Transformations

Remaining lectures:

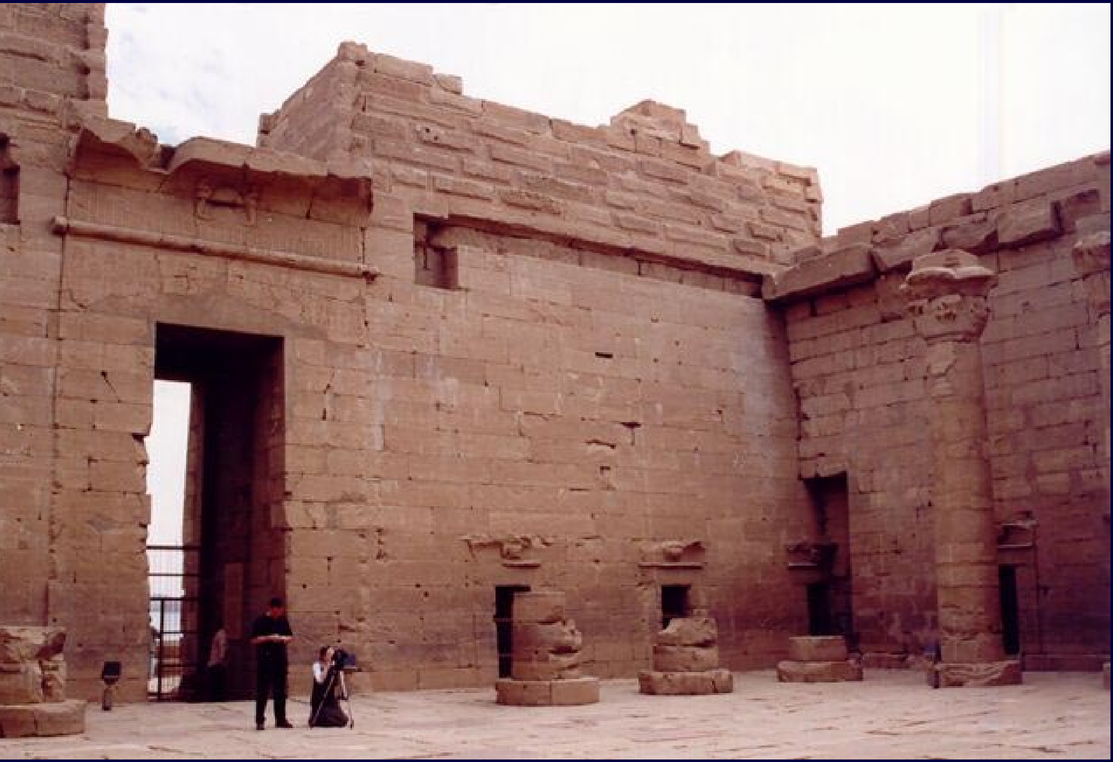
- Transformations - 2D and 3D
- 3D Projections
  - Orthographic
  - Perspective

Approximating Triangles Example

Shrek Example

From Nvidia

Photorealism



Translation

A translation is applied to an object by repositioning it along a straight path from one coordinate location to another.

$$x' = x + t_x$$
$$y' = y + t_y$$

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$$y' = y + t_y$$

In matrix form:

$P' = P + T$ where $P' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ $P = \begin{pmatrix} x \\ y \end{pmatrix}$, $and\ T = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$

Example: Triangle at $(20,0),(60,0),(40,100)$ is translated 100 units to the right and 10 units up ($t_x = 100, t_y = 10$). The new vertices are at $(120,10),(160,10),(140,110)$.

Rotation

A 2D Rotation is applied to an object by repositioning it along a circular path in the XY plane.

Rotation for the counter clockwise direction about the origin in the XY plane:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

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$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

In matrix form:

$P' = R\ .\ P$ where $P' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ $P = \begin{pmatrix} x \\ y \end{pmatrix}$, $and\ R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Example: Triangle at $(20,0),(60,0),(40,100)$ rotated 45^o clockwise about the origin is $(14.14,-14.14),(42.43,-42.43),(98.99,42.43)$.

Rotation

A 2D Rotation is applied to an object by repositioning it along a circular path in the XY plane.

Rotation for the counter clockwise direction about an arbitrary pivot point in the XY plane:

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$

Rotation

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Rotation for the counter clockwise direction about an arbitrary pivot point in the XY plane:

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$

In matrix form: $P' = R \,.\, P + T$

where $P = \begin{pmatrix} x - x_r \\ y - y_r \end{pmatrix}, T = \begin{pmatrix} x_r \\ y_r \end{pmatrix}$. P' and R are as before.

Scaling

A Scaling transformation alters the size of the object.

$$x' = x \ . \ s_x$$
$$y' = y \ . \ s_y$$

Scaling

A Scaling transformation alters the size of the object.

$$x' = x \ . \ s_x$$
$$y' = y \ . \ s_y$$

In matrix form:

$$P' = S \ . \ P \quad \text{where} \quad P' = \begin{pmatrix} x' \\ y' \end{pmatrix} \ P = \begin{pmatrix} x \\ y \end{pmatrix}, \ and \ S = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

Example: Triangle at $(20,0),(60,0),(40,100)$ with $s_x = s_y = 2$ becomes $(40,0),(120,0),(80,200)$.

Unequal s_x and s_y cause distortion by elongating or shrinking.

Scaling

A Scaling transformation alters the size of the object. After scaling, an object is repositioned. We can choose a position, called the *fi ed point*, that will remain unchanged after the transformation, e.g. a vertex of the object or the object centroid.

$$x' = x_f + (x - x_f)s_x$$

$$y' = y_f + (y - y_f)s_y$$

Rewrite to get a constant additive term for all object points:

Homogeneous Transformations

Basic transformations have these matrix representations:

$$P' = T + P \qquad P' = R \,.\, P \qquad P' = S \,.\, P$$

Homogeneous Transformations

Basic transformations have these matrix representations:

$$P' = T + P \qquad P' = R \,.\, P \qquad P' = S \,.\, P$$

But this does not allow us to treat transformations in an efficient manner.

We need *Homogeneous Coordinates:*

- These allow all types of transformations to be treated the same way as matrix multiplications
- The (x, y) coordinate now becomes (x_h, y_h, h)
- Then $x_h = xh$ and $y_h = yh$

2D Homogeneous Transformations

Translation

$$P' = T \;.\; P \qquad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotation

$$P' = R \;.\; P \qquad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Scaling

$$P' = S \;.\; P \qquad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

3D Translation

A translation is applied to an object by repositioning it along a straight path from one coordinate location to another.

$$
\begin{aligned}
x' &= x + t_x \\
y' &= y + t_y \\
z' &= z + t_z
\end{aligned}
$$

In the homogeneous coordinate representation:

$$
P' = T \;.\; P \qquad \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
$$

Example of 3D Translation

A triangular object define by the points:

P1: (4, 2, 3)
P2: (5, 2, 6)
P3: (1, 3, 5)

is translated with $t_x = 2$, $t_y = 4$, and $t_z = 1$. For example:

$$\begin{pmatrix} x' & y' & z' & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 4 & 1 & 1 \end{pmatrix}$$

Final positions:

P1: (6, 6, 4)
P2: ?
P3: ?

3D Scaling

A Scaling transformation alters the size of the object.

$$x' = x \, . \, s_x$$
$$y' = y \, . \, s_y$$
$$z' = z \, . \, s_z$$

In the homogeneous coordinate representation:

$$P' = S \, . \, P \qquad \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Unequal s_x, s_y, and s_z cause distortion by elongating or shrinking.

Example of 3D Scaling

A triangular object define by the points:

P1: (4, 2, 3)
P2: (5, 2, 6)
P3: (1, 3, 5)

is scaled with $s_x = 2$, $s_y = 4$, and $s_z = 1$. For example:

$$\begin{pmatrix} x' & y' & z' & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Final positions:

P1: (8, 8, 3)
P2: ?
P3: ?

3D Rotation

Important points:

- Must specify an "axis of rotation"
- Rotation can be specifie   around any line in space
- Easiest rotations axes are those parallel to the coordinate axes
- Positive rotation angles produce counter clockwise rotations

3D Rotation

Z-axis rotation in the *counter clockwise* direction:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

In homogeneous matrix form:

$$P' = R_z(\theta)\ .\ P$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D Rotation

X-axis rotation in the *counter clockwise* direction:

$$x' = x$$
$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$

In homogeneous matrix form:

$$P' = R_x(\theta)\ .\ P$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

3D Rotation

Y-axis rotation in the *counter clockwise* direction:

$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$
$$z' = z\cos\theta - x\sin\theta$$

In homogeneous matrix form:

$$P' = R_y(\theta)\ .\ P$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Example of 3D Rotation

A triangular object define   by the points:

P1: (0, 0, 0)
P2: (5, 2, 6)
P3: (1, 3, 5)

is rotated anticlockwise about the X-axis by 90°, e.g.

$$\begin{pmatrix} x' & y' & z' & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Final positions:

P1: (0, 0, 0)
P2: (5, -6, 2)
P3: ?

Combining Transformations

Translations, Scalings and Rotations etc. can be combined in any fashion to produce a **composite transformation matrix**.

$$\begin{pmatrix} x' & y' & z' & 1 \end{pmatrix} = \begin{pmatrix} x & y & z & 1 \end{pmatrix} \mathbf{M}_1$$

and

$$\begin{pmatrix} x'' & y'' & z'' & 1 \end{pmatrix} = \begin{pmatrix} x' & y' & z' & 1 \end{pmatrix} \mathbf{M}_2$$

then:

$$\begin{pmatrix} x'' & y'' & z'' & 1 \end{pmatrix} = \begin{pmatrix} x & y & z & 1 \end{pmatrix} \mathbf{M}_3$$

where:

$$M_3 = M_1 M_2$$

BEWARE

Translations and Scalings are commutative

$$T_2 T_1 = T_1 T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_{1x} + T_{2x} & T_{1y} + T_{2y} & T_{1z} + T_{2z} & 1 \end{pmatrix}$$

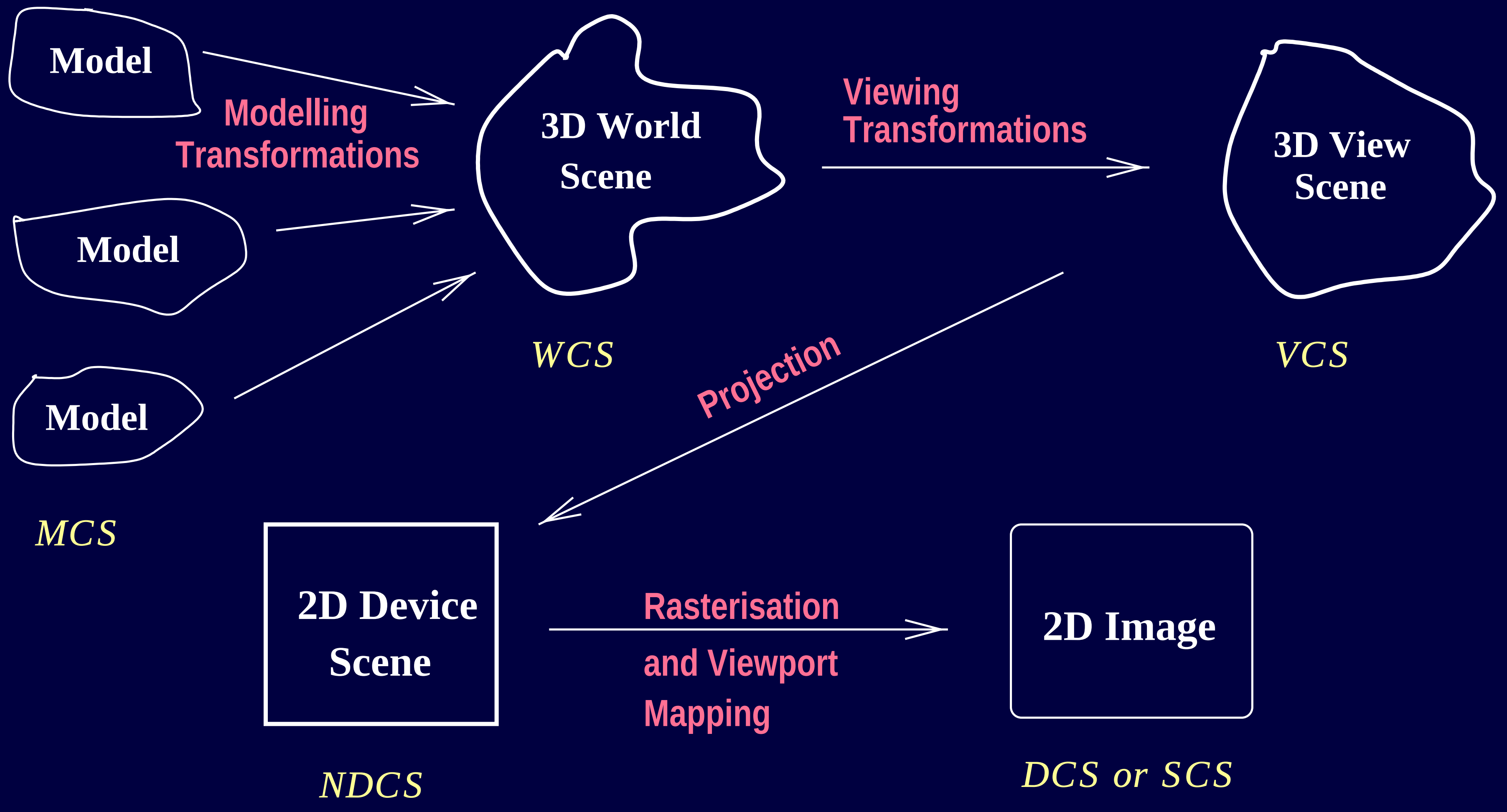
$$S_2 S_1 = S_1 S_2 = \begin{pmatrix} S_{1x} S_{2x} & 0 & 0 & 0 \\ 0 & S_{1y} S_{2y} & 0 & 0 \\ 0 & 0 & S_{1z} S_{2z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotations are NOT commutative

$$R_2 R_1 \neq R_1 R_2$$

3D Graphics

3D Graphics



Model
Model
Model
Modelling
Transformations
MCS
3D World
Scene
WCS
Viewing
Transformations
3D View
Scene
VCS
Projection
2D Device
Scene
NDCS
Rasterisation
and Viewport
Mapping
2D Image
DCS or SCS

3D Graphics: Viewing Coordinates

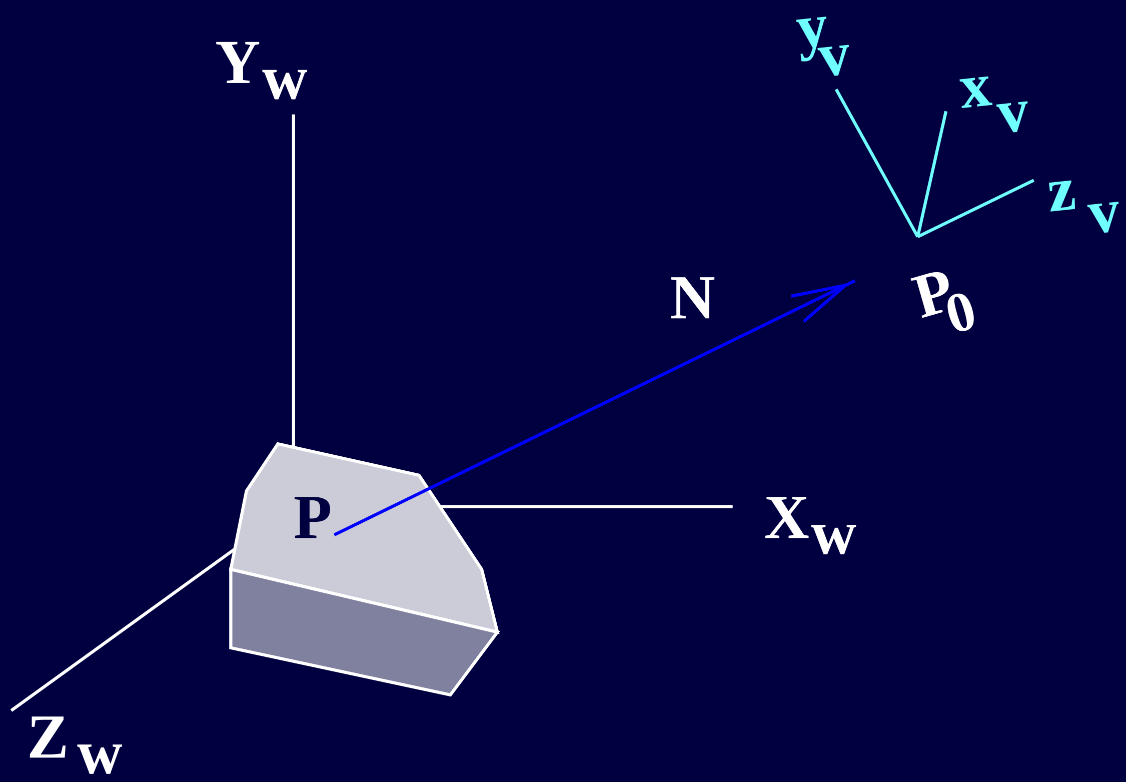
When photographing a scene, the projection onto the fla f lm surface depends on these camera parameters:

- the spatial position
- orientation
- aperture size

These ideas are incorporated into 3D graphics packages to generate views of a scene.

The Viewing Coordinate System

Orientation of the view plane for a specifie look relative to a right-handed viewing-coordinate system at P



Transformation from World to Viewing Coordinates

3D Graphics: Viewing Coordinates

Common practice for viewing coordinates:

- Right handed viewing system (x, y, z axes)
- View plane can be positioned anywhere along the z-axis.
- View plane is parallel to xy plane.
- z is called the viewing direction.
- N is called the view-plane normal vector.
- V (in the direction of y) is called the the view-up vector.
- U (in the direction of x) is easily obtained from y and z ($\perp$ to both)

Transformation from World to Viewing Coordinates

Conversion of object descriptions from world to viewing coordinates is equivalent to a transformation that superimposes the viewing reference frame onto the world frame.

- Translate view reference point to world origin
- apply rotations to align viewing and world axes

(Object descriptions can be projected to the view plane after they are transformed to viewing coordinates.)

Transformation from World to Viewing Coordinates

Depending on the direction of the normal N to the view plane up to three coordinate axes rotations will be required to make the superimposition.

For example if N is not aligned with any of the world axes, then the transformation sequence will involve R_z, R_y, R_x (as well as the initial translations T_z, T_y, T_x to reposition at origin).

How do we compute these rotations?

Transformation from World to Viewing Coordinates

A simple approach would be through the *uvn* unit vectors (mapped onto N and the view-up vectors)

$$R = \begin{pmatrix} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The complete world to viewing transformation is then:

$$M_{wc,vc} = R.T$$

Drawing as Projection

A painting based on a mythical tale as told by Pliny the Elder: a Corinthian traces the shadow of his departing lover.

Detail from The Invention of Drawing, 1830: Karl Friedrich Schinkel

Projection

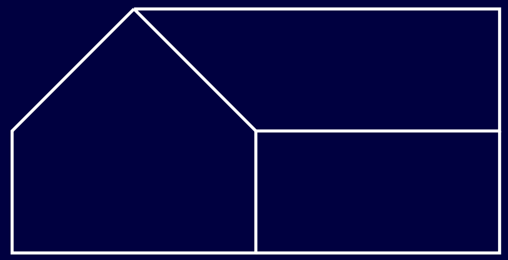
Fundamental to the production of a 2D display of a 3D scene is the notion of *projection*.

After converting world coordinate descriptions into viewing coordinates, the 3D objects can be *projected* onto the 2D view plane.

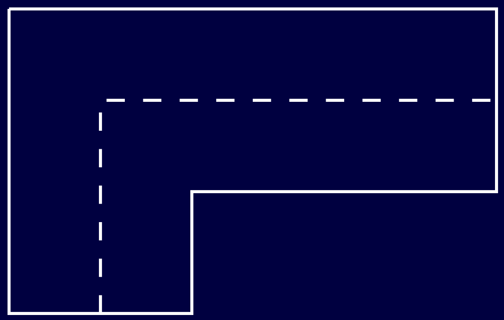
- Parallel Projection
  - Orthographic
  - Oblique
- Perspective Projection

Parallel Projection

A point on the screen is identifie   with a point in the 3D scene by a line $\perp$ to the screen.



Elevation view

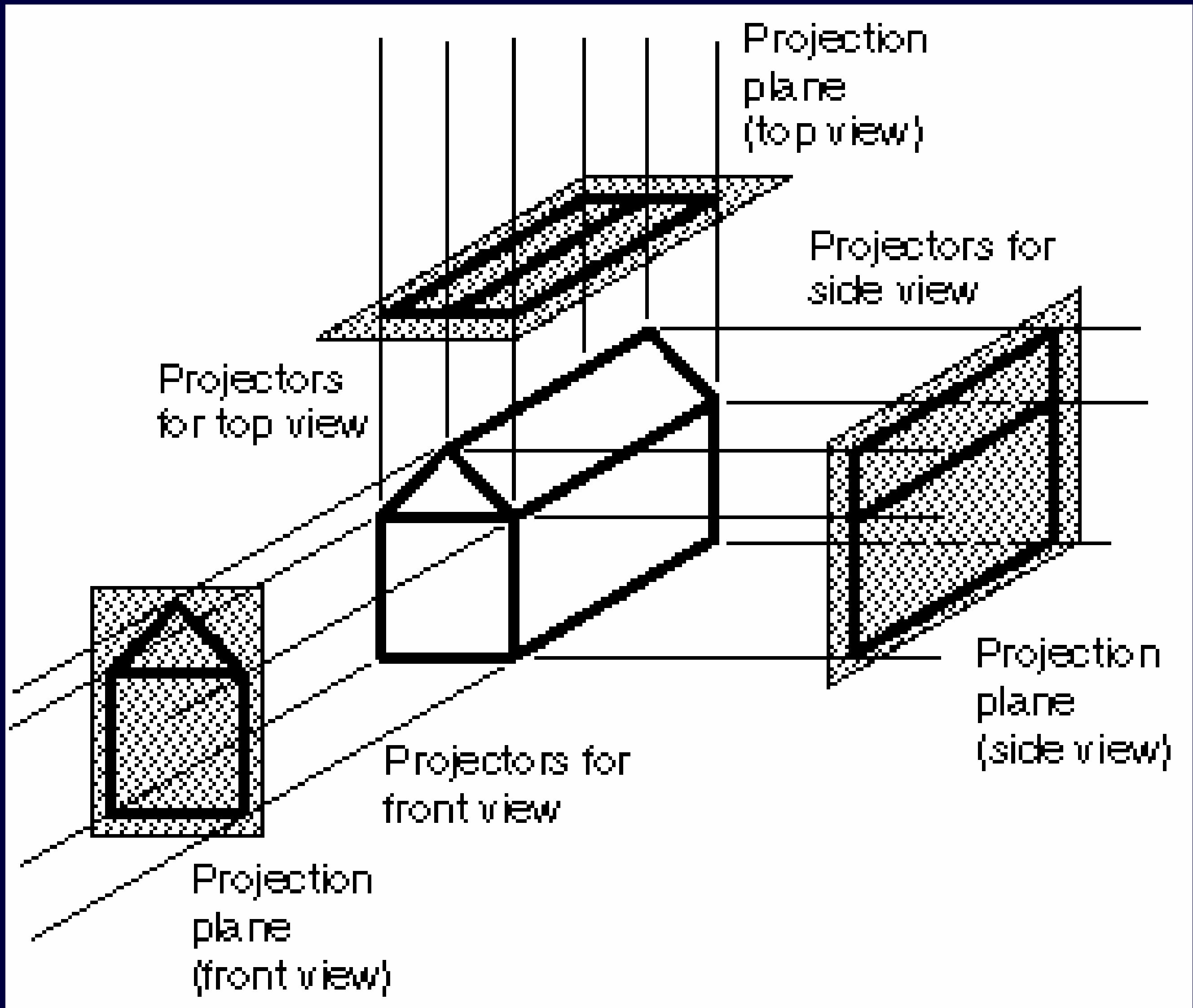


Plan view

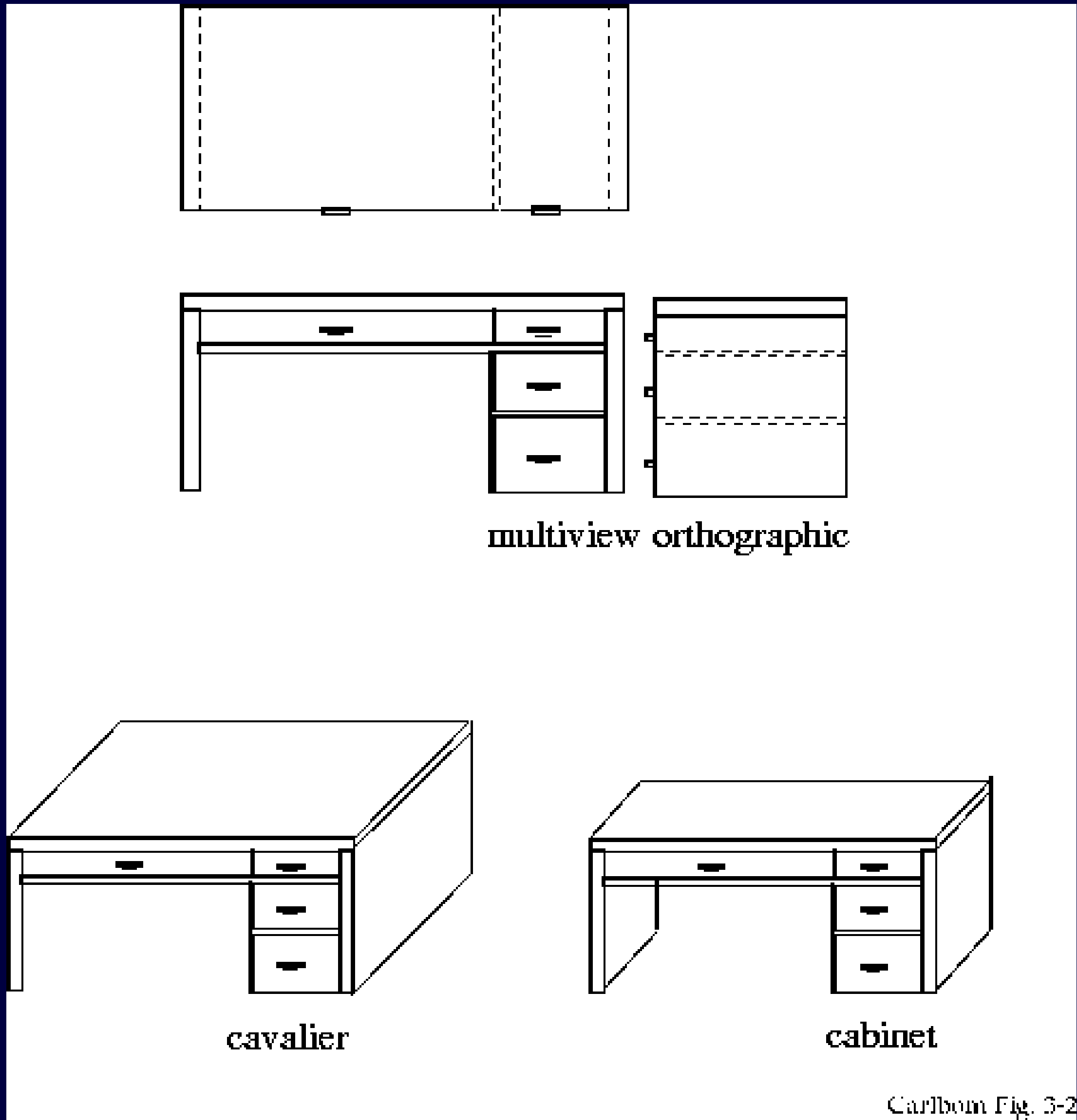
Viewer must try to *reconstruct* the scene.

The concept of depth does not (quite) exist.

Parallel Projections (orthographic)



Parallel Projections



Parallel Projection

Any parallel projection onto the XY viewing plane can be achieved with:

$$\begin{pmatrix} x_p \\ y_p \\ z_p \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & K\cos\theta & 0 \\ 0 & 1 & K\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

A Parallel Projection where the projection is $\perp$ to the viewing plane is called an Orthographic Projection. $(K = 0)$

A Parallel Projection where the projection is NOT $\perp$ to the viewing plane is called an Oblique Projection. $(K \neq 0)$

Parallel Oblique Projection

Let point (x, y, z) be projected to (x_p, y_p) on the view plane. The orthographic projection coordinates are (x, y). The oblique projection line makes angle α with the view plane or line L from (x_p, y_p) to (x, y). Line L is at an angle θ with the horizontal. The projection coordinates are then:

$$x_p = x + L\cos\theta \qquad y_p = y + L\sin\theta$$

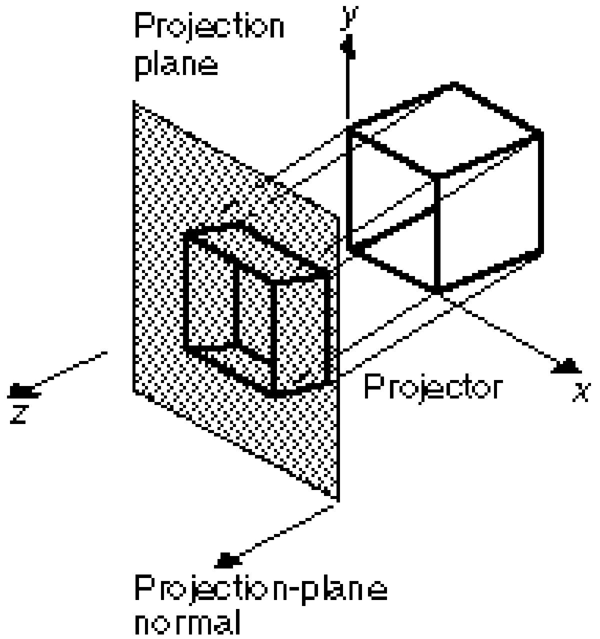
but $\tan\alpha = z/L$ or $L = zK$ where K is the inverse of $\tan\alpha$.

Thus, the oblique projection equations are:

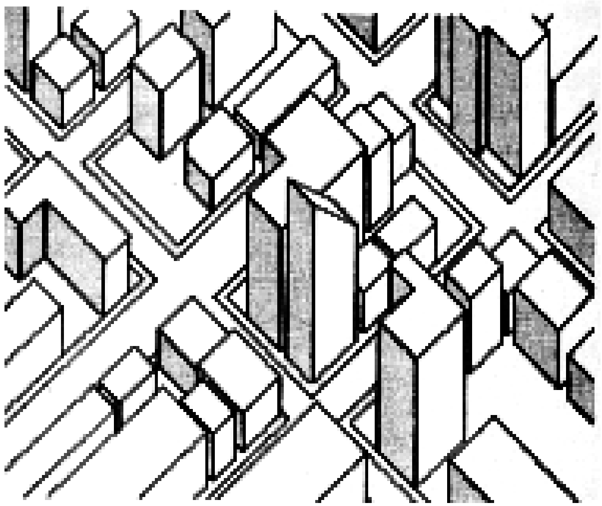
$$x_p = x + z(K\cos\theta) \qquad y_p = y + z(K\sin\theta)$$

When $K = 0$ (i.e. $\alpha = 90^\circ$) the projection becomes orthographic.

Parallel Projections (oblique)



Construction of an oblique parallel projection



(Carlbom Fig. 2-6)

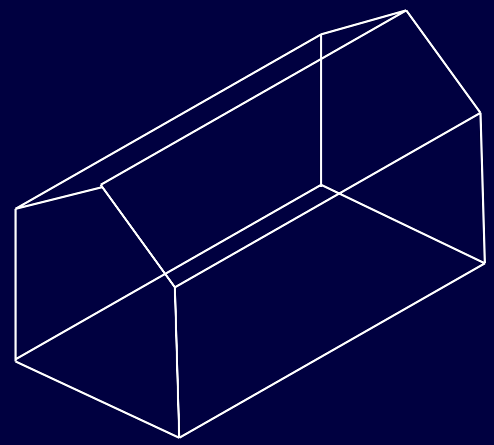
Example: plan oblique projection of a city

Perspective Projection

Most common technique: as in the eye or on film

Enables viewer to “perceive depth” in the two dimensional image.

Relative proportions are NOT preserved $\rightarrow$ conveys depth by making distant objects smaller than near ones.



Perspective view

Points are projected on the display along converging paths which meet at a point called the projection reference point or the centre of projection (COP).

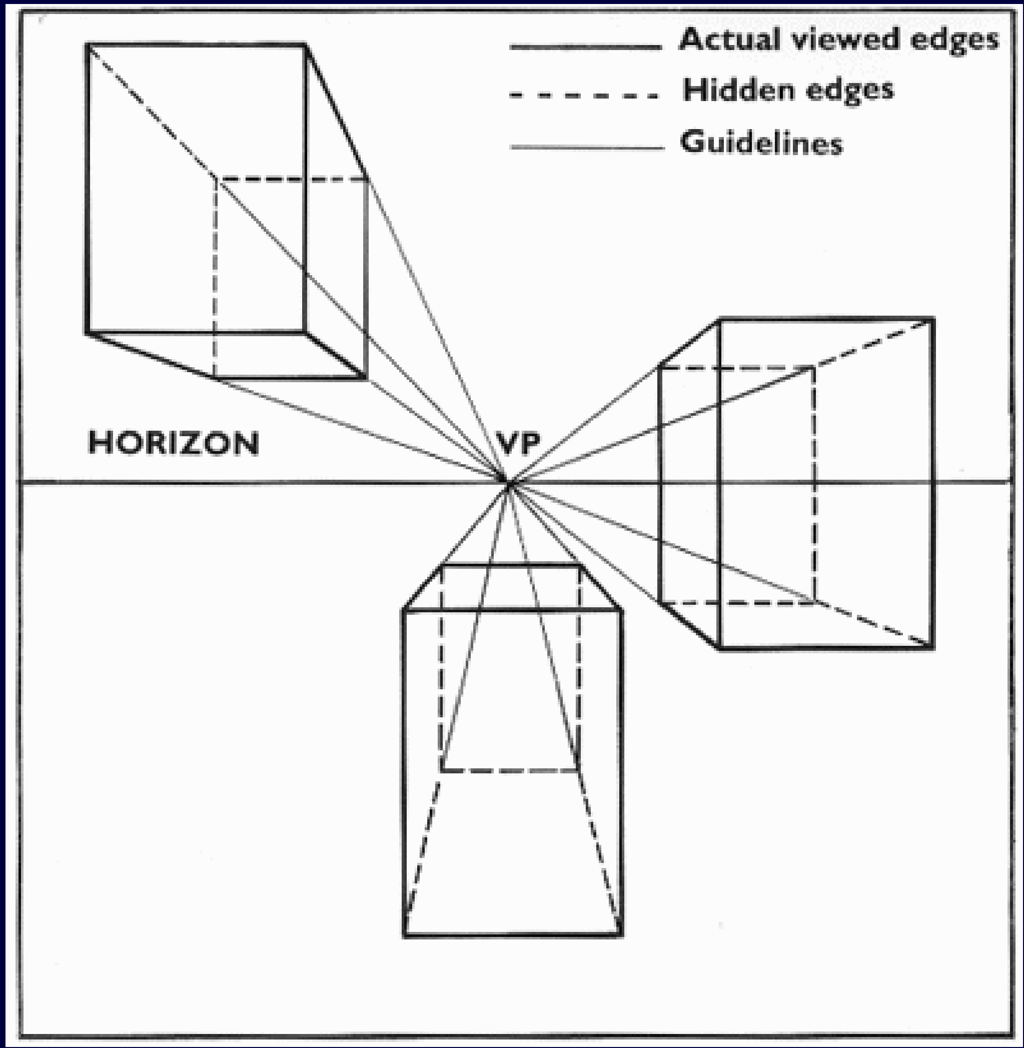
Vanishing Points

Parallel lines in a 3D object projected onto a view plane which are not parallel to the view plane are projected into converging lines. The point at which these parallel lines converge is called the vanishing point. A scene may have a number of vanishing points.

The principal vanishing point is the point at which lines parallel to one of the principal axes of an object meet.

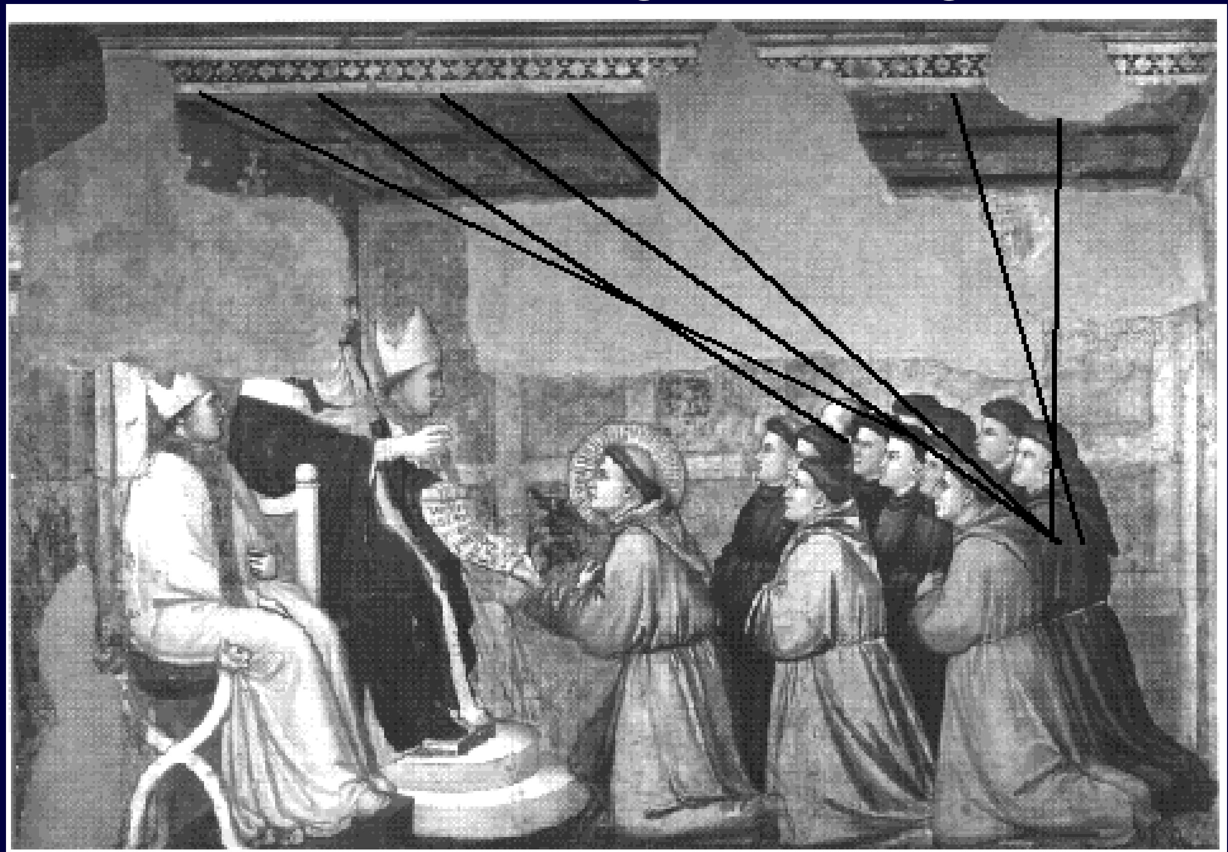
Lines parallel to the view plane will be projected as parallel lines.

Vanishing Points



Early Perspective

Invoking 3D space: rounded, volumetric forms suggested by shading, spatial depth of room suggested by converging lines.
Not systematic: lines do not converge to a single vanishing point.



Giotto, Confi mation of the Rule of Saint Francis, c.1325

Early Perspective

Invoking 3D space: rounded, volumetric forms suggested by shading, spatial depth of room suggested by converging lines.

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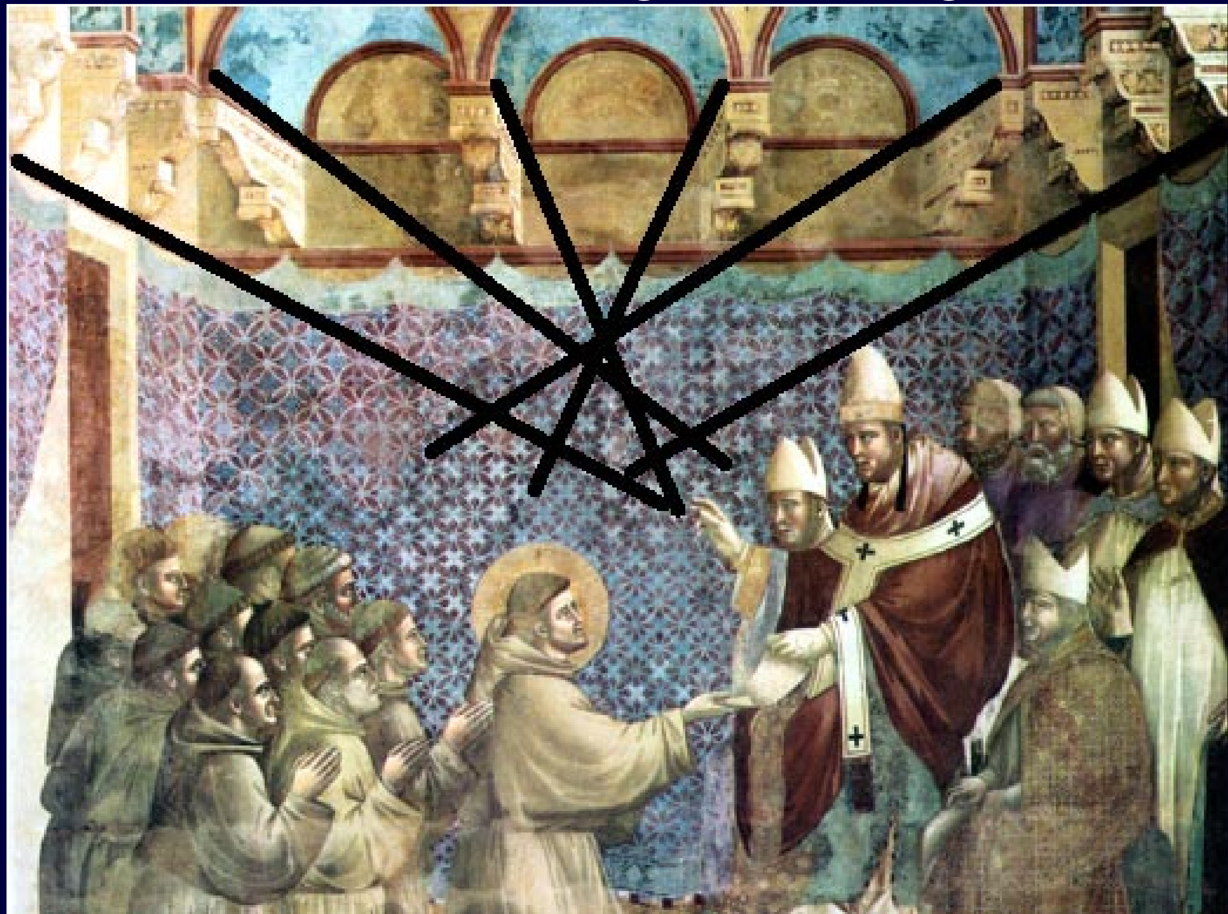
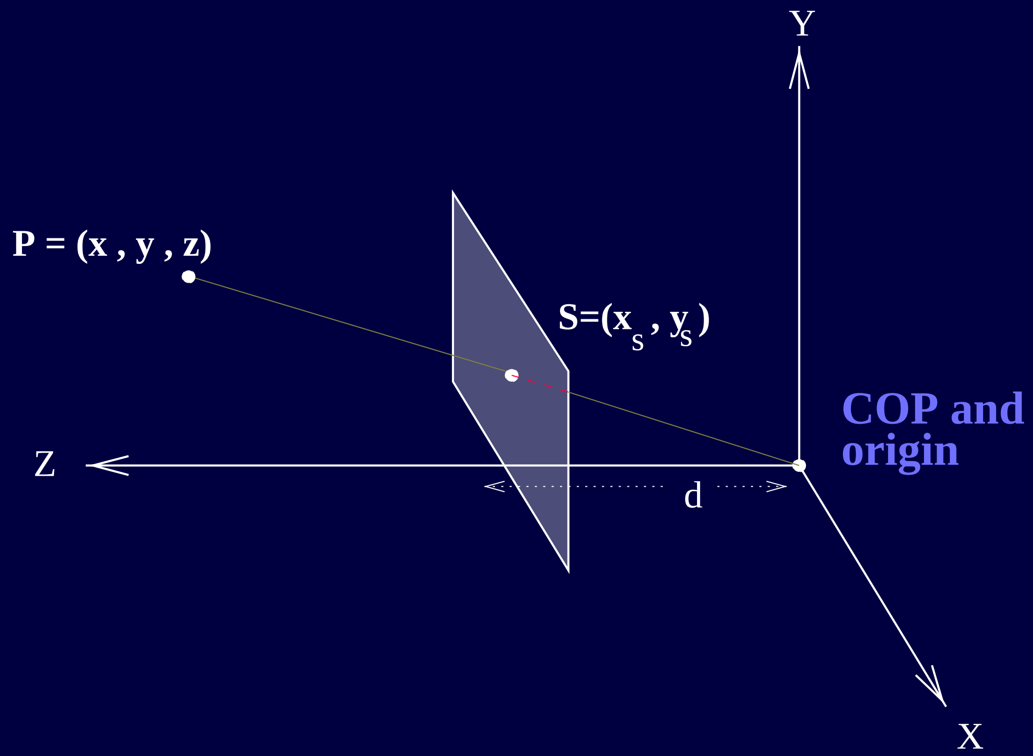


Giotto, Franciscan Rule approved, c.1300

Perspective Projection

Vertex point $P = (x, y, z)$ will be projected onto the viewing plane which (in this case) is placed *normal* to the z-axis with the origin at the point of intersection $\rightarrow$ this will produce vertex point $S = (x_s, y_s)$ on the display.



Perspective Projection

Using similar triangles,

$$\frac{x_s}{x} = \frac{d}{z}$$

$$\frac{y_s}{y} = \frac{d}{z}$$

Hence:

For the x-coordinate,

$$x_s = \frac{xd}{z}$$

For the y-coordinate,

$$y_s = \frac{yd}{z}$$

Example

Consider the point at $P=(33,52,180)$ in the world coordinate system. The view plane is parallel to the XY plane and positioned along the Z-axis at 70 units. The centre of projection is at $C=(0,0,0)$. Find the coordinates of $S=(x_s,y_s)$ as the perspective projection of point P on the view plane.

Using similar triangles:

For the x-coordinate,

$$x_s = \frac{xd}{z} = \frac{33*70}{180} = 12.83 \longrightarrow 13$$

For the y-coordinate,

$$y_s = \frac{yd}{z} = \frac{52*70}{180} = 20.22 \longrightarrow 20$$

Similar triangles

Concept of similar triangles described both geometrically and mechanically in treatise by Albrecht Durer (1471-1528)

Albrecht Durer, Artist Drawing a Lute, 1525

Perspective Projection

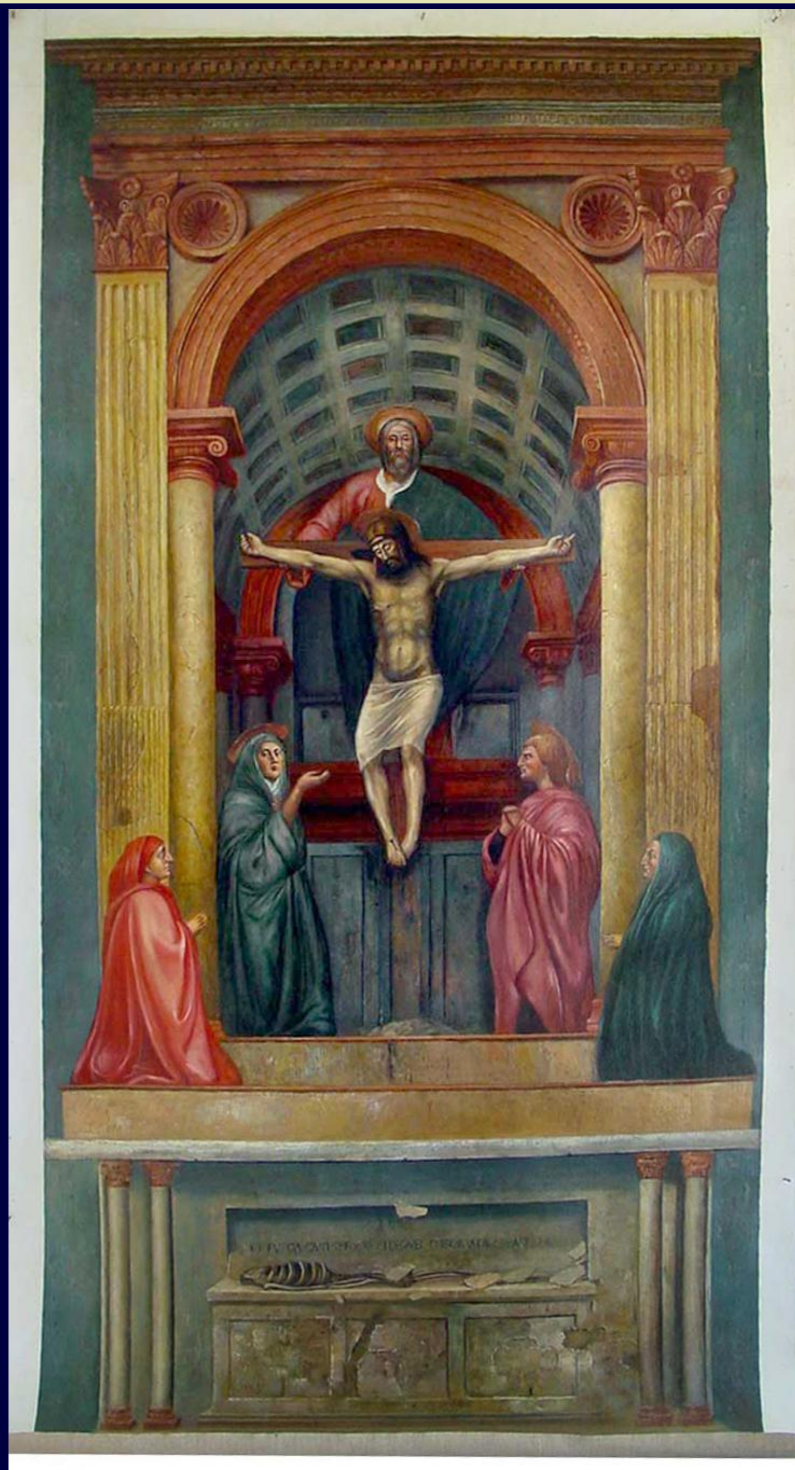
Consider a point $P = (x, y, z)$ is to be projected onto a view plane at z_p, at position $S = (x_s, y_s, z_p)$ given the projection reference point (COP) at position $(0, 0, z_r)$. Let $d = z_p - z_r$ be the distance of the view plane from the COP and let $h = z - z_r$ be the distance of the point from the COP. The perspective projection transformation matrix in the homogeneous coordinate representation is:

$$\begin{pmatrix} x_s \\ y_s \\ z_s \\ 1 \end{pmatrix} = \begin{pmatrix} d/h & 0 & 0 & 0 \\ 0 & d/h & 0 & 0 \\ 0 & 0 & 0 & z_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Perspective

First ever painting done in correct perspective.

Trinity with the Virgin, St John and Donors, c.1427, Masaccio



Projection Example

Orthographic

Perspective

In-Class Example

Consider the point at $P = (30, 50, 88)$ in the world coordinate system. The view plane is parallel to the XY plane and positioned along the Z-axis at -20 units. The centre of projection is at $C = (0, 0, -38)$. Find the coordinates of $S = (x_s, y_s)$ as the perspective projection of point P on the view plane.

Using similar triangles:

For the x-coordinate,

$$x_s =$$

For the y-coordinate,

$$y_s =$$

In-Class Example

Consider the point at $P = (30, 50, 88)$ in the world coordinate system. The view plane is parallel to the XY plane and positioned along the Z-axis at -20 units. The centre of projection is at $C = (0, 0, -38)$. Find the coordinates of $S = (x_s, y_s)$ as the perspective projection of point P on the view plane.

Using similar triangles:

For the x-coordinate,

$$x_s = 4.28 \longrightarrow 4$$

For the y-coordinate,

$$y_s = 7.14 \longrightarrow 7$$