Continued from last lecture ...

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Mechanical Derivation (2): Method #2 (Karnaugh map)

Example

Consider an example 3-input, 1-output function:

x	у	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Not	tes:			

- The red group spans columns 2 and 3, in row 1; provided x = 1 and z = 1 we specify just those cells, so the expression is x ∧ z, x = 1 restricts us to columns 2 and 3 (columns 0 and 1 have x = 0) and z = 1 restricts us to row 1 (row 0 has z = 0). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
- The green group spans column 3 and rows 0 and 1; provided x = 1 and y = 0 we specify just those cells, so the expression is $x \land \neg y$. That is, x = 1 and y = 0 restricts us to column 3 (columns 0, 1 and 2 have at least one of x = 0 or y = 1) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
- The blue group spans columns 0 and 1, in row 0; provided x = 0 and z = 0 we specify just those cells, so the expression is $\neg x \land \neg z$. x = 0 restricts us to columns 0 and 1 (columns 2 and 3 have x = 1) and z = 0 restricts us to row 0 (row 1 has z = 1). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.

Example

Consider an example 3-input, 1-output function:

х	y	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

r =

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Mechanical Derivation (2): Method #2 (Karnaugh map)

Example

Consider an example 3-input, 1-output function:

х	у	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

$$r = (x)$$

$$\wedge$$
 z)





Make

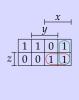
- The red group spans columns 2 and 3, in row 1, provided x = 1 and z = 1 we specify just those cells, so the expression is x ∧ z, x = 1 restricts us to columns 2 and 3 (columns 0 and 1 have x = 0) and z = 1 restricts us to row 1 (row 0 has z = 0). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
- The green group spans column 3 and rows 0 and 1; provided x = 1 and y = 0 we specify just those cells, so the expression is $x \land \neg y$. That is, x = 1 and y = 0 restricts us to column 3 (columns 0, 1 and 2 have at least one of x = 0 or y = 1) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
- The blue group spans columns 0 and 1, in row 0; provided x = 0 and z = 0 we specify just those cells, so the expression is $\neg x \land \neg z$. x = 0 restricts us to columns 0 and 1 (columns 2 and 3 have x = 1) and z = 0 restricts us to row 0 (row 1 has z = 1). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.

- The red group spans columns 2 and 3, in row 1; provided *x* = 1 and *z* = 1 we specify *just* those cells, so the expression is *x* ∧ *z*. *x* = 1 restricts us to columns 2 and 3 (columns 0 and 1 have *x* = 0) and *z* = 1 restricts us to row 1 (row 0 has *z* = 0). Note that the value of *y* doesn't matter: cells in the group hold the value 1 regardless of *y*.
 - The green group spans column 3 and rows 0 and 1; provided x = 1 and y = 0 we specify just those cells, so the expression is $x \land \neg y$. That is, x = 1 and y = 0 restricts us to column 3 (columns 0, 1 and 2 have at least one of x = 0 or y = 1) which is all we need because the group spans all rows. Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
 - The blue group spans columns 0 and 1, in row 0; provided x = 0 and z = 0 we specify just those cells, so the expression is ¬x ∧ ¬z.
 x = 0 restricts us to columns 0 and 1 (columns 2 and 3 have x = 1) and z = 0 restricts us to row 0 (row 1 has z = 1). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.

Example

Consider an example 3-input, 1-output function:

\boldsymbol{x}	y	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression $\,$

$$r = \begin{pmatrix} x & \wedge & z \end{pmatrix} \lor \begin{pmatrix} x & \wedge & \neg y \end{pmatrix}$$

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Mechanical Derivation (2): Method #2 (Karnaugh map)

Example

Consider an example 3-input, 1-output function:

x	у	z	r
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



Each group translates into one term of the SoP form expression

$$r = \left(\begin{array}{cccc} x & & \wedge & z \end{array}\right) \vee \left(\begin{array}{cccc} x & & \wedge & \neg y & & \\ (& \neg x & \wedge & \neg y & & \\ & & & \wedge & \neg z \end{array}\right) \vee$$

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Notes:

- The red group spans columns 2 and 3, in row 1; provided x = 1 and z = 1 we specify just those cells, so the expression is x A z, x = 1 restricts us to columns 2 and 3 (columns 0 and 1 have x = 0) and z = 1 restricts us to row 1 (row 0 has z = 0). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
- The green group spans column 3 and rows 0 and 1; provided x = 1 and y = 0 we specify just those cells, so the expression is $x \land \neg y$. That is, x = 1 and y = 0 restricts us to column 3 (columns 0, 1 and 2 have at least one of x = 0 or y = 1) which is all we need because the group spans *all* rows. Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
- The blue group spans columns 0 and 1, in row 0; provided x = 0 and z = 0 we specify just those cells, so the expression is $\neg x \land \neg z$. x = 0 restricts us to columns 0 and 1 (columns 2 and 3 have x = 1) and z = 0 restricts us to row 0 (row 1 has z = 1). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.

- The red group spans columns 2 and 3, in row 1; provided x = 1 and z = 1 we specify just those cells, so the expression is x ∧ z, x = 1 restricts us to columns 2 and 3 (columns 0 and 1 have x = 0) and z = 1 restricts us to row 1 (row 0 has z = 0). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.
 - The green group spans column 3 and rows 0 and 1; provided x = 1 and y = 0 we specify just those cells, so the expression is $x \land \neg y$. That is, x = 1 and y = 0 restricts us to column 3 (columns 0, 1 and 2 have at least one of x = 0 or y = 1) which is all we need because the group spans all rows. Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
 - The blue group spans columns 0 and 1, in row 0; provided x = 0 and z = 0 we specify just those cells, so the expression is ¬x ∧ ¬z.
 x = 0 restricts us to columns 0 and 1 (columns 2 and 3 have x = 1) and z = 0 restricts us to row 0 (row 1 has z = 1). Note that the value of y doesn't matter: cells in the group hold the value 1 regardless of y.

Example

Consider an example 4-input, 2-output function:

w	х	y	Z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0		1	0 1 ? 0 1 0 ?	?
0	1	1 0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0 0 0 0 0 0 1 1	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0		1	?	?
1	1	1 0	0	?	?
1 1 1	1	0	1	?	?
1	1	1	0	0 ? ? ? ? ?	1 0 ? 1 0 0 ? 0 0 1 ? ?
1	1	1	1	?	?

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Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$						
		х	у			r_0
0 0 0 1 0 1 0 0 1 0 1 0 0 0 1 1 ? ? 0 1 0 0 0 1 0 1 0 0 0 0 0 1 1 0 0 0 0 1 1 1 ? ? 1 0 0 0 1 0 1 0 1 1 0 0 1 0 1 1 ? ? 1 1 0 1 ? ? 1 1 0 1 ? ? 1 1 0 1 ? ? 1 1 1 1 ? ? 1 1 1 1 ? ?	0	0	0		0	0
0 0 1 0 1 0 0 0 0 1 1 ? ? 0 1 0 0 0 1 0 1 0 1 1 0 0 1 1 0 0 0 0 1 1 1 ? ? 1 0 0 0 1 0 0 1 0 1 0 0 1 1 0 1 1 ? ? 1 1 0 1 ? ? 1 1 0 1 ? ? 1 1 1 0 ? ? 1 1 1 1 ? ?	0	0	0	1	0	1
0 0 1 1 ? ? 0 1 0 0 0 1 0 1 0 0 0 0 0 1 1 0 0 0 0 1 1 0 0 0 1 0 0 0 1 0 0 1 0 1 0 0 1 1 0 1 1 ? ? 1 1 0 1 ? ? 1 1 0 1 ? ? 1 1 1 0 ? ? 1 1 1 1 ? ?	0	0	1	0	1	0
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	0		1	1	?	?
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	0		0	0	0	1
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	0	1	0	1	1	0
0 1 1 1 ? ? 1 0 0 0 1 0 1 0 0 1 0 0 1 0 1 0 0 1 1 0 1 1 ? ? 1 1 0 0 ? ? 1 1 0 0 ? ? 1 1 1 0 ? ? 1 1 1 1 ? ?	0	1	1	0	0	0
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	0	1	1			?
1 0 0 1 0 0 1 0 1 0 0 1 1 0 1 1 ? ? 1 1 0 0 ? ? 1 1 0 0 ? ? 1 1 1 0 ? ? 1 1 1 1 ? ?	1	0	0	0	1	0
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	1	0	0	1	0	0
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	1		1	0	0	1
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	1		1	1	?	?
1 1 0 1 ? ? 1 1 1 0 ? ? 1 1 1 1 ? ?	1	1	0	0	?	?
1 1 1 0 ? ? 1 1 1 1 ? ?	1	1	0		?	?
1 1 1 1 ? ?		1	1	0	?	?
	1	1	1	1	?	?

r_1)	r c	<u>v </u>
	0	0	?	1
$_{\pm z}$	0	1	?	0
y 2	?	?	?	?
9 -	1	0	?	0
_				

$$y = \begin{bmatrix} z \\ y \\ z \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ \hline 2 & 7 & 7 & 7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Each group translates into one term of the SoP form expressions

$$r_1 =$$

 $r_0 =$

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Notes:

- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is w ∧ −y ∧ −z. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
 - The green group spans column 1 and rows 2 and 3; provided w = 0, x = 0 and y = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
- The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \wedge z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and w don't matter: cells in the group hold the value 1 regardless of w and w and w don't matter.
- The magenta group spans columns 1 and $\frac{1}{2}$ in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is $x \land \neg y \land \neg z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is ¬w ∧ ¬x ∧ z. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is $w \land y$. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is w ∧ −y ∧ −z. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
 - The green group spans column 1 and rows 2 and 3; provided w=0, x=0 and y=1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w=0 and x=0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w=1 or x=1) and y=1 restricts us to rows 2 and 3 (rows 0 and 1 have y=0). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
 - The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \land z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and y don't matter: cells in the group hold the value 1 regardless of w and y.
 - The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is x ∧ ¬y ∧ ¬z. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
 - The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land z$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1 recardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is w ∧ y. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

Example

Consider an example 4-input, 2-output function:

w	х	y	z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1	1	1 ? 0 1 0 ? 1 0 0 ? ? ? ? ? ? ?	1 0 ? 1 0 0 ? 0 0 1 ? ?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1		?	?
1	1	0	1 0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	?	?

r_1		, ,	r	<i>v</i>
	0	0	?	1
_~ T	0	1	?	0
\sqrt{z}	?	?	?	?
91 -	1	0	?	0

Each group translates into one term of the SoP form expressions

$$r_1 = (w \land \neg y \land \neg z)$$

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Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	у	Z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1		1 ?	?
0	1	0	1 0	0	1
0 0 0	1	0	1	1	0
0	1	1	0	1 0 ?	0
0	1	1		?	?
1	0	0	1 0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	1 0 0 ? ? ?	1 0 ? 1 0 0 ? 0 0 1 ? ?
1	1	1	0	?	?
1	1	1	1	?	?

$$y = \begin{bmatrix} z \\ y \\ z \end{bmatrix}$$

Each group translates into one term of the SoP form expressions

$$r_1 = \left(\begin{array}{ccc} w & \wedge & \neg y & \wedge & \neg z \\ (y & \wedge & \neg w & \wedge & \neg x \end{array} \right) \vee \qquad \qquad r_0 = \left(\begin{array}{ccc} r_0 & -r_0 & -r_0 \\ -r_0 & -r_0 & -r_0 \end{array} \right)$$

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Notes:

- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is w ∧ −y ∧ −z. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
 - The green group spans column 1 and rows 2 and 3; provided w = 0, x = 0 and y = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
 - The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \land z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and y don't matter: cells in the group hold the value regardless of w and y.
 - The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is x ∧ ¬y ∧ ¬z. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is ¬w ∧ ¬x ∧ z. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is $w \land y$. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is w ∧ −y ∧ −z. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
- The green group spans column 1 and rows 2 and 3; provided w = 0, x = 0 and y = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
- The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \land z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and y don't matter: cells in the group hold the value 1 regardless of w and y.
- The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is $x \land \neg y \land \neg z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land z$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1 recardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is w \(\times \) y. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

Example

Consider an example 4-input, 2-output function:

w	х	y	Z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	
0	0	1	0	1	1 0
0	0	1	1	1 ? 0 1 0 ?	?
0	1	0	0	0	1
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	1	?	?
1	0	0	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0		1 0 0 ? ? ?	1 0 0 ? 0 0 1 ? ?
1	1	1	1 0	?	?
1	1	1	1	?	?

r_1		, ,	r c	<u>v</u>
	0	0	?	1
_~ T	0	1	Š	0
$\prod^z \rfloor$?	2	3/	?
	1	0	?	0

Each group translates into one term of the SoP form expressions

$$r_1 = \begin{pmatrix} w & \wedge & \neg y & \wedge & \neg z \end{pmatrix} \lor \\ \begin{pmatrix} y & \wedge & \neg w & \wedge & \neg x \end{pmatrix} \lor \\ \begin{pmatrix} x & \wedge & z \end{pmatrix}$$

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Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

w	x	у	Z	r_1	r_0
0	0	0	0	0	0
0	0	0	1	0	1
0	0	1	0	1	0
0	0	1		1 ?	?
0	1	0	1 0	0	1
0 0 0	1	0	1	1	0
0	1	1	0	1 0 ?	0
0	1	1		?	?
1	0	0	1 0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	1 0 0 ? ? ?	1 0 ? 1 0 0 ? 0 0 1 ? ?
1	1	1	0	?	?
1	1	1	1	?	?

$$y \begin{bmatrix} z \\ y \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} \begin{bmatrix} x \\ 0 & 0 & ? & 1 \\ 0 & 1 & ? & 0 \\ ? & 2 & ? & ? \\ 1 & 0 & ? & 0 \end{bmatrix}$$

Each group translates into one term of the SoP form expressions

$$r_1 = \left(\begin{array}{ccccc} w & \wedge & \neg y & \wedge & \neg z \\ (& y & \wedge & \neg w & \wedge & \neg x \end{array} \right) \vee \left(\begin{array}{ccccc} x & \wedge & z \end{array} \right) \vee \left(\begin{array}{ccccc} x & \wedge & z \end{array} \right)$$

$$r_0 = (x \wedge \neg y \wedge \neg z)$$

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Motoc

- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is w ∧ −y ∧ −z. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
 - The green group spans column 1 and rows 2 and 3; provided w = 0, x = 0 and y = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of z doesn't matter: cells in the group hold the value 1 regardless of z.
 - The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \land z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and y don't matter: cells in the group hold the value regardless of w and y.
 - The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is x ∧ ¬y ∧ ¬z. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is ¬w ∧ ¬x ∧ z. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is $w \land y$. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is w ∧ −y ∧ −z. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
 - The green group spans column 1 and rows 2 and 3; provided w = 0, x = 0 and y = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of z doesn't matter: cells in the group hold the value 1 reeardless of z.
- The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \wedge z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and w don't matter: cells in the group hold the value 1 regardless of w and w and w don't matter.
- The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is x ∧ ¬y ∧ ¬z. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land z$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1 restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w = 0 doesn't matter: cells in the group hold the value 1 regardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is w ∧ y. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have w = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

Example

Consider an example 4-input, 2-output function:

w	х	у	Z	r_1	r_0
0	0	0	0		0
0	0	0		Ō	1
0	0	1	1 0	1	0
	0	1		?	?
0 0 0 0	1	0	1 0	0	1
0	1	0	1	1	0
0	1	1	1 0	0	0
0	1	1	1	?	?
	0	0	0	1	0
1	0	0	1	0	ő
1 1 1	0	1	1 0	0	1
1	0	1	1	?	?
1	1	0	0	?	?
1	1	0	1	?	?
1	1	1	0	?	?
1	1	1	1	0 0 1 ? 0 1 0 ? 1 0 ? ? ?	1 0 ? 1 0 0 ? 0 0 1 ? ?



Each group translates into one term of the SoP form expressions

$$r_0 = \begin{pmatrix} x & \wedge & \neg y & \wedge & \neg z \\ z & \wedge & \neg w & \wedge & \neg x \end{pmatrix} \lor$$

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Mechanical Derivation (3): Method #2 (Karnaugh map)

Example

Consider an example 4-input, 2-output function:

	w	х	y	Z	r_1	r_0
	0	0	0	0	0	0
	0	0	0	1	0	1
	0	0	1	0		0
	0	0	1	1	1 ?	1 0 ?
	0	1	0	0	0	1
	0	1	0	1	1	0
	0	1	1	0	0	0
	0	1	1	1	?	?
	1	0	0	0	1	0
ı	1	0	0	1	0	0
	1	0	1	0	0	1
	1	0	1	1	?	?
	1	1	0	0	?	?
	1	1	0	1	0 ? ? ? ? ?	1 0 0 ? 0 0 1 ? ?
	1	1	1	0	?	?
	1	1	1	1	?	?

$$y = \begin{bmatrix} z \\ y \\ z \end{bmatrix} \begin{bmatrix} x \\ 0 & 0 & ? & 1 \\ 0 & 1 & ? & 0 \\ ? & 2 & ? & ? \\ 1 & 0 & ? & 0 \end{bmatrix}$$

Each group translates into one term of the SoP form expressions



- The red group spans columns 2 and 3 in row 0; provided w = 1, y = 0 and z = 0 we specify just those cells, so the expression is $w \land \neg y \land \neg z$. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1). Note that the value of x doesn't matter: cells in the group hold the value 1
 - The green group spans column 1 and rows 2 and 3; provided w = 0, x = 0 and y = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land y$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of z doesn't matter: cells in the group hold the value 1
 - The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \land z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and y don't matter: cells in the group hold the value 1 regardless of w and y.
 - The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is $x \land \neg y \land \neg z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless
- The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land z$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless of w.
- The orange group spans columns 2 and 3 and rows 2 and 3; provided w = 1 and y = 1 we specify just those cells, so the expression is $w \wedge y$. That is, w = 1 restricts us to columns 2 and 3 (columns 0 and 1 have w = 0) and y = 1 restricts us to rows 2 and 3 (rows 0 and 1 have y = 0). Note that the value of x and z don't matter: cells in the group hold the value 1 regardless of x and z.

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 - The blue group spans columns 1 and 2 and rows 1 and 2; provided x = 1 and z = 1 we specify just those cells, so the expression is $x \land z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and z = 1 restricts us to rows 1 and 2 (rows 0 and 3 have z = 0). Note that the value of w and y don't matter: cells in the group hold the value 1 regardless of w and y.
 - The magenta group spans columns 1 and 2 in row 0; provided x = 1, y = 0 and z = 0 we specify just those cells, so the expression is $x \land \neg y \land \neg z$. That is, x = 1 restricts us to columns 1 and 2 (columns 0 and 3 have x = 0) and y = 0 and z = 0 restricts us to row 0 (rows 1, 2 and 3 have at least one of y = 1 or z = 1) Note that the value of w doesn't matter: cells in the group hold the value 1 regardless
 - The yellow group spans column 0 and rows 1 and 2; provided w = 0, x = 0 and z = 1 we specify just those cells, so the expression is $\neg w \land \neg x \land z$. That is, w = 0 and x = 0 restricts us to column 0 (columns 1, 2 and 3 have at least one of w = 1 or x = 1) and y = 1restricts us to rows 1 and 2 (rows 0 and 1 have y = 0) Note that the value of w doesn't matter: cells in the group hold the value 1
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"Building Block" Components (1) - Choice

- ► The idea of choice is crucial in constructing larger components:
- 1. a **multiplexer** continuously drives one of many inputs onto a single output depending on a control signal, and
- 2. a **demultiplexer** continuously drives a single input onto one of many outputs depending on a control signal.
- ► An *m*-input (resp. *m*-output), *n*-bit multiplexer (resp. demultiplexer) has
- 1. *m* inputs (resp. outputs), each having *n* bits, and
- 2. a ($\lceil \log_2(m) \rceil$)-bit control signal that selects between the inputs (resp. outputs).



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Notes:

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"Building Block" Components (2) – Choice

► As an analogy, the C switch statement

```
Listing (C)

1 switch( c ) {
2   case 0 : r = w; break;
3   case 1 : r = x; break;
4   case 2 : r = y; break;
5   case 3 : r = z; break;
6 }
```

acts similarly to a 4-input multiplexer.

Likewise,

```
Listing (C)

1 switch(c) {
2 case 0 : r0 = x; break;
3 case 1 : r1 = x; break;
4 case 2 : r2 = x; break;
5 case 3 : r3 = x; break;
6 }
```

acts similarly to a 4-output demultiplexer.

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"Building Block" Components (3) – Choice

Definition (2-input, 1-bit multiplexer)

The behaviour of the component

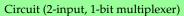


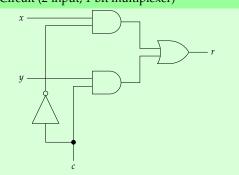
is described by the truth table

MUX2				
С	х	y	r	
0	0	?	0	
0	1	?	1	
1	?	0	0	
1	?	1	1	

which can be used to derive the following implemen-

$$r = (\neg c \land x) \lor (c \land y)$$





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"Building Block" Components (4) – Choice

Definition (2-output, 1-bit demultiplexer)

The behaviour of the component



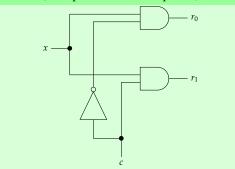
is described by the truth table

DEMUX2					
С	х	r_1	r_0		
0	0	?	0		
0	1	?	1		
1	0	0	?		
1	1	1	?		

which can be used to derive the following implementation:

$$\begin{array}{ccc} r_0 & = & \neg c \wedge x \\ r_1 & = & c \wedge x \end{array}$$

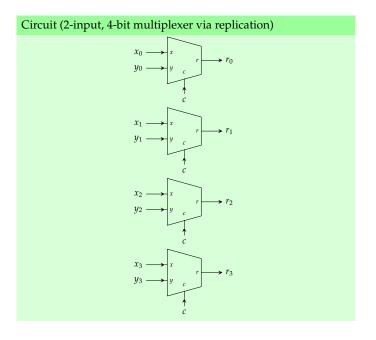
Circuit (2-output, 1-bit demultiplexer)



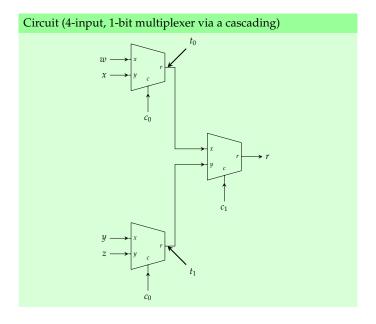


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An Aside: Applying our design patterns



An Aside: Applying our design patterns





Notes:

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- ▶ Given two 1-bit operands *x* and *y*,
- 1. a **half-adder** computes x + y to produce a sum s and a carry-out co,
- 2. a **full-adder** computes x + y + ci, where ci is a 1-bit carry-in, to produce a sum s and a carry-out
- 3. an **equality comparator** computes

$$r = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and

4. a **less than comparator** computes

$$r = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

all of which are used to create larger, more general components.



"Building Block" Components (8) – Arithmetic

Definition (half-adder)

The behaviour of the component



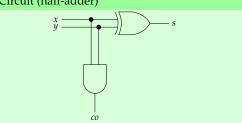
is described by the truth table

Half-Adder					
x y co s					
0	0	0	0		
0	1	0	1		
1	0	0	1		
1	1	1	0		

which can be used to derive the following implementation:

$$\begin{array}{ccc} co & = & x \wedge y \\ s & = & x \oplus y \end{array}$$

Circuit (half-adder)



Notes:

"Building Block" Components (9) – Arithmetic

Definition (full-adder)

The behaviour of the component

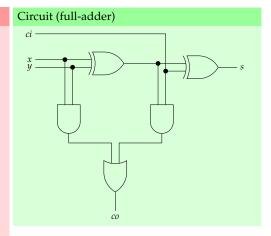


is described by the truth table

Full-Adder						
ci	ci x y co s					
0	0	0	0	0		
0	0	1	0	1		
0	1	0	0	1		
0	1	1	1	0		
1	0	0	0	1		
1	0	1	1	0		
1	1	0	1	0		
1	1	1	1	1		

which can be used to derive the following implementation:

$$\begin{array}{lll} co & = & (x \wedge y) \vee ((x \oplus y) \wedge ci) \\ s & = & x \oplus y \oplus ci \end{array}$$

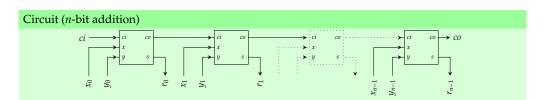


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"Building Block" Components (10) – Arithmetic



Notes:		

Definition (equality comparator)

The behaviour of the component



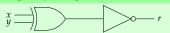
is described by the truth table

	Equal					
I	х	у	r			
ĺ	0	0	1			
ı	0	1	0			
ı	1	0	0			
l	1	1	1			

which can be used to derive the following implementation:

$$r = \neg(x \oplus y)$$

Circuit (equality comparator)



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"Building Block" Components (12) – Arithmetic

Definition (less than comparator)

The behaviour of the component



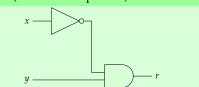
is described by the truth table

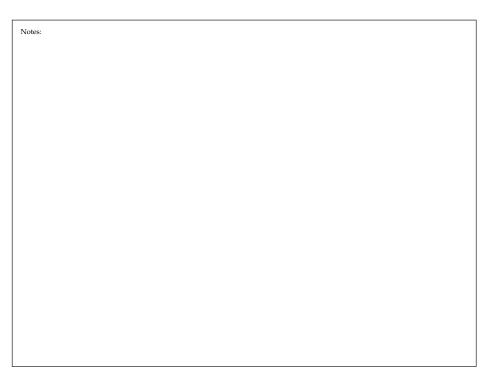
Less-Than				
x	y	r		
0	0	0		
0	1	1		
1	0	0		
1	1	0		

which can be used to derive the following implementation:

$$r = \neg x \wedge y$$

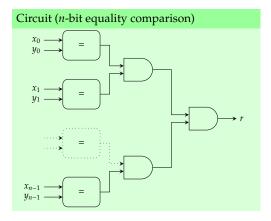
Circuit (less than comparator)



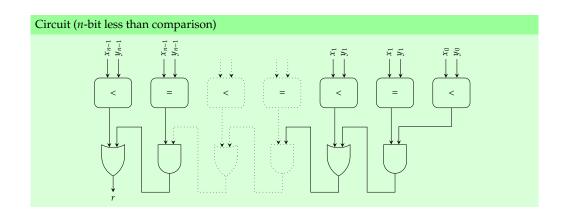


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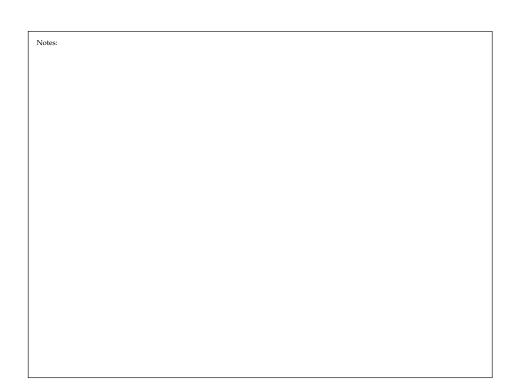
"Building Block" Components (13) – Arithmetic



"Building Block" Components (14) – Arithmetic



Notes:		



Conclusions

► Take away points:

- 1. There are a *huge* number of issues to consider when designing even simple components, e.g.,
 - how do we describe what the circuit should do?
 - what sort of standard library do we use?
 - do we aim for the fewest gates?
 - do we aim for shortest critical path?
 - how do we cope with propagation delay and fan-out?
- 2. The design patterns, and mechanical techniques are the key concepts to focus on: these allow you do produce an effective implementation most easily.
- 3. In many cases, use of appropriate **Electronic Design Automation (EDA)** tools can provide (semi-)automatic solutions.

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