COMS10003 Work Sheet 23

Linear Algebra: Eigenvalues and eigenvectors

1. Compute the eigenvalues and eigenvectors of the following matrices

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

   Answer:
   $\lambda = 1,\ v = (-0.8944, 0.4472)$
   $\lambda = 4,\ v = (0.707, 0.707)$

   $\lambda = 1,\ v = (-0.5547, 0.8321)$
   $\lambda = 6,\ v = (0.707, 0.707)$

   $\lambda = 2,\ v = (-0.8018, 0.5345, 0.2673)$
   $\lambda = 6,\ v = (0.4082, 0.8165, 0.4082)$
   $\lambda = 2,\ v = (-0.2023, -0.5839, 0.7862)$

2. Compute the eigenvectors of the following symmetric matrix A and show that they are orthogonal. Hence find the matrix P such that the matrix $B = P^{-1}AP$ is diagonal. Show that the diagonal components are equal to the eigenvalues of A.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   Answer:
   $\lambda = -0.4142,\ v = (0.5, -0.707, 0.5)$
   $\lambda = 1,\ v = (-0.707, 0.0, 0.707)$
   $\lambda = 2.4142,\ v = (0.5, 0.707, 0.5)$

$$P = \begin{bmatrix} 0.5 & -0.707 & 0.5 \\ -0.707 & 0 & 0.707 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}$$

3. For the matrix A above, compute A^3, first by making use of $B = P^{-1}AP$ and then by doing it the long way, i.e. $A \times A \times A$. Make sure that you get the same answer.

   Answer:

$$A^3 = PB^3P^{-1} = \begin{bmatrix} 4 & 5 & 3 \\ 5 & 7 & 5 \\ 3 & 5 & 4 \end{bmatrix}$$

4. Show that two similar matrices have the same eigenvalues.

   Answer: $A = PBP^{-1}$ *and* $A\mathbf{v} = \lambda\mathbf{v} \rightarrow PBP^{-1}\mathbf{v} = \lambda\mathbf{v} \rightarrow BP^{-1}\mathbf{v} = \lambda P^{-1}\mathbf{v}$ *hence* λ *is eigenvalue of* B.

5. The 2-D data points below have zero mean. Determine the covariance matrix for the points and compute the two principal axes corresponding to the eigenvectors of the covariance matrix. Hence compute the two principal components for each data point. Determine the total error if the first principal component (corresponding to the principal axis with the largest eigenvalue) is used to represent the data.

x	1	2	3	-1	-2	-3
y	1	1	2	-1	-1	-2

Plots the points on a graph along with the principal axes and the first principal components.

Answer:

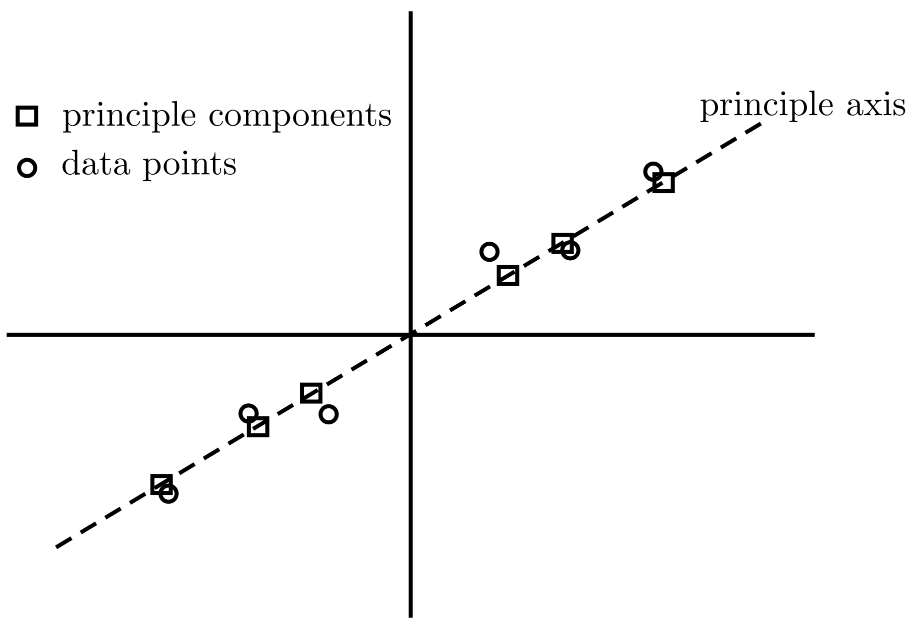
$$C = \begin{bmatrix} 28 & 18 \\ 18 & 12 \end{bmatrix}$$

$\lambda_1 = 0.3\ \mathbf{v}_1 = (0.5449, -0.8385)$
$\lambda_2 = 39.6977,\ \mathbf{v}_2 = (-0.8385, -0.5449)$

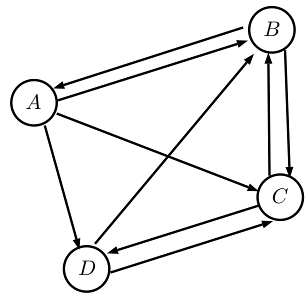
Prin Comp = dot product between data point and eigenvector:

x	*1*	*2*	*3*	*-1*	*-2*	*-3*
y	*1*	*1*	*2*	*-1*	*-1*	*-2*
P1	*-0.2936*	*0.2513*	*-0.0422*	*0.2936*	*-0.2513*	*0.0422*
P2	*-1.3834*	*-2.2219*	*-3.6053*	*1.3834*	*2.2219*	*3.6053*

$$Error = \sqrt{\sum_i |(x_i, y_i) - P1 * \mathbf{v}_2|^2} = 0.3023$$



6. The graph below represents web pages that link to each other, where the directed edges indicate that a page contains a link to another page. Determine the transition matrix for the network and hence compute the Pagerank vector. Starting from an assignment of equal importance to all 4 pages, demonstrate that repeated application of the transition matrix provides a good approximation to the Pagerank vector. For example, if T is the transition matrix and $\mathbf{x}_1 = (1/4, 1/4, 1/4, 1/4)$, say, then compute $\mathbf{x}_2 = T\mathbf{x}_1$, $\mathbf{x}_3 = T\mathbf{x}_2$, etc.



Answer:

$$T = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 1/2 & 0 \end{bmatrix}$$

Pagerank vector = eigenvector corresponding to largest eigenvalue of $T =$ *(0.3,0.6,0.6,0.4)*