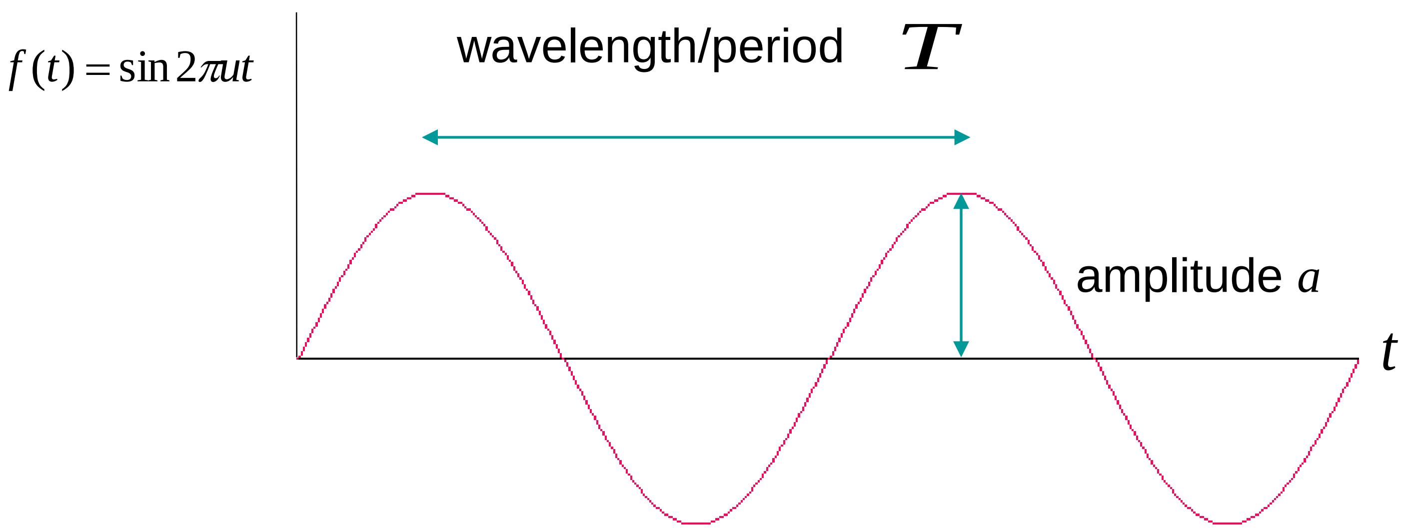
Signals and Functions

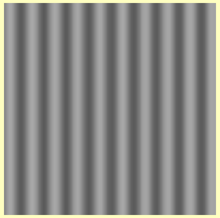
- Frequency - allows us to characterise signals.
- Example: sine function - $a \sin 2\pi ut$
- Repeats over regular intervals - period = T
- Frequency is $u = \dfrac{1}{T}$ cycles/sec (Hz)

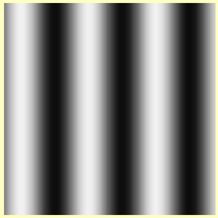


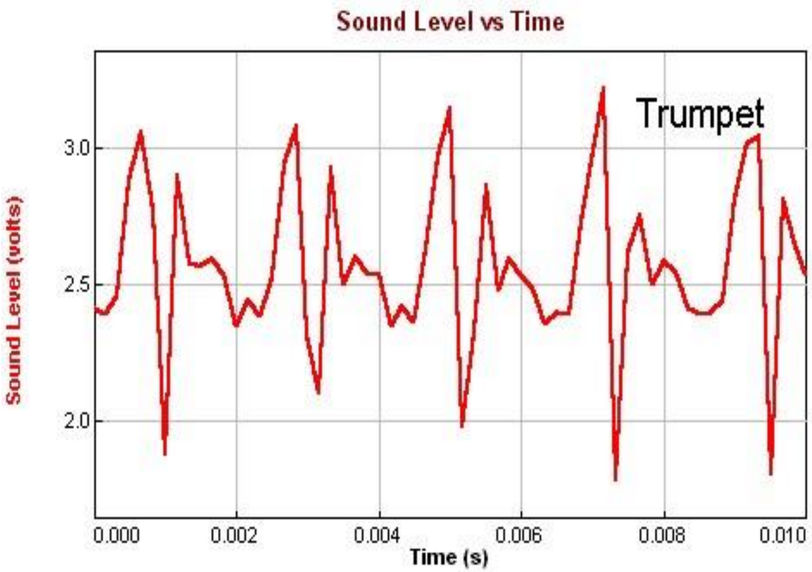
Let’s practice: How do you interpret these musical instrument signals?

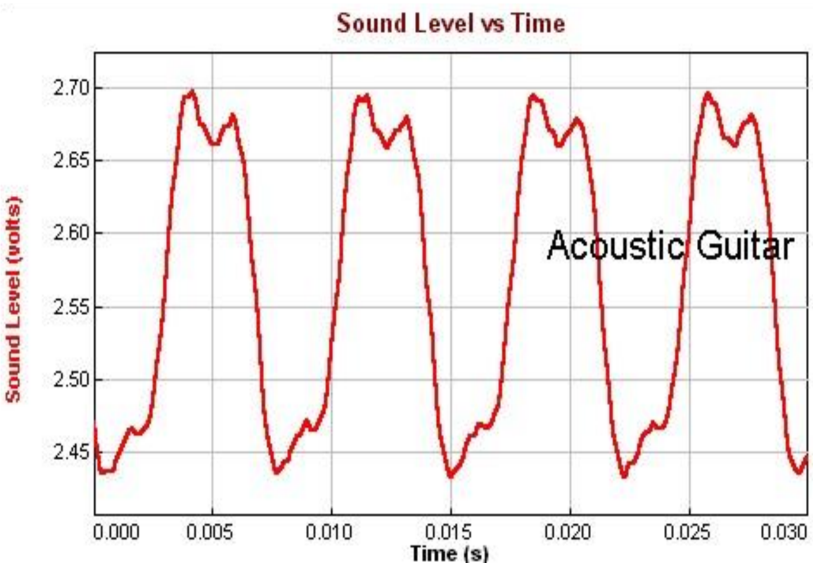
Characteristics of sound in audio signals:

- High pitch - rapidly varying signal
- Low pitch - slowly varying signal



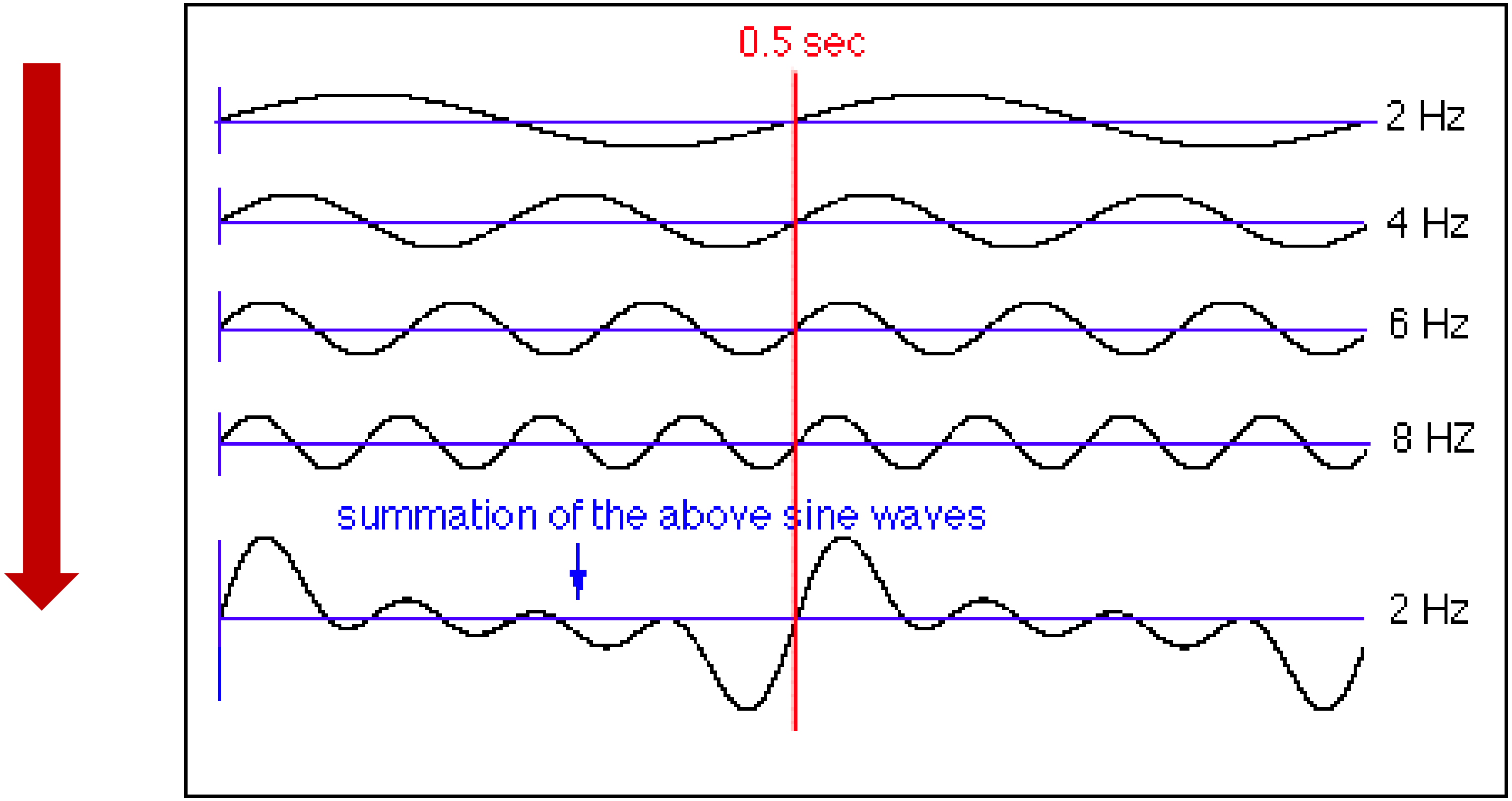






Reminder: Linear Systems

- *For a linear system, output of the linear combination of many input signals is the same linear combination of the outputs →superposition*



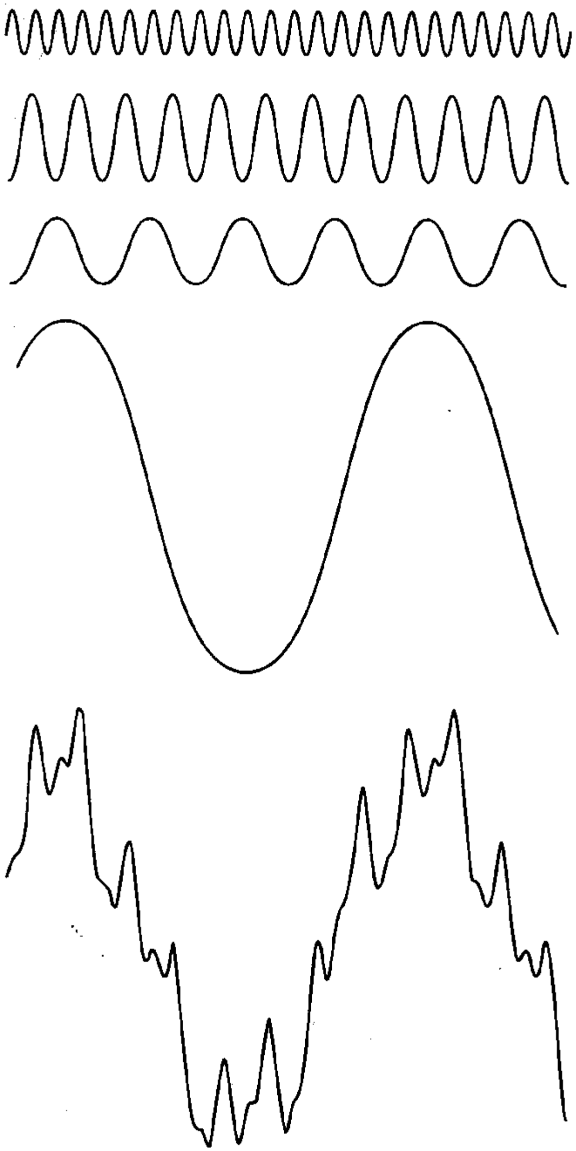
Frequency Analysis

Fourier Series: Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. → *Jean Baptiste Joseph Fourier* (1822).

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)$$

- Thus a function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.
- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

Expressing a periodic function as a sum of sinusoids



Fourier Series: once more…

A ***Fourier series*** **is an expansion of a periodic function** ***f(x)*****. This expansion is in terms of an infinite sum of sines and cosines.**

$$f\left(x\right)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2\pi nx}{T}\right)\ +\ b_n\sin\left(\frac{2\pi nx}{T}\right)$$

cf. with slide 4

This allows any arbitrary periodic function to be broken into a set of simple terms that can be solved individually, and then combined to obtain the solution to the original problem or an approximation to it.

Fourier Series

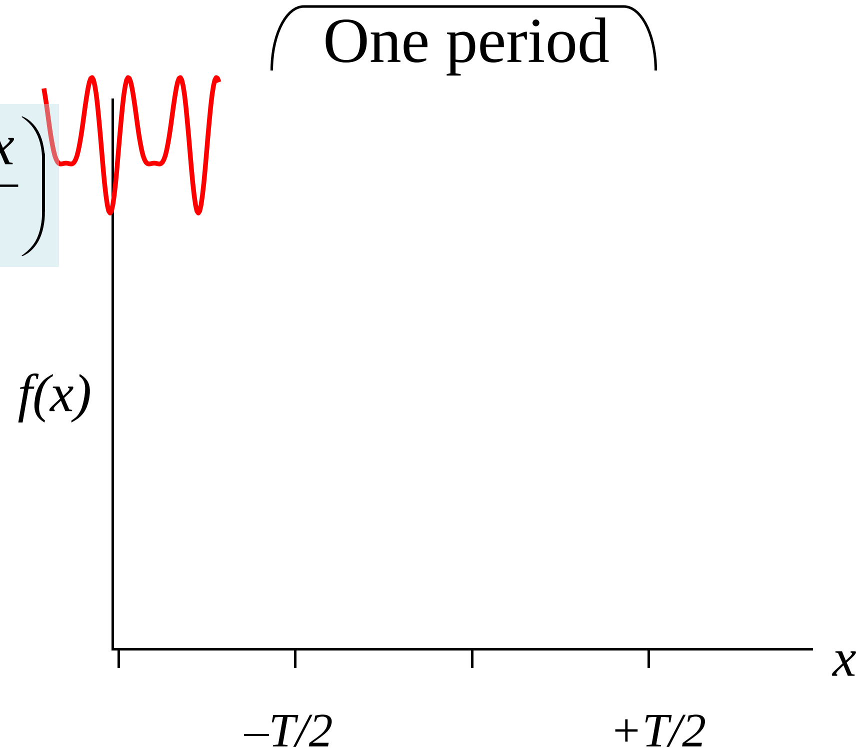
A *Fourier series* provides an equivalent representation of the function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)$$

The coefficients are:

$$a_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(x) \cos\left(\frac{2\pi nx}{T}\right) dx$$

$$b_n = \frac{1}{T} \int_{-T/2}^{+T/2} f(x) \sin\left(\frac{2\pi nx}{T}\right) dx$$



Example periodic function on -$T/2$, +$T/2$

Fourier Series Example: Square Wave

- *f(x)* is a square wave

$$a_n = \frac{1}{T}\int_{-T/2}^{+T/2} f(x)\cos(2\pi nx/T)\,dx$$

$$= \frac{1}{T}\int_{-T/2}^{0} \cos(2\pi nx/T)\,dx - \frac{1}{T}\int_{0}^{+T/2} \cos(2\pi nx/T)\,dx = 0$$

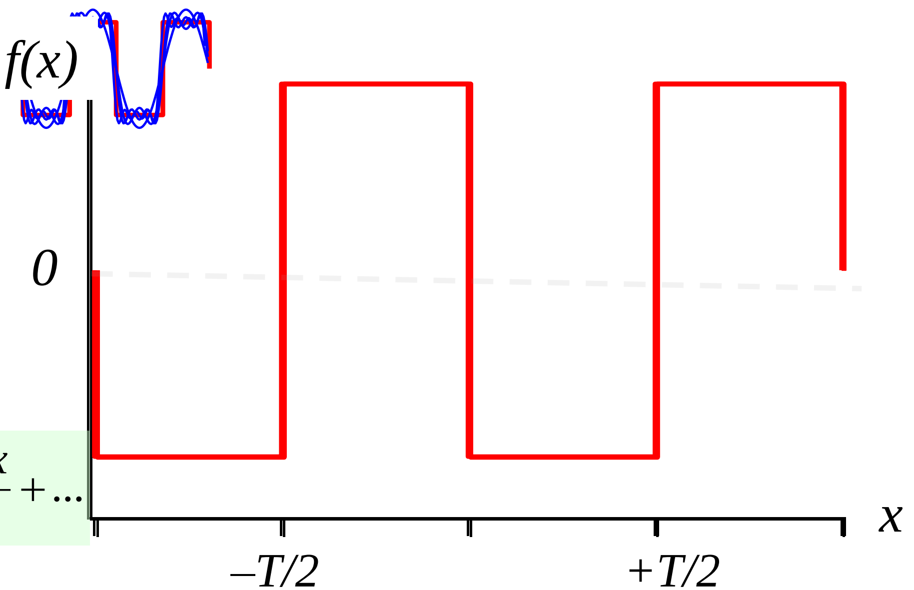
$$f(x) = \begin{cases} +1 & \frac{-T}{2} \le x < 0 \\ -1 & 0 \le x < \frac{T}{2} \end{cases}$$

$$b_n = \frac{1}{T}\int_{-T/2}^{+T/2} f(x)\sin(2\pi nx/T)\,dx$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

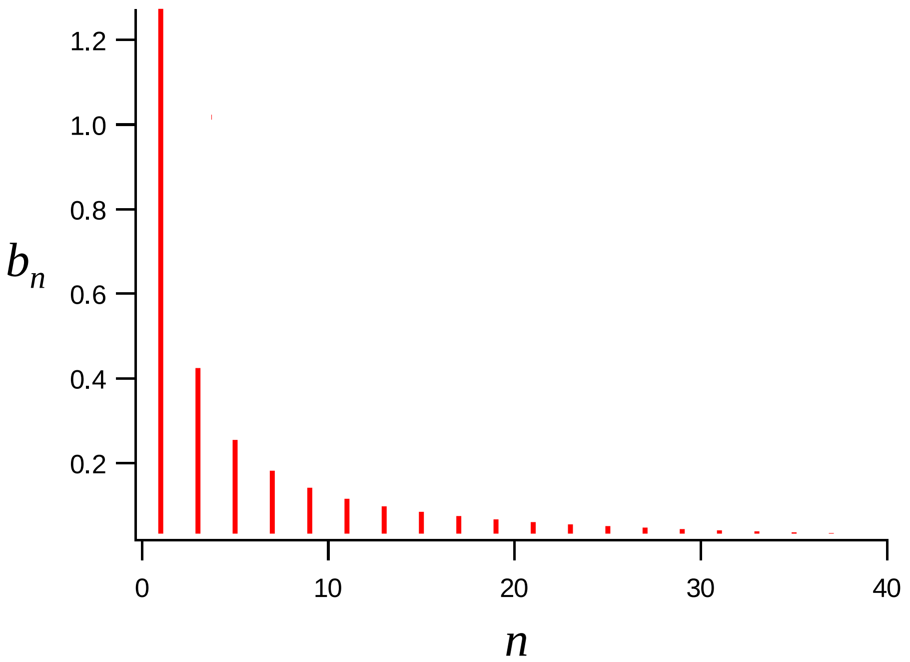
n = 1,3,5,7,…

$$f(x) = \frac{4}{\pi}\cdot\sin\frac{2\pi x}{T} + \frac{4}{3\pi}\cdot\sin 3\cdot\frac{2\pi x}{T} + \frac{4}{5\pi}\cdot\sin 5\cdot\frac{2\pi x}{T} + \ldots$$



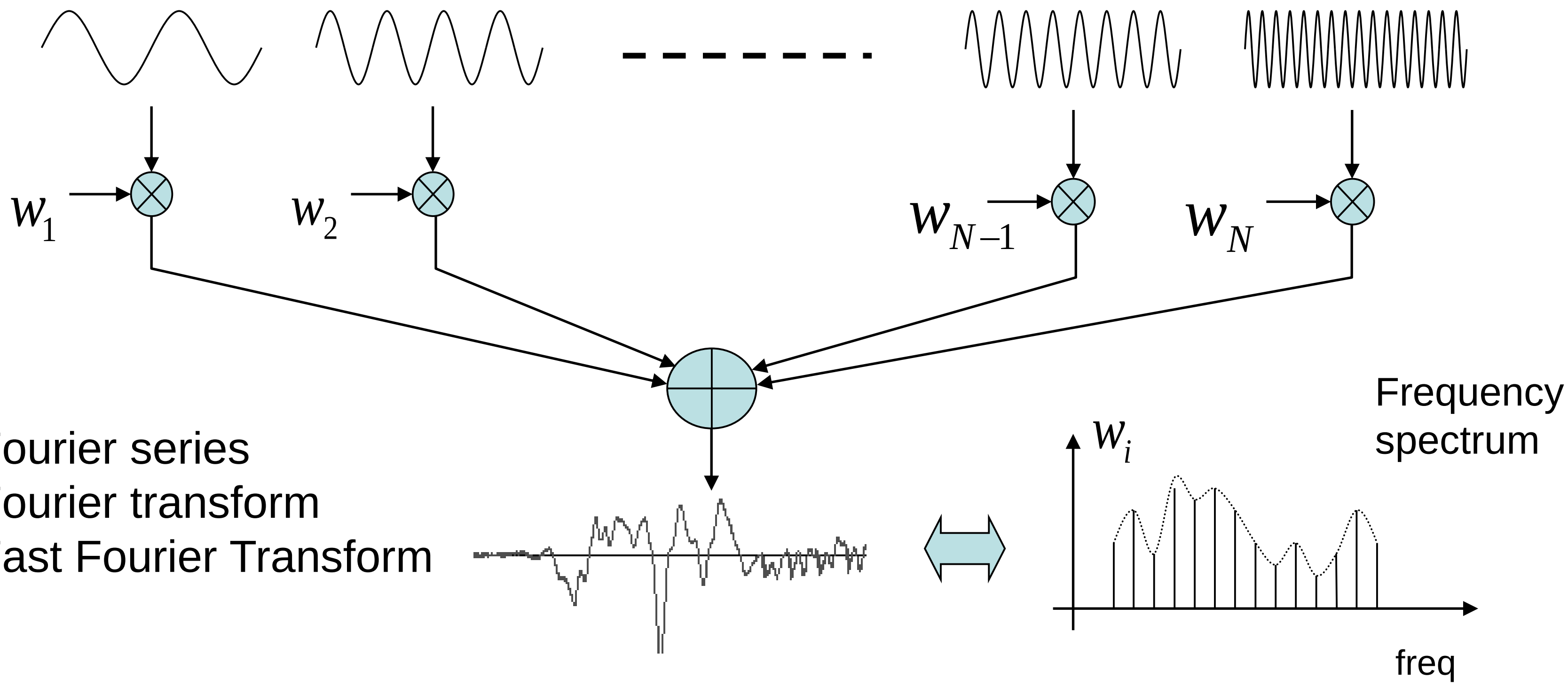
Fourier Series Example: Square Wave

- The set of *Fourier-space* coefficients b_n contain complete information about the function
- Although *f(x)* is periodic to infinity, b_n is negligible beyond a finite range
- Sometimes the Fourier representation is more convenient to use, or just view



Yet another look: Frequency Decomposition

- Every signal can be represented as a summation of sine and cosine waves – **Fourier analysis**



Fourier series
Fourier transform
Fast Fourier Transform

Frequency Analysis

- The aim of processing a signal using Fourier analysis is to *manipulate the spectrum of a signal* rather than manipulating the signal itself.
- Example: simple compression

- Functions that are ***not periodic*** can also be expressed as the integral of sines and/or cosines weighted by a coefficient. In this case we have the Fourier transform.
- The Fourier transform provides a way of representing a signal in a different space - i.e., in the frequency domain.

Fourier Transform Applications

- Applications wide ranging and ever present in modern life:
  - *Telecomms/Electronics/IT* - GSM/cellular phones, digital cameras, satellites, etc.
  - *Entertainment* - music, audio, multimedia devices
  - *Industry* - X-ray spectrometry, Car ABS, chemical analysis, radar design
  - *Medical* - PET, CAT, & MRI machines
  - *Image and Speech analysis* (voice activated “devices”, biometry, …)
  - and many other fields...

1D Fourier Transform

- The Fourier Transform of a single variable continuous function $f(x)$ is:

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi ux} dx$$

- Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} F(u)\, e^{j2\pi ux} du$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

1D Fourier Transform: Discrete Form

- The Fourier Transform of a discrete function of one variable, $f(x), x=0,1,2...,N-1$ is:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-j2\pi ux}{N}} \qquad \text{for } u = 0,1,2,\ldots,N-1.$$

- Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x) = \sum_{u=0}^{N-1} F(u)\, e^{\frac{j2\pi ux}{N}} \qquad \text{for } x = 0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

1D Fourier Domain

- The concept of the frequency domain follows from Euler's Formula:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Thus each term of the Fourier Transform is composed of the sum of *all* values of the function $f(x)$ multiplied by sines and cosines of various frequencies:

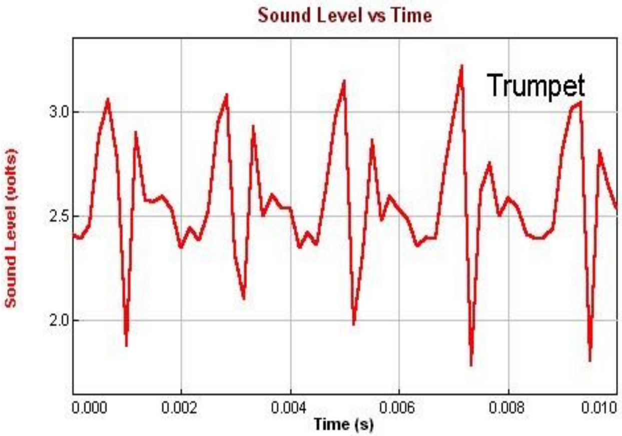
$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - j\sin\left(\frac{2\pi ux}{N}\right)\right]$$

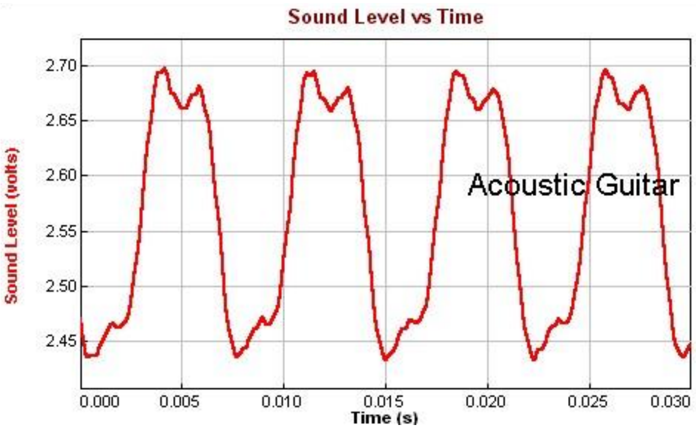
$$\text{for}\, u = 0,1,2,\ldots,N-1.$$

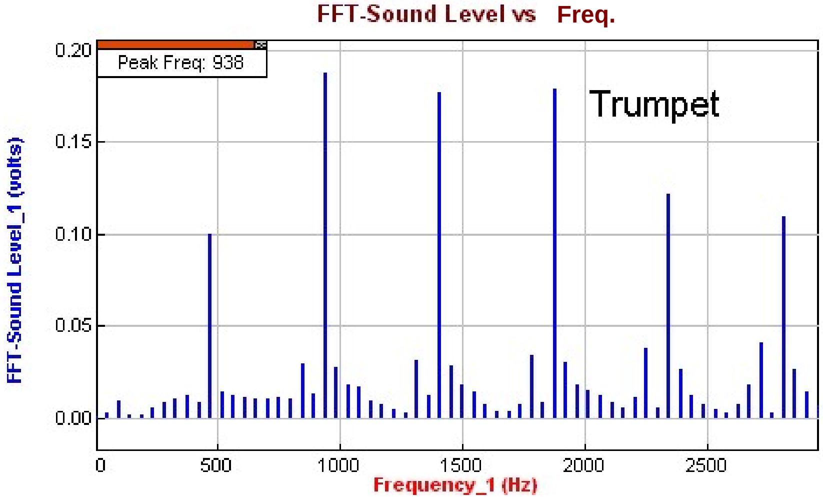
We have transformed from a time domain to a frequency domain representation.

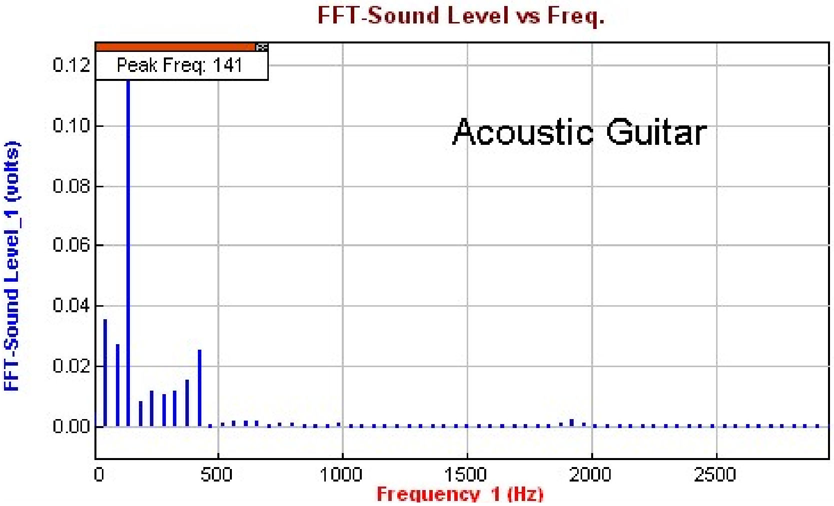
Example: Low and High Frequency

Characteristics of sound in audio signals.



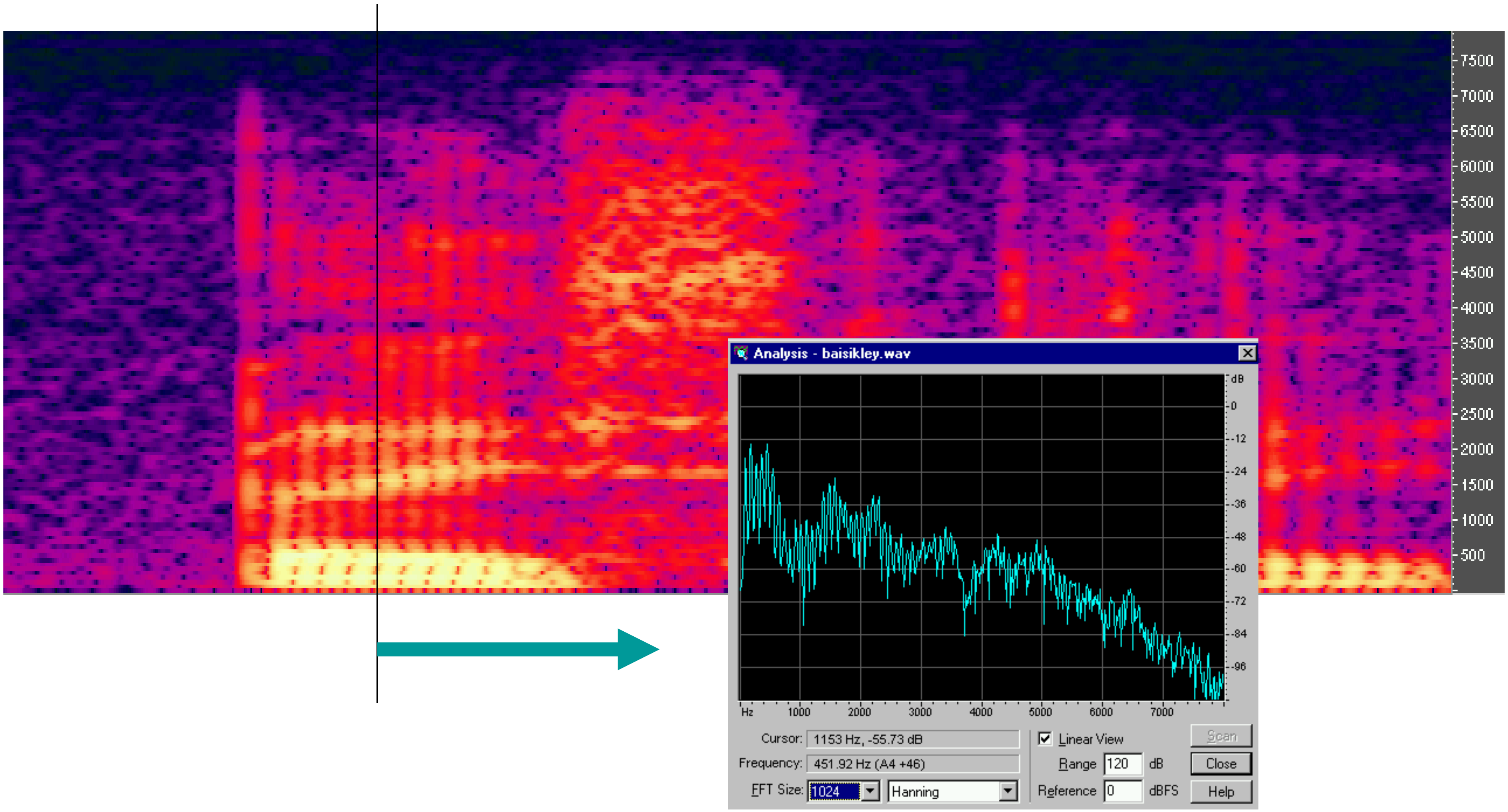






Example: Acoustic Data Analysis

Spectrogram



1D Fourier Transform

- $F(u)$ is a complex number & has real and imaginary parts:

$$F(u) = R(u) + jI(u)$$

- *Magnitude* or *spectrum* of the FT:

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- Phase angle or phase spectrum:

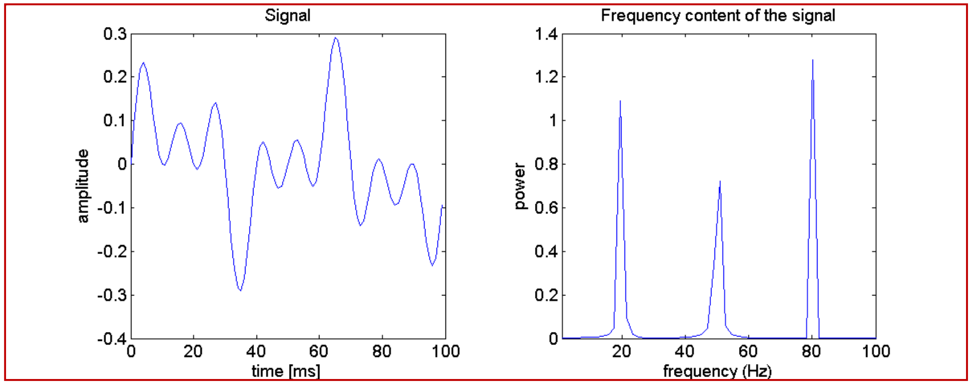
$$\phi(u) = \tan^{-1} \frac{I(u)}{R(u)}$$

- Expressing $F(u)$ in polar coordinates:

$$F(u) = |F(u)| e^{j\phi(u)}$$

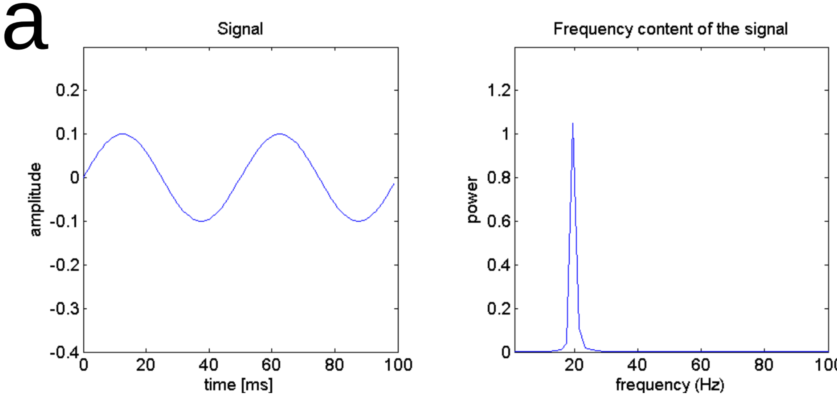
Simple 1D example

$$d = a + b + c$$



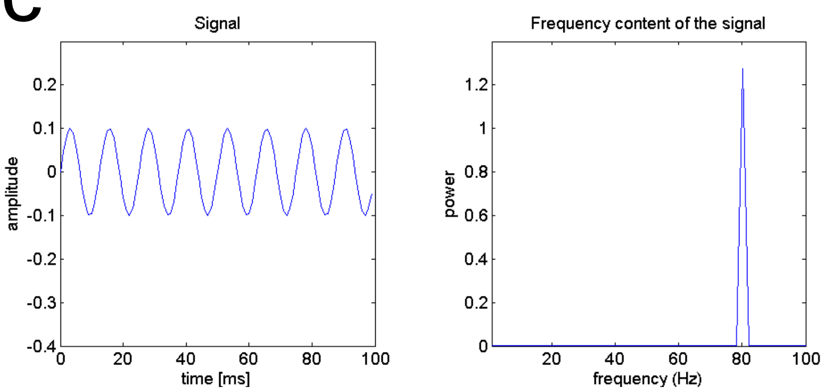
time domain frequency domain

a



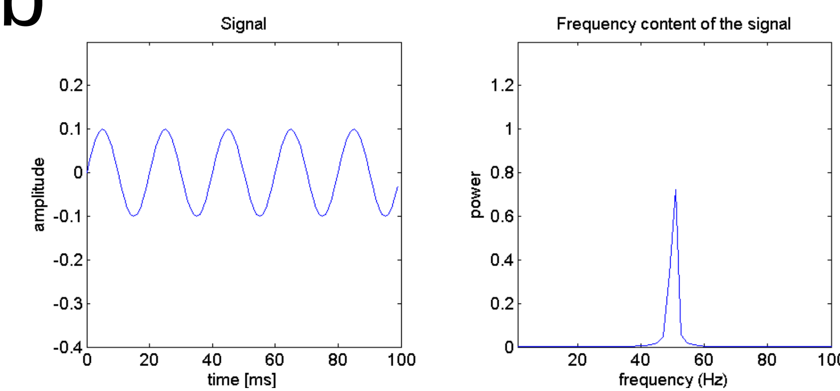
time domain frequency domain

c



time domain frequency domain

b



time domain frequency domain

Frequency Spectrum

- Distribution of $|F(u)|$ → frequency spectrum of signal.
- Slowly changing signals → spectrum concentrated around low frequencies.
- Rapidly changing signals → spectrum concentrated around high frequencies.
- Hence low and high frequency signals.
- Also bandlimited signals → frequency content confined within some frequency band.

Very Simple Application example

- Automatic speech recognition between two speech utterances $x(n)$ and $y(n)$.
- Naïve approach:

$$E = \sum_{\forall n} \left(x(n) - y(n) \right)^2$$

Problems with this approach?

$x(n) = K\, y(n)$, yet $E \neq 0$ *(K being a scaling parameter)*

$x(n) = y(n-m)$, yet $E \neq 0$ *(m causing a delay shift)*

Frequency domain features

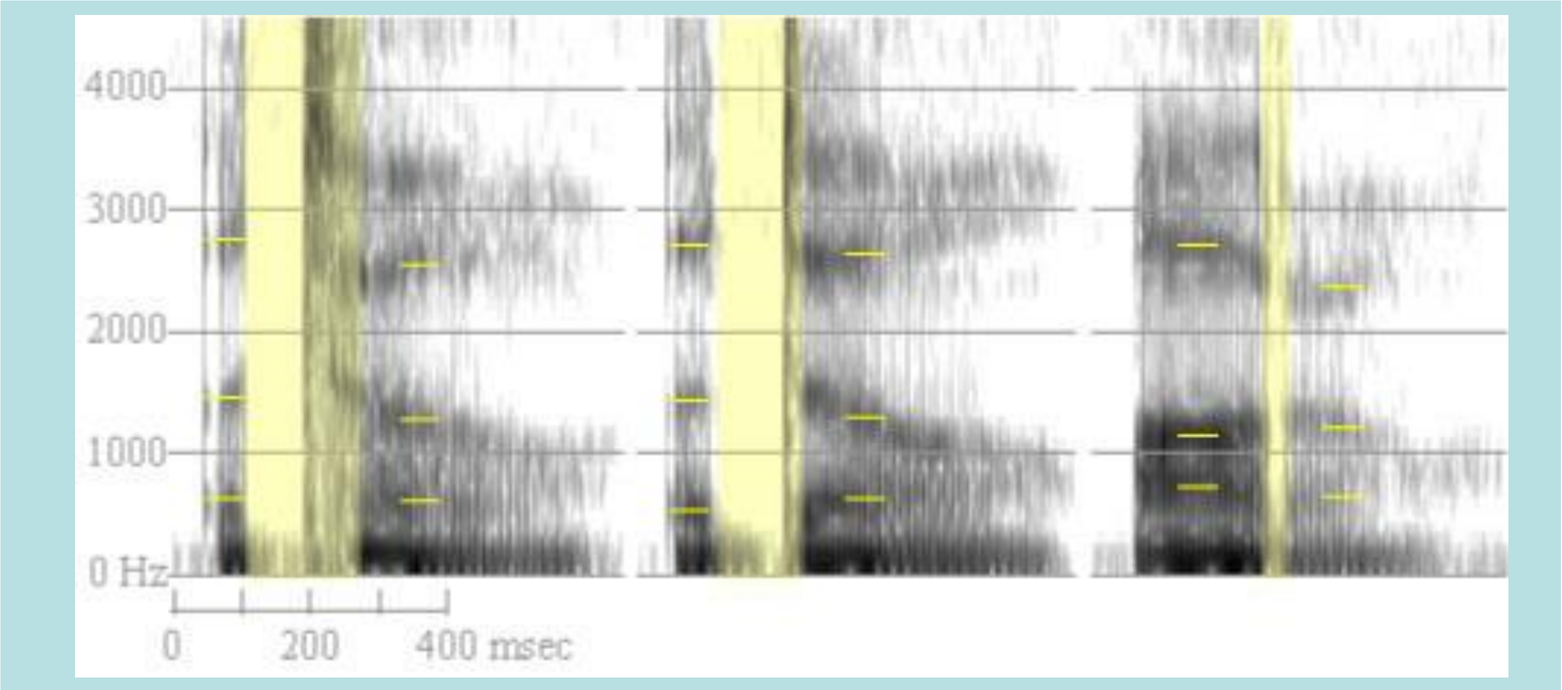
- Take the Fourier transform of both utterances to get $X(u)$ and $Y(u)$.

- Then consider the Euclidean distance between their magnitude spectrums: $|X(u)|$ and $|Y(u)|$:

$$d_E = \sum_{\forall u} \left(|X(u)| - |Y(u)| \right)^2$$

Frequency domain analysis

- Still a difficult task even in the frequency domain.



a tot *a dot* *otto*

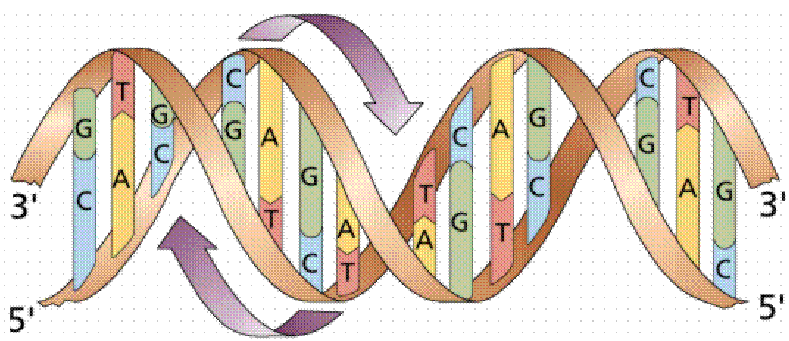
Chinese Year of the Horse

Jan.31,2014-Feb.18,2015

Ooops!

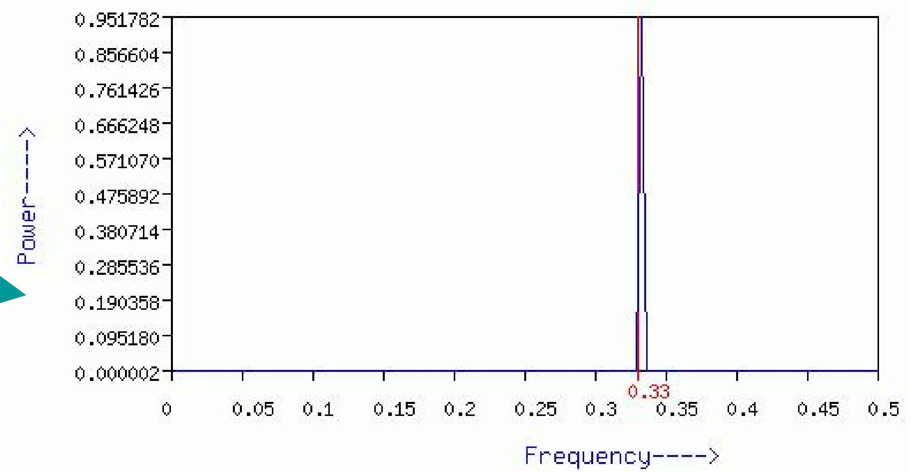
DNA sequence FT example

- The analysis of correlations in DNA sequences is used to identify protein coding genes in genomic DNA.
- Locating and characterizing repeats and periodic clusters provides certain information about the structural and functional characteristics of the molecule.
- DNA sequences are represented by letters, A, C, G or T, and - .
- e.g. ACAATG-GCCATAAT-ATGTGAAC--GCTCA…

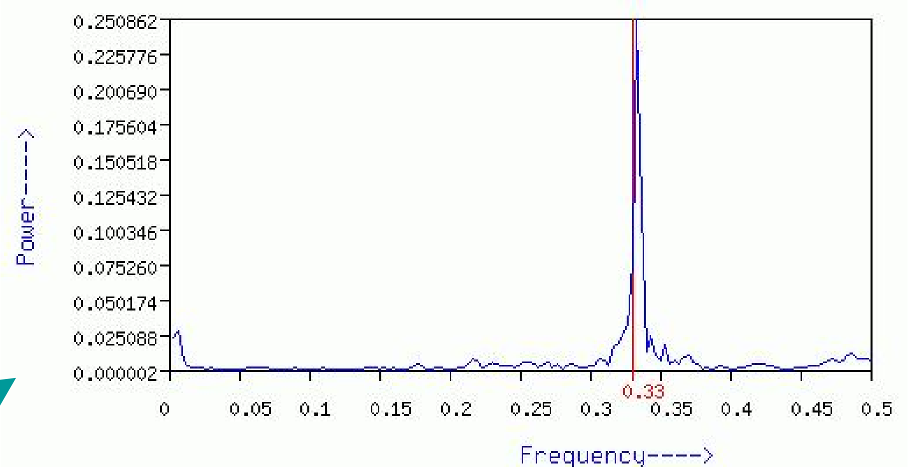


DNA sequence FT example

- Consider the periodic sequence A--A--A--A--..... where blanks can be filled randomly by A, C, G or T. This shows a periodicity of 3.



- The spectral density of such a sequence is significantly non-zero only at one frequency (0.33) which corresponds to the perfect periodicity of base A (1/0.333=3.0).



- Destroy the perfect repetition by randomly replacing the A's with all letters…

Let’s practice: DNA sequence analysis

- The computation of Fourier and other linear transforms of *symbolic data* is a big problem.
- The simplest solution is to map each symbol to a number. The difficulty with this approach is the dependence on the particular labeling adopted.

Consider, for example, the following symbolic periodic sequence:
s = (ATAGACATAGAC . . .).

The mapping
A → 1,
T → 0,
G → 0,
C → 0,
→ Period Two

The mapping
A → 1,
T → 2,
G → 3,
C → 4,
→ Period Six

- This clearly shows that some of the relevant harmonic structure can be hidden (or exposed) by the symbolic-to-numeric labelling.