COMS10003 Work Sheet 20

Linear Algebra: Vector Spaces, Span and Basis

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1. Define the set of vectors which are spanned by the following set of vectors

$$\{(2,-1,0,3),(1,3,-1,2),(1,-1,1,-2)\}$$

Answer:

$(2a+b+c,-a+3b-c,c-b,3a+2b-2c)$

2. Determine whether each of the following sets of vectors are dependent or independent. Hence determine the dimension of the space spanned by each set.

   (a) $\{(-1,2,2.4),(1.25,-2.5,-3)\}$

   (b) $\{(1,-1,2),(2,-1,-2)\}$

   (c) $\{(1,0,-1,1),(0,2,-1,1),(-2,4,0,0),(1,-4,1,-1\}$

Answer:

Solve for a, b, etc in linear equations derived from equating linear combination to 0:

(a) dependent, dimension=1

(b) independent, dimension=2

(c) 2 dependent vectors, dimension=2

3. Show that the coordinates of a vector projected onto a subspace are given by the projection of the vector onto each vector of an orthonormal basis for the subspace.

Answer:

Projection $\mathbf{v}' = \sum_i a_i \mathbf{b}_i$

$\mathbf{b}_i.\mathbf{b}_i = 1 \quad \mathbf{b}_i.\mathbf{b}_j = 0 \;\; i \neq j$

Projection satisfies $\mathbf{b}_j.(\mathbf{v}-\mathbf{v}') = 0$, *for all* j, *ie difference vector is orthogonal to all vectors in the subspace, which includes the basis vectors.*

Thus we require: $\mathbf{b}_j.\mathbf{v} = \mathbf{b}_j.\mathbf{v}'$

$\mathbf{b}_j.\mathbf{v}' = \sum_i a_i \mathbf{b}_j.\mathbf{b}_i) = a_j$

Hence $a_j = \mathbf{b}_j.\mathbf{v}$. *Bingo.*

4. Determine the projection of the vector $\mathbf{v} = (2,-1,3,-2)$ onto the subspace spanned by the orthogonal vectors $(1,1,1,0)$, $(0,-1,1,1)$, $(-1,1,0,1)$

Answer:

Coordinates of projection vector wrt orthonormal vectors given by projection of vector onto each orthonormal vector, ie

$c_1 = (2, -1, 3, -2).(1, 1, 1, 0)/3 = 4/3$

$c_2 = (2, -1, 3, -2).(0, -1, 1, 1)/3 = 2/3$

$c_2 = (2, -1, 3, -2).(-1, 1, 0, 1)/3 = -5/3$

$\mathbf{v}' = 4(1, 1, 1, 0)/3 + 2(0, -1, 1, 1)/3 - 5(-1, 1, 0, 1)/3 = (3, -1, 2, -1)$

5. Extend the following sets of vectors to be orthogonal bases for 2-D and 3-D subspaces of $\mathcal{R}^3$ and $\mathcal{R}^4$, respectively. How many such subspaces are there in each case?

   (a) $\{(1, -1, 2)\}$

   (b) $\{(1, 1, -1, -1), (2, -1, 1, 0)\}$

   Answer:

   (a) Eg $(a, b + 2, 1)$

   (b) Eg $(1/3, a, a - 2/3, 1)$

   there are an infinite number of such spaces.

6. For the basis sets and vectors below, determine an orthogonal basis for the subspace spanned by the basis set $\mathcal{B}$ (keeping $\mathbf{v}_1$ in the basis) and hence determine the projection of the vector onto the subspace. Compute the distance between the vector and the subspace and confirm that the error vector between the projection and the vector is orthogonal to the projection.

   (a) Basis set: $\mathcal{B} = \{\mathbf{v}_1 = (1, 1, 1), \mathbf{v}_2 = (1, 0, 1)\}$ and vector $v = (1, 1, 4)$.

   (b) Basis set: $\mathcal{B} = \{\mathbf{v}_1 = (1, 1, 0, 1), \mathbf{v}_2 = (2, -1, 1, 0)\}$ and vector $v = (-1, -1, 2, -1)$.

   Answer:

   (a) Project $\mathbf{v}_2$ *onto direction of* $\mathbf{v}_1$ *to find difference vector*

   Difference vector $\mathbf{w} = \mathbf{v}_2 - (\mathbf{v}_2.\mathbf{v}_1)\mathbf{v}_1/|\mathbf{v}_1|^2$

   $\mathbf{v}_2.\mathbf{v}_1 = (1, 0, 1).(1, 1, 1) = 2$

   $|\mathbf{v}_1|^2 = 3$

   $\mathbf{w} = (1, 0, 1) - 2(1, 1, 1)/3 = (1/3, -2/3, 1/3)$

   $|\mathbf{w}|^2 = 2/3$

   Let projection onto subspace be $\mathbf{v}' = c1\mathbf{v}_1 + c2\mathbf{w}$

   $c1 = \mathbf{v}.\mathbf{v}_1/|\mathbf{v}_1|^2 = (1, 1, 4).(1, 1, 1)/3 = 2$

   $c2 = \mathbf{v}.\mathbf{w}/|\mathbf{w}|^2 = 3(1, 1, 4).(1, -2, 1)/(3 * 2) = 1.5$

   $\mathbf{v}' = 2(1, 1, 1) + (1, -2, 1)/2 = (2.5, 1, 2.5)$

   Difference vector: $(\mathbf{v} - \mathbf{v}') = (1, 1, 4) - (2.5, 1, 2.5) = (-1.5, 0, 1.5)$

   Distance from subspace=$|\mathbf{v} - \mathbf{v}'| = \sqrt{4.5} = 2.12$

   Check for orgononality with $\mathbf{v}'$*:* $(-1.5, 0, 1.5).(2.5, 1, 2.5) = 0$*. Bingo.*

   (b) Project $\mathbf{v}_2$ *onto direction of* $\mathbf{v}_1$ *to find difference vector*

Difference vector $\mathbf{w} = \mathbf{v}_2 - (\mathbf{v}_2.\mathbf{v}_1)\mathbf{v}_1/|\mathbf{v}_1|^2$

$\mathbf{v}_2.\mathbf{v}_1 = (2,-1,1,0).(1,1,0,1) = 1$

$|\mathbf{v}_1|^2 = 3$

$\mathbf{w} = (2,-1,1,0) - (1,1,0,1)/3 = (5/3,-4/3,3/3,-1/3)$

$|\mathbf{w}|^2 = 51/9 =$

Let projection onto subspace be $\mathbf{v}' = c1\mathbf{v}_1 + c2\mathbf{w}$

$c1 = \mathbf{v}.\mathbf{v}_1/|\mathbf{v}_1|^2 = (-1,-1,2,-1).(1,1,0,1)/3 = -1$

$c2 = \mathbf{v}.\mathbf{w}/|\mathbf{w}|^2 = 9*(-1,-1,2,-1).(5,-4,3,-1)/(3*51) = 3*6/51 = 18/51$

$\mathbf{v}' = (-1,-1,0,-1)+18(5,-4,3,-1)(3*51) = (-1,-1,0,-1)+(30,-24,18,-6)/51 = (-0.41,-1.47,0.35,-1.12)$

Difference vector: $(\mathbf{v}-\mathbf{v}') = (-1,-1,2,-1)-(-0.41,-1.47,0.35,-1.12) = (-0.59,0.47,1.65,0.12)$

Distance from subspace$=|\mathbf{v}-\mathbf{v}'| = \sqrt{3.29} = 1.82$

Check for orgononality with $\mathbf{v}'$*:* $(-0.59,0.47,1.65,0.12).(-0.41,-1.47,0.35,-1.12) = 0.2419 - 0.6909 + 0.5775 - 0.1344 = -0.0059 -- > 0$. *Bingo.*

7. *Given two vectors $\mathbf{u} = \frac{1}{\sqrt{3}}(1,-1,1)$ and $\mathbf{v} = \frac{1}{\sqrt{6}}(-1,1,2)$, answer the following:

   (a) Show that the vectors are unit vectors and that they are orthogonal, i.e. that they are orthonormal

   (b) Determine the projections of the vectors $\mathbf{w} = (3,-3,0)$ and $\mathbf{z} = (0,1,3)$ onto the subspace spanned by $\mathbf{u}$ and $\mathbf{v}$.

   (c) Which of the two vectors lies within the subspace? Explain how you arrived at your answer. Use the other vector and its projection to determine a vector which is orthogonal to the subspace. Show that it is orthogonal to the subspace.

*This question is taken from the 2013-14 exam.

Answer: *Use the dot product to show that* $\mathbf{u}.\mathbf{u} = \mathbf{v}\mathbf{v} = 1$ *and that* $\mathbf{u}.\mathbf{v} = 0$.

The key point is that as the vectors are orthonormal then projection onto the subspace can be found by individual projection onto each (basis) vector using the dot product [**1 mark**]. *Let* $proj(\mathbf{w}) = a\mathbf{u} + b\mathbf{v}$, *then*

$$a = \mathbf{w}.\mathbf{u} = 6/\sqrt{3} \qquad b = \mathbf{w}.\mathbf{v} = -6/\sqrt{6}$$

giving $proj(\mathbf{w}) = \frac{6}{\sqrt{3}}\mathbf{u} - \frac{6}{\sqrt{6}}\mathbf{v} = 2(1,-1,1) - (-1,1,2) = (3,-3,0)$, *ie* $\mathbf{w}$. [**1 mark**]
Similarly for $\mathbf{z}$, $proj(\mathbf{z}) = c\mathbf{u} + d\mathbf{v}$

$$c = \mathbf{z}.\mathbf{u} = 2/\sqrt{3} \qquad b = \mathbf{z}.\mathbf{v} = 7/\sqrt{6}$$

giving $proj(\mathbf{z}) = \frac{2}{\sqrt{3}}\mathbf{u} + \frac{7}{\sqrt{6}}\mathbf{v} = 4(1,-1,1)/6 + 7(-1,1,2)/6 = (-1/2,1/2,3)$. [**1 mark**]

$\mathbf{w}$ *lies in the subspace since its projection is itself - it can be defined as a linear combination of the the two bases vectors.* [**1 mark**] $\mathbf{z}$ *is outside the subspace and* *subtracting its projection then gives a vector orthogonal to the subspace, ie* $(0,1,3)-(-0.5,0.5,3)=(0.5,0.5,0)$. **[1 mark]** *We can check by seeing if it is orthogonal to the bases*

$$(0.5,0.5,0).(1,-1,1)=0 \qquad (0.5,0.5,0).(-1,1,2)=0$$

which it is. **[1 mark]**