# COMS12200 hand-out: Boolean algebra quick reference

## Axioms

### Axioms in AND-form

$$
\begin{array}{llcl}
\text{commutativity} & x \wedge y & \equiv & y \wedge x \\
\text{association} & (x \wedge y) \wedge z & \equiv & x \wedge (y \wedge z) \\
\text{distribution} & x \wedge (y \vee z) & \equiv & (x \wedge y) \vee (x \wedge z) \\
\text{identity} & x \wedge 1 & \equiv & x \\
\text{null} & x \wedge 0 & \equiv & 0 \\
\text{idempotency} & x \wedge x & \equiv & x \\
\text{inverse} & x \wedge \neg x & \equiv & 0 \\
\text{absorption} & x \wedge (x \vee y) & \equiv & x \\
\text{de Morgan} & \neg(x \wedge y) & \equiv & \neg x \vee \neg y
\end{array}
$$

### Axioms in OR-form

$$
\begin{array}{llcl}
\text{commutativity} & x \vee y & \equiv & y \vee x \\
\text{association} & (x \vee y) \vee z & \equiv & x \vee (y \vee z) \\
\text{distribution} & x \vee (y \wedge z) & \equiv & (x \vee y) \wedge (x \vee z) \\
\text{identity} & x \vee 0 & \equiv & x \\
\text{null} & x \vee 1 & \equiv & 1 \\
\text{idempotency} & x \vee x & \equiv & x \\
\text{inverse} & x \vee \neg x & \equiv & 1 \\
\text{absorption} & x \vee (x \wedge y) & \equiv & x \\
\text{de Morgan} & \neg(x \vee y) & \equiv & \neg x \wedge \neg y
\end{array}
$$

### Misc

$$
\begin{array}{llcl}
\text{equivalence} & x \equiv y & \equiv & (x \Rightarrow y) \wedge (y \Rightarrow x) \\
\text{implication} & x \Rightarrow y & \equiv & \neg x \vee y \\
\text{involution} & \neg\neg x & \equiv & x
\end{array}
$$

## Transformations and standard forms

**Definition 1** *The fact there are AND and OR forms of most axioms hints at a more general underlying* **principle of duality***. Consider a Boolean expression $e$: the* **dual expression** *$e^D$ is formed by*

1. *leaving each variable as is,*
2. *swapping each $\wedge$ with $\vee$ and vice versa, and*
3. *swapping each $0$ with $1$ and vice versa.*

**Definition 2** *The de Morgan axiom can be generalised into a* **principle of complements***. Consider a Boolean expression $e$: the* **complement expression** *$\neg e$ is formed by*

1. *swapping each variable $x$ with the complement $\neg x$,*
2. *swapping each $\wedge$ with $\vee$ and vice versa, and*
3. *swapping each $0$ with $1$ and vice versa.*

**Definition 3** *Consider a Boolean function $f$ with $n$ inputs. When the expression for $f$ is written as a sum (i.e., OR) of terms which each comprise the product (i.e., AND) of a number of inputs, it is said to be in* **disjunctive normal form** *or* **Sum of Products (SoP)** *form; the terms in this expression are called the* **minterms***. For example,*

$$\underbrace{(a \wedge b \wedge c)}_{\textit{minterm}} \vee (d \wedge e \wedge f),$$

*is in SoP form. Note that each variable can exist as-is* or *complemented using NOT, meaning*

$$\underbrace{(\neg a \wedge b \wedge c)}_{minterm} \vee (d \wedge \neg e \wedge f),$$

*is also a valid SoP expression.*

*Conversely, when the expression for $f$ is written as a product (i.e., AND) of terms which each comprise the sum (i.e., OR) of a number of inputs, it is said to be in* **conjunctive normal form** *or* **Product of Sums (PoS)** *form; the terms in this expression are called the* **maxterms***. For example,*

$$\underbrace{(a \vee b \vee c)}_{maxterm} \wedge (d \vee e \vee f),$$

*is in PoS form. As above, each variable can exist as-is* or *complemented using NOT.*