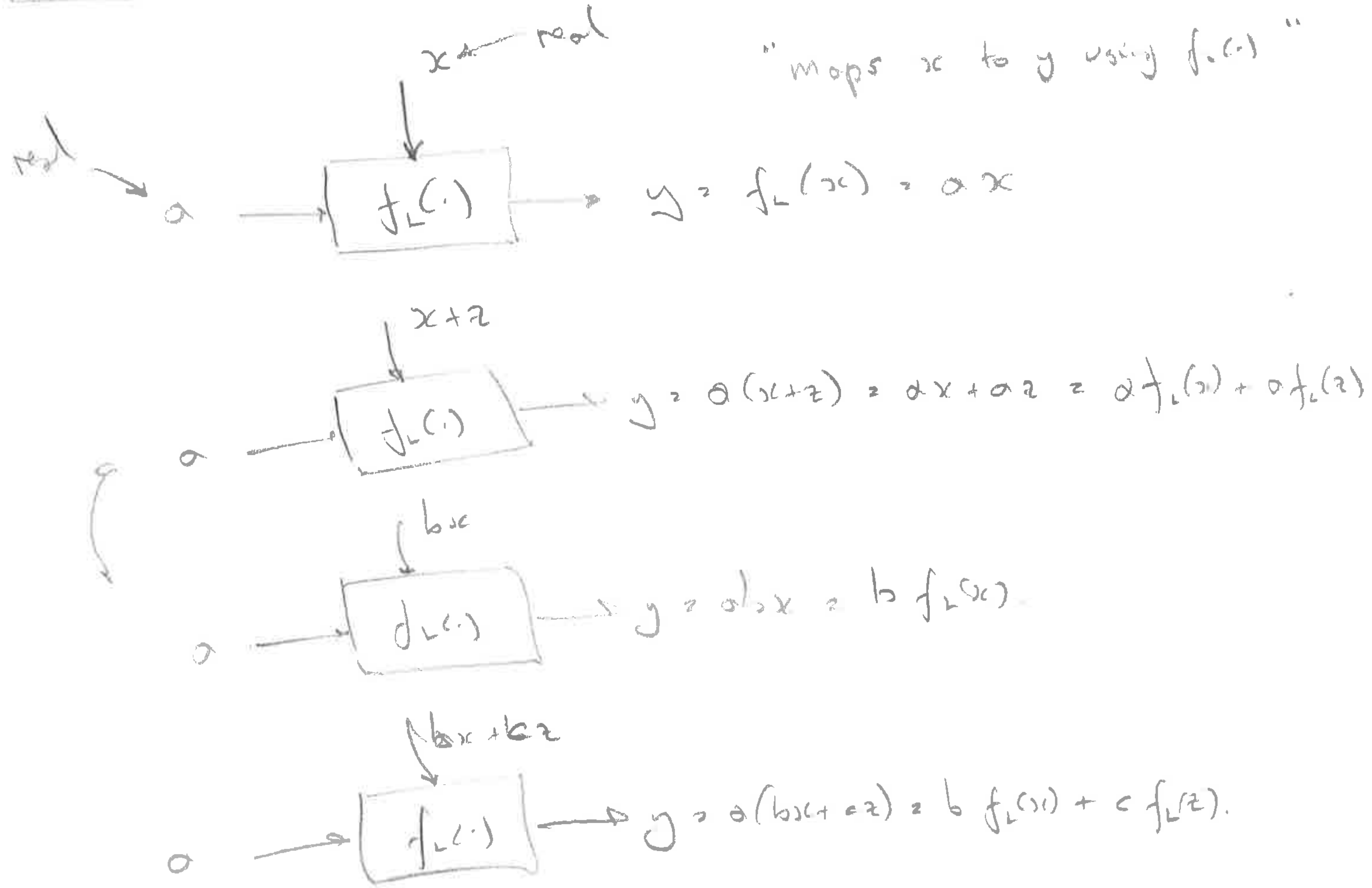
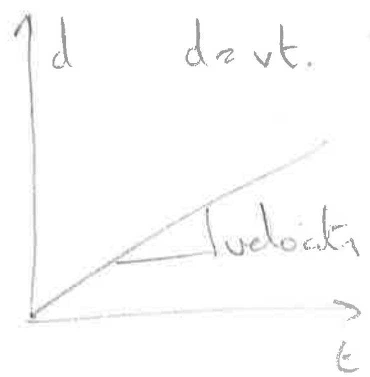
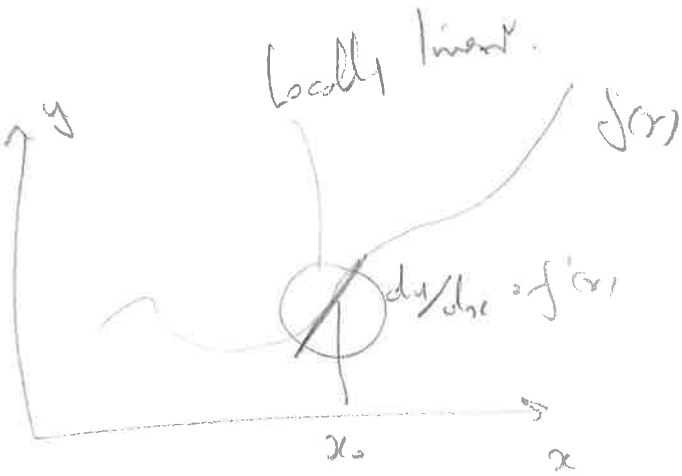
Linear Algebra - Introduction

Linear functions (or Linear Mappings)

"maps x to y using $f_L(\cdot)$"







$\therefore$ Linear models
$\longrightarrow$ simple and useful

$$y \simeq f(x_0) + f'(x_0)(x - x_0) + \ldots$$

Taylor Series

Vectors

(2)

$\vec{x}$

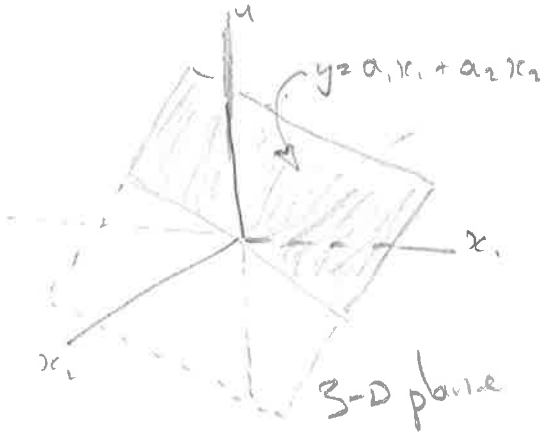
Vectors $\rightarrow (x_1, x_2 \ldots x_n)$ magnitude (length) + direction

$(a_1, a_2 \ldots a_n) \rightarrow \boxed{f_L(\cdot)} \rightarrow y = a_1 x_1 + a_2 x_2 + \ldots a_n x_n$

$\vec{a}$

dot (or scalar) product

fundamental operation in LA.



Shorthand: $y = \vec{a} \cdot \vec{x}$

Matrices and Vectors

(3)

$m \times n$ Matrix

$\vec{x} = (x_1, x_2 \ldots x_n)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \rightarrow \boxed{f_L(\cdot)} \rightarrow \begin{aligned} y_1 &= a_{11} x_1 + a_{12} x_2 \ldots a_{1n} x_n \\ &\vdots \\ y_m &= a_{m1} x_1 + a_{m2} x_2 \ldots a_{mn} x_n \end{aligned}$$

$\vec{y}$ $\vec{a}_2$

$\Downarrow$

$\vec{y} = \sum_i x_i \vec{a}_i$ $\vec{y} = A \vec{x}$

$\bar{y}$ is weighted sum of the columns of A

System of linear eqns

A transforms $\vec{x}$ to $\vec{y}$

Matrices and Matrices

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$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ \vdots & & & \\ x_{n1} & \cdots & & x_{nm} \end{pmatrix} \quad n \times m$$

$$\underset{p \times n}{A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{p1} & & \cdots & a_{pn} \end{pmatrix}} \xrightarrow{f_L(\cdot)} \underset{p \times m}{Y = \begin{pmatrix} (a_{11}x_{11} + a_{12}x_{21} \ldots) & \cdots & (a_{11}x_{1m} + a_{12}x_{2m} \cdots a_{1n}x_{nm}) \\ \vdots & & \\ (a_{p1}x_{11} + a_{p2}x_{21} \cdots a_{pn}x_{n1}) & \cdots & (a_{p1}x_{1m} + \cdots a_{pn}x_{nm}) \end{pmatrix}}$$

complicated!

easy!

$Y = AX$ matrix $\times$ matrix $\to$ matrix

Interpretation and Representation.

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$y = ax$

$y = \vec{a} . \vec{x}$

$\vec{y} = A\vec{x}$

$Y = AX$

$y = \Sigma a_i x_i$ linear function of x_i with coefficients a_i

$y \to$ dot product of $\vec{a}$ and $\vec{x}$

$\vec{y} \to$ linear combination of the columns of A with weights x_i

A transforms $\vec{x}$ to $\vec{y}$
$\to$ linear mapping.

$\vec{x}$ is the solution to a set of Linear Equations

Questions and Topics

$y = \bar{a} \cdot \bar{x}$

$\vec{y} = A\bar{x}$

$Y = AX$

what $\bar{x}$ are we dealing with all in n-D? a subset? If the latter what does it tell us about $\vec{y}$? → vector spaces and subspaces

If we can write $\bar{x} = c_1\bar{u}_1 + c_2\bar{u}_2$, say, for all vectors, then

$A\bar{x} = c_1A\bar{u}_1 + c_2A\bar{u}_2$

All transformations can be written as linear combinations of transformations of $\bar{u}_1$ and $\bar{u}_2$.

Where do $\bar{u}_1$ and $\bar{u}_2$ come from? properties? Best $\bar{u}_1 + \bar{u}_2$? → Span Basis

What properties of A can we reason about? what transformations do they represent? Do the underlying systems of eqns have solutions? Are the transformations from $\bar{x}$ to $\bar{y}$ invertible?

Matrices + Matrix Inverse

For a given A, do some $\bar{x}$ transform in a special way? Eg what is significance of vectors $\bar{x}$ st $A\bar{x} = \lambda\bar{x}$?

Eigenvalues & Eigenvectors

What does $\bar{x}$ represent?

Example

$$2x_1 + x_2 = 1 \qquad -3x_2 = -3 \qquad x_2 = 1$$
$$x_1 + 2x_2 = 2 \qquad\qquad\qquad\qquad x_1 = 0$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Transformation $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Eigenvalue $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} - \\ - \end{pmatrix} = 3\begin{pmatrix} - \\ - \end{pmatrix}$$

axes of transformation

Inverse

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \qquad \underbrace{\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{A^{-1}}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$