Data Structures and Algorithms – COMS21103

2015/2016

Line Segment Intersections

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slides inspired by Marc van Kreveld

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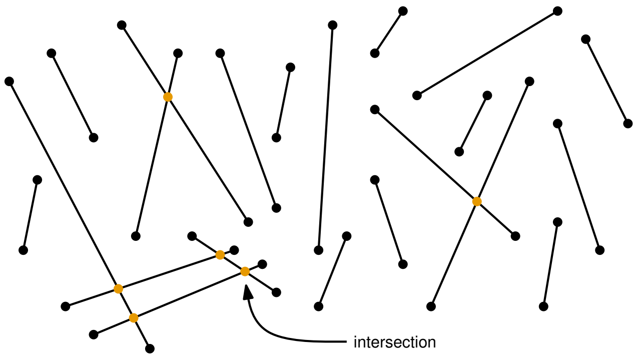
Introduction

Problem Given n line segments, find all the intersections...

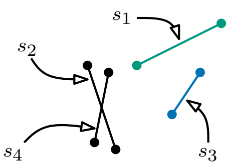


In the input, each line segment is given by
the coordinates of its end points

A simple algorithm

One simple approach to this problem is to test every pair of line segments...

Let s_i denote the i-th line intersection




```
For i = 1,2,...,n
  For j= 1,2,...,n
    If (s_i intersects s_j) and (i ≠ j)
      output (i,j)
```

Given two line segments s_i and s_j described by their end point coordinates
deciding whether (and where) they intersect
takes $O(1)$ time

Why?

Any computation on two objects with $O(1)$ space descriptions takes $O(1)$ time

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A simple algorithm

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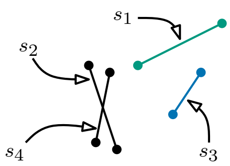


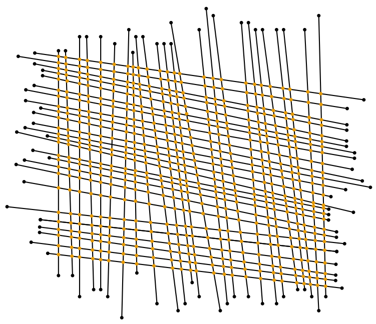

```
For i = 1,2,...,n
  For j= 1,2,...,n
    If (s_i intersects s_j) and (i ≠ j)
      output (i,j)
```

This algorithm runs in $O(n^2)$ time

(because checking pair of lines takes $O(1)$ time)

...can we do better?

If there are n line segments...
how many intersections can there be?



Here there are 50 line segments

and 625 intersections

In general, there could be more than $(\frac{n}{2})^2$ intersections

If we want to output all the intersections,
we can't possibly expect to do better than $O(n^2)$ time in the worst-case

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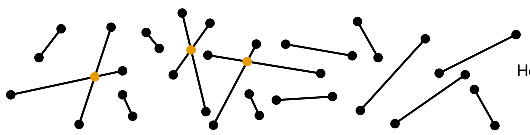
Output sensitive algorithms

The time complexities of the algorithms we have seen so far (in this course)
have only depended on the size on the input

The time complexity of the algorithm we will see in this lecture
also depends on the size of the output



Here $n = 17$
and $k = 3$

Let k denote the number of line segment intersections
(k is not provided in the input)

We will see an algorithm for line segment intersection which takes
$O(n \log n + k \log n)$ time in the worst-case

Output sensitive algorithms

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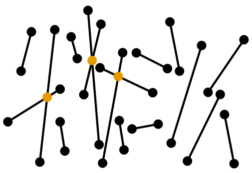
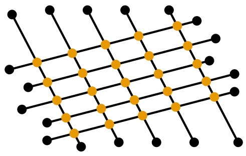
If k is small...



For example if $k \leqslant 2n$ we get an

$O(n \log n)$ worst-case time

If k is big...



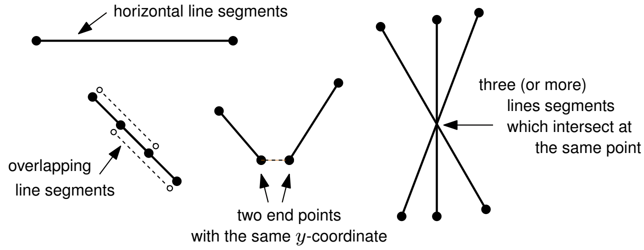
For example if $k \geqslant (\frac{n}{2})^2$ we get an

$O(n^2 \log n)$ worst-case time

(which is worse than the simple algorithm)

Some simplifying restrictions

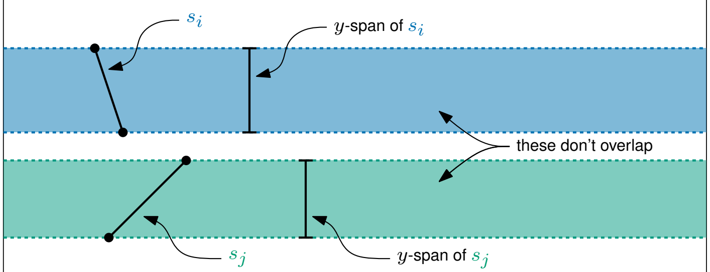
In the interest of simplicity, we don't allow the input to contain any of the following:



All of these restrictions can be removed making the algorithm slightly more involved

(without changing the time complexity)

A first observation



If s_i and s_j don't have overlapping y-spans

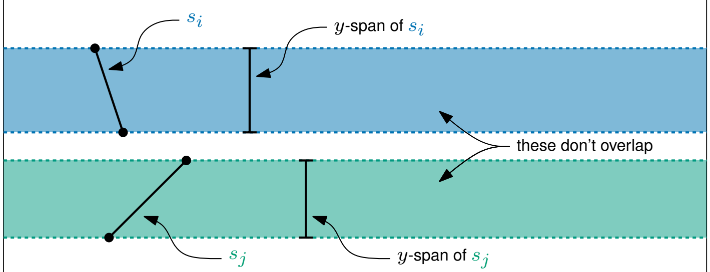
they don't intersect

This suggests an overall approach to the problem...

sweep a horizontal line through the plane from top to bottom

finding intersections as we go

A first observation



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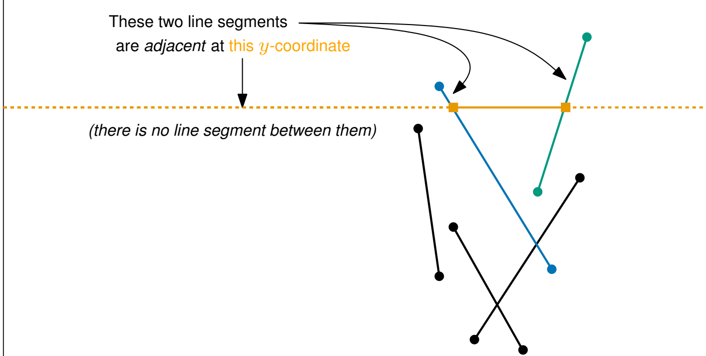
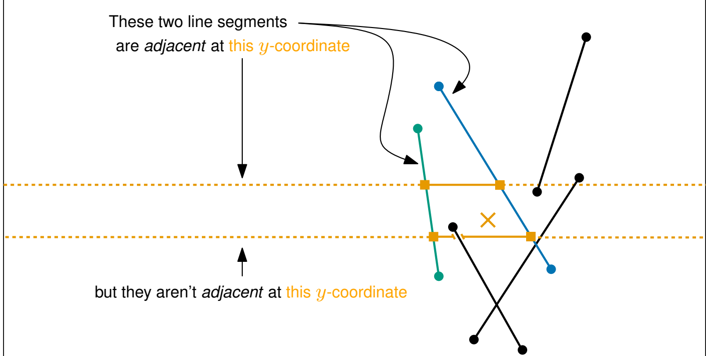
Adjacent line segments and a second observation



Adjacent line segments and a second observation



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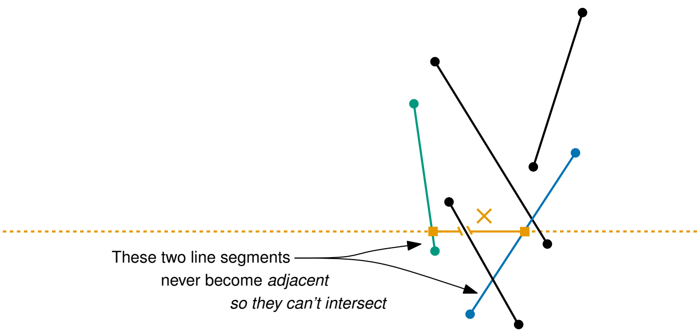
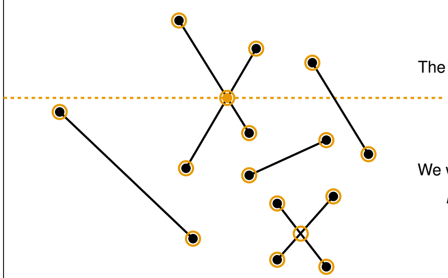


Two line segments s_i and s_j which are never adjacent don't intersect

The overall approach

The overall approach is to
imagine a horizontal line passing through the plane from the top to the bottom
this is called a plane sweep

We cannot hope to process the sweep line at every y-coordinate
so the sweep line jumps between *interesting positions*
called event points



The event points are
the end points of line segments
and the *intersection points*

We will have to detect the
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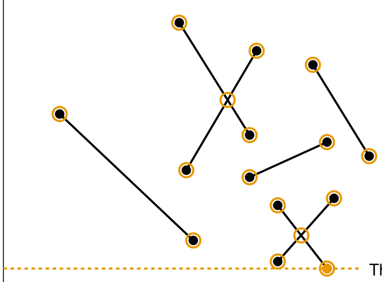

The event points are
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We will have to detect the
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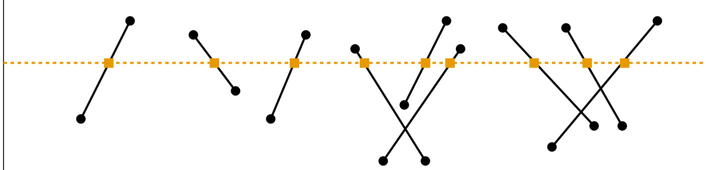
The number of event points is $O(n+k)$

The status of the sweep line

The *status* of the sweep line
is the set of line segments which currently intersect the sweep line
ordered from left to right by where they intersect
(i.e. in the order given by the ■)



Fact the status of the sweep line can only change at event points
(i.e. at an end point or an intersection)

the ■ *move but the order stays the same*

We will store the status of the sweep line in a data structure
which allows efficient updates (more details later)

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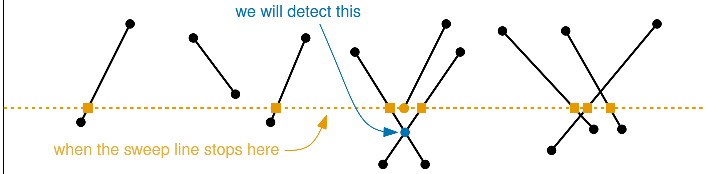


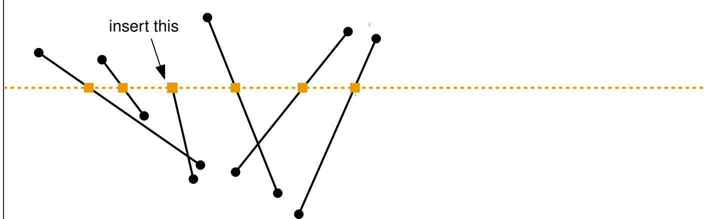
The status of the sweep line tells us which line segments are currently adjacent

Why is this useful?
two line segments which **never** become adjacent cannot intersect

We will detect each upcoming intersection when
the corresponding line segments first become adjacent

Updating the sweep line

Every time the sweep line moves to the next event point, we update
the status data structure



If the event point is
the top of a line segment, we insert it into the status data structure
(at the appropriate place)

We then check whether this segment will intersect either of the adjacent segments

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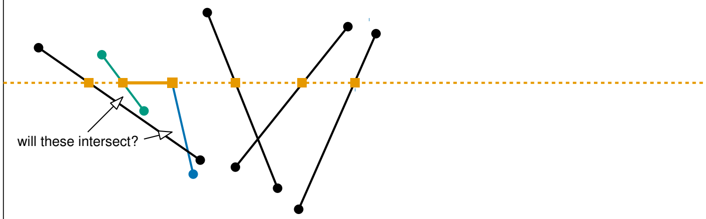

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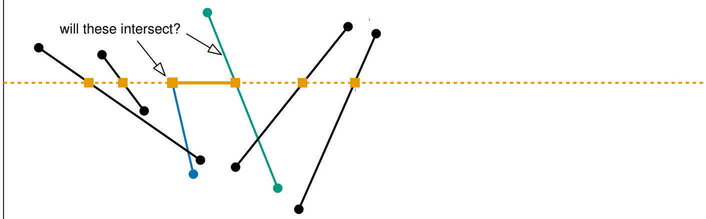

If the event point is
the top of a line segment, we insert it into the status data structure
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We then check whether this segment will intersect either of the adjacent segments

If either pair will intersect, we've found a new event point

Updating the sweep line

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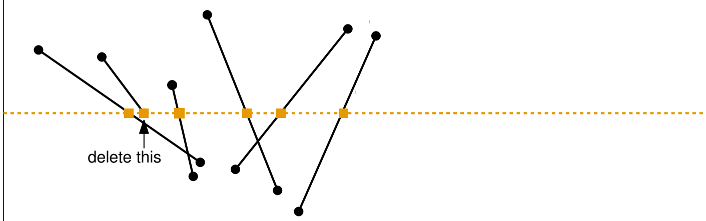

If the event point is
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Updating the sweep line

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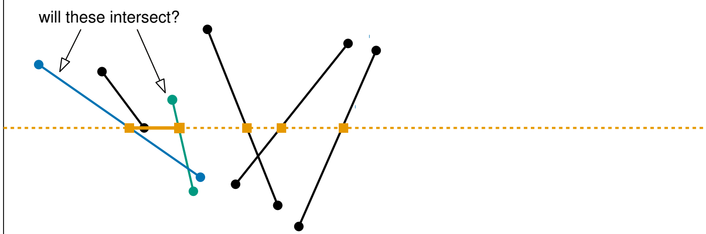

If the event point is
the bottom of a line segment, we delete it from the status data structure

This makes a pair of segments adjacent, so we check whether they will intersect

Updating the sweep line

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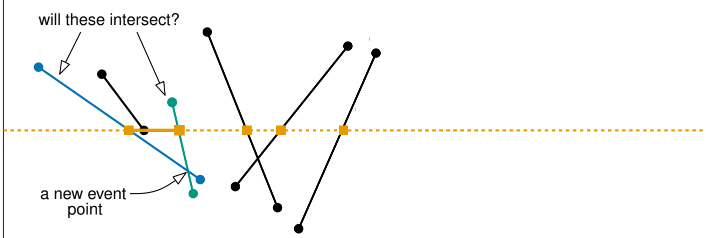

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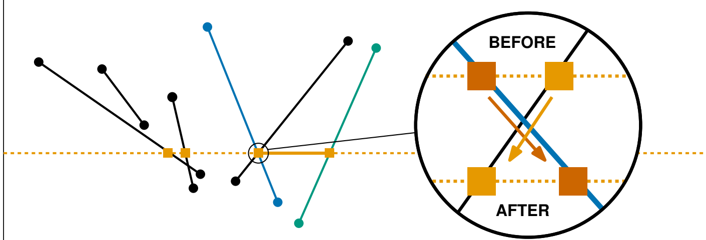

If the event point is
an intersection point, we **swap** the two line segments in the status data structure

This gives two new pairs of adjacent segments to check

If either pair will intersect, we've found a new event point

Updating the sweep line

Every time the sweep line moves to the next event point, we update
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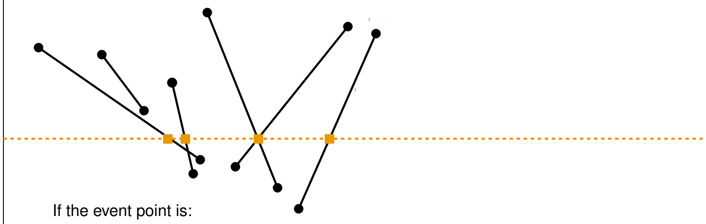


If the event point is:

the top of a line segment, we **insert** it into the status data structure

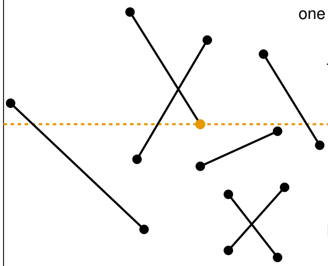
the bottom of a line segment, we **delete** it from the status data structure

an intersection point, we **swap** the two line segments in the status data structure

and we always check whether we have discovered any new event points
(specifically intersections)

How do we keep track of the event points?

At the start of the algorithm,
we are aware of $2n$ event points,
one for each end of each line segment



The sweep line starts at the top event point
(the one with the highest y value)

After processing an event point,
the sweep line moves down
to the next event point

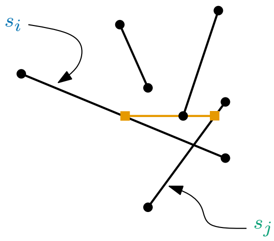
However, in processing an event point,
the algorithm may discover new event points
(specifically intersections)

We keep track of the event points using a Priority Queue

Every event point is INSERTED as it is discovered (with its y value as the key)

We can then use DELETEMIN to recover the next event point

Can we miss out on an intersection?



If s_i and s_j **intersect**
they must become adjacent at some y-coordinate
(before they intersect)

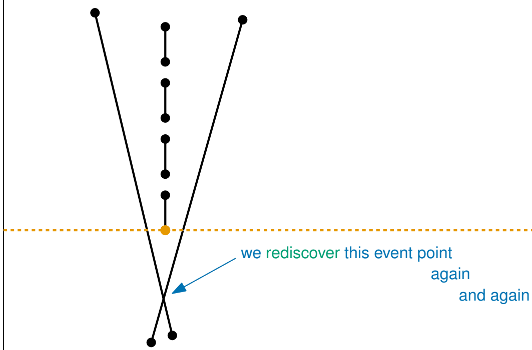
In particular, they must become adjacent at some *event point*
with a higher y-coordinate

This is because the status of the sweep line doesn't change between event points

The formal proof then follows by induction

Can we find the same event point twice?

Consider the line segments shown...



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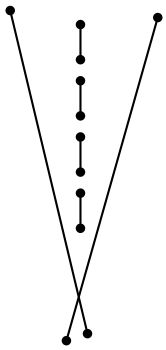


That is, we *can* discover
the same event point more than once

There are (at least) two ways to deal with this:

1. Check whether we already found the new point
   - by looking it up in the priority queue

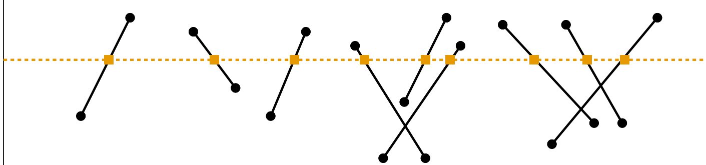
2. Don't worry about it (INSERT it anyway)
   but make sure that when we process
   an event point, we only process it once
   - by checking that the priority queue didn't
   return the same event point as last time

either approach gives the same time complexity

How do we implement the status data structure?

We need a data structure to store the *status* of the sweep line

i.e. the set of line segments which currently intersect the sweep line
ordered from left to right by where they intersect
(i.e. in the order given by the ■)



We insert each line segment s_i into the self-balancing search tree using
the description of s_i (i.e. its end points) as the key

This may seem odd as we normally think of a key as being an integer

Actually, all we require is that the keys have an order
and we can compare two keys in $O(1)$ time

Time Complexity (sketch)

The algorithm moves the sweep line $O(n+k)$ times,
once for each event point

If the status data structure and priority queue structures
are implemented so that their operations take $O(\log n)$ time

(e.g. with a self-balancing tree and a binary heap, respectively)

The overall complexity then becomes $O(n \log n + k \log n)$
as claimed

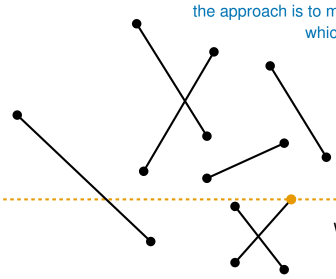
This is because we do a $O(n+k)$ operations on each data structure
while moving the sweep line

Summary

We have seen an algorithm for line segment intersection
which runs in $O(n \log n + k \log n)$ time

where n is the number of line segments and k is the number of intersections

the approach is to move a horizontal line through the plane
which jumps between all the *interesting positions*



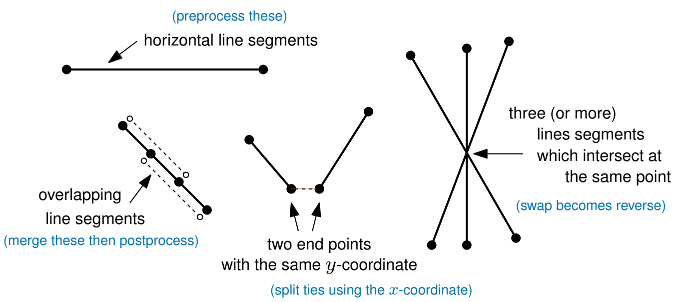
The efficiency relies on using
a Self-balancing search tree
and
a Binary Heap

We put quite a few restrictions on the input,
fixing these is fiddly but not difficult

In the original paper, they suggest adding random noise to the points
to avoid the restrictions

Dealing with the restrictions

In the interest of simplicity, we didn't allow the input to contain any of the following:



All of these restrictions can be removed making the algorithm slightly more involved
(hints are given for the interested)