Answers 5 - Fibonacci

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Week 5

1.
```
/* Recursive Fibonacci */
int f(int n) {
    if (n==0) return 0;
    if (n==1) return 1;
    else return f(n-1)+f(n-2);
}
```

2.
```
/* Iterative Fibonacci */
int g(int n) {
    int x=0, y=1;
    while (n>0) {
        int z=x+y;
        x=y;
        y=z;
        n--;
    }
    return x;
}
```

3. The 46'th Fibonacci number, 1836311903, is the largest computed before an integer overflow.

4. **Theorem.**

$$f(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \textit{ for all } n \geq 0$$

Proof. By induction on n

Base Cases

First we need to show that the theorem holds for $n = 0$

$$RHS = \frac{(1+\sqrt{5})^0 - (1-\sqrt{5})^0}{2^0\sqrt{5}} = \frac{1-1}{1\sqrt{5}} = \frac{0}{\sqrt{5}} = 0 = f(0) = LHS$$

Then we need to show that the theorem holds for $n = 1$

$$RHS = \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}}$$

$$= \frac{1+\sqrt{5} - 1 + \sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 = f(1) = LHS$$

Induction Step

Assuming the theorem holds for all $0 \leq n < k$ for some $k > 1$, we need to show that it holds for $n = k$

$$LHS = f(k) = f(k-1) + f(k-2)$$

$$= \frac{(1+\sqrt{5})^{k-1} - (1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}} + \frac{(1+\sqrt{5})^{k-2} - (1-\sqrt{5})^{k-2}}{2^{k-2}\sqrt{5}}$$

$$= \frac{2^1(1+\sqrt{5})^{k-1} - 2^1(1-\sqrt{5})^{k-1}}{2^1 2^{k-1}\sqrt{5}} + \frac{2^2(1+\sqrt{5})^{k-2} - 2^2(1-\sqrt{5})^{k-2}}{2^2 2^{k-2}\sqrt{5}}$$

$$= \frac{2(1+\sqrt{5})^{k-1} - 2(1-\sqrt{5})^{k-1} + 4(1+\sqrt{5})^{k-2} - 4(1-\sqrt{5})^{k-2}}{2^k\sqrt{5}}$$

$$= \frac{2(1+\sqrt{5})(1+\sqrt{5})^{k-2} - 2(1-\sqrt{5})(1-\sqrt{5})^{k-2} + 4(1+\sqrt{5})^{k-2} - 4(1-\sqrt{5})^{k-2}}{2^k\sqrt{5}}$$

$$= \frac{(2+2\sqrt{5})(1+\sqrt{5})^{k-2} - (2-2\sqrt{5})(1-\sqrt{5})^{k-2} + 4(1+\sqrt{5})^{k-2} - 4(1-\sqrt{5})^{k-2}}{2^k\sqrt{5}}$$

$$= \frac{(4+2+2\sqrt{5})(1+\sqrt{5})^{k-2} - (4+2-2\sqrt{5})(1-\sqrt{5})^{k-2}}{2^k\sqrt{5}}$$

$$= \frac{(6+2\sqrt{5})(1+\sqrt{5})^{k-2} - (6-2\sqrt{5})(1-\sqrt{5})^{k-2}}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^2(1+\sqrt{5})^{k-2} - (1-\sqrt{5})^2(1-\sqrt{5})^{k-2}}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}} = RHS$$

And this completes the proof by induction.

$\square$

EXTRAS

1. On my laptop this number was computed recursively in 16 seconds and iteratively in 1 microsecond.

2. 476 Fibonacci numbers are computable using doubles.

3. **Theorem.**

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f(n+1) & f(n) \\ f(n) & f(n-1) \end{pmatrix} \textit{ for all } n \geq 1$$

Proof. By induction on n

Base Case

We need to show that the theorem holds for $n = 1$

$$LHS = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$RHS = \begin{pmatrix} f(1+1) & f(1) \\ f(1) & f(1-1) \end{pmatrix} = \begin{pmatrix} f(2) & f(1) \\ f(1) & f(0) \end{pmatrix}$$

$$= \begin{pmatrix} f(1)+f(0) & f(1) \\ f(1) & f(0) \end{pmatrix} = \begin{pmatrix} 1+0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus LHS=RHS when $n = 1$

Induction Step

Assuming the theorem holds for all $1 \leq n < k$ for some $k > 1$, we need to show that it holds for $n = k$

$$LHS = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(k) & f(k-1) \\ f(k-1) & f(k-2) \end{pmatrix}$$

$$= \begin{pmatrix} f(k)+f(k-1) & f(k-1)+f(k-2) \\ f(k) & f(k-1) \end{pmatrix}$$

$$= \begin{pmatrix} f(k+1) & f(k) \\ f(k) & f(k-1) \end{pmatrix} = RHS$$

Thus LHS=RHS when $n > 1$

And this completes the proof by induction.

□

4. This simply follows from the fact that $f(-1) = 1$ and the 0'th power of any square matrix is the identity matrix.