COMS10003: Workshop on Proof

Proof Strategies

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Introduction

This worksheet contains problems that require you to use different proof strategies, including direct proof, indirect proof, proof by contradiction, existence proofs, proof by exhaustion and proof by induction.

For some problems, more than one proof strategy can be applied. In these cases, try all of those applicable and see which leads to the most elegant, the most natural, the simplest or the shortest proof. **Always state the method of proof you are using.**

It is also important that you practice how to write up proofs. This requires you to clearly lay the proof out on paper, use proper English sentences and to justify each step explicitly.

Proposed method of working:

1. **Draft:** Outline your argument based on a proof strategy, indicate how the steps follow from each other. You don't need to pay attention to the notation at this stage. Instead, focus on the logic of the argument.
2. **Write up:** Based on the draft, write it up as a proper proof with all steps itemized. Formalize the argument using formal notation. Formalize your assumptions. Justify each step, clearly indicating how it follows from which previous steps. Use full sentences where appropriate. Do not forget to state the conclusion of your proof.
3. For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Get your write up checked over by a Teaching Assistant before you move to the next proof.

Task 1: Basic Proof Strategies

Prove or disprove the following statements. For each proof, state clearly which proof strategy you are using and justify your choice of strategy. Which other strategies work? Attempt different strategies and discuss the differences and similarities in the resulting proofs.

1. The sum of an even integer and an odd integer is an odd integer.
2. Two integers are said to have the same parity if they are both odd or both even. Show that if x and y are two integers for which $x + y$ is even, then x and y have the same parity.
3. All primes are odd.
4. For all integers n, $n^3 - n$ is even.
5. The square of any odd integer has the form $8m + 1$ for some integer m.
6. For all real numbers a and b, if $a^2 = b^2$, then $a = b$.

Task 2:

Prove that the square of an even integer is an even integer, using:

1. a direct proof,
2. an indirect proof,
3. a proof by contradiction.

Task 3:

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even using:

1. a direct proof,
2. an indirect proof,
3. a proof by contradiction.

Task 4:

Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Hint: Formalize this statement using a biconditional. Then prove each implication separately. Note that there are also different cases.

Task 5:

Prove that $n^2 = (n-1)^2 + 2n - 1$ using:

1. direct proof,
2. induction.

Hint: Make sure that you use the assumption in the inductive proof. If the assumpiotn is not used, the proof structure is incorrect, even though the proof result may still be correct (due to the direct proof you've used.)

Task 6:

Prove that every positive integer greater than or equal to 2 has a prime decomposition, i.e. can be written as a product of prime numbers.

Clearly state the proof strategy you use. You may need to experiment with different proof strategies. If you get stuck, ask a Teaching Assistant to help you find a proof strategy that works.

Task 7:

Prove that $\sqrt{2}$ is irrational. Which proof strategy did you use?

Task 8: Resolution

Research the foundations of resolution as a proof method. State the inference rule on which resolution is based.

Using resolution, prove the statement: "Sam will drive to the meeting place or Sam will get a lift." given the following facts:

- Sam will not take the bus or Sam will get unwell.
- Sam will take the bus or Sam will drive to the meeting place.
- Sam will not get unwell or Sam will get a lift.

Task 9:

Prove that $n^4 - 1$ is divisible by 5 when n is not divisible by 5. Which proof strategy did you use?