Amortized Analysis

He Sun

University of BRISTOL

Motivating Example: Stack

Stack Operations

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- **PUSH**(S, x)

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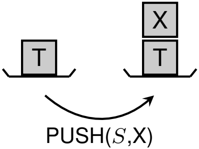
  - pushes object x onto stack S



Motivating Example: Stack

Stack Operations

- **PUSH**(S, x)
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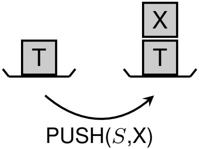

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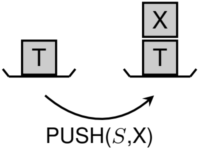

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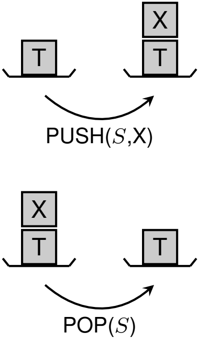

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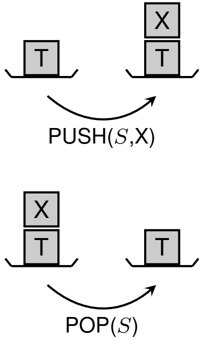

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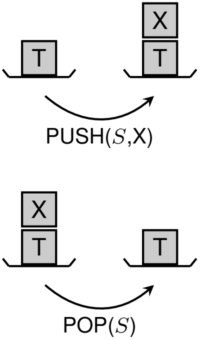

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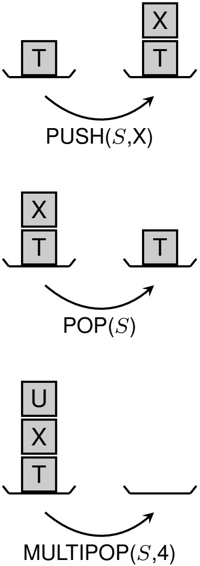

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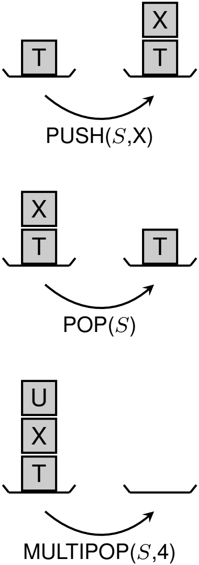

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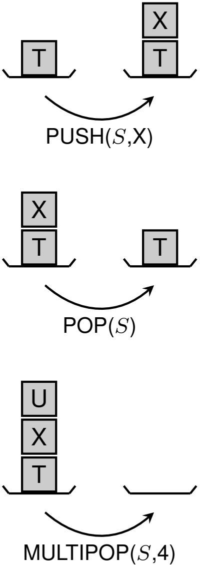
```
MULTIPOP(S, k)
while not S.empty() and k > 0
    POP(S)
    k = k - 1
```



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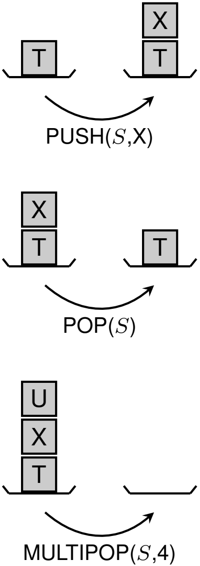

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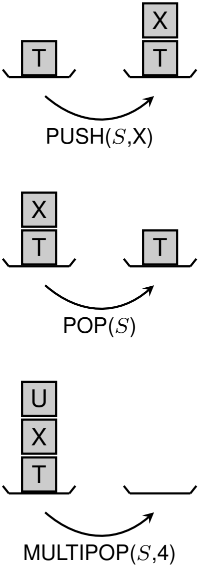

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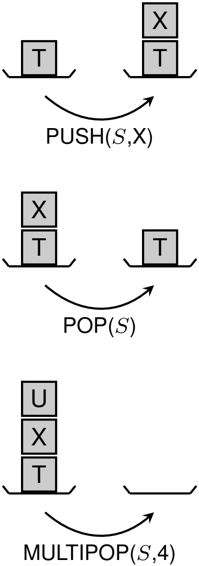

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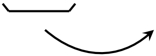

PUSH(T)

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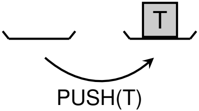

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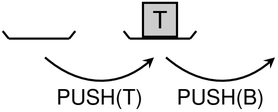

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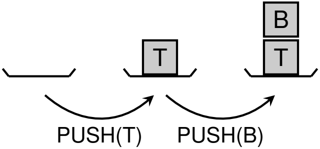

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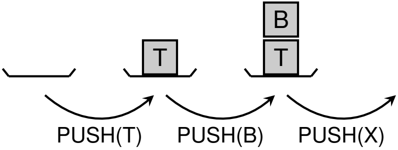

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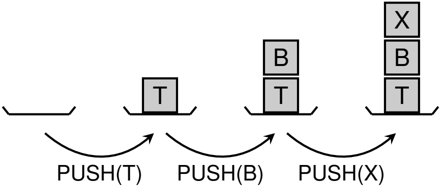

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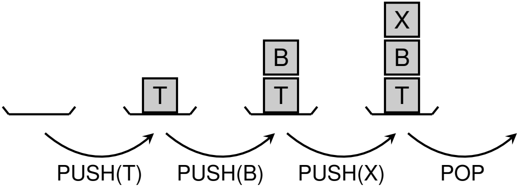

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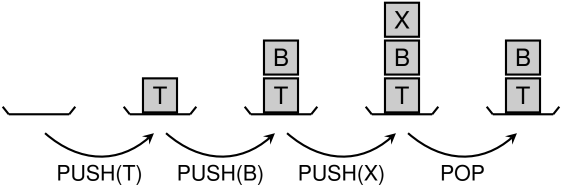

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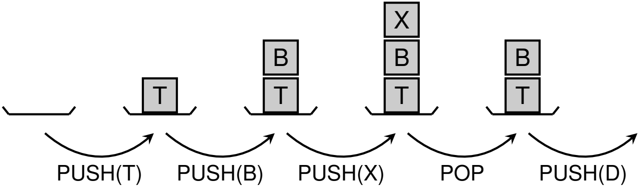

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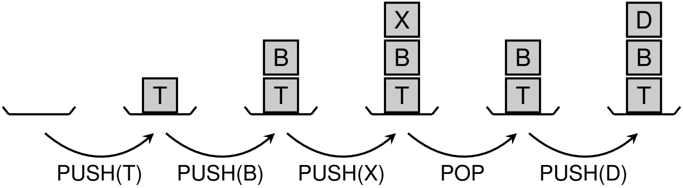

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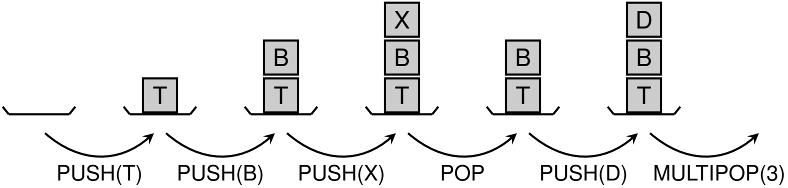

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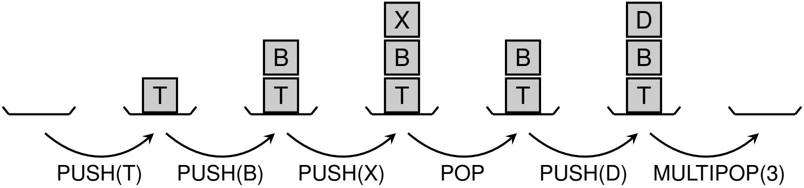

A new Analysis Tool: Amortized Analysis

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- analyse a sequence of operations

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Data structure operations (Heap, Stack, Queue etc.)

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Even though operations may be of different types/costs

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Simple Worst-Case Bound:

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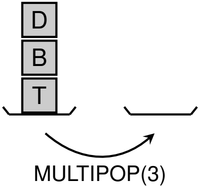

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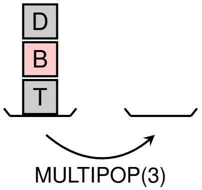

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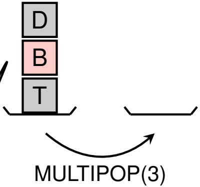

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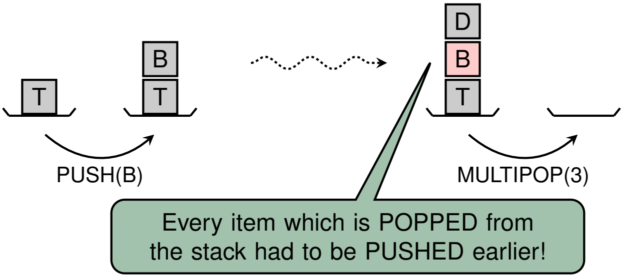

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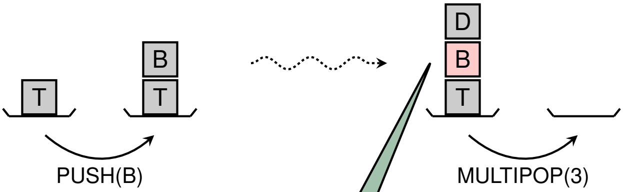

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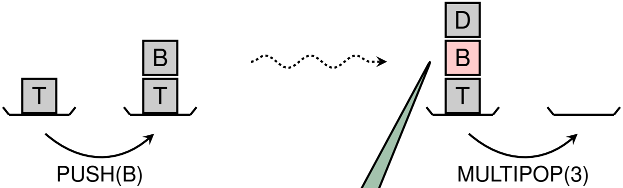

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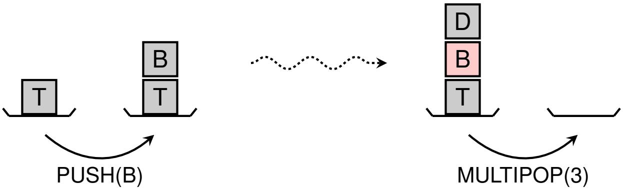

$$T(n) \leq T_{POP}(n) + T_{PUSH}(n)$$

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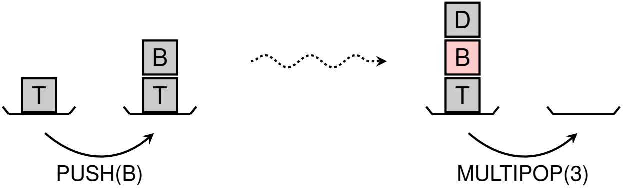

MULTIPOP(k) contributes $\min\{k, |S|\}$ to $T_{POP}(n)$

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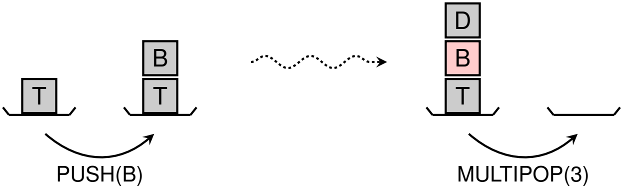

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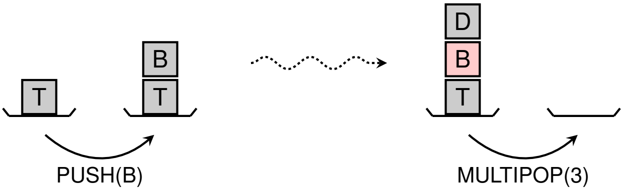

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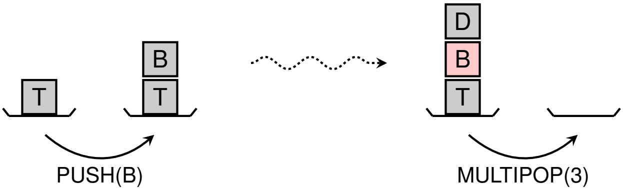

Aggregate Analysis: The amortized cost per operation is $\frac{T(n)}{n} \leq 2$

$$T(n) \leq T_{POP}(n) + T_{PUSH}(n) \leq 2 \cdot T_{PUSH}(n) \leq 2 \cdot n.$$

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- allow different amortized costs

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Potential of a data structure can be also thought of as

- amount of potential energy stored
- distance from an ideal state

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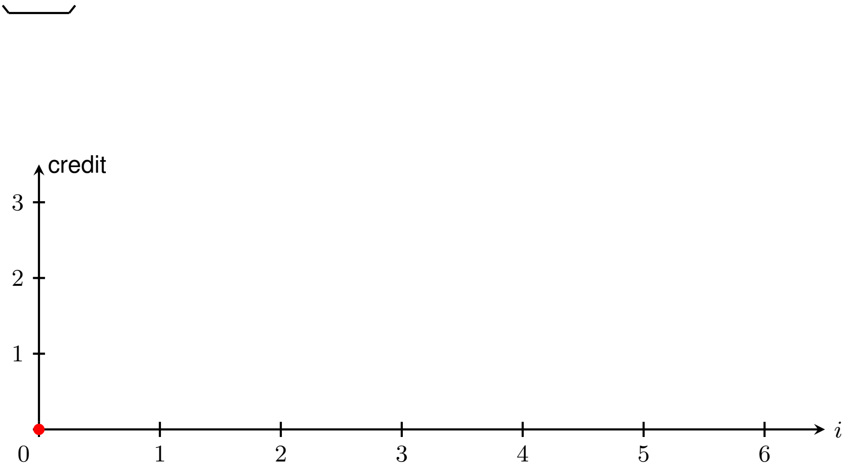
Stack and Coins



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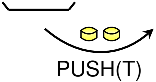

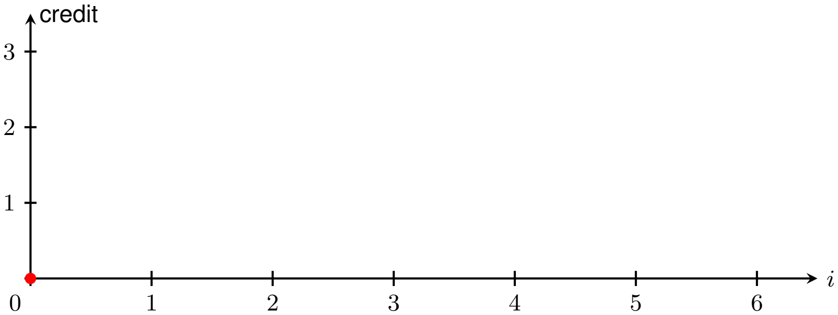

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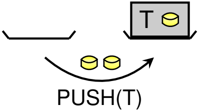

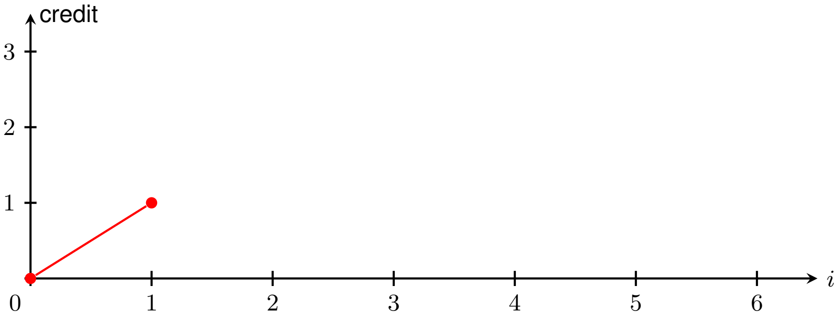

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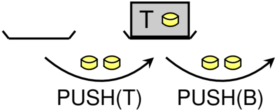

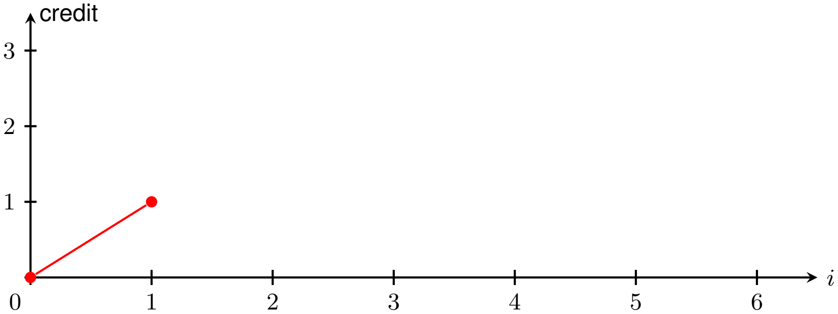

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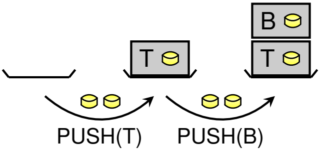

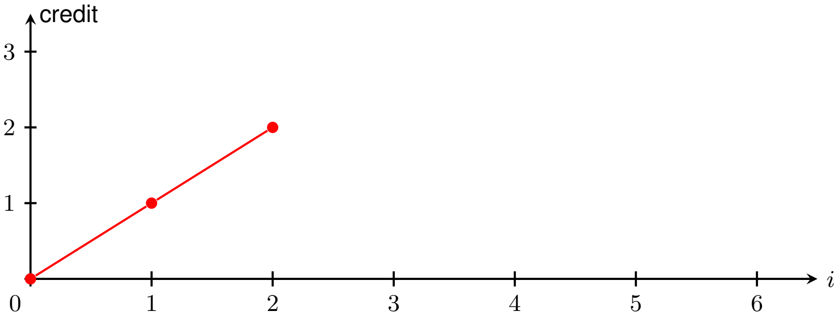

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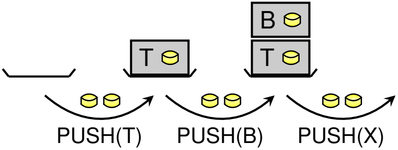

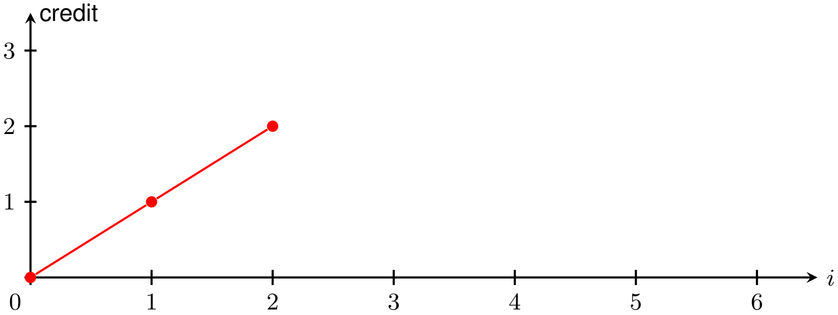

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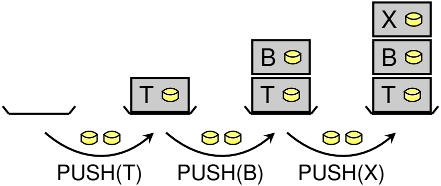

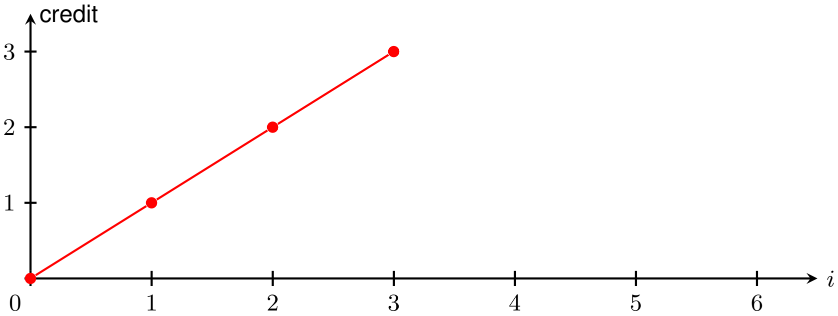

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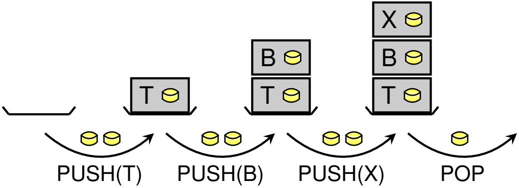

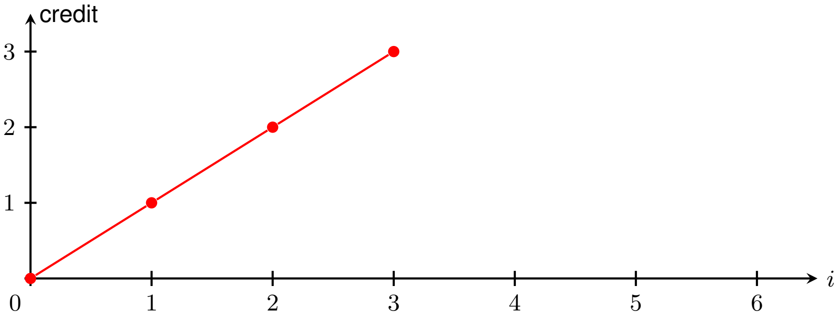

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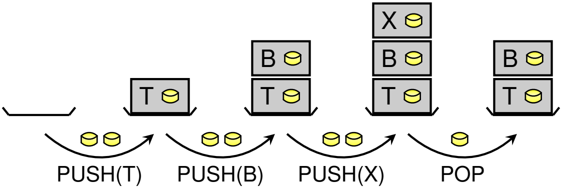

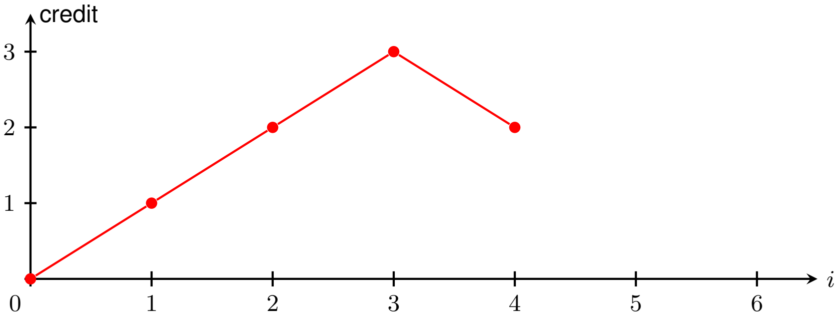

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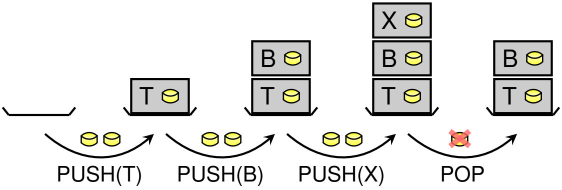

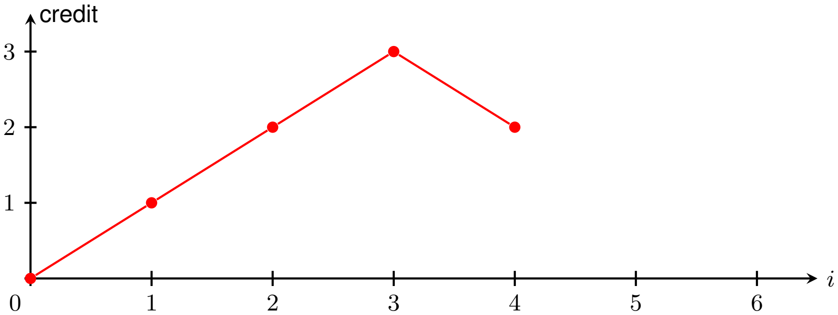

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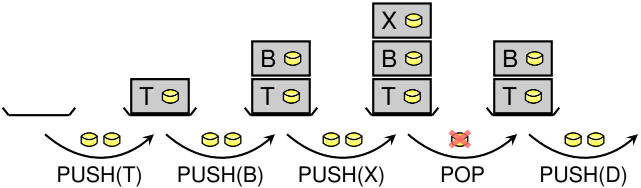

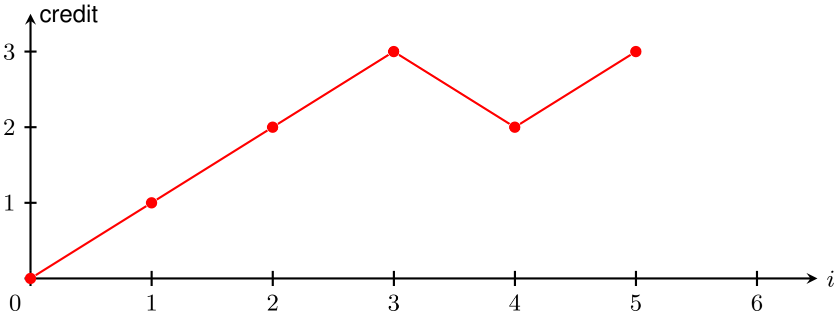

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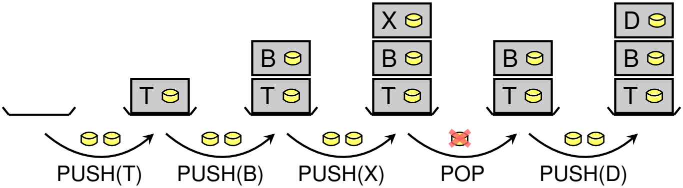

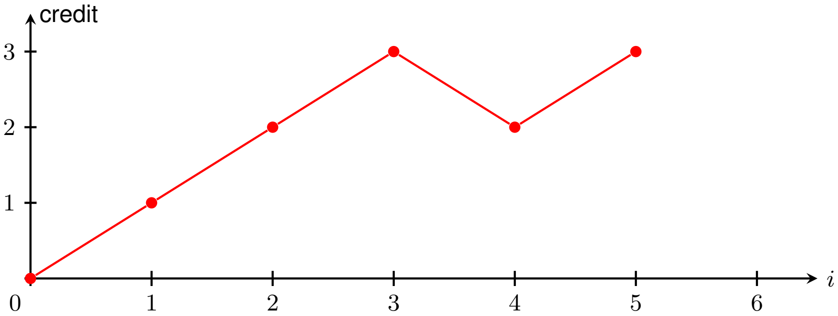

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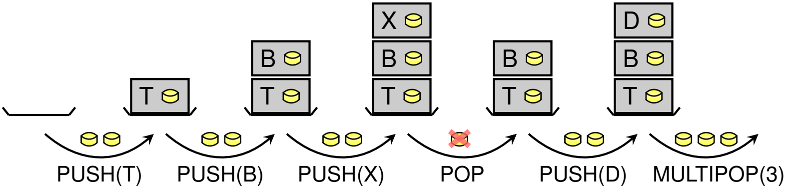
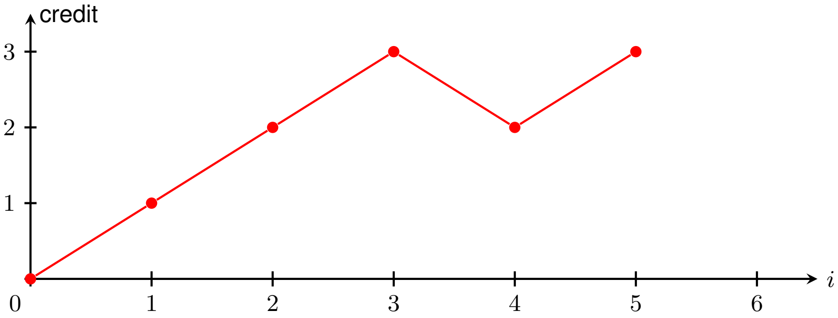
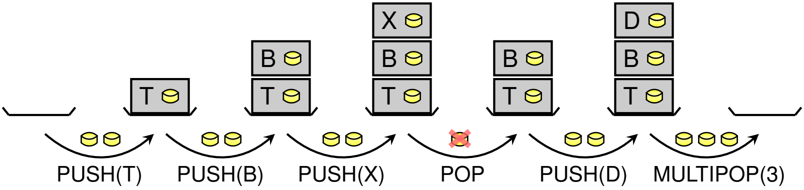
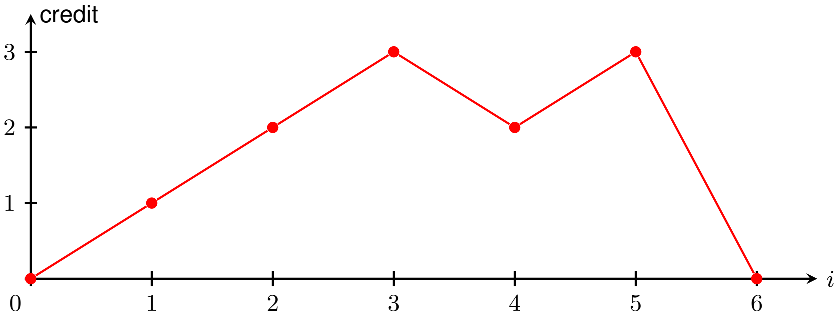




Stack and Coins





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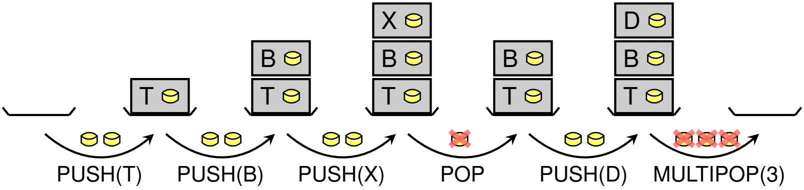

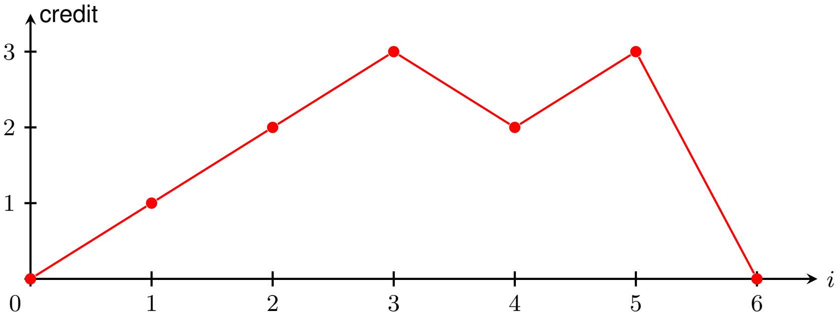

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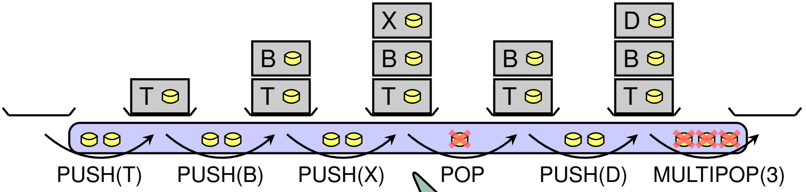
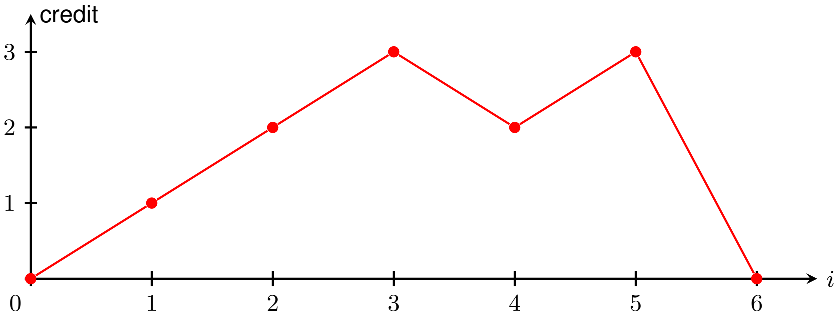


Every operation costs at most **two coins**!



Potential Method in Detail

- c_i is the actual cost of operation i

Potential Method in Detail

- c_i is the actual cost of operation i
- $\widetilde{c}_i$ is the amortized cost of operation i

Potential Method in Detail

$c_i < \widetilde{c}_i$, $c_i = \widetilde{c}_i$ or $c_i > \widetilde{c}_i$ are all possible!

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Function that maps states of the data structure to some value

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$$\Phi_i = \Phi_{i-1} + \widetilde{c}_i - c_i$$

$$\begin{aligned}\sum_{i=1}^{n} \widetilde{c}_i &= \sum_{i=1}^{n} \left(c_i + \Phi_i - \Phi_{i-1}\right) \\ &= \sum_{i=1}^{n} c_i + \Phi_n - \Phi_0\end{aligned}$$

If $\Phi_n \geq 0$ for all n, sum of amortized costs is an upper bound for the sum of actual costs!

Stack: Analysis via Potential Method

$$\Phi_i =$$

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Φ_i = # objects in the stack after ith operation $(=$ # coins$)$

Stack: Analysis via Potential Method

$\Phi_i =$ # objects in the stack after ith operation $(=$ # coins$)$

PUSH

Stack: Analysis via Potential Method

Φ_i = # objects in the stack after ith operation (= # coins)

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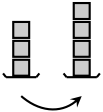
- actual cost: $c_i = 1$

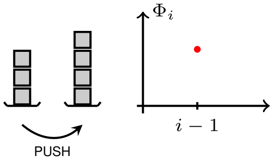


Stack: Analysis via Potential Method

Φ_i = # objects in the stack after ith operation (= # coins)

PUSH

- actual cost: $c_i = 1$
- potential change: $\Phi_i - \Phi_{i-1} =$



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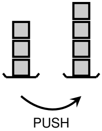

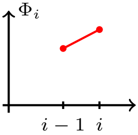

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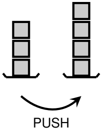

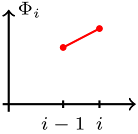

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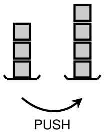

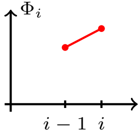

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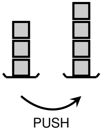

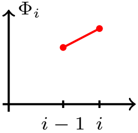

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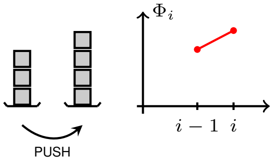

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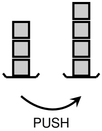

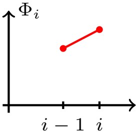

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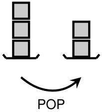

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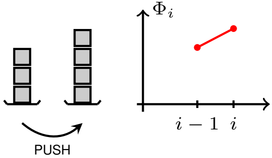

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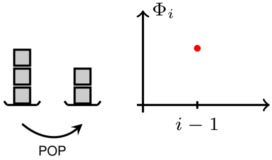

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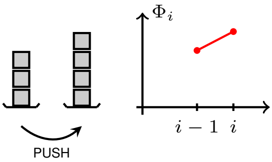

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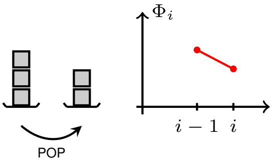

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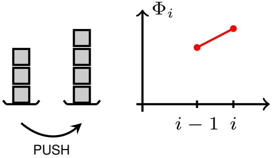

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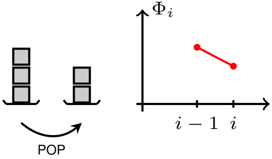

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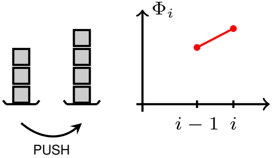

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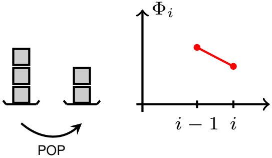

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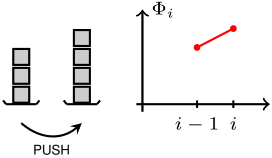

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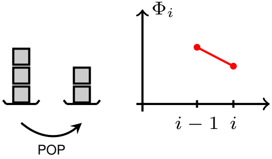
Stack is non-empty!



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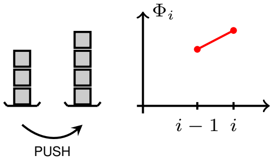

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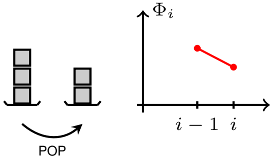

MULTIPOP(k)

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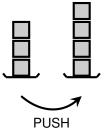

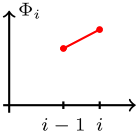

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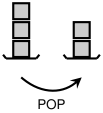
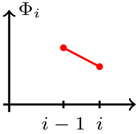
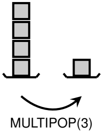




MULTIPOP(k)

- $c_i = \min\{k, |S|\}$



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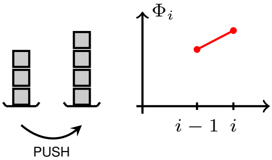

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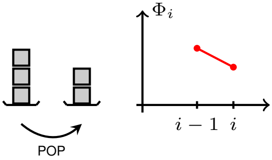
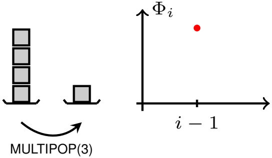
- $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) = 1 - 1 = 0$



MULTIPOP(k)

- $c_i = \min\{k, |S|\}$
- $\Phi_i - \Phi_{i-1} =$

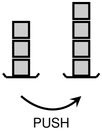


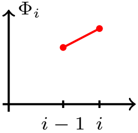
Stack: Analysis via Potential Method

Φ_i = # objects in the stack after ith operation $\quad$ (= # coins)

PUSH

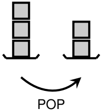
- actual cost: $c_i = 1$
- potential change: $\Phi_i - \Phi_{i-1} = 1$
- amortized cost: $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) = 1 + 1 = 2$

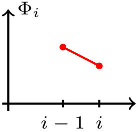




POP

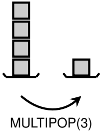
- $c_i = 1$
- $\Phi_i - \Phi_{i-1} = -1$
- $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) = 1 - 1 = 0$

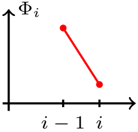




MULTIPOP(k)

- $c_i = \min\{k, |S|\}$
- $\Phi_i - \Phi_{i-1} = -\min\{k, |S|\}$





Stack: Analysis via Potential Method

Φ_i = # objects in the stack after ith operation $(=$ # coins$)$

PUSH

- actual cost: $c_i = 1$
- potential change: $\Phi_i - \Phi_{i-1} = 1$
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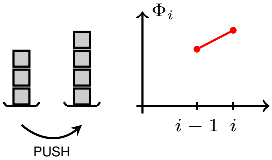

POP

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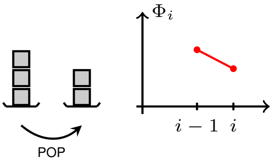
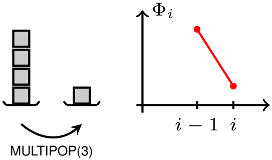


MULTIPOP(k)

- $c_i = \min\{k, |S|\}$
- $\Phi_i - \Phi_{i-1} = -\min\{k, |S|\}$
- $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) =$

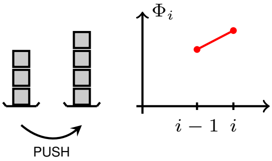


Stack: Analysis via Potential Method

Φ_i = # objects in the stack after ith operation (= # coins)

PUSH

- actual cost: $c_i = 1$
- potential change: $\Phi_i - \Phi_{i-1} = 1$
- amortized cost: $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) = 1 + 1 = 2$



POP

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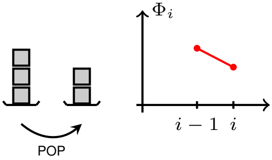

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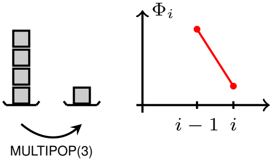

Stack: Analysis via Potential Method

Φ_i = # objects in the stack after ith operation (= # coins)

PUSH

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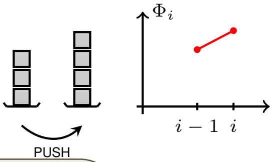
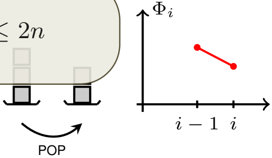


POP

- $c_i = 1$
- $\Phi_i - \Phi_{i-1} = -1$
- $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) = 1 - 1 = 0$

Amortized Cost $\leq 2 \Rightarrow T(n) \leq 2n$



MULTIPOP(k)

- $c_i = \min\{k, |S|\}$
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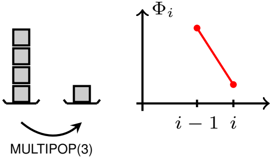

Stack: Analysis via Potential Method

$\Phi_i =$ # objects in the stack after ith operation $(=$ # coins$)$

PUSH

- actual cost: $c_i = 1$
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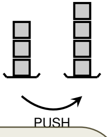
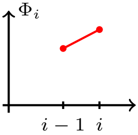
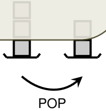


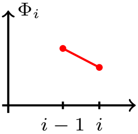


$n/2$ PUSH, $n/2$ POP $\Rightarrow T(n) \leq n$

POP

- $c_i = 1$
- $\Phi_i - \Phi_{i-1} = -1$
- $\widehat{c_i} = c_i + (\Phi_i - \Phi_{i-1}) = 1 - 1 = 0$





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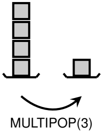

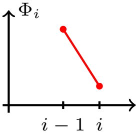

Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
1	0	1	1	11

Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
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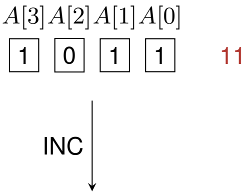
Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
- only operation: **`INC`**



Second Example: Binary Counter

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- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
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  - increases the counter by one

$A[3]\ A[2]\ A[1]\ A[0]$

$\boxed{1}\ \boxed{0}\ \boxed{1}\ \boxed{1}$ 11

$\Big\downarrow$ INC

Second Example: Binary Counter

Binary Counter

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$$\Big\downarrow \text{INC}$$

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
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Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
- only operation: **INC**
  - increases the counter by one

```
INC(A)
i = 0
while i < k and A[i]==1
    A[i] = 0
    i = i + 1
A[i] = 1
```

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
1	0	1	1	11

$\downarrow$ INC

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
1	1	0	0	12

Second Example: Binary Counter

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- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
- only operation: **INC**
  - increases the counter by one
  - total cost: ??

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INC(A)
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    A[i] = 0
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```

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
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$\downarrow$ INC

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Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
- only operation: **INC**
  - increases the counter by one
  - total cost: $\leq k$

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INC(A)
i = 0
while i < k and A[i]==1
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    i = i + 1
A[i] = 1
```

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
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INC

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
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Binary Counter

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- Use array for counting from 0 to $2^k - 1$
- only operation: **INC**
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  - total cost: number of flips (smallest index of a zero)

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1	0	1	1	11

INC $\downarrow$

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
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Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
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  - increases the counter by one
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$A[3]\,A[2]\,A[1]\,A[0]$

$\boxed{1}\ \boxed{0}\ \boxed{1}\ \boxed{1}$ 11

$\Big\downarrow$ INC

$A[3]\,A[2]\,A[1]\,A[0]$

$\boxed{1}\ \boxed{1}\ \boxed{0}\ \boxed{0}$ 12

> What is the total cost of a sequence of n INC operations?

Second Example: Binary Counter

Binary Counter

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$\boxed{1}\ \boxed{0}\ \boxed{1}\ \boxed{1}$ 11

$\Big\downarrow$ INC

$A[3]\,A[2]\,A[1]\,A[0]$

$\boxed{1}\ \boxed{1}\ \boxed{0}\ \boxed{0}$ 12

What is the total cost of a sequence of n INC operations?

Simple Worst-Case Bound:

- largest cost of an operation: k
- cost is at most $n \cdot k$

Second Example: Binary Counter

Binary Counter

- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
- Use array for counting from 0 to $2^k - 1$
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What is the total cost of a sequence of n INC operations?

Simple Worst-Case Bound:

- largest cost of an operation: k
- cost is at most $n \cdot k$ **(correct, but not tight!)**

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
1	0	1	1	11

$\Big\downarrow$ INC

$A[3]$	$A[2]$	$A[1]$	$A[0]$	
1	1	0	0	12

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
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Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8

Incrementing a Binary Counter

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0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
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4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8

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1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments
- In a sequence of n increments from 0, bit $A[i]$ is flipped $\lfloor \frac{n}{2^i} \rfloor$ times

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments
- In a sequence of n increments from 0, bit $A[i]$ is flipped $\lfloor \frac{n}{2^i} \rfloor$ times

$$T(n) \leq$$

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments
- In a sequence of n increments from 0, bit $A[i]$ is flipped $\lfloor \frac{n}{2^i} \rfloor$ times

$$T(n) \leq \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor$$

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments
- In a sequence of n increments from 0, bit $A[i]$ is flipped $\lfloor \frac{n}{2^i} \rfloor$ times

$$T(n) \leq \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor \leq \sum_{i=0}^{k-1} \frac{n}{2^i}$$

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

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- Bit $A[i]$ is only flipped every 2^i increments

Aggregate Analysis: The amortized cost per operation is $\frac{T(n)}{n} \leq 2$.

$$T(n) \leq \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor \leq \sum_{i=0}^{k-1} \frac{n}{2^i} = n \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right) \leq 2 \cdot n.$$

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Increment without Carry-Over

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- actual cost: $c_i = 1$

1 1 0 0

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$$\begin{array}{c} 1\ 1\ 0\ \boxed{0} \\ \big\downarrow \text{INC} \\ 1\ 1\ 0\ 1 \end{array}$$

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- actual cost: $c_i = 1$
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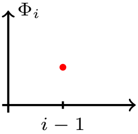
1 1 0 0

$\downarrow$ INC

1 1 0 1



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↓ INC

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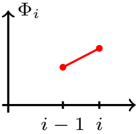

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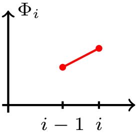
1 1 0 0

$\downarrow$ INC

1 1 0 1



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1 1 0 0

↓ INC

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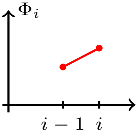

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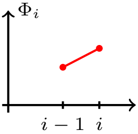
1 1 0 0

$\downarrow$ INC

1 1 0 1



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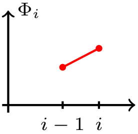

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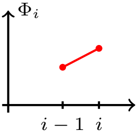
1 1 0 0

INC ↓

1 1 0 1



Increment with Carry-Over

- $c_i = x + 1$, (x lowest index of a zero)

0 1 1 1

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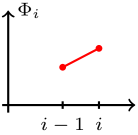

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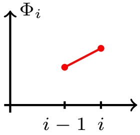

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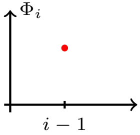

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1 1 0 0

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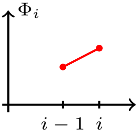

Increment with Carry-Over

- $c_i = x + 1$, (x lowest index of a zero)
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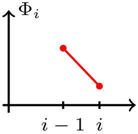
0 1 1 1

↓ INC

1 0 0 0



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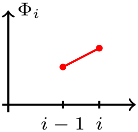

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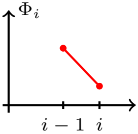

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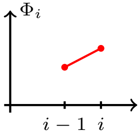

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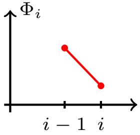

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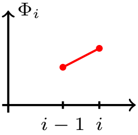
↓ INC

1 1 0 1



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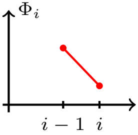
↓ INC

1 0 0 0



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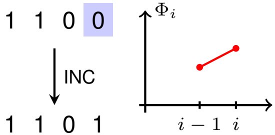

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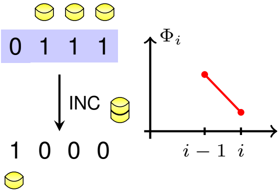

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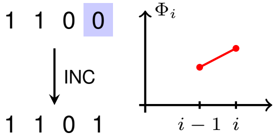

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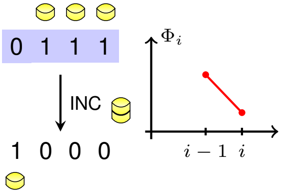

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Amortized Cost $= 2 \Rightarrow T(n) \leq 2n$

Increment without Carry-Over

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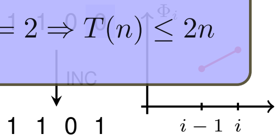

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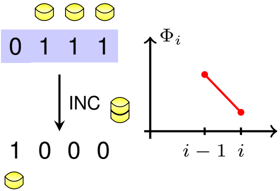

Summary

Amortized Analysis

- Average costs over a sequence of n operations
- overcharge cheap operations and undercharge expensive operations
- no probability/average case analysis involved!

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- Determine an absolute upper bound $T(n)$

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E.g. by bounding the number of expensive operations

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- Determine an absolute upper bound $T(n)$
- every operation has amortized cost $\frac{T(n)}{n}$

$T(n)$ 

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Potential Method

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- use savings from cheap operations to compensate for expensive ones

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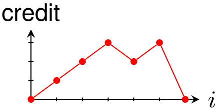

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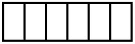
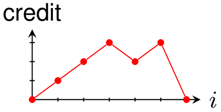
$T(n)$ 

$T(n)$ 

Potential Method

- use savings from cheap operations to compensate for expensive ones
- operations may have different amortized cost



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$T(n)$ 

$T(n)$ 

Full power of this method will become clear later!

Potential Method

- use savings from cheap operations to compensate for expensive ones
- operations may have different amortized cost