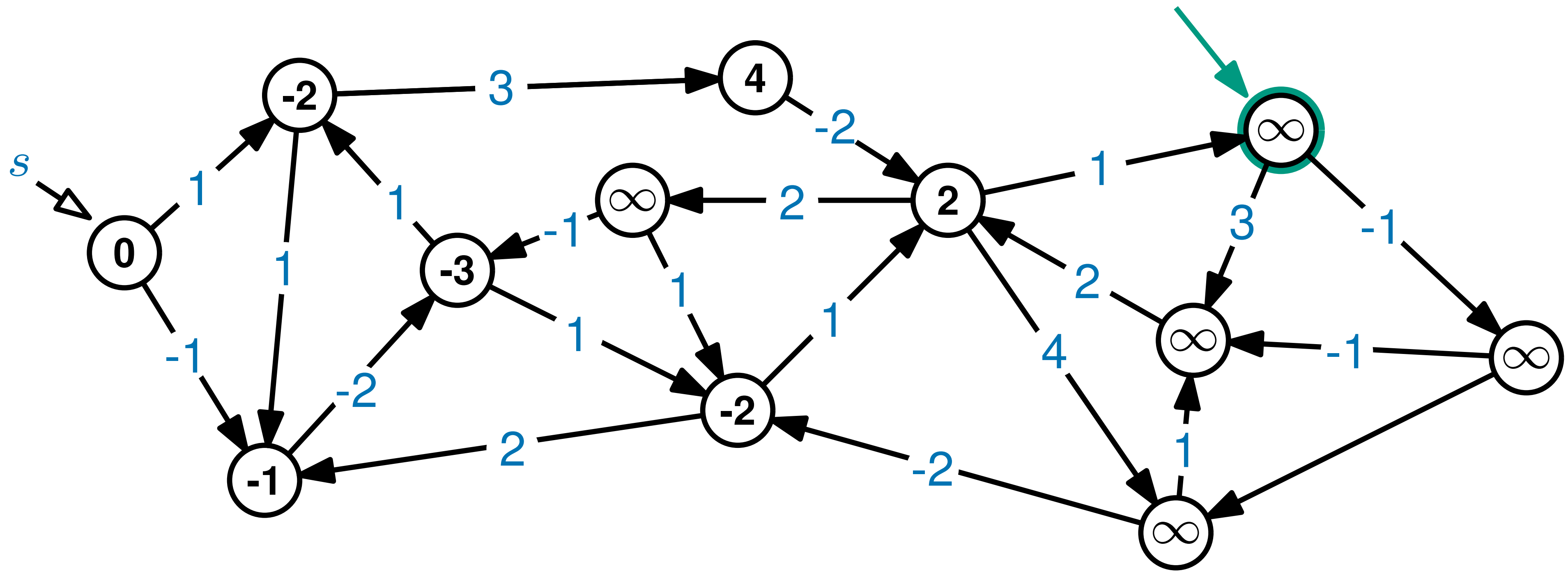
We're going to simulate

MOSTOFBELLMAN-FORD(s)

How are things looking?

this doesn't look so good



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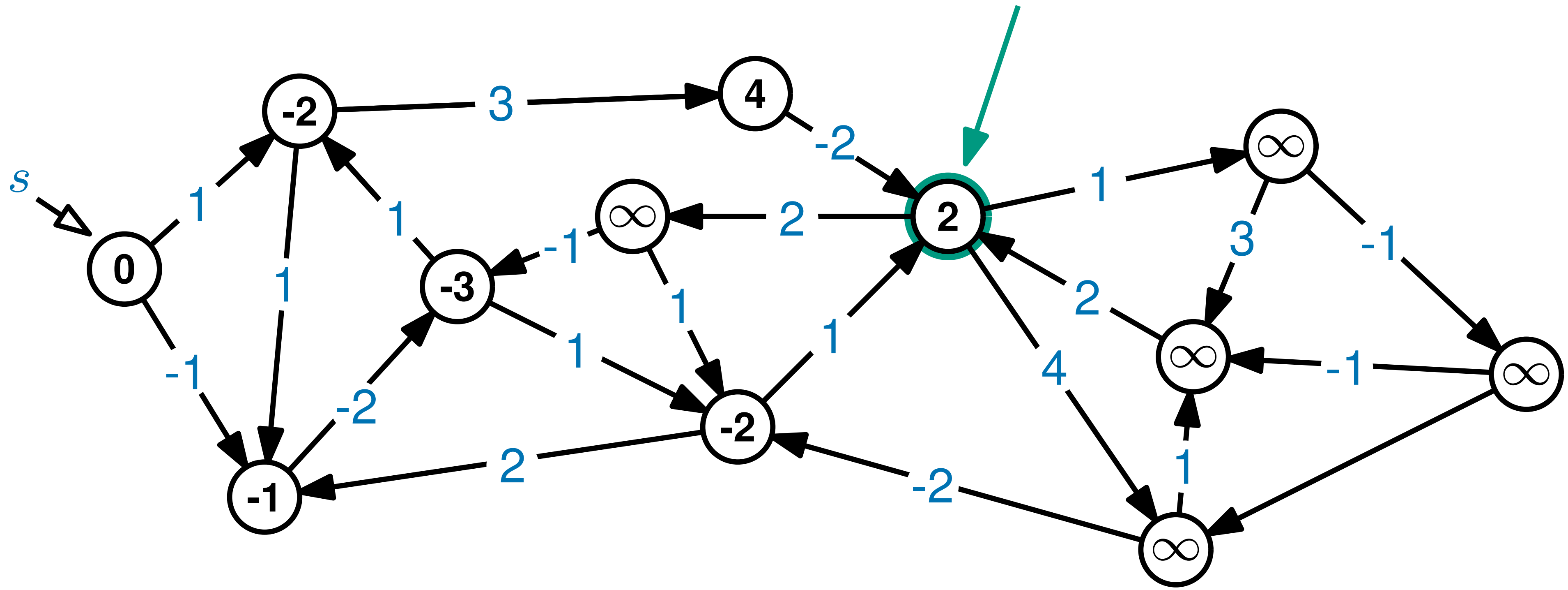
We're going to simulate

MOSTOFBELLMAN-FORD(s)

How are things looking?

neither does this



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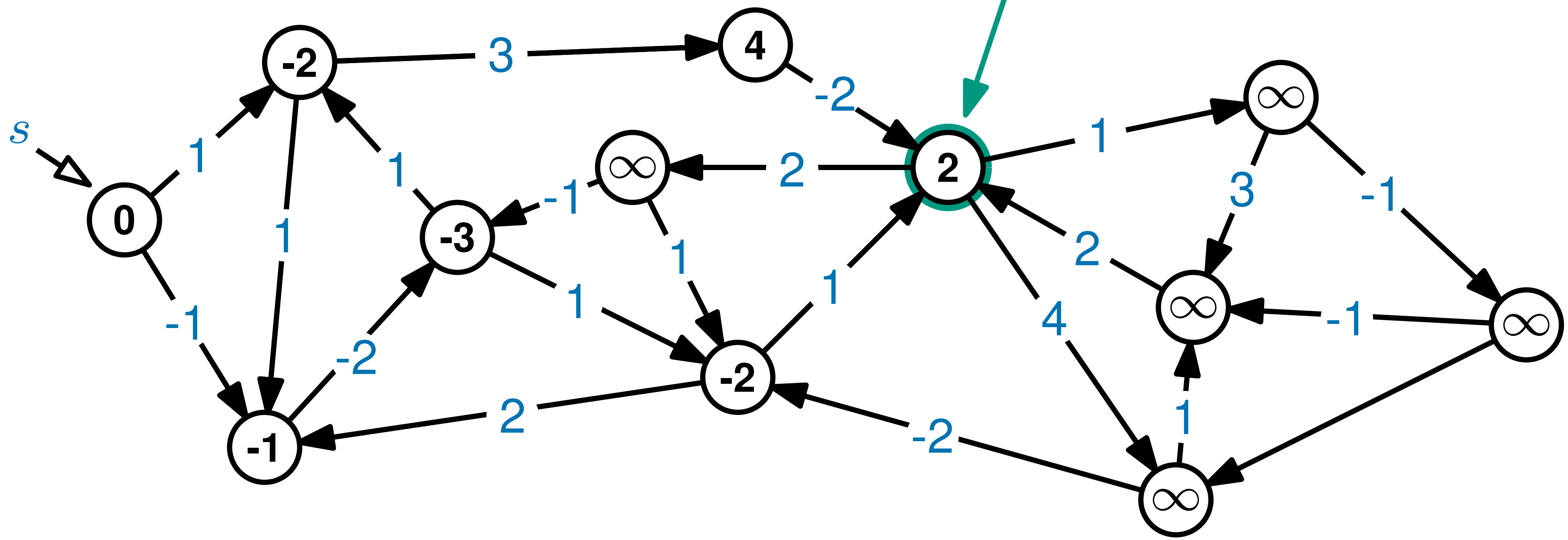
We're going to simulate

MOSTOFBELLMAN-FORD(s)

How are things looking?

neither does this

(it's not the shortest path length)



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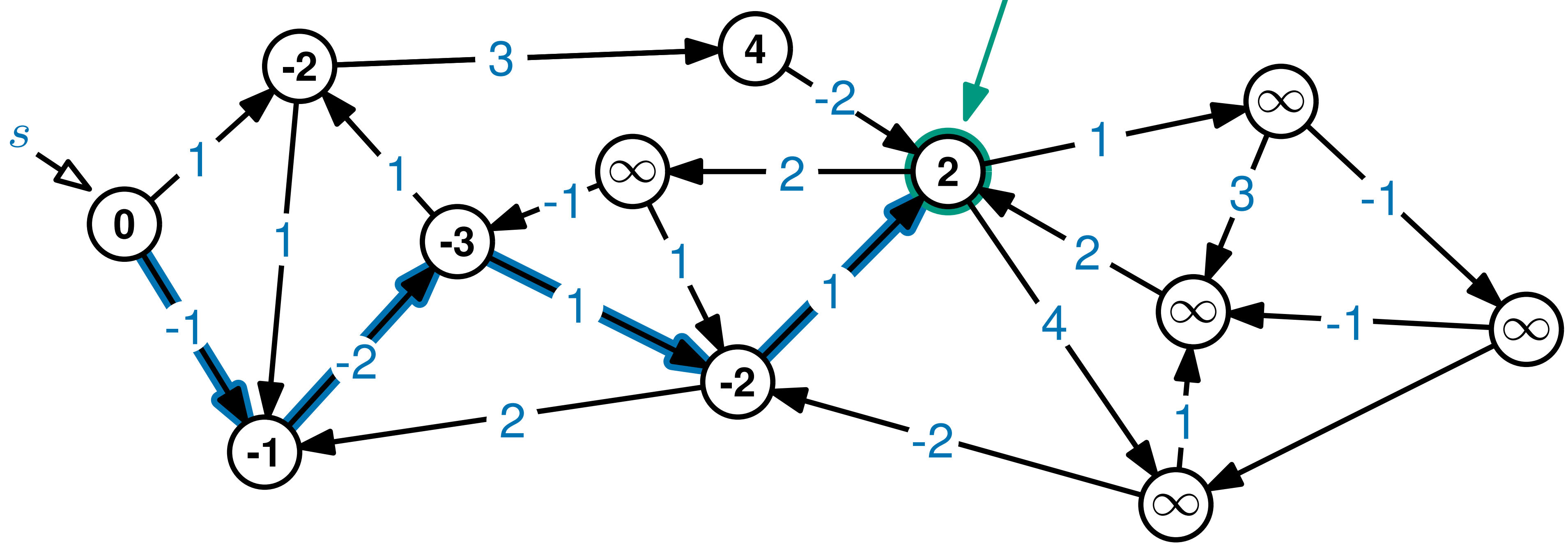

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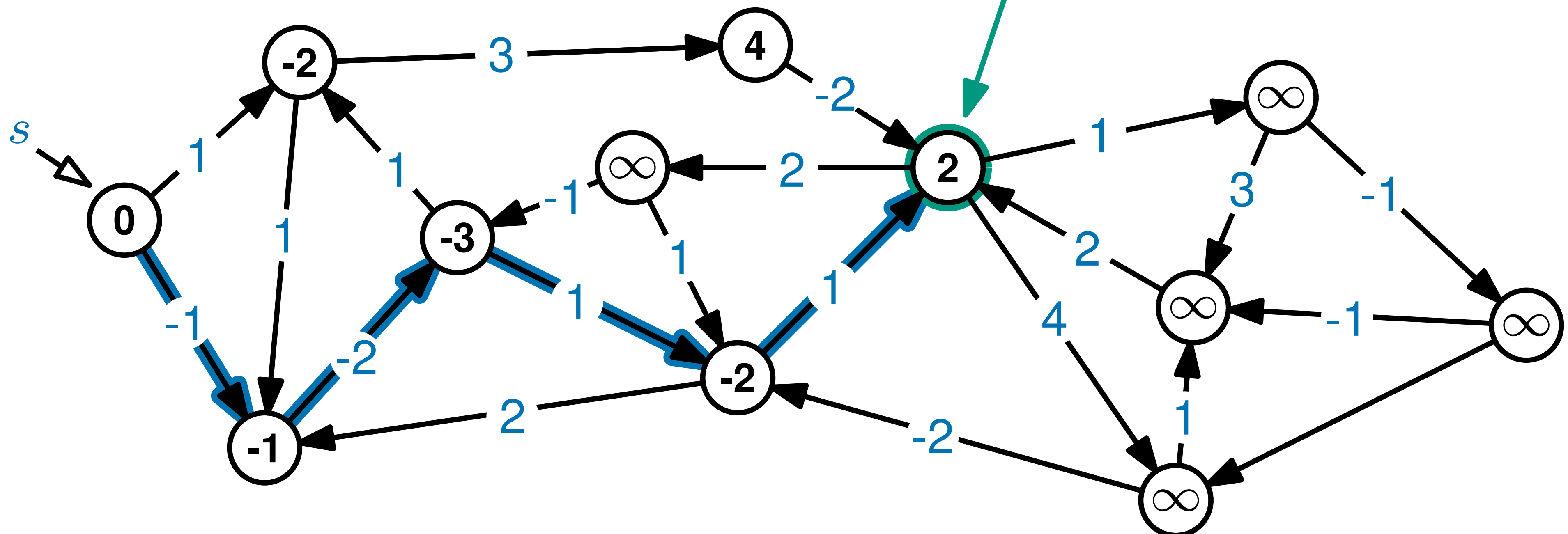

This path has length -1

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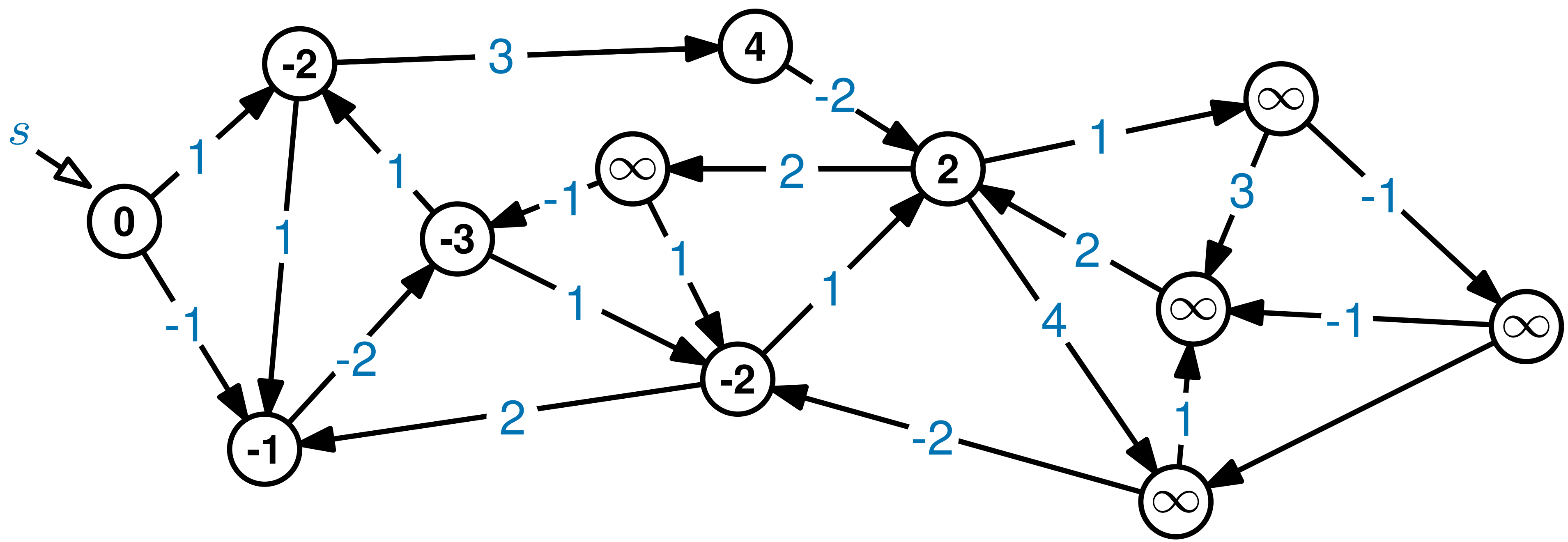

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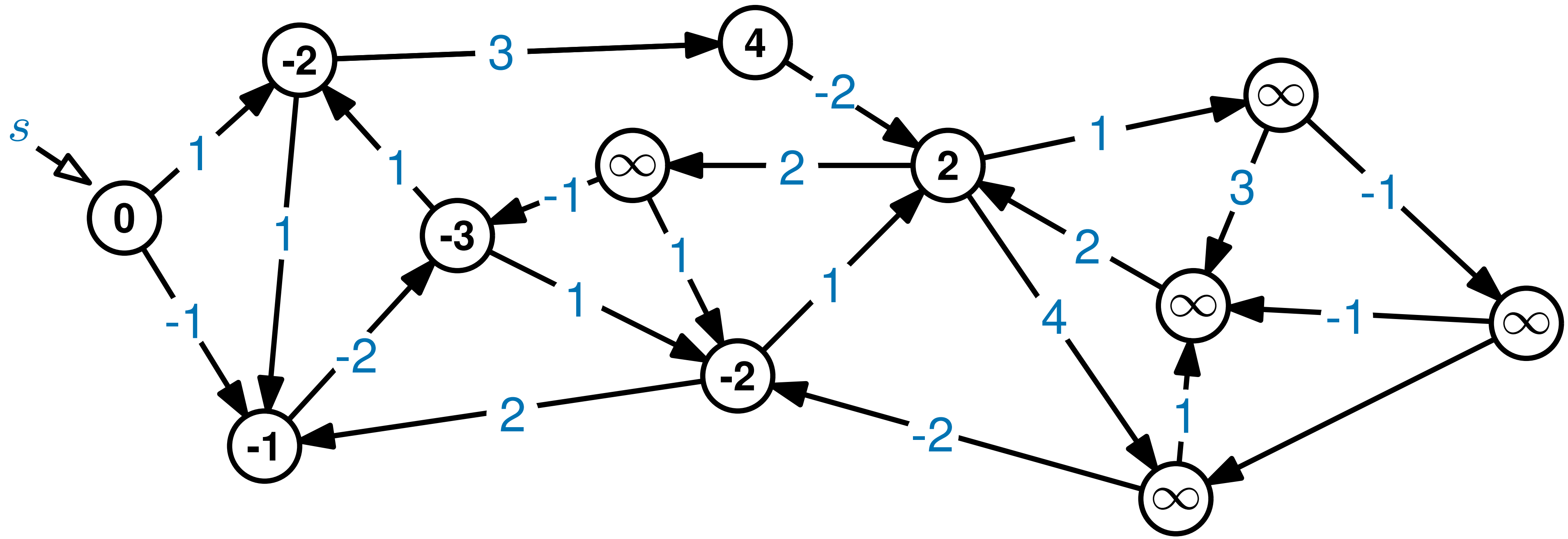
We're going to simulate

MostOfBellman-Ford(s)

How are things looking?

we aren't done



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Relax(u, v)

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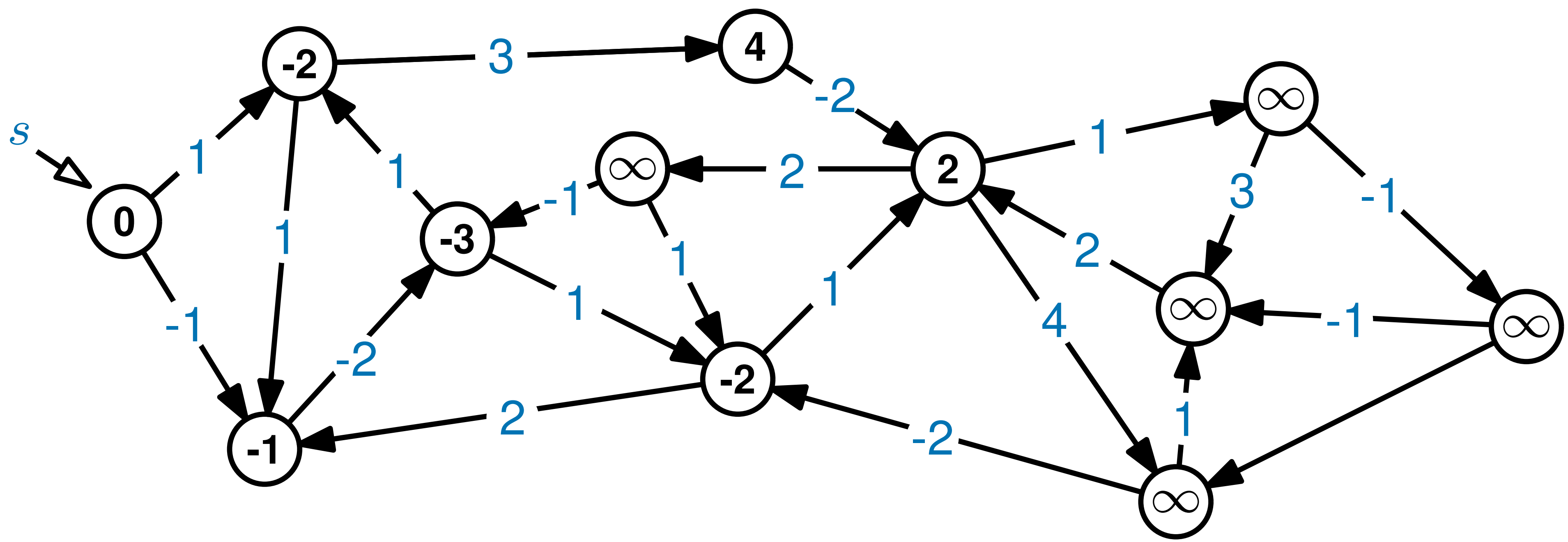
- the length of the best path between s and v, found so far

We're going to simulate

MostOfBellman-Ford(s)

How are things looking?

we aren't done *but it seems like we made progress*



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vertex v

$\texttt{dist}(v)$ *- the length of the best path between s and v, found so far*

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MOSTOFBELLMAN-FORD(s)

How are things looking?

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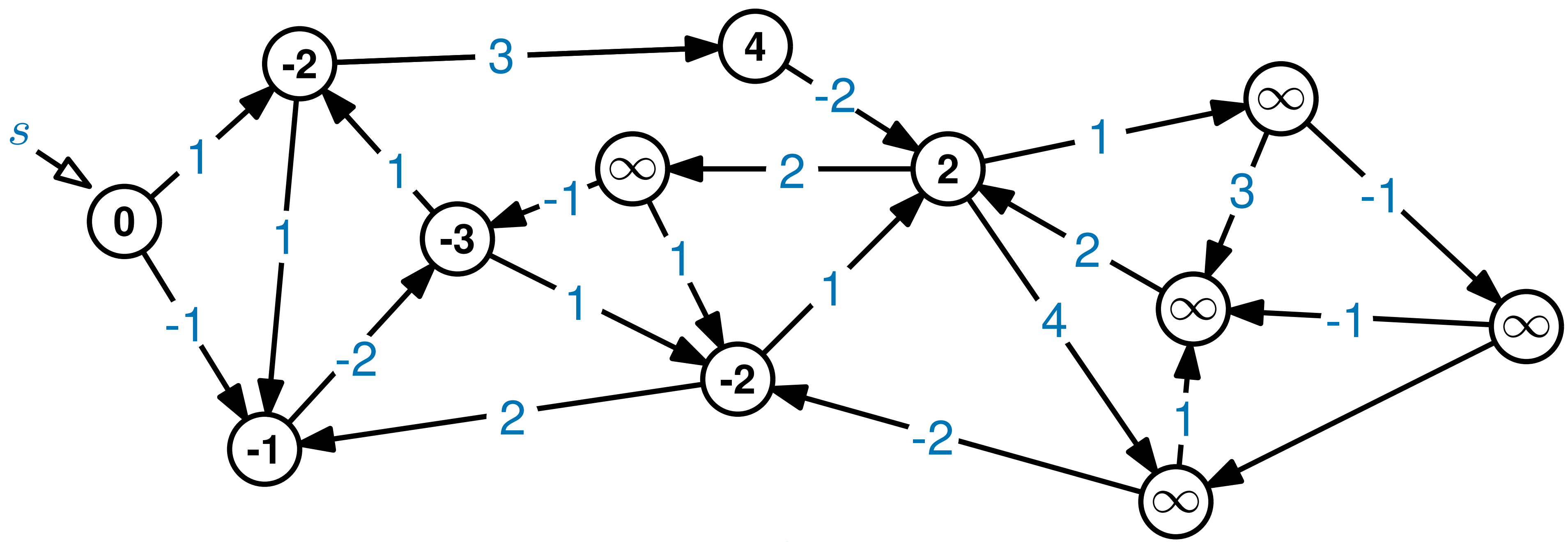

does every iteration make progress?

MOSTOFBELLMAN-FORD runs $|V|$ iterations,

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We're going to simulate MOSTOFBELLMAN-FORD(s)

How are things looking?

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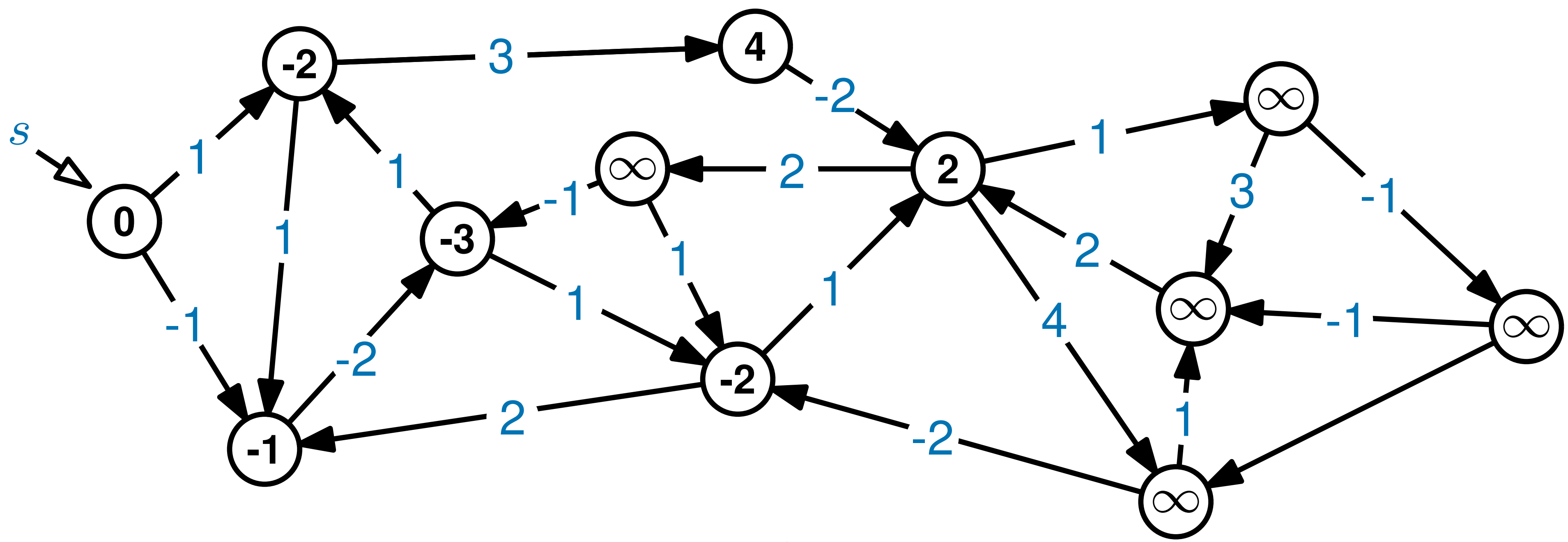

does every iteration make progress?

are $|V|$ iterations enough?

MOSTOFBELLMAN-FORD runs $|V|$ iterations,

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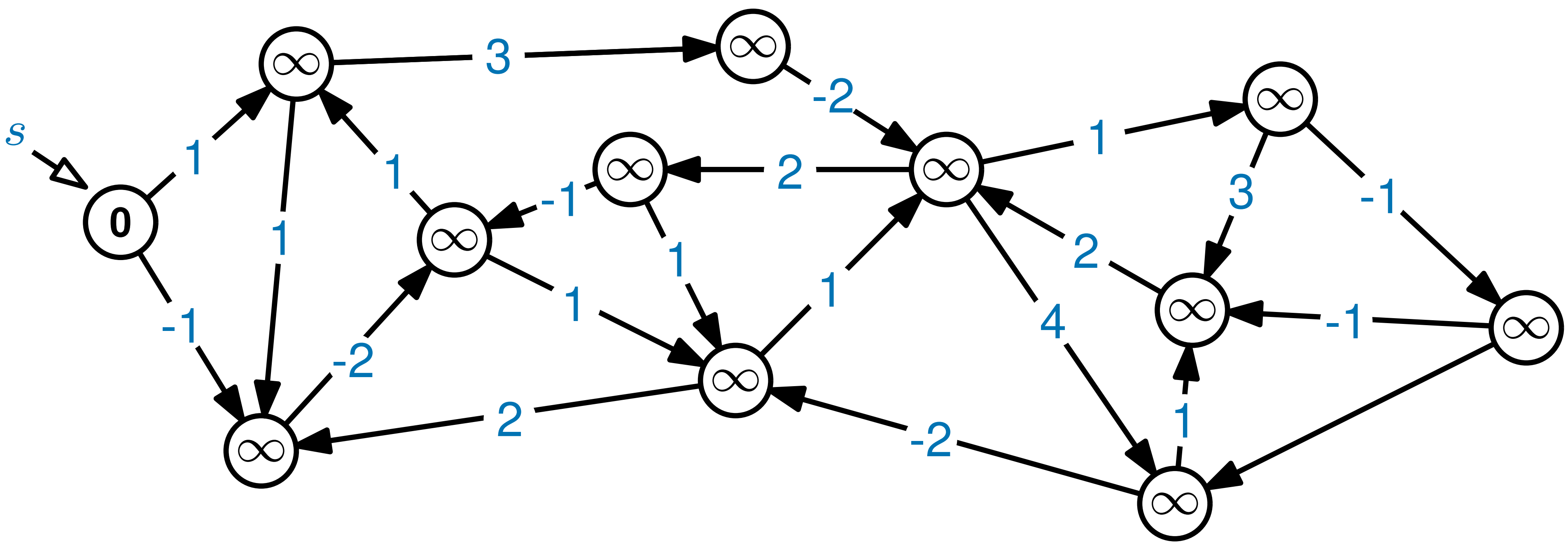
The proof idea

Imagine a different algorithm where in each iteration...

you only relax one edge *(rather than all edges)*



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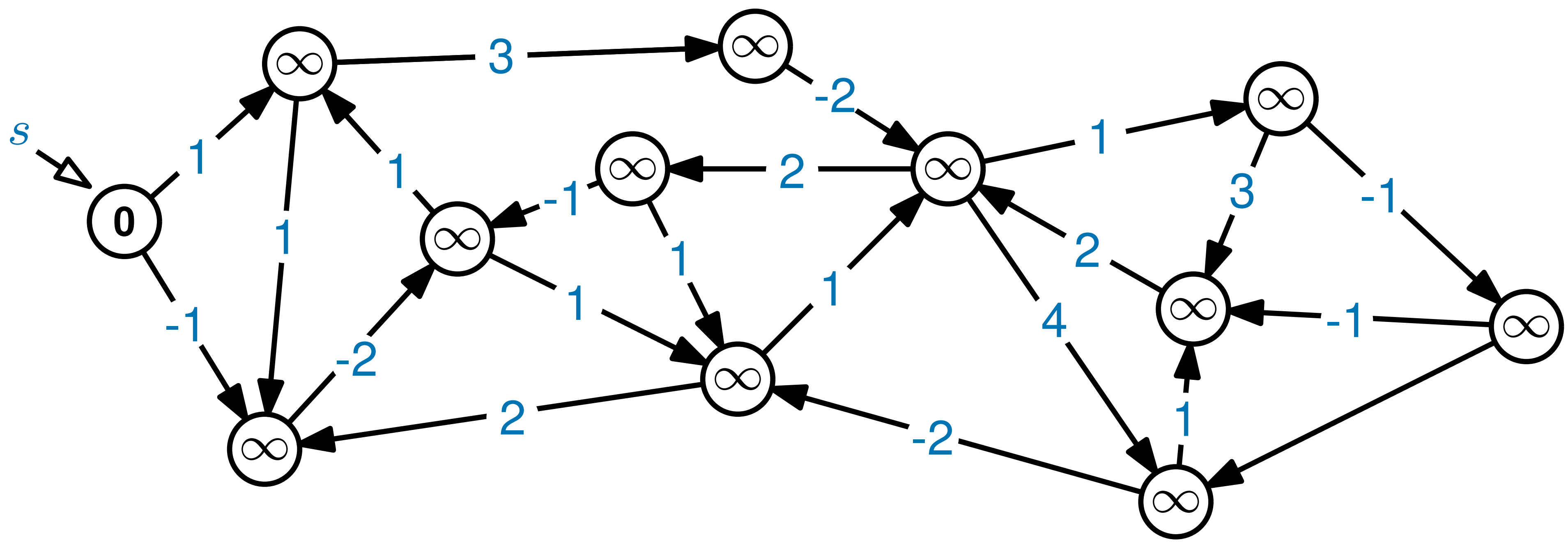

Further, imagine that (magically or otherwise), the edge that you relax in iteration i

is the i-th edge in a shortest path from s to some vertex t

The proof idea

Imagine a different algorithm where in each iteration. . .

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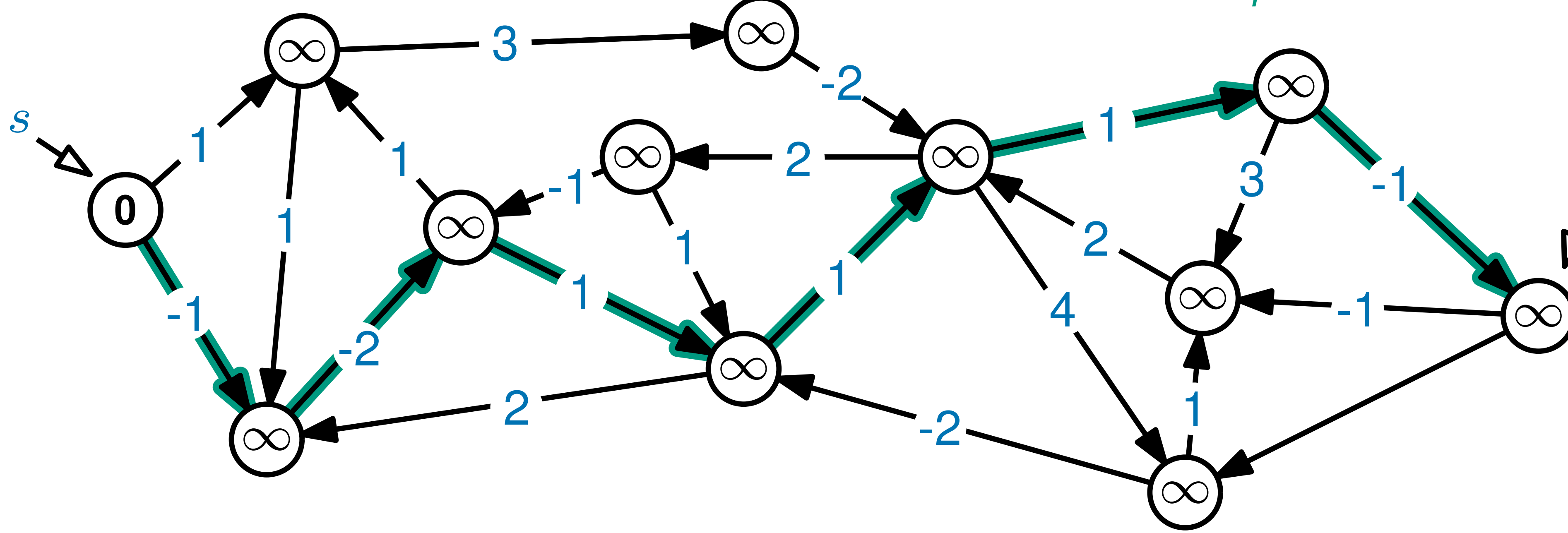
This is a shortest path from s to t



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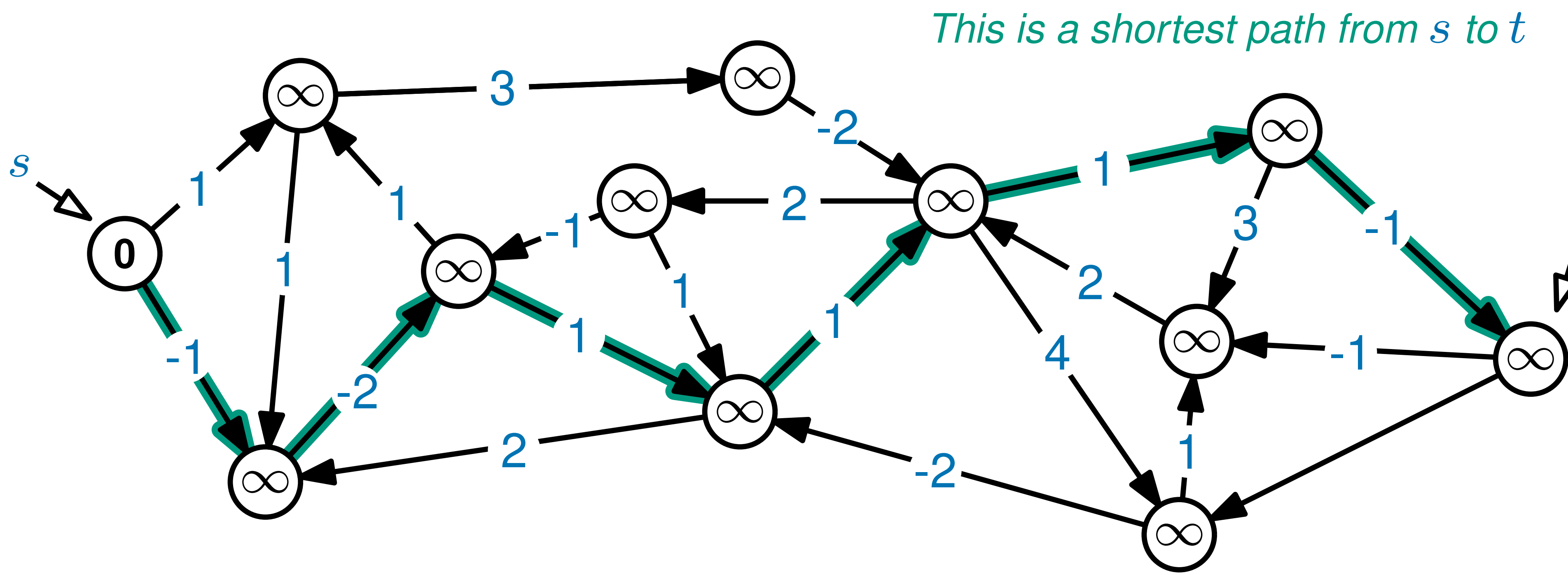

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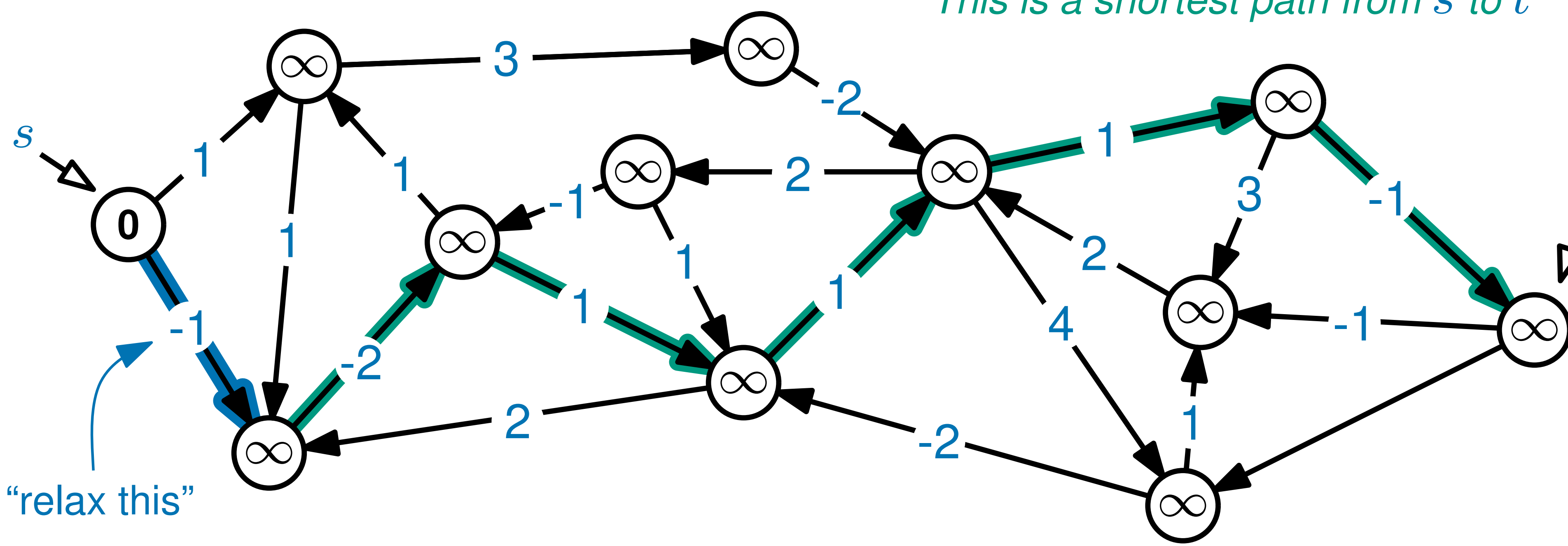

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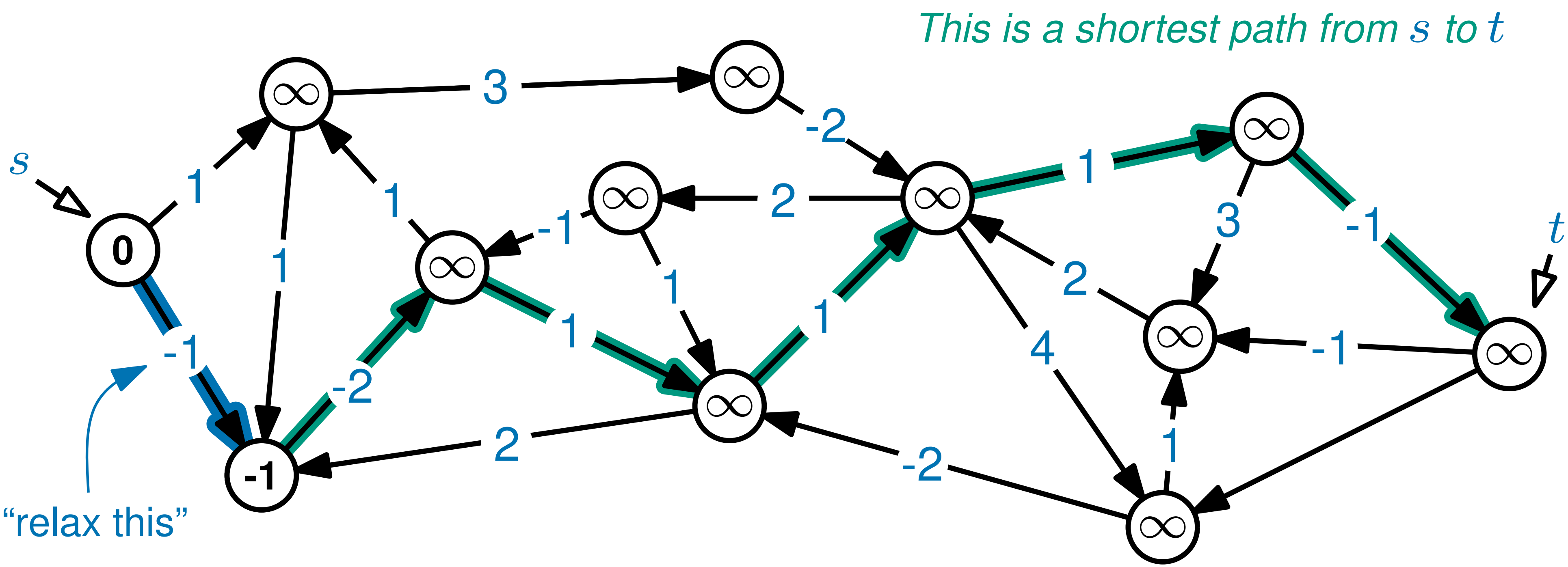

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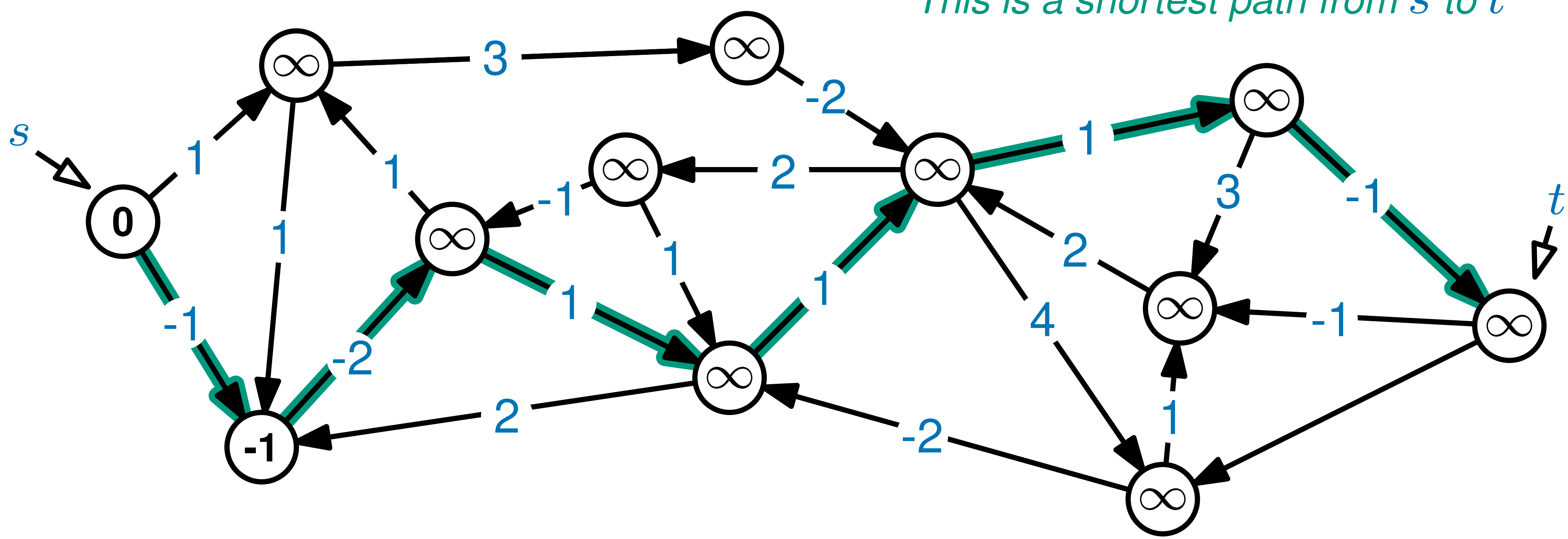

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Iteration 2:

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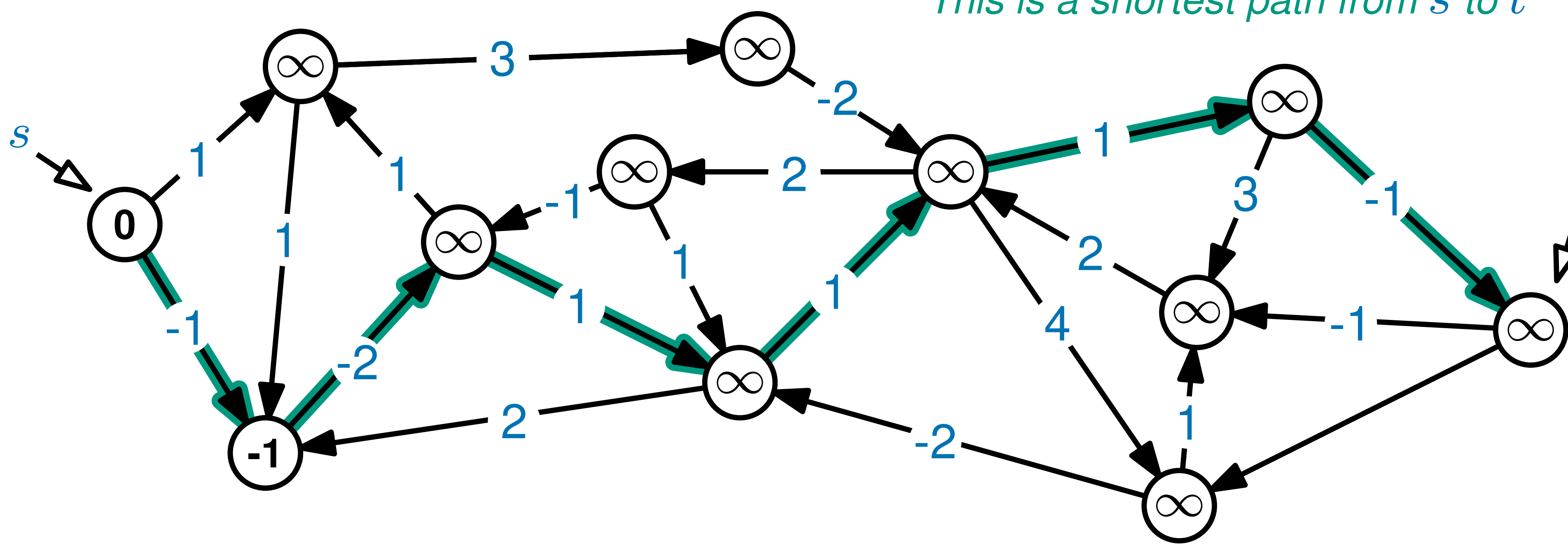

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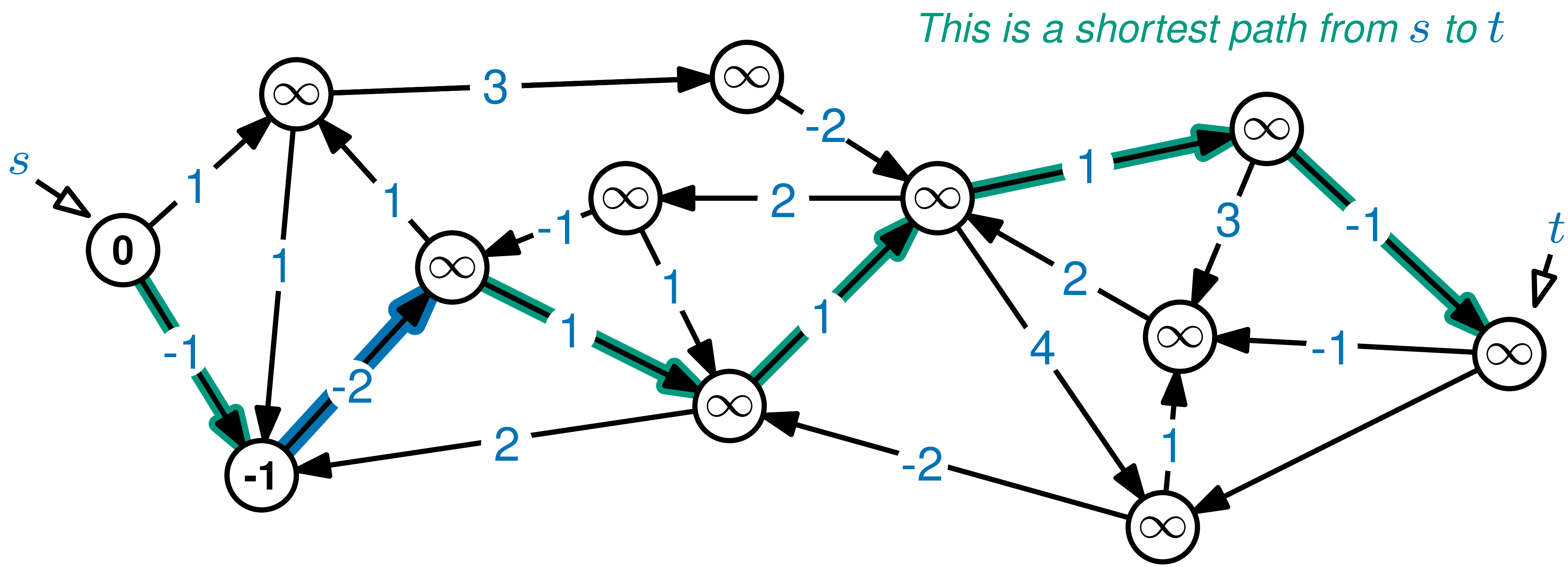

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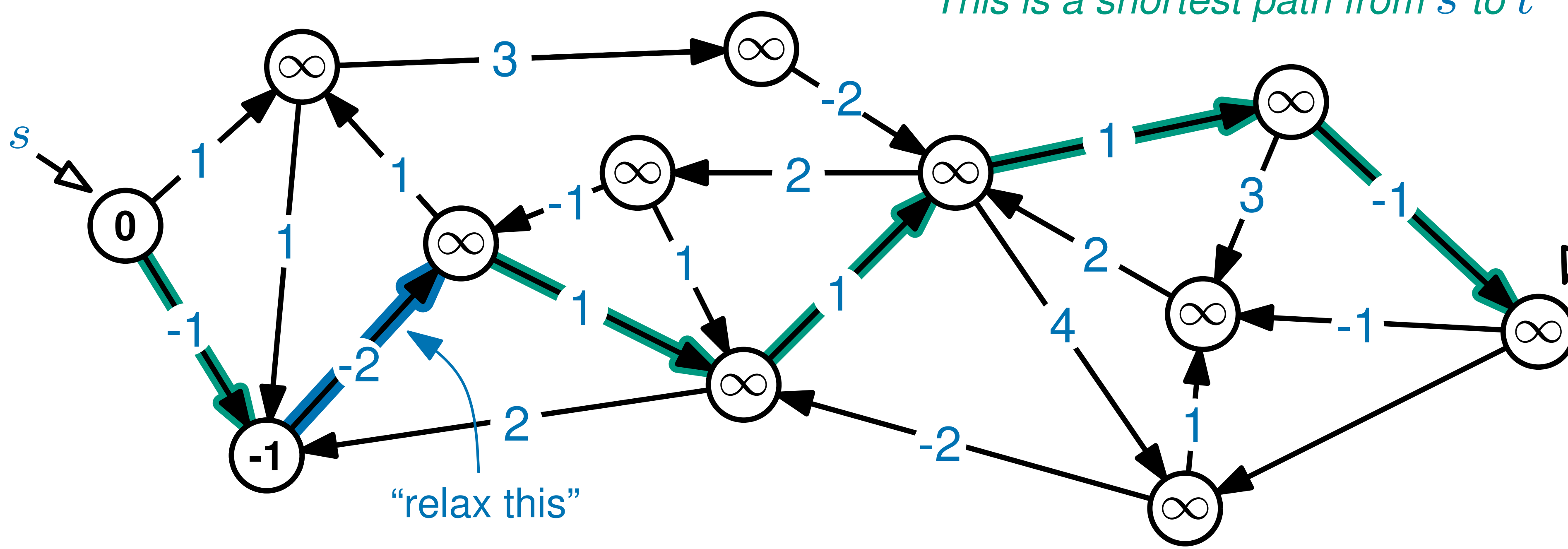

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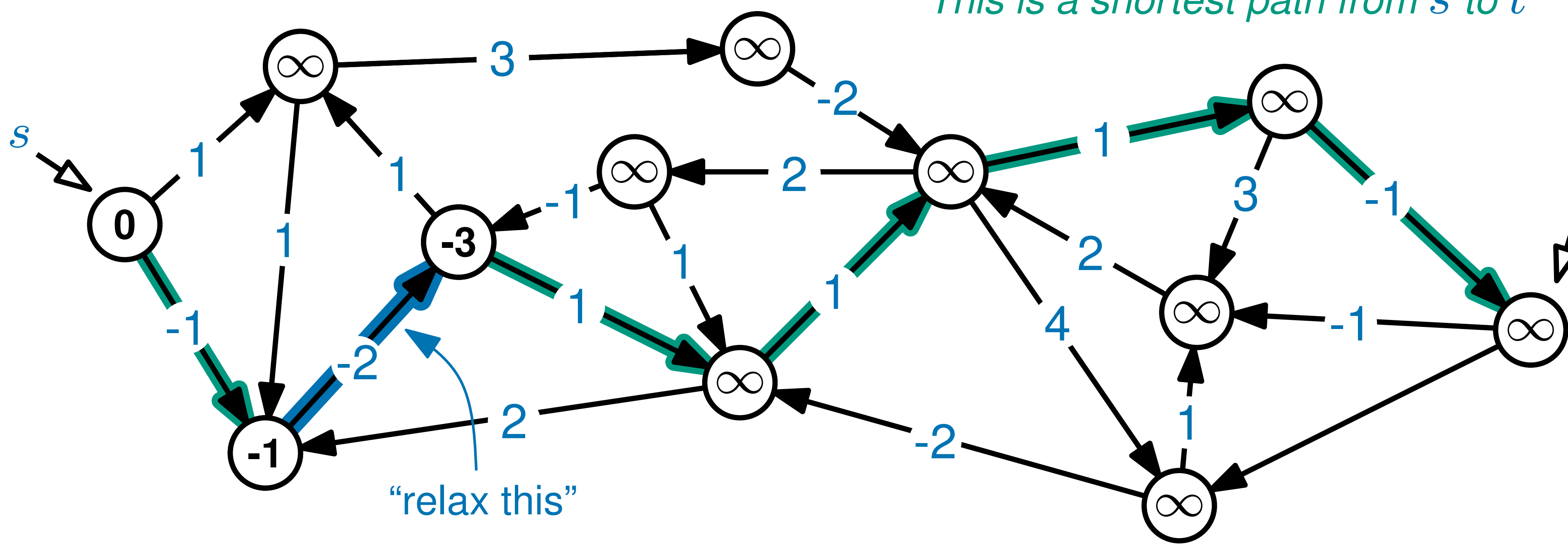

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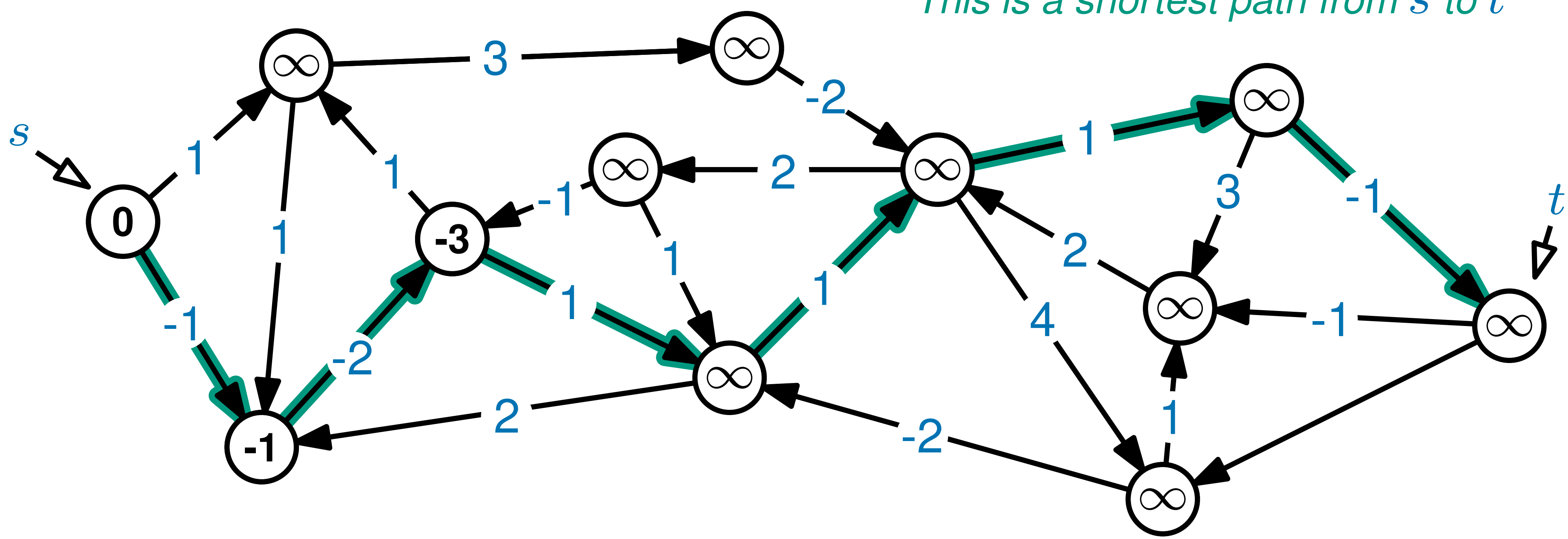

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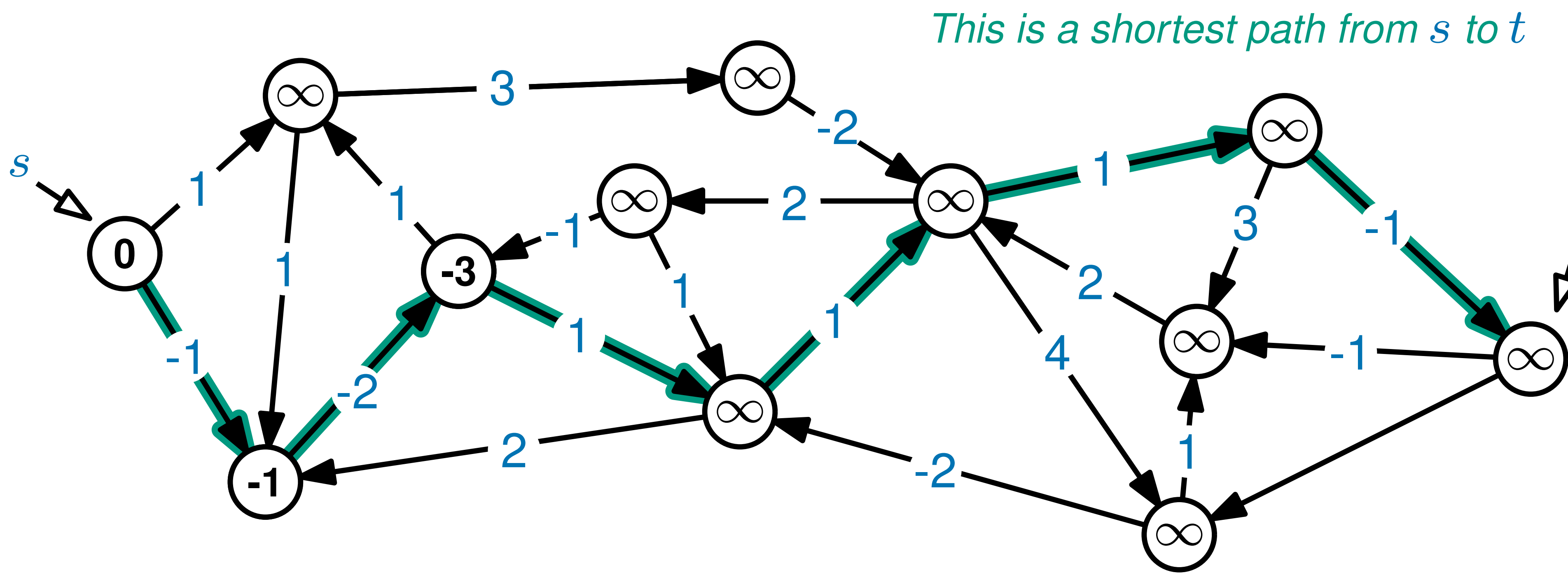
Iteration 3:



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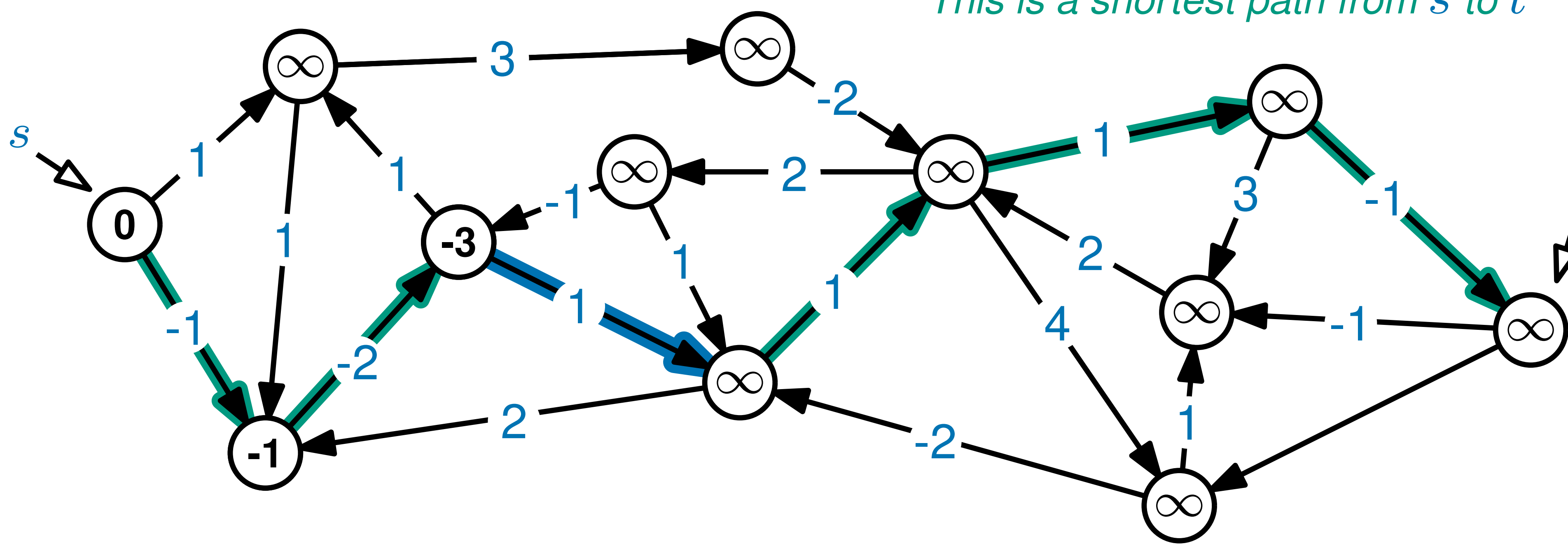

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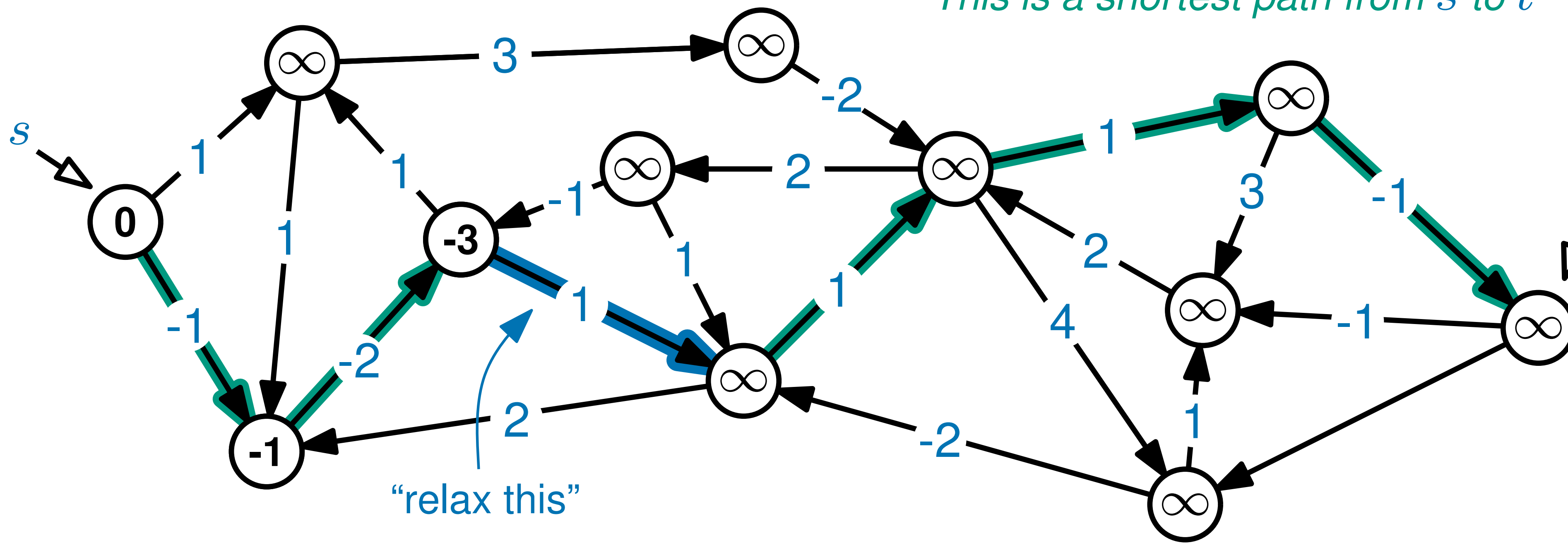

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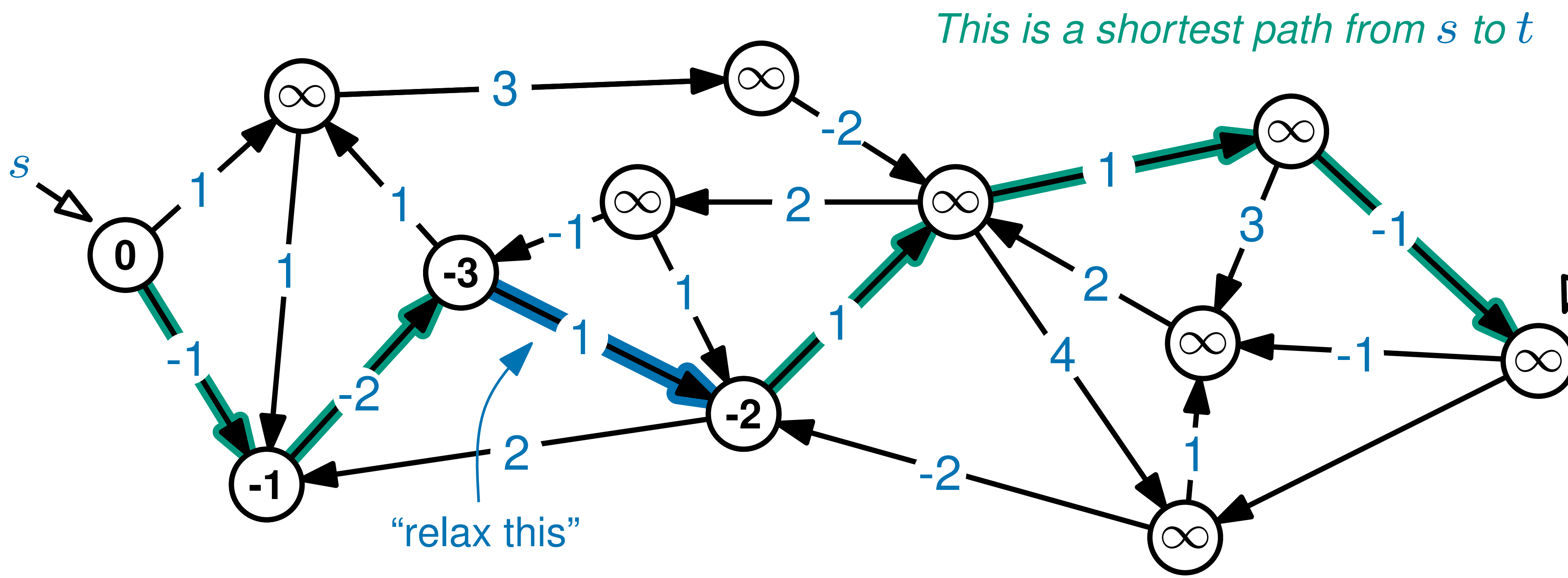

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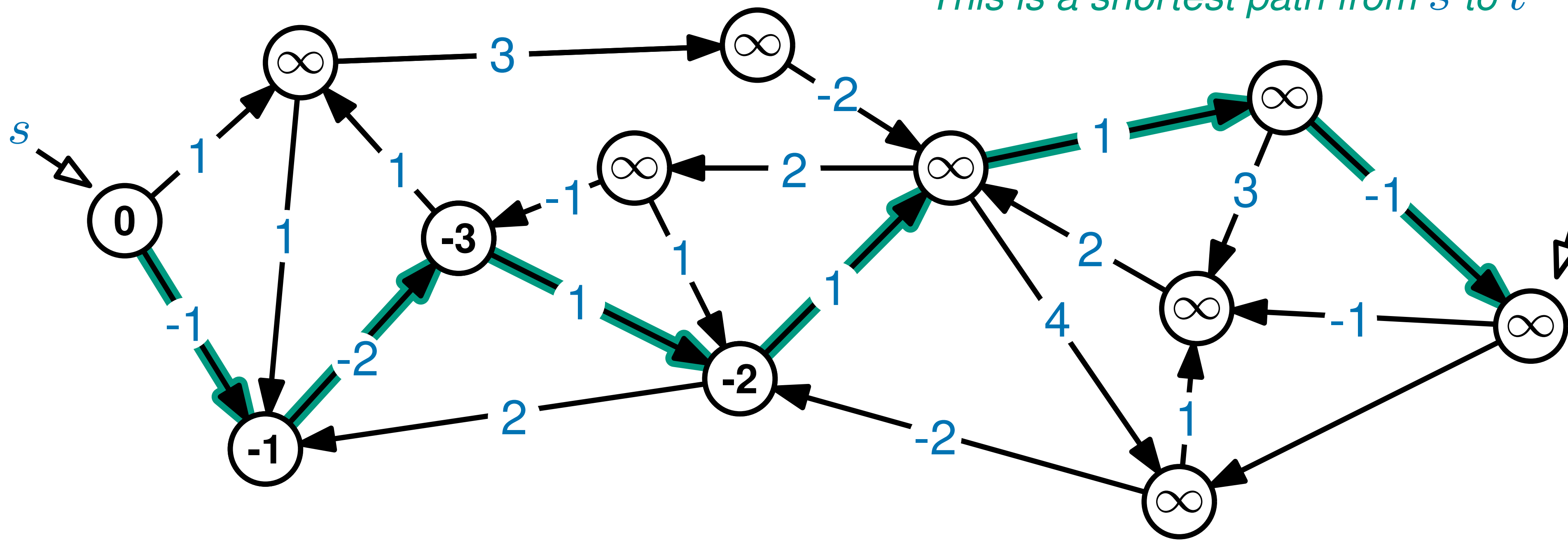

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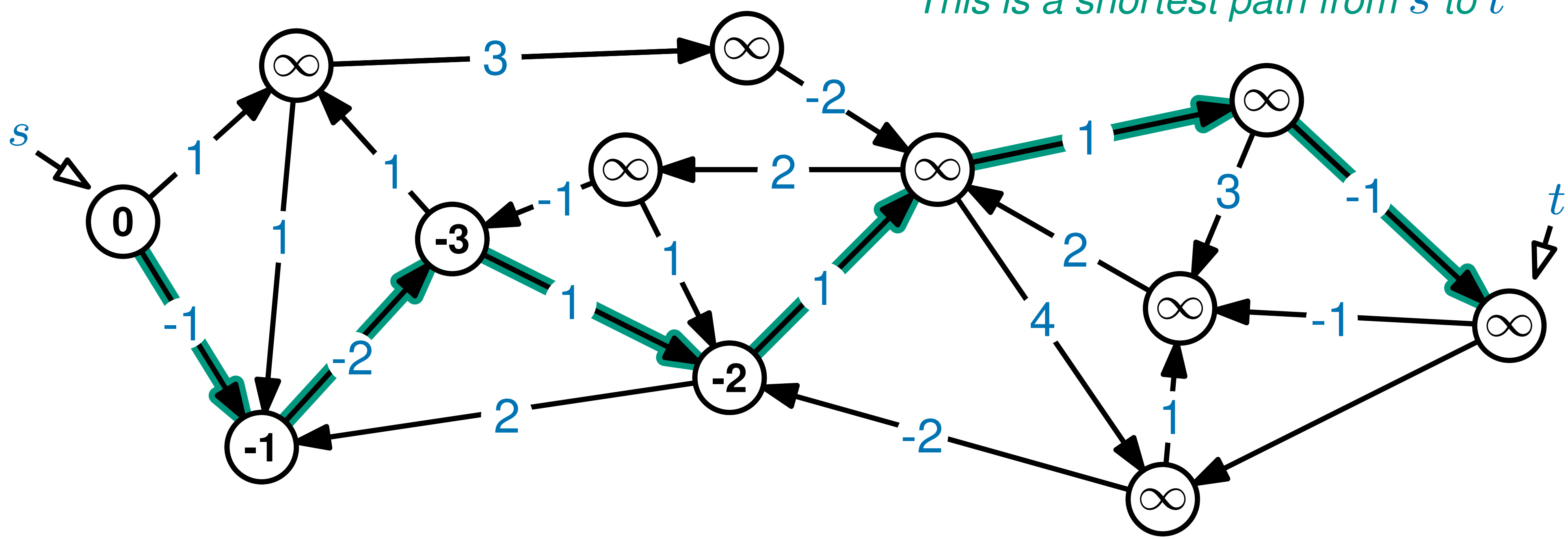
Iteration 4:

This is a shortest path from s to t



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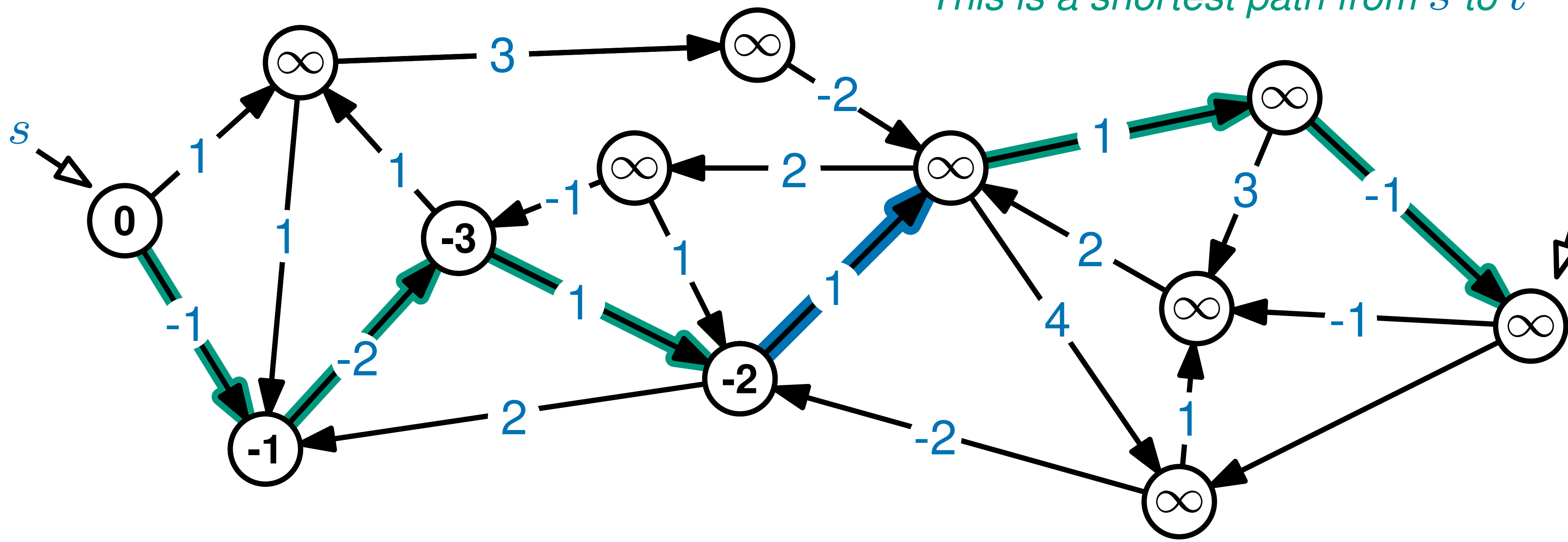

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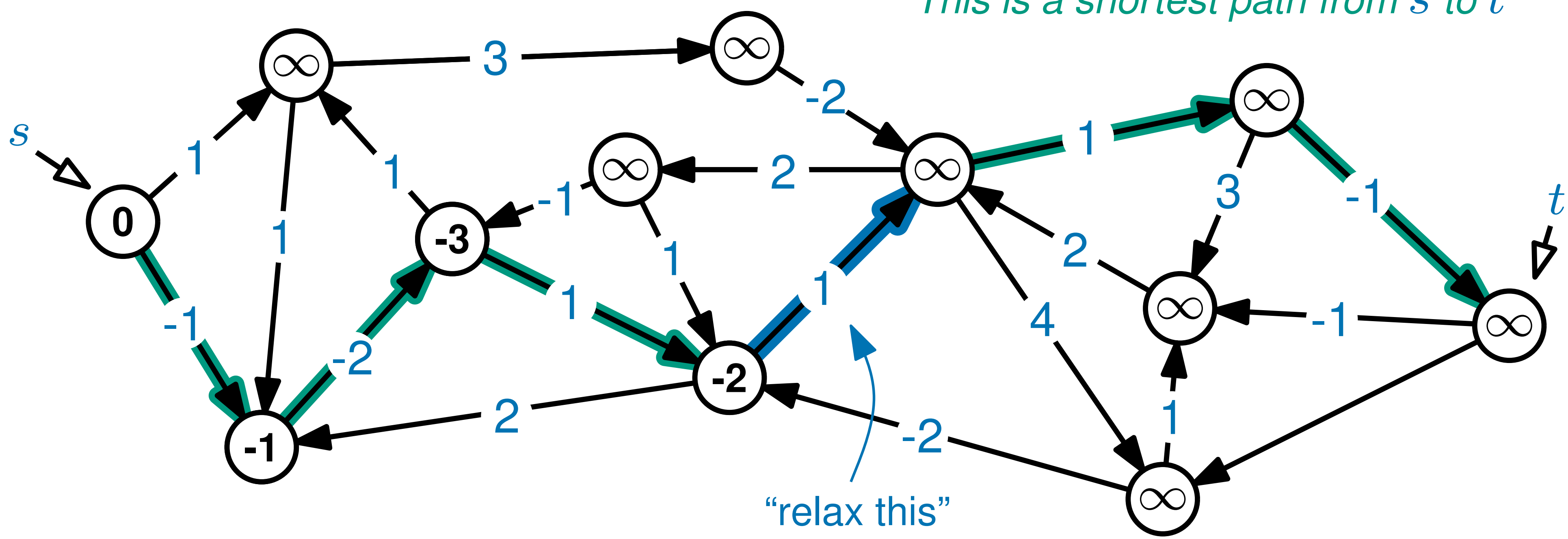

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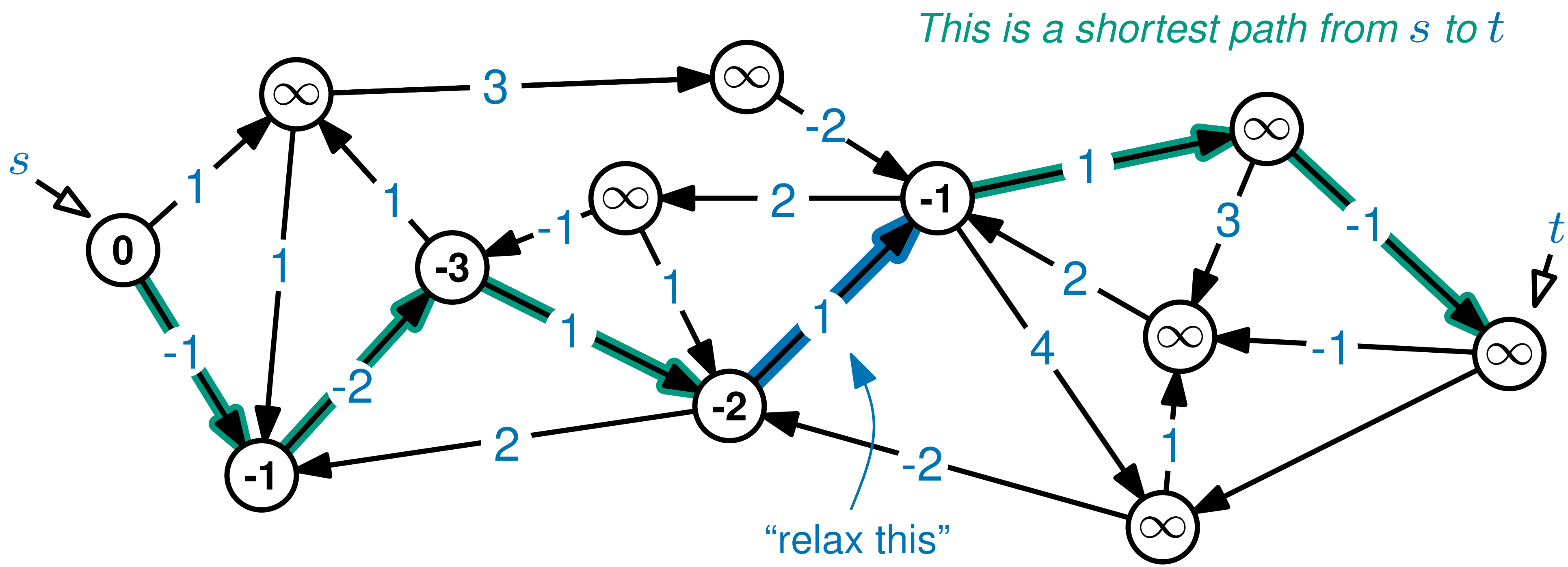

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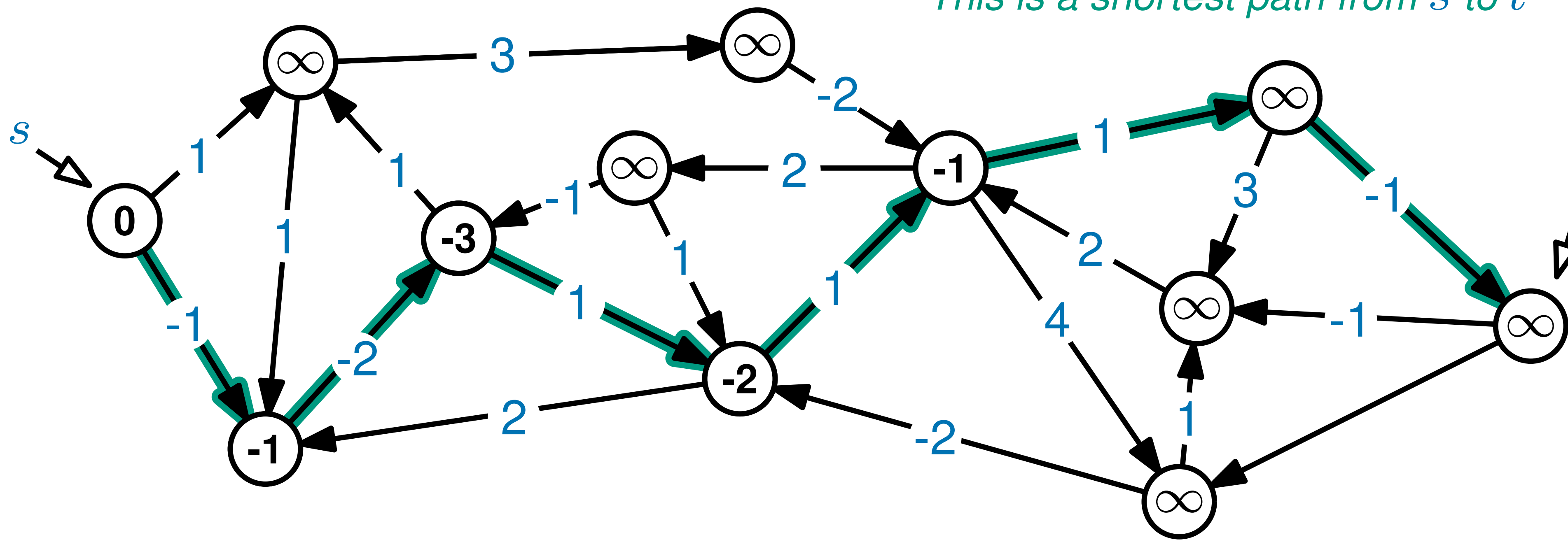

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The proof idea

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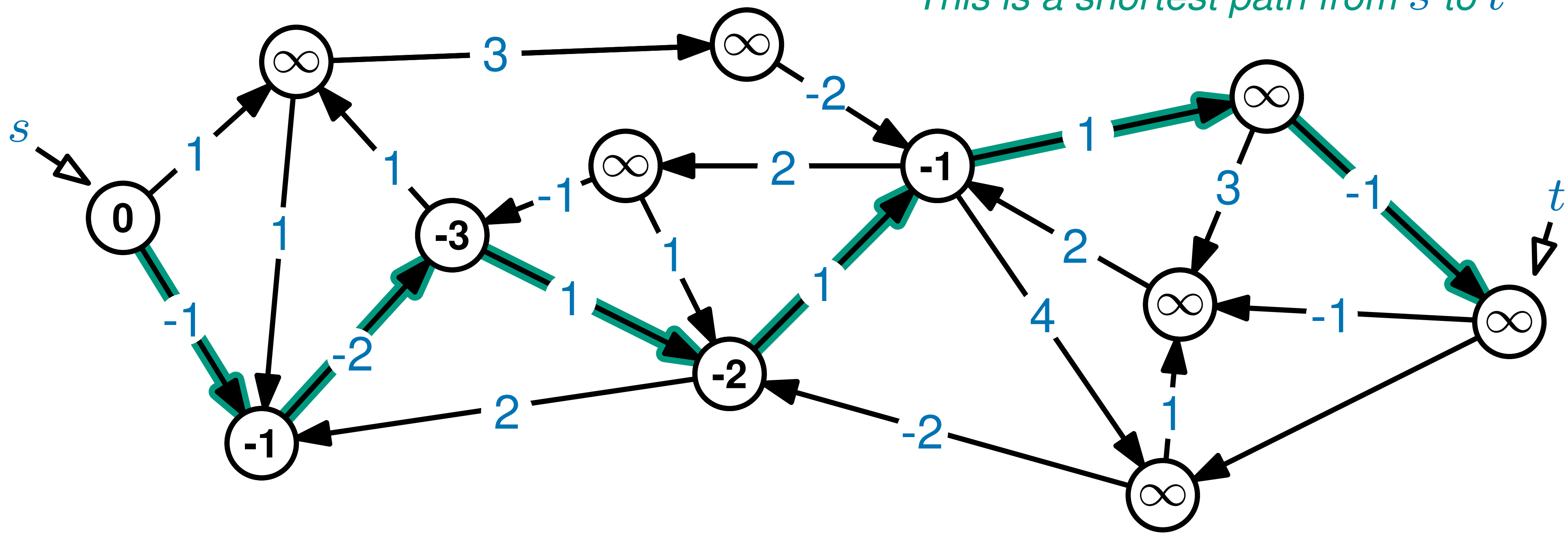
Iteration 5:

This is a shortest path from s to t



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The proof idea

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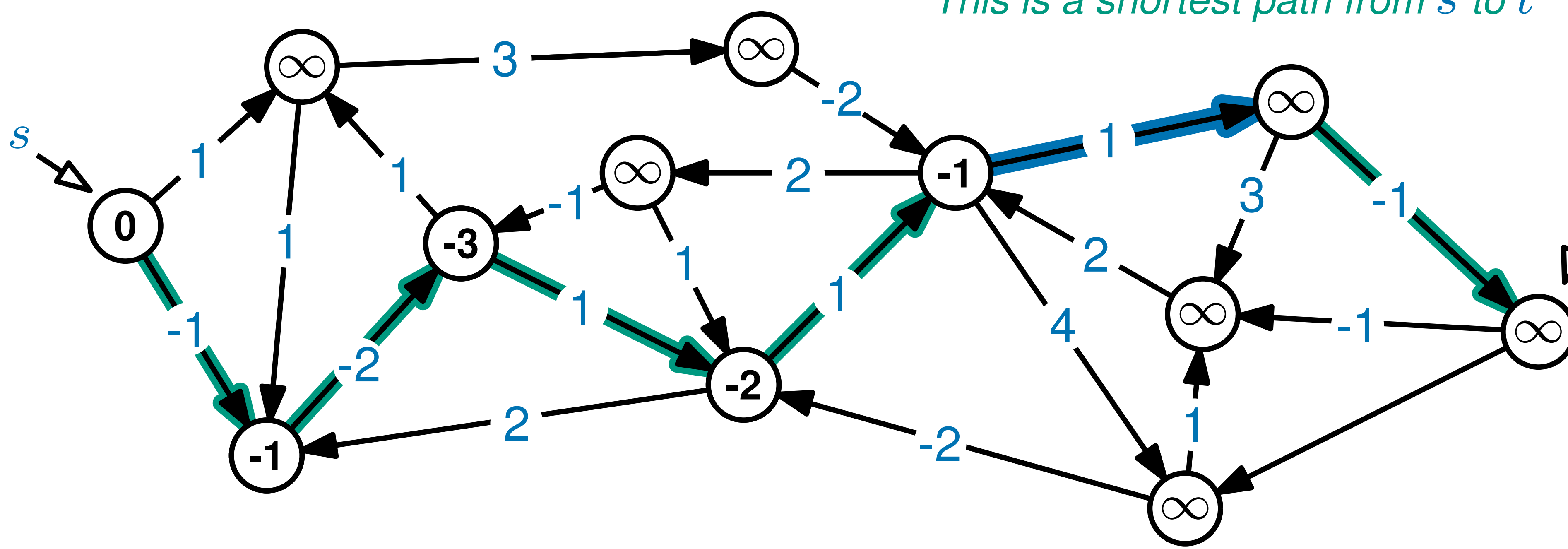
Iteration 5:

This is a shortest path from s to t



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Imagine a different algorithm where in each iteration. . .

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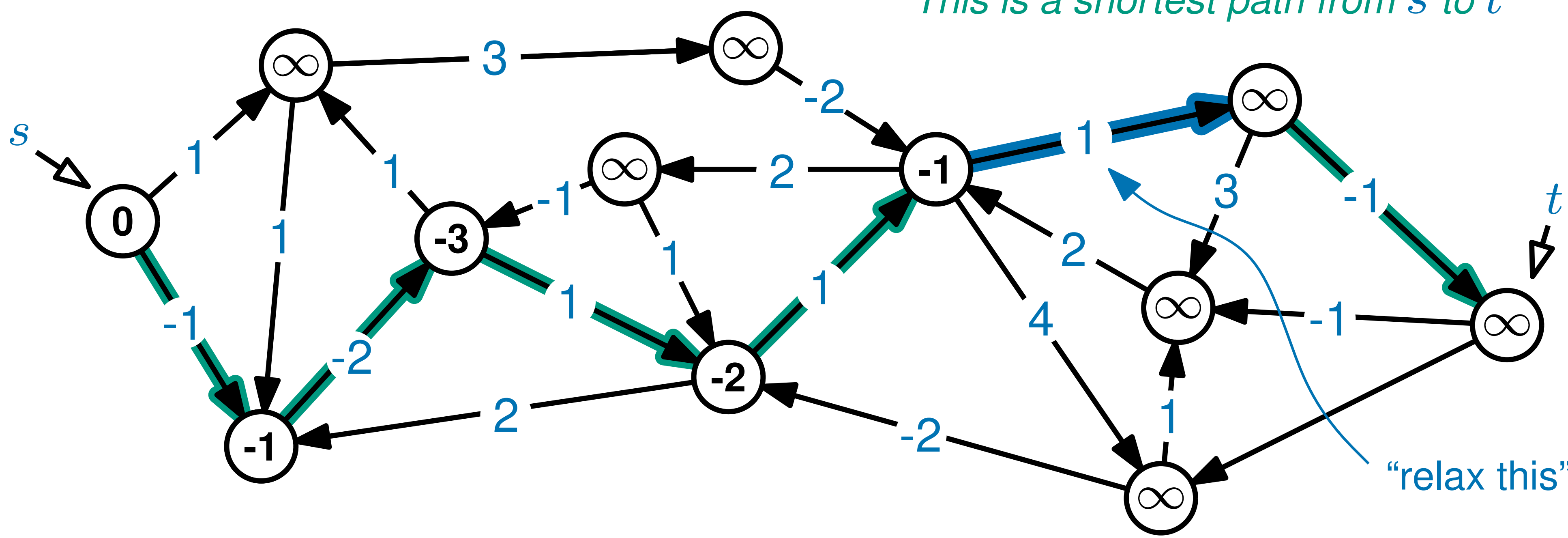
Iteration 5:

This is a shortest path from s to t



Further, imagine that (magically or otherwise), the edge that you relax in iteration i

is the i-th edge in a shortest path from s to some vertex t

The proof idea

Imagine a different algorithm where in each iteration. . .

you only relax one edge *(rather than all edges)*

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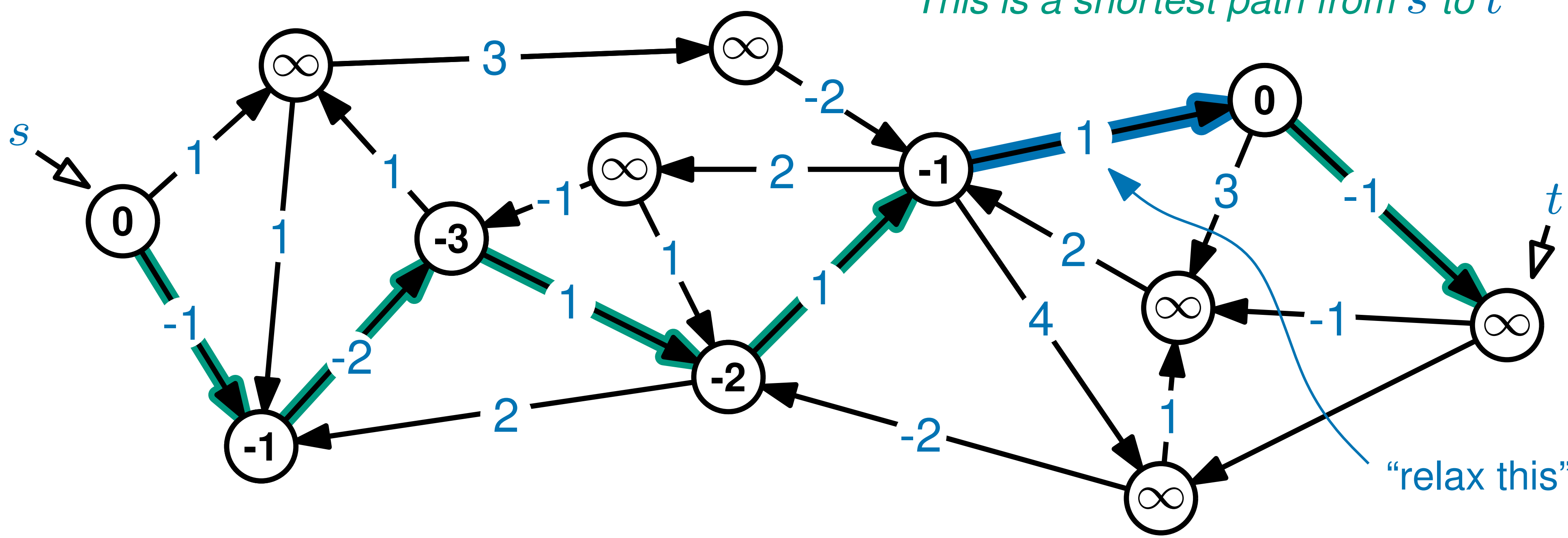

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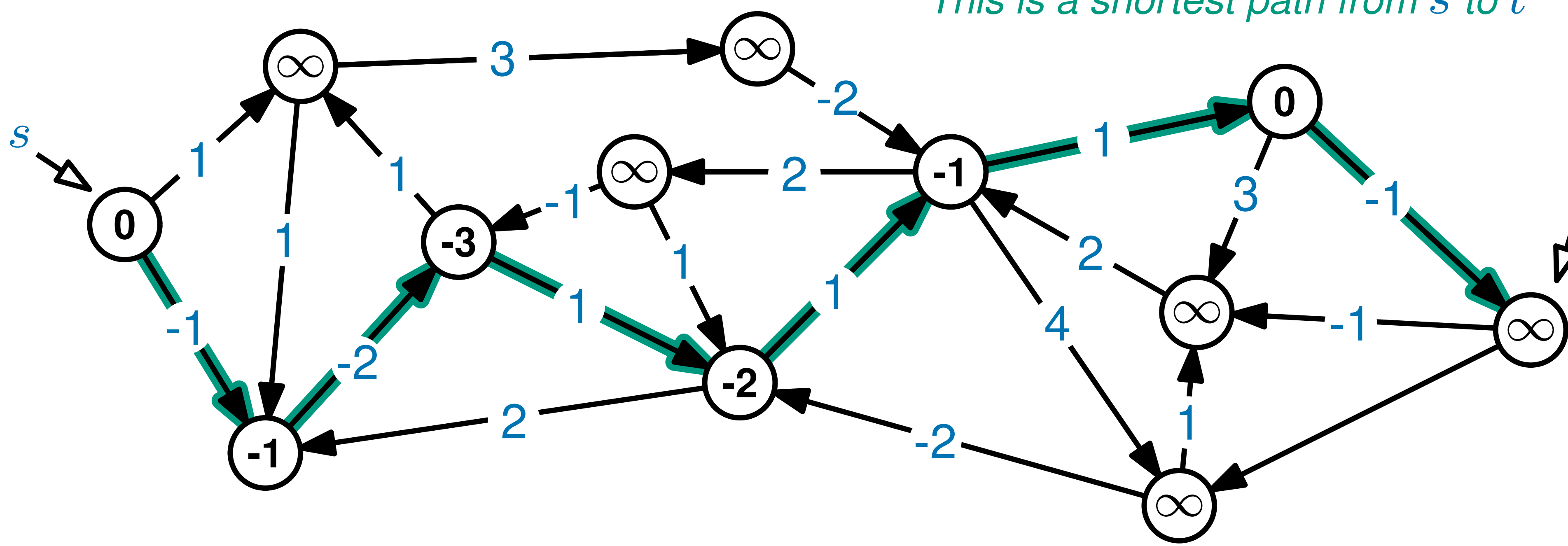

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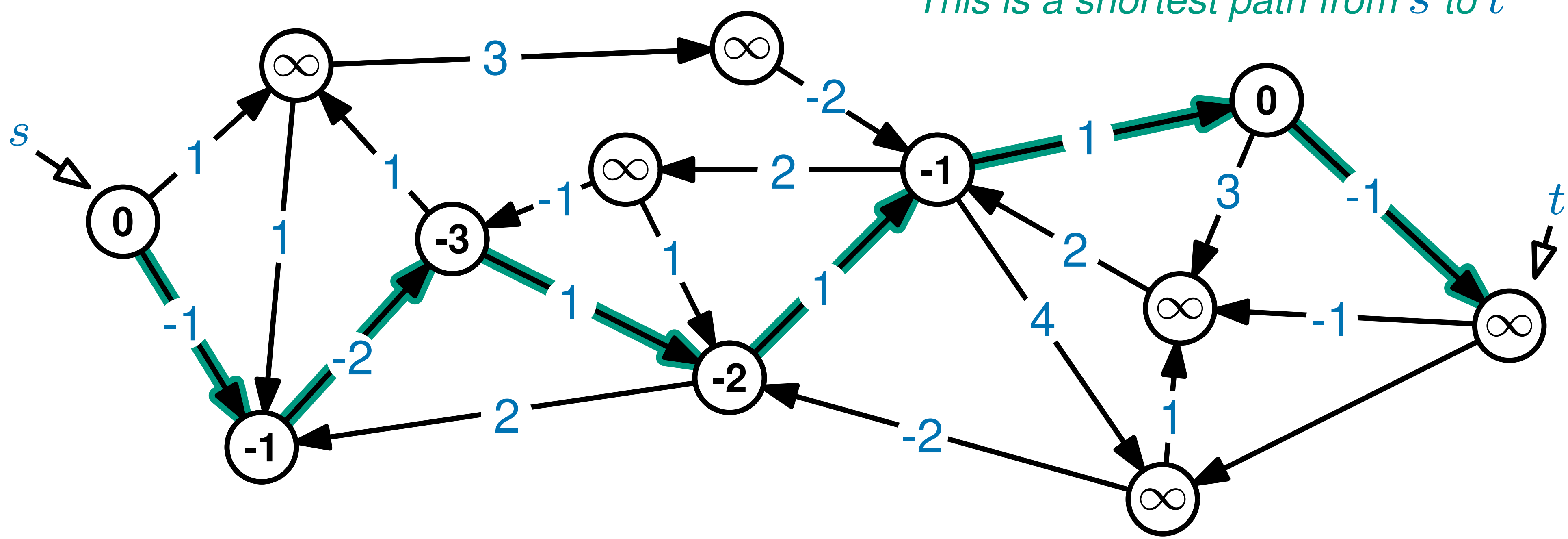

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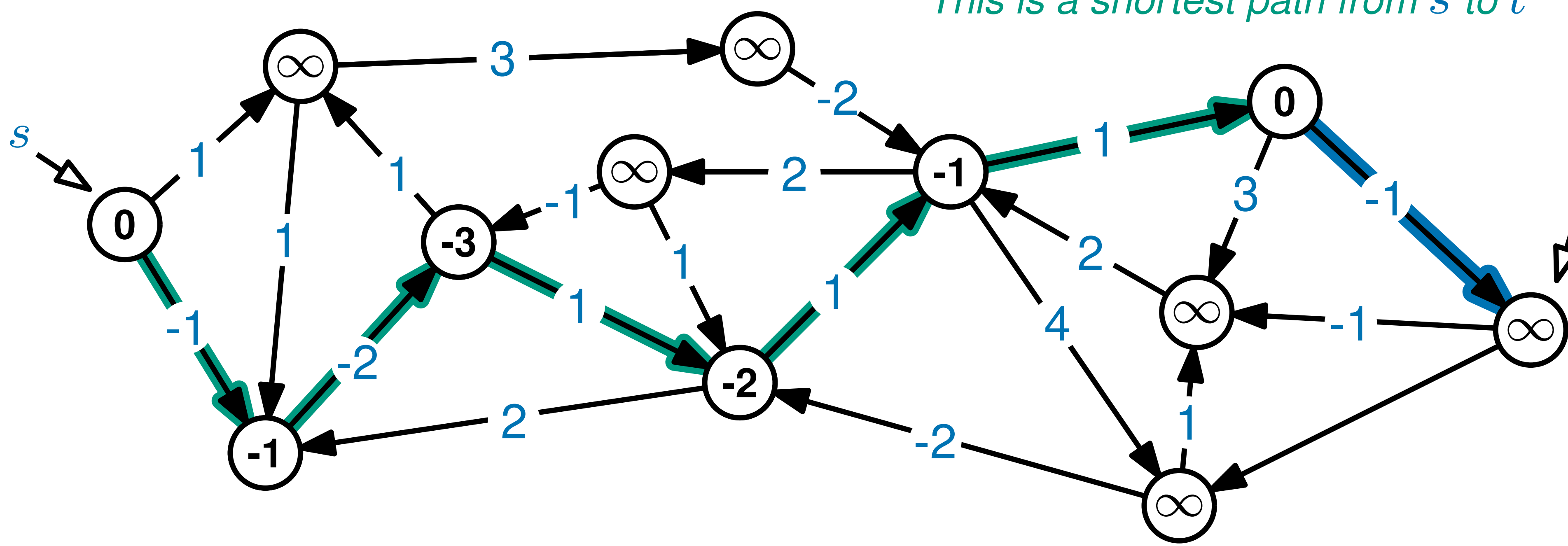

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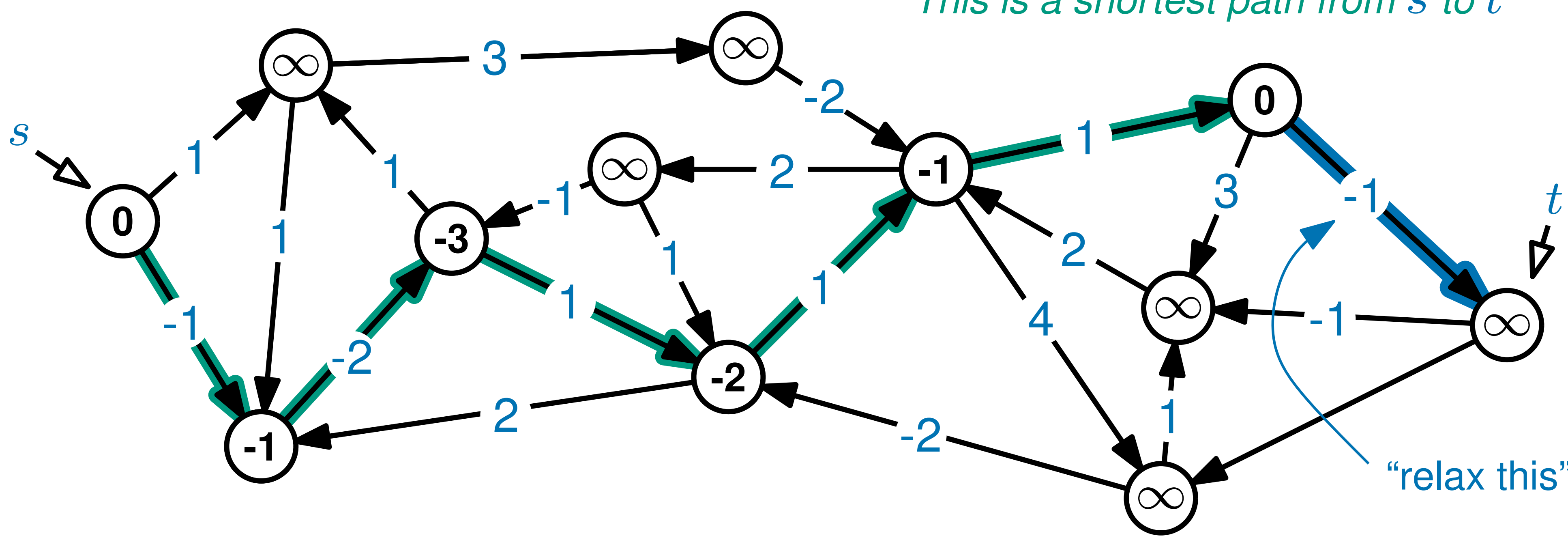

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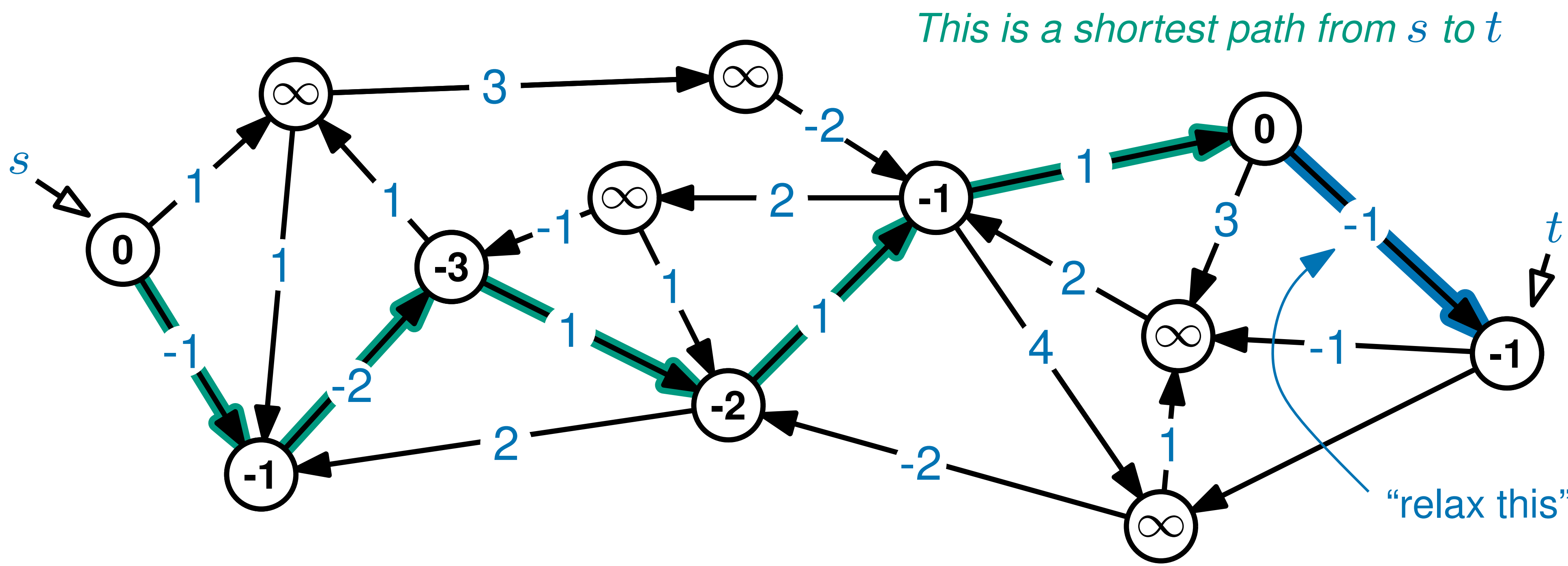

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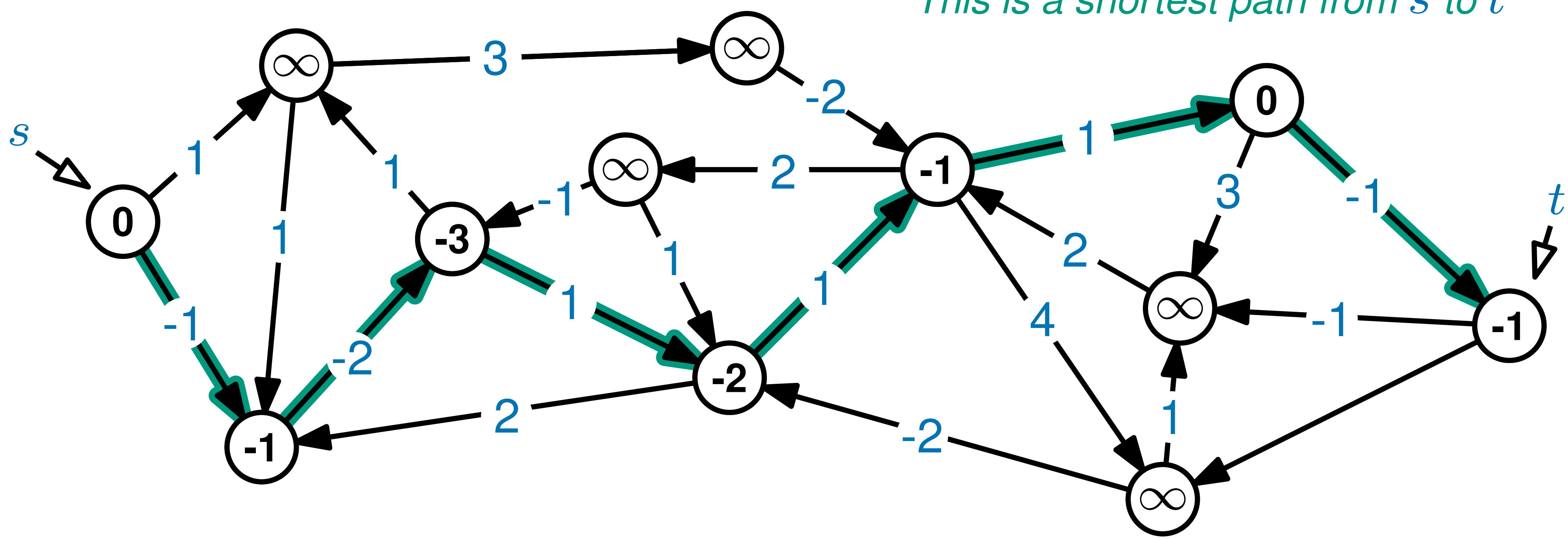

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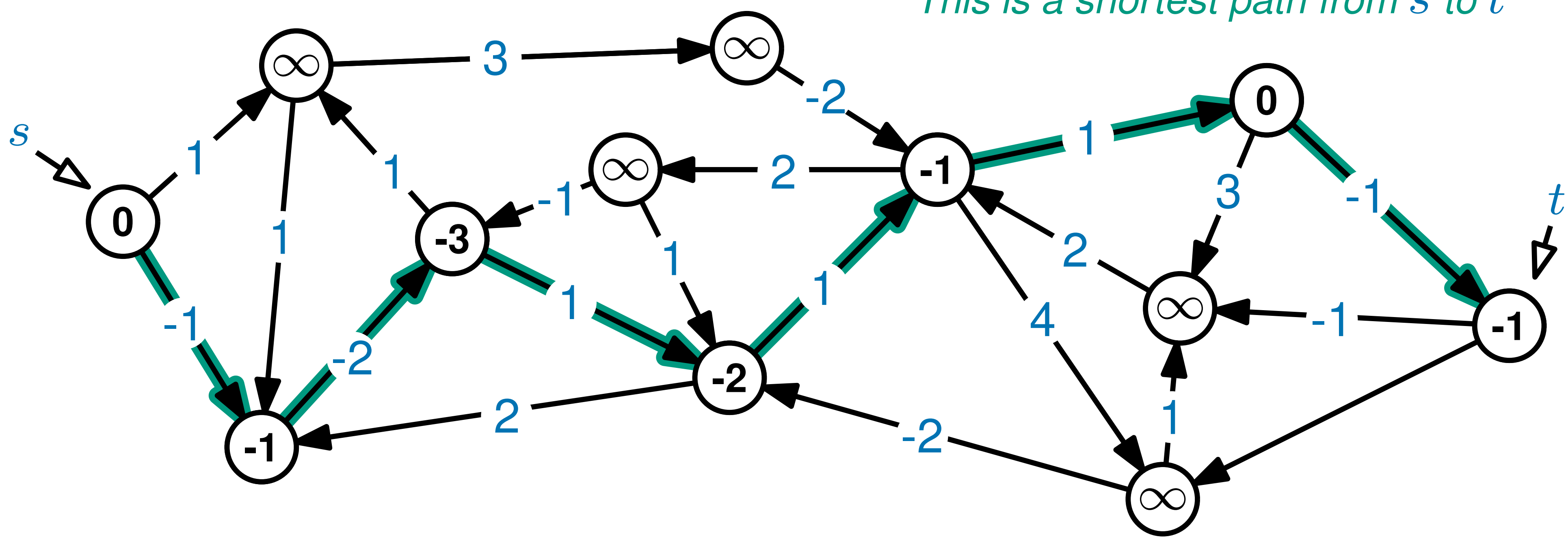

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When this algorithm terminates...

$\texttt{dist}(t)$ is the length of the shortest path from s to t

The proof idea

Now consider the MOSTOFBELLMAN-FORD algorithm where in each iteration...

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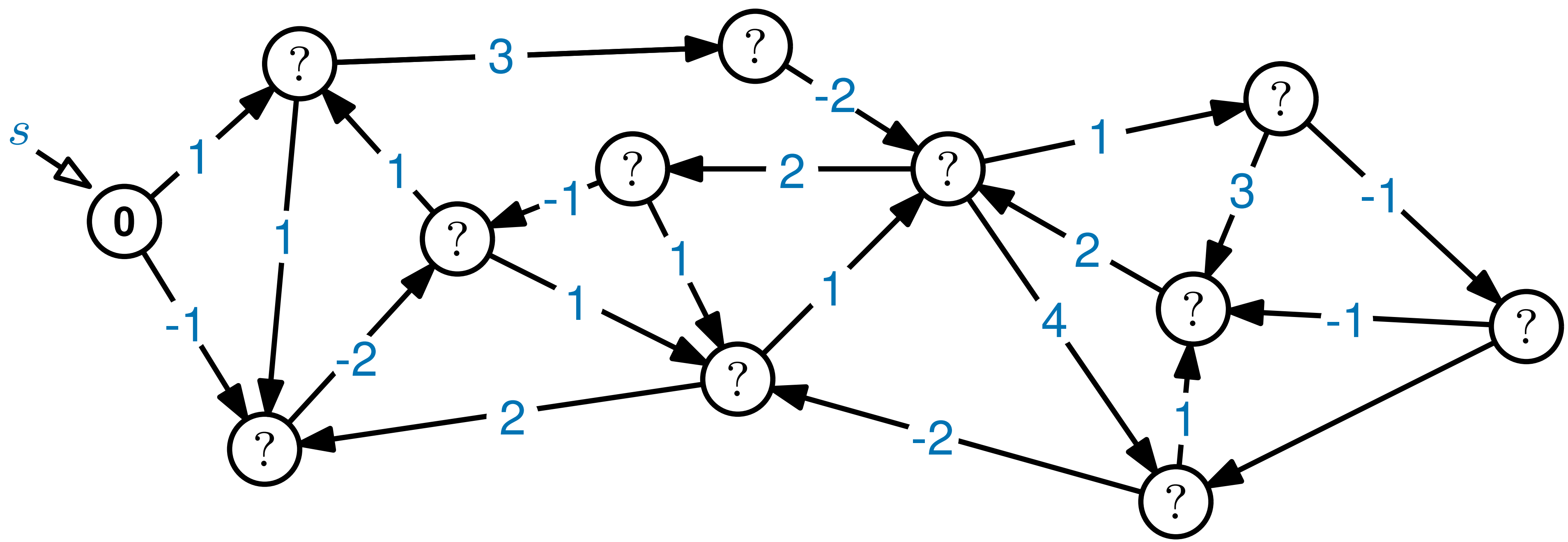

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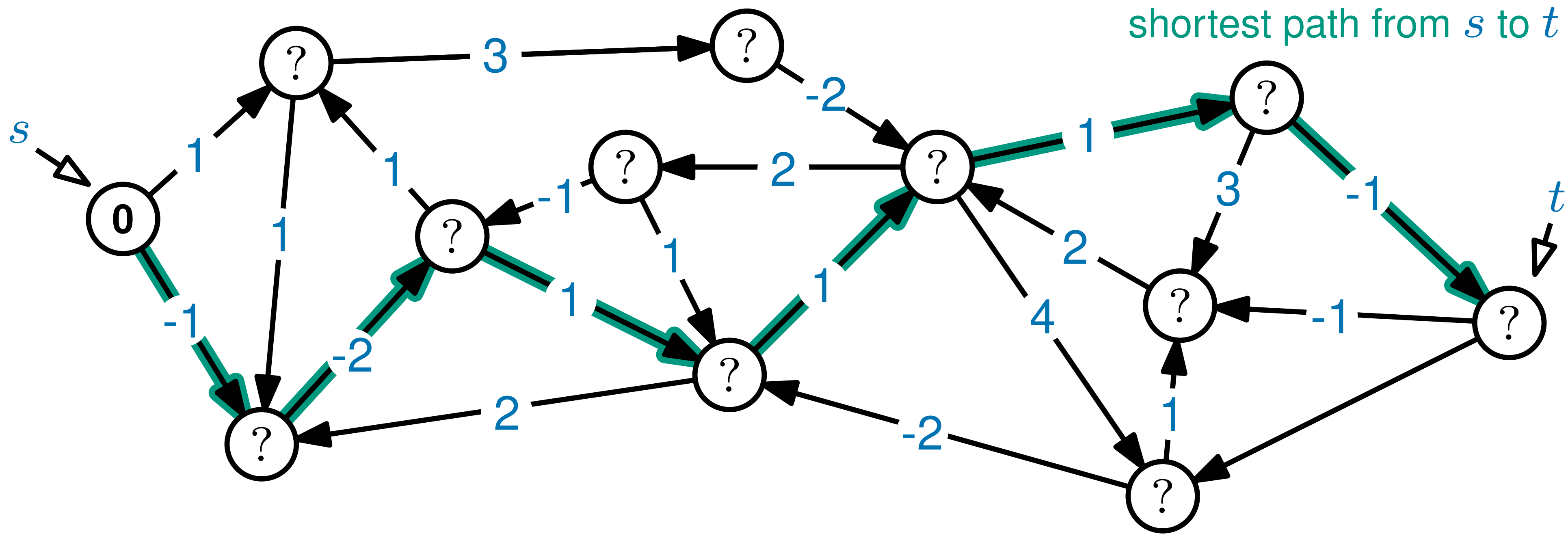

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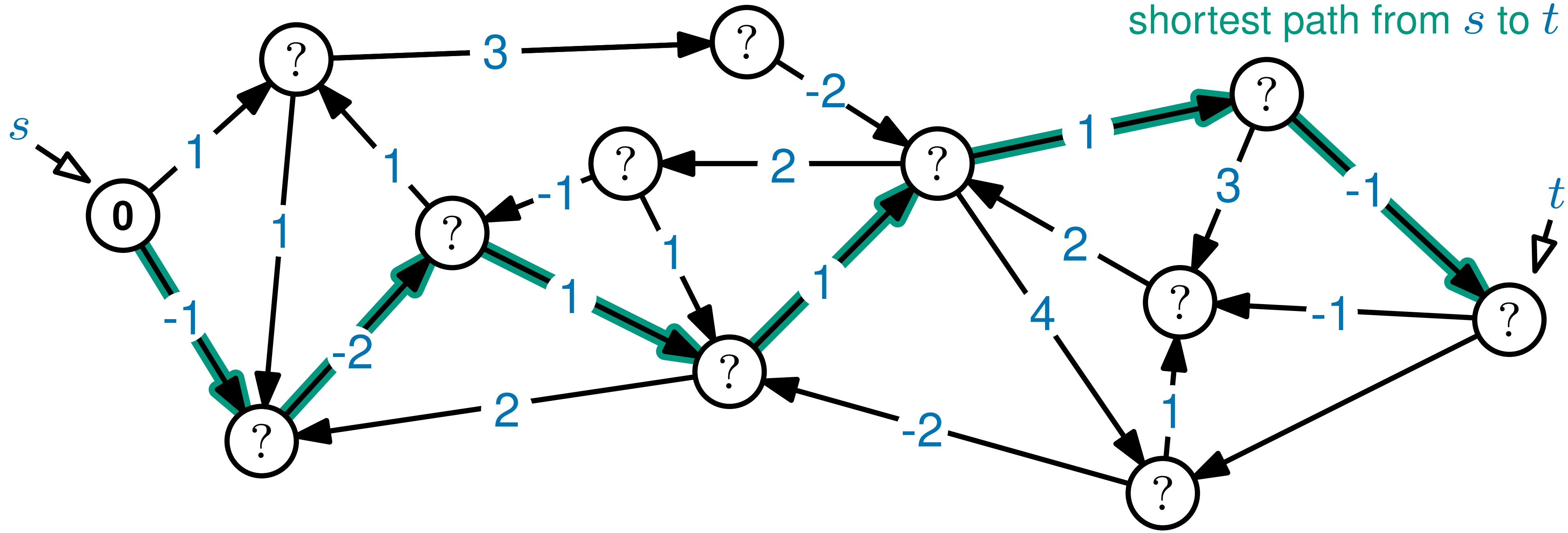

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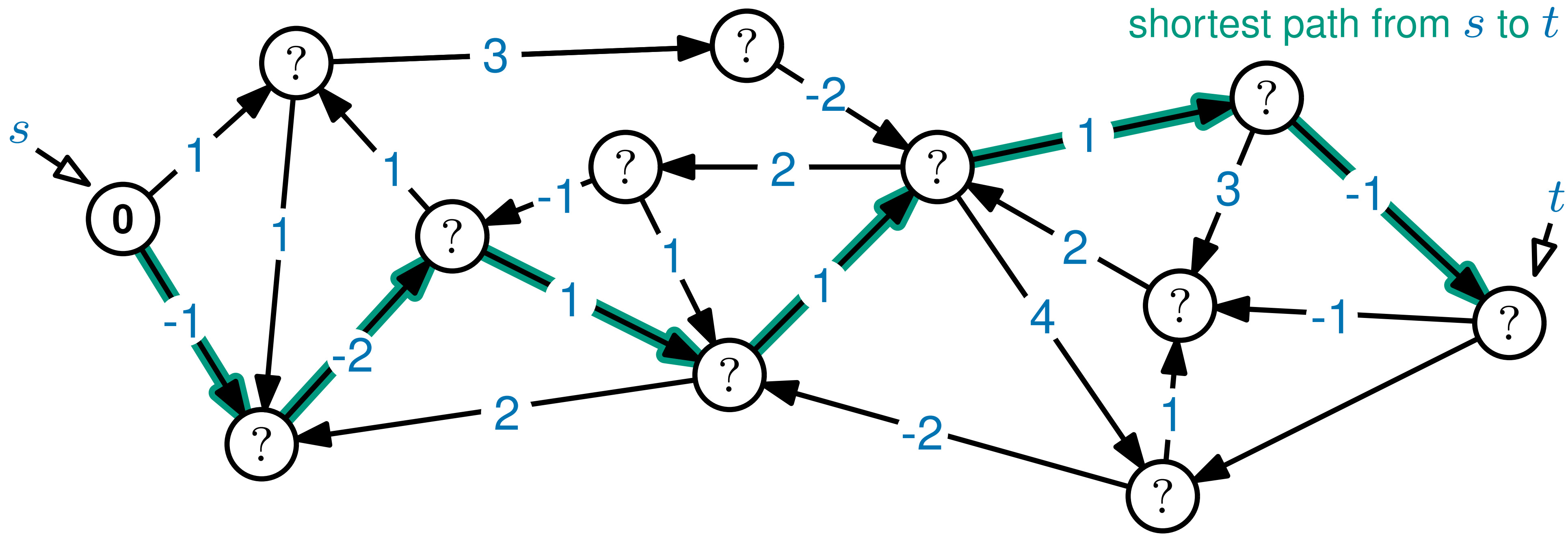

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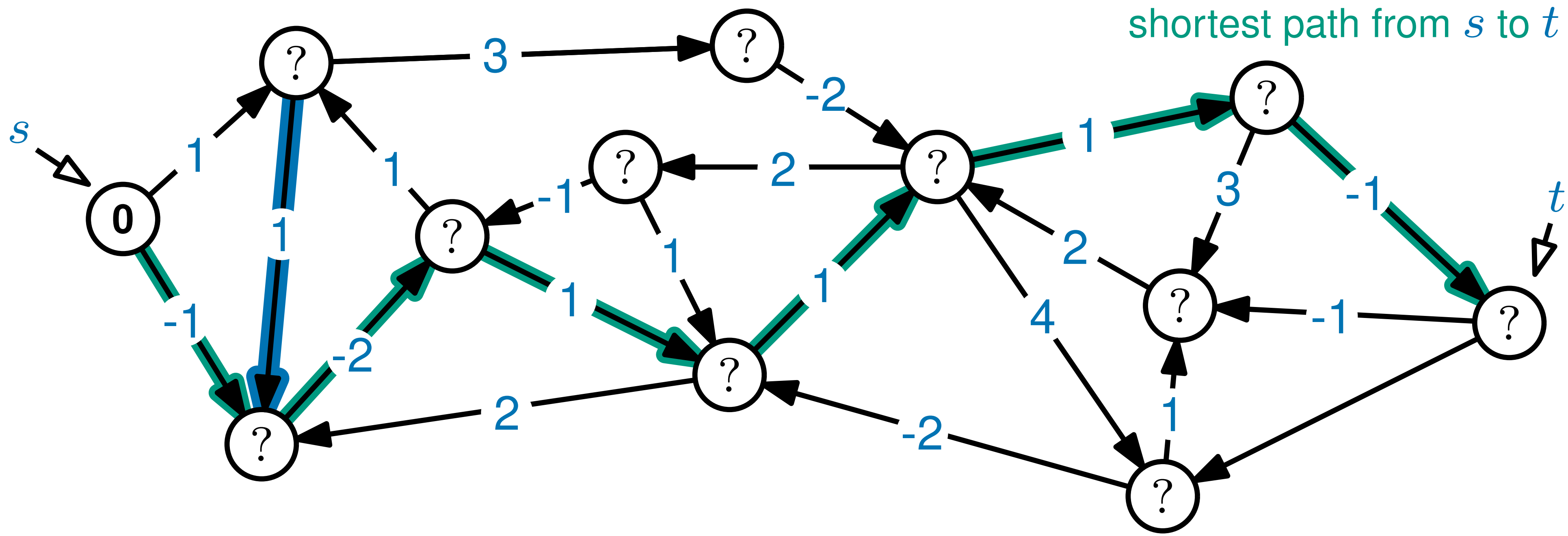

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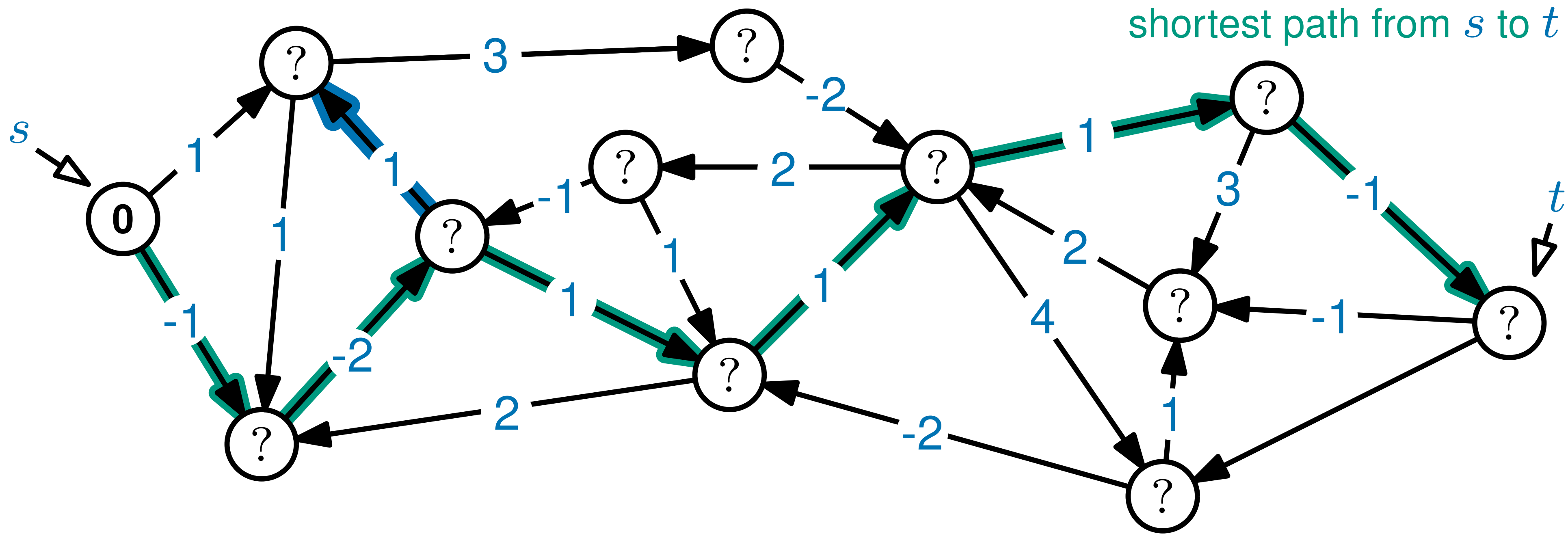

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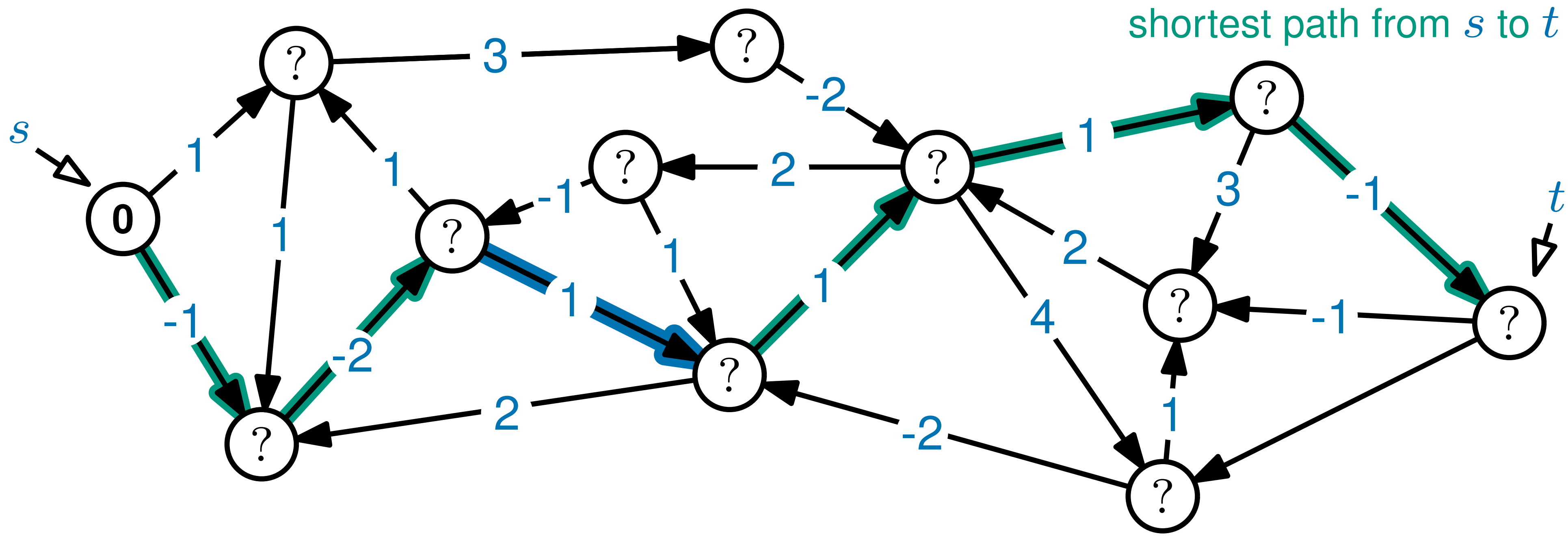

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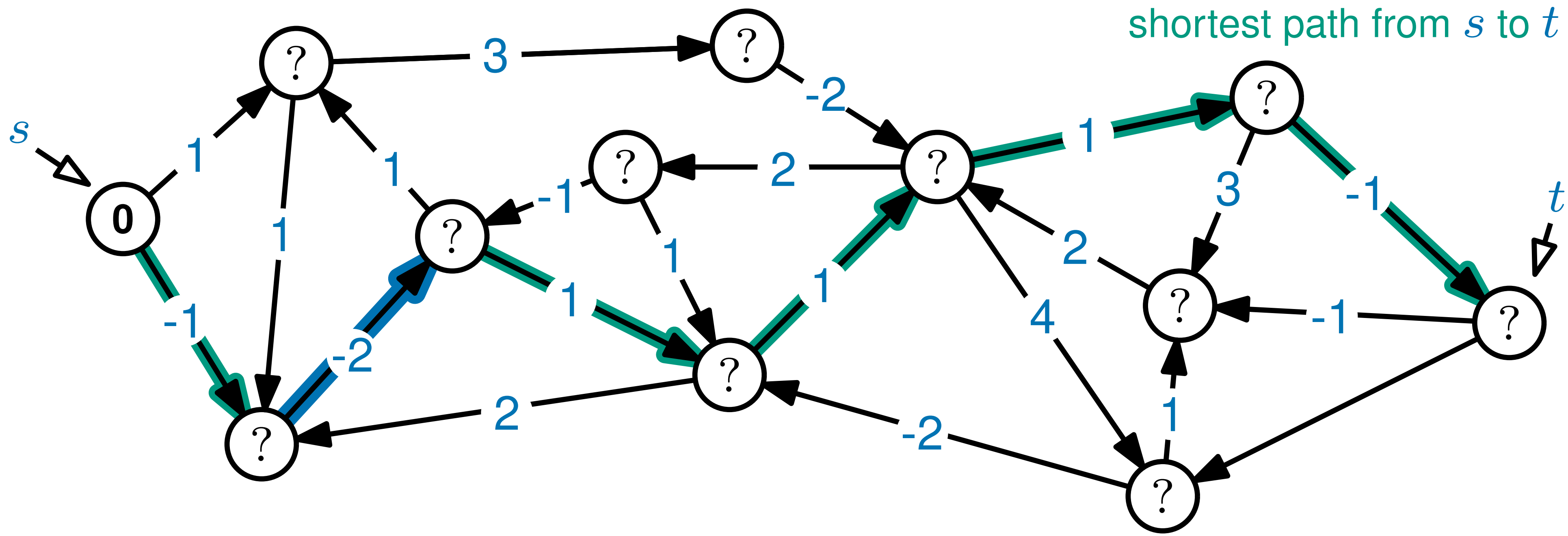

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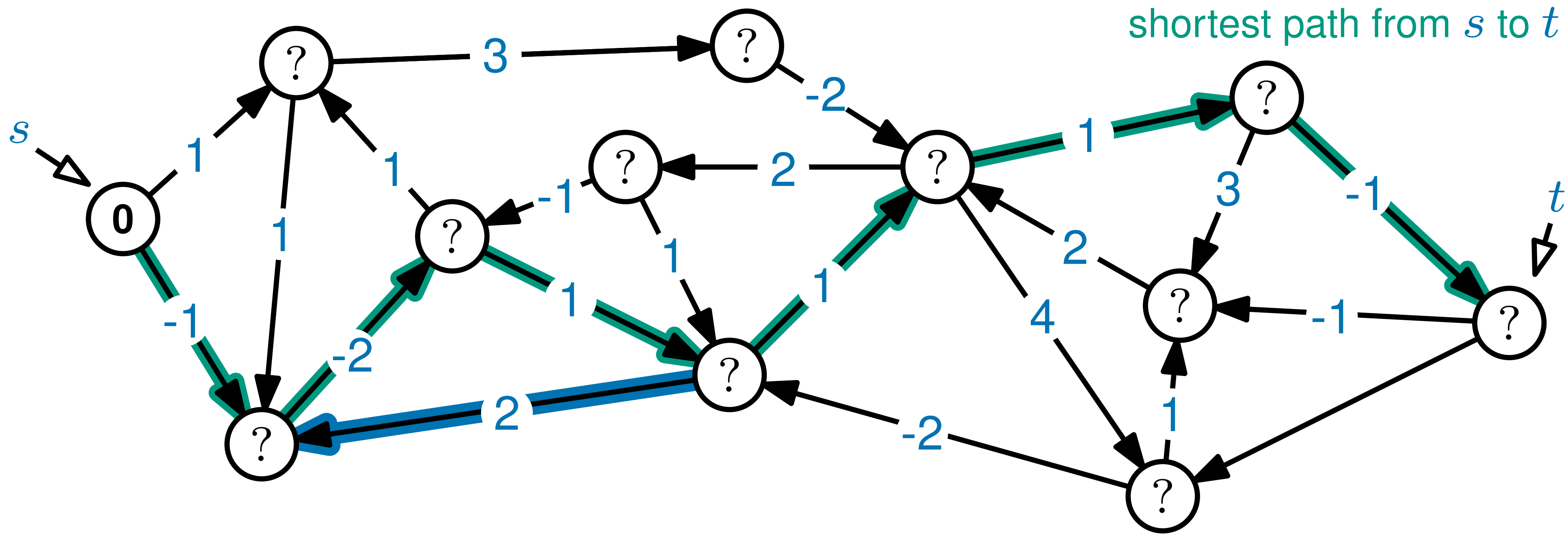

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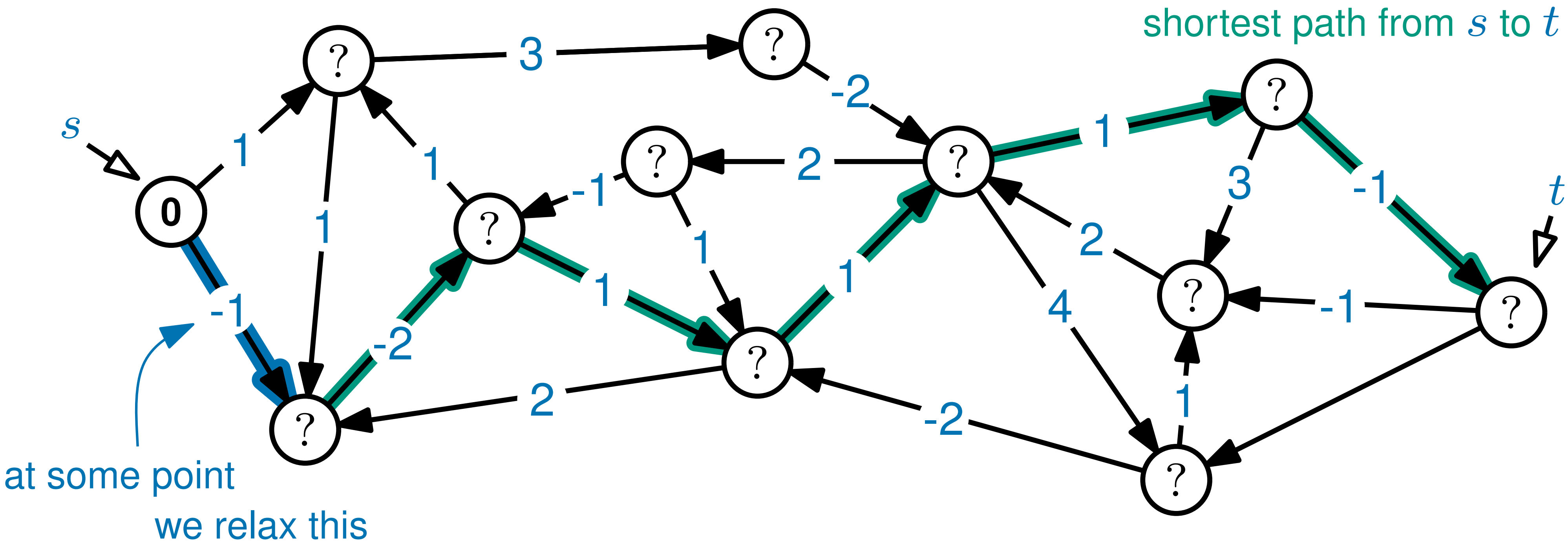

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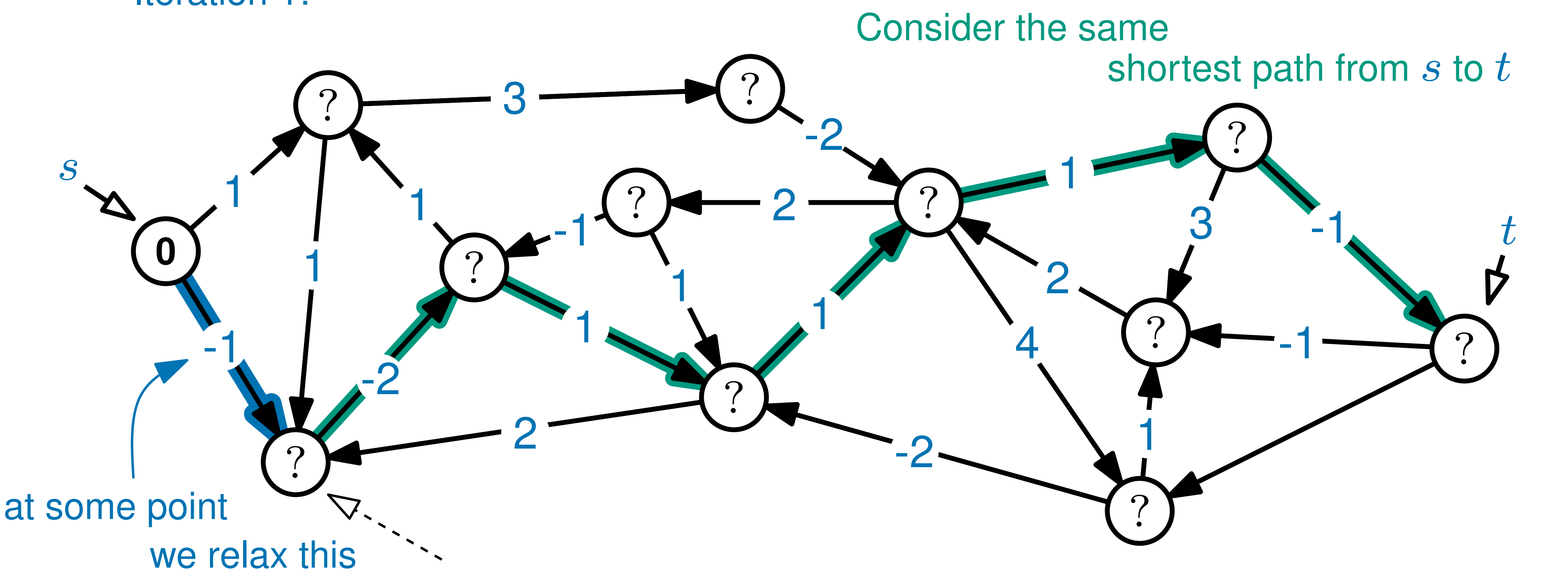

Don't worry about what ? was before. Relax picks the smaller of

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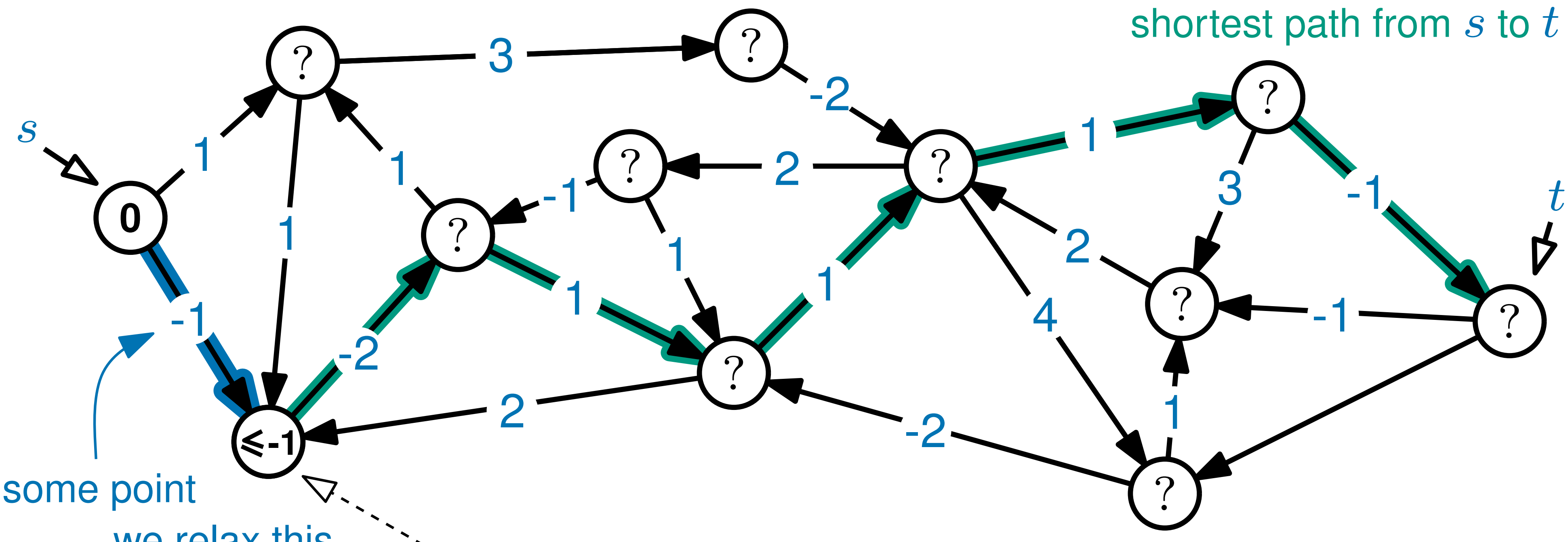

at some point we relax this

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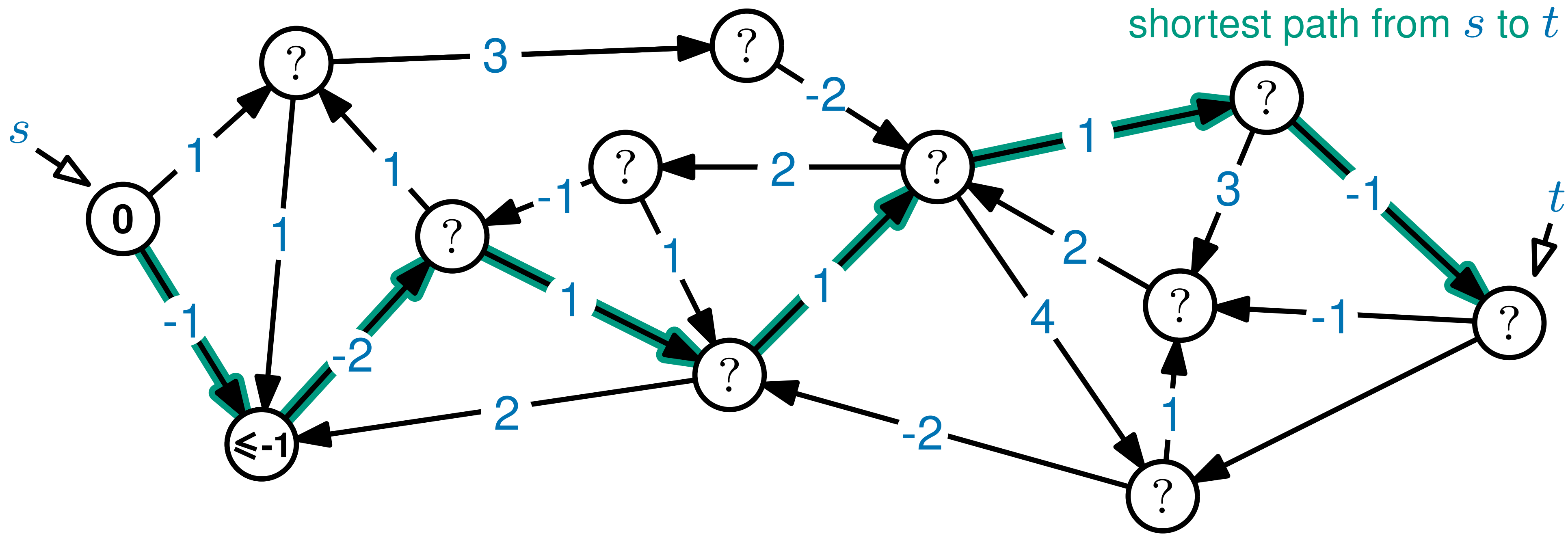

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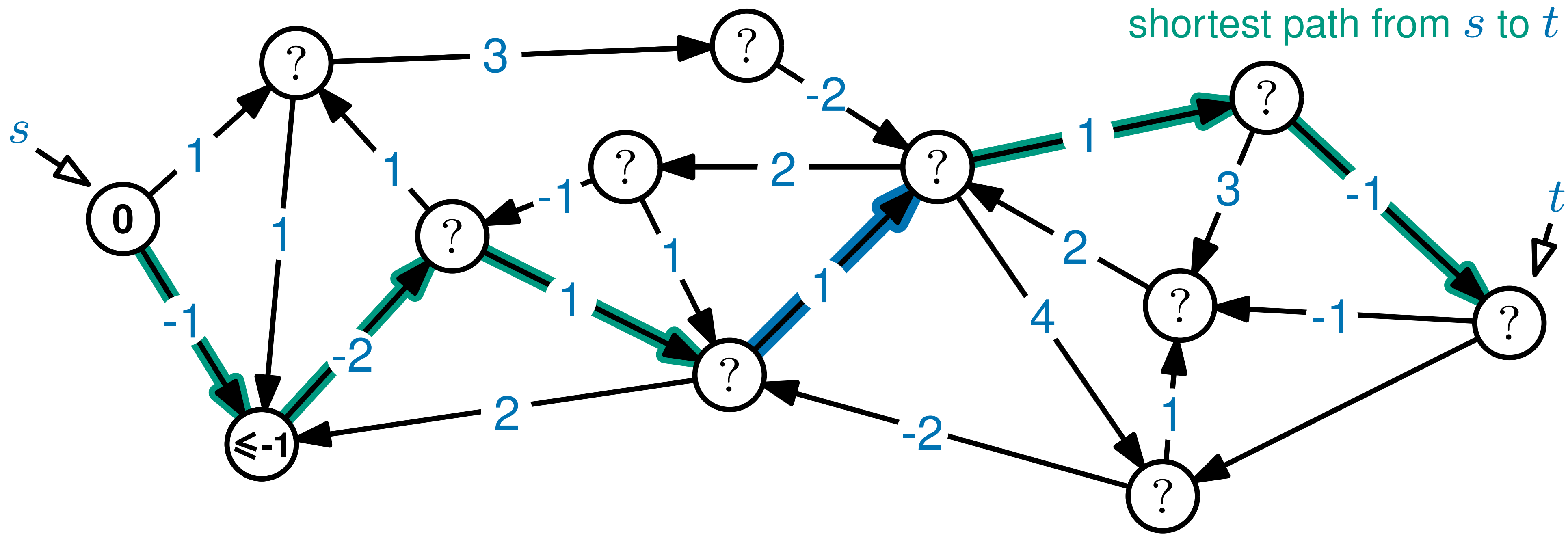

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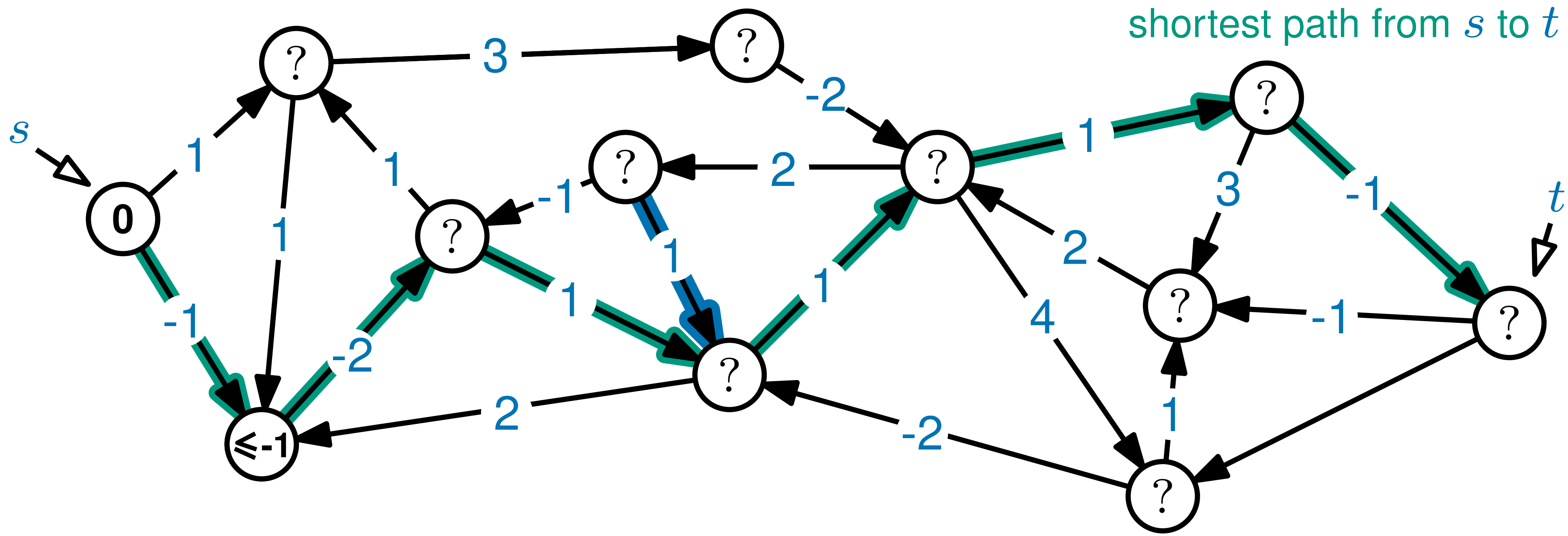

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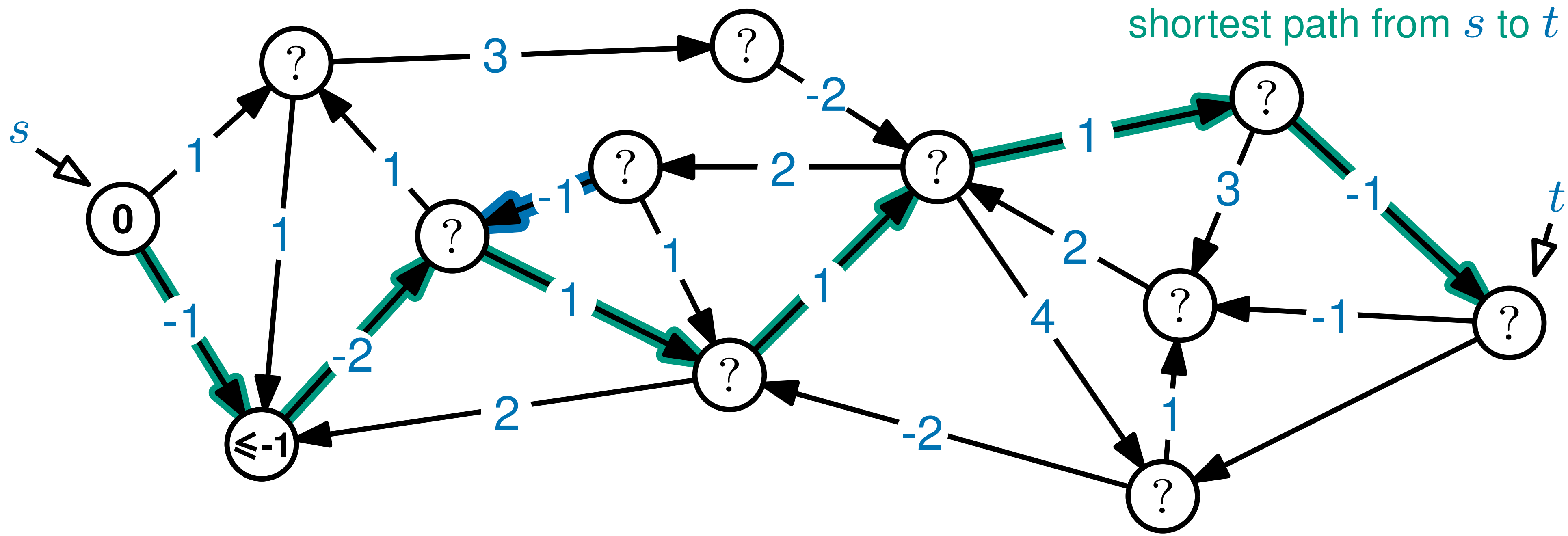

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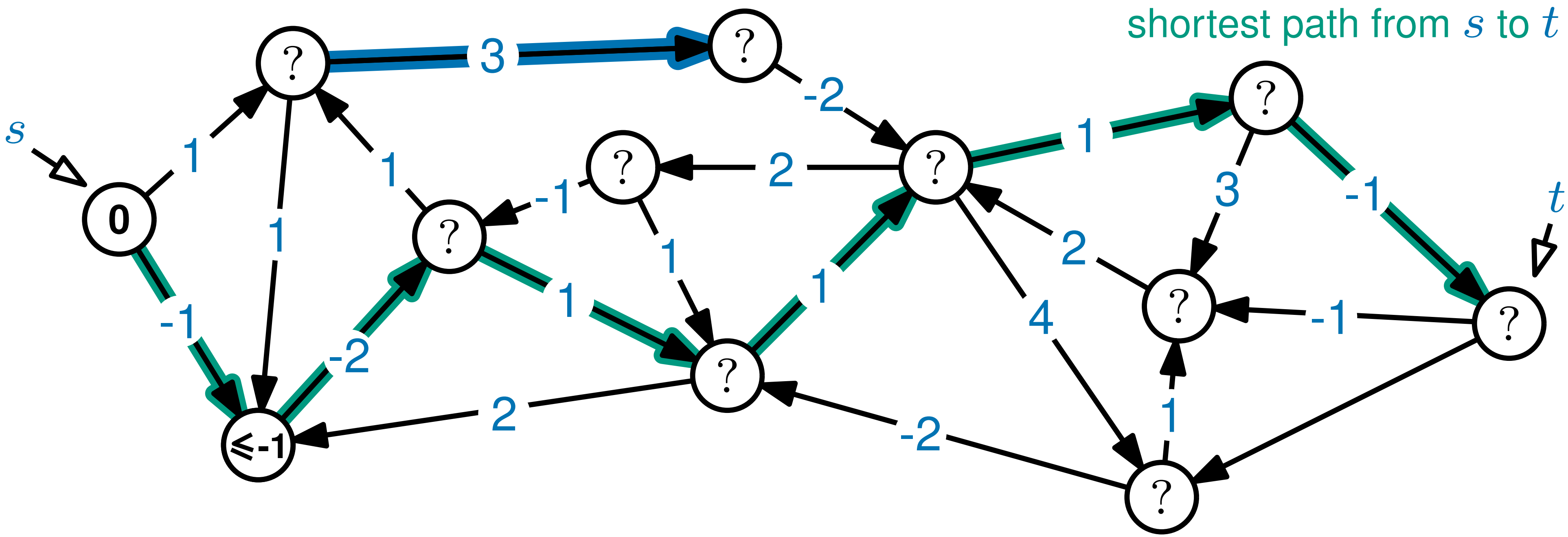

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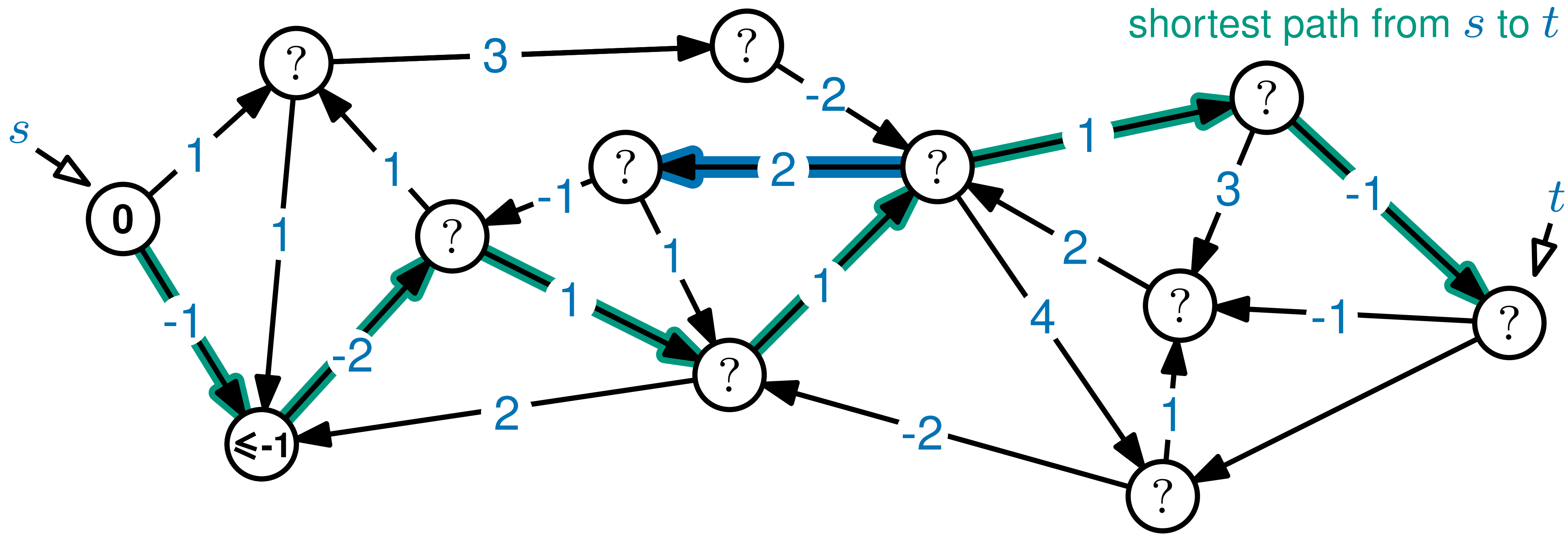

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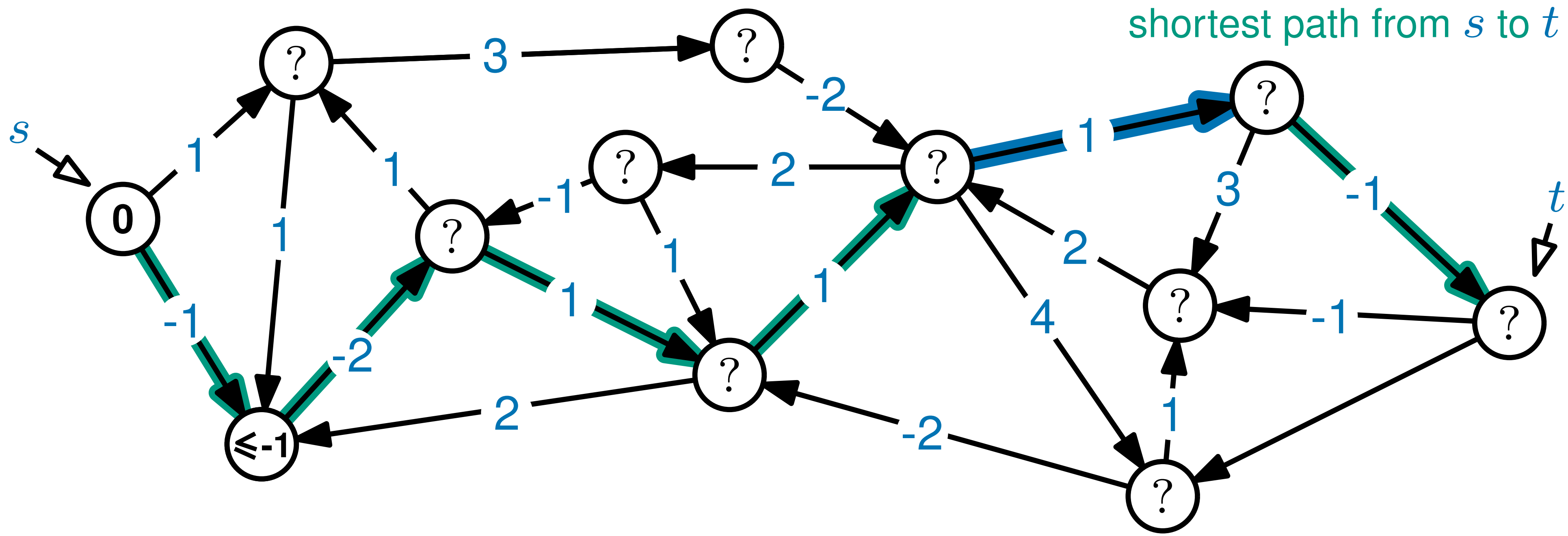

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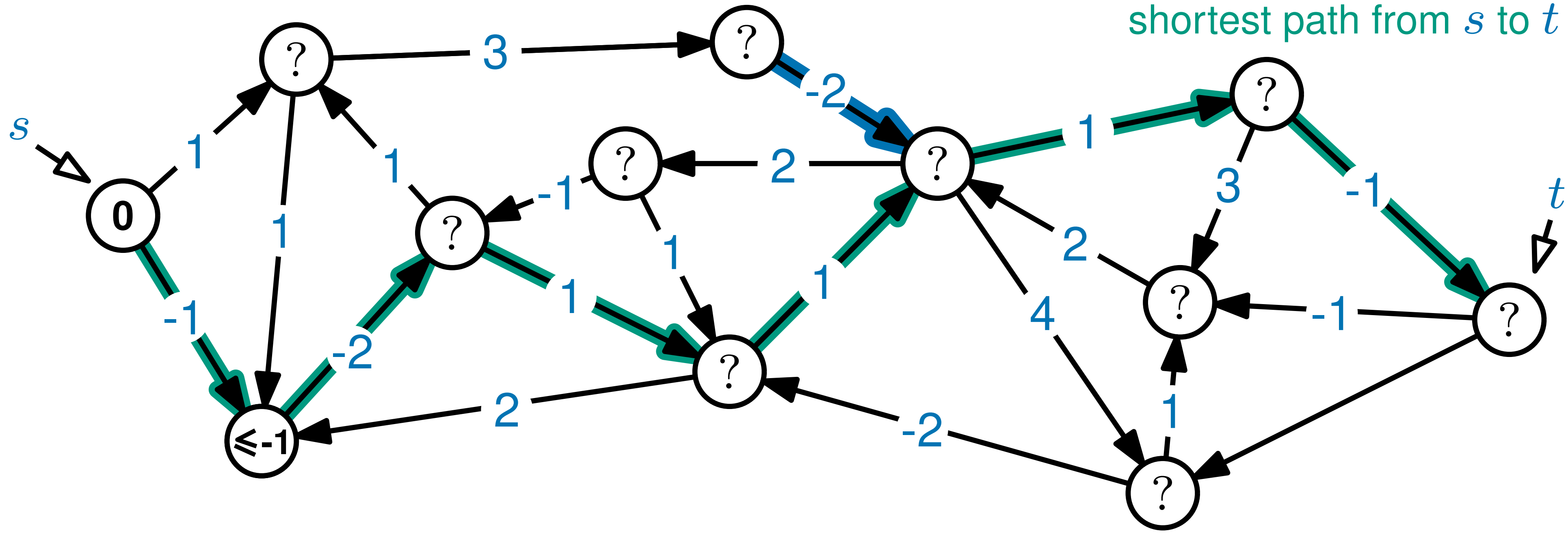

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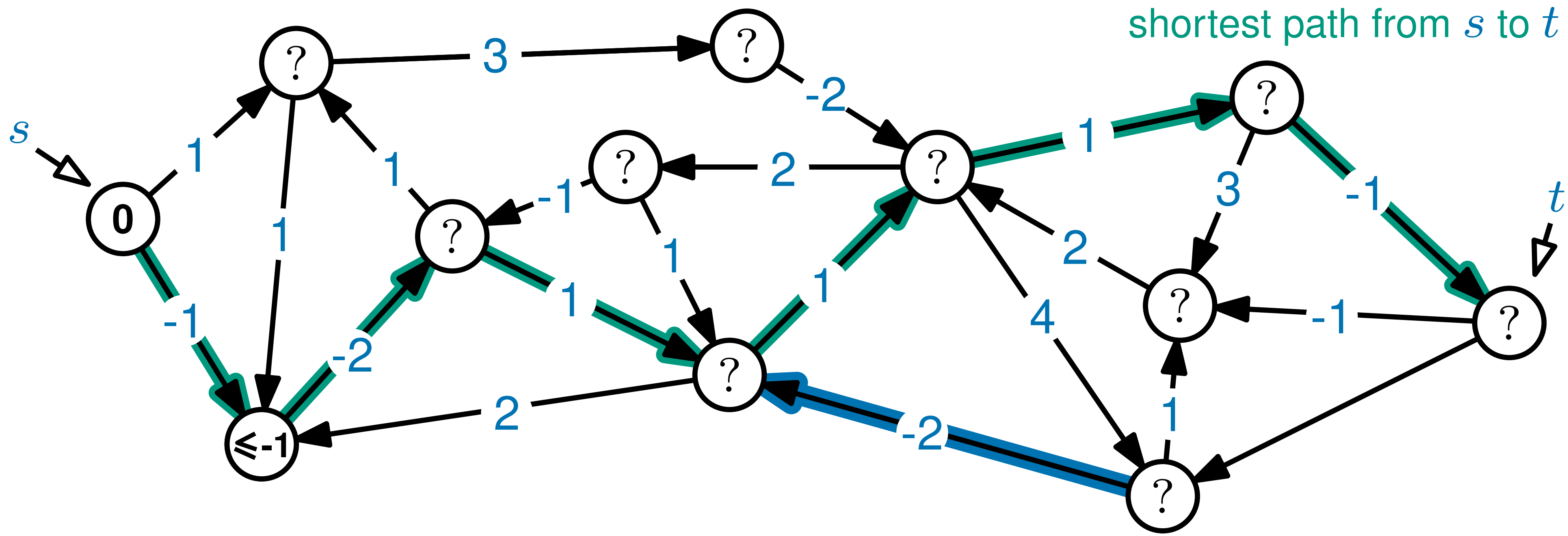

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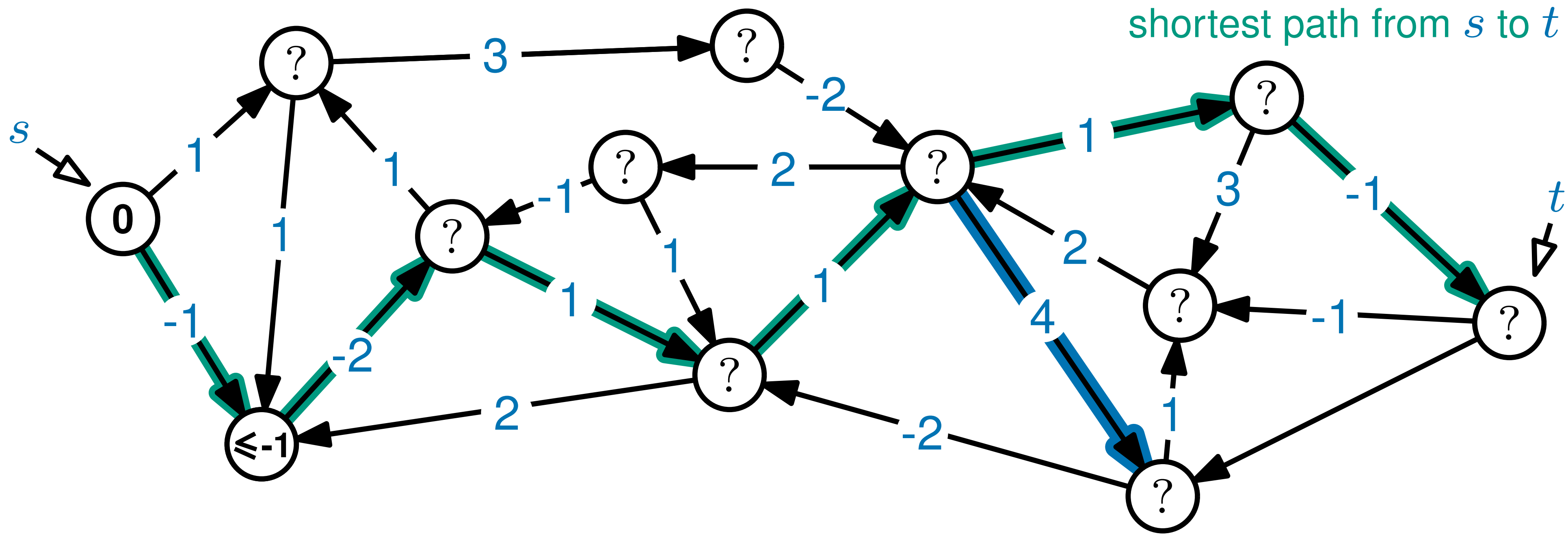

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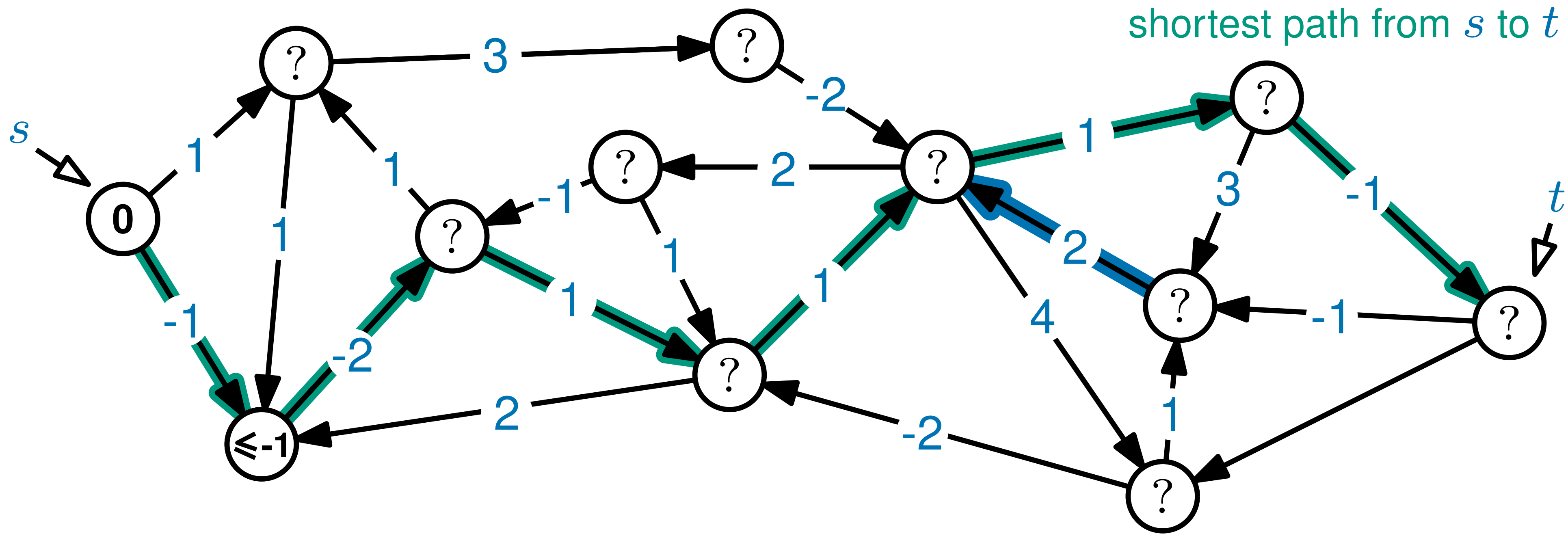

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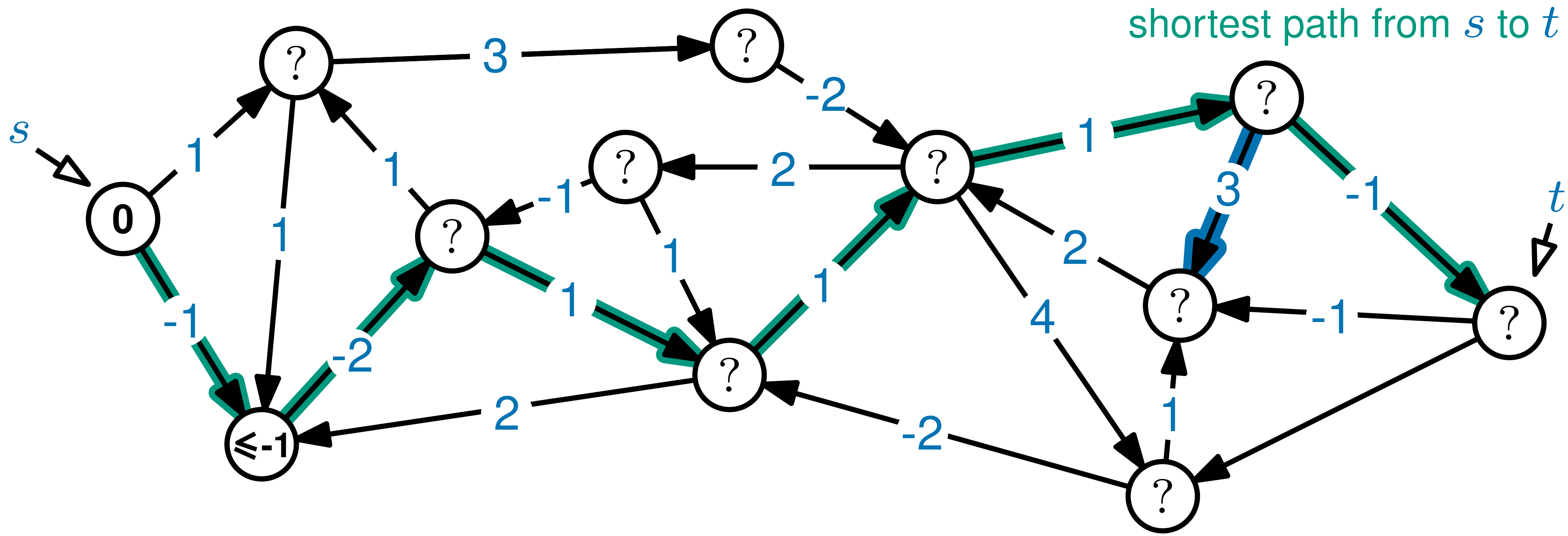

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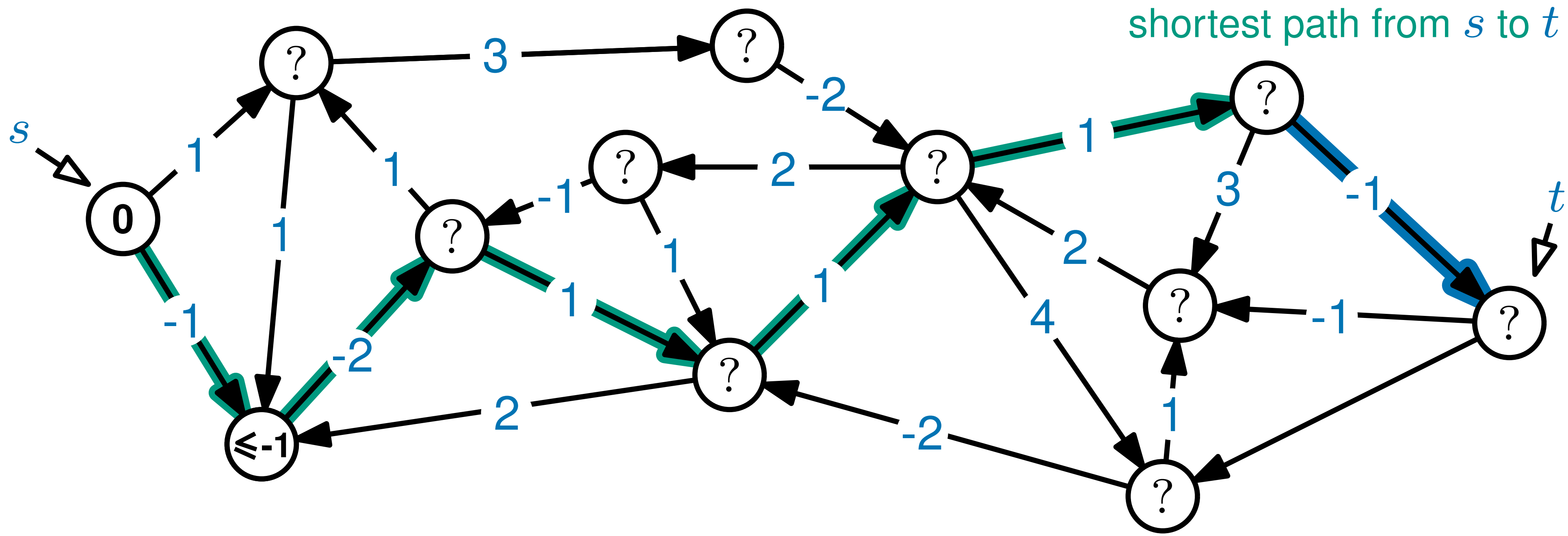

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University of BRISTOL

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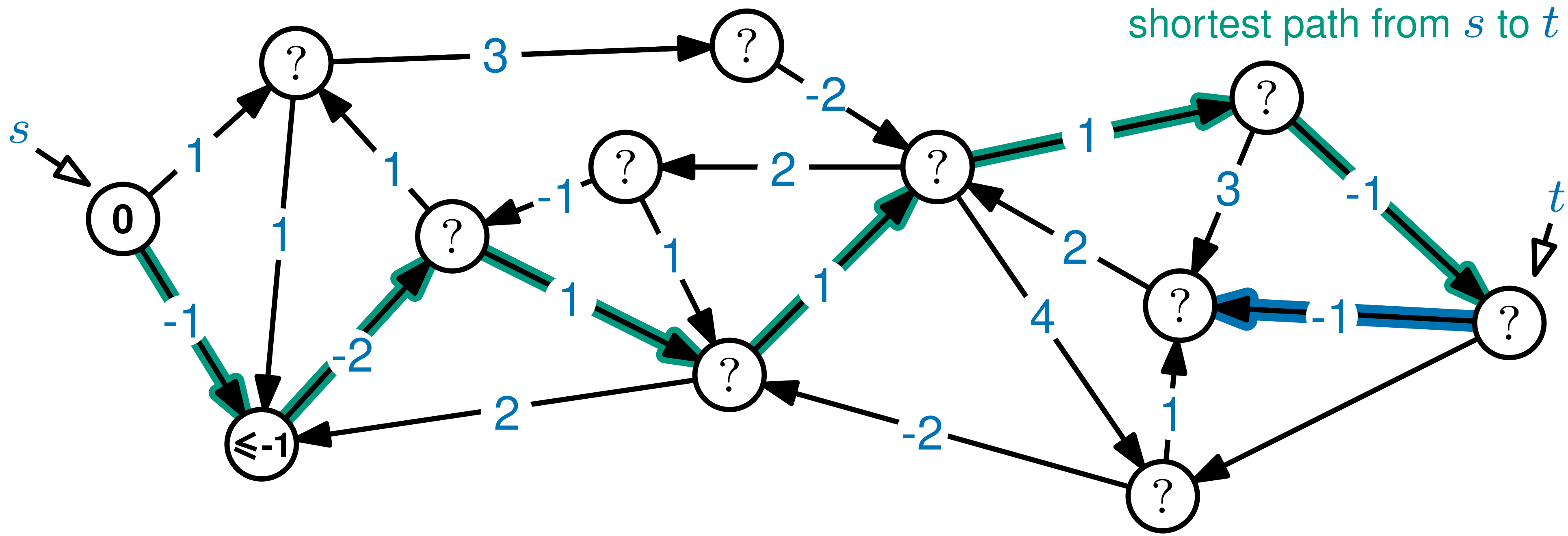

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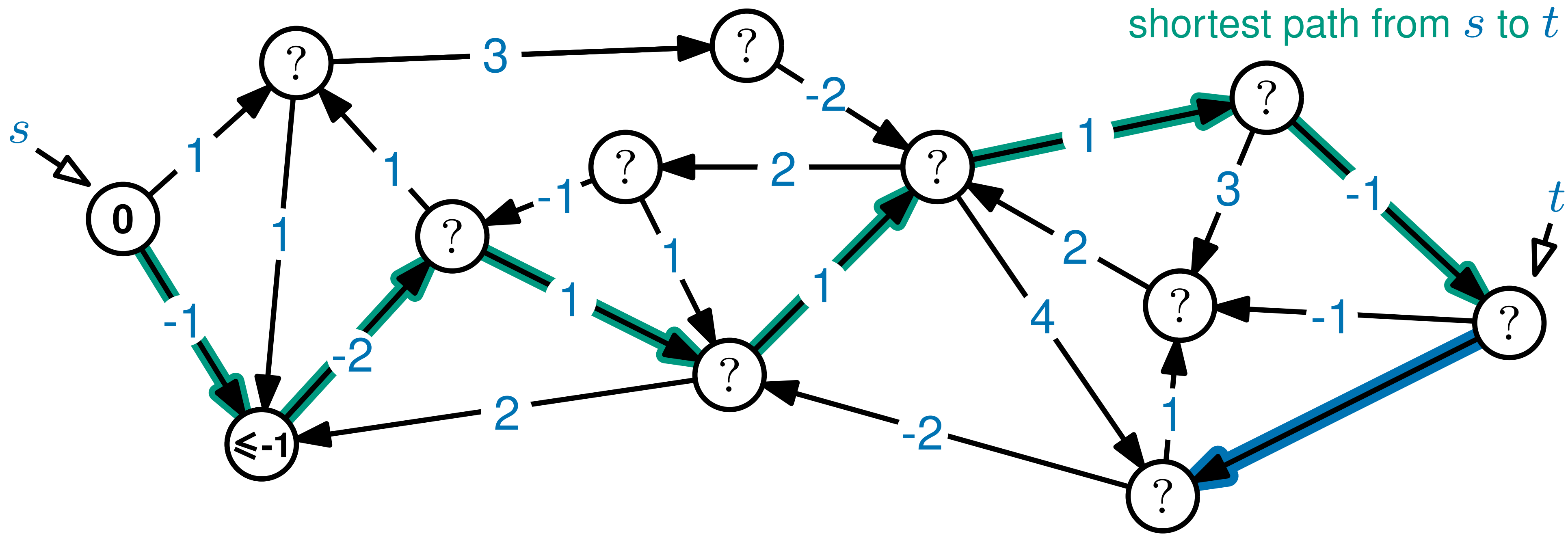

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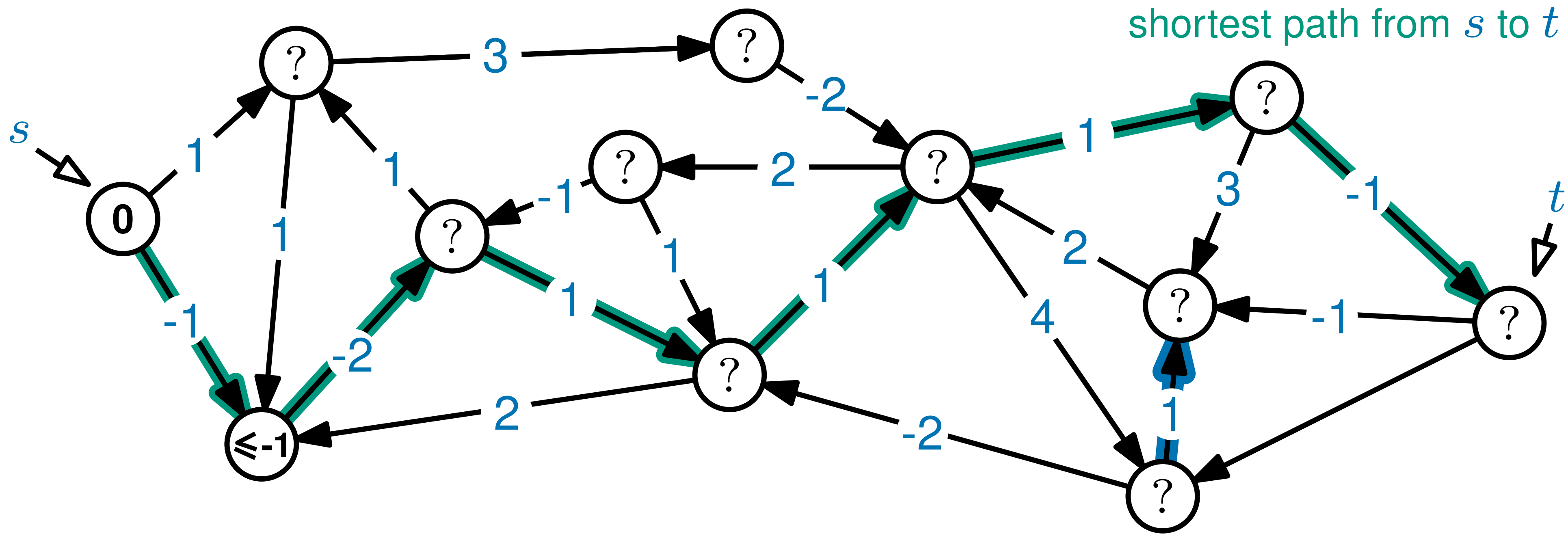

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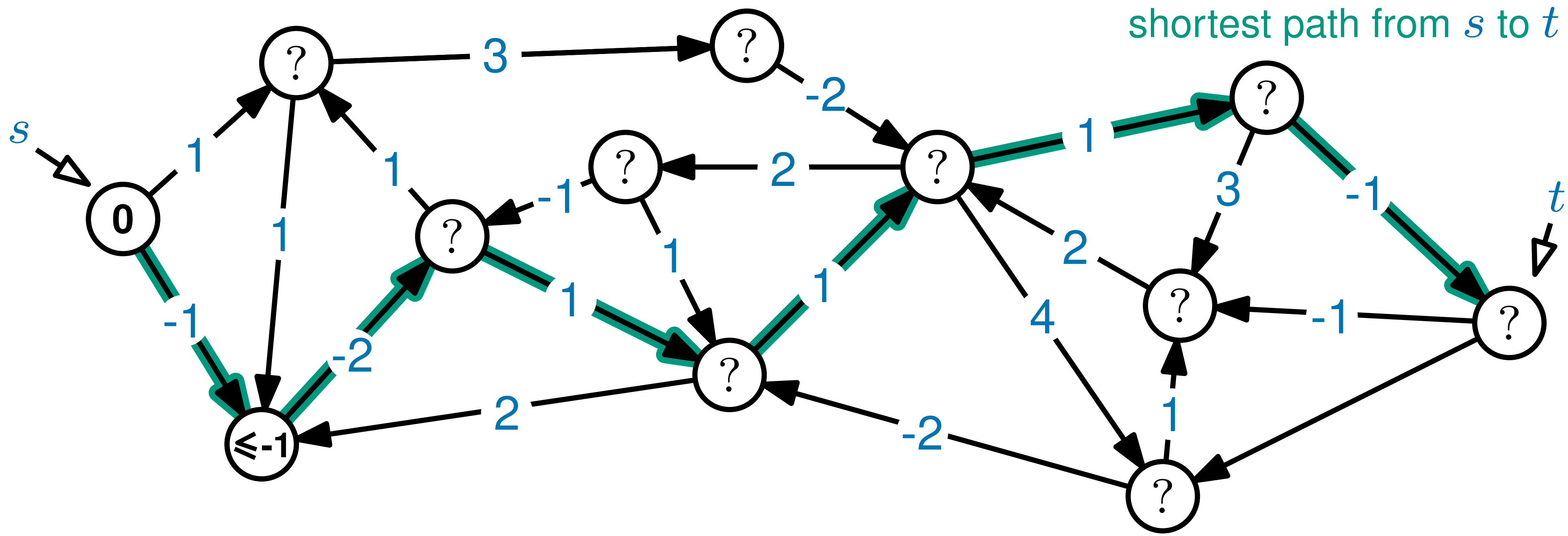

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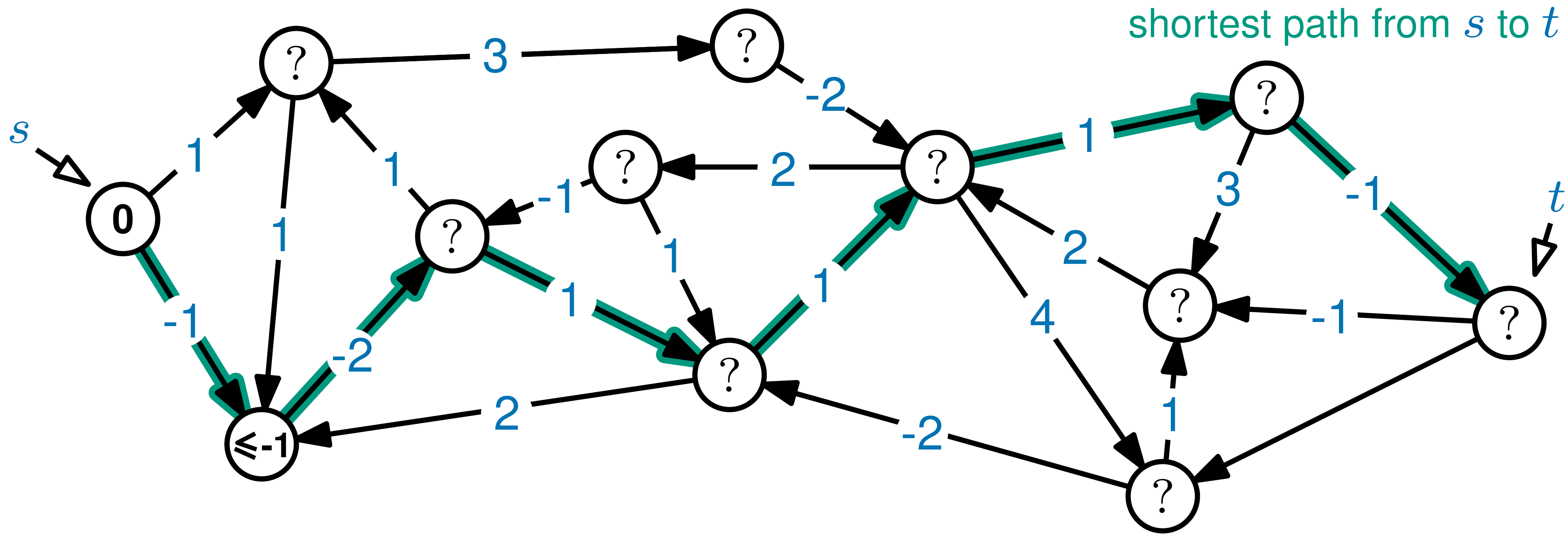

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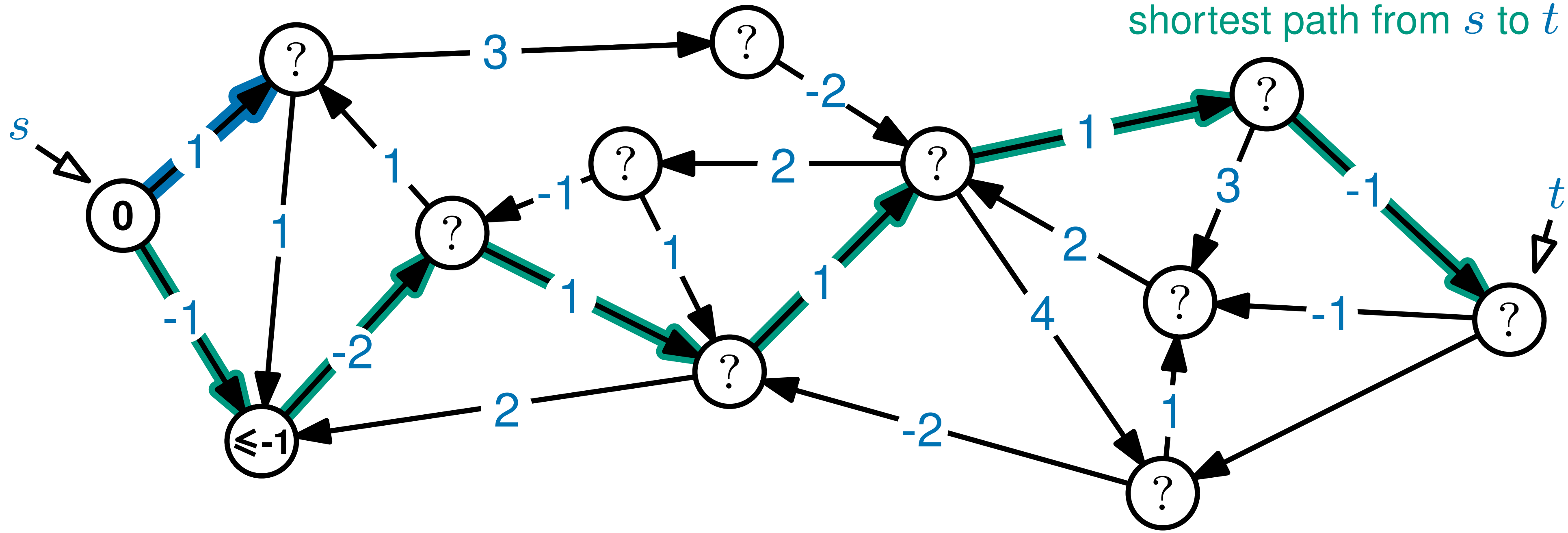

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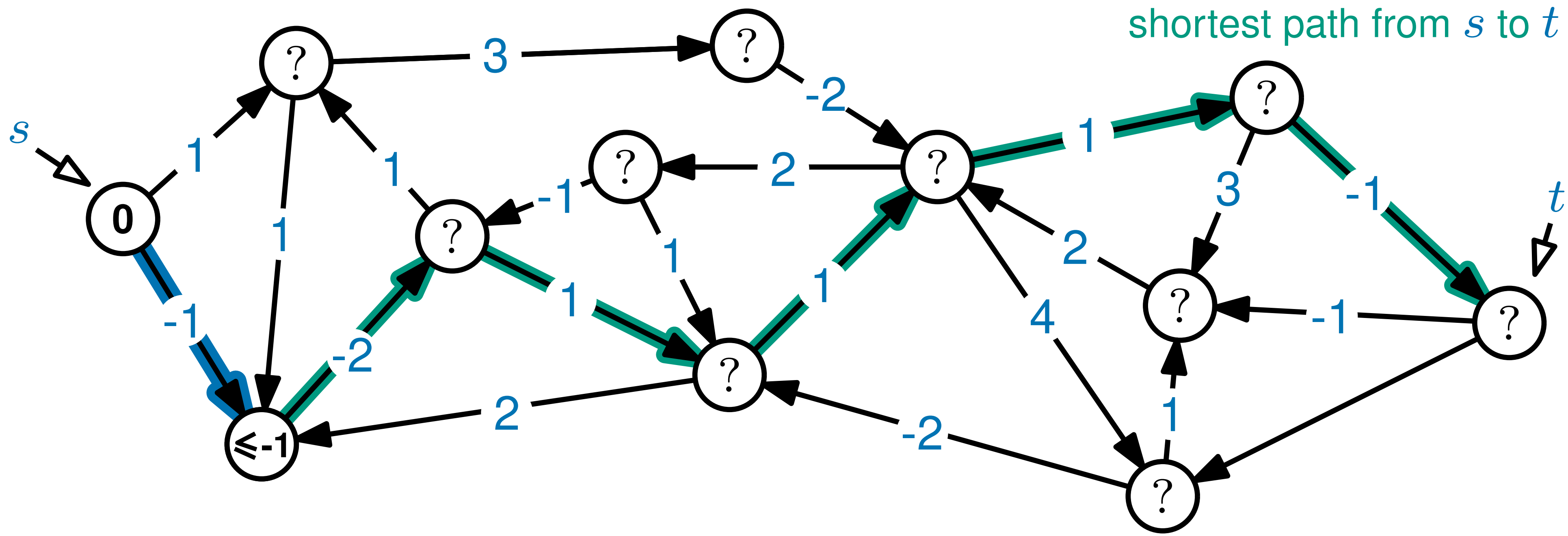

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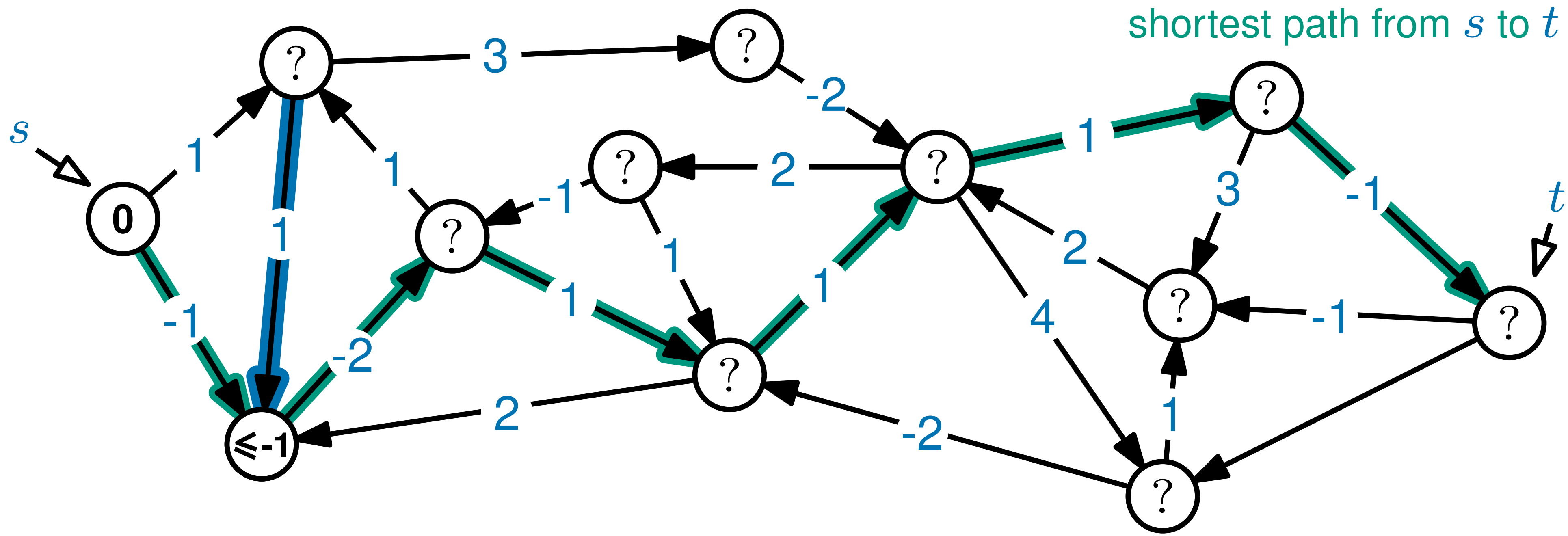

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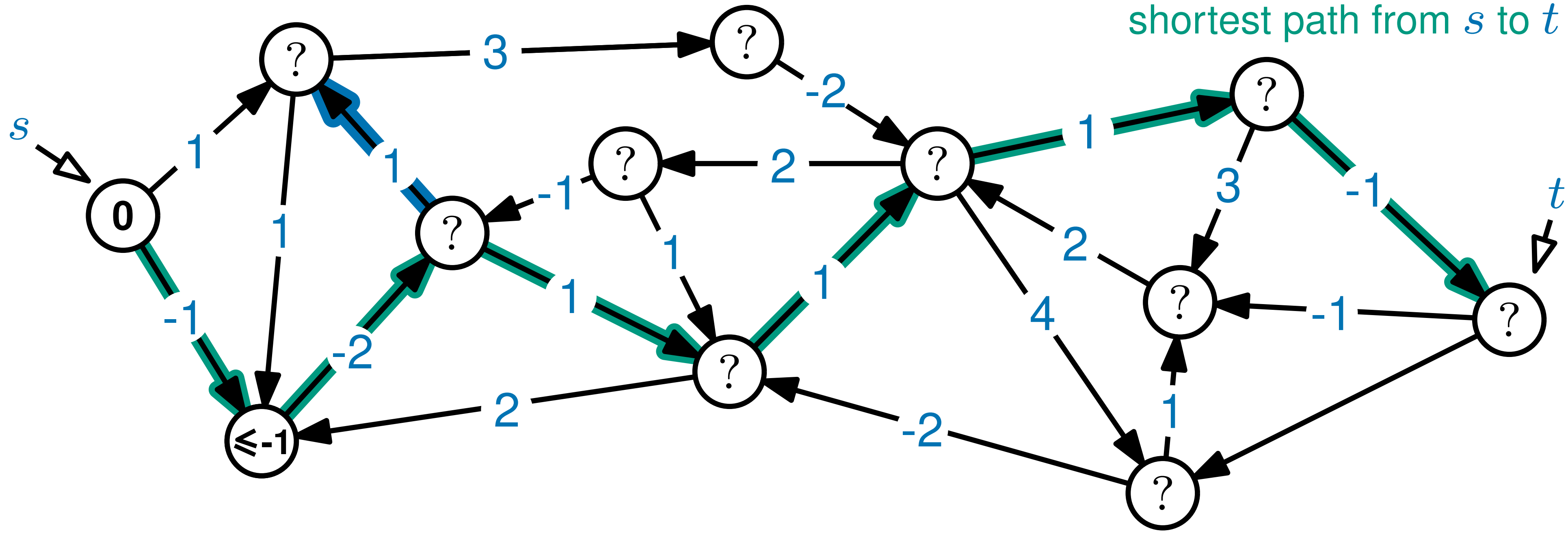

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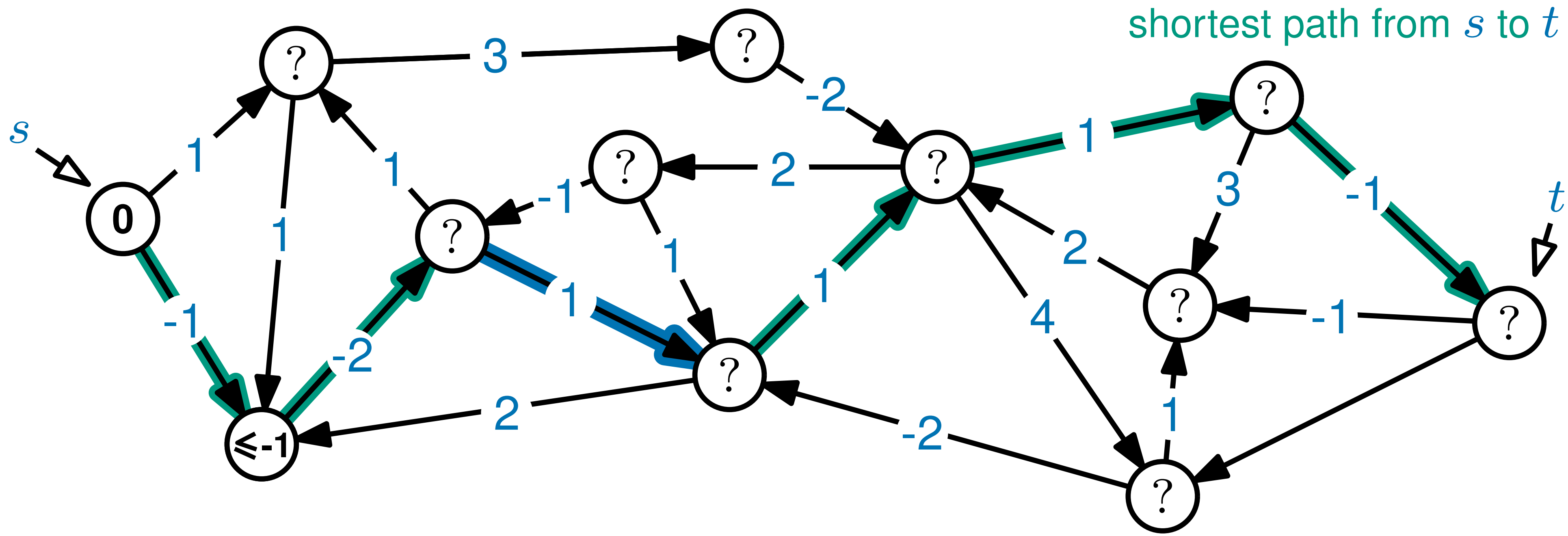

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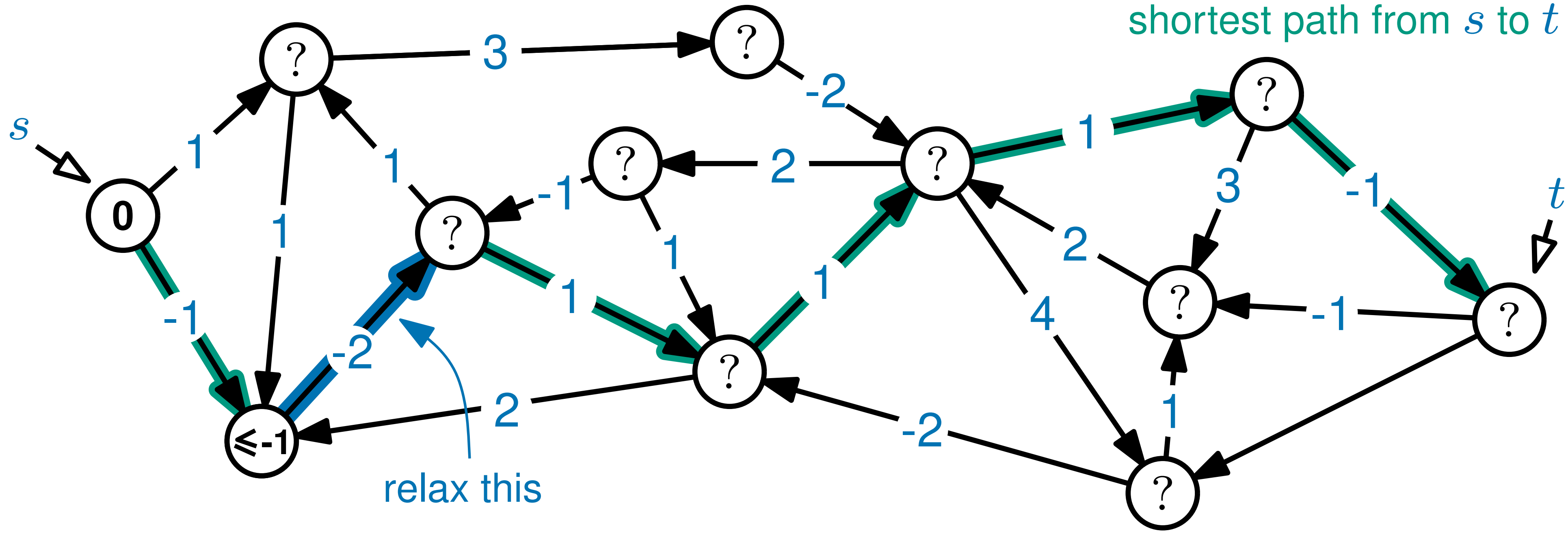

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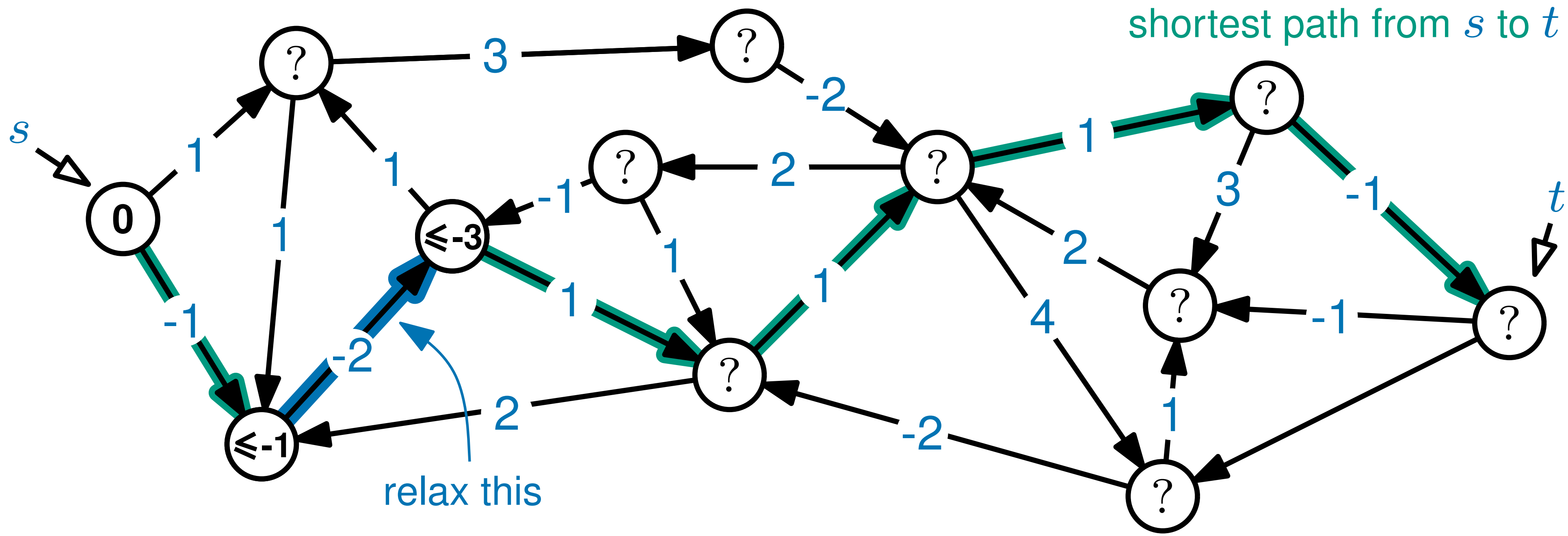

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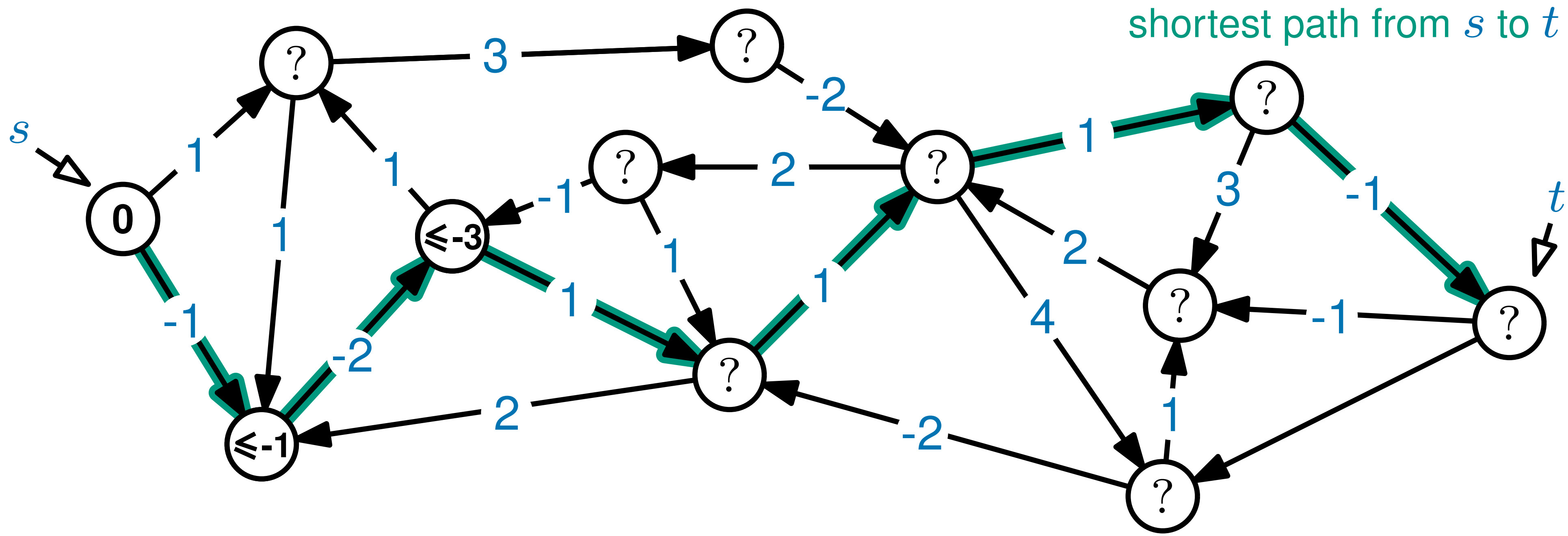

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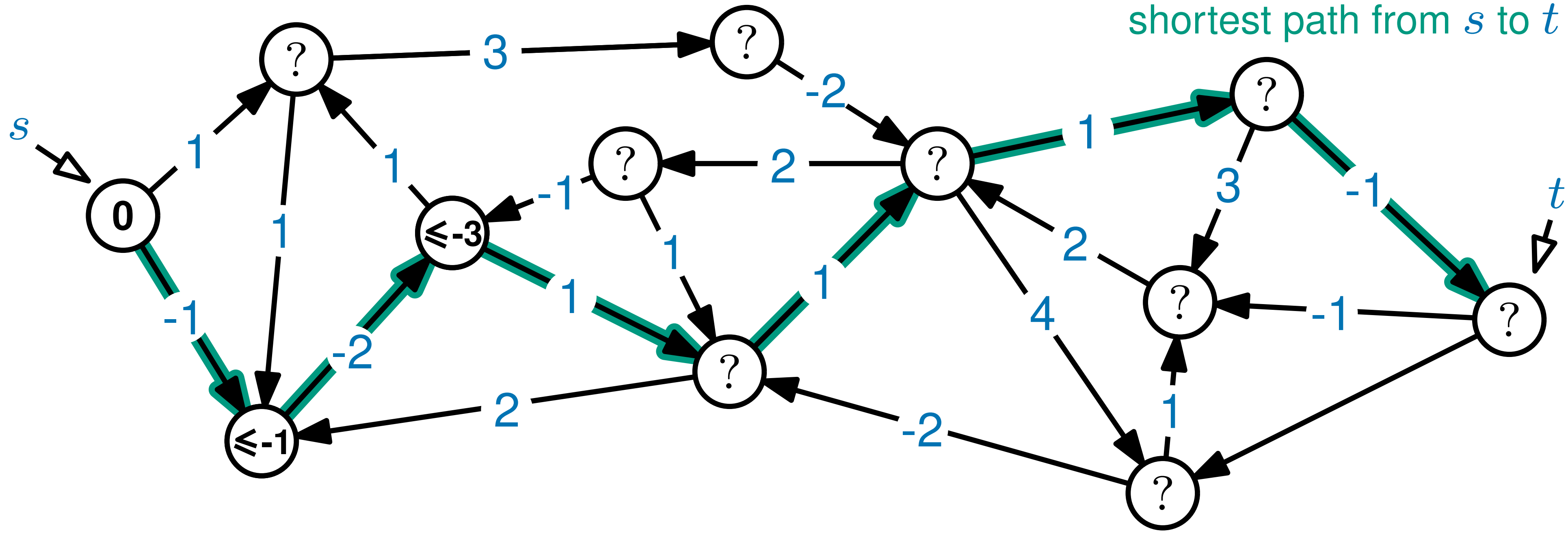

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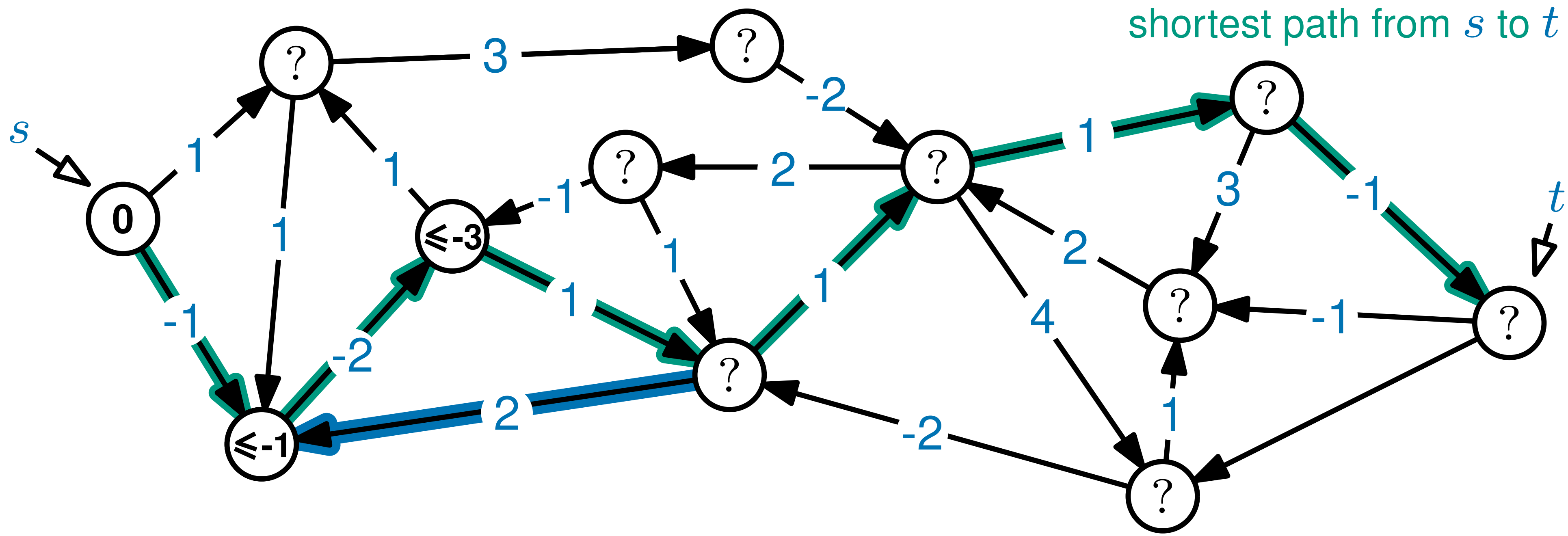

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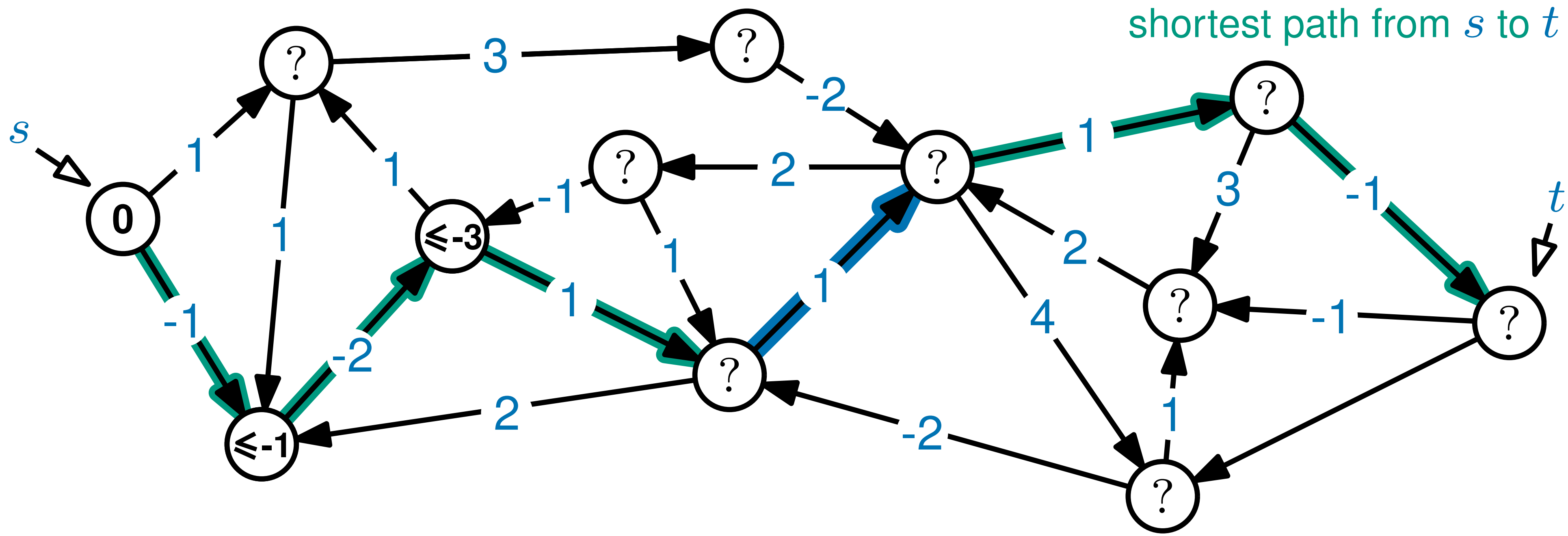

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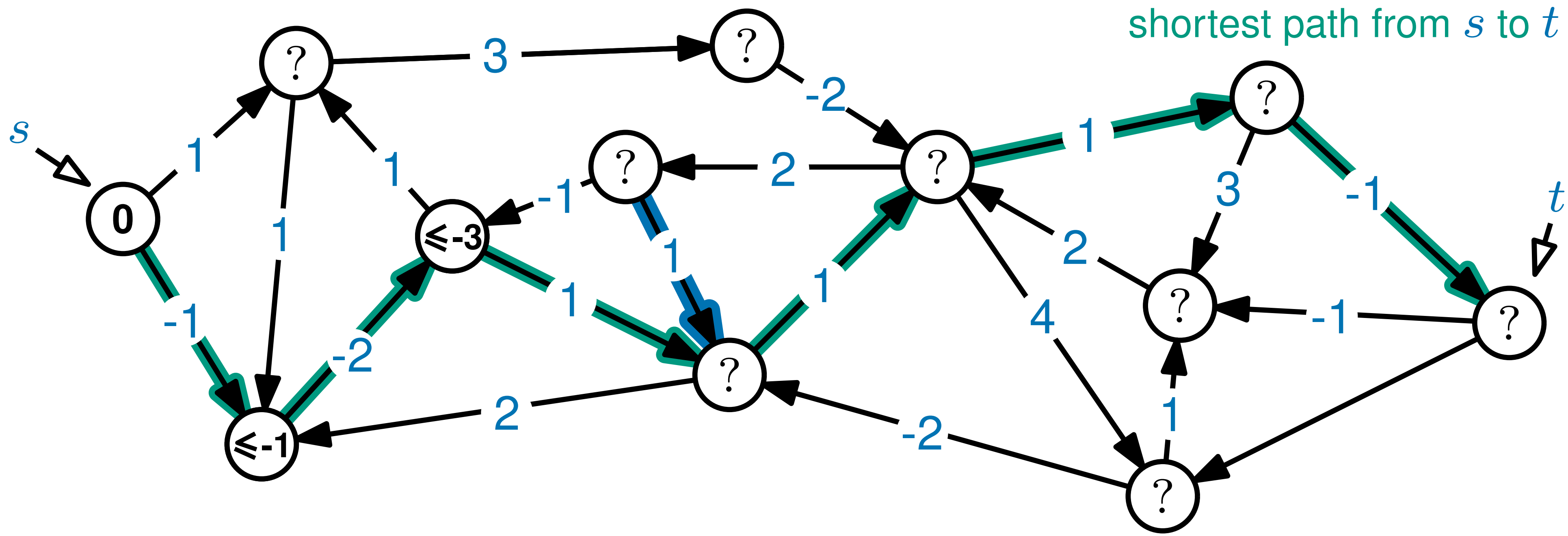

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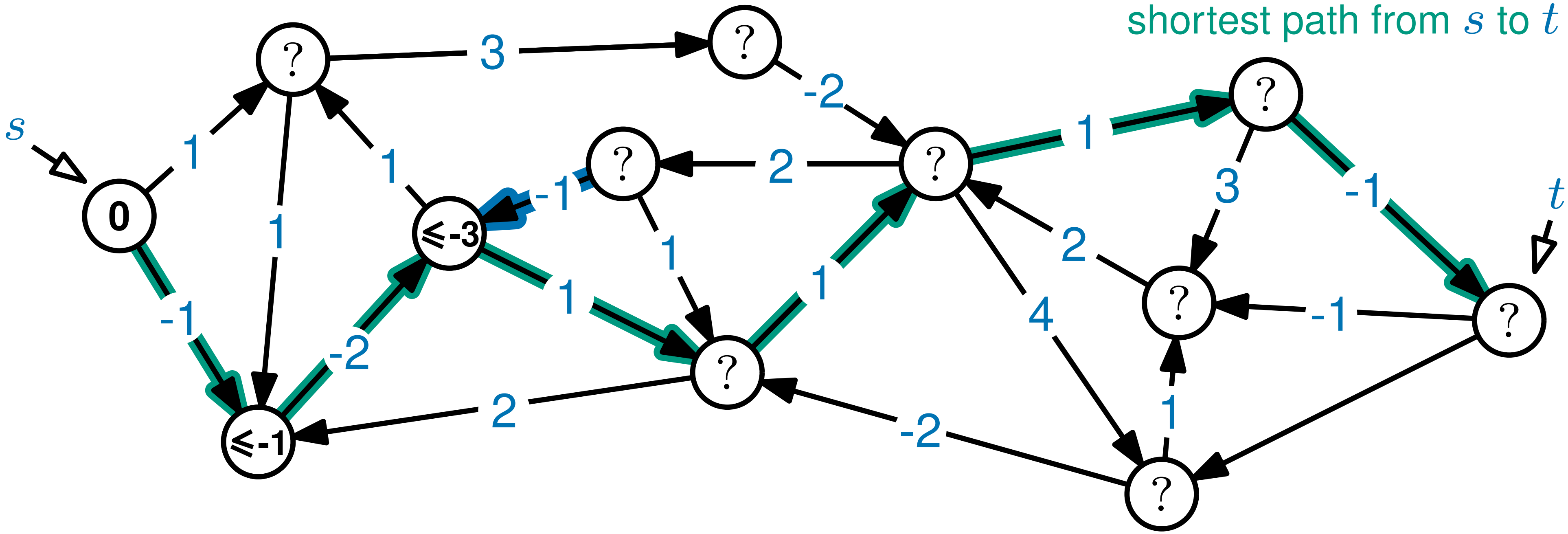

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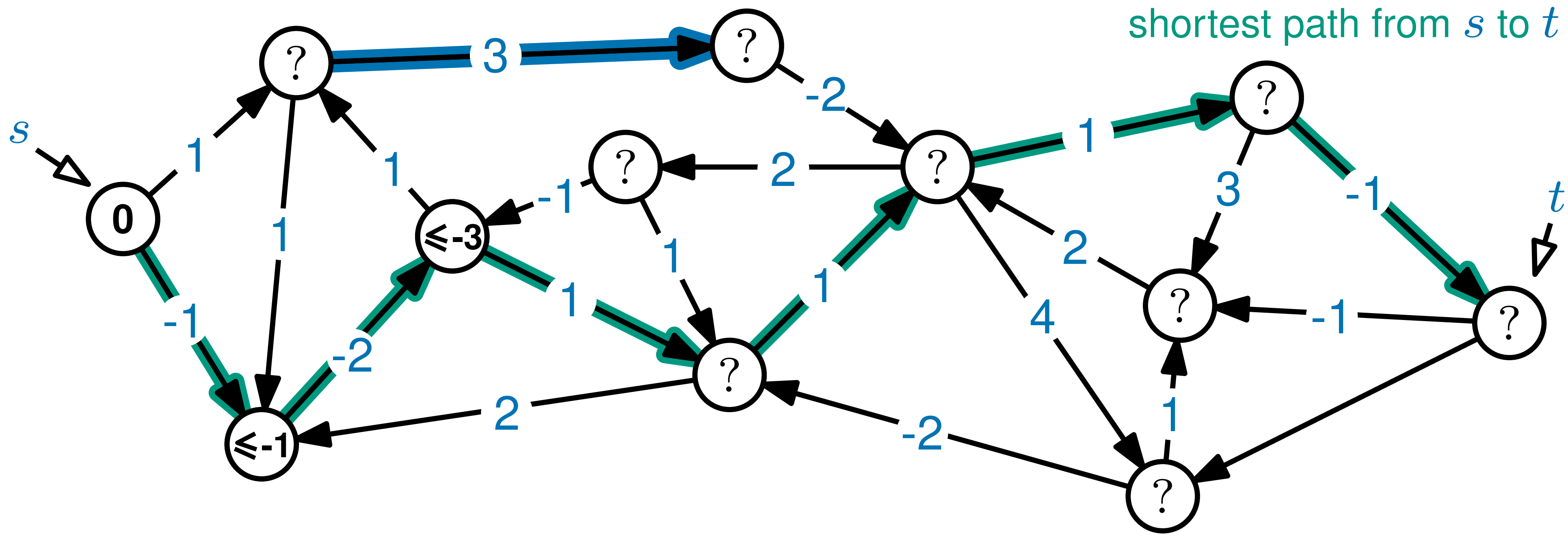

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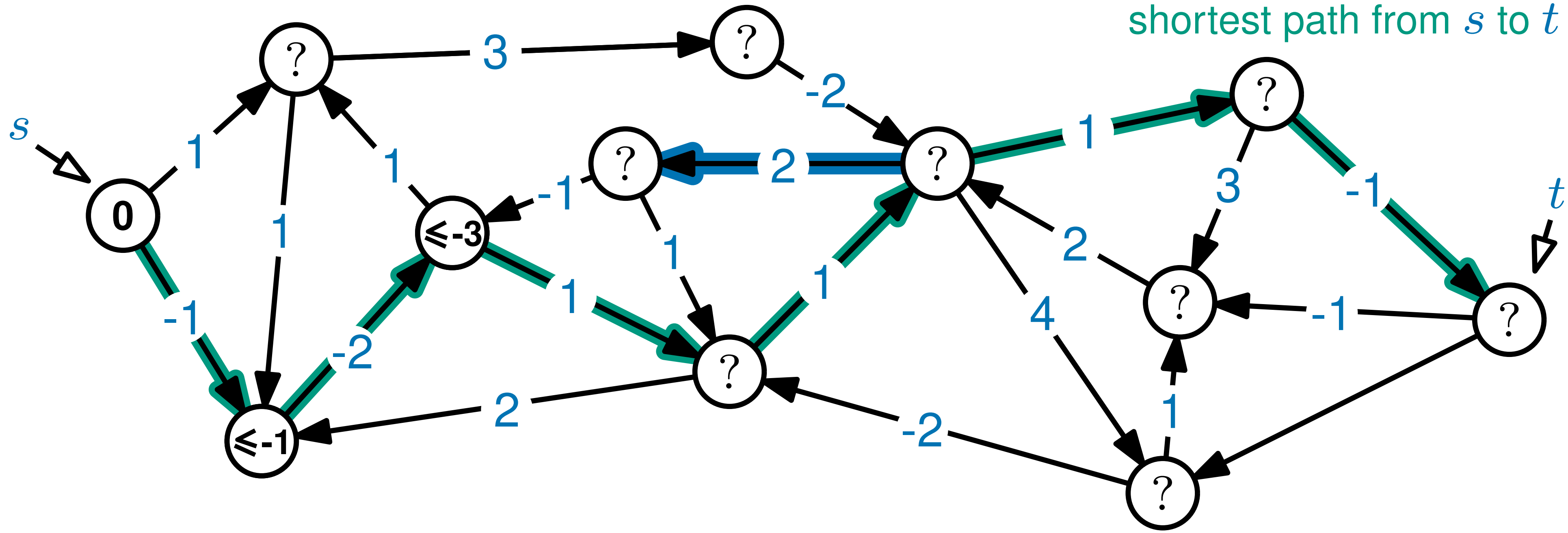

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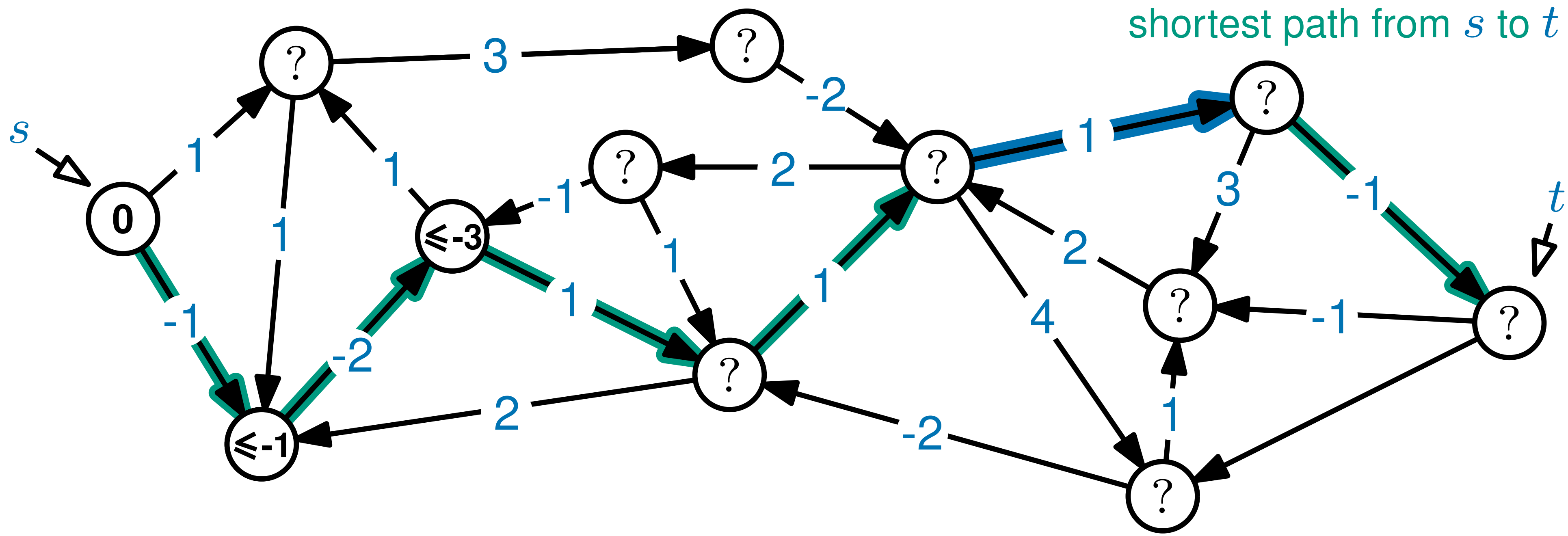

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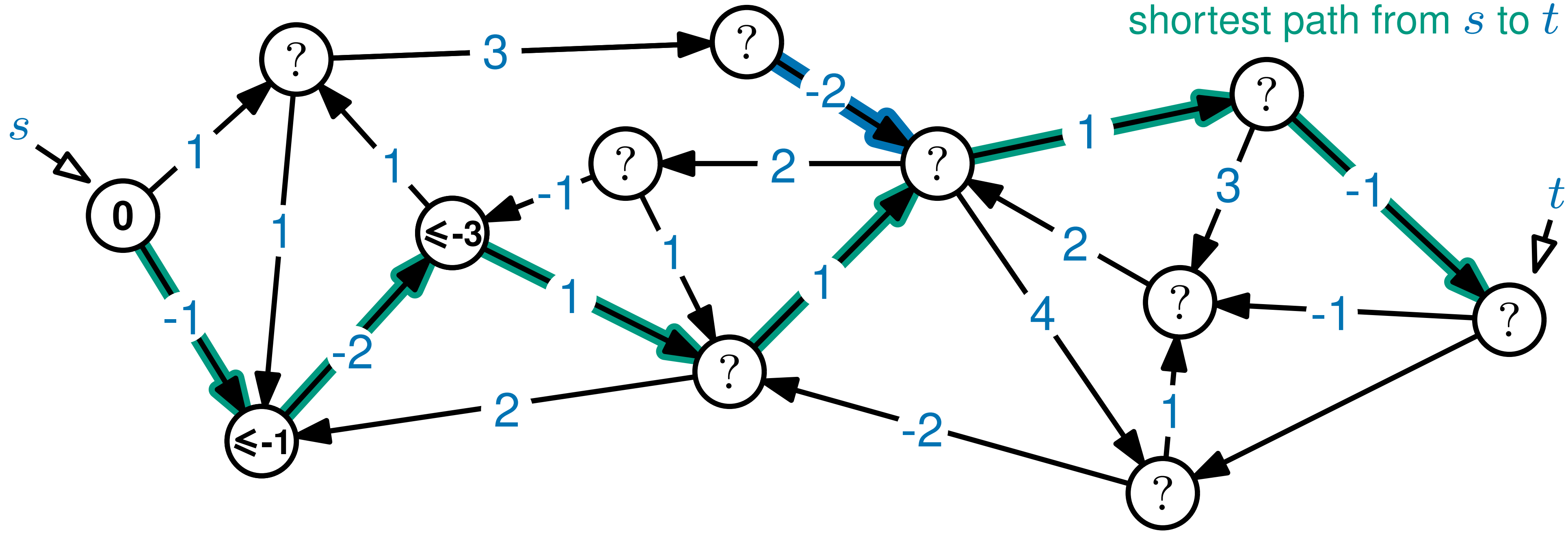

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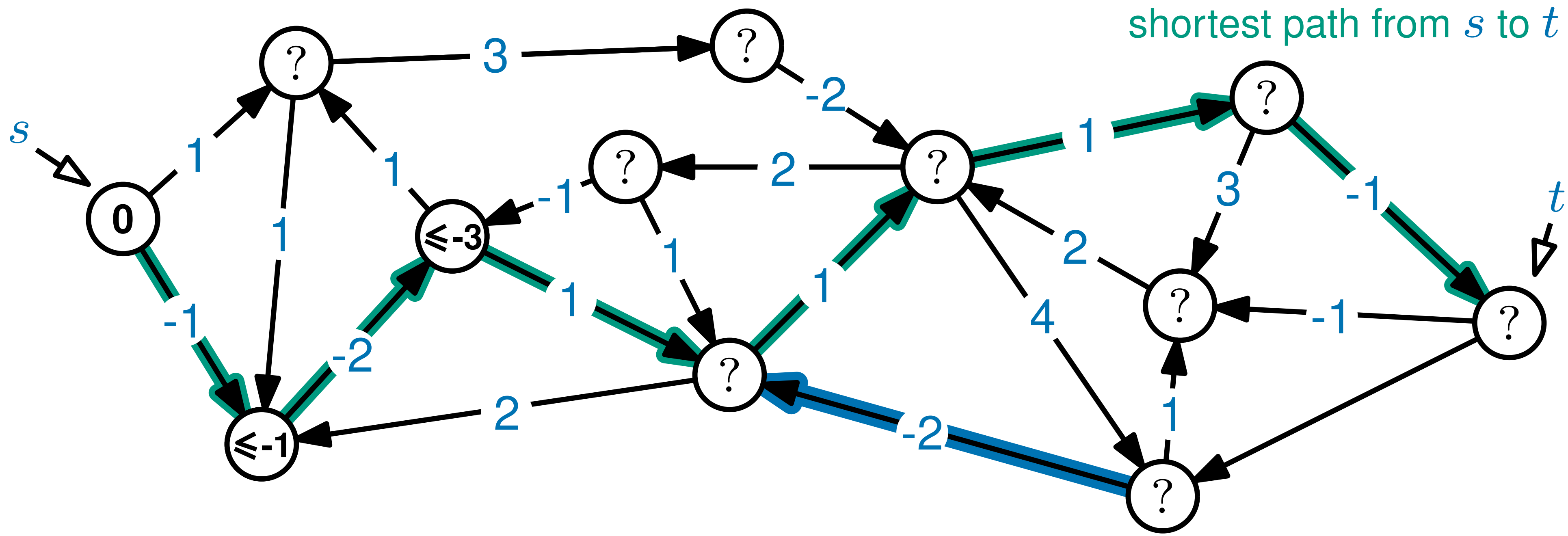

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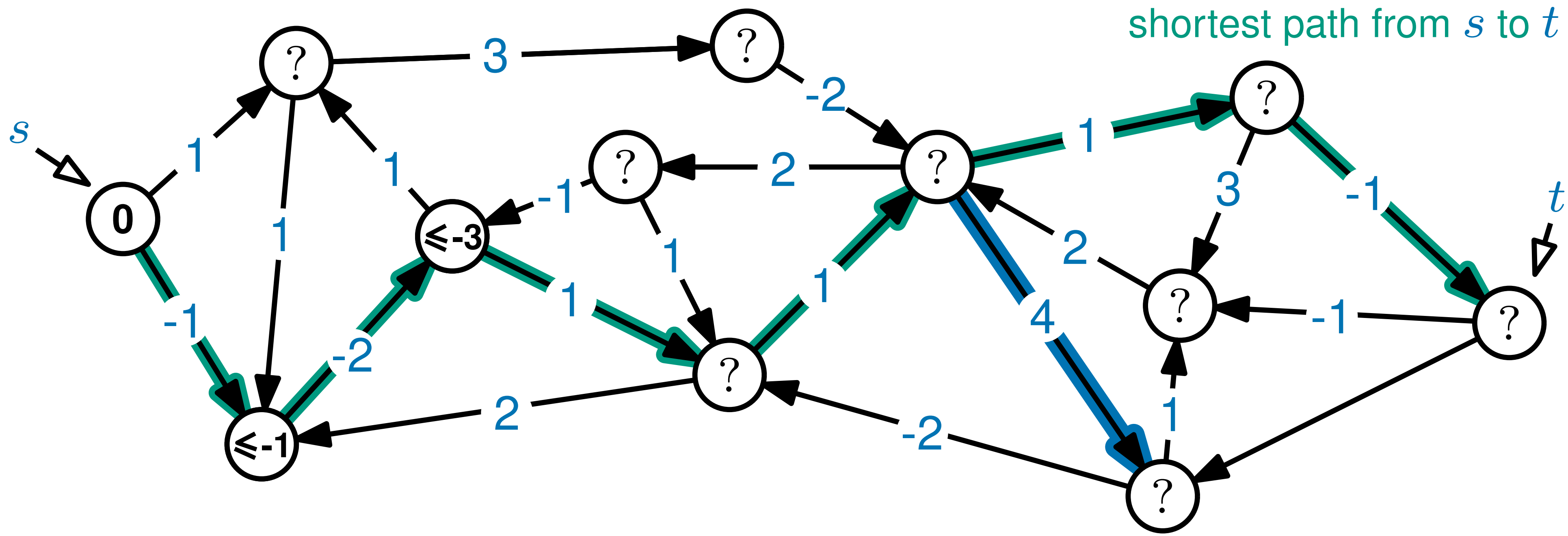

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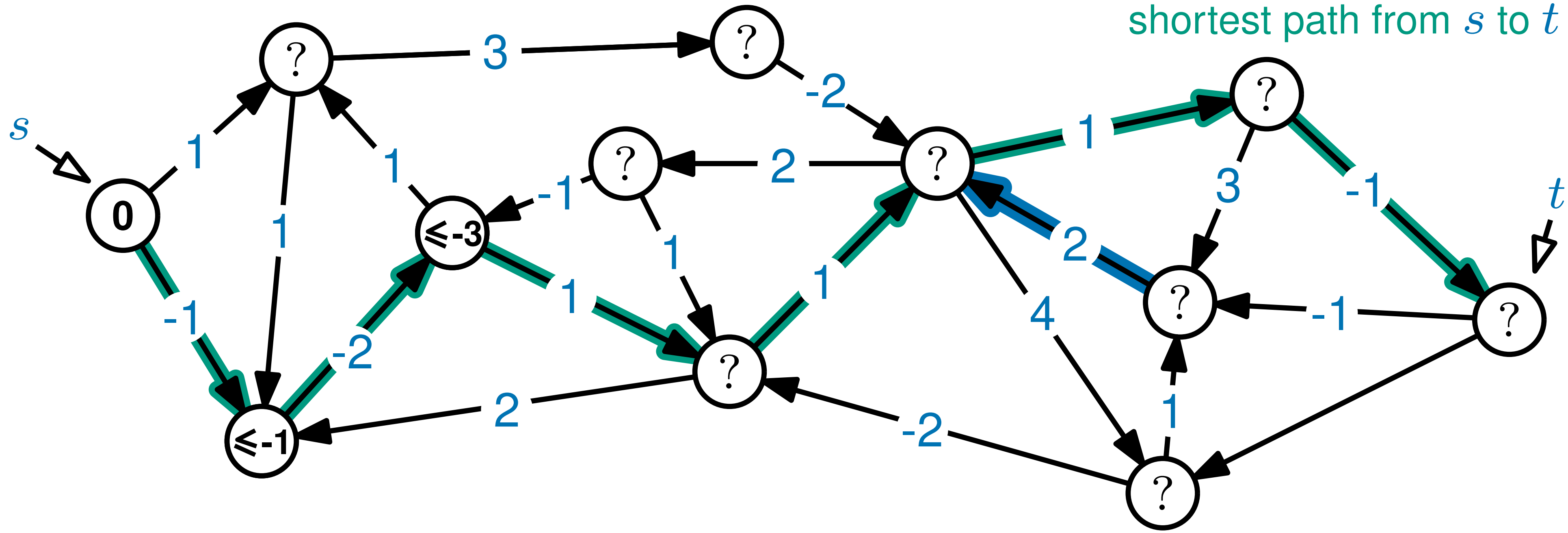

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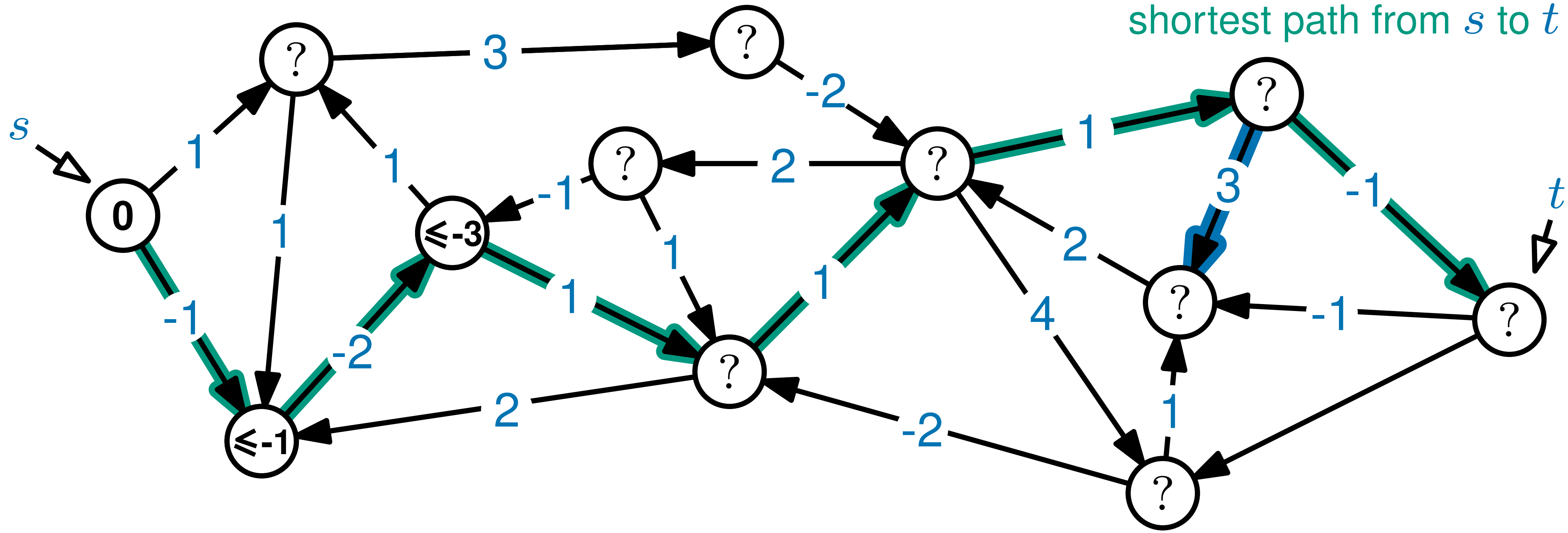

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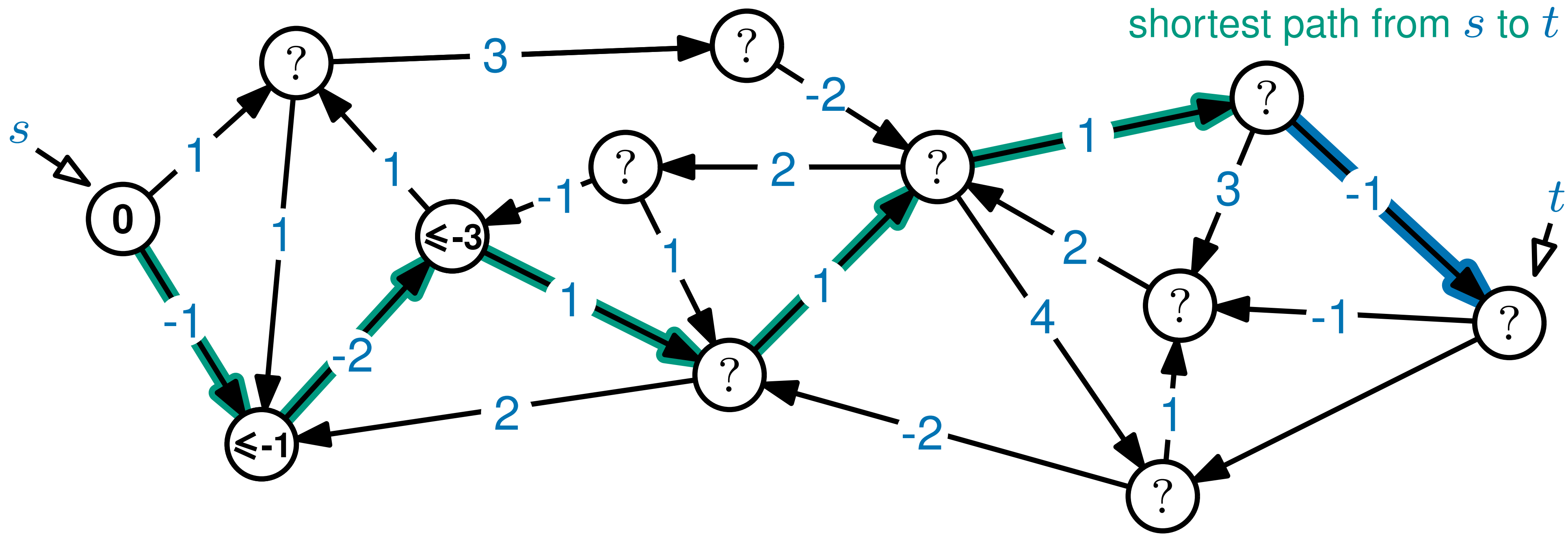

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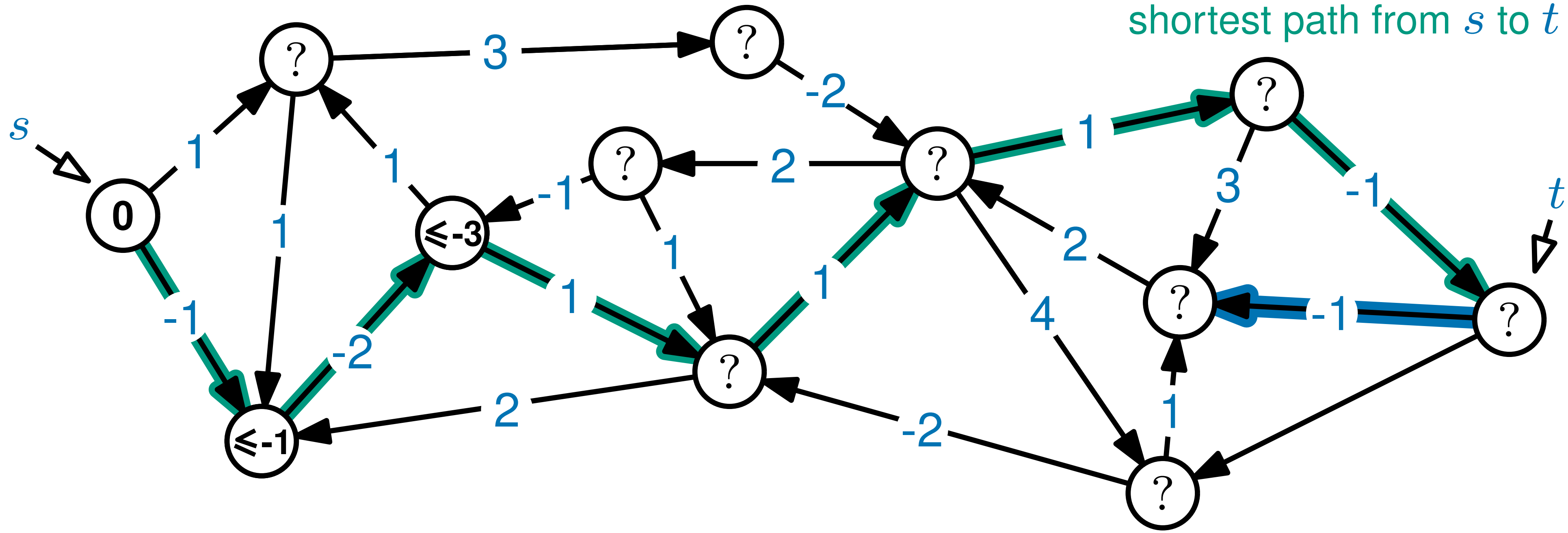

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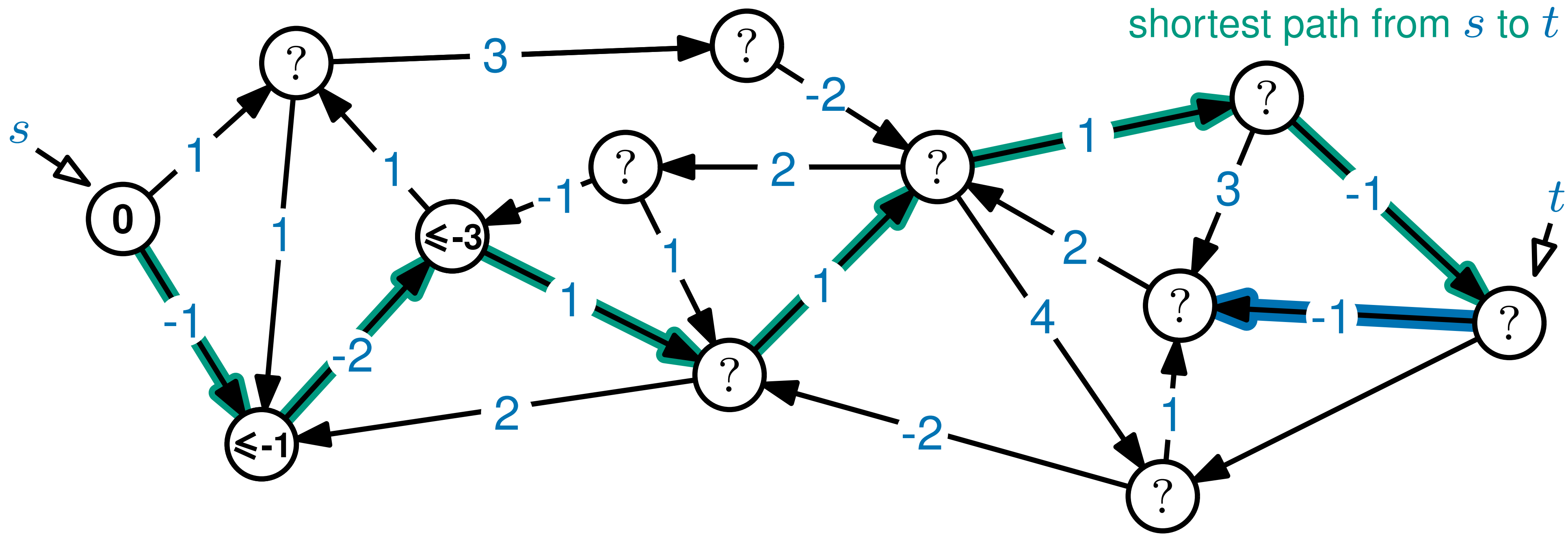

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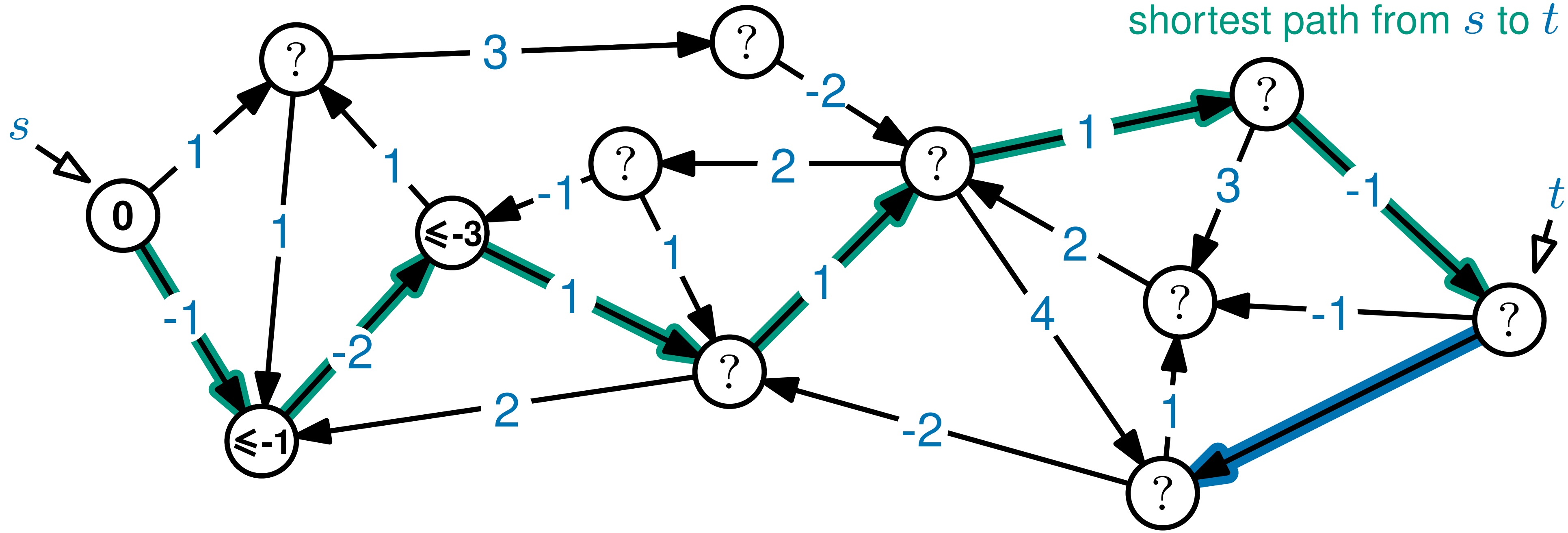

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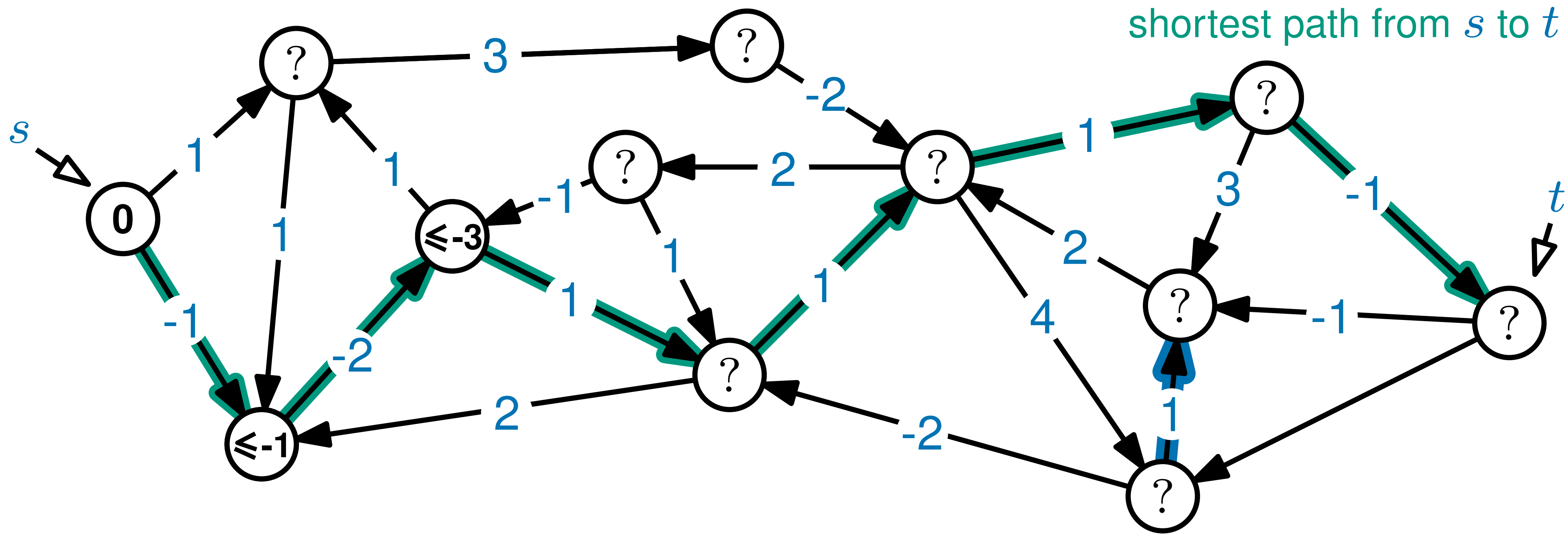

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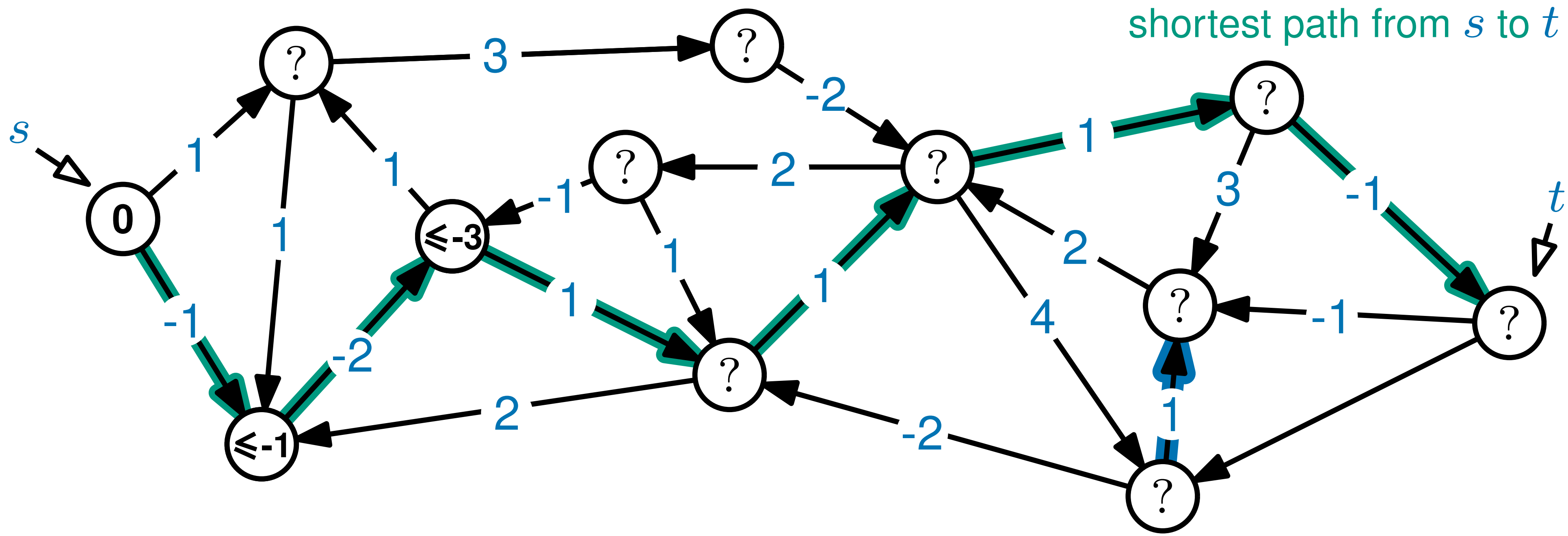

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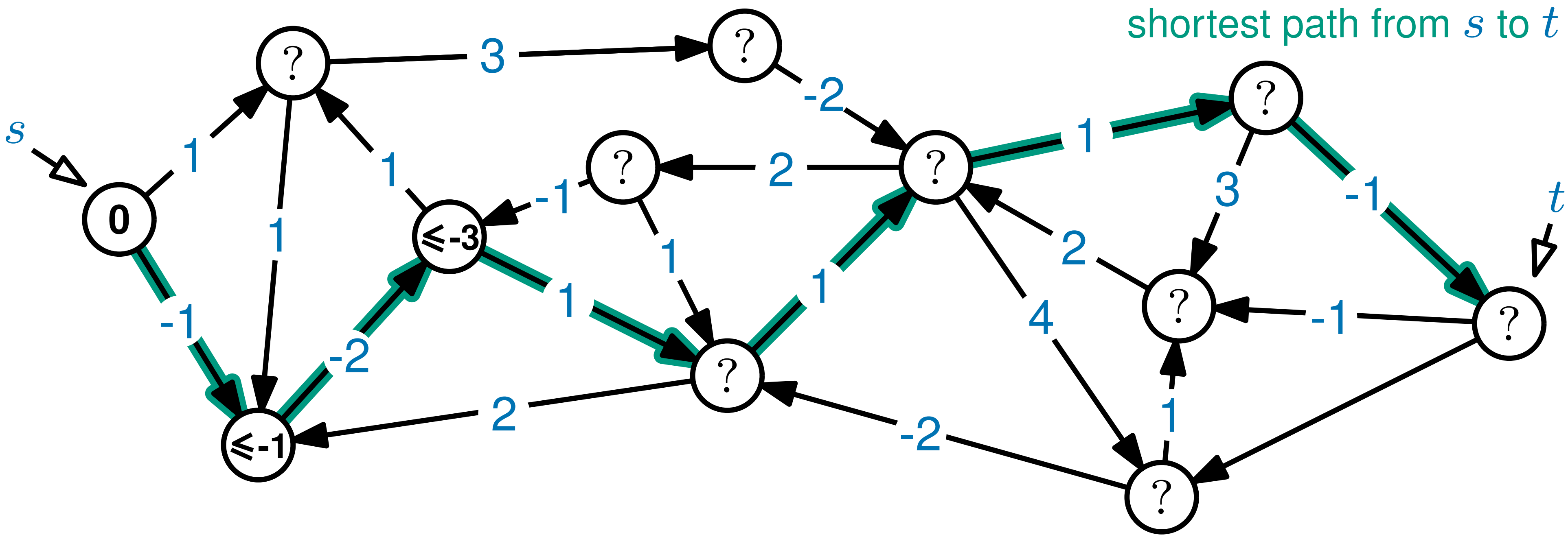

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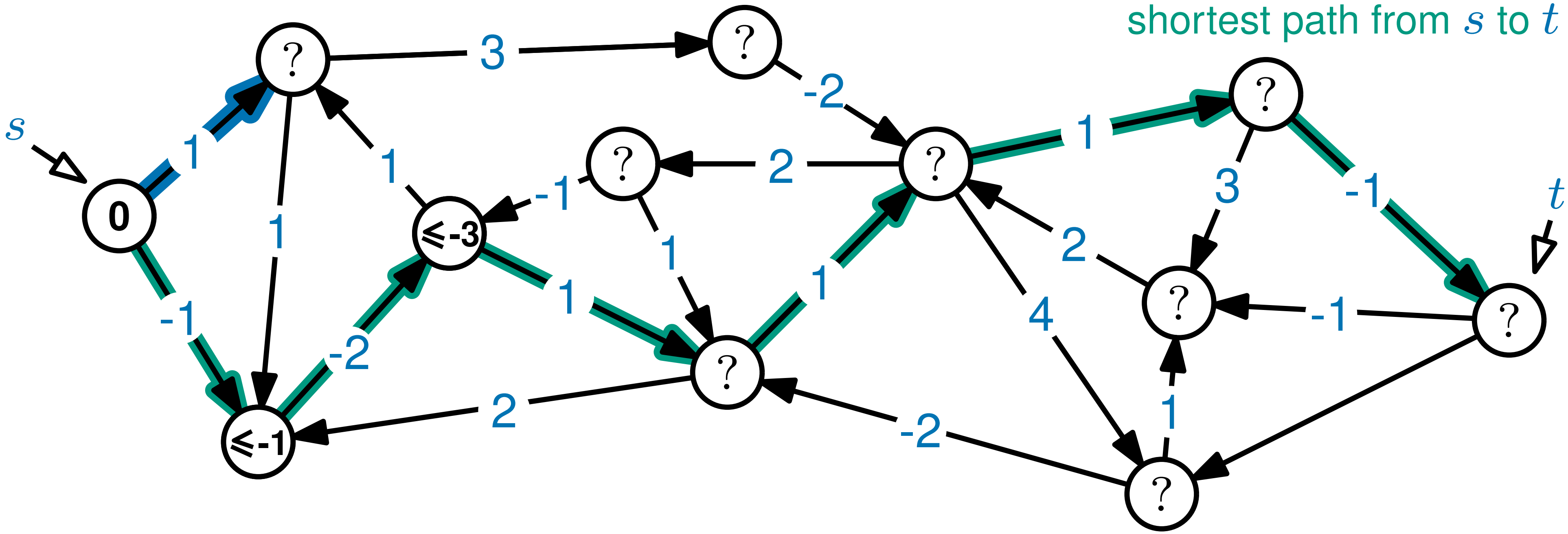

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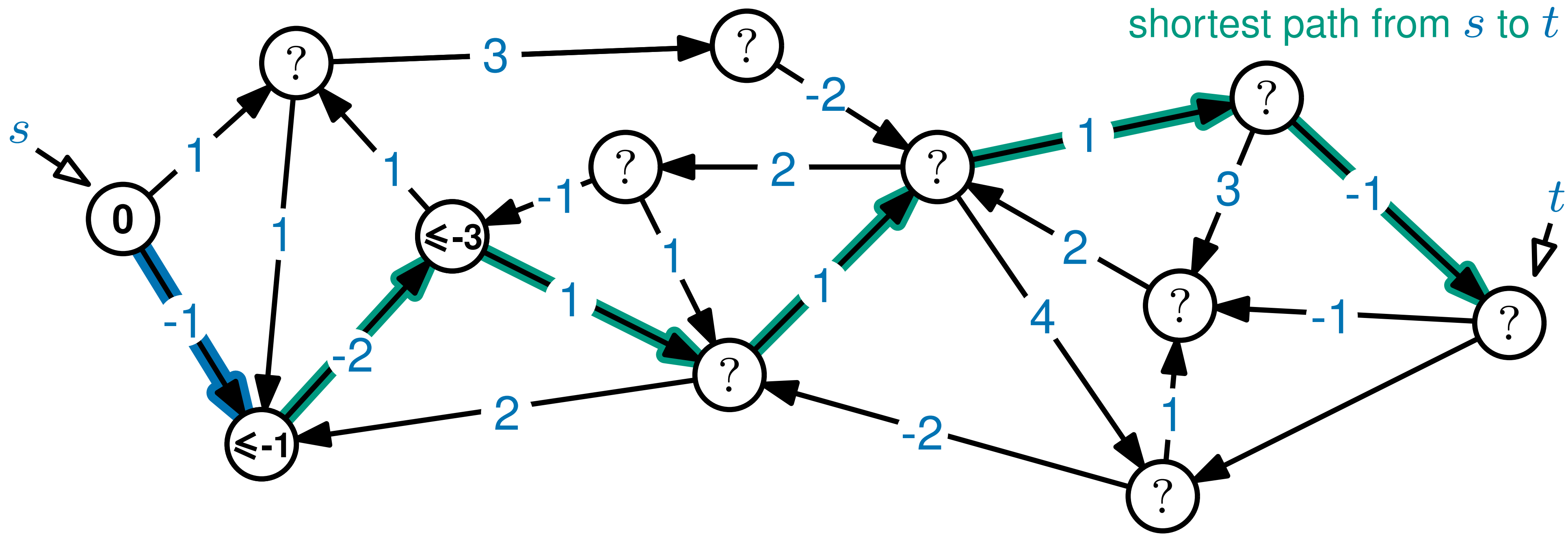

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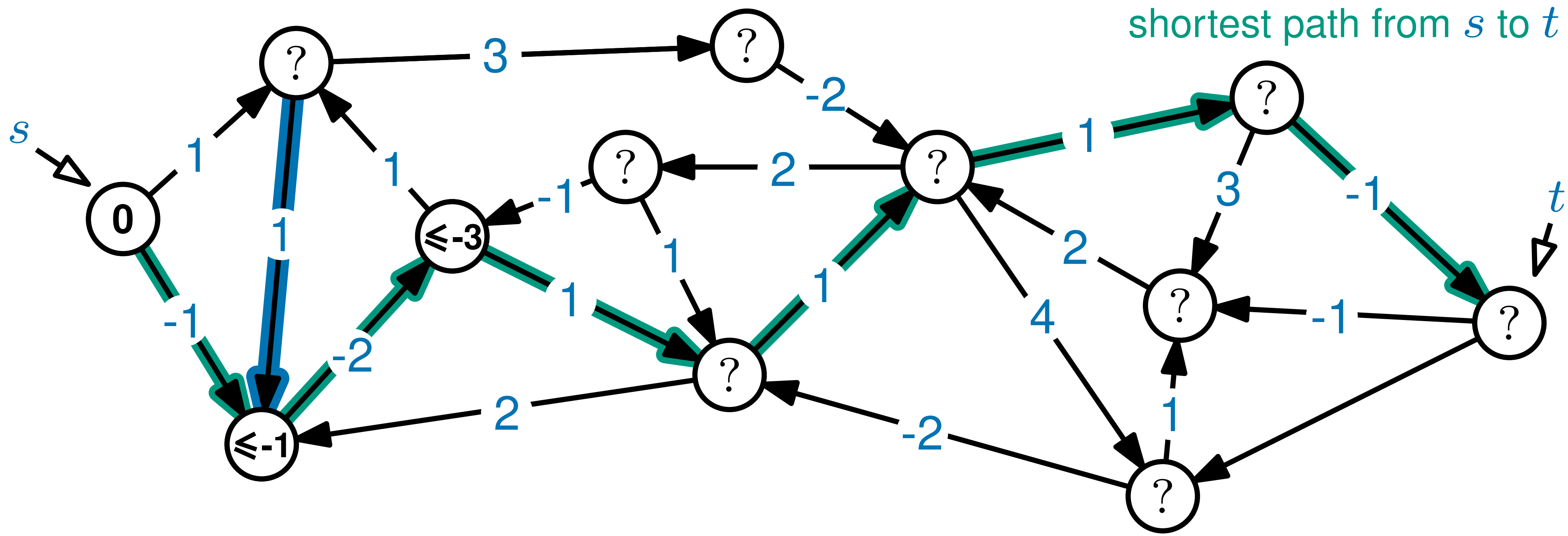

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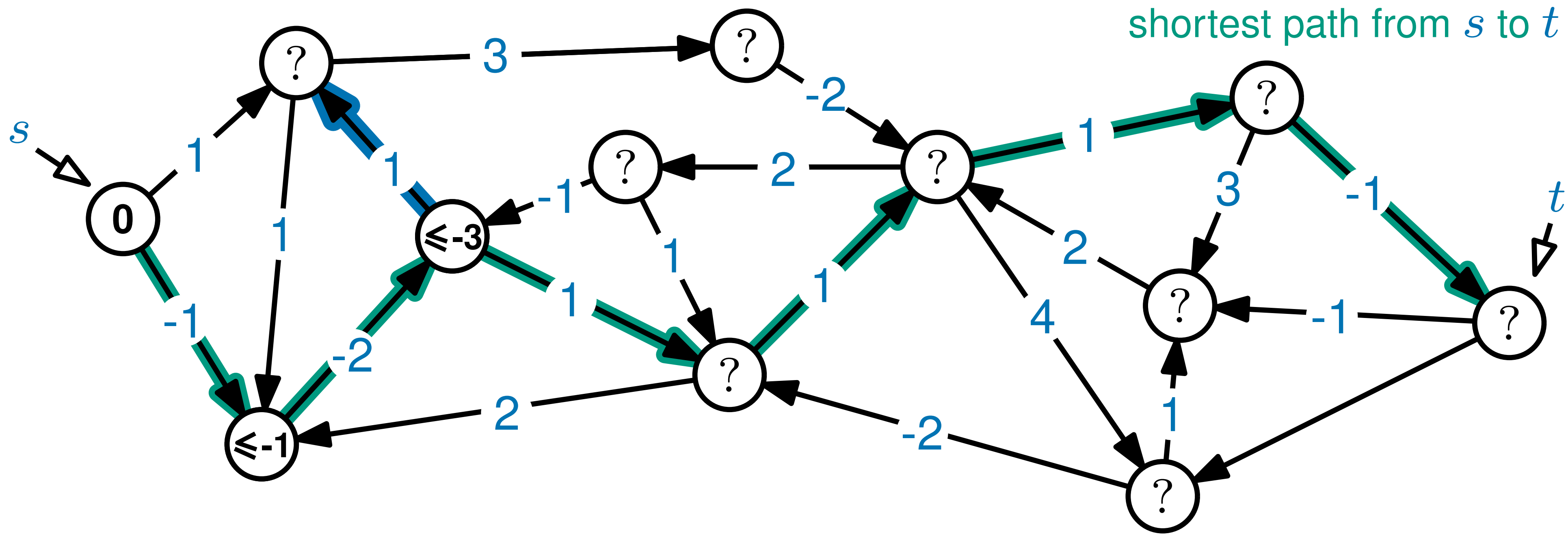

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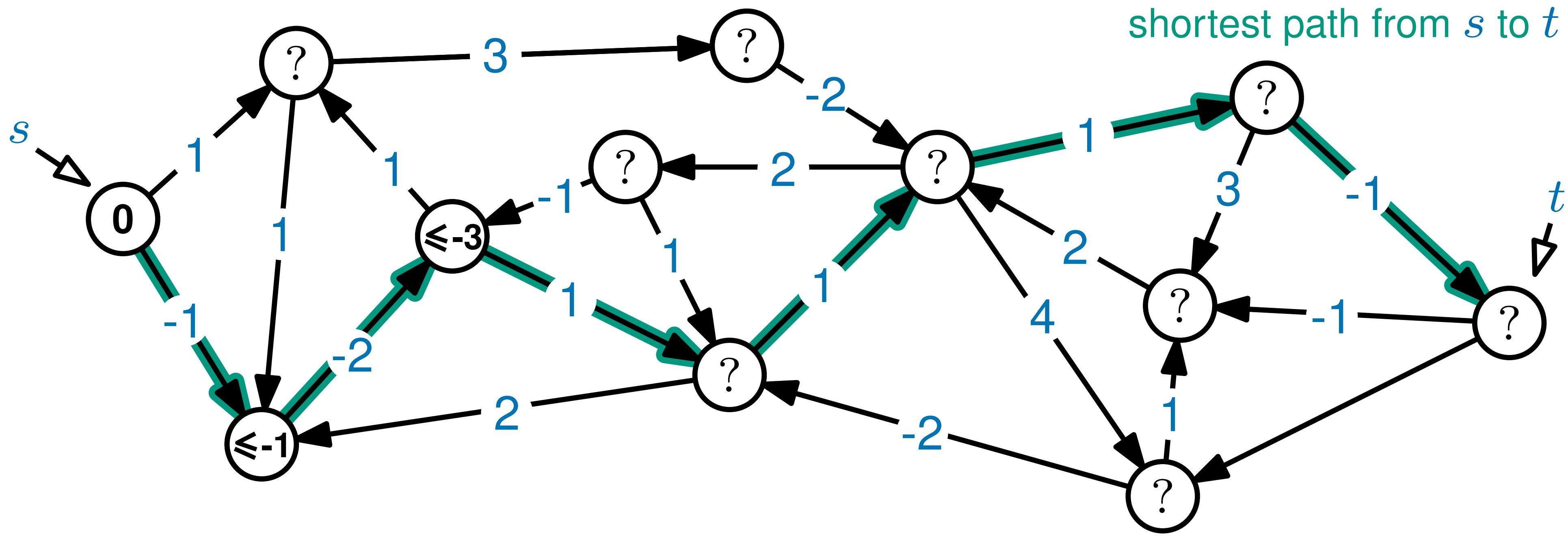

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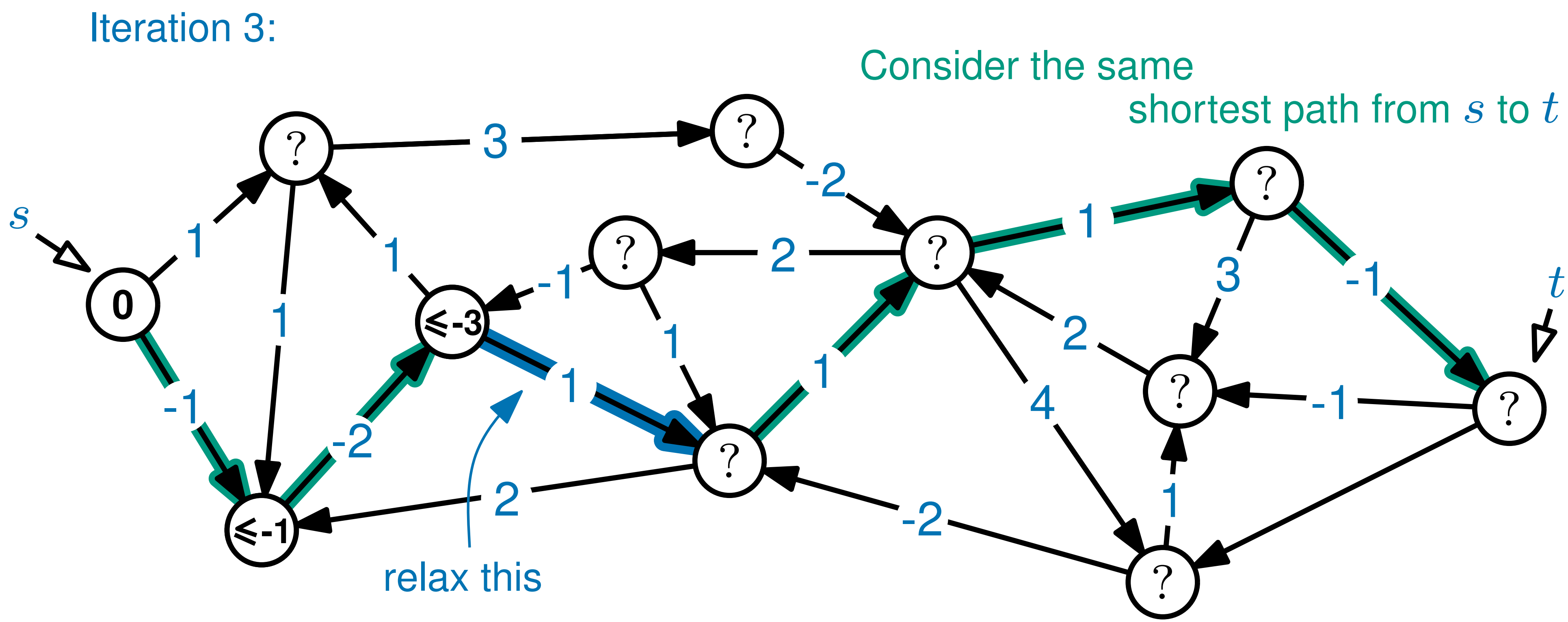

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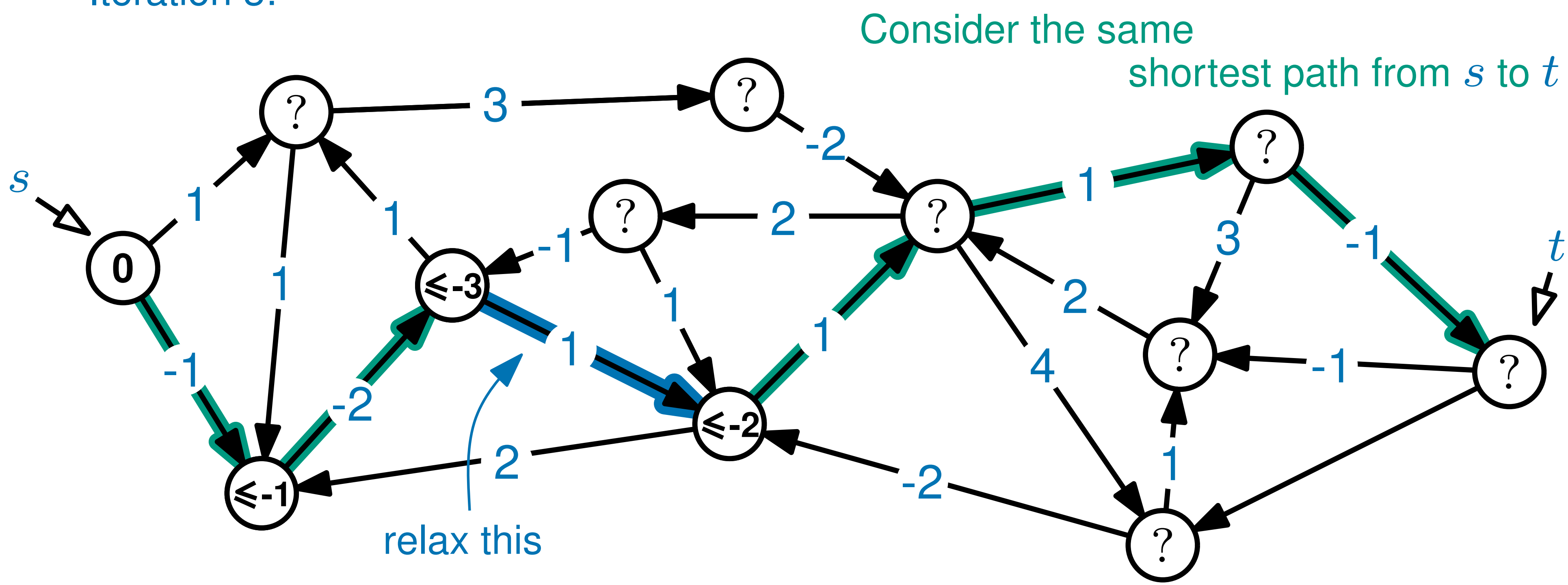

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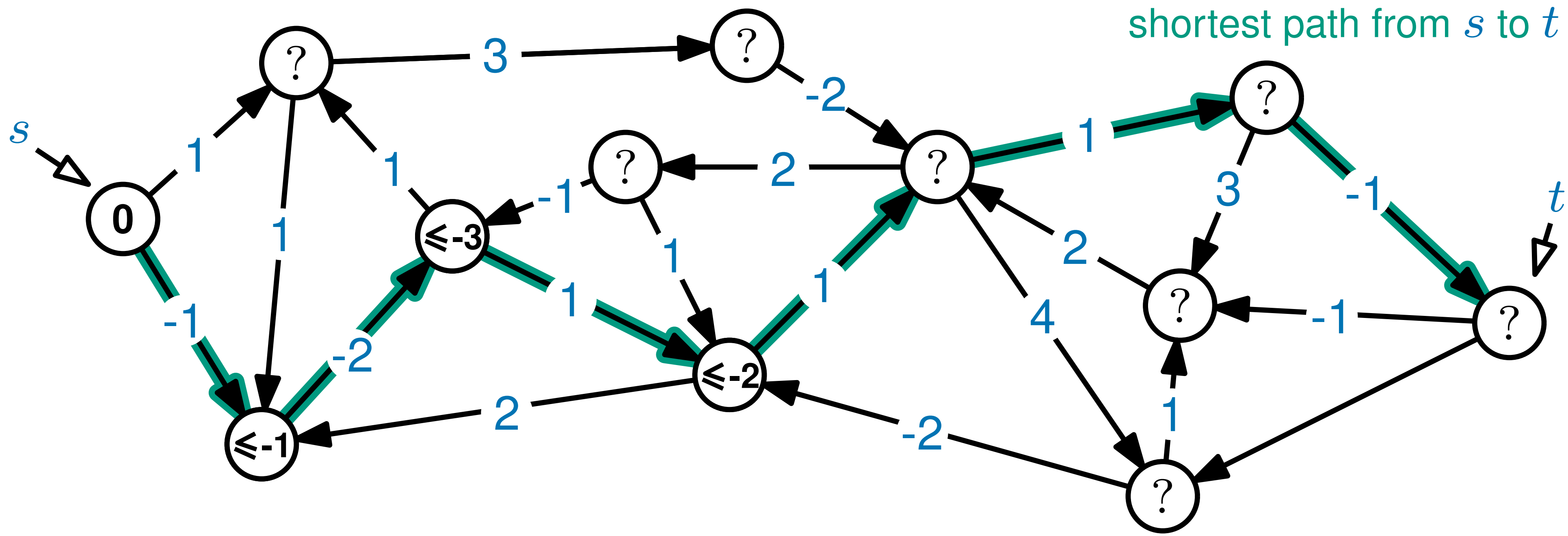

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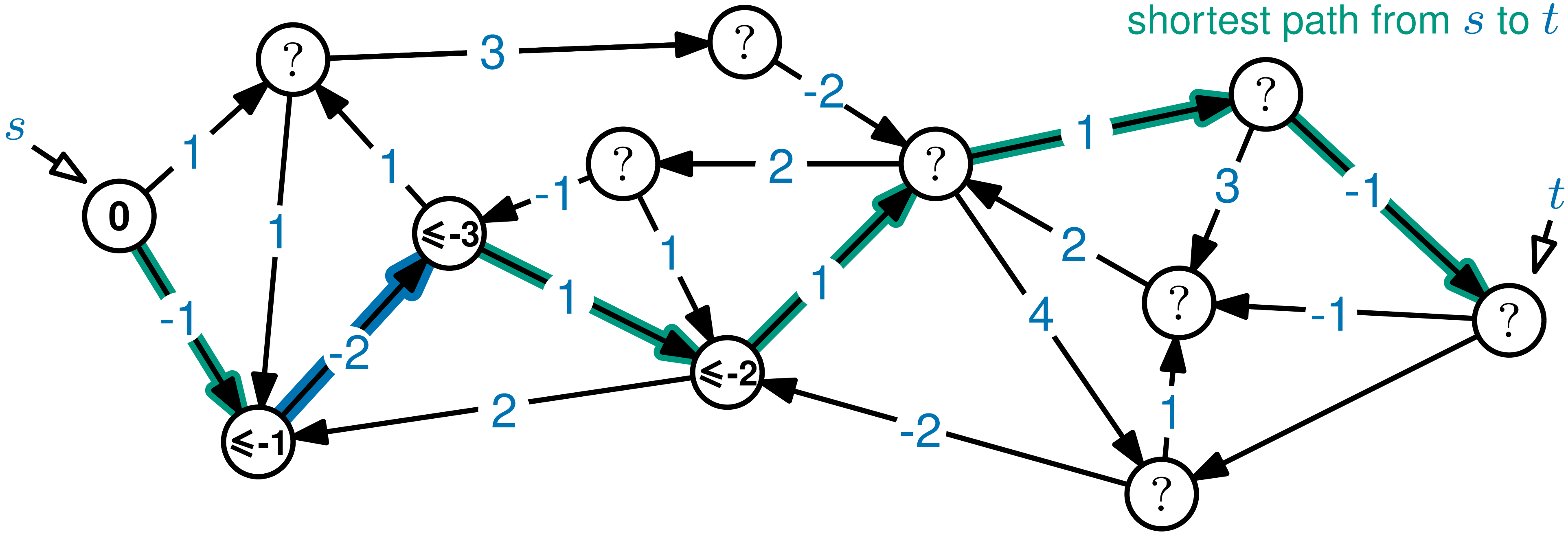

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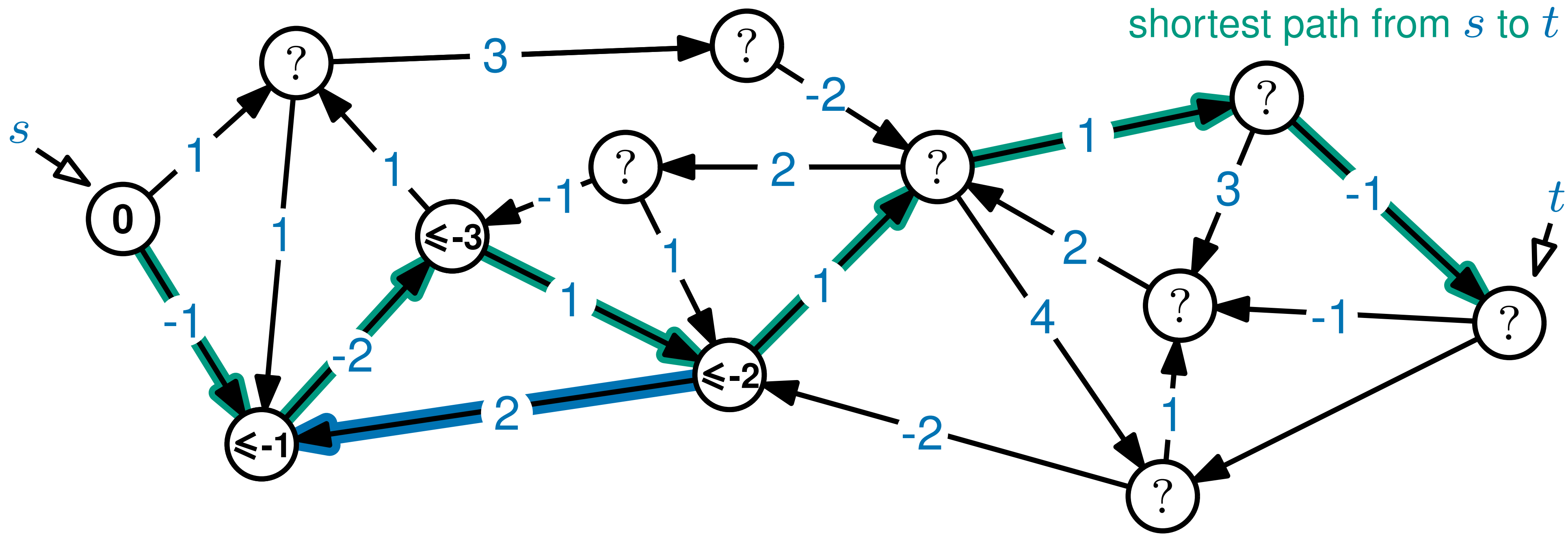

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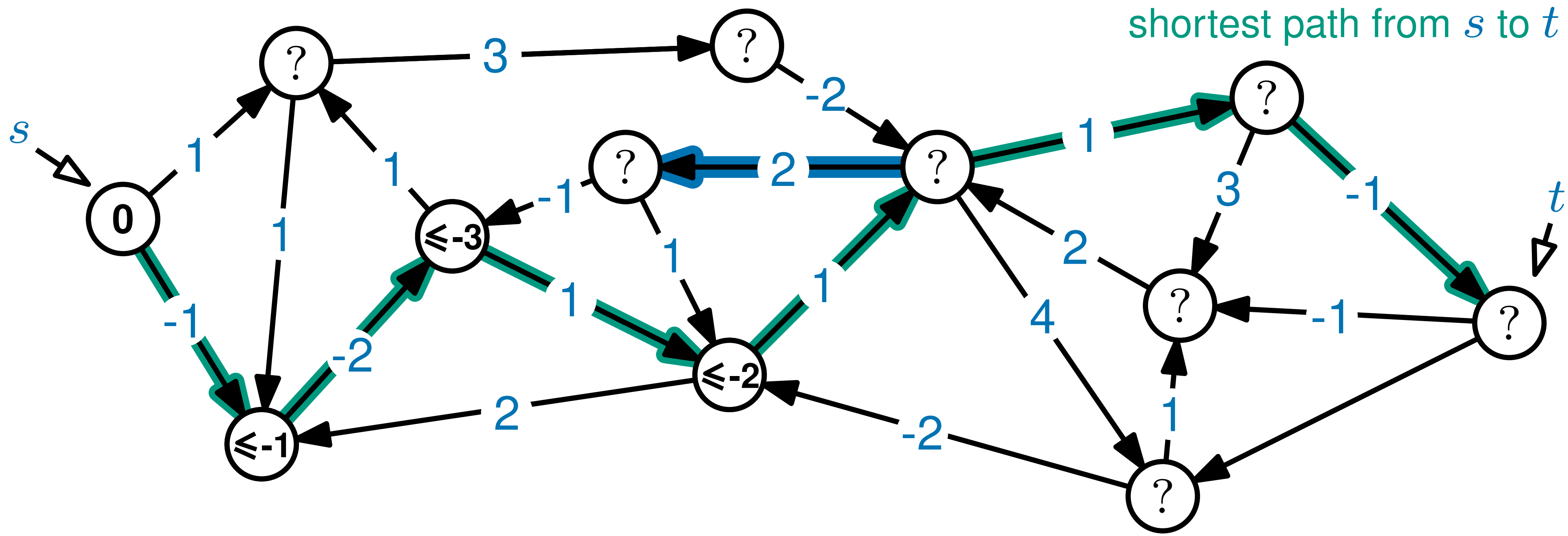

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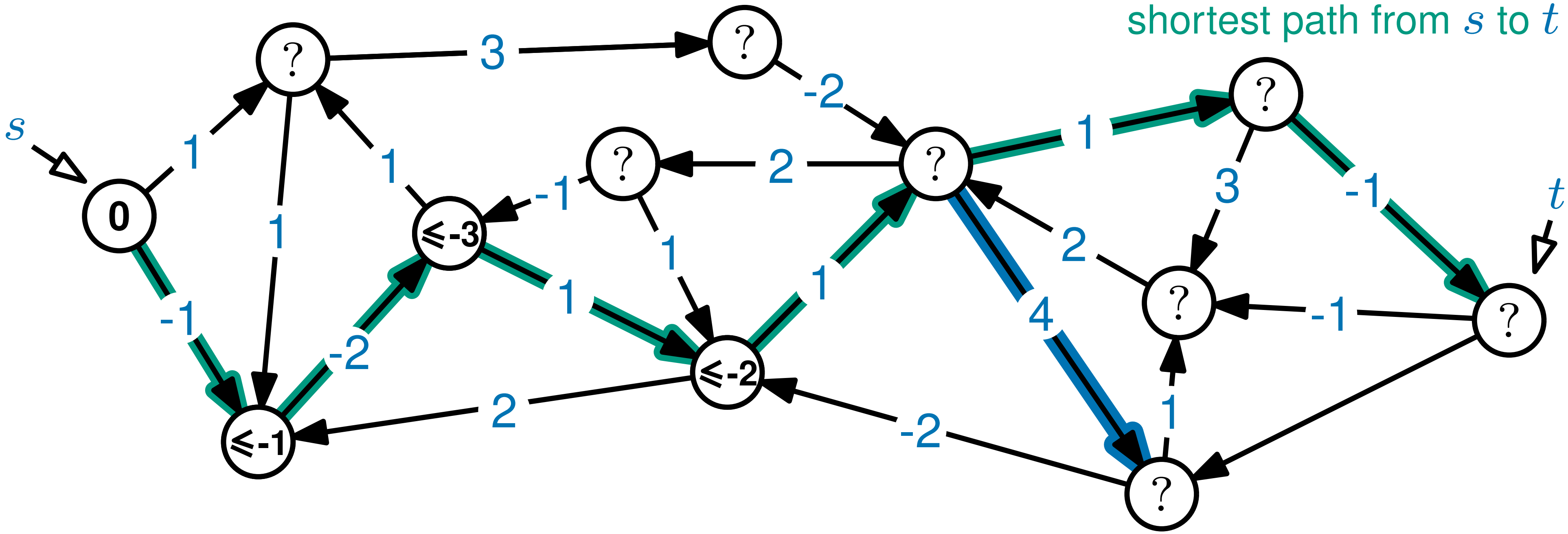

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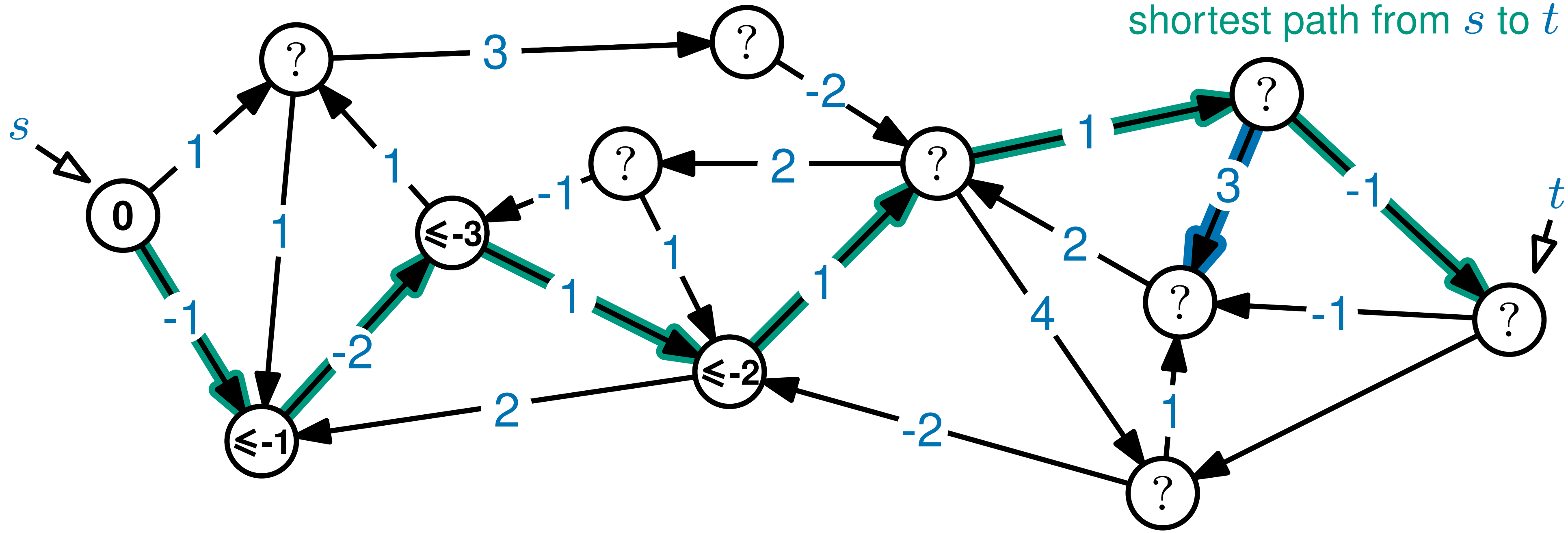

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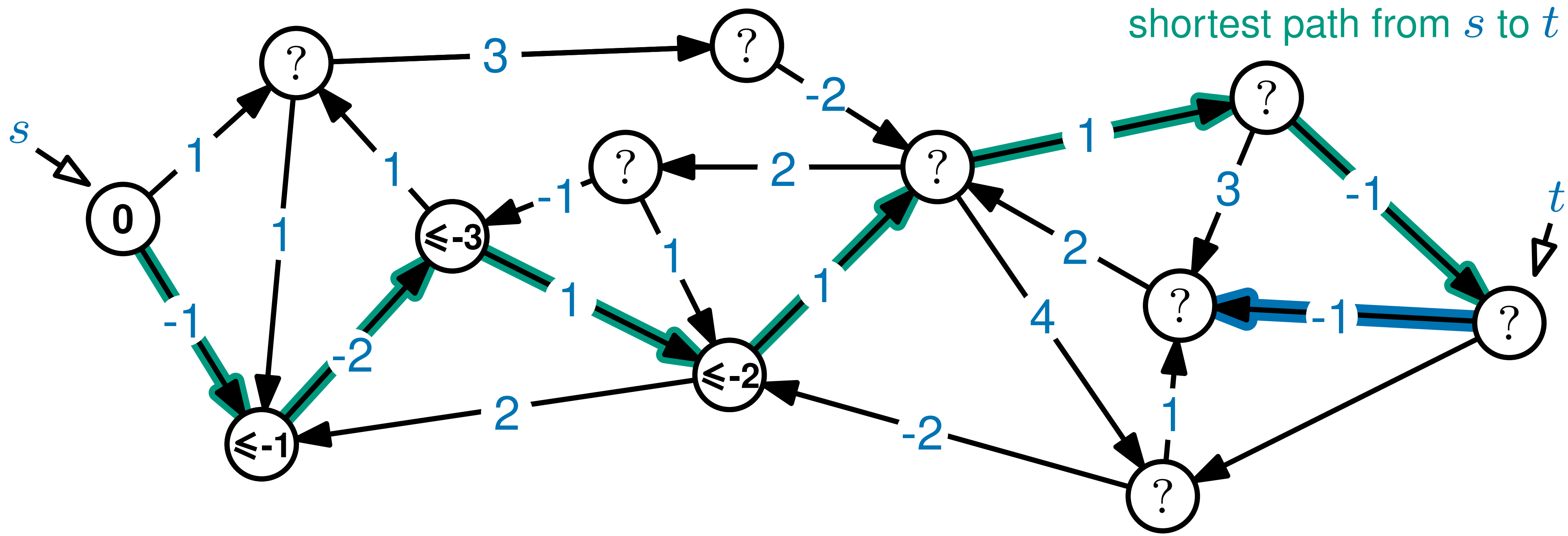

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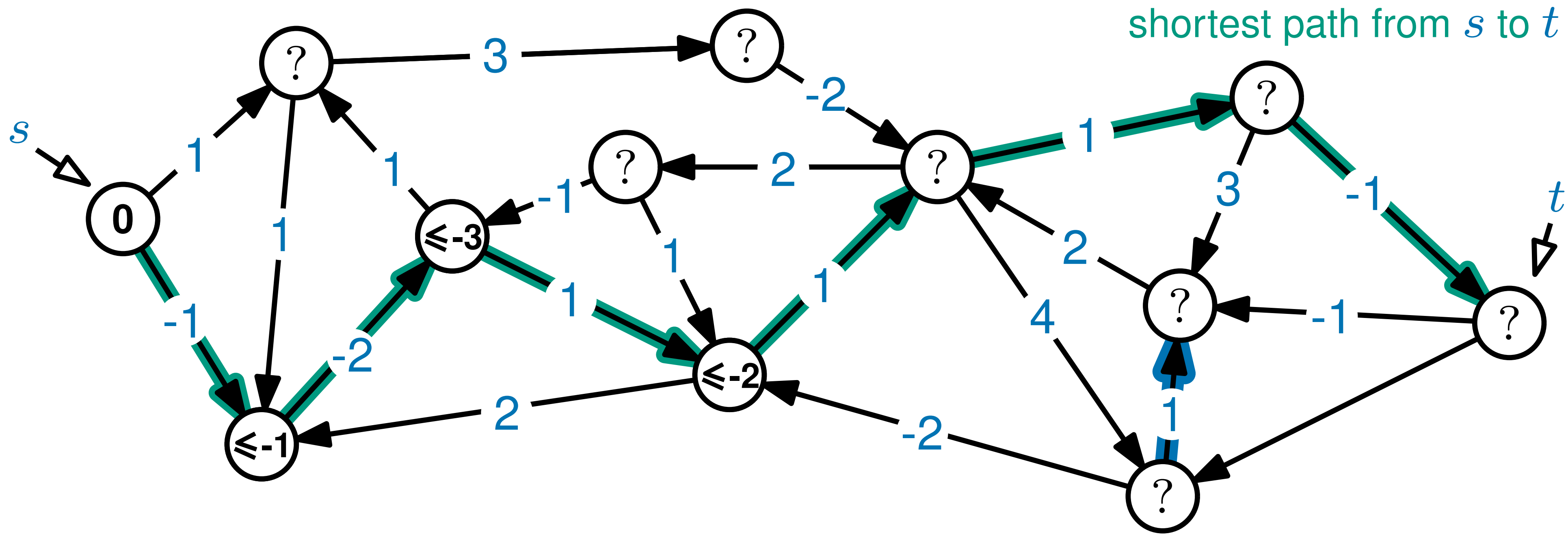

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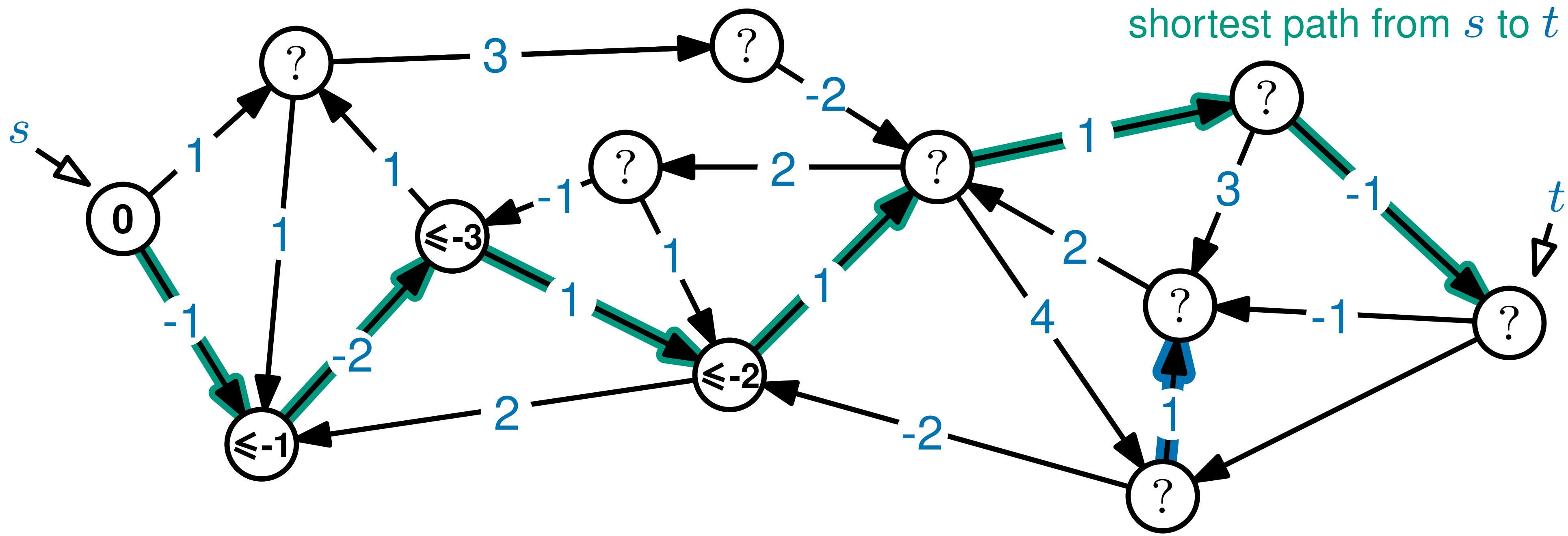

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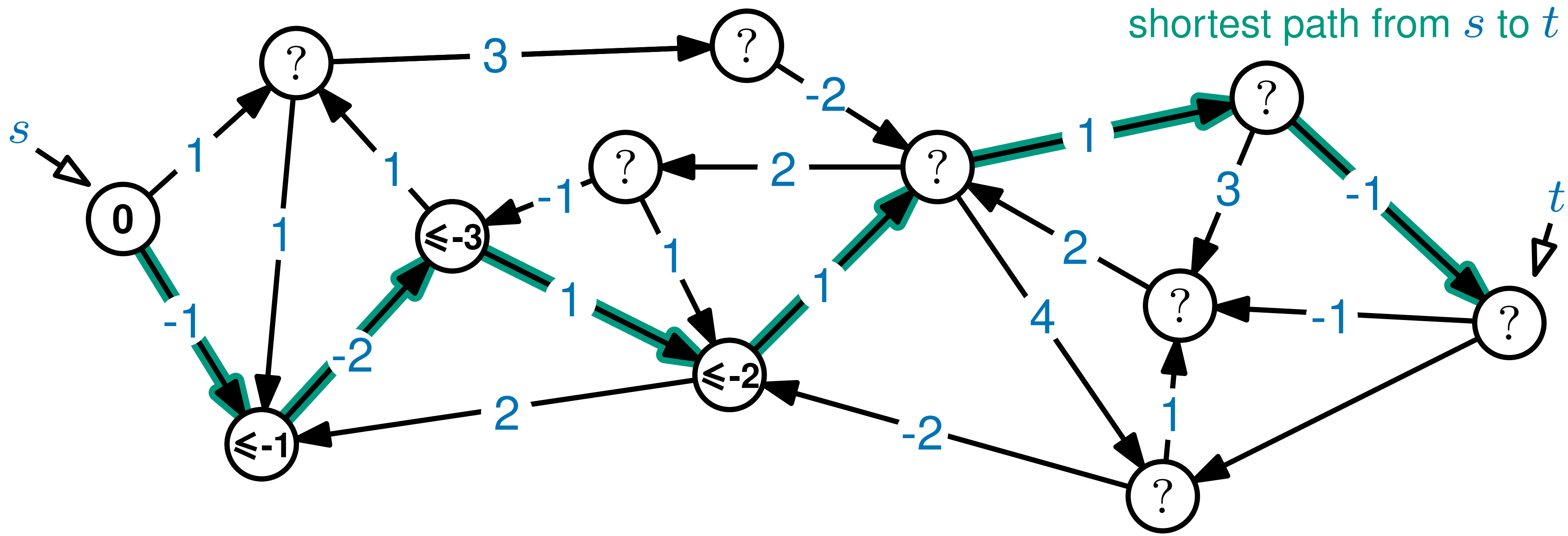

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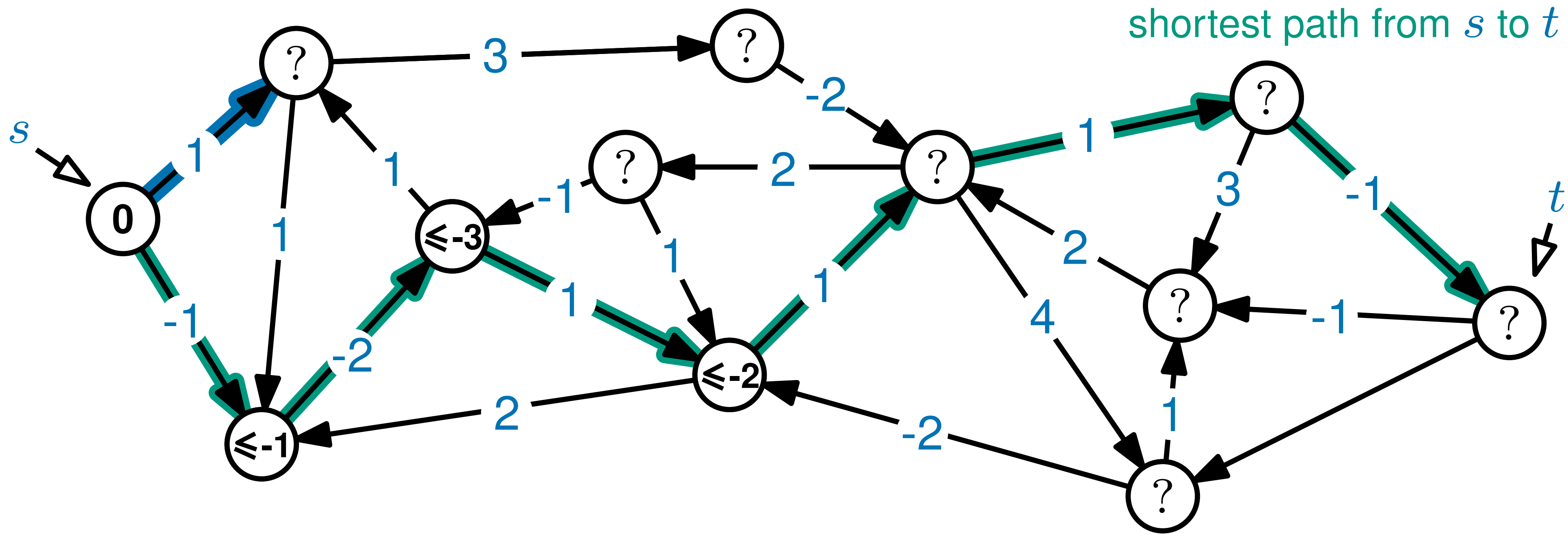

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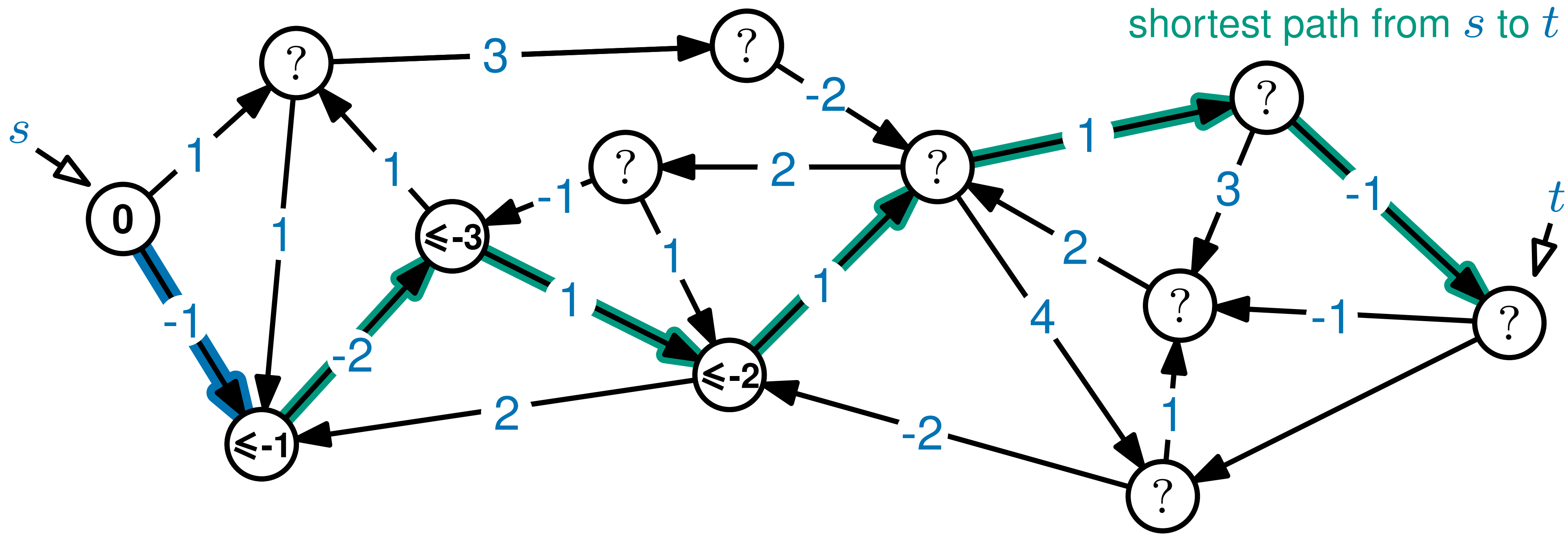

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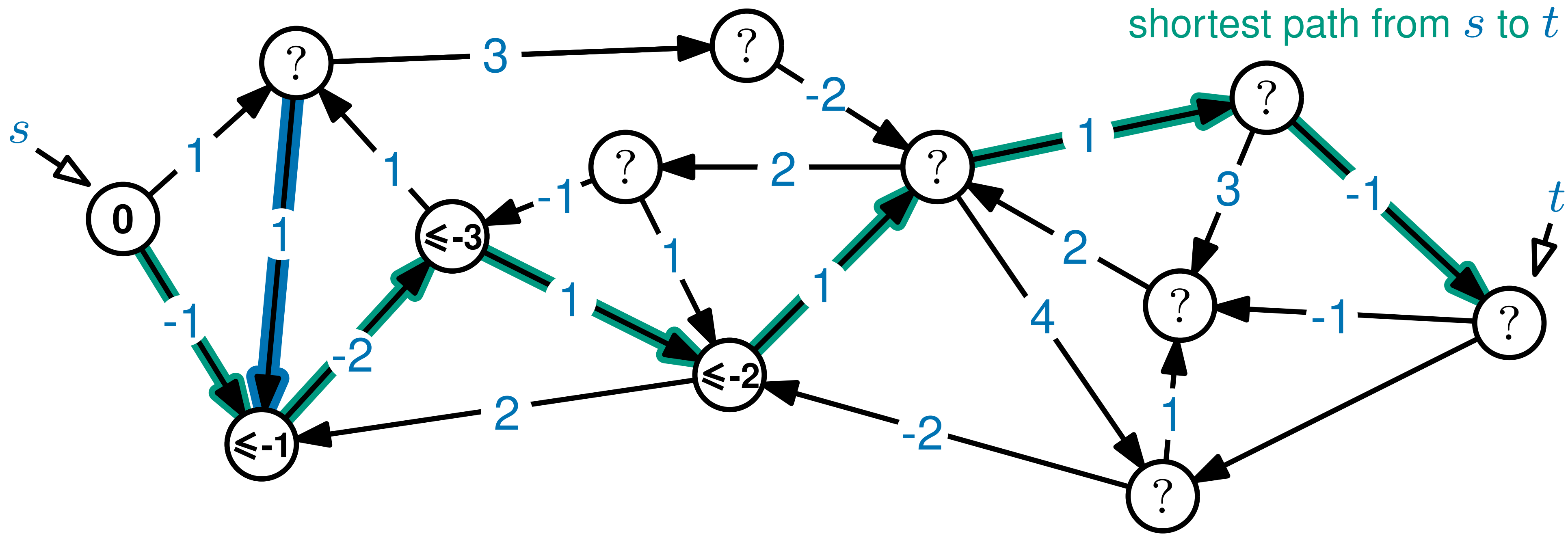

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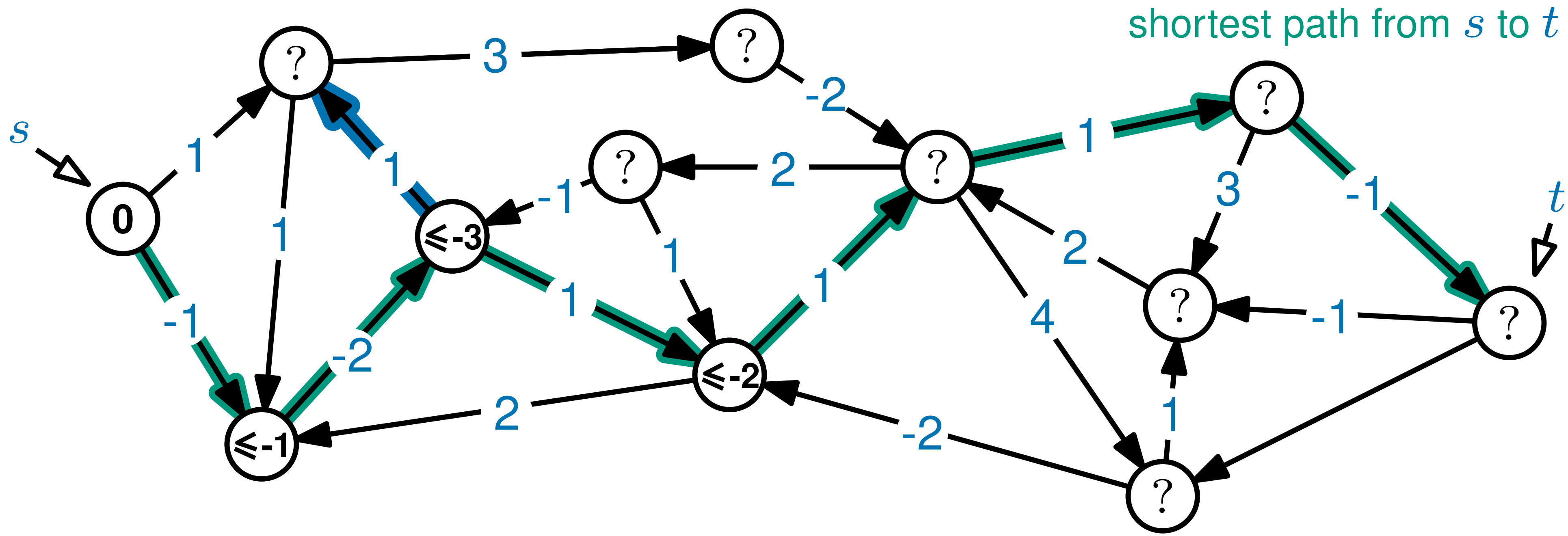

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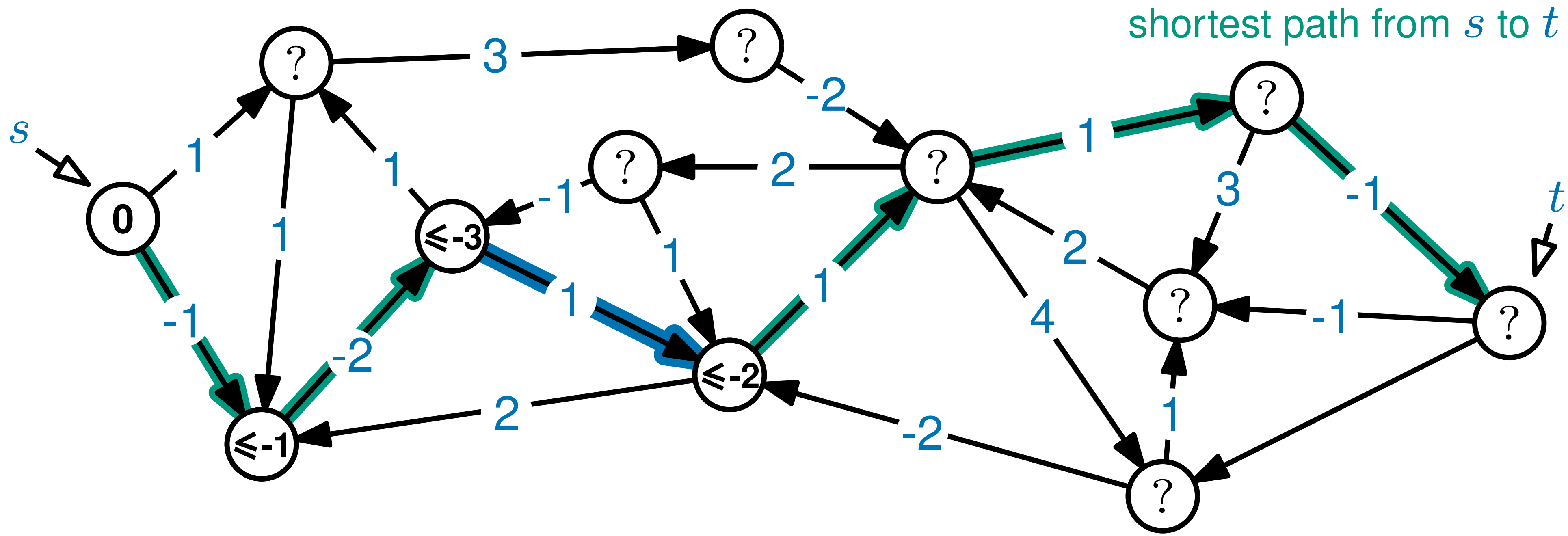

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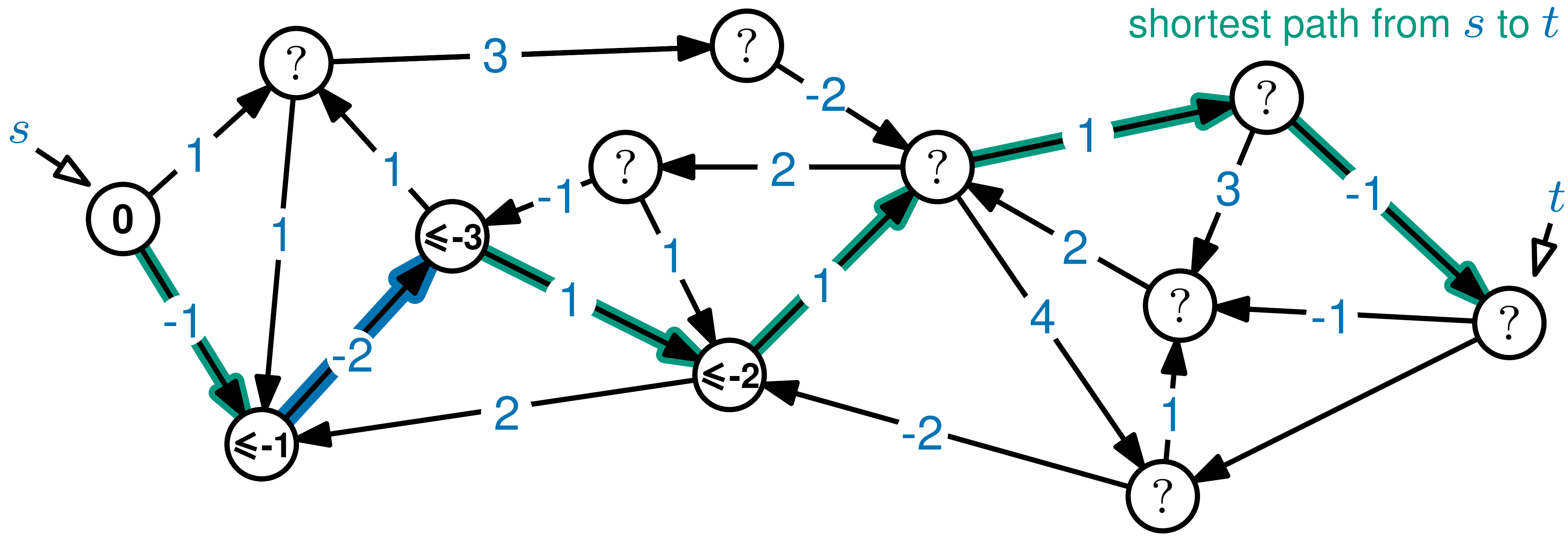

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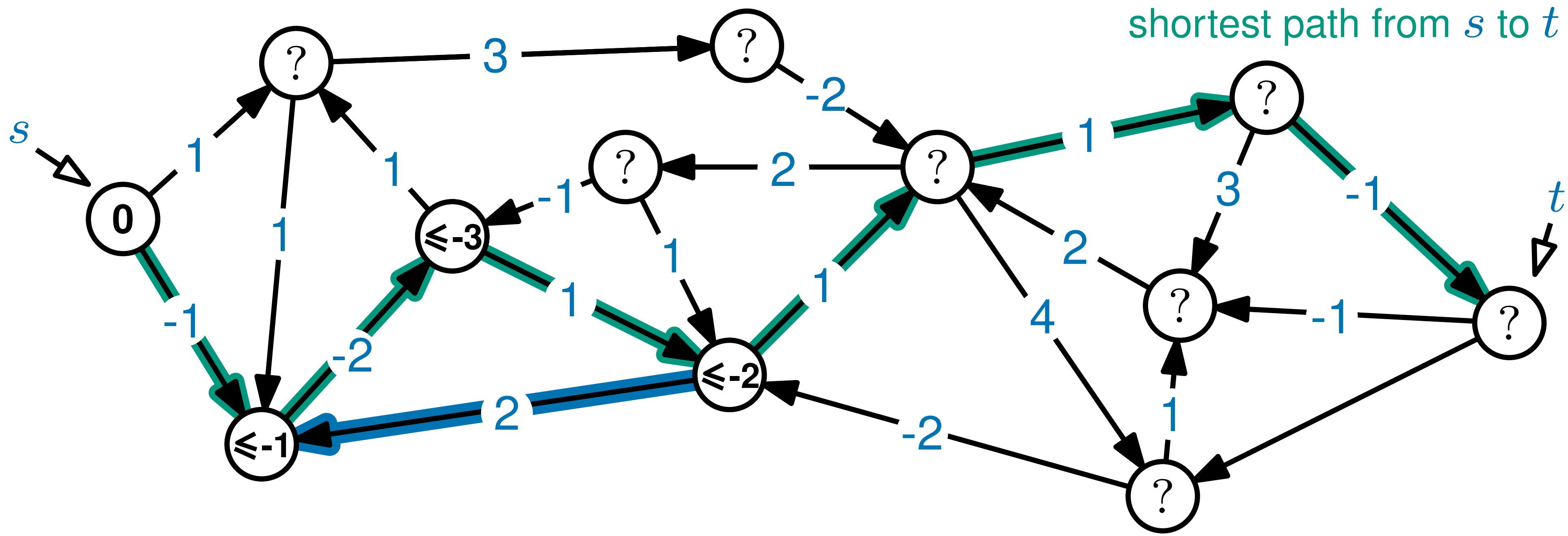

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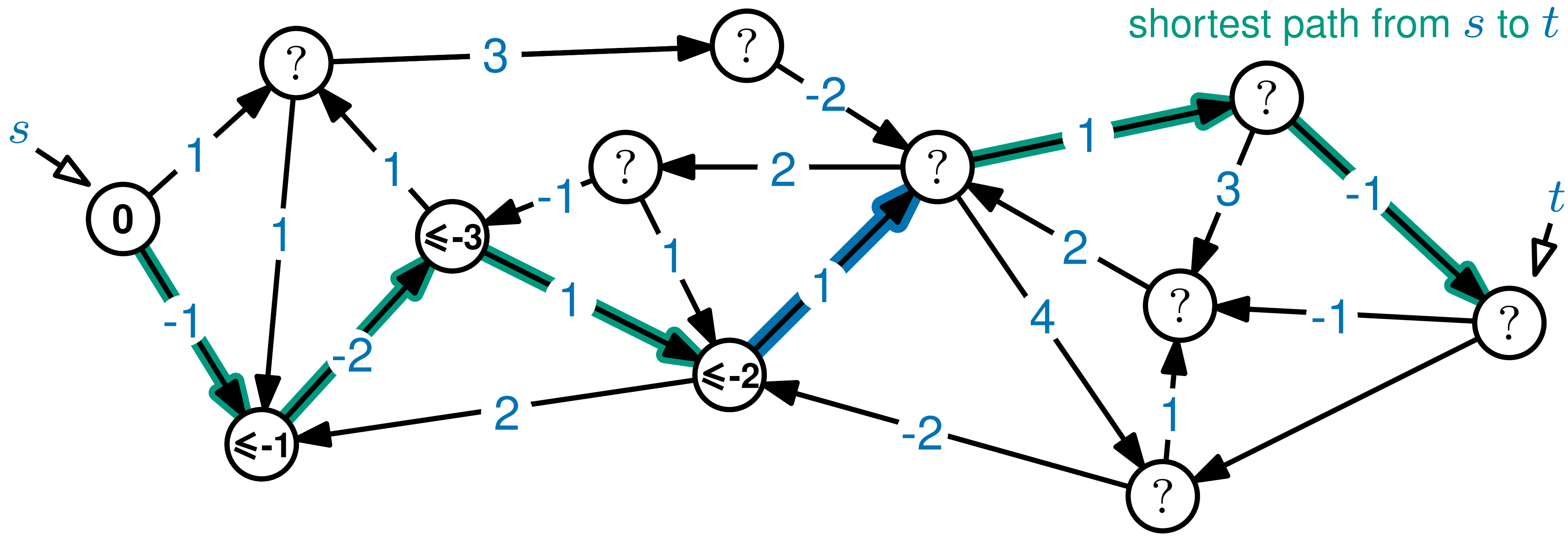

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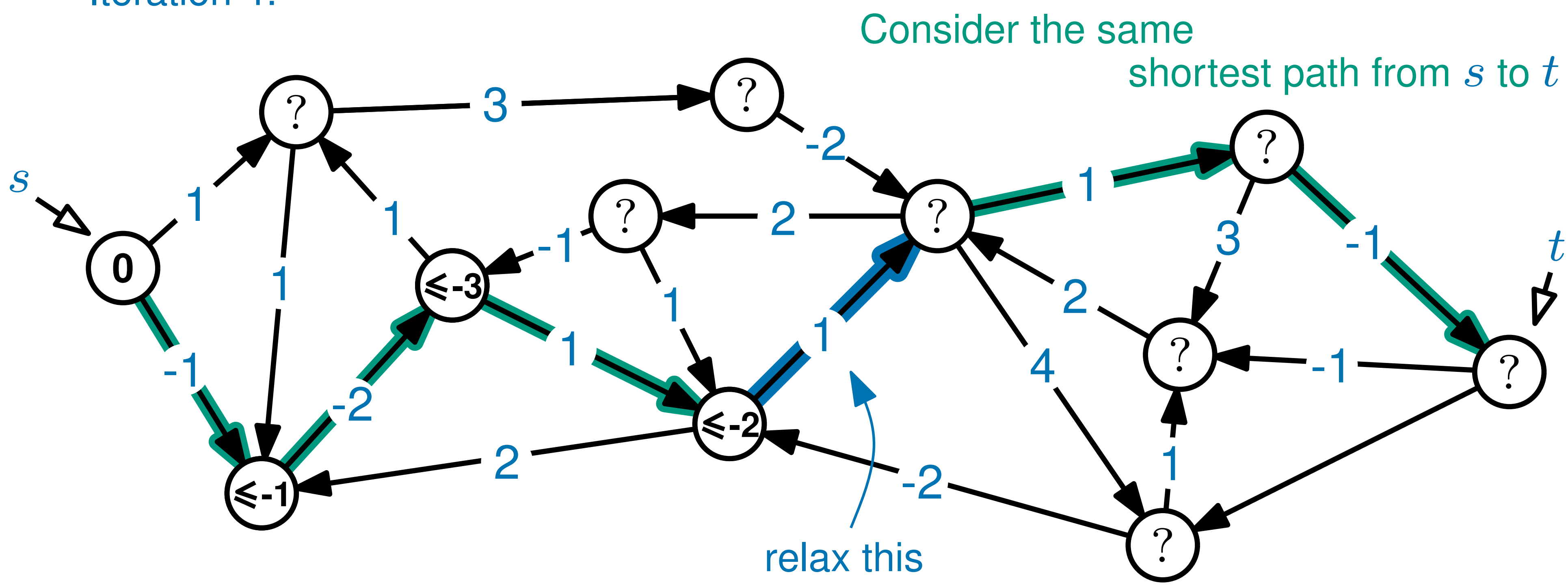

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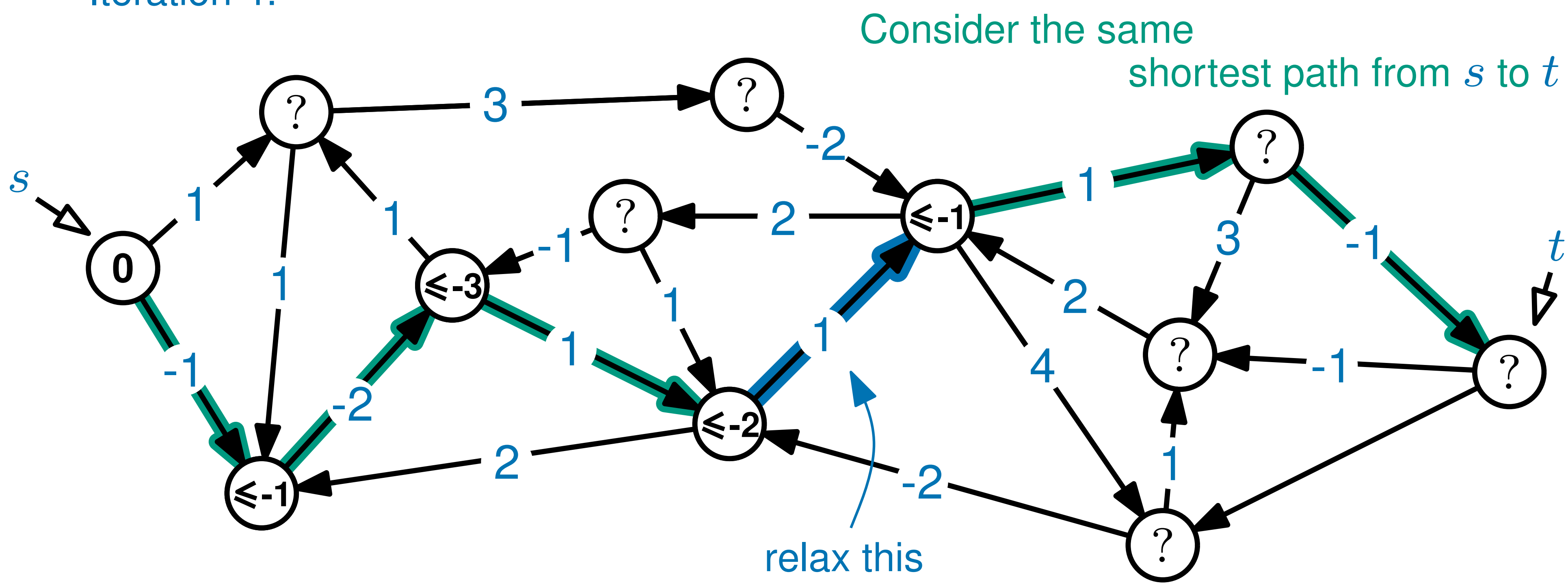

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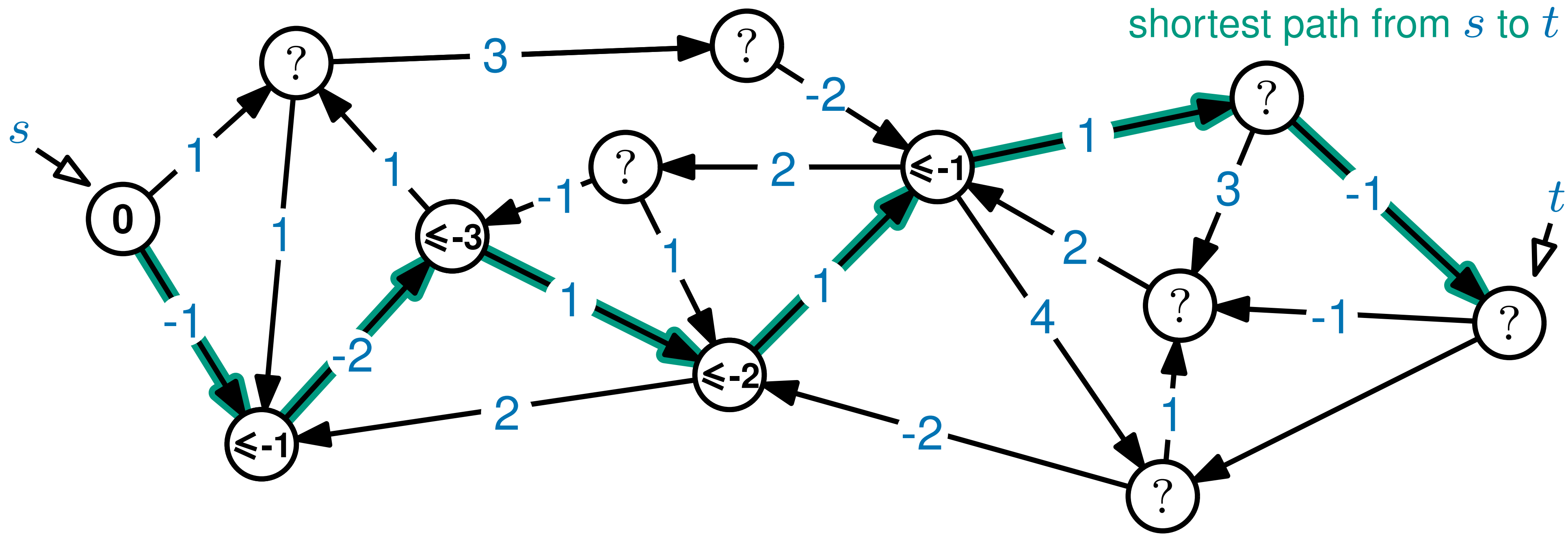

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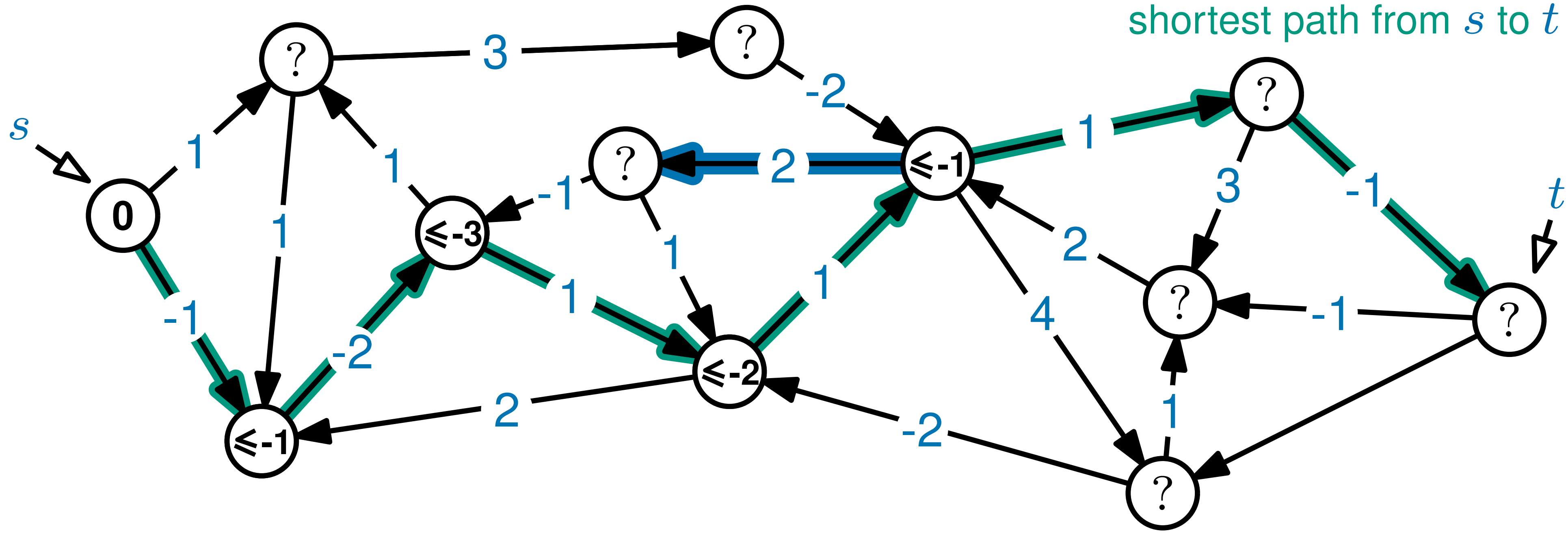

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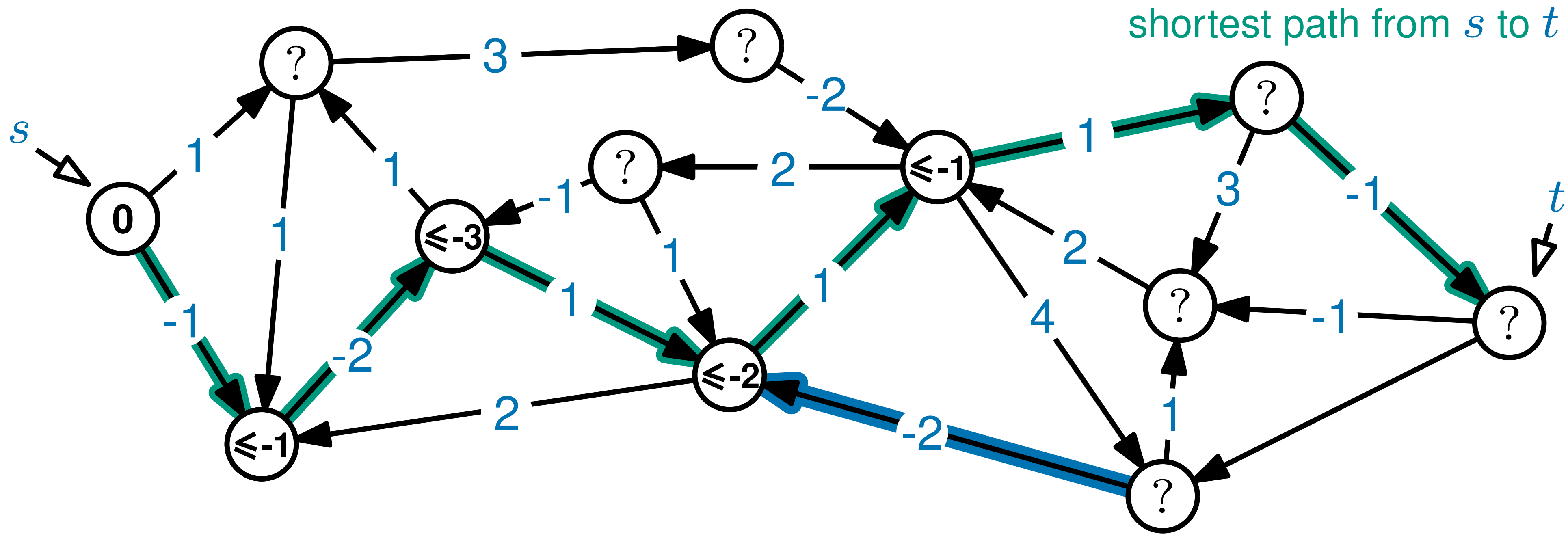

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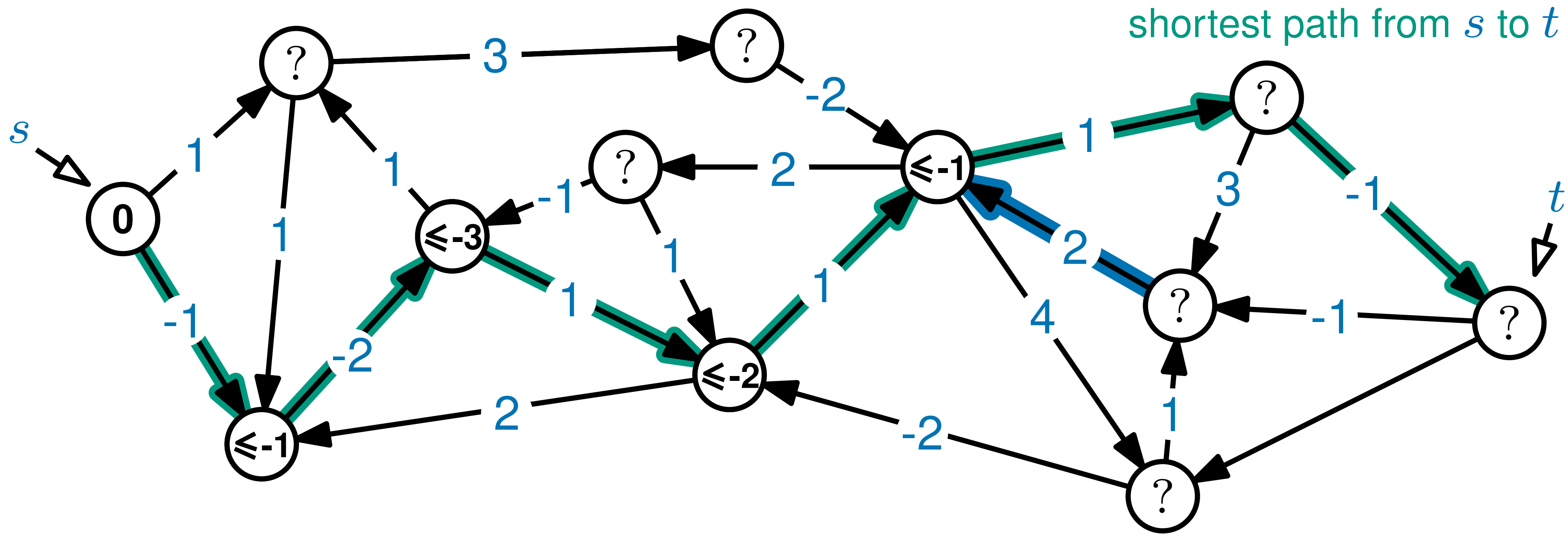

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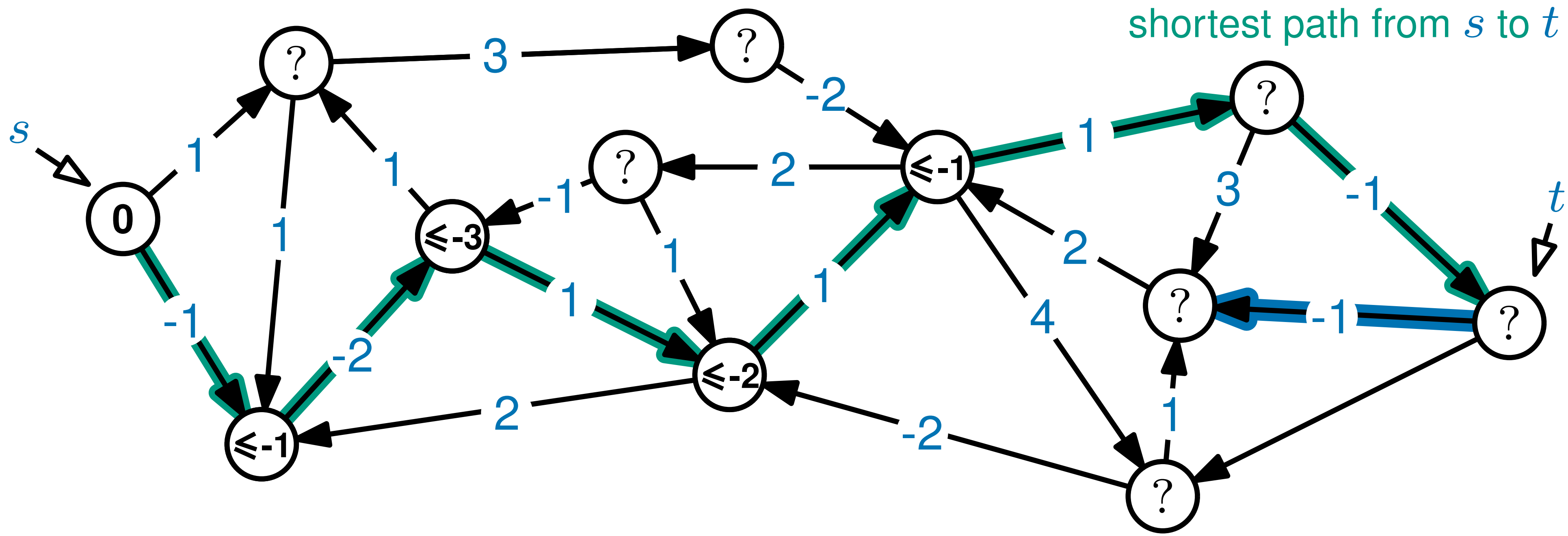

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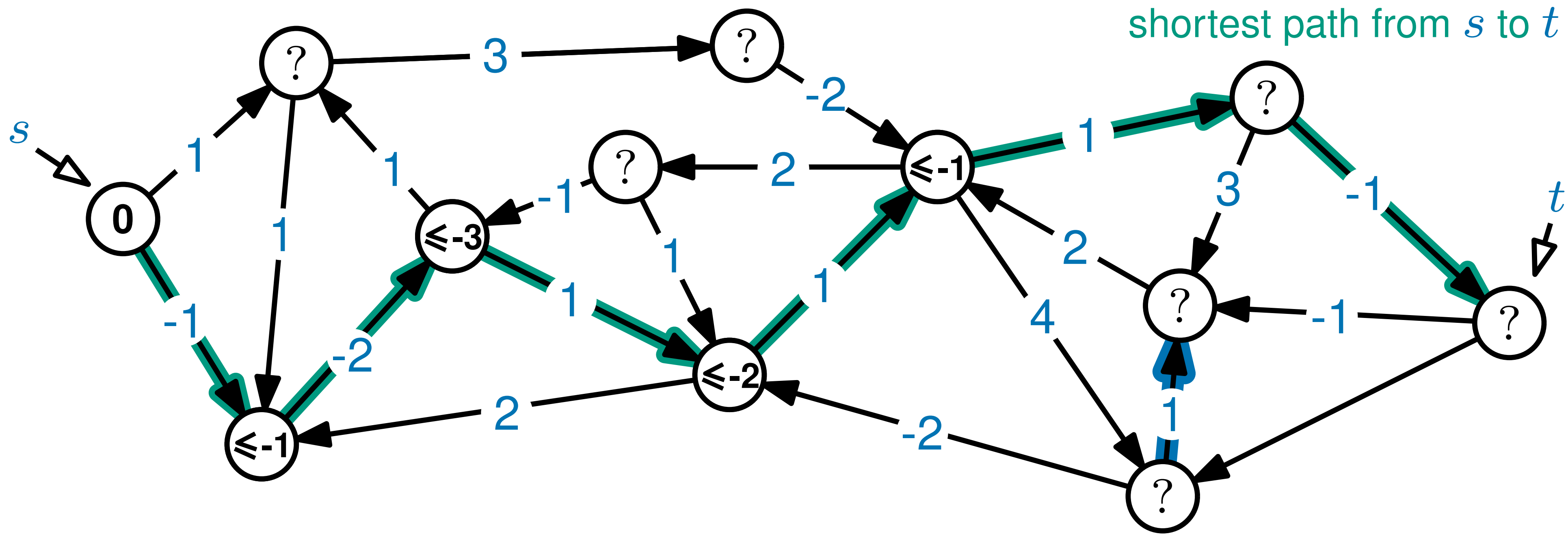

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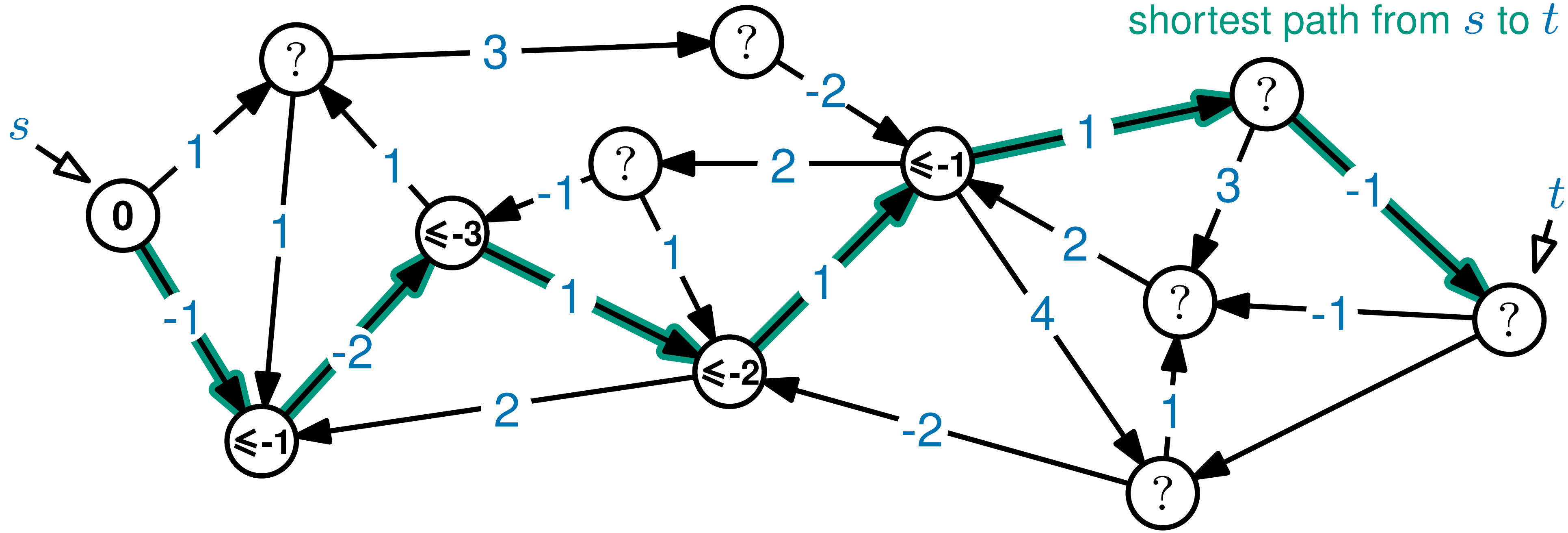

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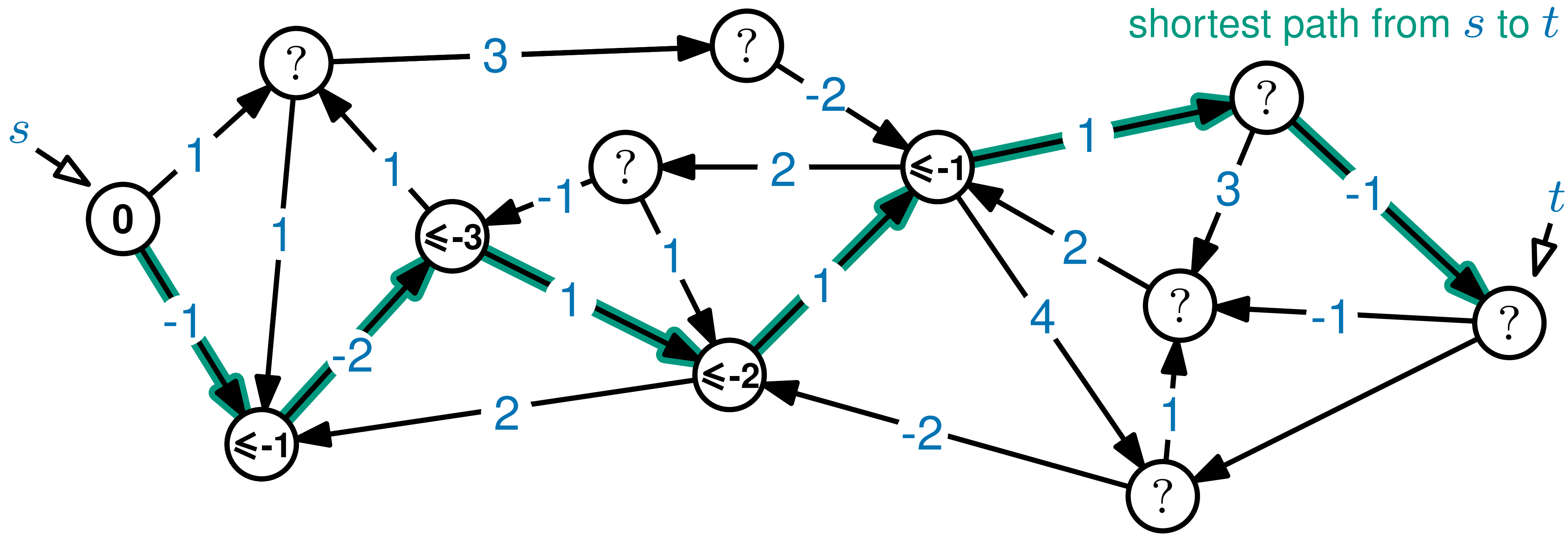

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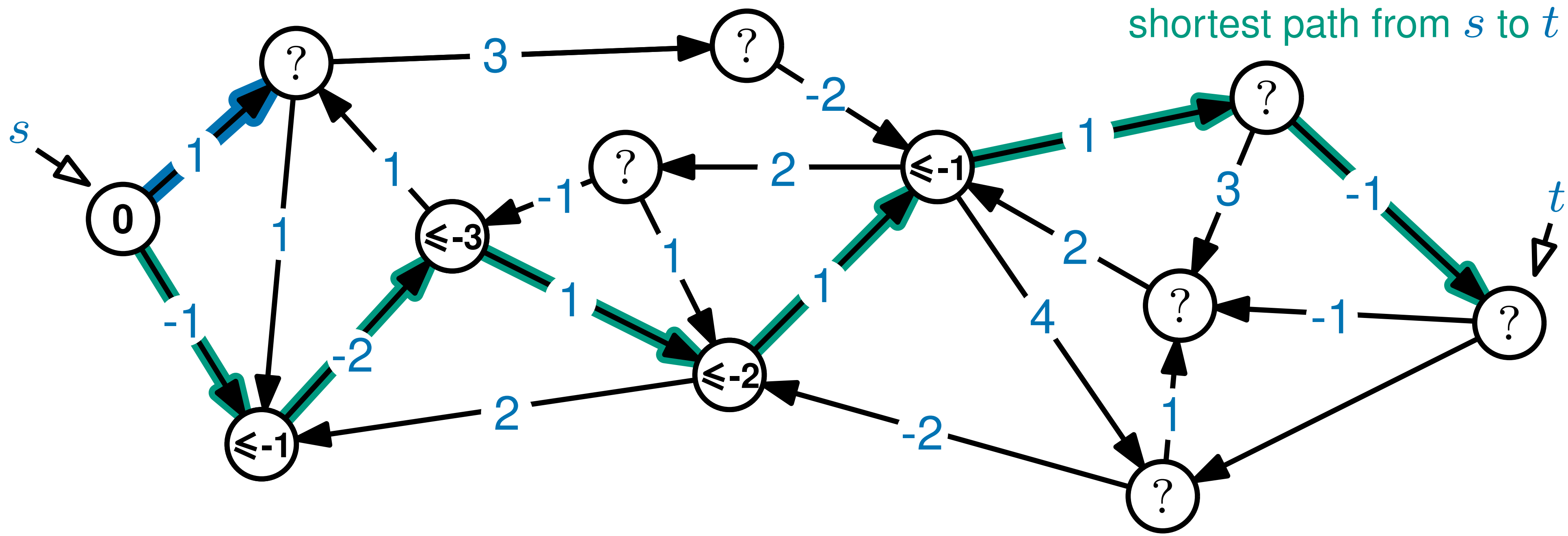

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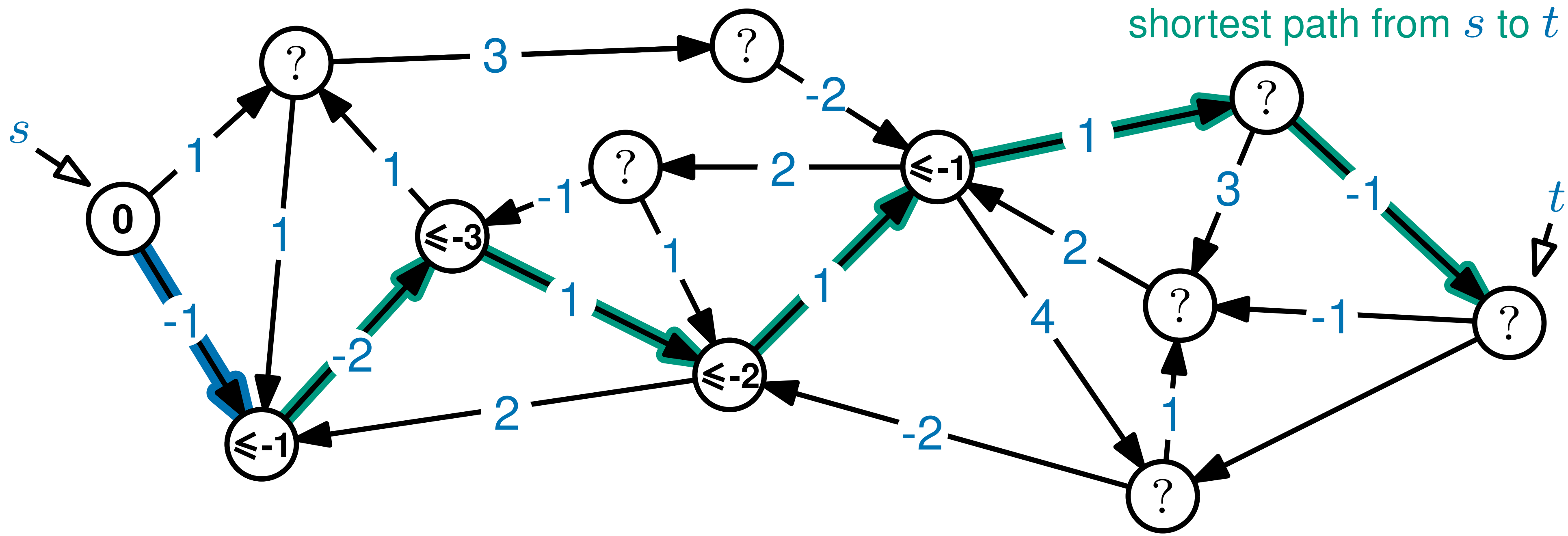

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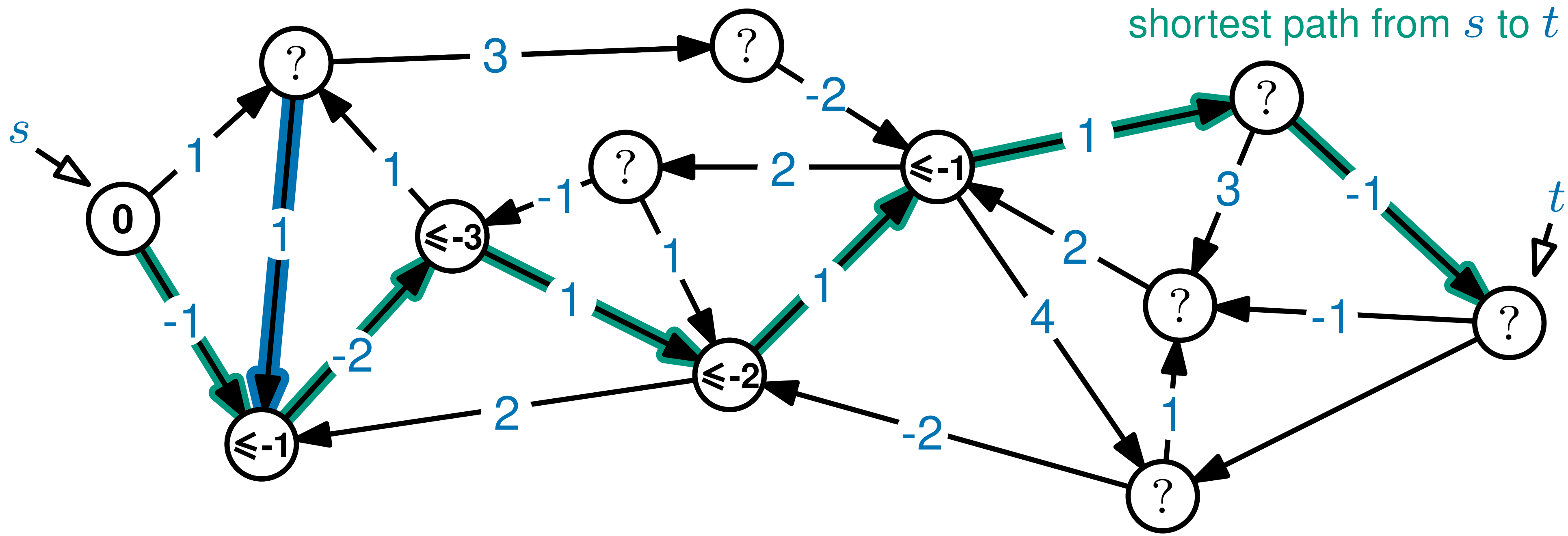

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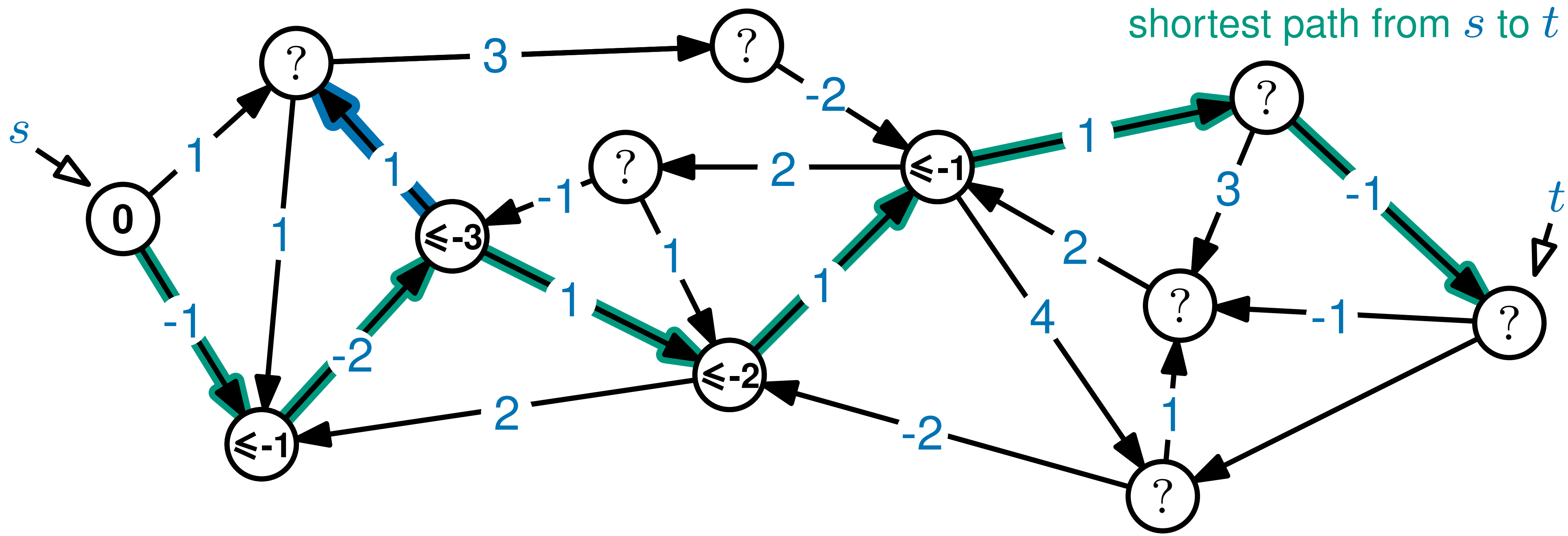

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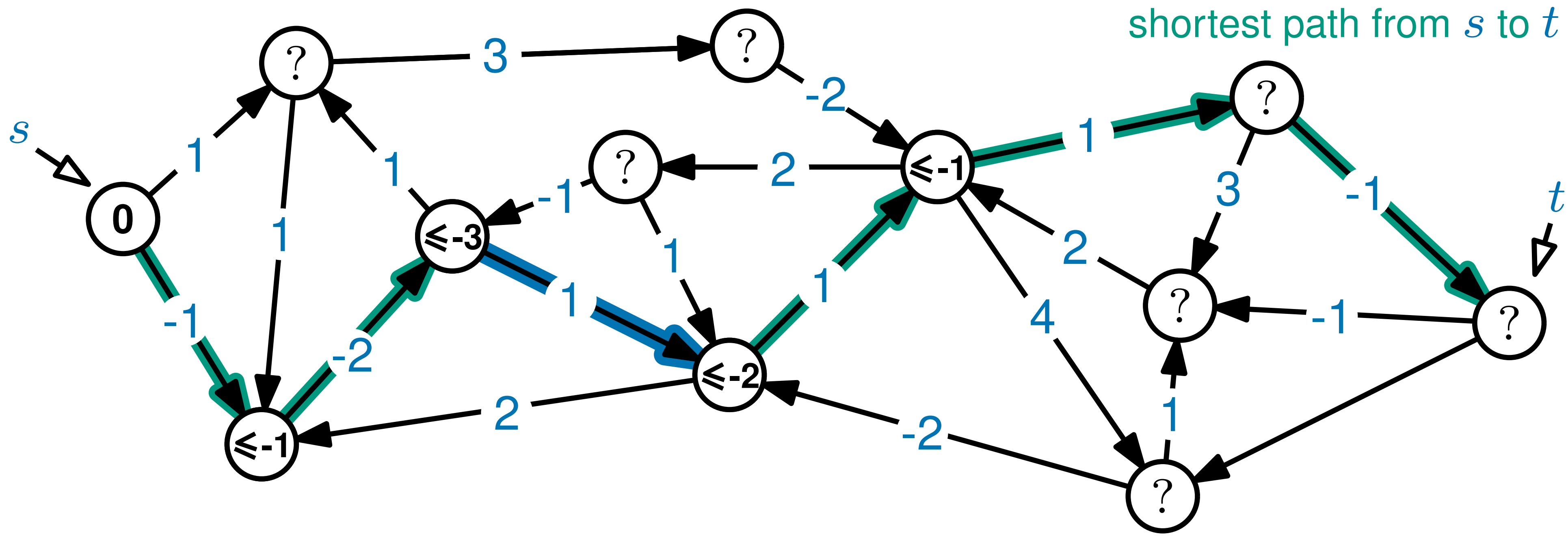

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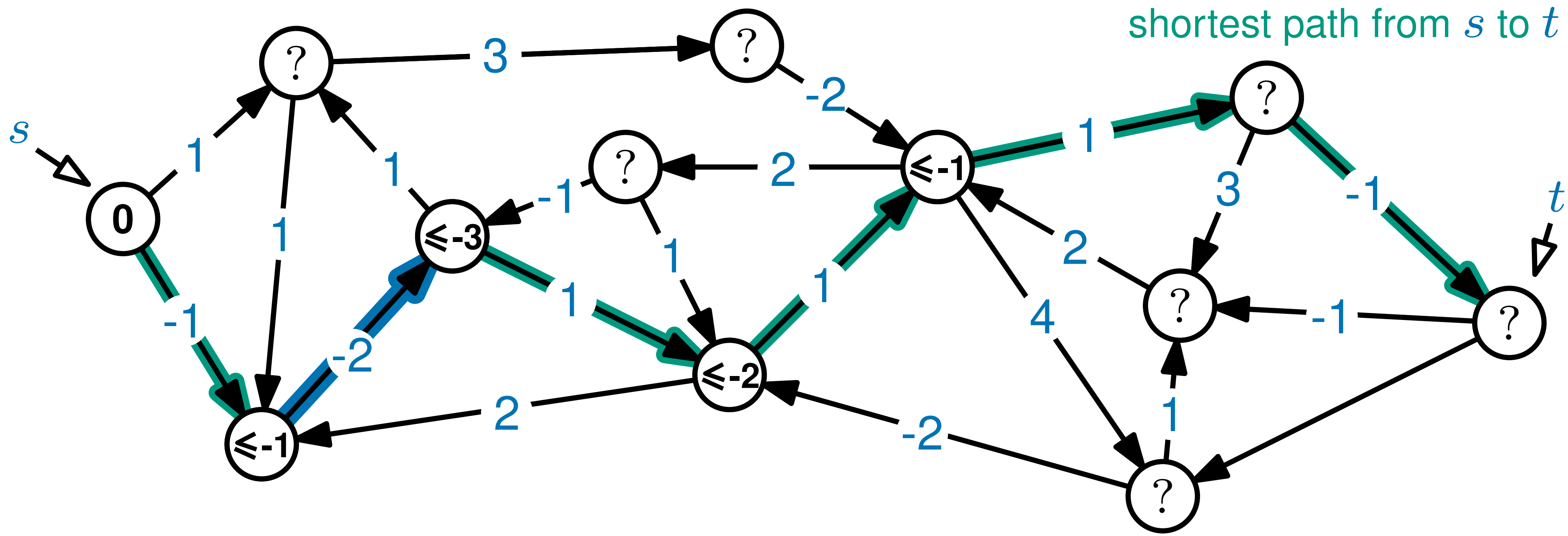

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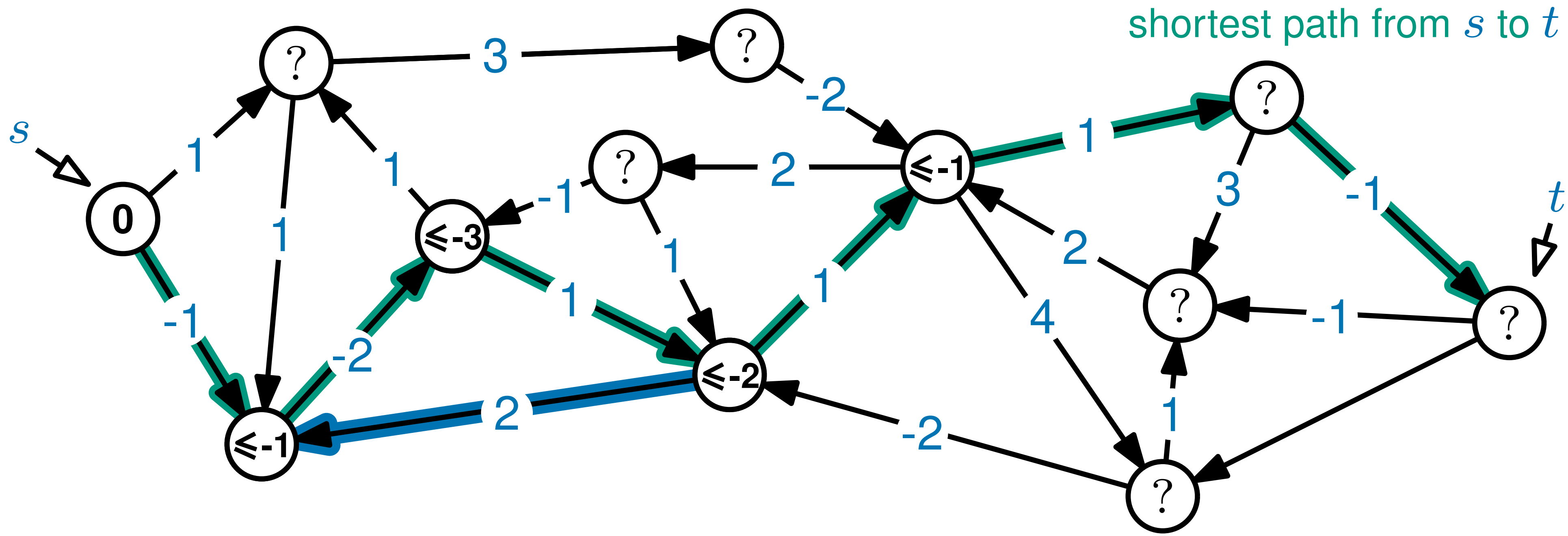

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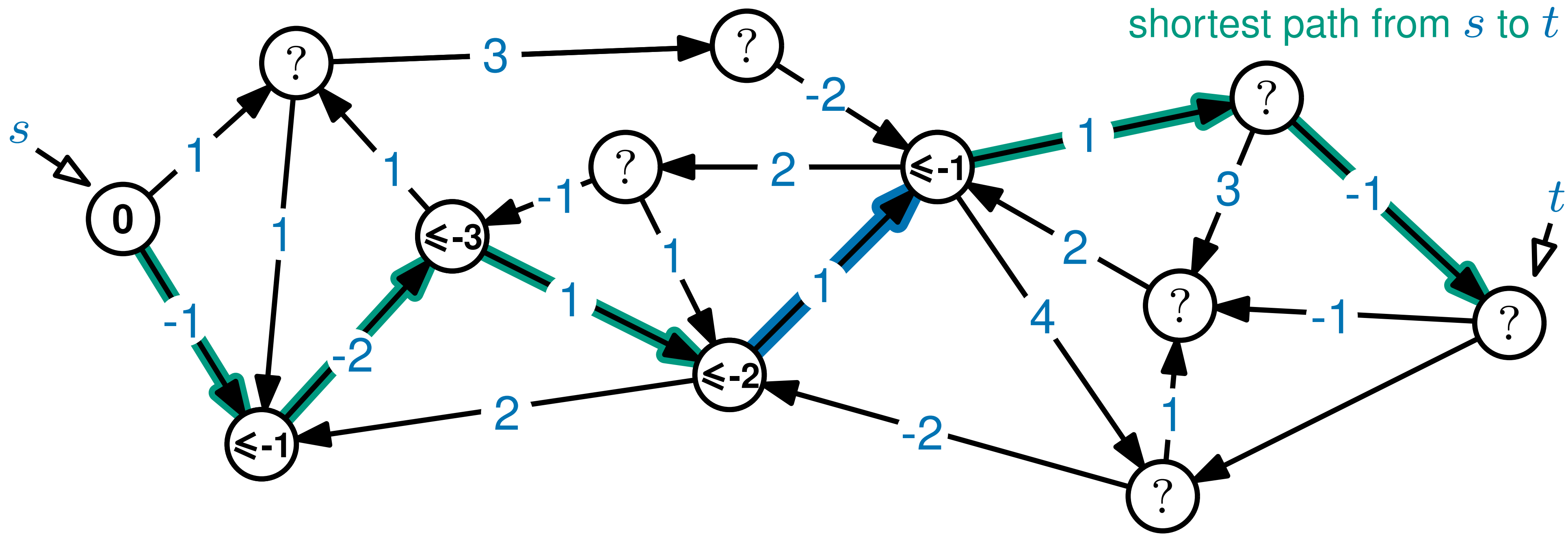

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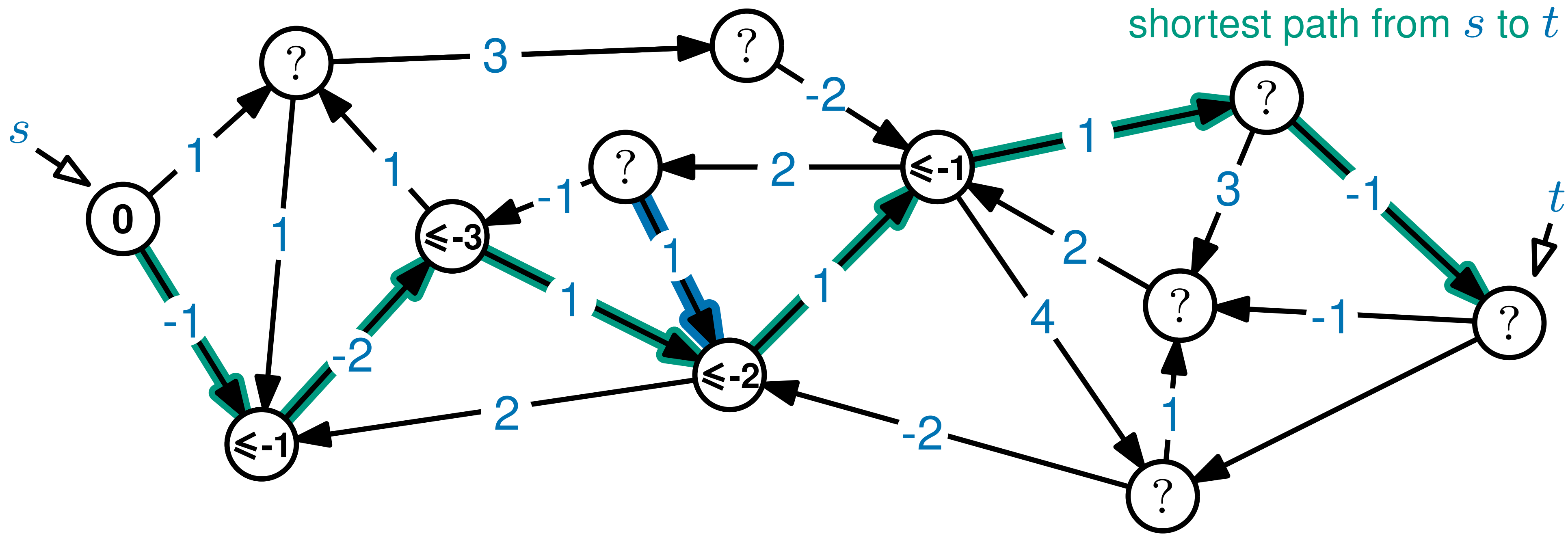

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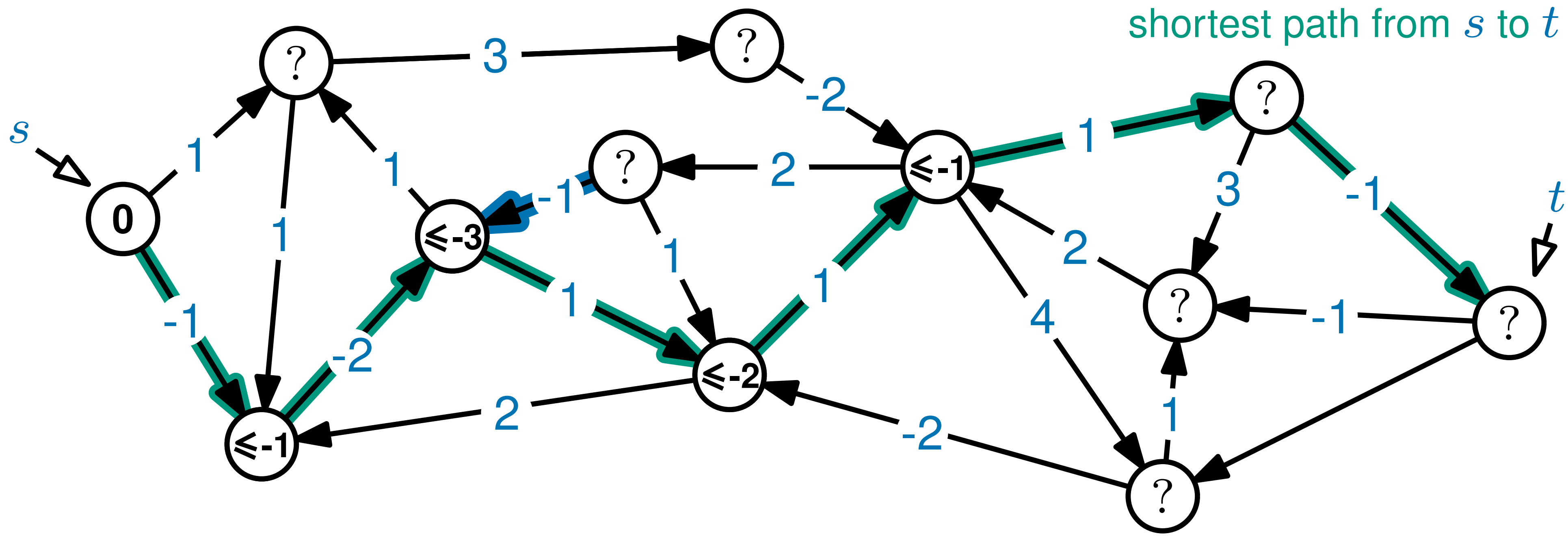

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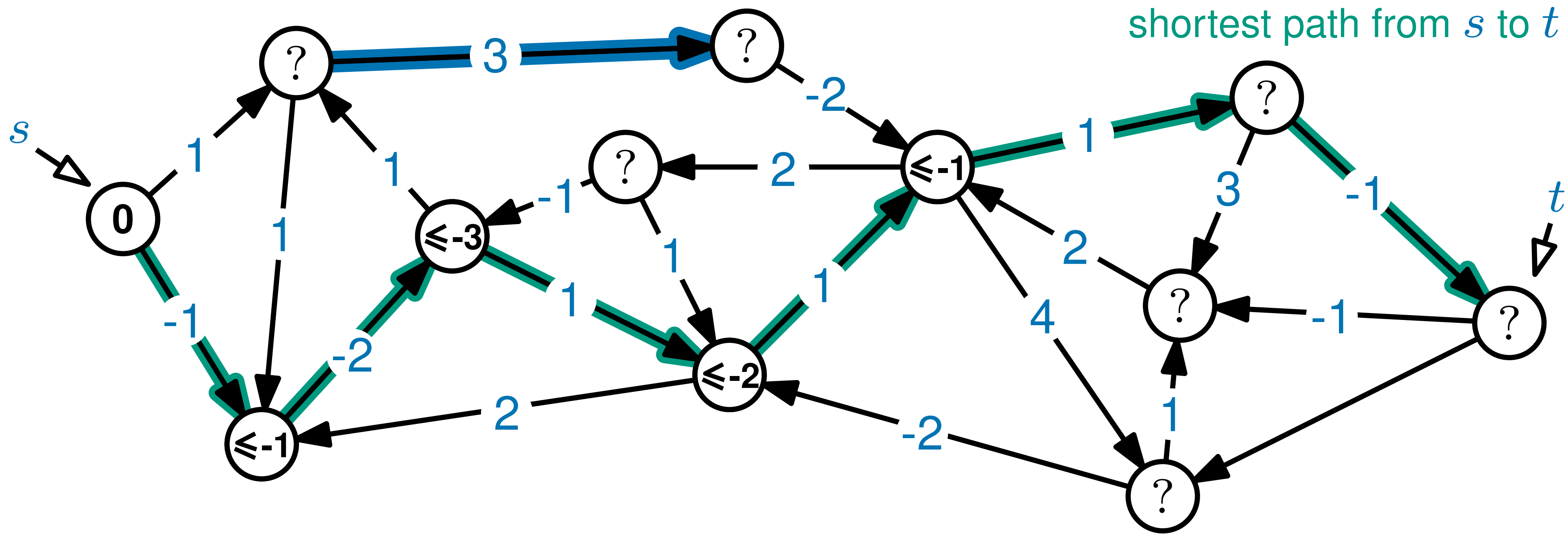

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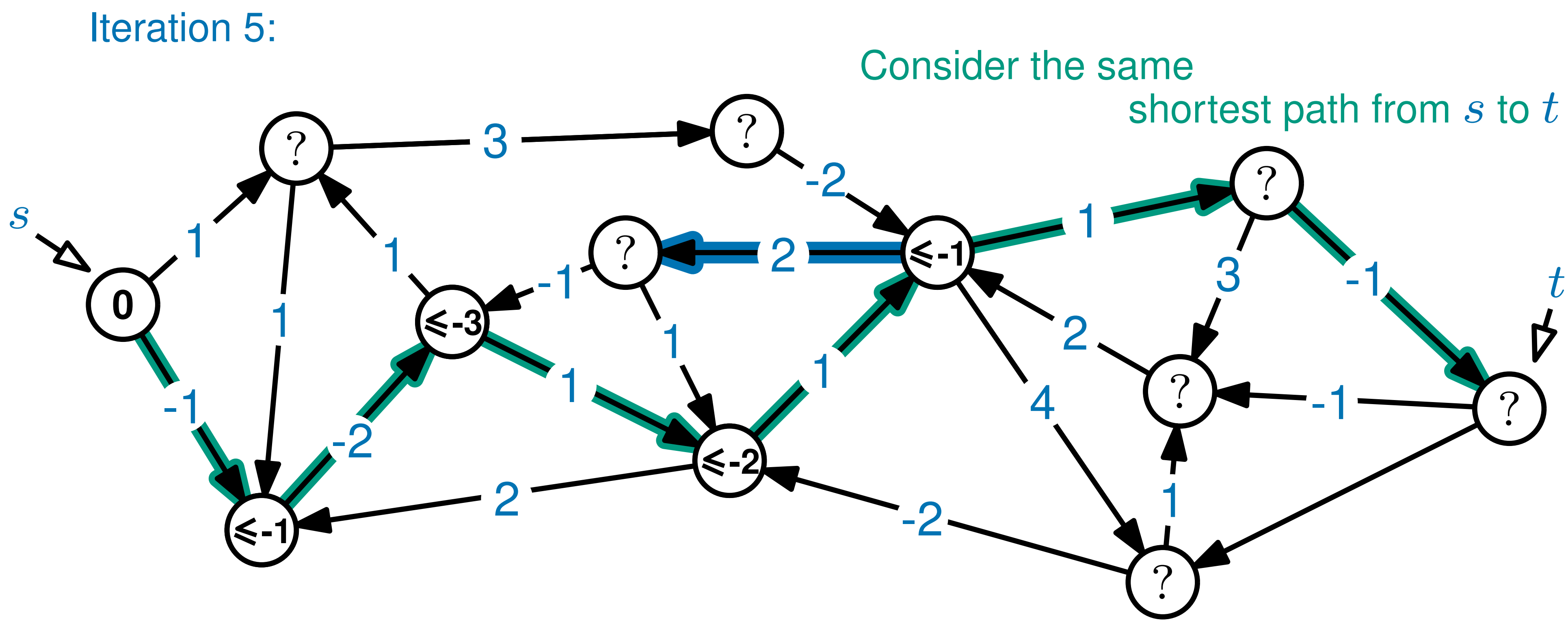

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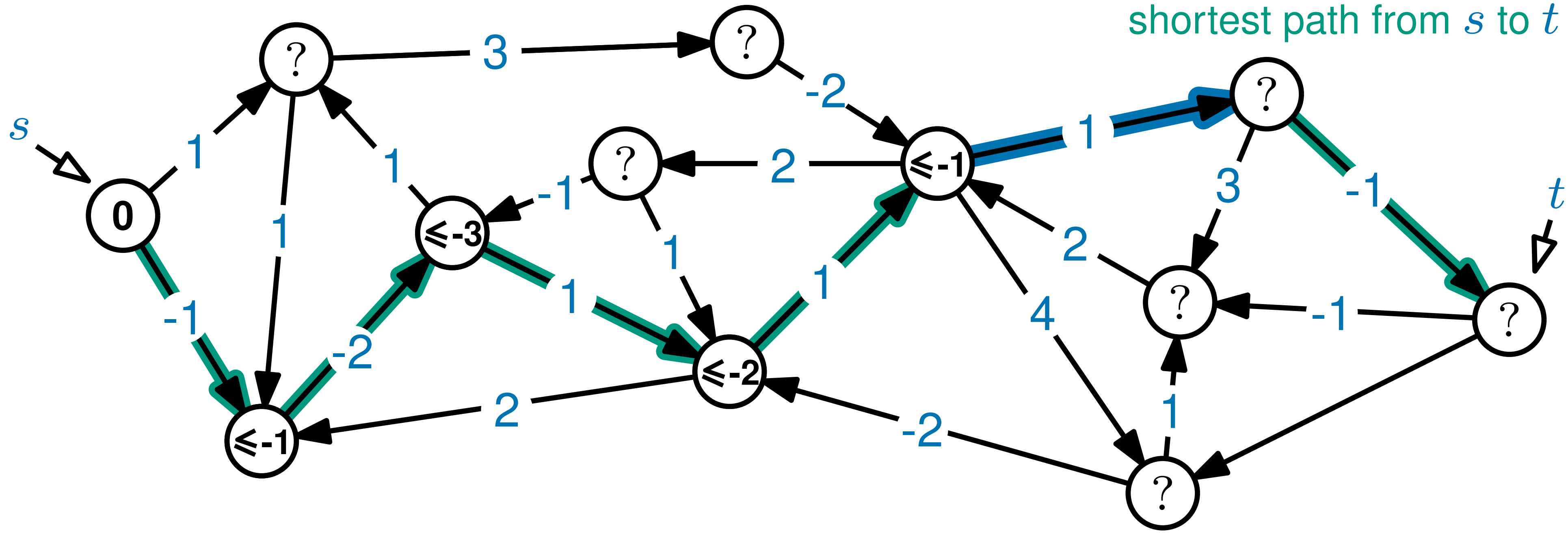

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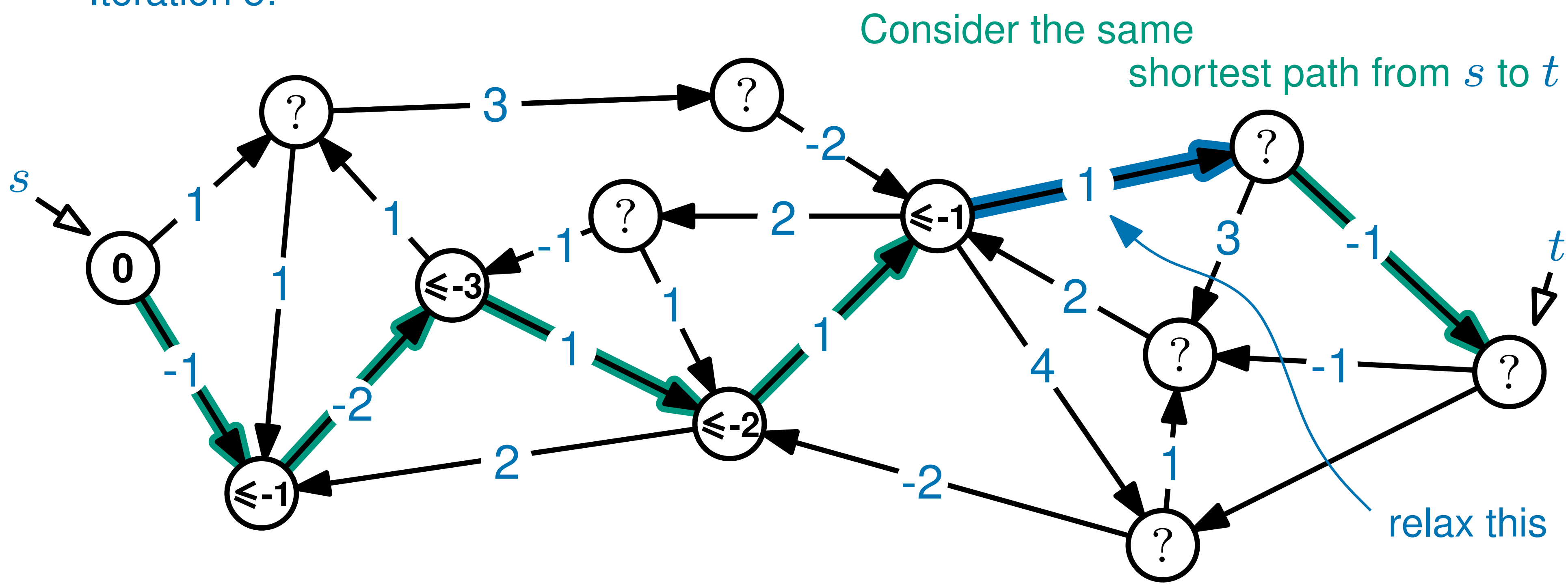

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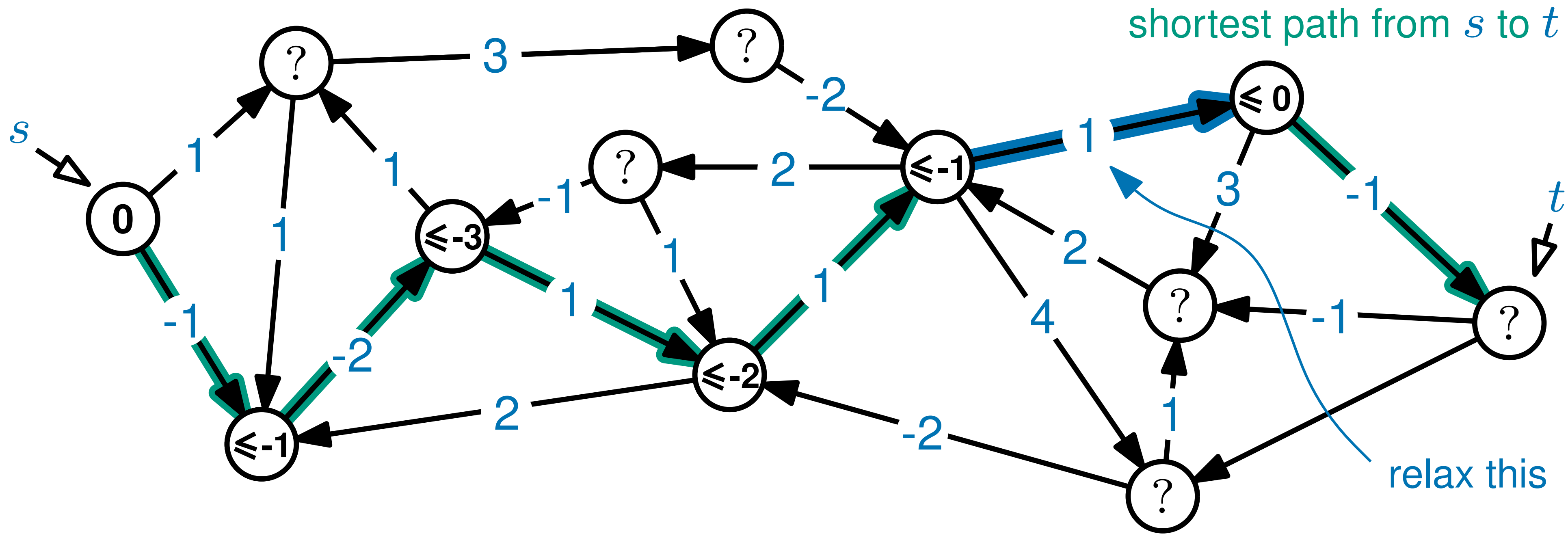

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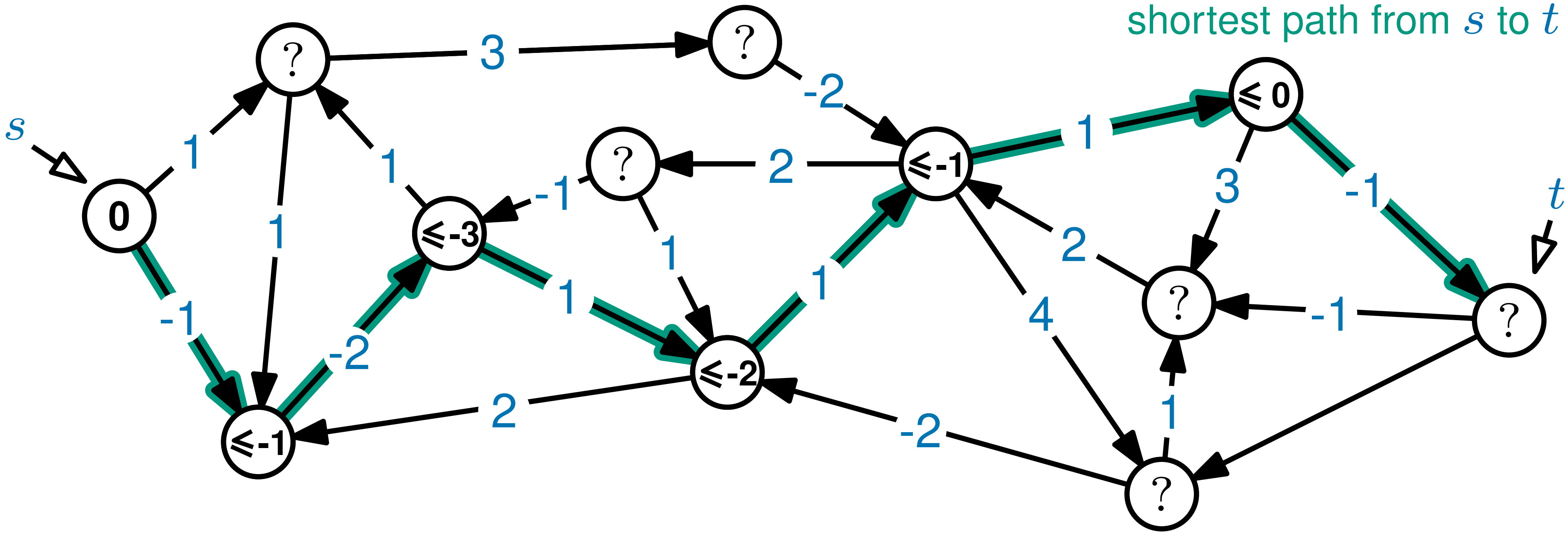

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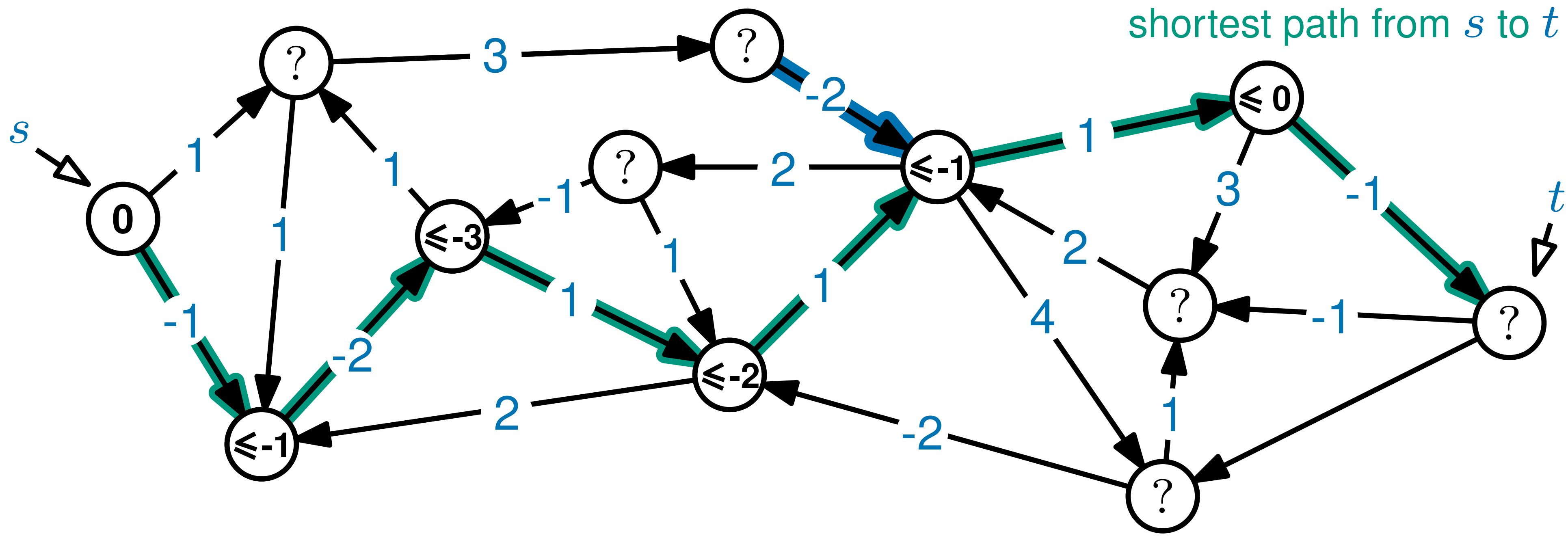

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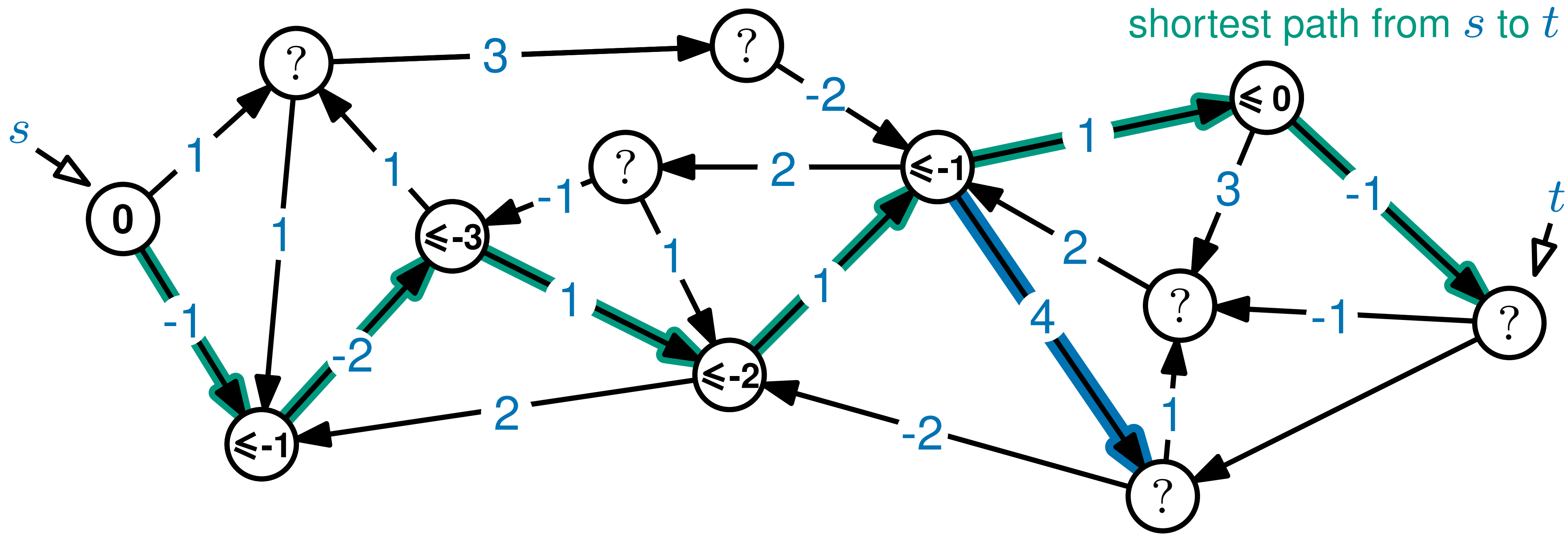

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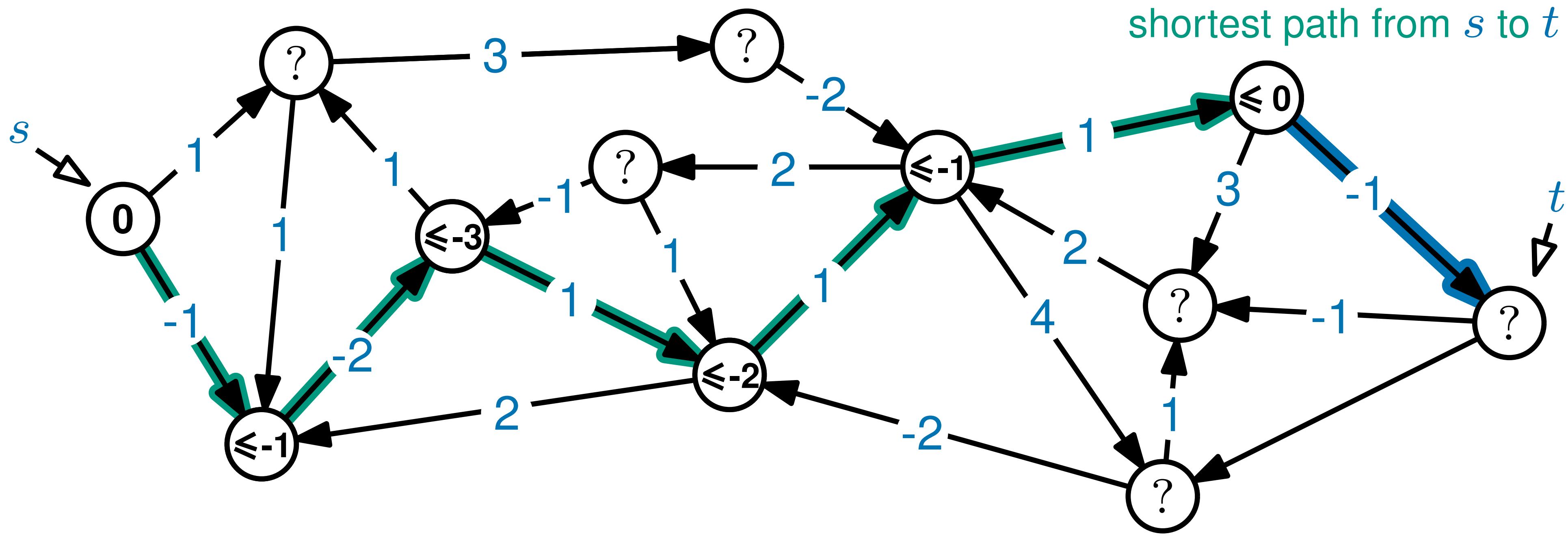

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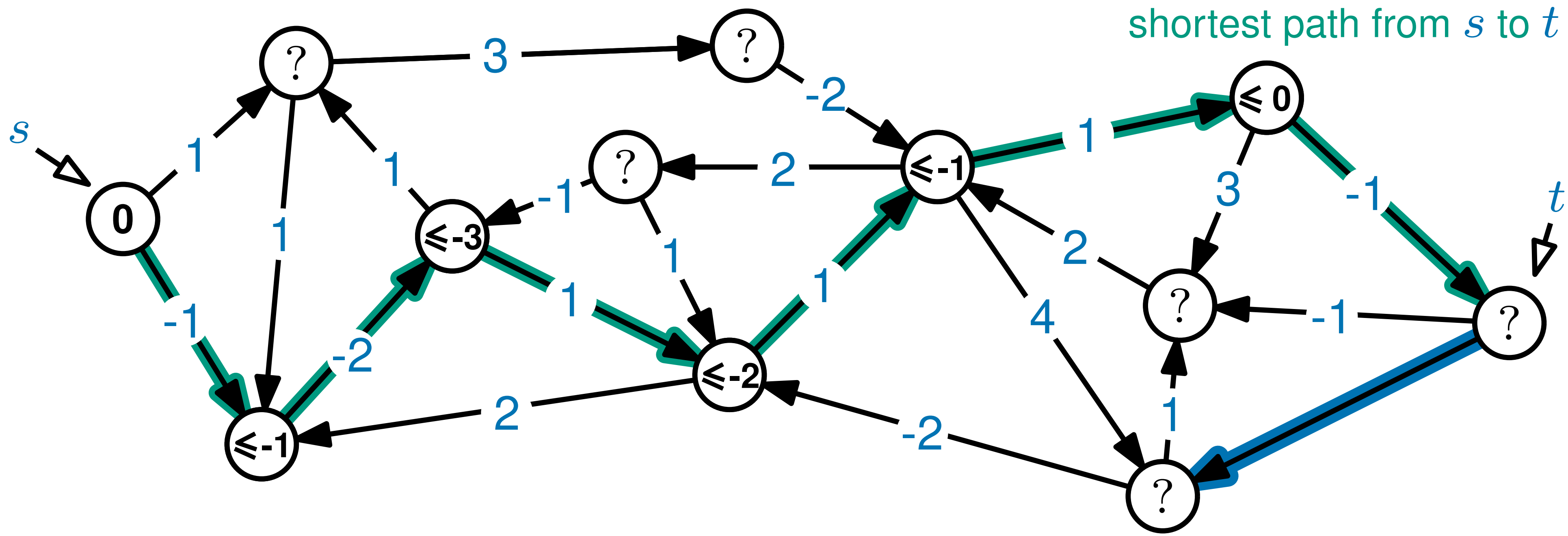

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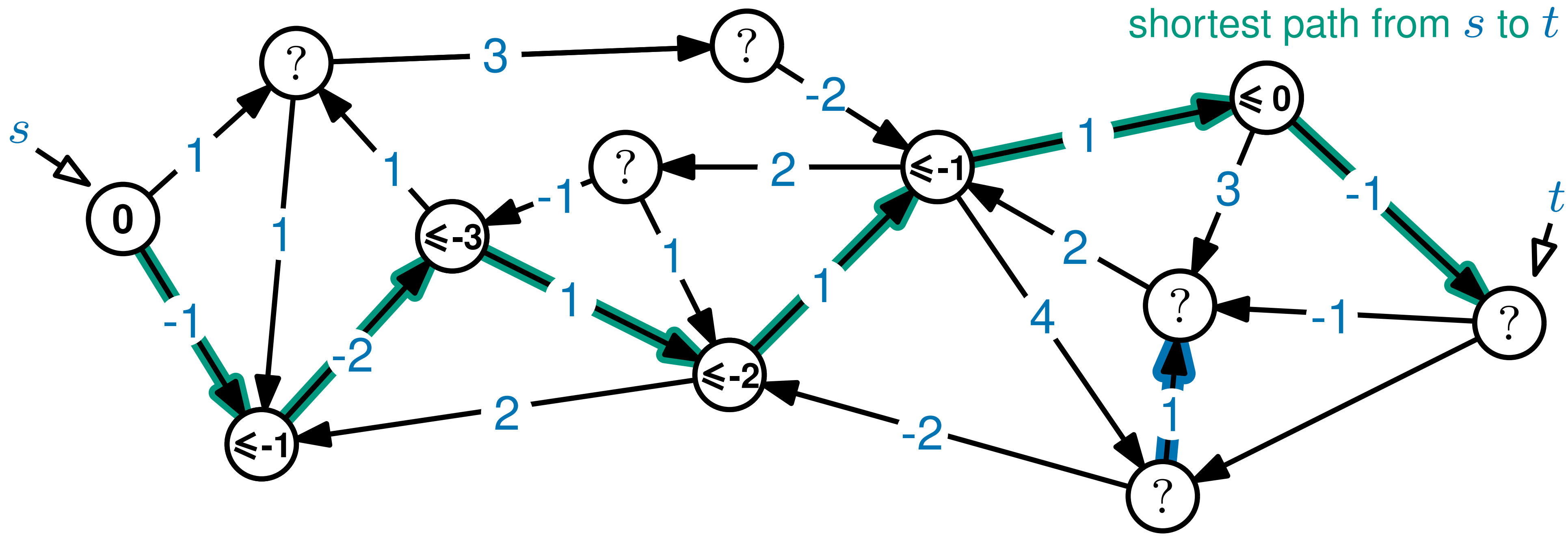

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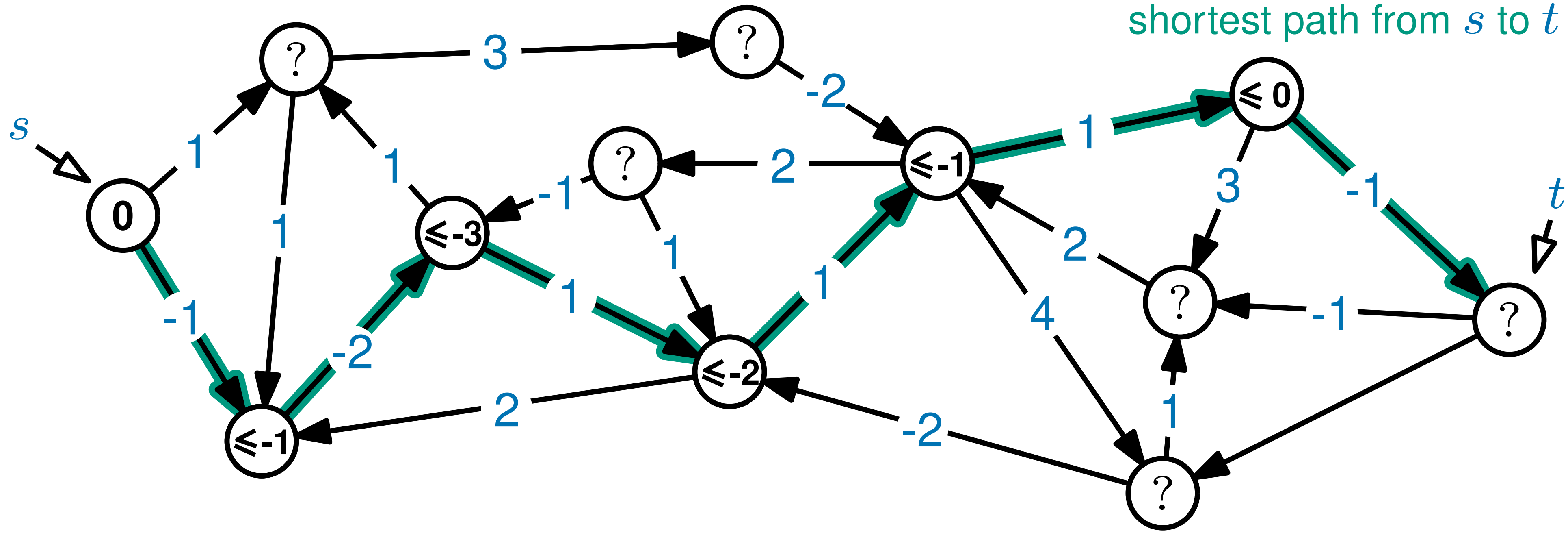

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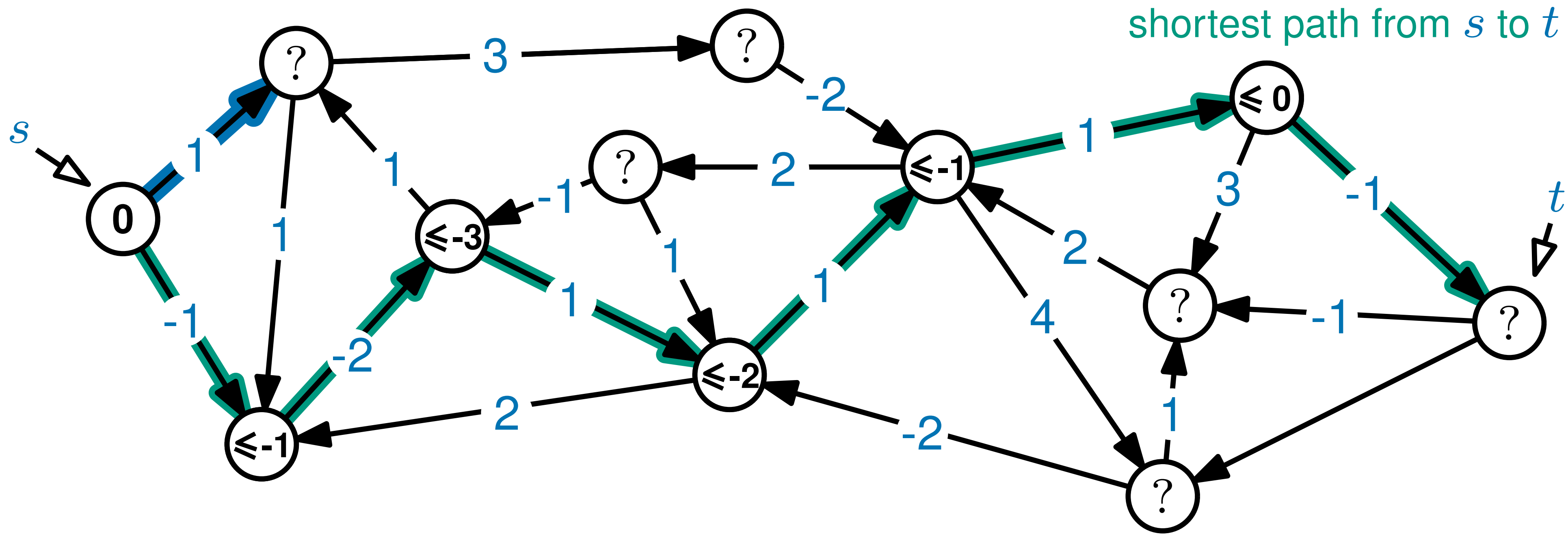

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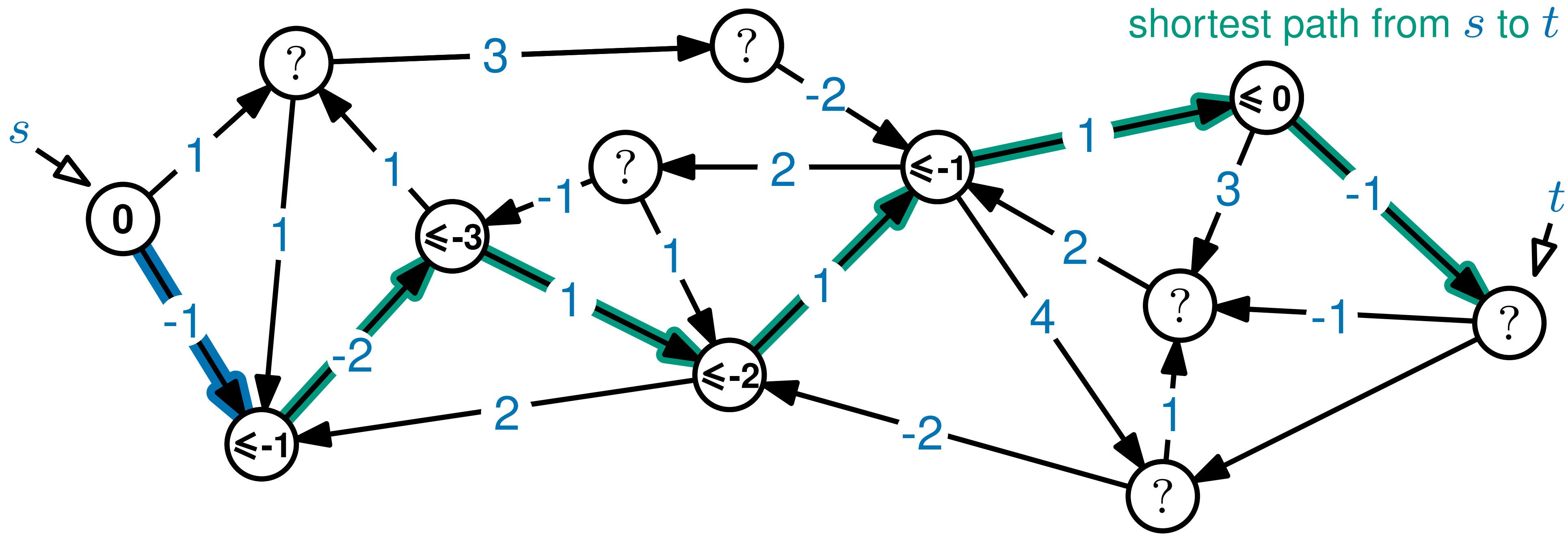

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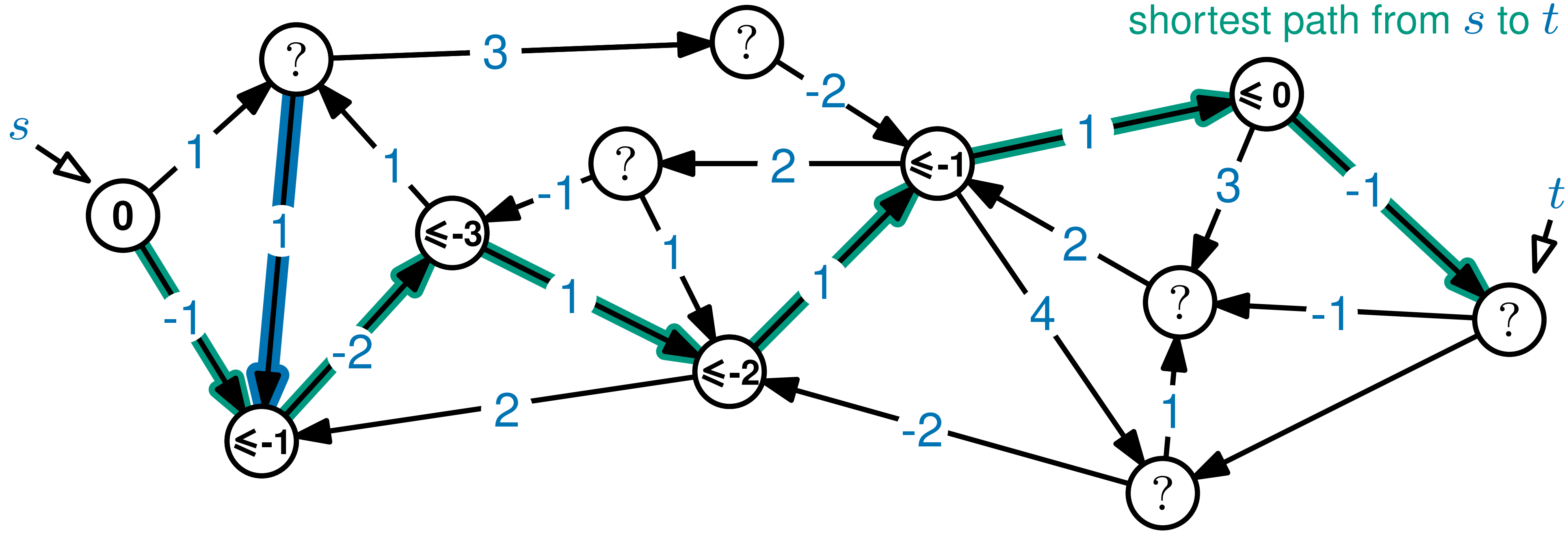

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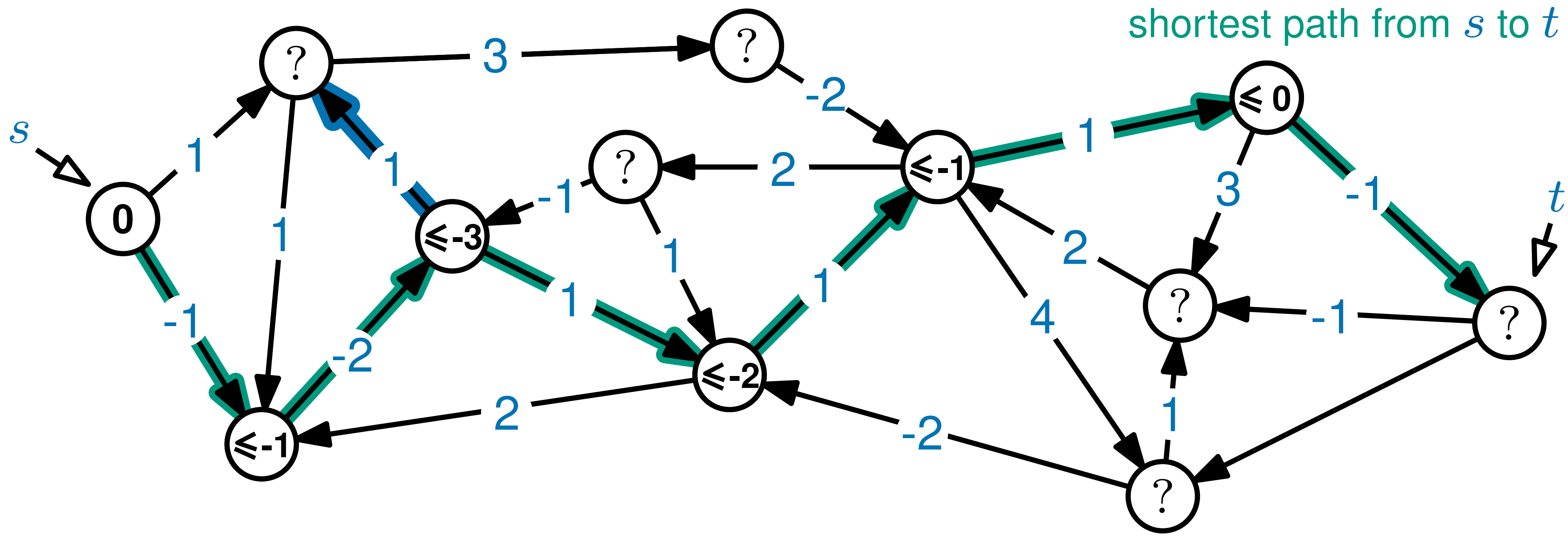

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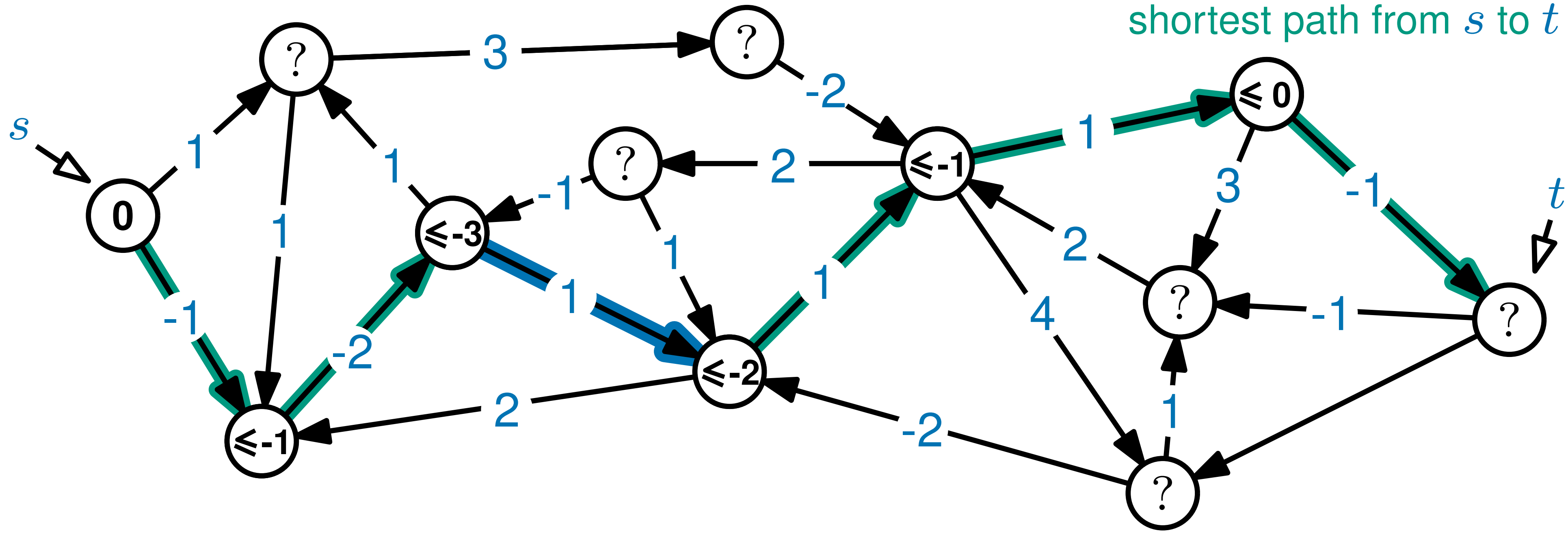

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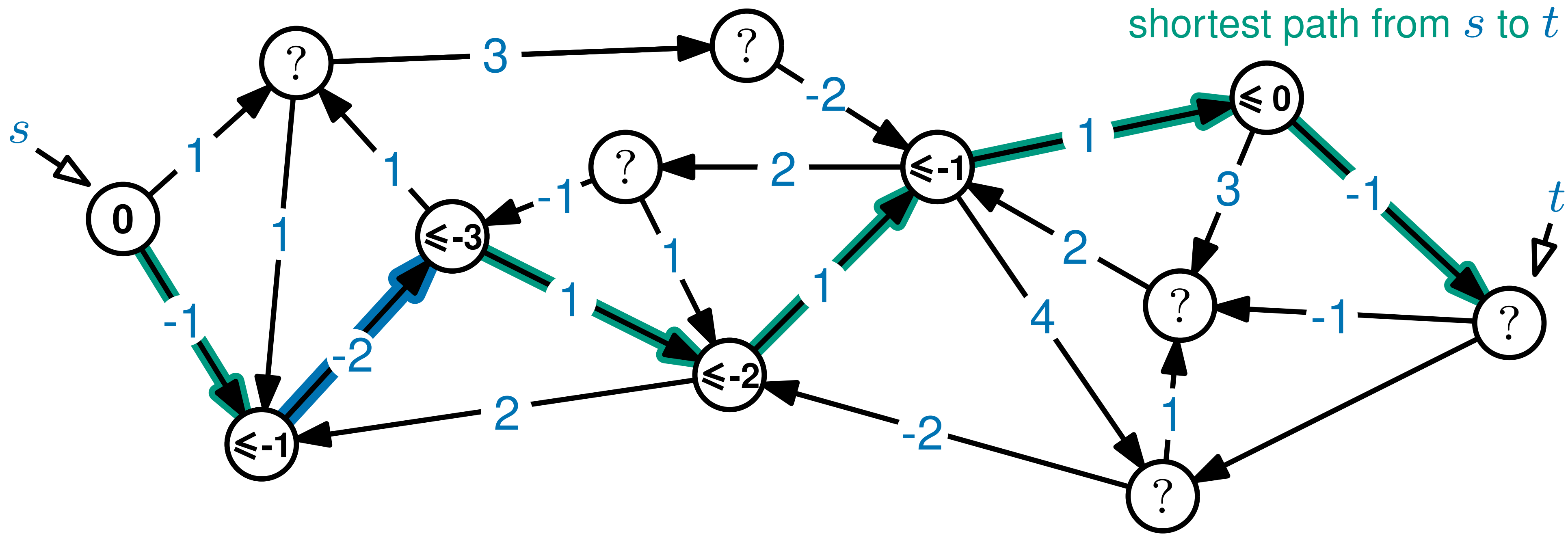

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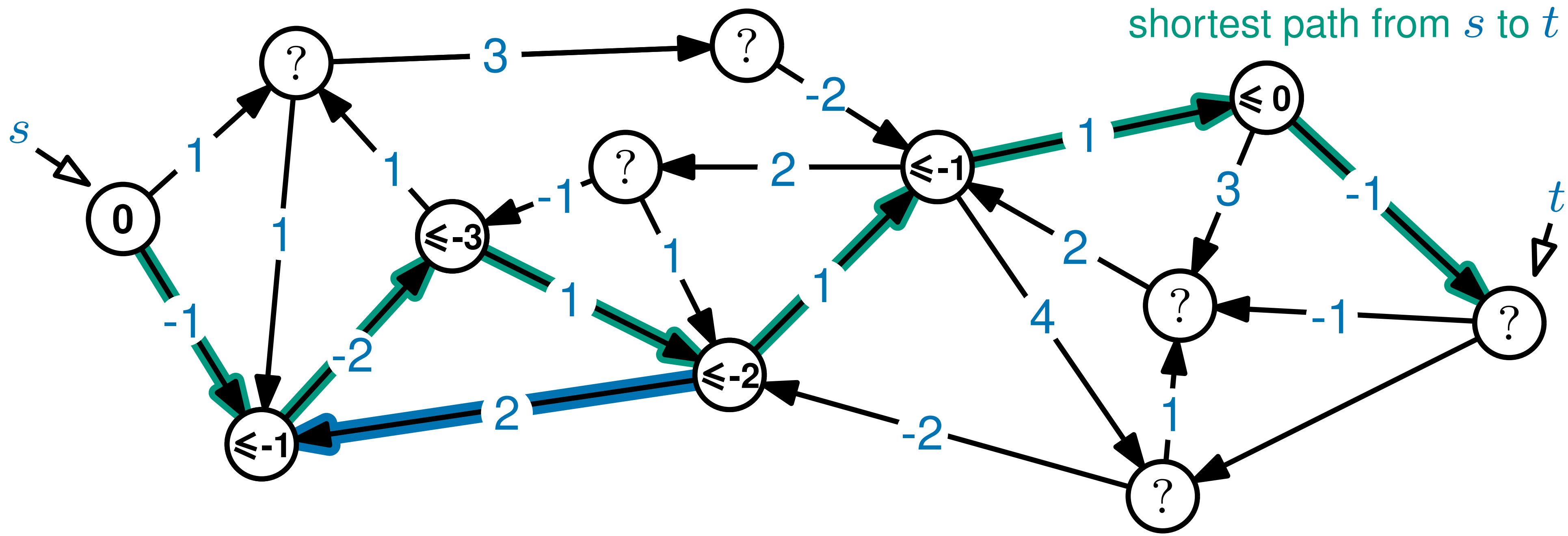

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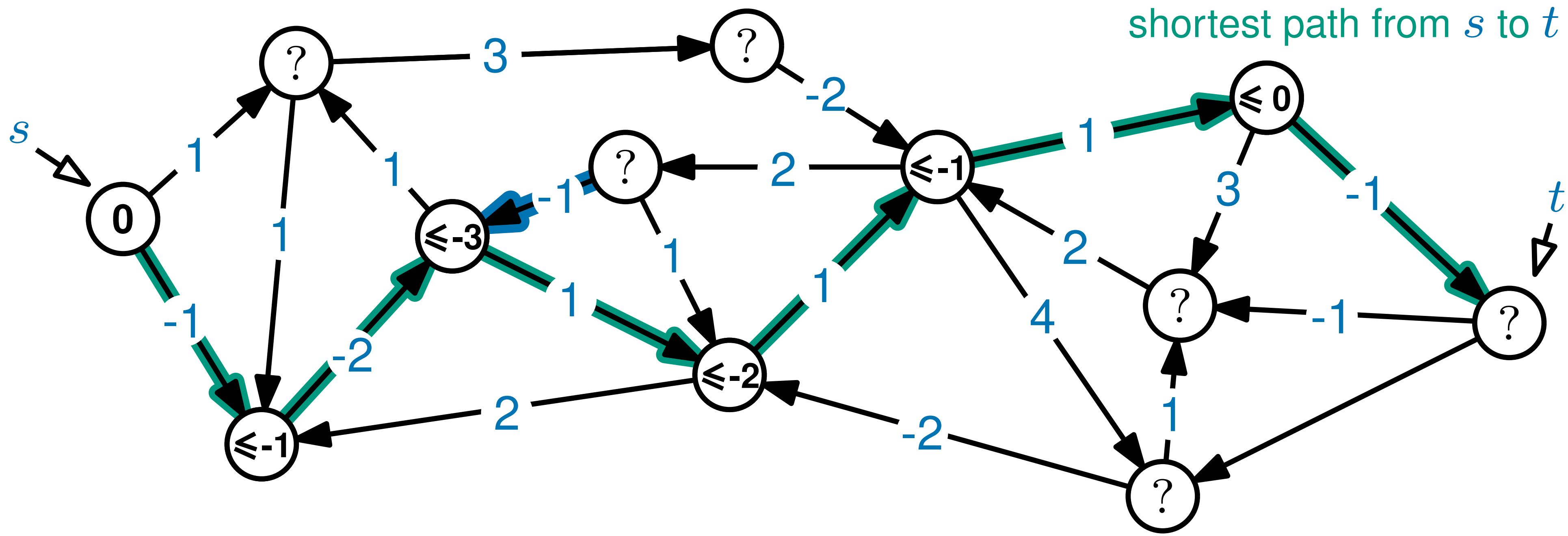

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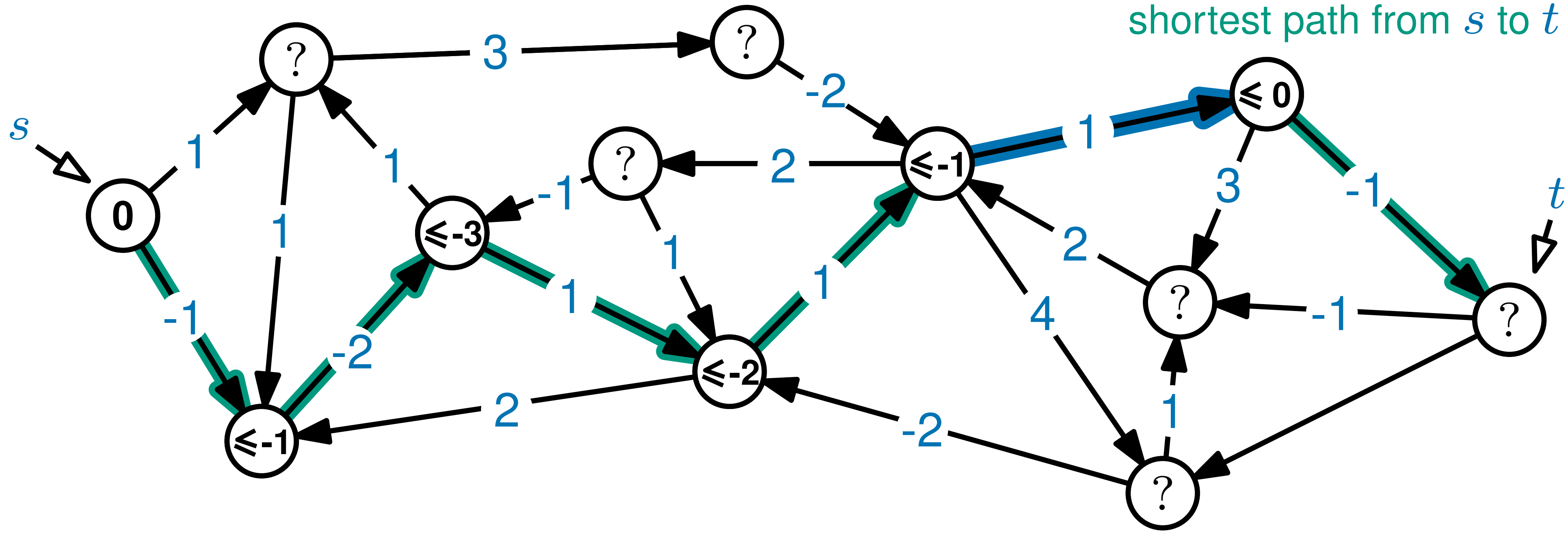

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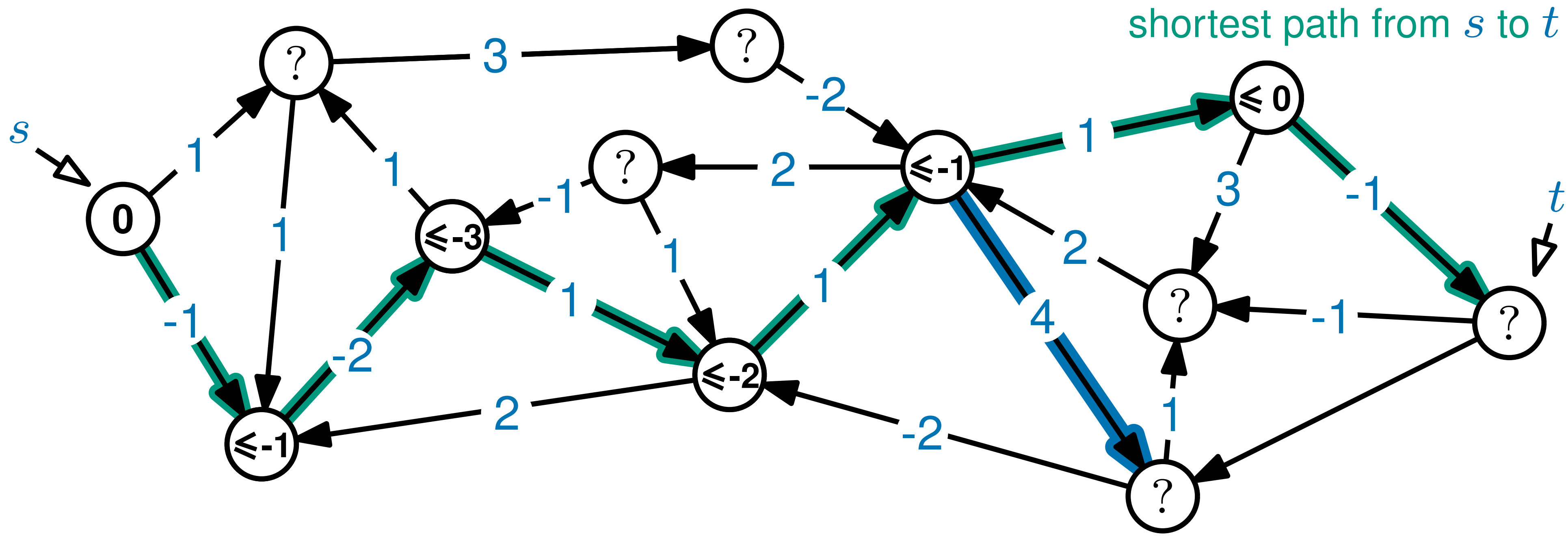

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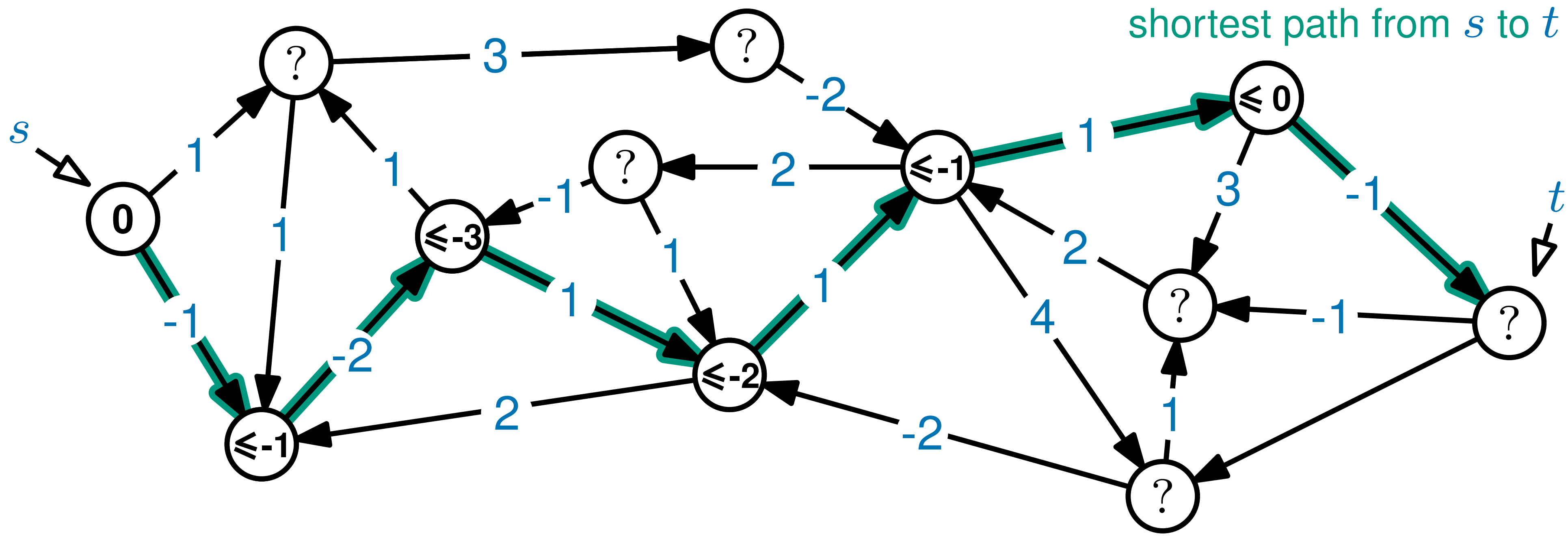

At some point in iteration i

you relax the i-th edge in the shortest path from s to t. . .

The proof idea

Now consider the MOSTOFBELLMAN-FORD algorithm where in each iteration...

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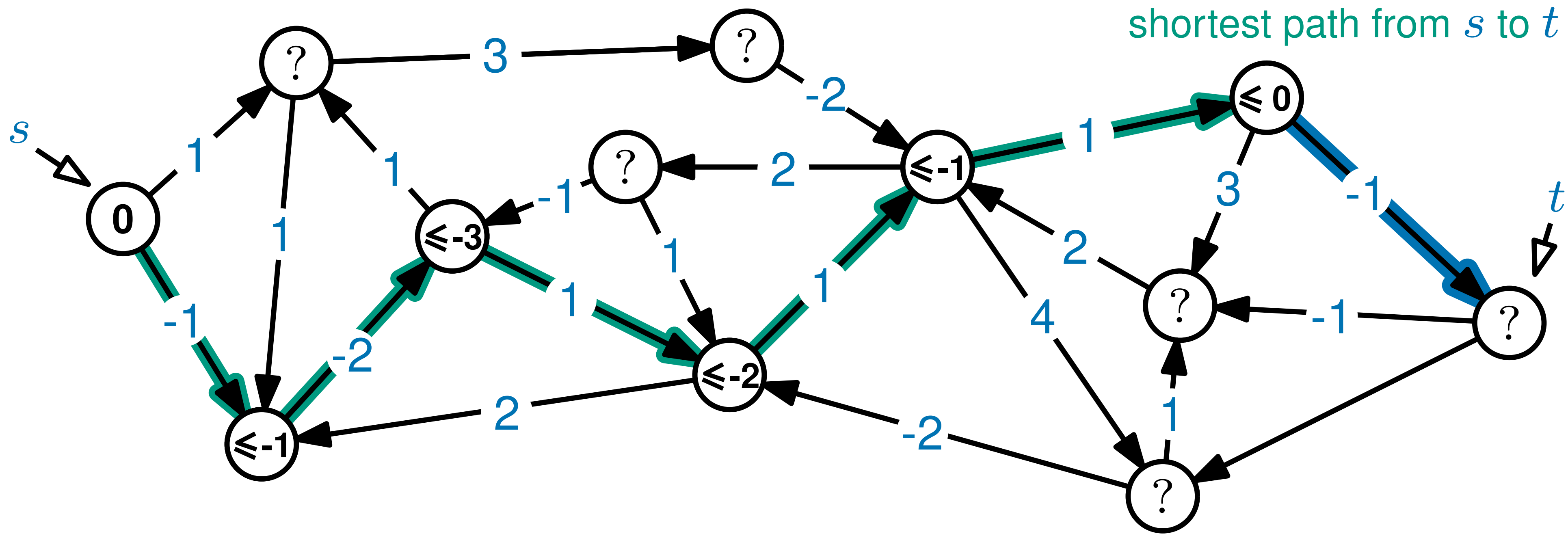

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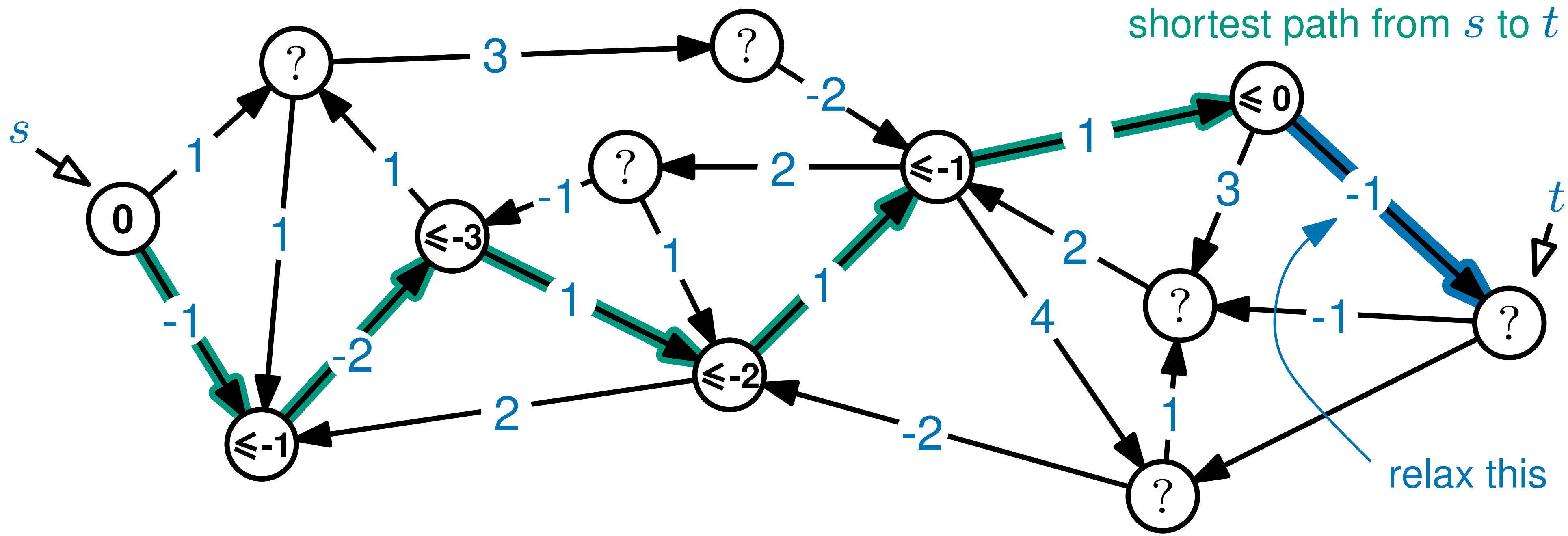

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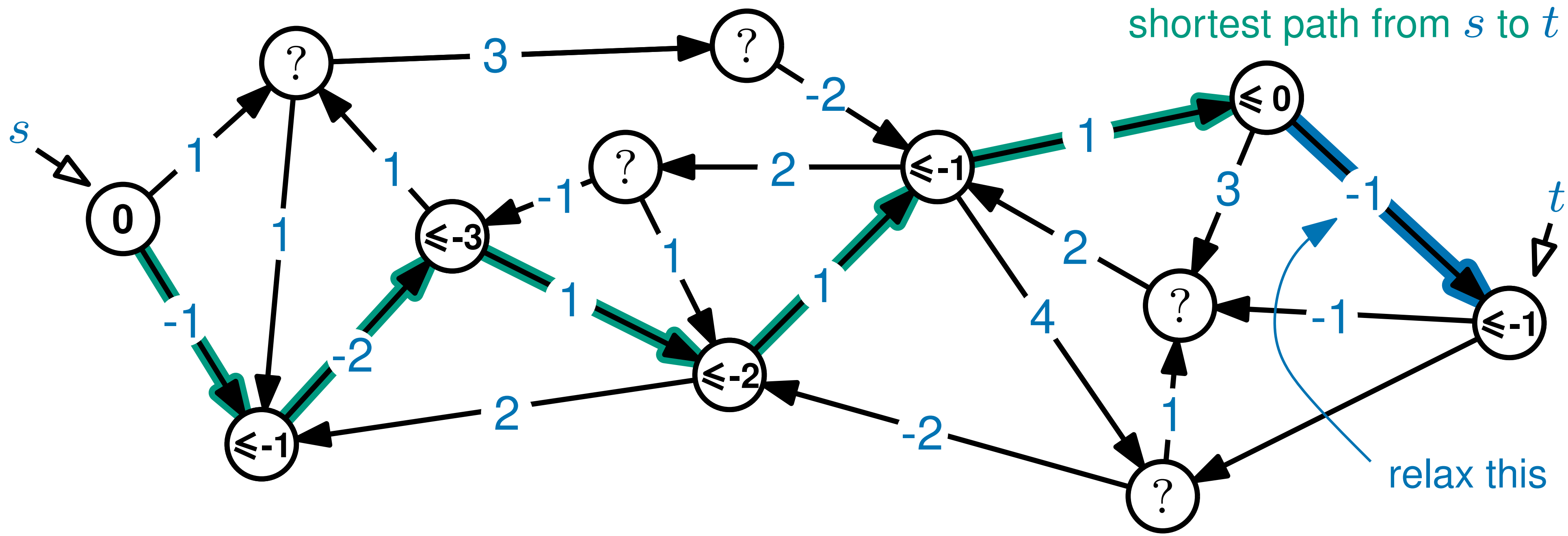

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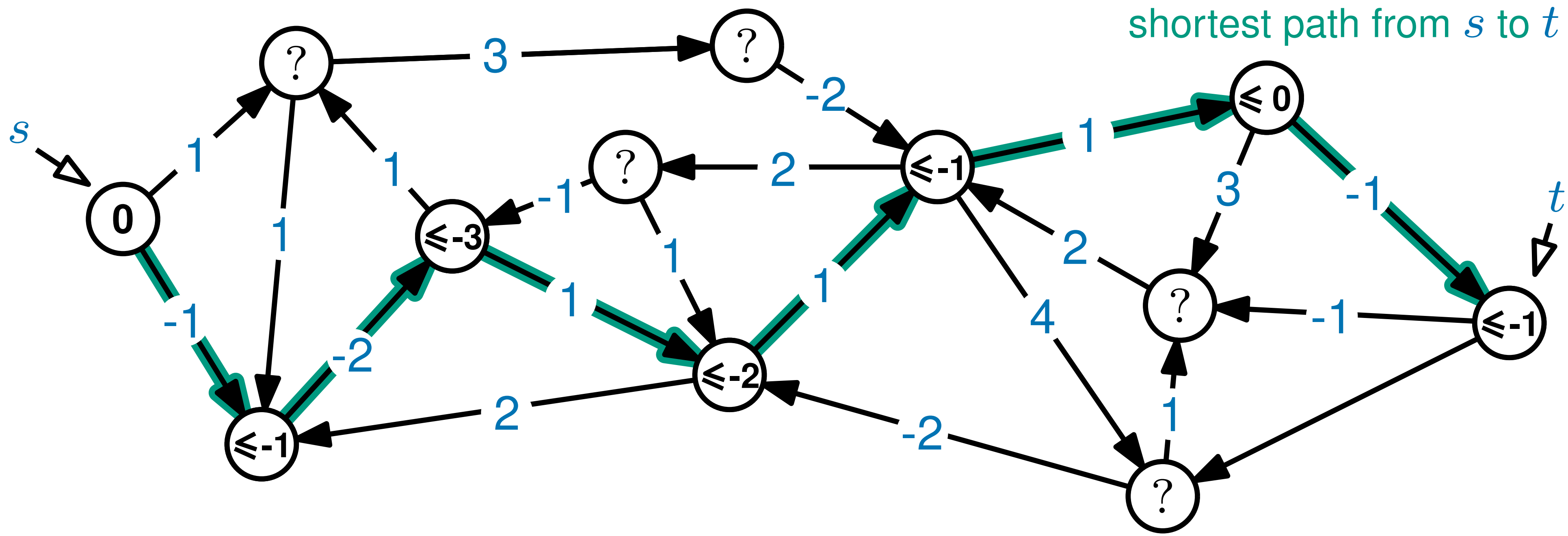

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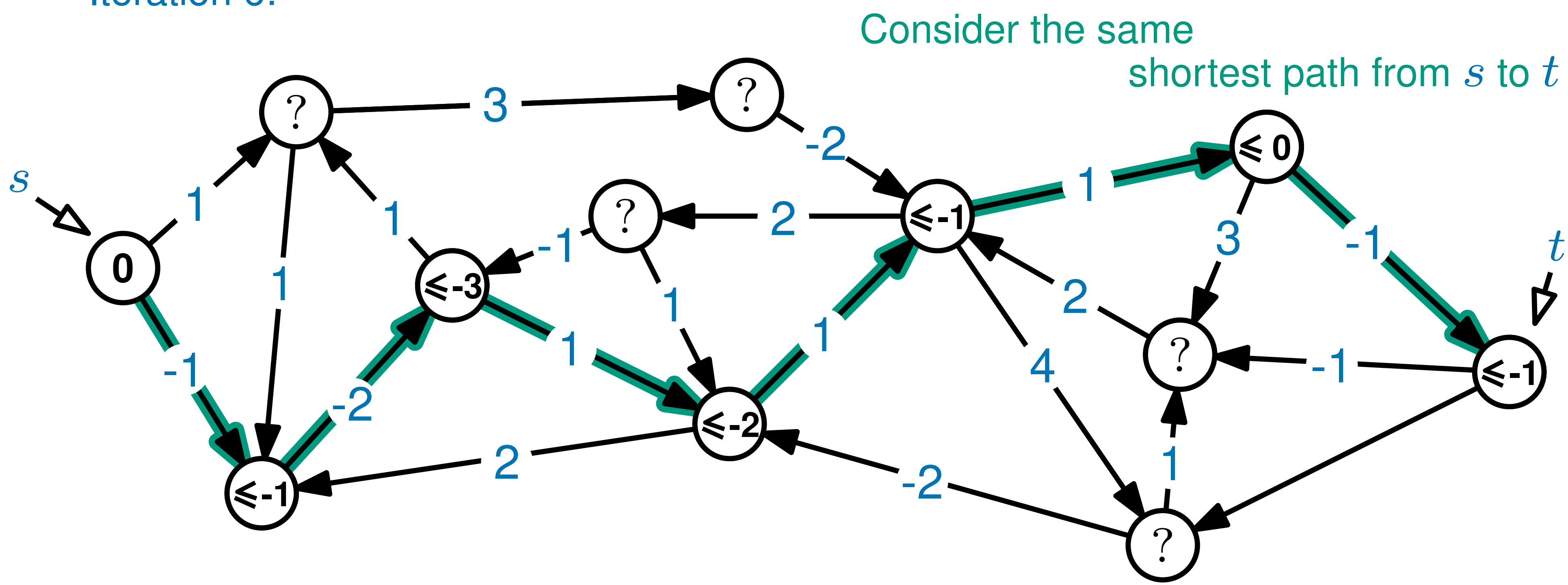

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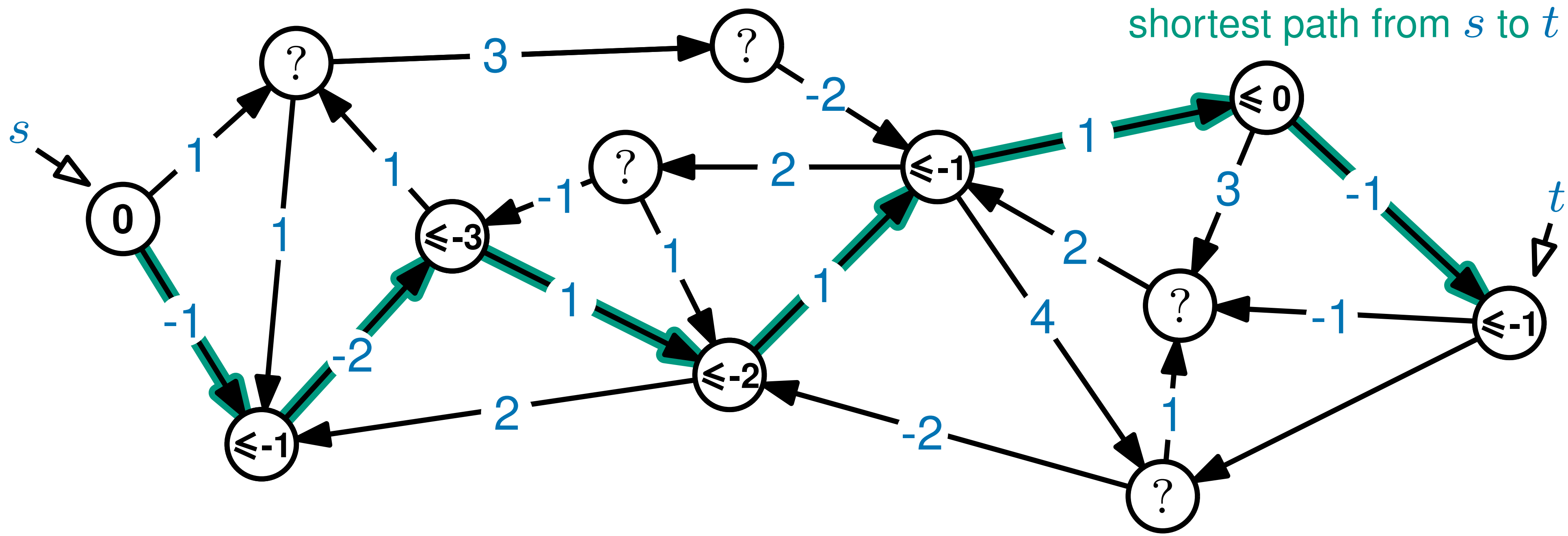

So after *enough* iterations...

$\texttt{dist}(t)$ is the length of a path from s to t

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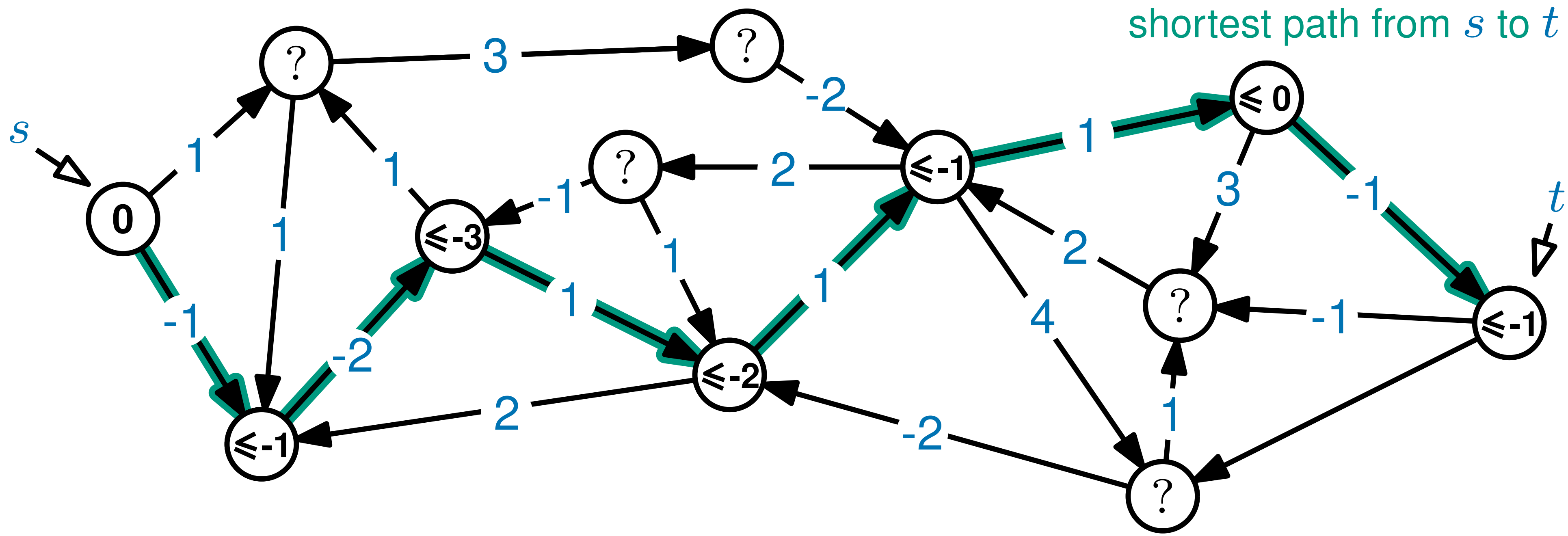

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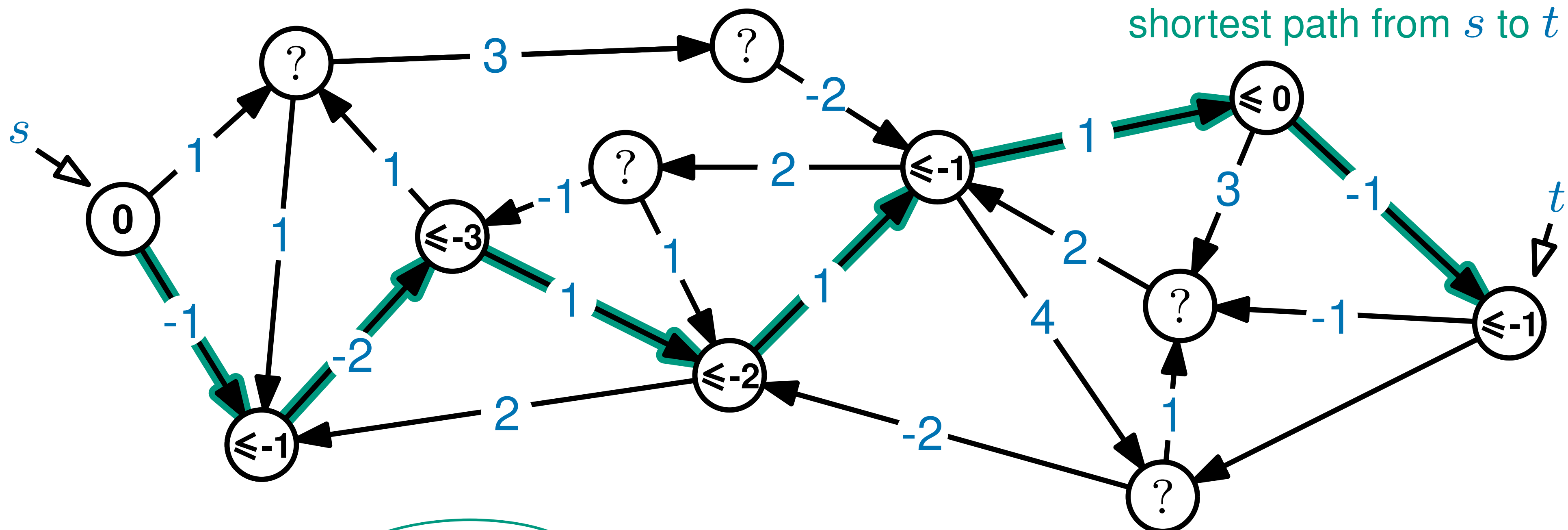

how many iterations are needed?

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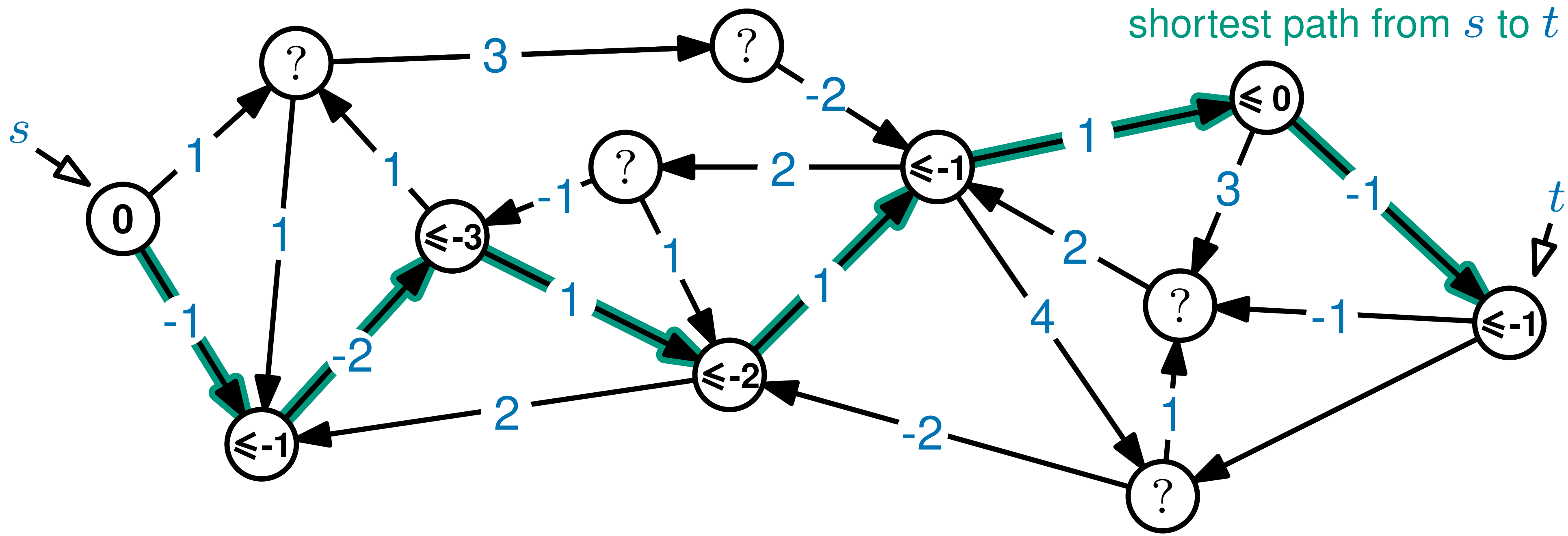

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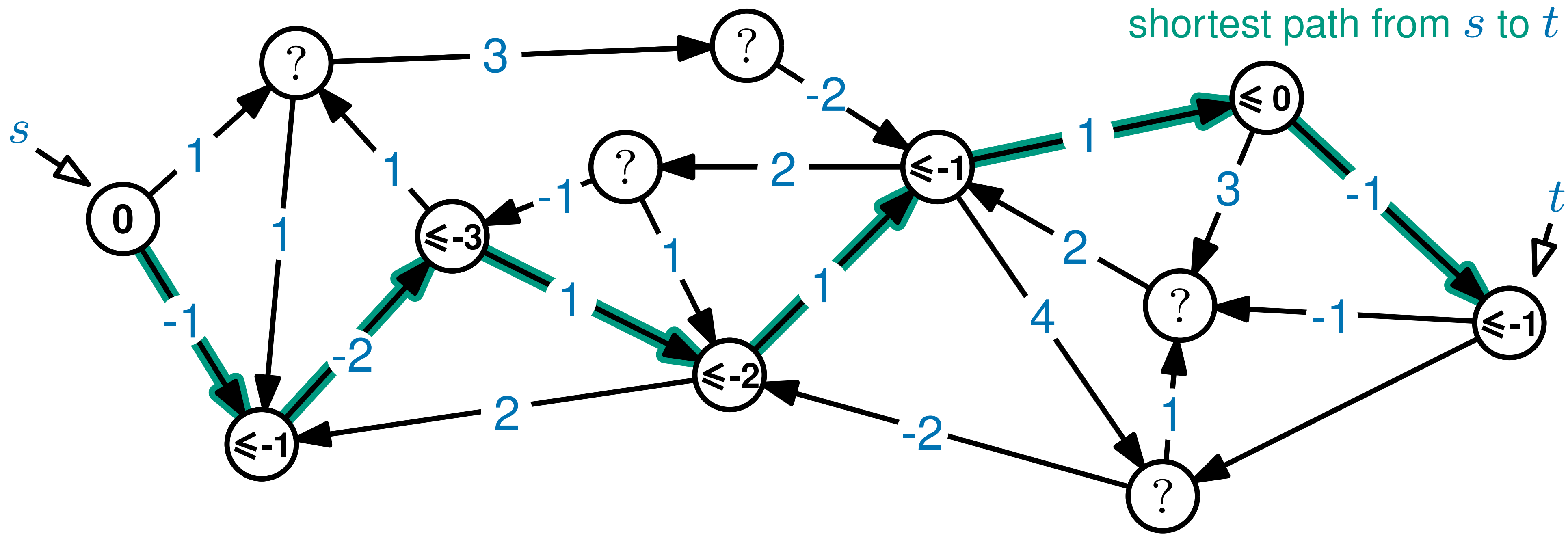

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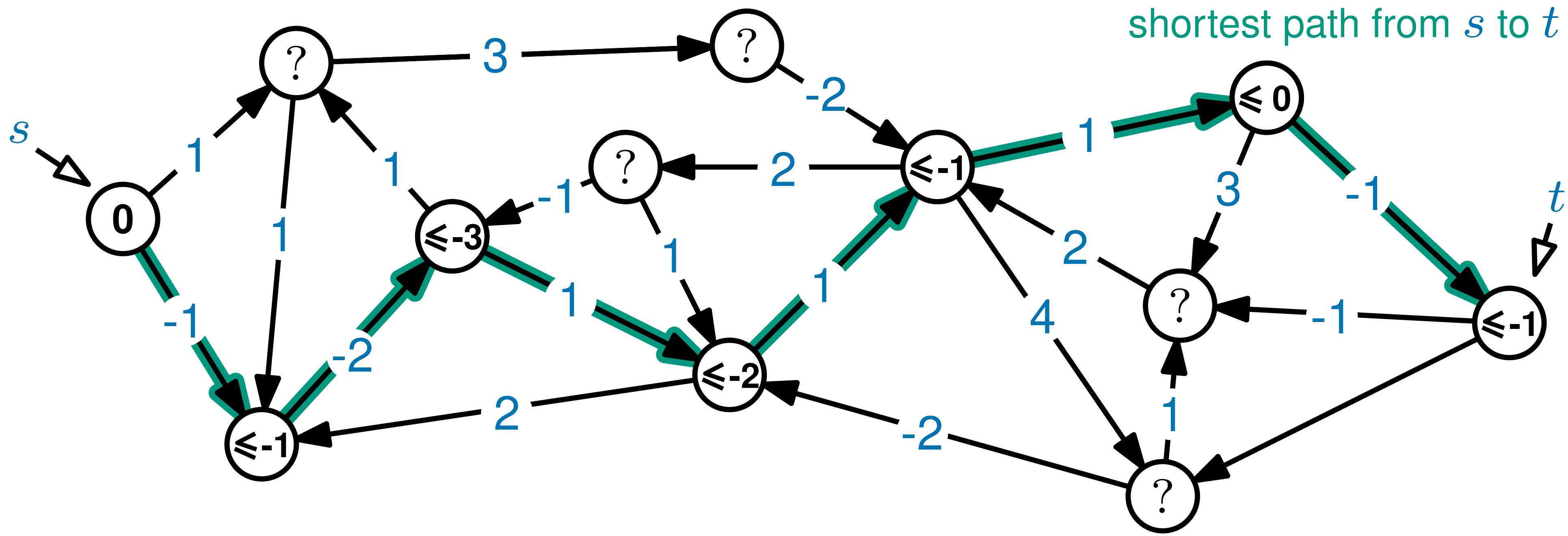

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and therefore $|V|$ iterations will be enough

Claim *if there are no negative weight cycles* in the graph,

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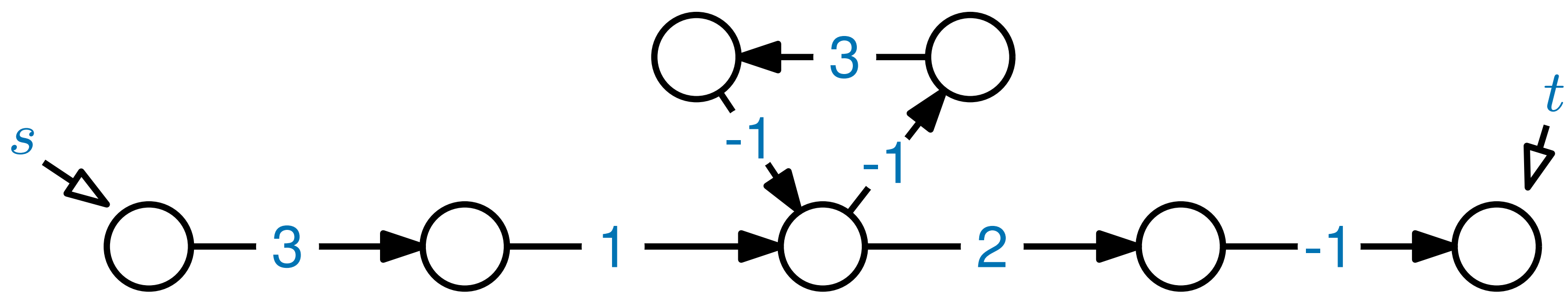

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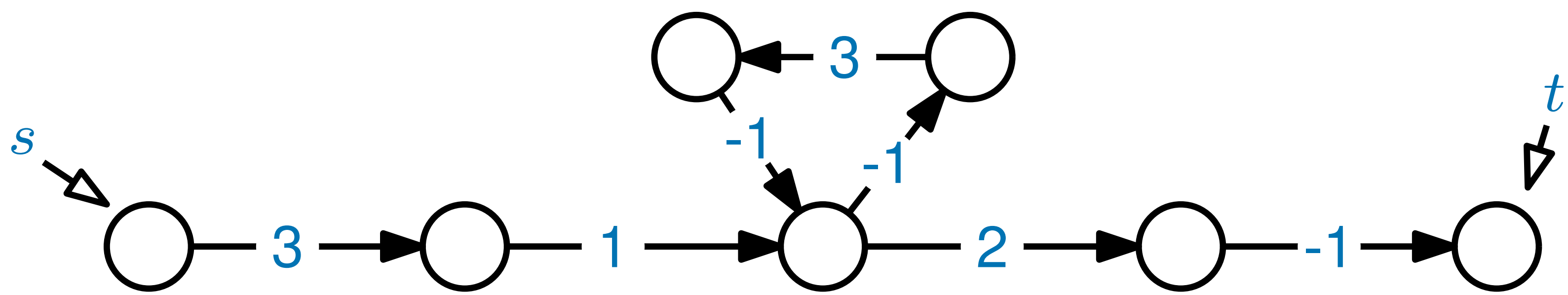

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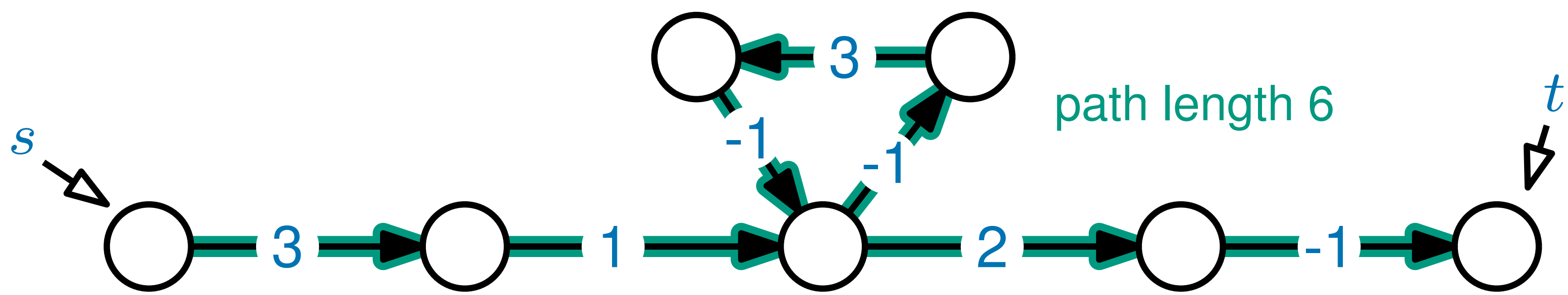

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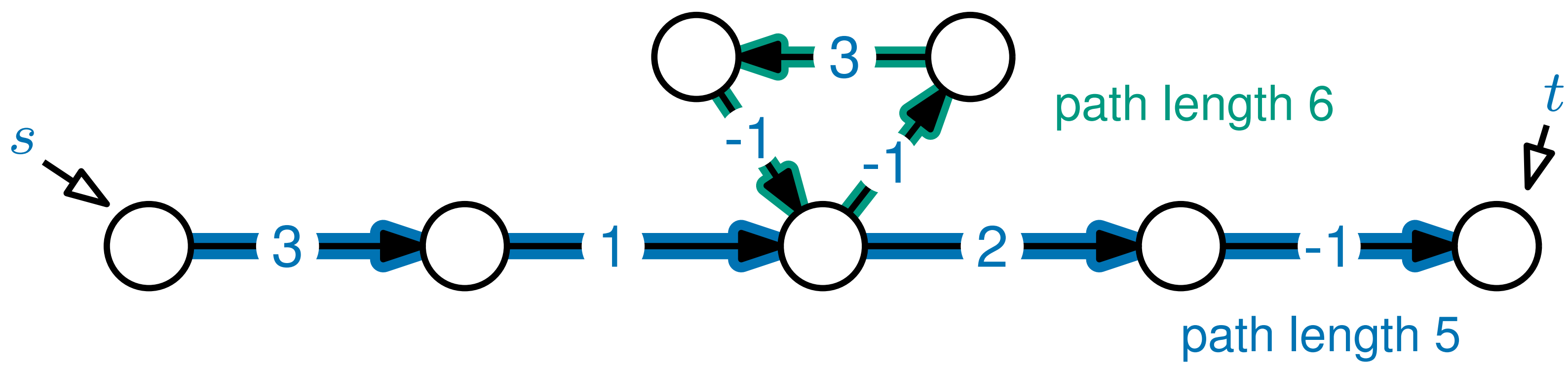

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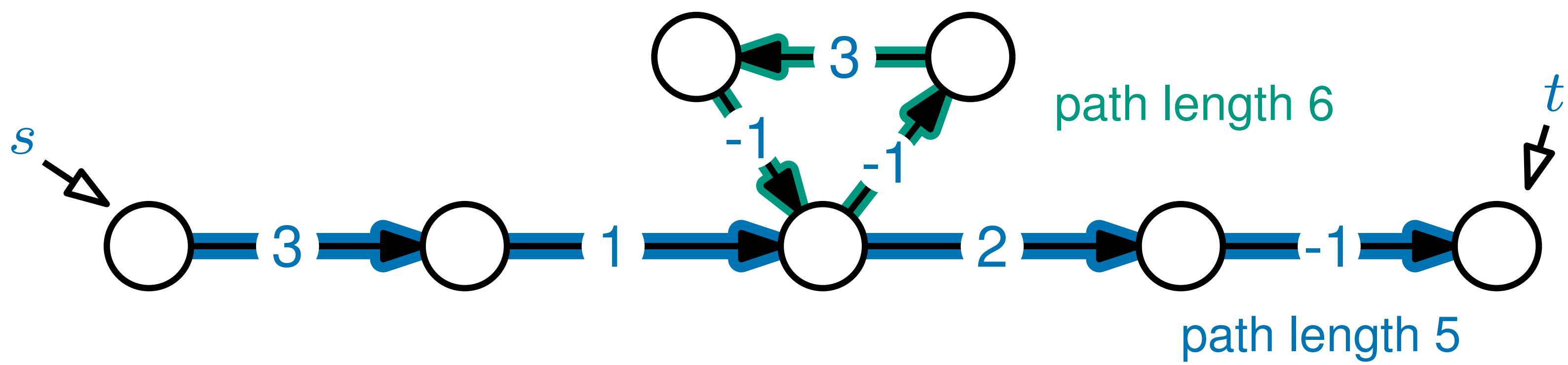

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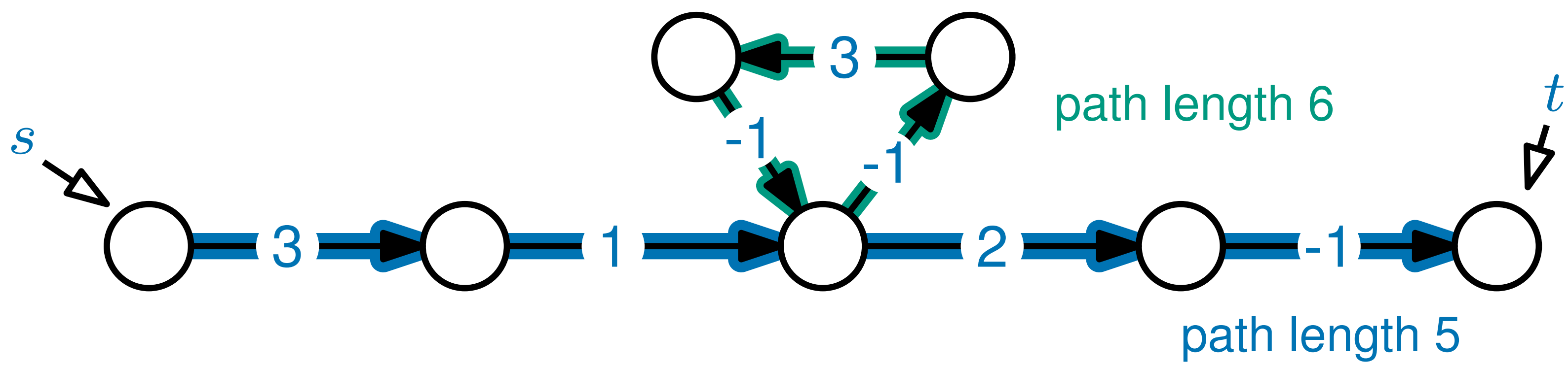

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A path with no cycles enters each vertex at most once

so contains at most $|V|$ edges

MostOfBellman-Ford Summary

Claim When the MostOfBellman-Ford algorithm terminates,

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After $|V|$ iterations of the algorithm
$\texttt{dist}(v)$ is the length of the shortest path between s and v

So what is the rest of the algorithm?

BELLMAN-FORD(s)

```
For all v, set dist(v) = ∞
set dist(s) = 0
For i = 1, 2, ..., |V|,
  For every edge (u, v) ∈ E
    if dist(v) > dist(u) + weight(u, v)
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We will prove that this is happens if and only if there is a negative weight cycle

Bellman-Ford without negative weight cycles

We first argue that the algorithm still works when there is no negative weight cycle

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If the final check outputs: 'Negative weight cycle found'

then there is a path from s to some v (via u) with length $\texttt{dist}(u) + \texttt{weight}(u, v) < \texttt{dist}(v)$

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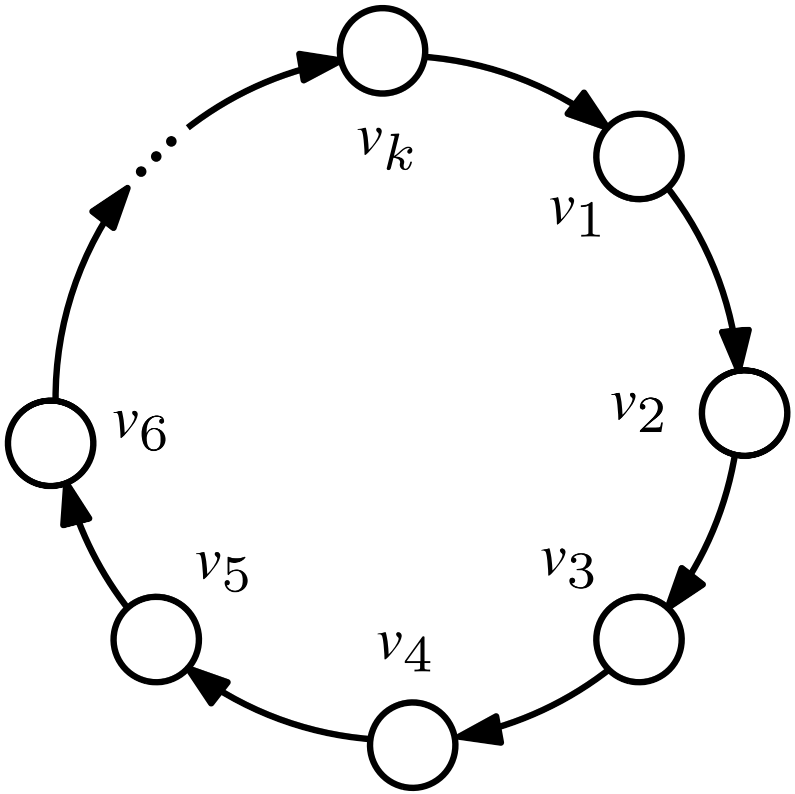
i.e. a path which is shorter than the shortest path *this is a contradiction.*

BELLMAN-FORD *with* a negative weight cycle

Let $v_1, v_2, \ldots v_k \in V$ be a negative weight cycle
and assume for a contradiction, that it wasn't reported by the algorithm



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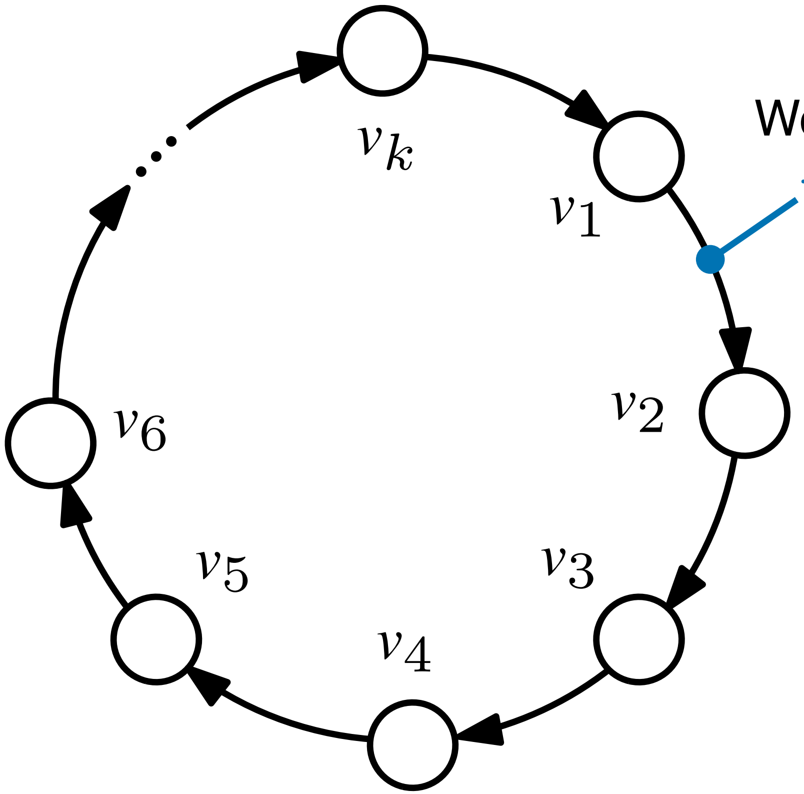

We have that,
$\texttt{weight}(v_1, v_2) \geqslant \texttt{dist}(v_2) - \texttt{dist}(v_1)$

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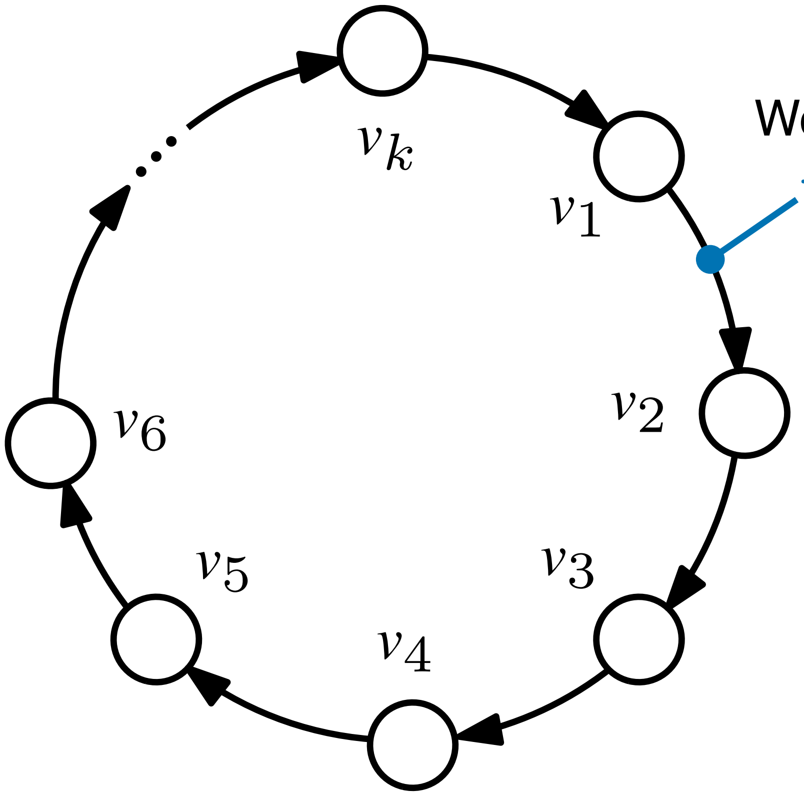

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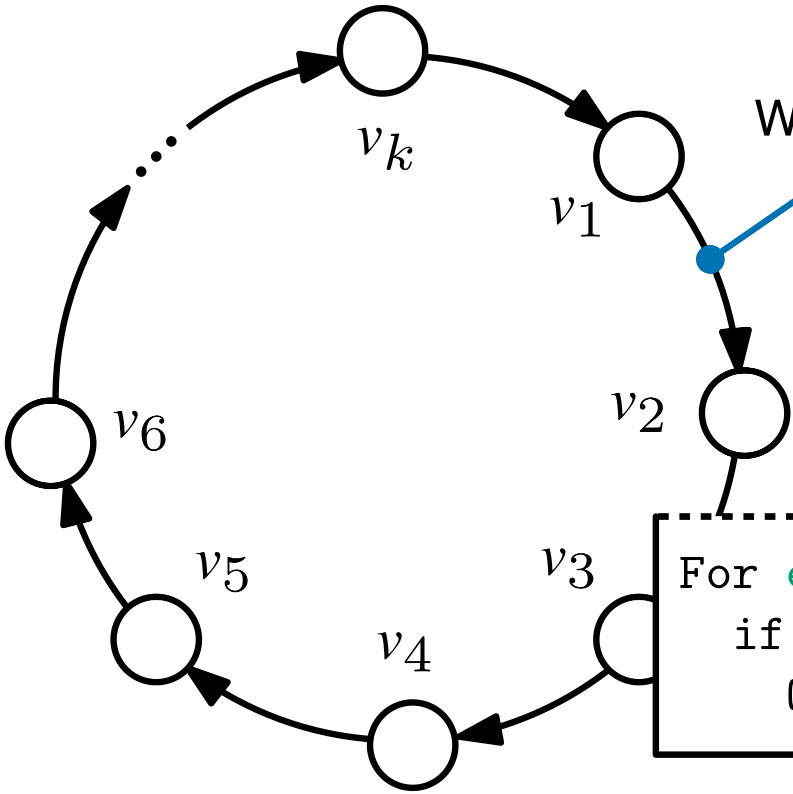

We have that,

$$\texttt{weight}(v_1, v_2) \geqslant \texttt{dist}(v_2) - \texttt{dist}(v_1)$$

because we ran the following checks
but found no negative weight cycle:

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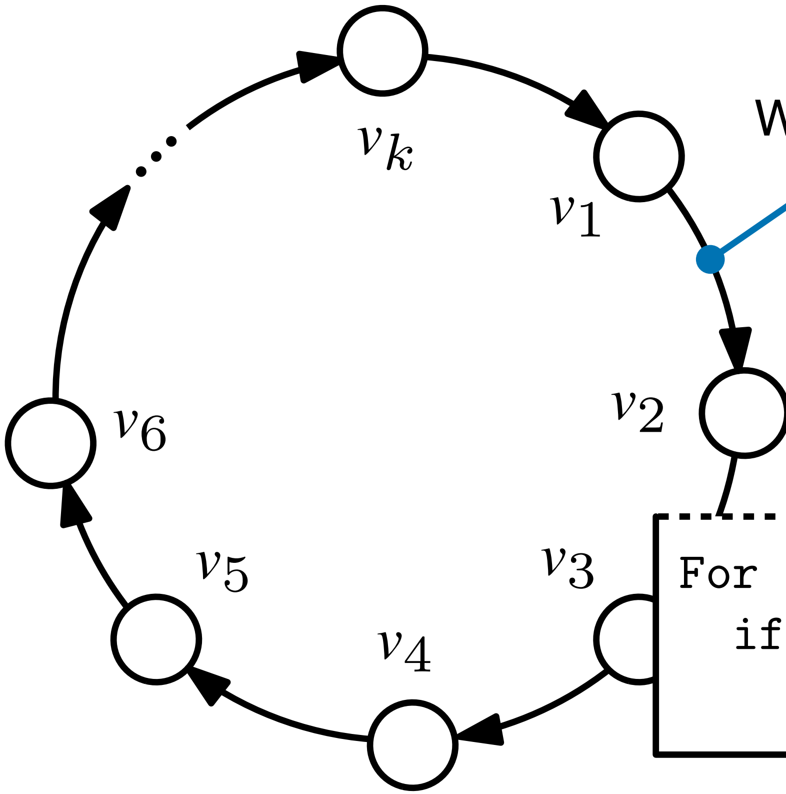

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So,

$$\texttt{dist}(v_2) \leqslant \texttt{dist}(v_1) + \texttt{weight}(v_1, v_2)$$

(then rearrange)

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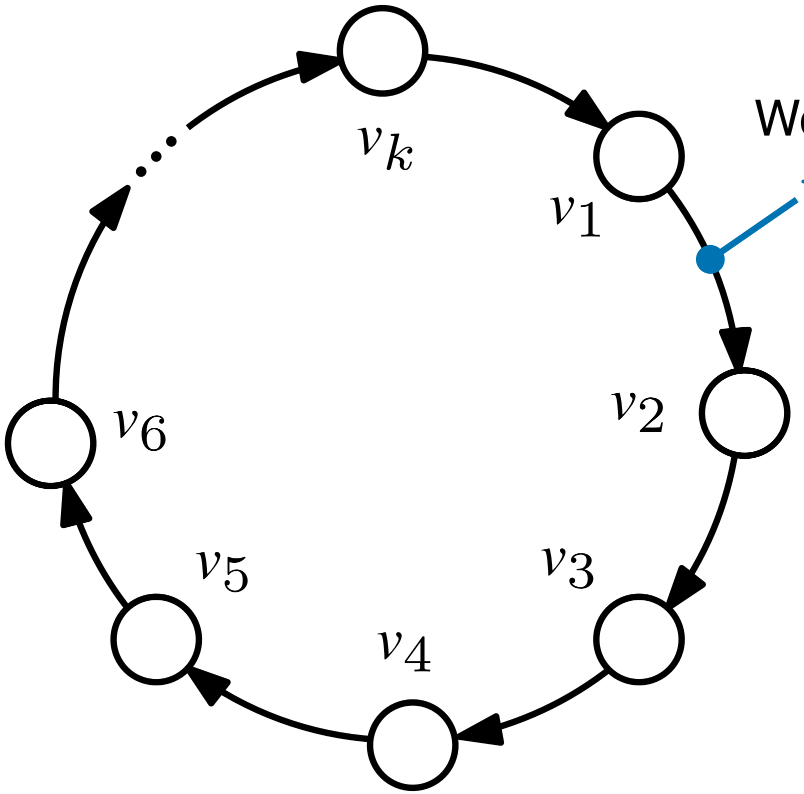

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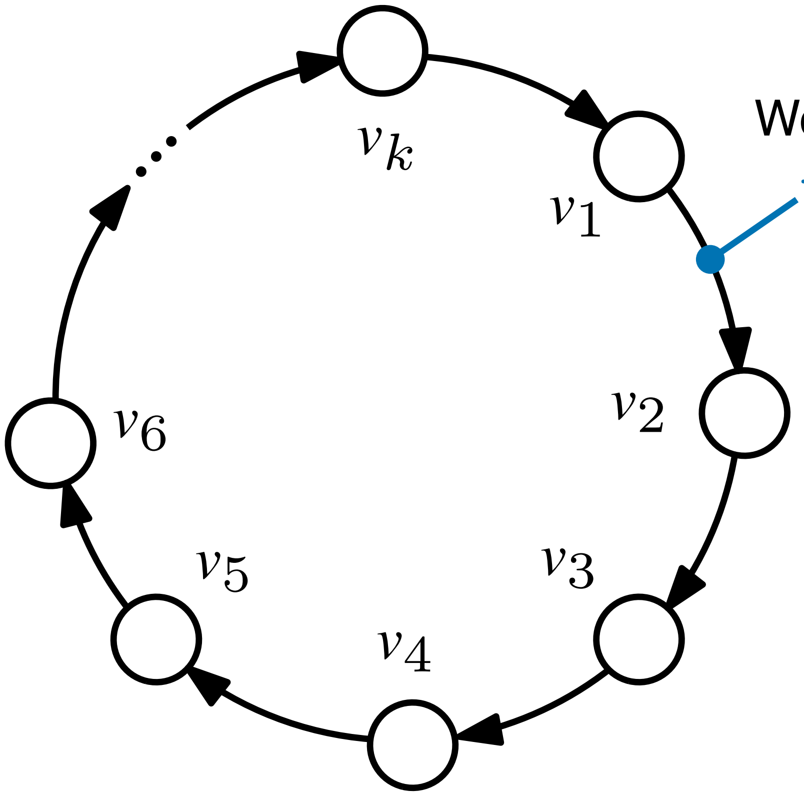

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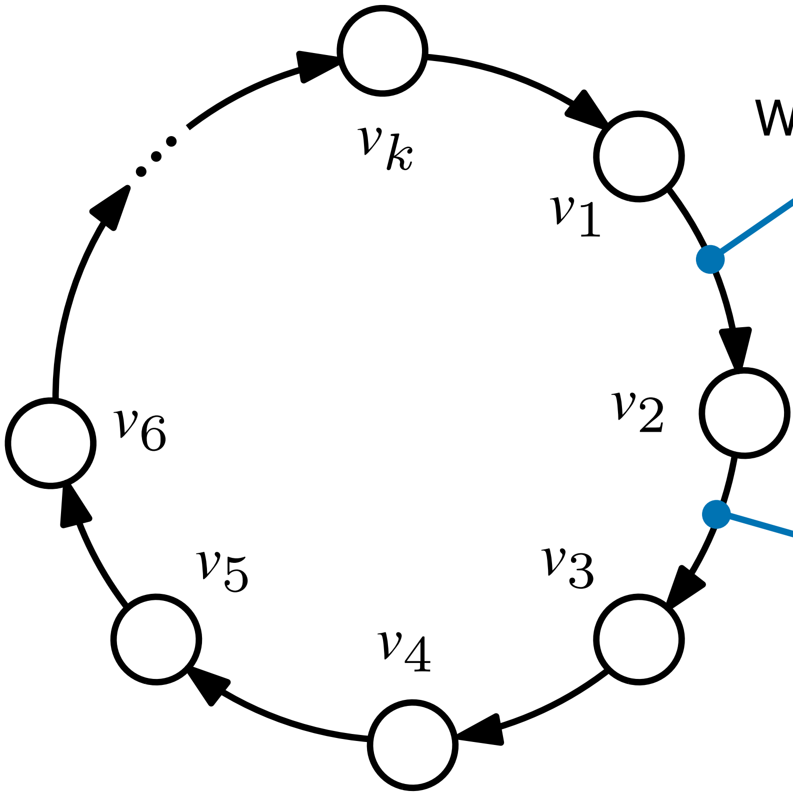

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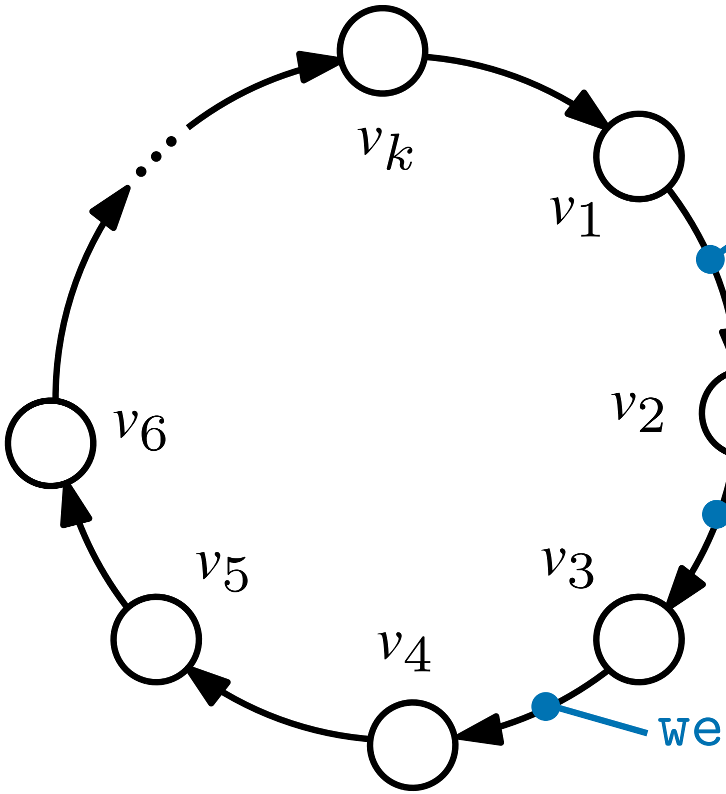

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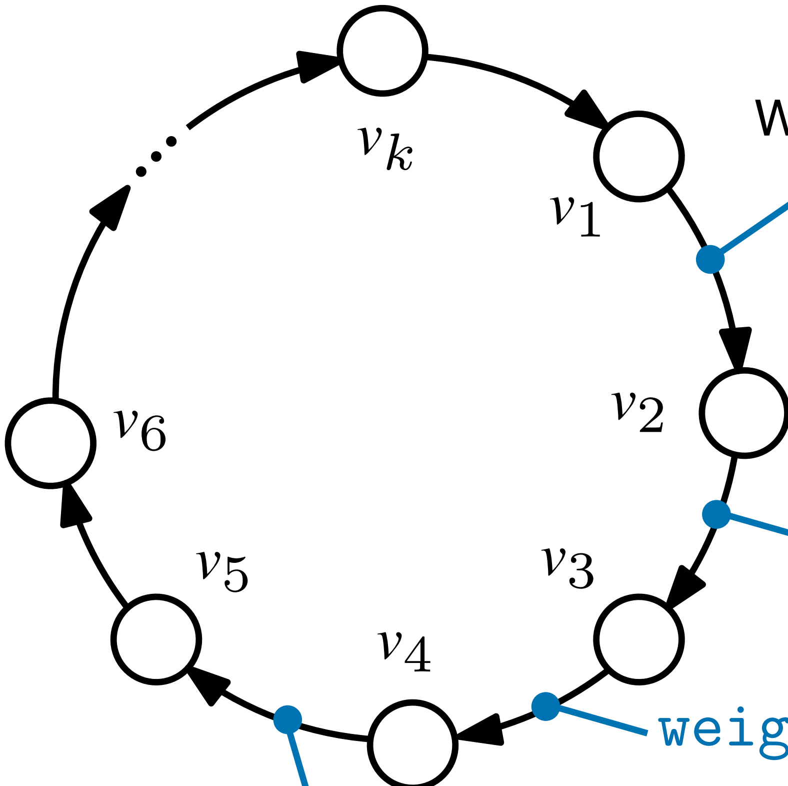
BELLMAN-FORD *with* a negative weight cycle

Let $v_1, v_2, \ldots v_k \in V$ be a negative weight cycle
and assume for a contradiction, that it wasn't reported by the algorithm



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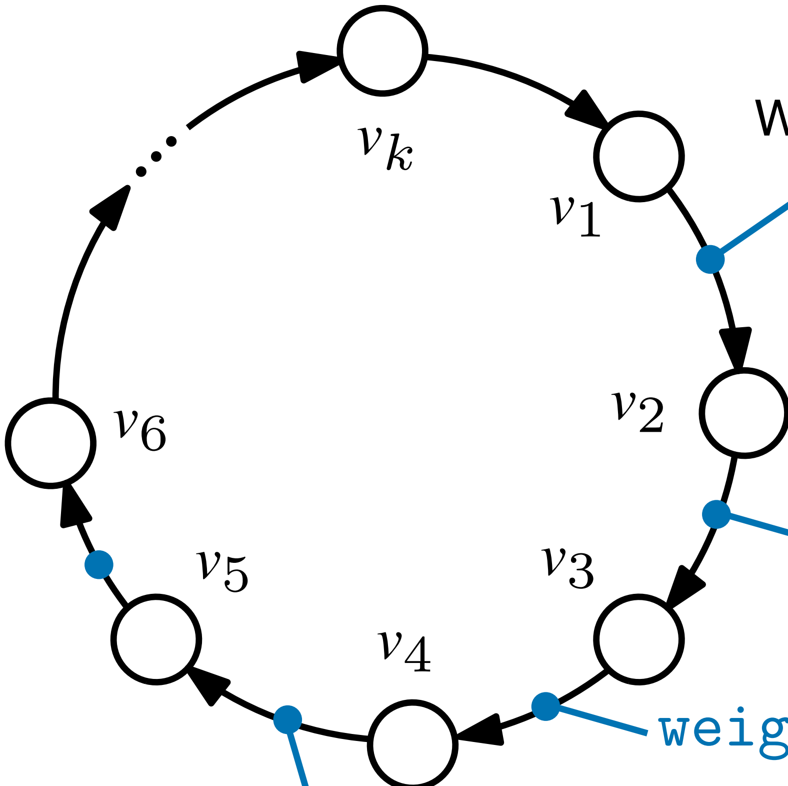

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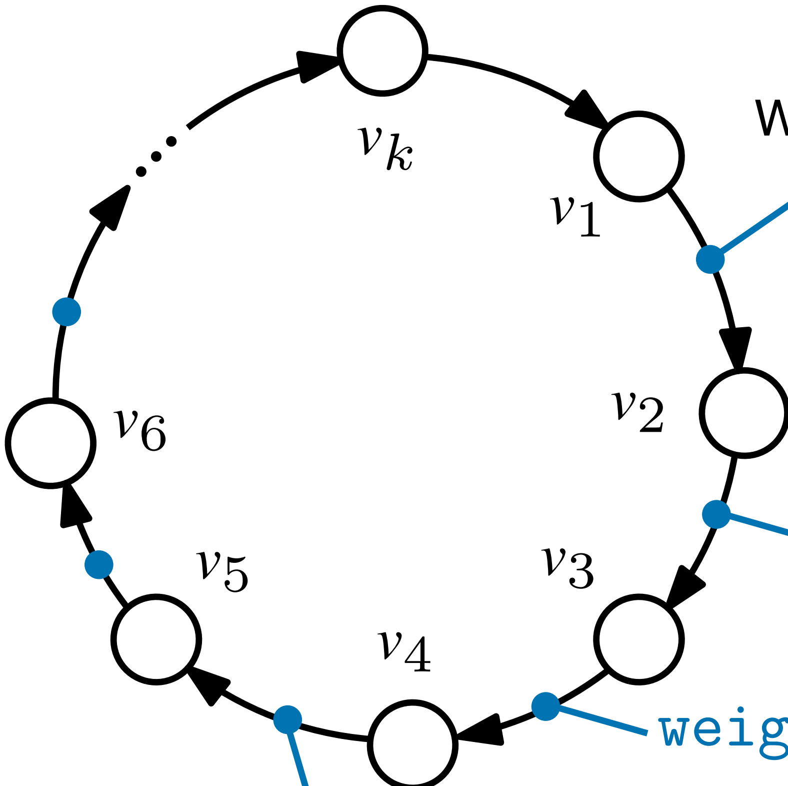

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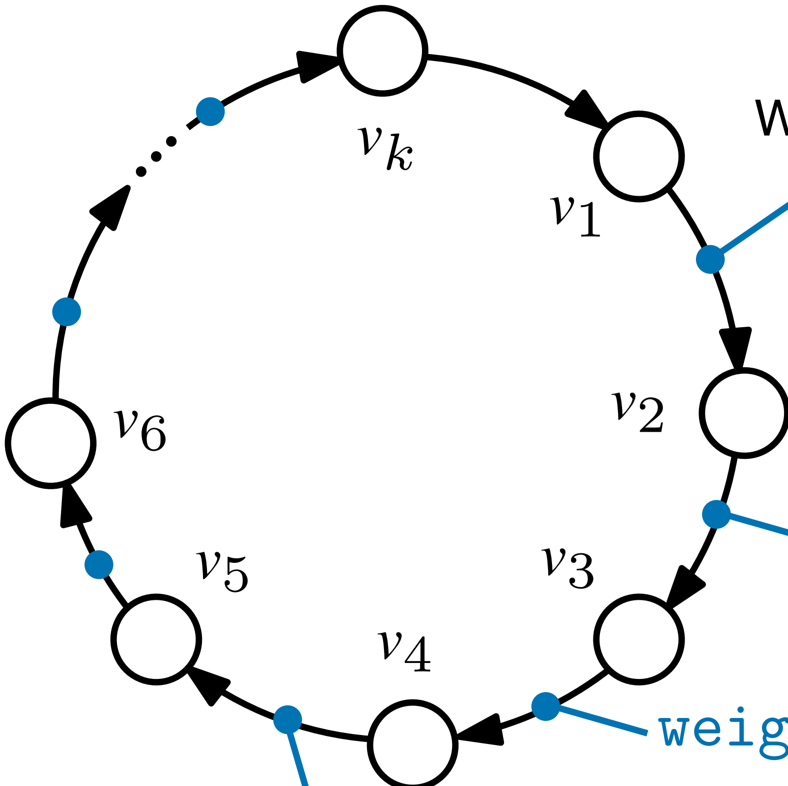

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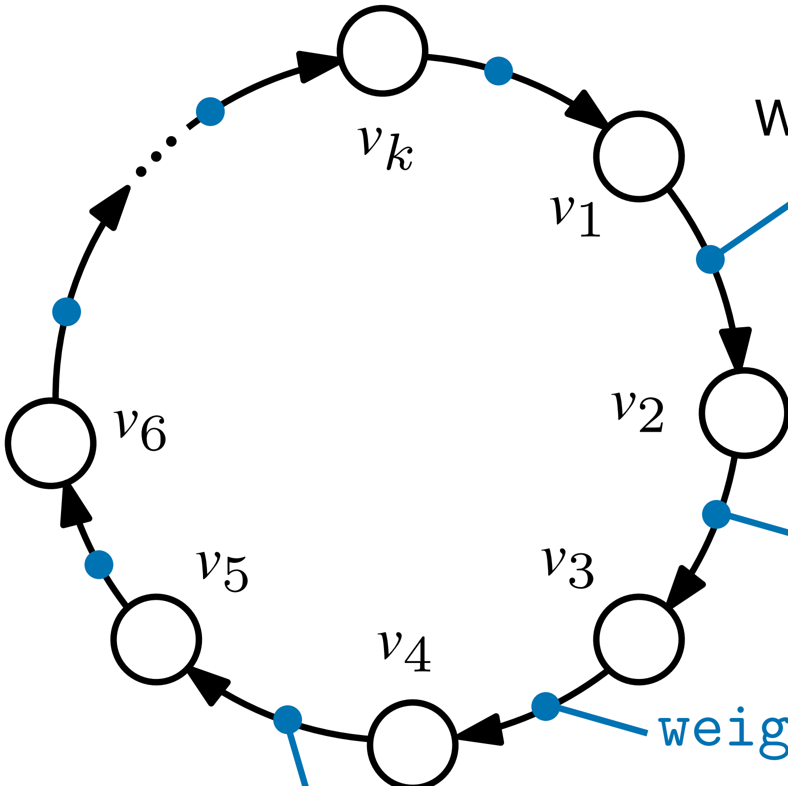

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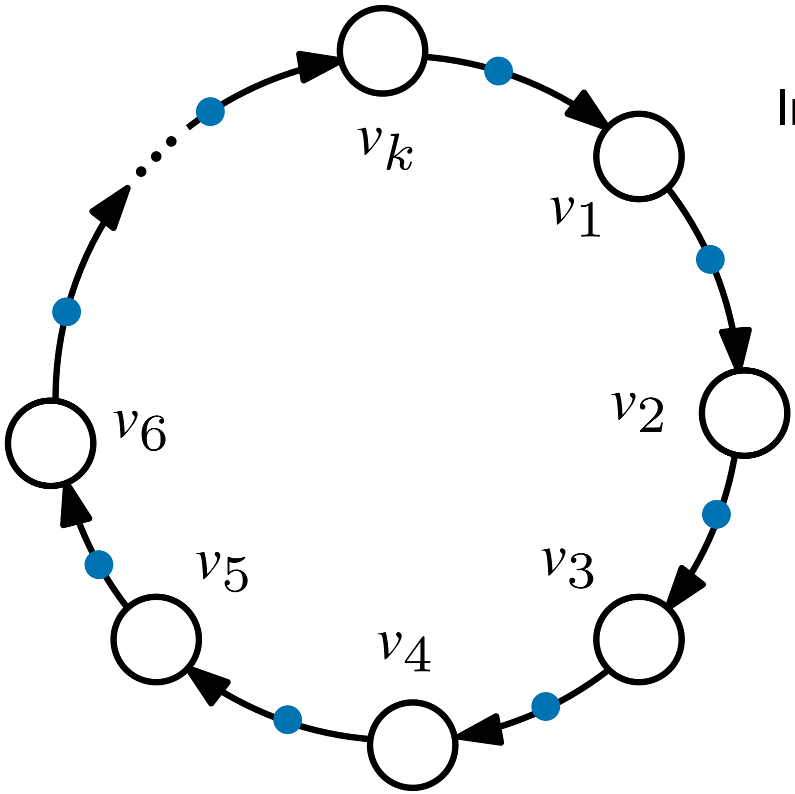

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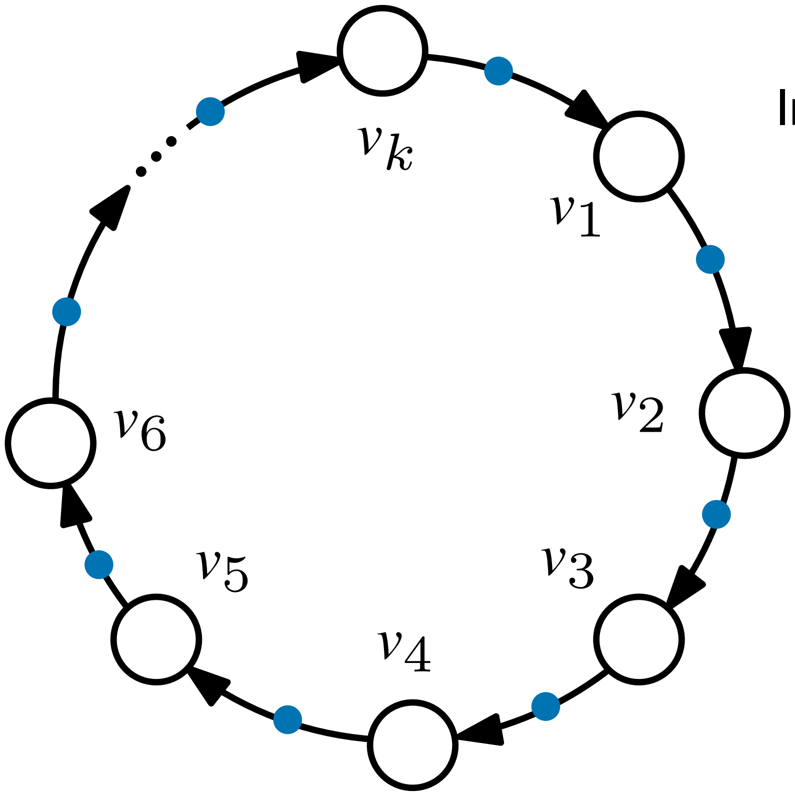

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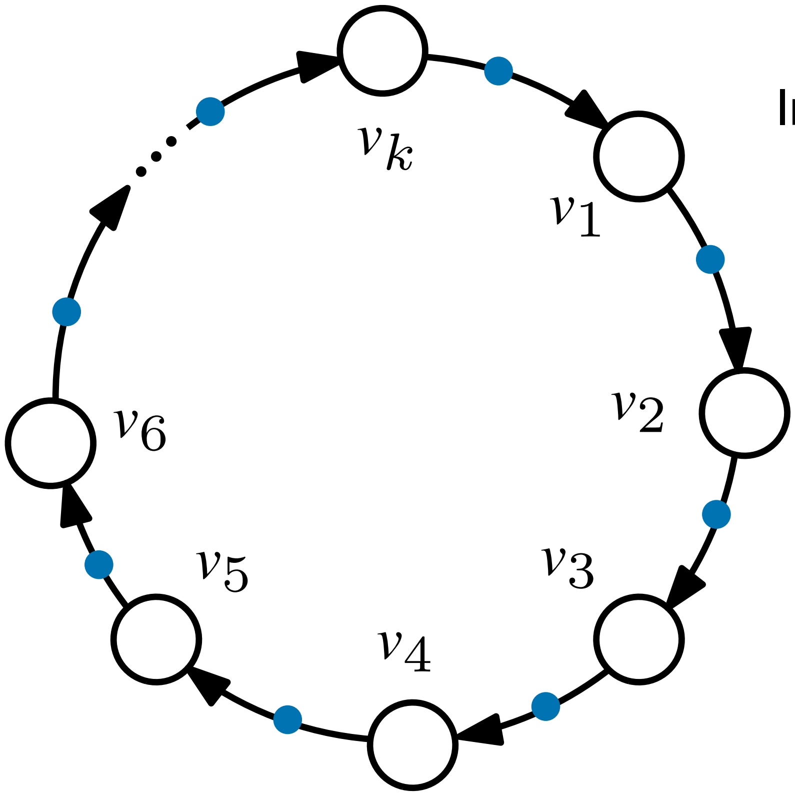

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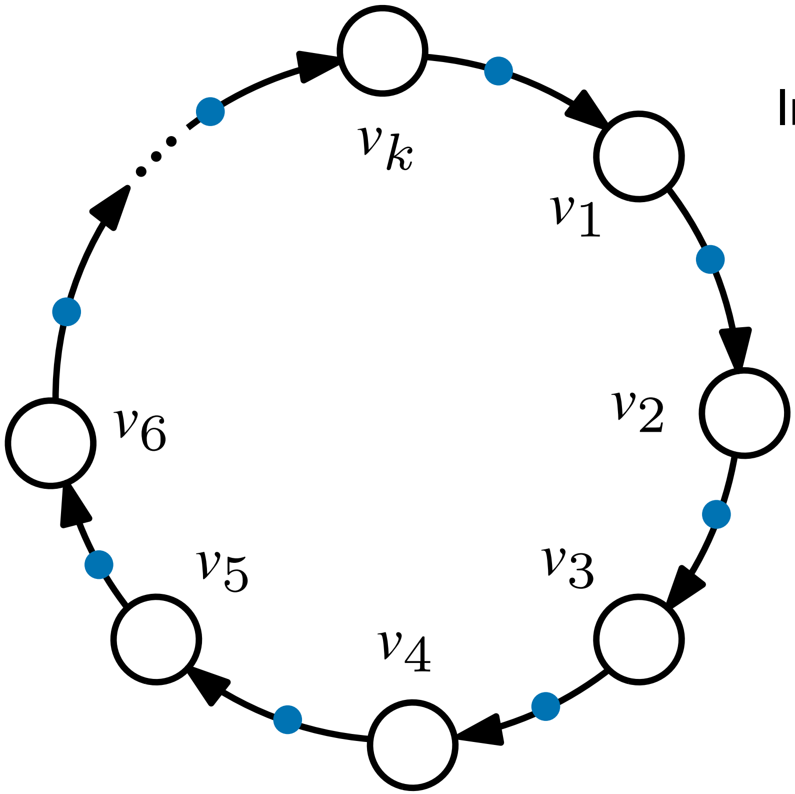

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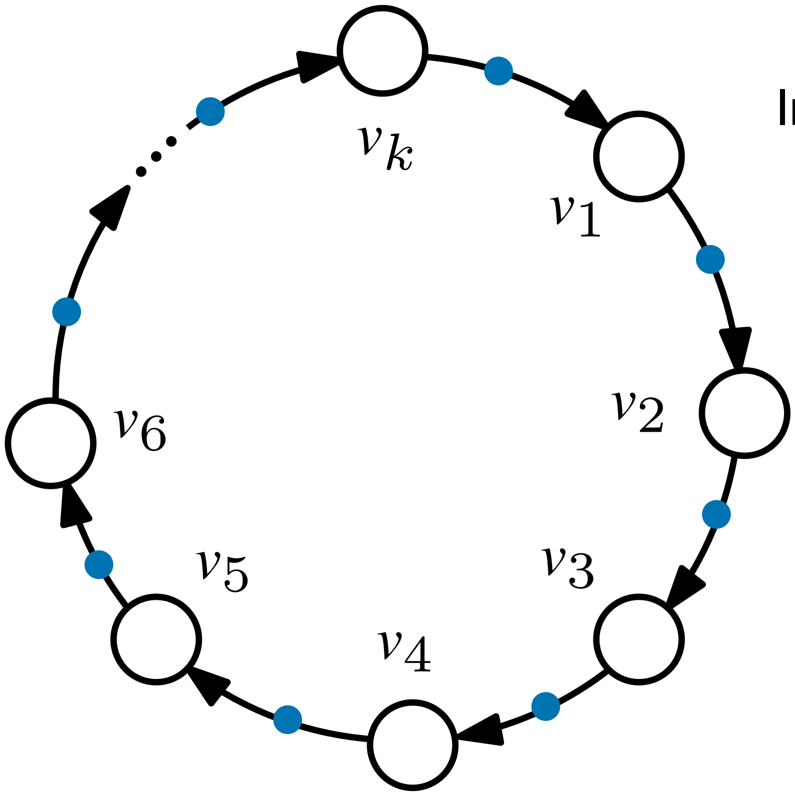

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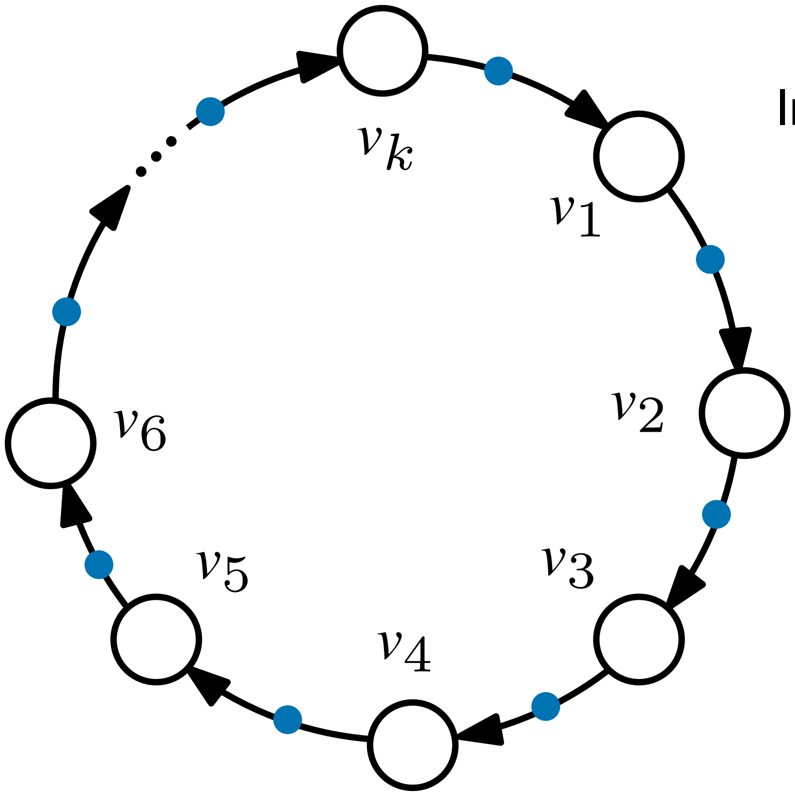

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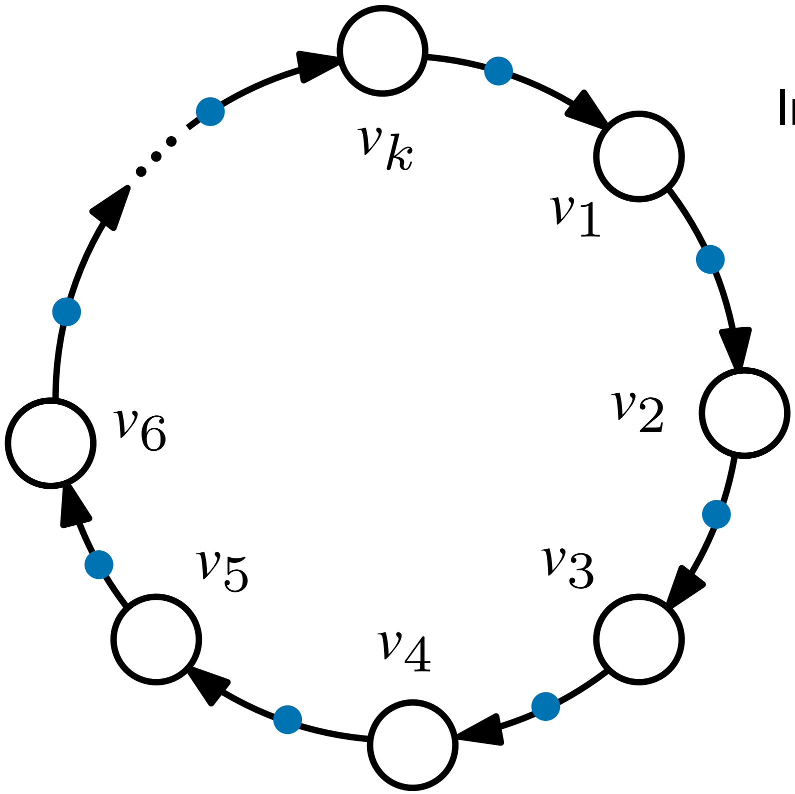

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Contradiction!

BELLMAN-FORD Summary

When the BELLMAN-FORD algorithm terminates,

for each vertex v, $\texttt{dist}(v)$ is the distance between s and v

(or it reports that there is a negative weight cycle)

BELLMAN-FORD(s)

For all v, set $\texttt{dist}(v) = \infty$
set $\texttt{dist}(s) = 0$
For $i = 1, 2, \ldots, |V|$,
  For every edge $(u, v) \in E$
    if $\texttt{dist}(v) > \texttt{dist}(u) + \texttt{weight}(u, v)$
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What is the time complexity of the BELLMAN-FORD algorithm?

Time Complexity

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$O(|V|)$ time

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(*we store* dist *in an array of length* $|V|$)

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$O(|V|)$ time

(*we store* $\texttt{dist}$ *in an array of length* $|V|$)

$O(1)$ time

$O(|E|)$ time for each iteration

Time Complexity

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$O(|V|)$ time

(*we store* dist *in an array of length* $|V|$)

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…and there are $|V|$ iterations

Time Complexity

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$O(|V|)$ time

(*we store* $\texttt{dist}$ *in an array of length* $|V|$)

$O(1)$ time

$O(|V||E|)$ time overall

$O(|E|)$ time

so overall the BELLMAN-FORD algorithm takes $O(|V||E|)$ time

Summary

The Bellman-Ford algorithm solves the **single source shortest paths** problem in a directed, weighted graph with **positive and negative edge weights**

If there is a negative weight cycle, the algorithm reports this and aborts *(in this case, the problem is not well-defined)*

The Bellman-Ford algorithm runs in $O(|V||E|)$ time

and uses no non-elementary data structures

This is not as good as Dijkstra's algorithm which runs in $O((|V| + |E|) \log |V|)$ time

(when implemented using a binary heap)

However, Dijkstra's algorithm only works for graphs with **non-negative edge weights**

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End of part one

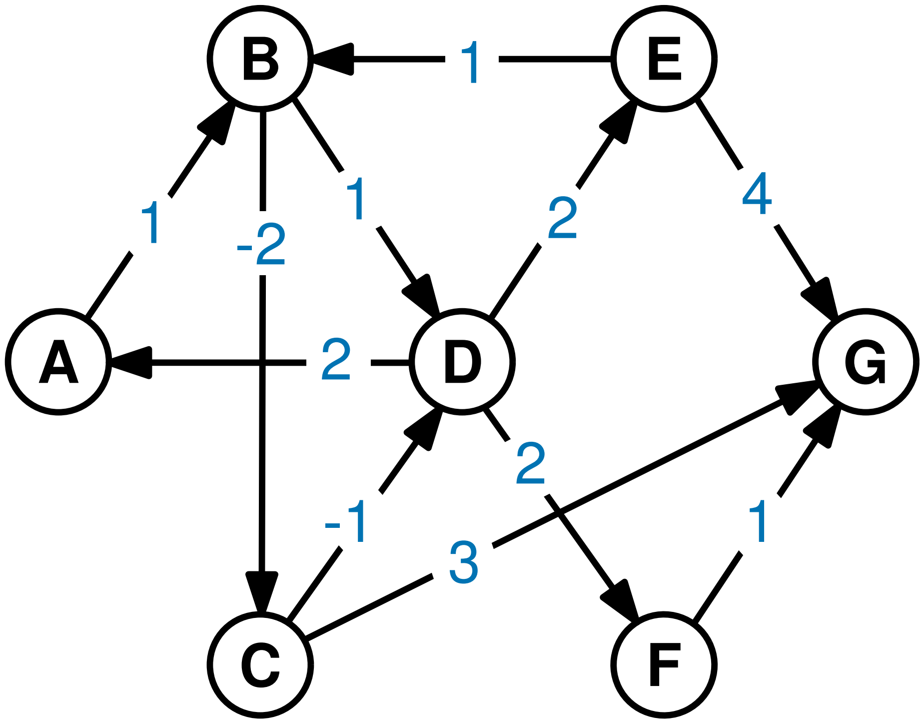
Part two

All-Pairs Shortest Paths

All-Pairs Shortest Paths

In previous lectures, we have focused on the **single source** **shortest paths** problem

in a **weighted**, directed graph. . .

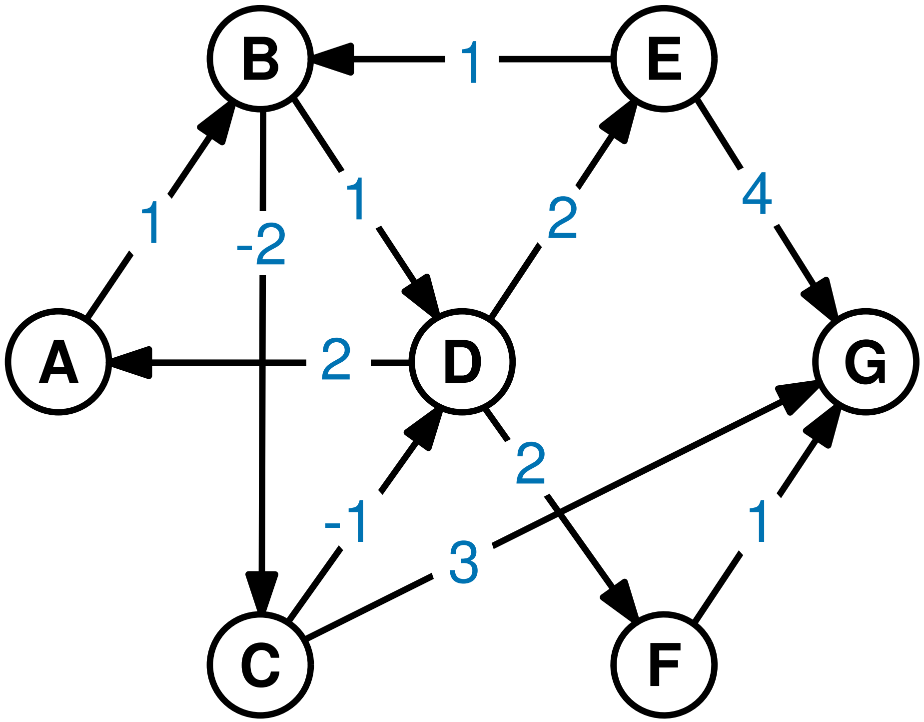


All-Pairs Shortest Paths

In previous lectures, we have focused on the **single source** **shortest paths** problem

in a **weighted**, directed graph...

i.e. in finding the shortest paths from a single given source vertex
to every other vertex



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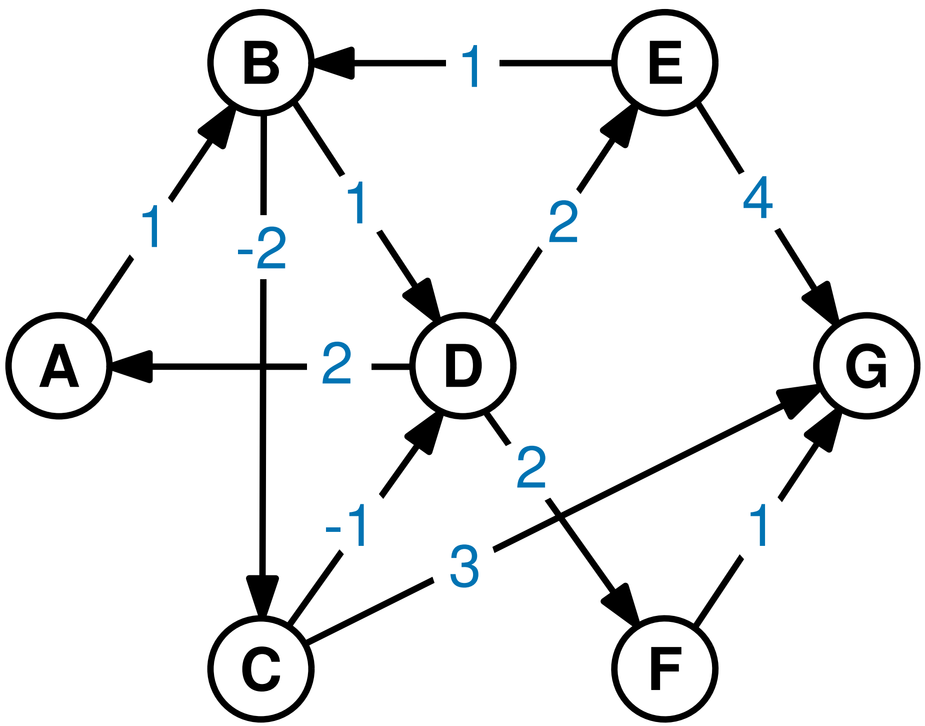

What should we do if we want to find the shortest paths from every vertex
to every other vertex?

Use something we saw before

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What should we do if we want to find the shortest paths from every vertex to every other vertex?

We already have two options:

If the graph has **non-negative edge weights**,

we could run DIJKSTRA's algorithm $|V|$ times, once with each vertex as the source

this takes $O(|V||E| \log |V|)$ time

(if the priority queue is a binary heap)

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this takes $O(|V||E| \log |V|)$ time

(if the priority queue is a binary heap)

If the graph has **positive and negative edge weights**,

we could run BELLMAN-FORD's algorithm $|V|$ times, once for each vertex

this takes $O(|V|^2|E|)$ time

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What should we do if we want to find the shortest paths from every vertex to every other vertex?

If the graph has **non-negative edge weights**,

repeatedly running DIJKSTRA's algorithm takes $O(|V||E| \log |V|)$ time

If the graph has **positive and negative edge weights**,

repeatedly running BELLMAN-FORD's algorithm takes $O(|V|^2|E|)$ time

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Are these any good? *Can we do better?* *How does* $|V|$ *compare to* $|E|$*?*

The output contains the length of the shortest path between every pair of vertices

What should we do if we want to find the shortest paths from every vertex to every other vertex?

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There are $|V| \cdot (|V| - 1)$ pairs of vertices

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The output contains the length of the shortest path between every pair of vertices

There are $|V| \cdot (|V| - 1)$ pairs of vertices

so we can't expect to do better than $O(|V|^2)$ time

What should we do if we want to find the shortest paths from every vertex to every other vertex?

If the graph has **non-negative edge weights**,

repeatedly running DIJKSTRA's algorithm takes $O(|V||E|\log|V|)$ time

If the graph has **positive and negative edge weights**,

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Imagine that $|E| \approx \frac{|V|^2}{4}$ (the graph is very dense)

e.g. each vertex has an edge to about half of the other vertices

What should we do if we want to find the shortest paths from every vertex to every other vertex?

If the graph has **non-negative edge weights**,

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$O(|V|^2 \log |V|)$ is *a lot better* than $O(|V|^3)$

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In this lecture, we will discuss JOHNSON's algorithm for all-pairs shortest paths in graphs with **positive and negative edge weights** which takes $O(|V||E|\log|V|)$ time

It will use *both* DIJKSTRA's algorithm (implemented with a binary heap) and BELLMAN-FORD to achieve this

JOHNSON's algorithm - the overall approach

We have already seen one algorithm for all-pairs shortest paths
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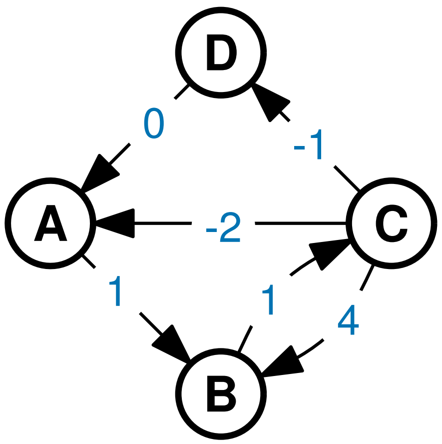

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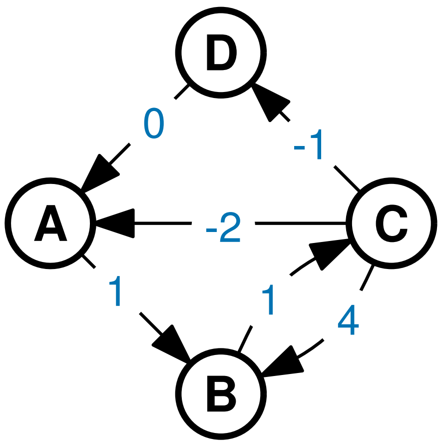

The approach employed by JOHNSON's algorithm is to **reweight** the edges

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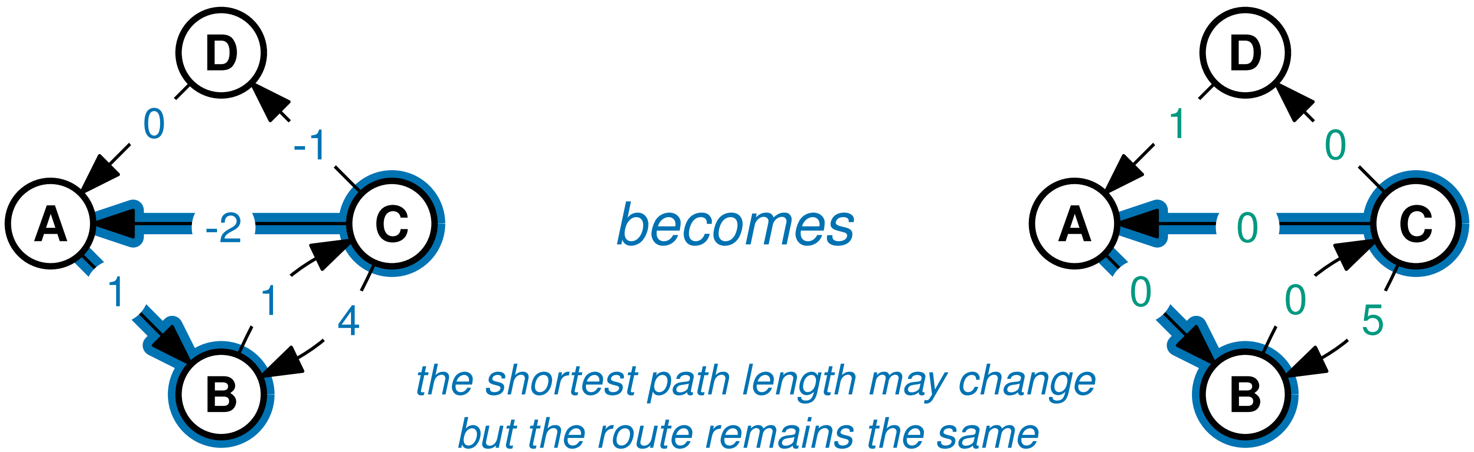

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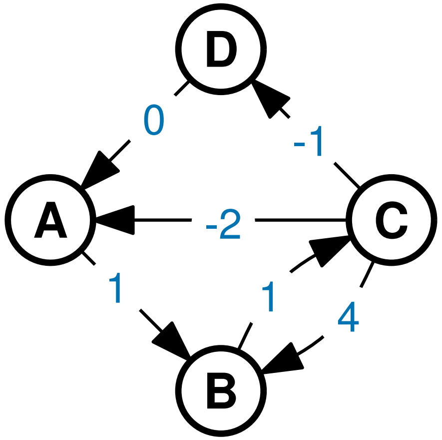

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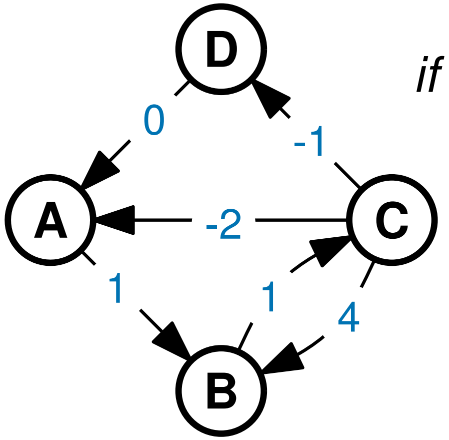
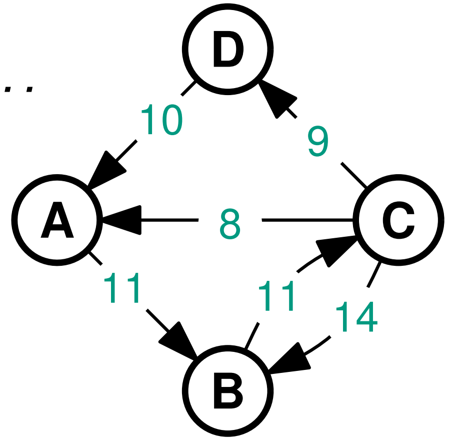


if we add 10 *to every edge. . .*

becomes



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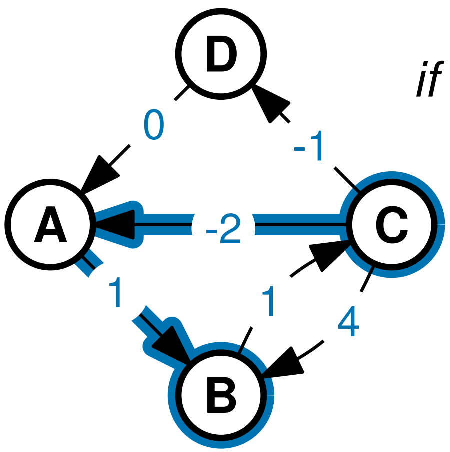

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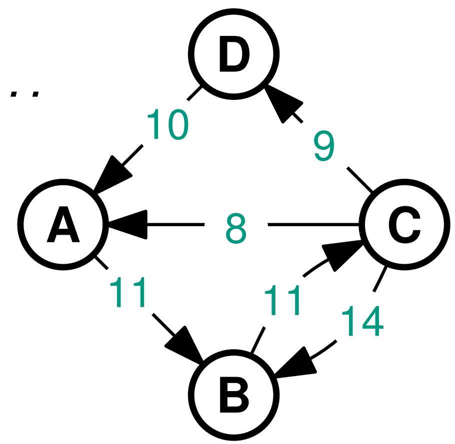

The shortest path from **C** to **B** goes this way in the original graph

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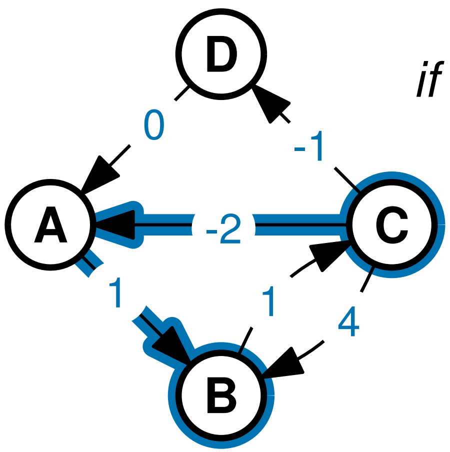

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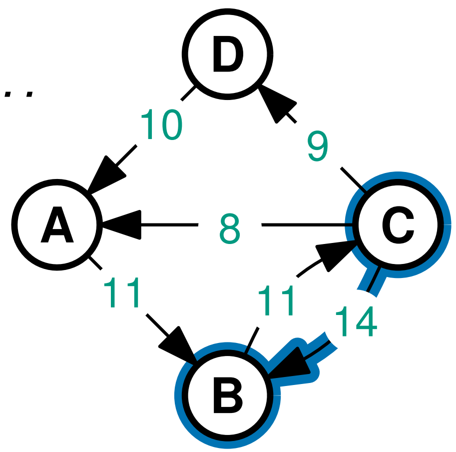

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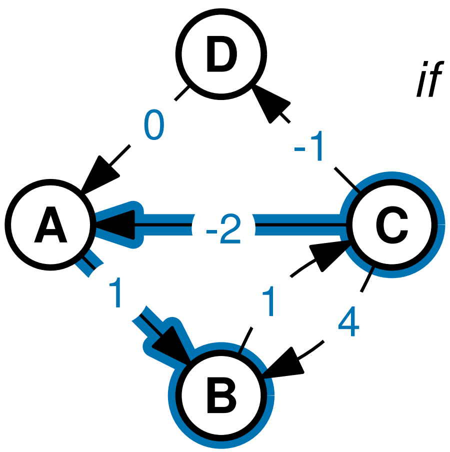

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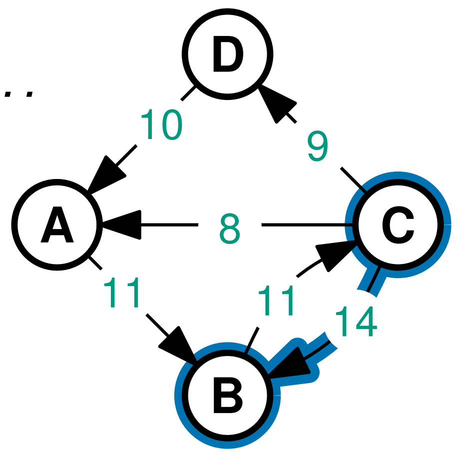

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Unfortunately, if we reweight the graph like this, both the shortest paths lengths and the routes might change

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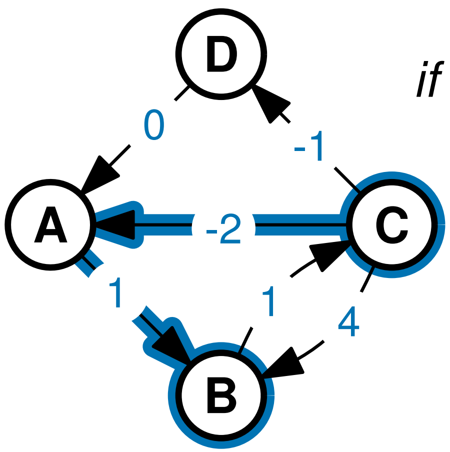

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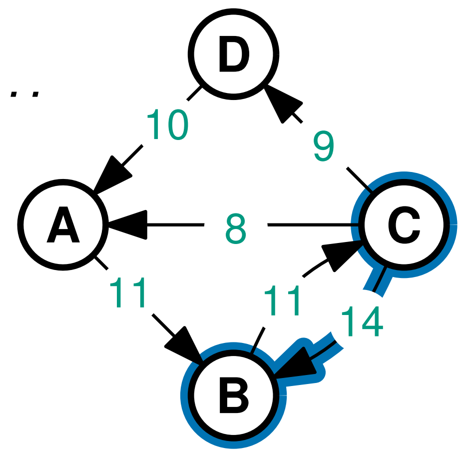

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The shortest path from **C** to **B** goes this way in the reweighted graph

Unfortunately, if we reweight the graph like this,
both the shortest paths lengths and the routes might change

this doesn't look good

Reweighting based on vertex potential

To overcome this, we are going to reweight each edge differently

the new weight of each edge will depend on which vertices it connects

We are going to associate a value $\mathsf{h}(v)$ with each vertex $v \in V$

- we will call this the potential of v

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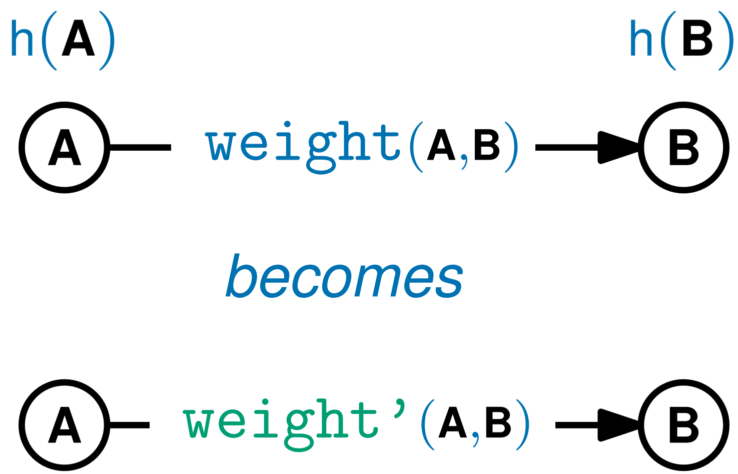

where $\texttt{weight'}(\mathbf{A},\mathbf{B}) = \texttt{weight}(\mathbf{A},\mathbf{B}) + \mathsf{h}(\mathbf{A}) - \mathsf{h}(\mathbf{B})$

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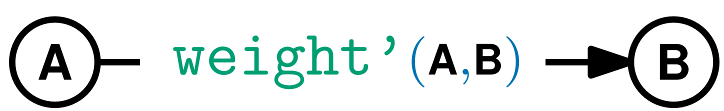

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Why is this better than the first attempt?

Reweighted paths

Each each vertex $v \in V$ has a value $\mathsf{h}(v)$ - called the potential of v

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(for now you can think of the values as arbitrary)

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Reweight the whole graph using `weight'` *(we don't just reweight this path)*...

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where $\texttt{weight'}(v_i, v_{i+1}) = \texttt{weight}(v_i, v_{i+1}) + \mathsf{h}(v_i) - \mathsf{h}(v_{i+1})$

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Reweight the whole graph using $\texttt{weight'}$ *(we don't just reweight this path)*. . .

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$$= (-2) + (5) - (2) = 1$$

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Reweight the whole graph using `weight'` *(we don't just reweight this path)*...

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we will pick these values carefully later

Consider the following path as an example. . .

$$v_1 \xrightarrow{-2} v_2 \xrightarrow{3} v_3 \xrightarrow{1} v_4 \xrightarrow{7} v_5 \xrightarrow{4} v_6$$

$\mathsf{h}(v_1) = 5$ $\mathsf{h}(v_2) = 2$ $\mathsf{h}(v_3) = 0$ $\mathsf{h}(v_4) = 1$ $\mathsf{h}(v_5) = 2$ $\mathsf{h}(v_6) = 4$

- *this path has length* 13

Reweight the whole graph using $\texttt{weight'}$ *(we don't just reweight this path)*. . .

$$v_1 \xrightarrow{1} v_2 \xrightarrow{5} v_3 \xrightarrow{0} v_4 \xrightarrow{6} v_5 \xrightarrow{2} v_6$$

where $\texttt{weight'}(v_i, v_{i+1}) = \texttt{weight}(v_i, v_{i+1}) + \mathsf{h}(v_i) - \mathsf{h}(v_{i+1})$

Reweighted paths

Each each vertex $v \in V$ has a value $\mathsf{h}(v)$ - called the potential of v

we will pick these values carefully later

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$$v_1 \xrightarrow{1} v_2 \xrightarrow{5} v_3 \xrightarrow{0} v_4 \xrightarrow{6} v_5 \xrightarrow{2} v_6$$

- *this path has length* 14

where $\texttt{weight'}(v_i, v_{i+1}) = \texttt{weight}(v_i, v_{i+1}) + \mathsf{h}(v_i) - \mathsf{h}(v_{i+1})$

Reweighted paths

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we will pick these values carefully later

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- *this path has length* $14 = 13 + 5 - 4 = 13 + h(v_1) - h(v_6)$

where $\texttt{weight'}(v_i, v_{i+1}) = \texttt{weight}(v_i, v_{i+1}) + \mathsf{h}(v_i) - \mathsf{h}(v_{i+1})$

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is this a coincidence?

where $\mathtt{weight'}(v_i, v_{i+1}) = \mathtt{weight}(v_i, v_{i+1}) + \mathtt{h}(v_i) - \mathtt{h}(v_{i+1})$

Reweighted paths

Consider an arbitrary path...

$$v_1 \xrightarrow{?} v_2 \xrightarrow{?} v_3 \xrightarrow{?} v_4 \xrightarrow{?} v_5 \xrightarrow{\cdots} v_k$$

$$\mathsf{h}(v_1) \qquad \mathsf{h}(v_2) \qquad \mathsf{h}(v_3) \qquad \mathsf{h}(v_4) \qquad \mathsf{h}(v_5) \qquad \mathsf{h}(v_k)$$

Reweighted paths

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In the original graph, the path length is $\displaystyle\sum_{i=1}^{k-1} \mathtt{weight}(v_i, v_{i+1})$

Reweighted paths

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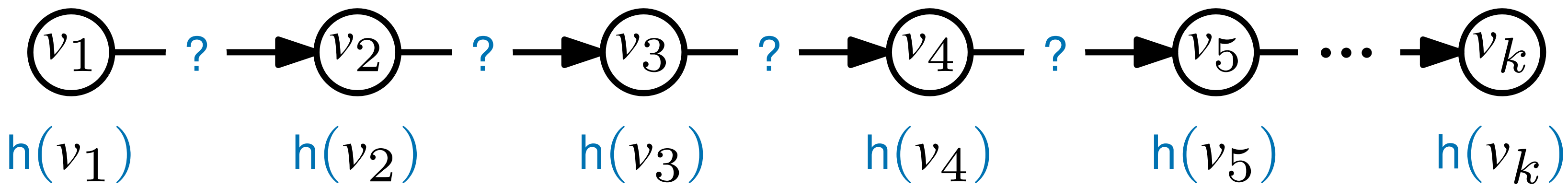

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Reweighted paths

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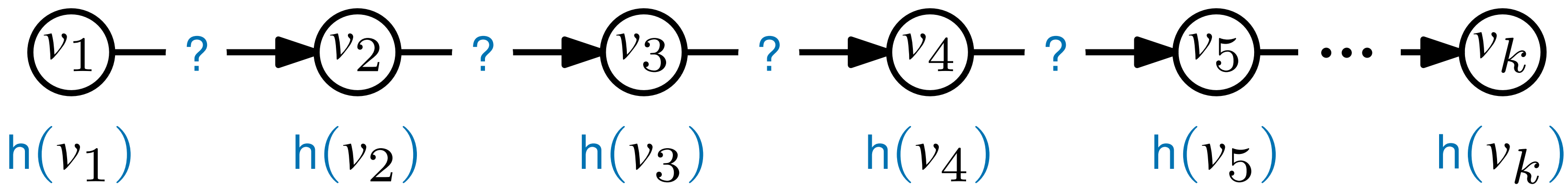

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In the reweighted graph, the path length is

$$\sum_{i=1}^{k-1} \texttt{weight'}(v_i, v_{i+1}) = \sum_{i=1}^{k-1} \big(\texttt{weight}(v_i, v_{i+1}) + \mathsf{h}(v_i) - \mathsf{h}(v_{i+1})\big)$$

Reweighted paths

Consider an arbitrary path...

$$v_1 \xrightarrow{\ ?\ } v_2 \xrightarrow{\ ?\ } v_3 \xrightarrow{\ ?\ } v_4 \xrightarrow{\ ?\ } v_5 \cdots \to v_k$$

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$$= \sum_{i=1}^{k-1} \big(\texttt{weight}(v_i, v_{i+1})\big) + \mathsf{h}(v_1) - \mathsf{h}(v_k)$$

So the weight of a path only changes by the potential values of the end points...

Reweighted shortest paths

Let the function $\mathtt{h}$ give a value $\mathtt{h}(v)$ for each vertex $v \in V$

Change the weight of every edge (u, v) to be

$$\mathtt{weight'}(u,v) = \mathtt{weight}(u,v) + \mathtt{h}(u) - \mathtt{h}(v)$$

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Lemma any path is a shortest path in the original graph
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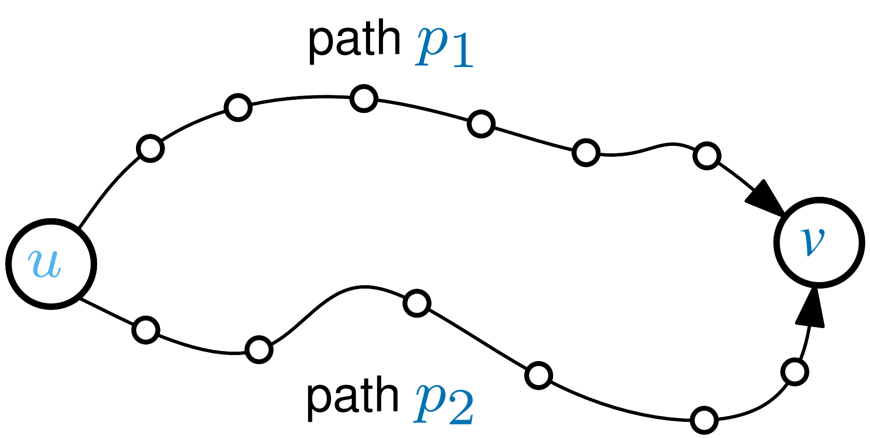

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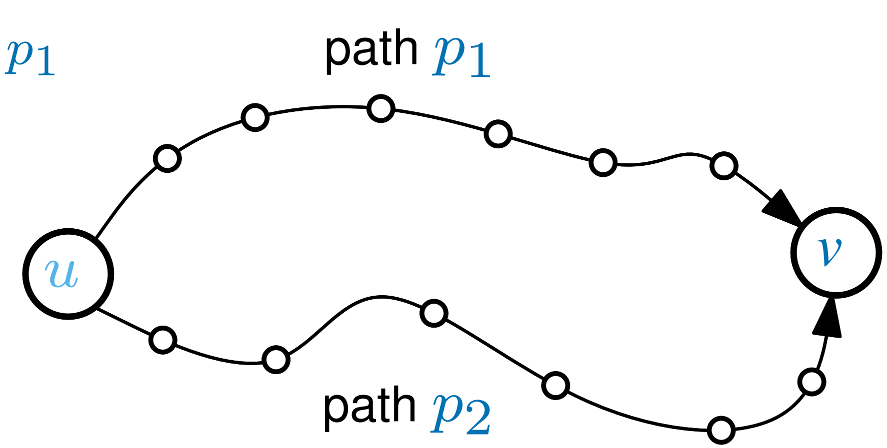
Let l_1 be the length of p_1
in the original graph



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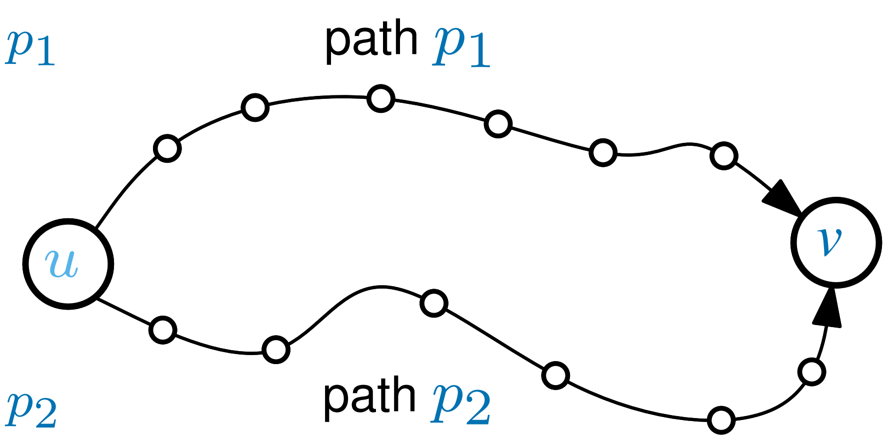

Let l_2 be the length of p_2
in the original graph

Reweighted shortest paths

Let the function h give a value h(v) for each vertex $v \in V$

Change the weight of every edge (u, v) to be

$$\texttt{weight'}(u,v) = \texttt{weight}(u,v) + \mathsf{h}(u) - \mathsf{h}(v)$$

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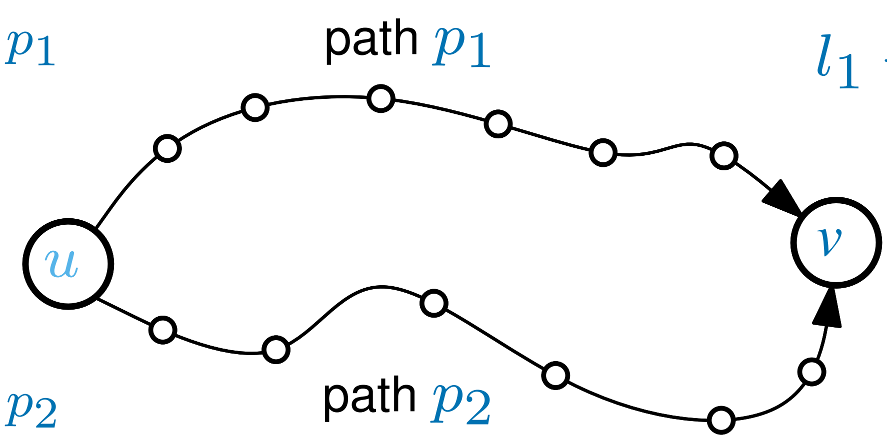

$l_1 + h(u) - h(v)$
is the length of p_1
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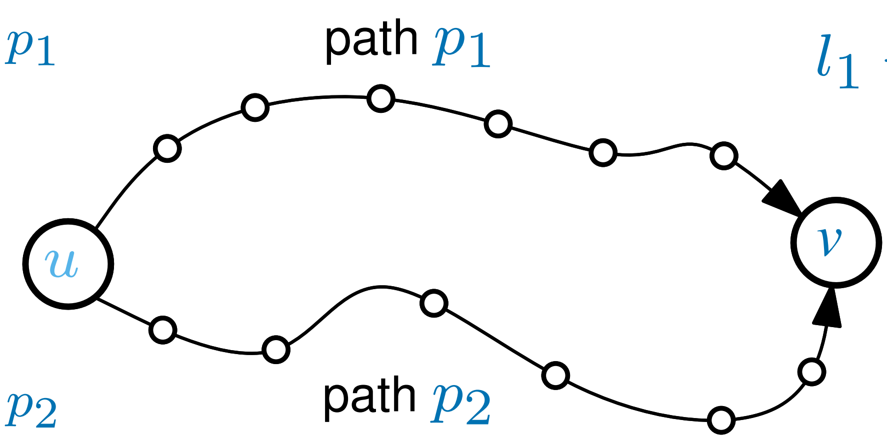

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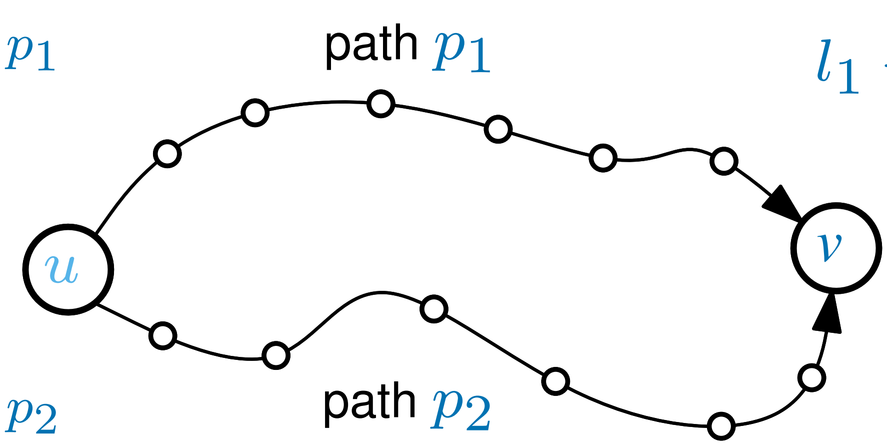

$l_1 + h(u) - h(v)$
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$l_2 + h(u) - h(v)$
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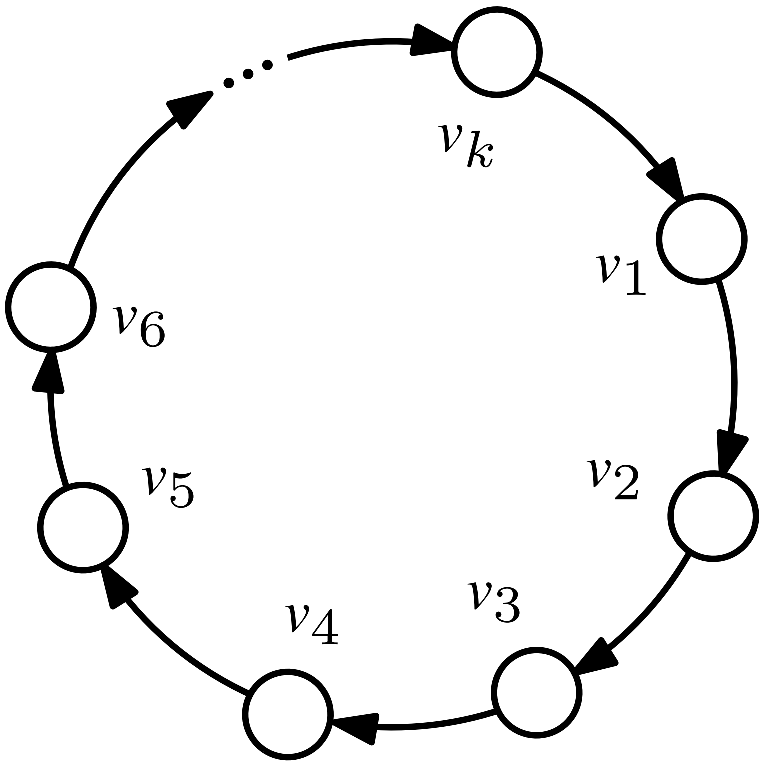
$l_1 \leqslant l_2$ if and only if

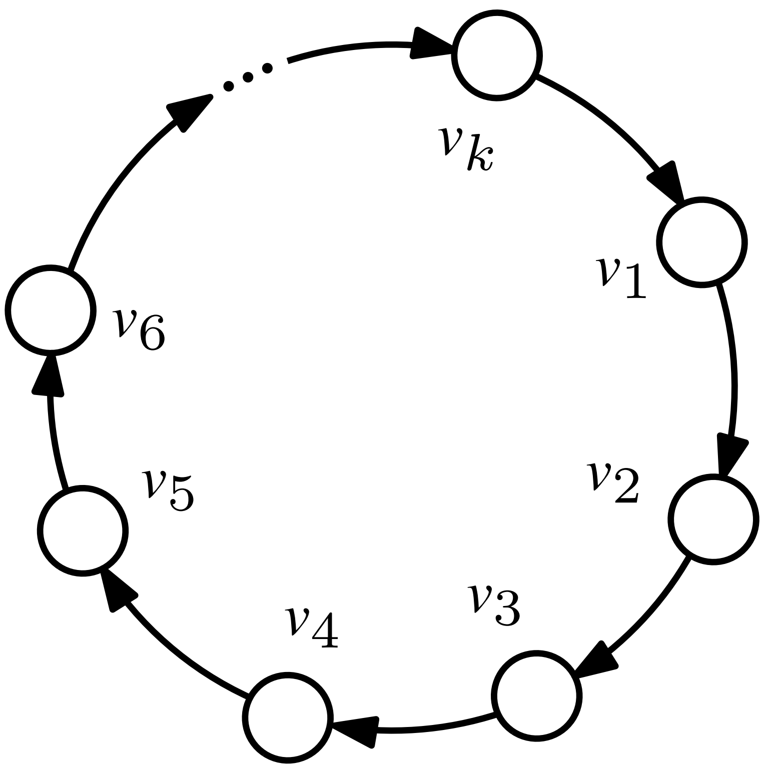
$$l_1 + h(u) - h(v) \leqslant l_2 + h(u) - h(v)$$

Reweighted negative cycles



Let $v_1, v_2, \ldots v_k \in V$ be a negative weight cycle

Reweighted negative cycles



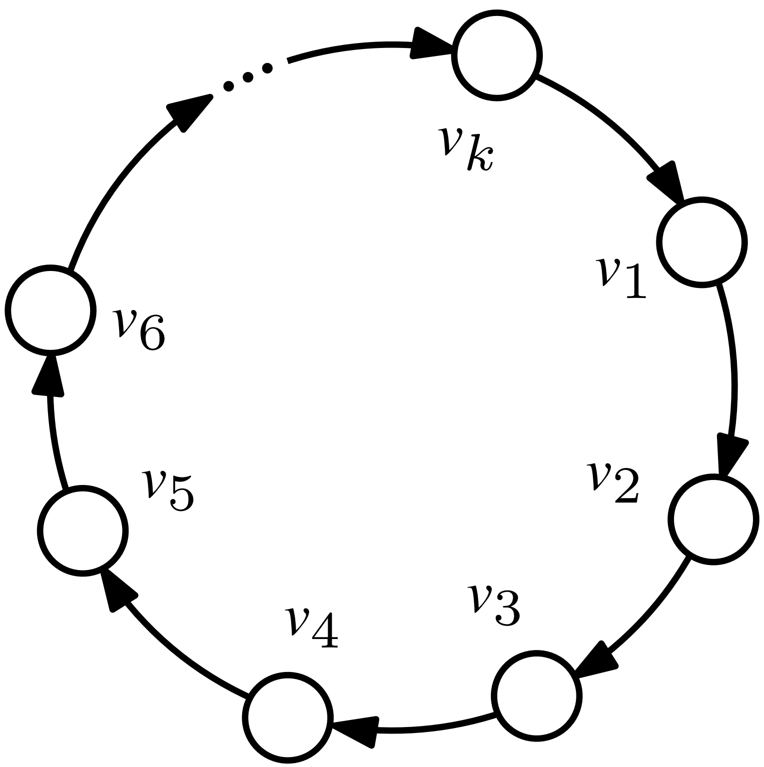
Let $v_1, v_2, \ldots v_k \in V$ be a negative weight cycle

The weight of this cycle in the original graph is,

$$\sum_{i=1}^{k} \texttt{weight}(v_i, v_{i+1}) < 0$$

(where $v_{k+1} = v_1$)

Reweighted negative cycles



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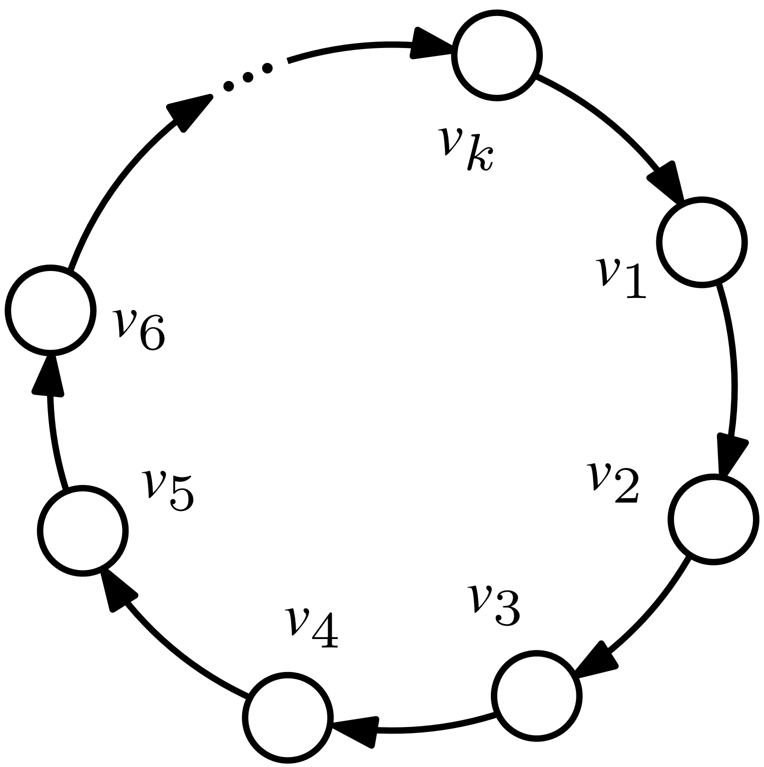

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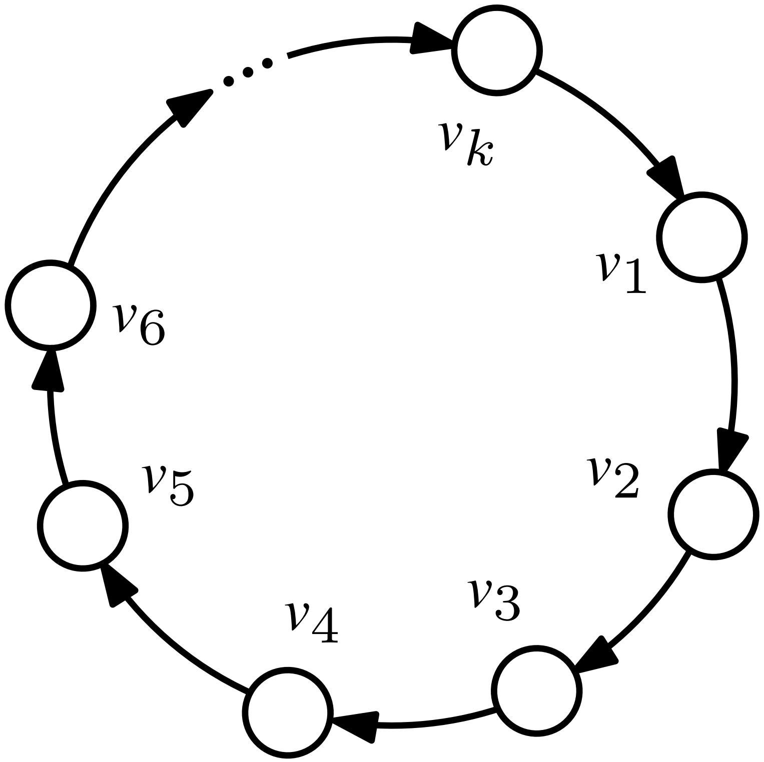
Reweighted negative cycles



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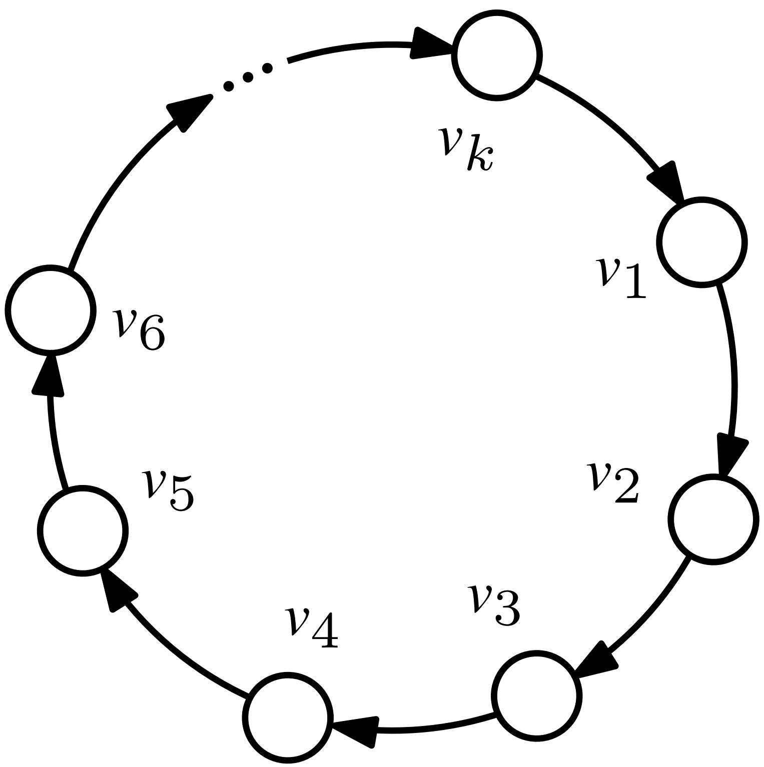
Reweighted negative cycles



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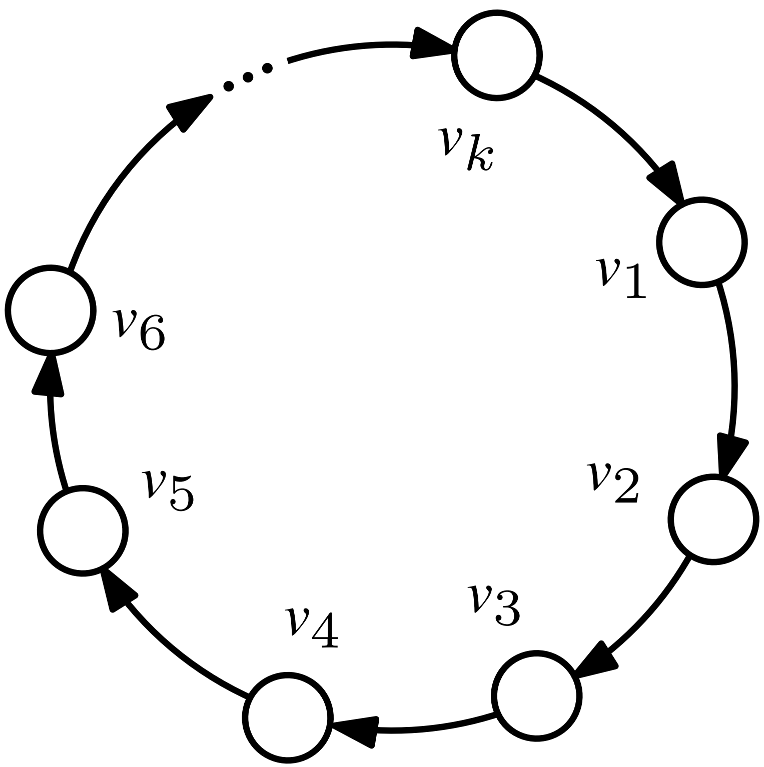

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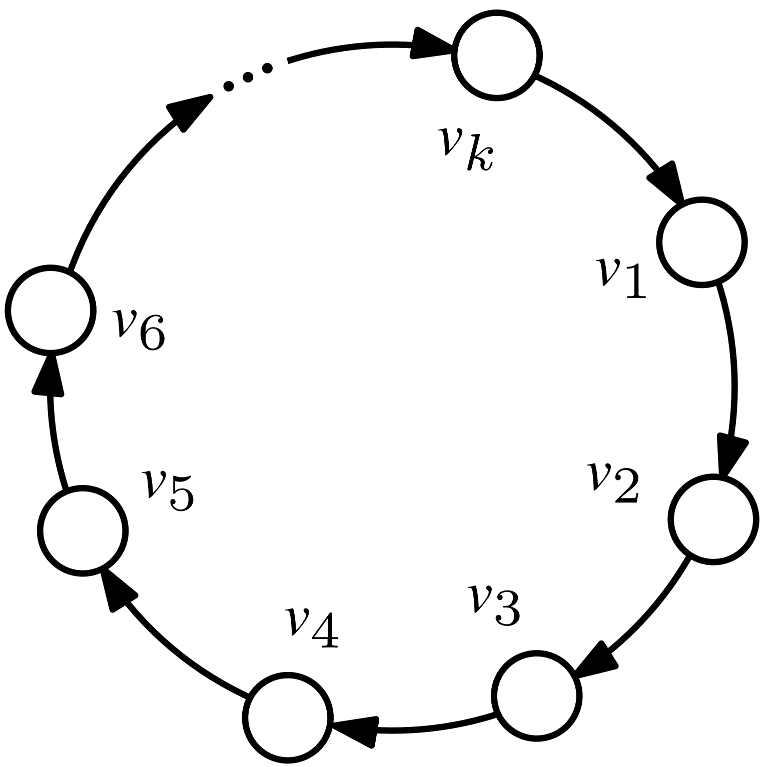

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So reweighting doesn't affect negative cycles

Reweighting summary

Let the function h give a value $\mathsf{h}(v)$ for each vertex $v \in V$

Change the weight of every edge (u, v) to be

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*So if we solve the all-pairs shortest paths problem on the reweighted graph. . .
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So if we solve the all-pairs shortest paths problem on the reweighted graph. . .
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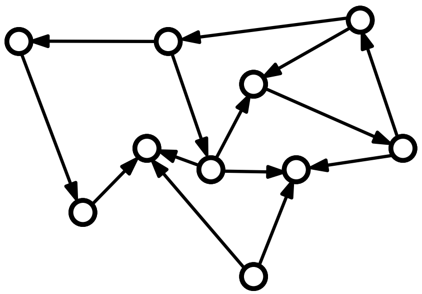
To take advantage of this, we need to make all the edge weights *non-negative*

How do we choose h?



We first add one additional vertex called s to the original graph

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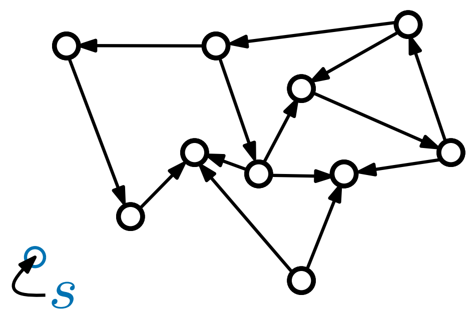

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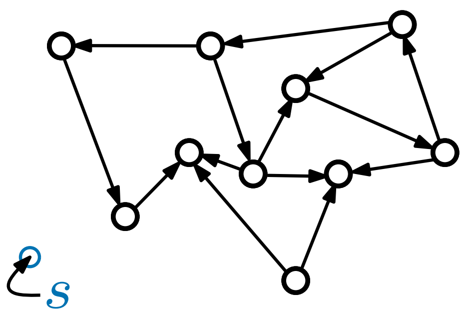
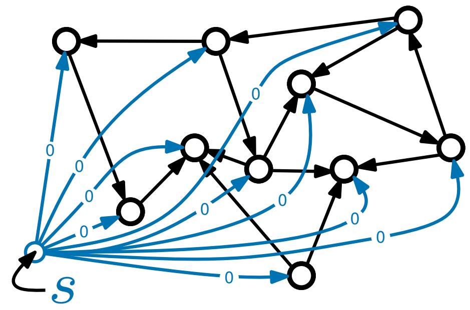


We first add one additional vertex called s to the original graph

We also add an edge (s, v) from s to each other vertex $v \in V$

How do we choose h?



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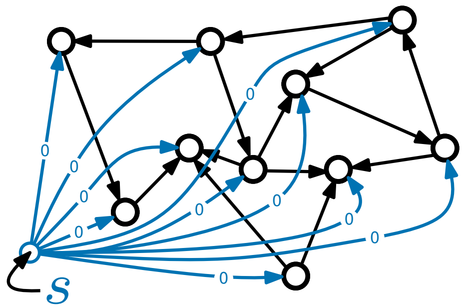

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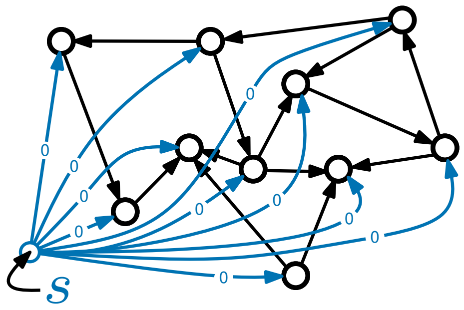

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This does not introduce any new negative weight cycles

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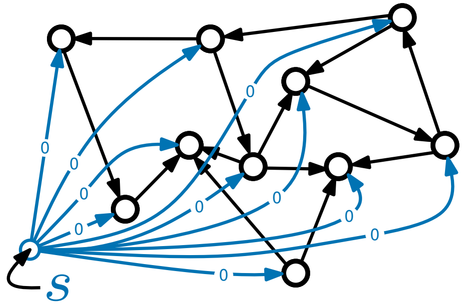

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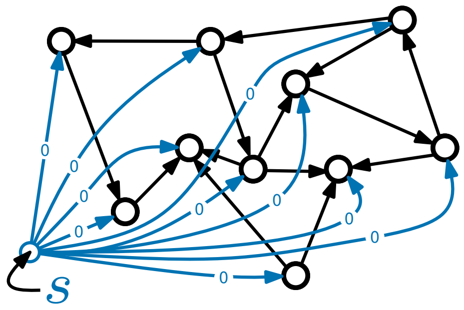

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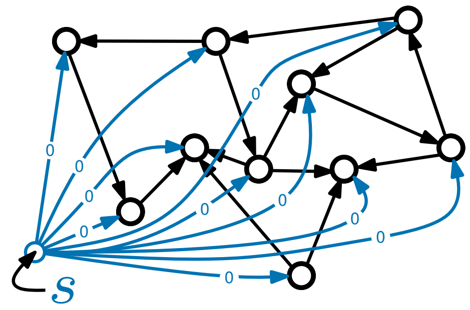

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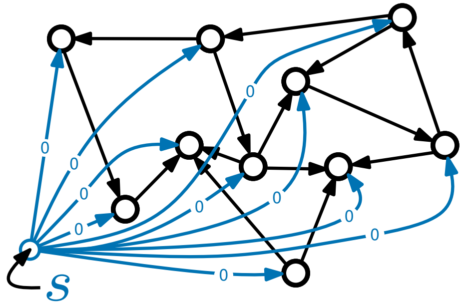
JOHNSON's *algorithm will detect this and abort*

Let's continue under the assumption that there is no negative weight cycle

How do we choose h?



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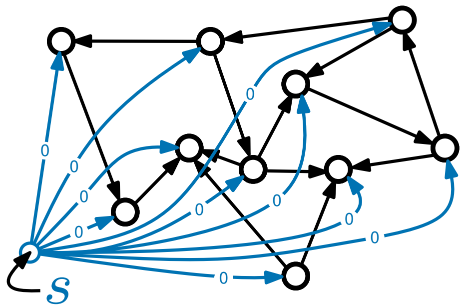

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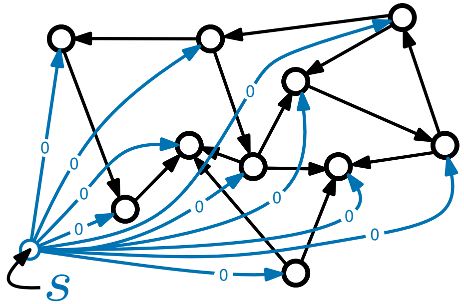

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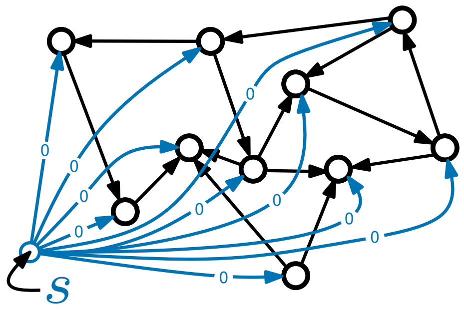

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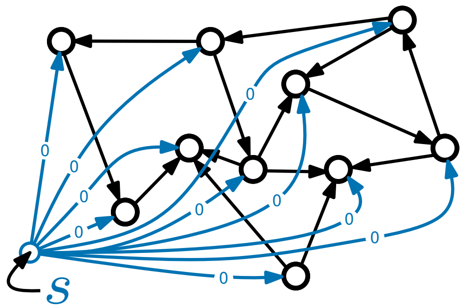

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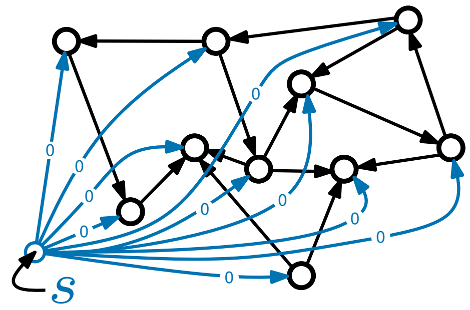

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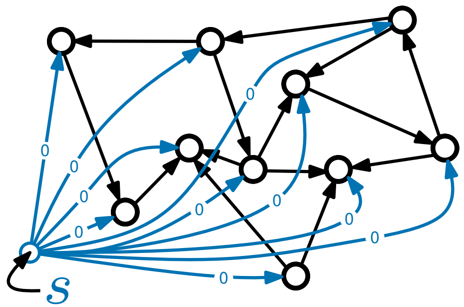

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with length $\delta(s, u) + \texttt{weight}(u, v)$

so the shortest path can't be longer

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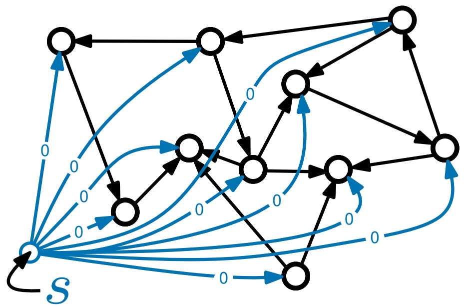

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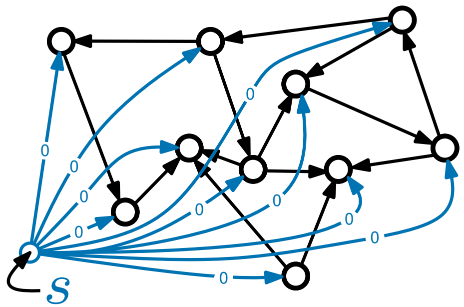

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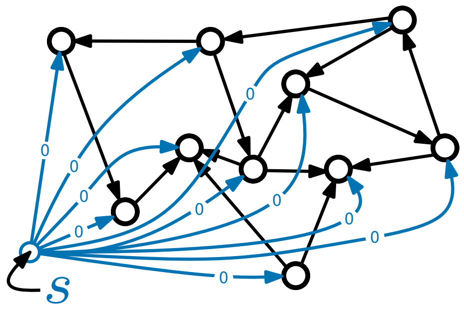

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The new weight of an edge (u, v) then becomes

$$\texttt{weight'}(u,v) = \texttt{weight}(u,v) + \mathsf{h}(u) - \mathsf{h}(v)$$

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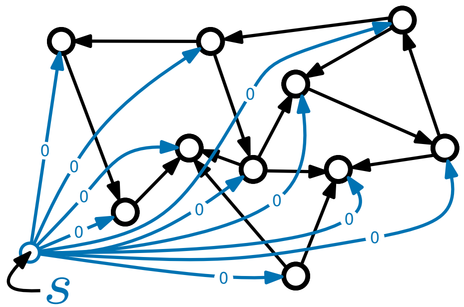

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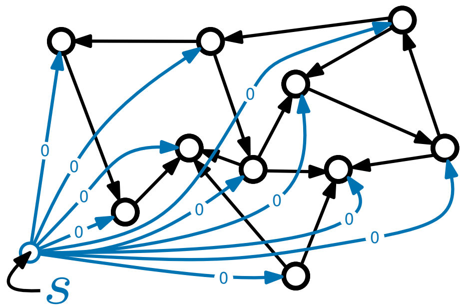

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$$= \texttt{weight}(u, v) + \delta(s, u) - \delta(s, v) \geqslant 0$$

So all the reweighted edge weights are non-negative

JOHNSON's algorithm

We can now piece together JOHNSON's algorithm which operates as follows:

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Step 4: Reweight each edge $(u, v) \in E$ so that,

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Time Complexity

How long does all this take?

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Step 2: For each vertex, add an edge (s, v) with weight 0

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$O(1)$ time

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Step 1: Add one additional vertex called s to the original graph

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How long does all this take?

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Step 2: For each vertex, add an edge (s, v) with weight 0 — $O(|V|)$ time

Step 3: Run the BELLMAN-FORD algorithm with source s — $O(|V||E|)$ time

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Step 5: For each vertex u, run DIJKSTRA's algorithm with source $s = u$. — $O(|E| \log |V|)$ time per iteration *(using a binary heap)*

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$$\delta(u, v) = \delta'(u, v) + \mathsf{h}(v) - \mathsf{h}(u)$$

$O(|V|^2)$ time

(The previous version of this slide said $O(|E|)$ *for* **Step 6** *which was a typo)*

Time Complexity

How long does all this take?

$O(1)$ time

Step 1: Add one additional vertex called s to the original graph

$O(|V|)$ time

Step 2: For each vertex, add an edge (s, v) with weight 0

$O(|V||E|)$ time

Step 3: Run the BELLMAN-FORD algorithm with source s

$O(|E|)$ time

Step 4: Reweight each edge (u, v) so that,

$$\texttt{weight'}(u,v) = \texttt{weight}(u,v) + \mathsf{h}(u) - \mathsf{h}(v)$$

Step 5: For each vertex u, run DIJKSTRA's algorithm with source $s = u$.

$O(|V||E|\log|V|)$ time overall

(using a binary heap)

Step 6: For each pair of vertices u, v, compute

$$\delta(u, v) = \delta'(u, v) + \mathsf{h}(v) - \mathsf{h}(u)$$

$O(|V|^2)$ time

(The previous version of this slide said $O(|E|)$ for **Step 6** *which was a typo)*

So the overall time complexity is $O(|V||E|\log|V|)$

see next slide

This is the complexity for (strongly) connected graphs (actually any graph without isolated vertices)...

where every pair of vertices is connected

In such graphs we have that $|E| > |V|/2$ so $O(|V|^2) = O(|V||E|)$

For unconnected graphs there is a (non-examinable and fiddly) fix...

Johnson's algorithm on graphs with $|E| < |V|/2$

This slide contains non-examinable material and was added post-lecture.

It covers the corner case that Johnson's algorithm is run on a graph with $|E| < |V|/2$.

The simple answer: In this case, the algorithm (as discussed) takes $O(|V|^2 + |V||E|\log|V|)$ time

We can't "hide" the $O(|V|^2)$ term because $|V|$ could be much much larger than $|E|$

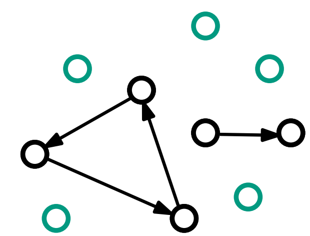
A more convoluted answer: With a simple preprocessing step, we can improve the time complexity to

$O(|V|^2 + |V||E|\log|V|)$ for this case

Sketch Proof

We define a vertex to be *isolated* if it contains no incoming and no outgoing edges.

(in the diagram the ○ are isolated and the ○ are not)



Add a new step to the start of Johnson's algorithm,

Step 0: Delete every isolated vertex ← $O(|V| + |E|)$ time

In the new graph $|E'| = |E|$, $|V'| \leqslant |V|$ and $|E'| \geqslant |V'|/2$

each remaining vertex is at one end of at least one edge

Therefore the modified algorithm takes

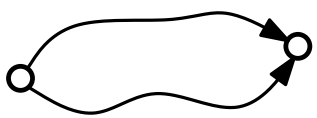
$$O(|V| + |E| + |V'||E'|\log|V'|) = O(|V||E|\log|V|) \text{ time}$$

Notice that the isolated vertices have distince ∞ to everywhere so deleting them doesn't change anything

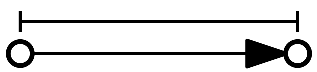
JOHNSON's algorithm summary

The overall approach taken by JOHNSON's algorithm
is to *reweight* the edges in the graph
(using new weights picked using BELLMAN-FORD*)*

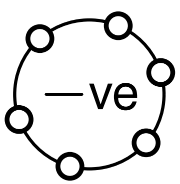
The key properties of the reweighted graph are:



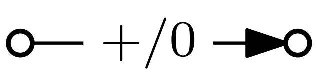
Lemma Any path is a shortest path in the original graph
if and only if it is a shortest path in the reweighted graph



Fact The length of a shortest path in the original graph
can be calculated from its length in the reweighted graph



Fact A cycle has negative weight in the original graph
if and only if it has negative weight in the reweighted graph



Fact If there are no negative weight cycles in the original graph,
all of the edges in the reweighted graph have non-negative weights

After reweighting, all-pairs shortest paths are calculated
using DIJSKTRAS algorithm, once with each vertex v as the source

Overall this takes $O(|V||E|\log|V|)$ time

Shortest paths algorithms: the summary

To compute **single source** shortest paths in a directed graph which is/has:

unweighted: Use Breadth First Search in $O(|V| + |E|)$ time

non-negative edge weights: Use DIJKSTRA's algorithm
which takes $O((|V| + |E|) \log |V|)$ time *(when implemented using a binary heap)*

positive and negative edge weights: Use BELLMAN-FORD which takes $O(|V||E|)$ time

To compute **all-pairs** shortest paths in a directed graph which is/has:

unweighted: Use Breadth First Search once for each vertex in $O(|V|^2 + |V||E|)$ time

non-negative edge weights: Use DIJKSTRA's algorithm once for each vertex,
which takes $O(|V||E| \log |V|)$ time *(when implemented using a binary heap)*

positive and negative edge weights: Use JOHNSON'S algorithm
which takes $O(|V||E| \log |V|)$ time
(when DIJKSTRA's algorithm is implemented using a binary heap)