Worksheet Fibonacci - Part IV

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Week 8

This worksheet continues the Fibonacci case study. You will find the time complexity and show correctness of the following functions (from three weeks ago) which both compute the n'th Fibonacci number $f(n)$ for some $n \geq 0$.

```
int f(int n) {
    if (n==0) return 0;
    if (n==1) return 1;
    else return f(n-1)+f(n-2);
}
```

```
int i(int n) {
    int x=0, y=1;
    while (n>0) {int z=x+y; x=y; y=z; n--;}
    return x;
}
```

1. Find the number of arithmetic operations done by `f` and `i` on input n.

   Hint: For the former consider expressions of the form $a.f(n \pm b) \pm c$.

2. Prove that `f(n)` and `i(n)` both return $f(n)$ for all $n \geq 0$.

   Hint: For the latter follow the format of the slide "Proving our Loop Invariant" in the notes on correctness: i) write the program, line by line, leaving space for PRE, POST and MID conditions; ii) add the PRE and POST conditions (taking care to remember the initial value n_0 of n); iii) use forwards and backwards reasoning to identify the conditions before and after the loop; iv) guess the loop invariant to set up the conditions immediately before and after the loop and its body; v) use backward reasoning through the body of loop; vi) logically prove initialisation, maintenance and termination.

ANSWERS

1a The function `f` performs $3f(n+1) - 3$ arithmetic operations (which can be proven by induction using the fact there is 1 addition and 2 subtractions on each recursive case).

n	arith. ops.
0	0
1	0
2	3+0+0=3
3	3+3+0=6
4	3+6+3=12
5	3+12+6=21
6	3+21+12=36
...	...

1b The function `i` performs $2n$ arithmetic operations (1 addition and 1 subtraction on every iteration of the loop).

n	arith. ops.
0	0
1	2
2	4
3	6
4	8
5	10
6	12
...	...

2a Trivial proof by induction.

2b int i (int n) {

$// \boxed{\text{given } n = n_0 \geq 0}$ PRE

```
int x=0, y=1;
```

$// \ x = 0 \wedge y = 1 \wedge n = n_0 \geq 0$

$\downarrow_{INIT}$

$// \boxed{x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0}$ INV

```
while (n>0) {
```

$// \boxed{x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0} \wedge \boxed{n > 0}$

$\downarrow_{MAINT}$

$// \ y = f(n_0 - n + 1) \wedge (x + y) = f(n_0 - n + 2) \wedge n \geq 1$

```
int z=x+y;
```

$// \ y = f(n_0 - n + 1) \wedge z = f(n_0 - n + 2) \wedge n \geq 1$

```
x=y;
```

$// \ x = f(n_0 - n + 1) \wedge z = f(n_0 - n + 2) \wedge n \geq 1$

```
y=z;
```

$// \ x = f(n_0 - n + 1) \wedge y = f(n_0 - n + 2) \wedge n \geq 1$

$// \ x = f(n_0 - (n-1)) \wedge y = f(n_0 - (n-1) + 1) \wedge (n-1) \geq 0$

```
n--;
```

$// \boxed{x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0}$

```
}
```

$// \boxed{x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0} \wedge \boxed{\neg(n > 0)}$

$\downarrow_{TERM}$

$// \ x = f(n_0)$

```
return x;
```

$\boxed{// \text{ return } f(n_0)}$ POST

```
}
```

For INIT we need to show

$x = 0 \wedge y = 1 \wedge n = n_0 \geq 0 \models x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0$

$$
\begin{aligned}
x &= 0 && \text{given} && (1)\\
y &= 1 && \text{given} && (2)\\
n &= n_0 && \text{given} && (3)\\
n_0 &\geq 0 && \text{given} && (4)\\
n_0 &= n && \text{from 3 using symmetry of } = && (5)\\
n_0 - n = n - n &= 0 && \text{from 5 subtracting } n \text{ from both sides} && (6)\\
f(n_0 - n) = f(0) &= 0 && \text{from 6 using definition of } f && (7)\\
\boxed{x = f(n_0 - n)} & && \text{from 1,7 using transitivity of } = && (8)\\
n_0 - n + 1 = n - n + 1 &= 1 && \text{from 5 adding 1 to both sides} && (9)\\
f(n_0 - n + 1) = f(1) &= 1 && \text{from 9 using definition of } f && (10)\\
\boxed{y = f(n_0 - n + 1)} & && \text{from 2,10 using transitivity of } = && (11)\\
n &\geq n_0 && \text{from 3 by definition of of } \geq && (12)\\
\boxed{n \geq 0} & && \text{from 12,4 using transitivity of } \geq && (13)
\end{aligned}
$$

For MAINT we need to show

$x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0 \wedge n > 0 \models$
$y = f(n_0 - n + 1) \wedge (x + y) = f(n_0 - n + 2) \wedge n \geq 1$

$$
\begin{aligned}
x &= f(n_0 - n) && \text{given} && (1)\\
\boxed{y = f(n_0 - n + 1)} & && \text{given} && (2)\\
n &\geq 0 && \text{given} && (3)\\
n &> 0 && \text{given} && (4)\\
x + y &= f(n_0 - n) + f(n_0 - n + 1) && \text{from 1,2} && (5)\\
\boxed{x + y = f(n_0 - n + 2)} & && \text{from 5 by definition of } f && (6)\\
\boxed{n \geq 1} & && \text{from 4} && (7)
\end{aligned}
$$

For TERM we need to show

$x = f(n_0 - n) \wedge y = f(n_0 - n + 1) \wedge n \geq 0 \wedge \neg(n > 0) \models x = f(n_0)$

$x = f(n_0 - n)$	given	(1)
$y = f(n_0 - n + 1)$	given	(2)
$n \geq 0$	given	(3)
$\neg(n > 0)$	given	(4)
$n \leq 0$	from 4	(5)
$n = 0$	from 3,5	(6)
$f(n_0 - n) = f(n_0 - 0) = f(n_0)$	from 6	(7)
$\boxed{x = f(n_0)}$	from 1,7 using transitivity of =	(8)