COMS10003 Workshop Sheet 17.

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Introduction

This worksheet is about multidimensional calculus and the gradient vector.

Useful facts

- Partial derivative

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \\ \frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} \end{aligned} \tag{1}$$

  In practise this means that to take the partial derivative with respect to x you treat y like a constant and visa versa.

- The gradient of $f(x,y)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \tag{2}$$

- Directional derivative

$$\nabla_{\mathbf{n}} f = \nabla f \cdot \mathbf{n} \tag{3}$$

- To *normalize* a vector is to rescale it so it have length one.

Work sheet

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ for
   (a) $f(x,y) = xy\sin xy$
   (b) $f(x,y) = e^{x^2+y^2}$
   (c) $f(x,y) = xe^{xy}$
   (d) $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$

2. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ for
   (a) $f(x,y,z) = xy\ln z$
   (b) $f(x,y,z) = x^2 + y^2 + z^2$
   (c) $f(x,y,z) = x\sin xyz$

3. For $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$ work out the directional derivative in the $(2,1)$ direction at $(1,0)$; don't forget to normalize the direction vector.
4. Find the gradient of $f(x,y) = 3x + y^2 + x^2$ and $f(x,y) = \frac{1}{x^2+y^2}$.
5. Going to three-dimensions in the obvious way, what is the gradient of
$$f(x,y,z) = \sin x + \cos y + \sin z \tag{4}$$
at $(\pi, 0, \pi)$.
6. The divergence is a differential operator acting on a vector field to give a scalar, that's the other way around to the gradient which acts on a scalar to give a vector field. It is defined by
$$\operatorname{div}\mathbf{v}(x,y) = \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{5}$$
What is the divergence of (x,y)? What about $(y,-x)$?
7. The Laplacian operator $\Box f = \operatorname{div}(\operatorname{grad} f)$. Write down the formula for $\Box f$ in terms of partial derivatives.
8. If a surface is given by $f(x,y,z) = c$ where c is a constant, then $\operatorname{grad} f$ is perpendicular to the surface. Examine the two-dimensional version by considering $x^2 + y^2 = 1$. What is the gradient? On $x^2 + y^2 = 1$ we can write $x = \cos\theta$ and $y = \sin\theta$ since these satisfy $x^2 + y^2 = 1$. What does the gradient look like on the surface? Can you find a vector perpendicular to it, and therefore parallel to the surface.

Exercise sheet

The difference between the work sheet and the exercise sheet is that the solutions to the exercise sheet won't be given and the problems are designed to be more suited to working on on your own.

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ for
   (a) $f(x,y) = xy\ln xy$
   (b) $f(x,y) = 12x^4y + y^2$
   (c) $f(x,y) = xe^{y^2}$
2. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ for
   (a) $f(x,y,z) = xyz$
   (b) $f(x,y,z) = (x-y)^2 + (y-z)^2 + (z-x)^2$
3. For $f(x,y) = xy$ work out the directional derivative in the $(1,3)$ direction at $(1,1)$; don't forget to normalize the direction vector.

4. The third differential operator is curl; it acts on vector fields to give another vector field. It is only defined in three dimensions and has quite a complicated form

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \tag{6}$$

Show $\operatorname{grad}(\operatorname{curl}\mathbf{v}) = 0$.