Data Structures and Algorithms – COMS21103

2015/2016

Dynamic Programming

Largest Empty Square and Weighted Interval Scheduling

Benjamin Sach

University of BRISTOL

The name

Dynamic Programming is an approach to algorithm design...

why does it sound like an alternative to Agile Software Development?

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Serious answer:

- Richard Bellman invented Dynamic programming around 1950

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Dynamic Programming is an approach to algorithm design. . .

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Real answer:

“The 1950s were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word, research... His face would suffuse, he would turn red, and he would get violent if people used the term, research, in his presence. You can imagine how he felt, then, about the term, mathematical... I thought dynamic programming was a good name. It was something not even a Congressman could object to.”

- Richard Bellman

What problems can Dynamic Programming solve?

- Longest Common Subsequence
  (used heavily in Bioinformatics for DNA similarity)
- Edit Distance
  (used heavily in Bioinformatics for sequence alignment)
- Text justification
- Seam Carving
  (Google this later, it's really awesome)
- Solving the Towers of Hanoi
- Predicting cricket scores
- Assembly Line Scheduling
- Matrix Chain Multiplication
- Playing Tetris perfectly
- Dynamic Time Warping
  (used extensively in computer vision)
- Finding optimal Binary Search Trees
  (when you know the likely frequencies of searches)
- The Travelling Salesman Problem *(though still slowly)*
- Knapsack *(though still slowly)*

and **loads** of other problems

Introduction

Dynamic programming is a technique for finding efficient algorithms for problems which can be broken down into simpler, overlapping subproblems.

The basic idea:

1. Find a recursive formula for the problem
- in terms of answers to subproblems.

(typically this is the hard bit)

2. Write down a naive recursive algorithm

(typically this algorithm will take exponential time)

3. Speed it up by storing the solutions to subproblems

(to avoid recomputing the same thing over and over)

4. Derive an iterative algorithm by solving the subproblems in a good order

(iterative algorithms are often better in practice, easier to analyse and prettier)

in other words. . .

Dynamic programming is *recursion without repetition*

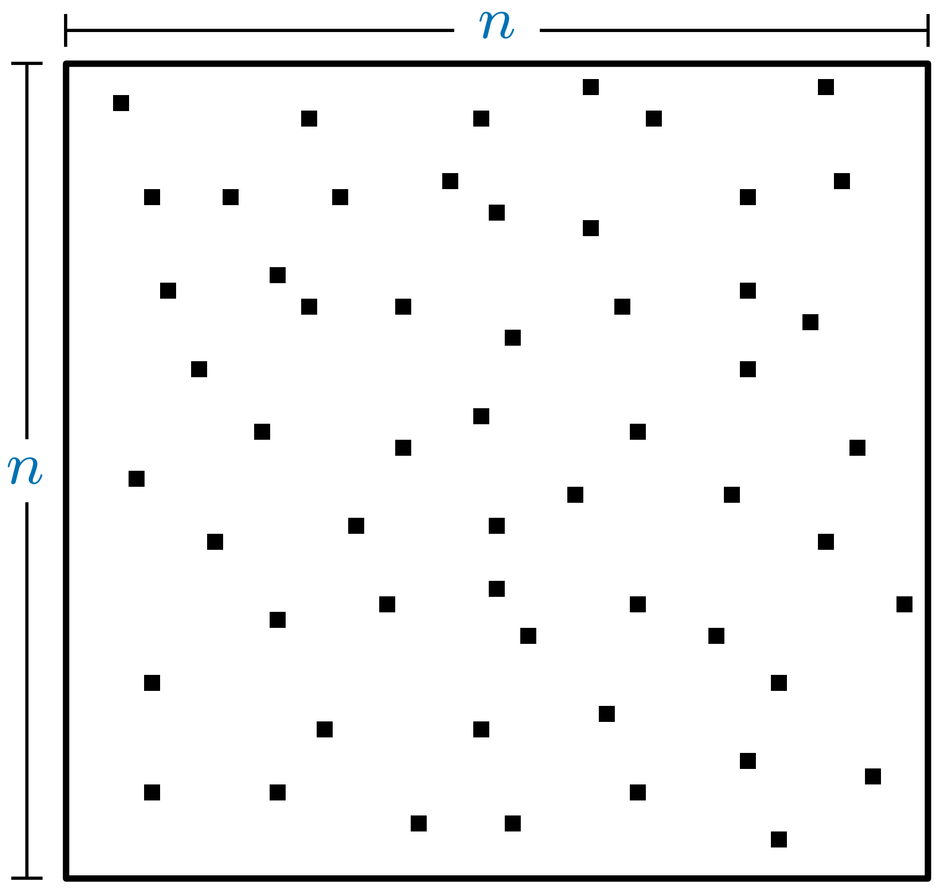
Part one

Largest Empty Square

Largest Empty Square

Problem Given an $n \times n$ monochrome image, find the largest empty square.

i.e. without any black pixels



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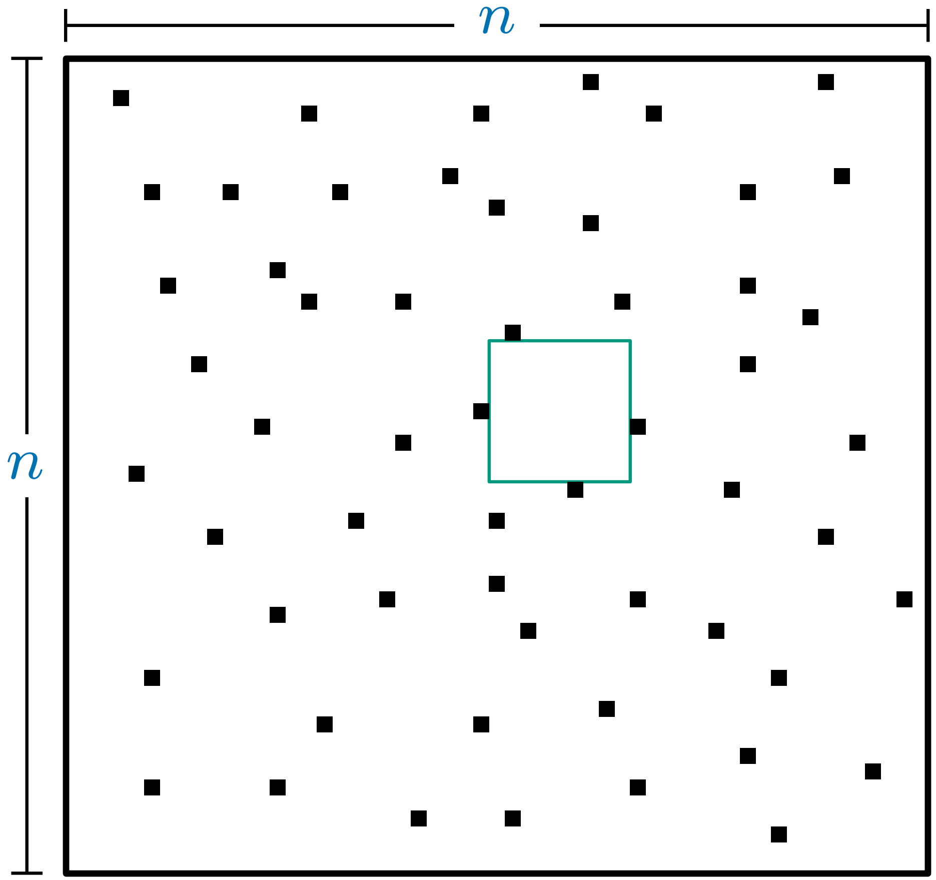

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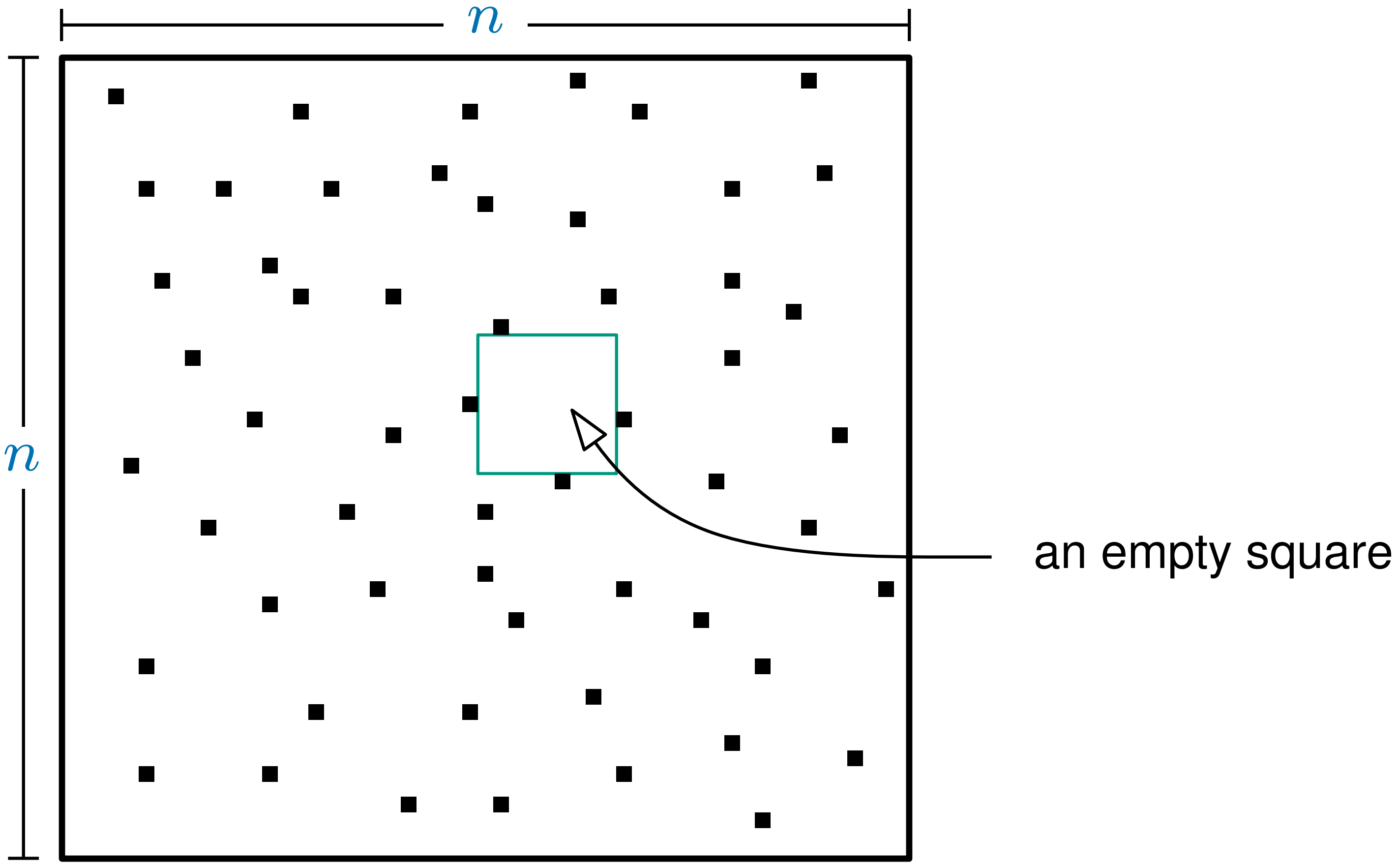

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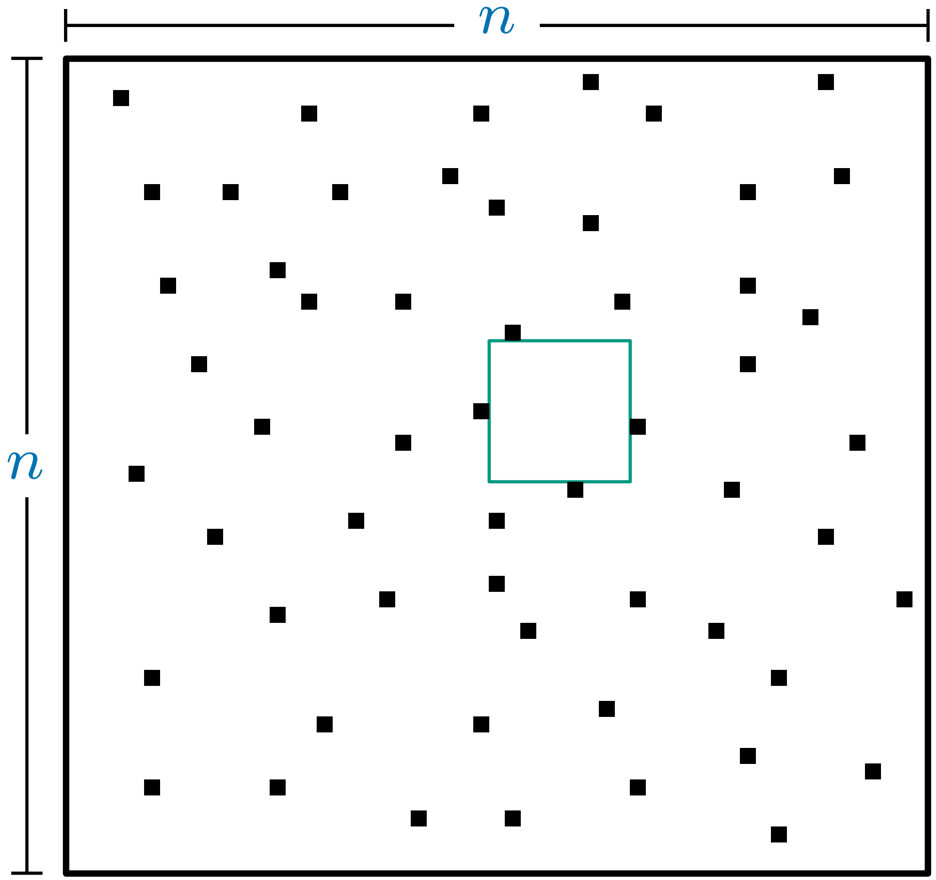

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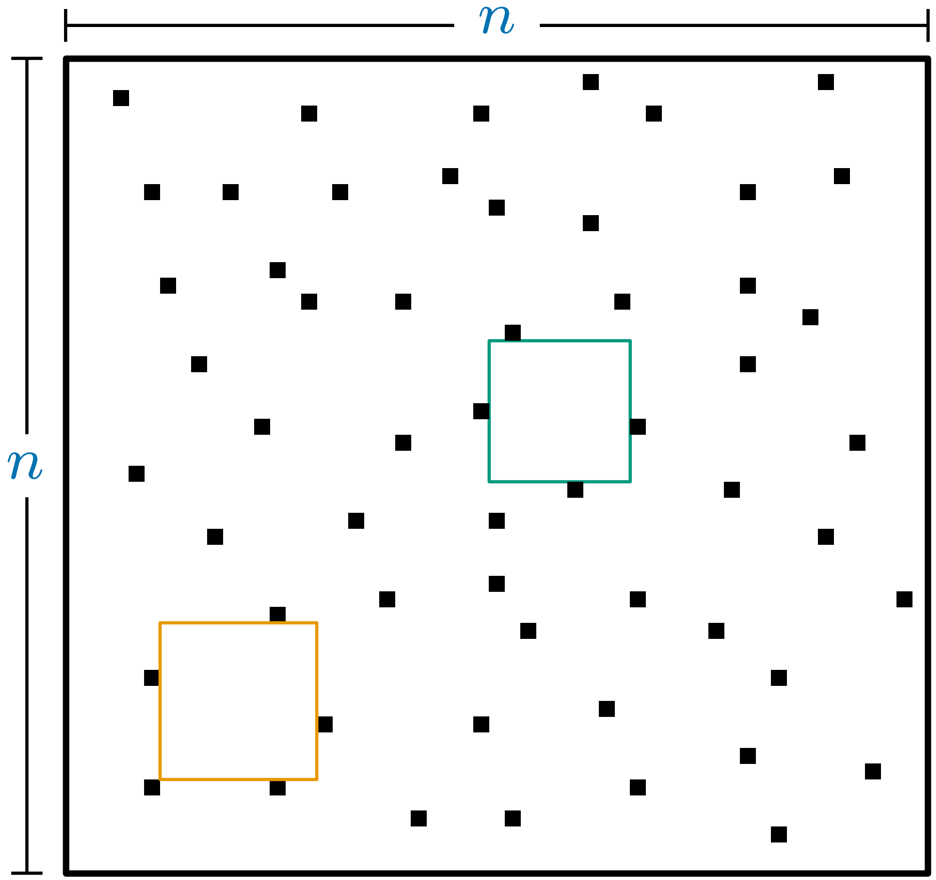

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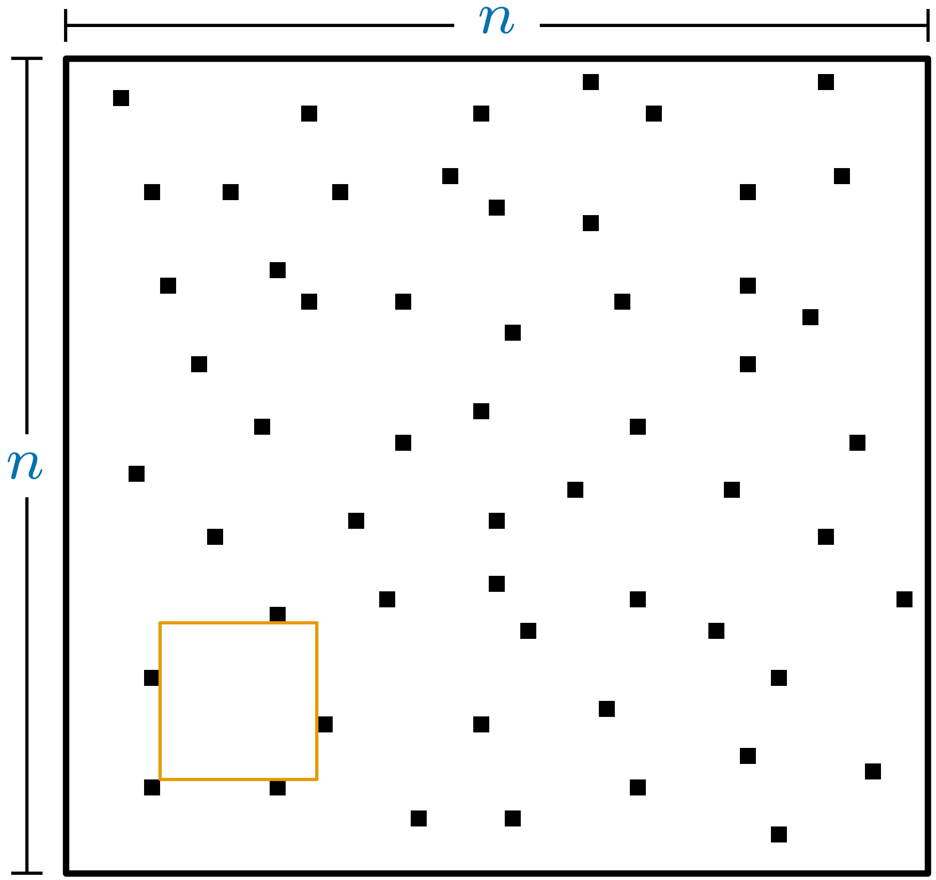

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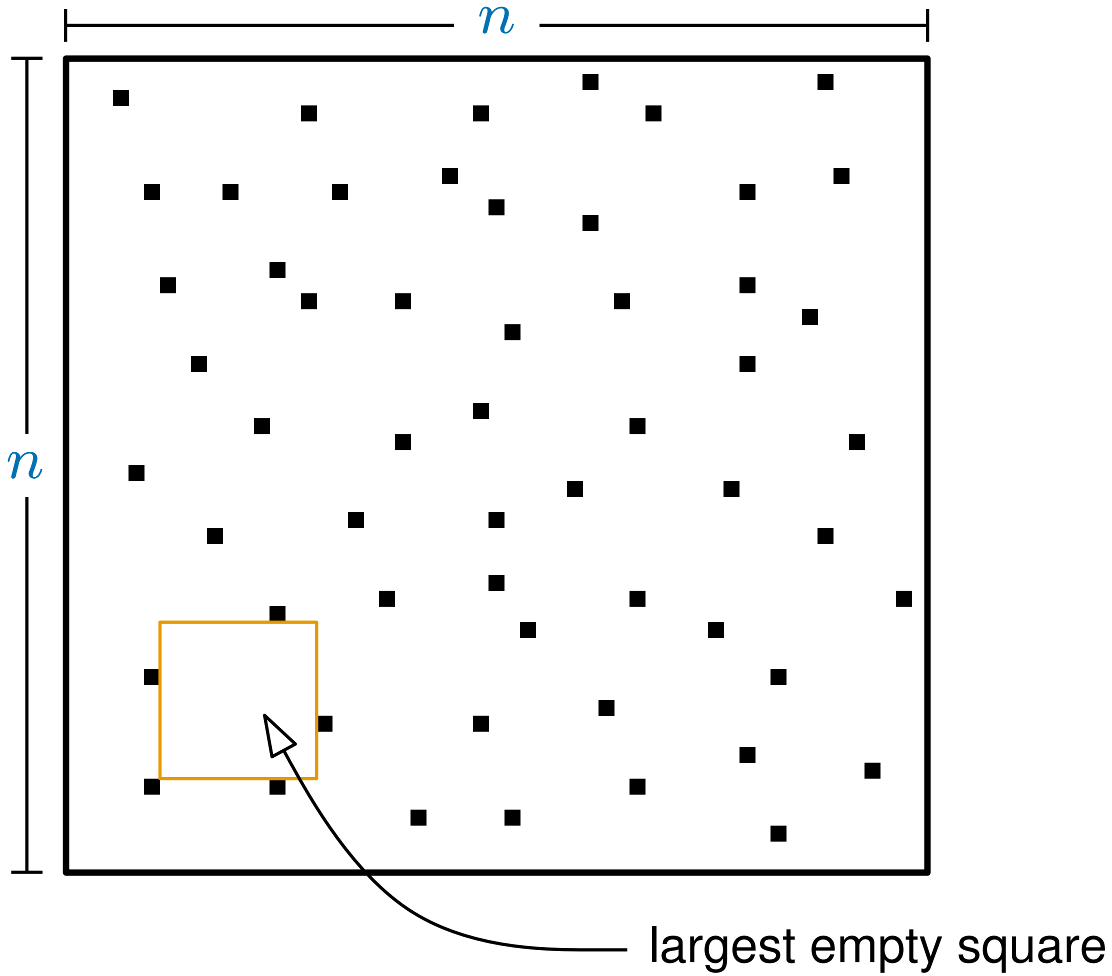

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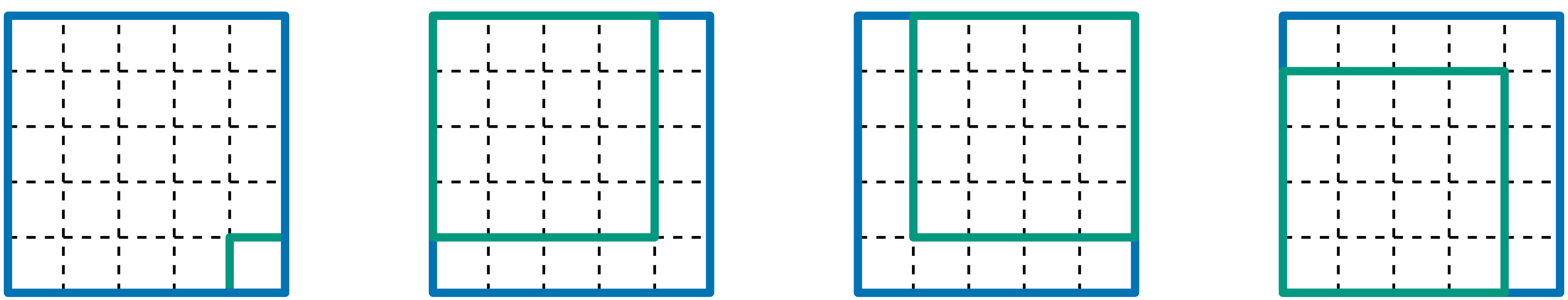

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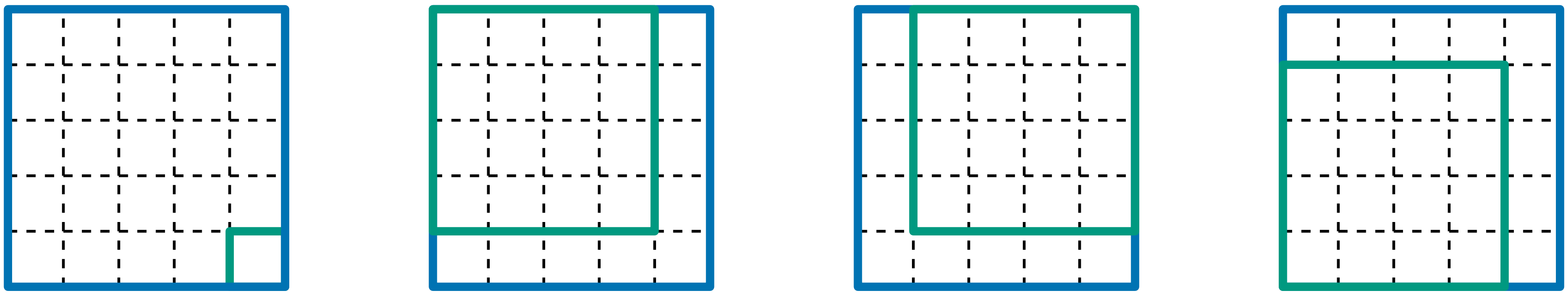

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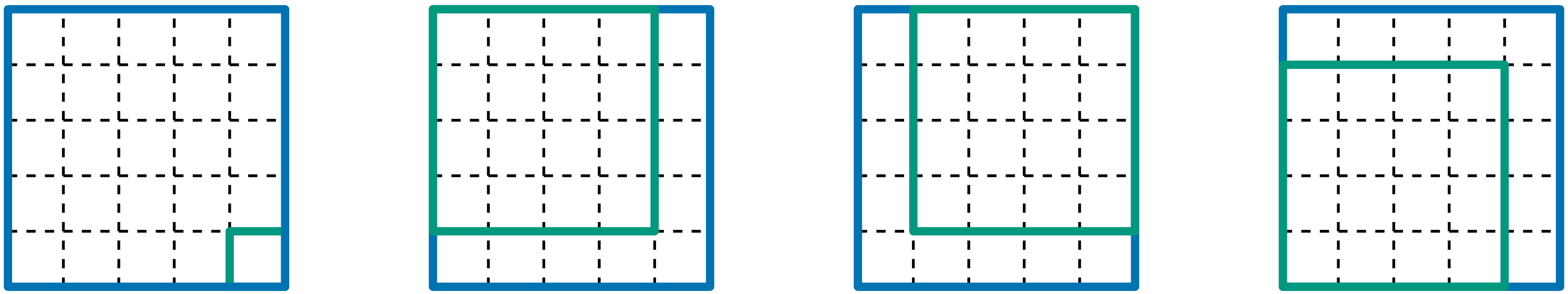

If S is empty then all four ☐ are empty

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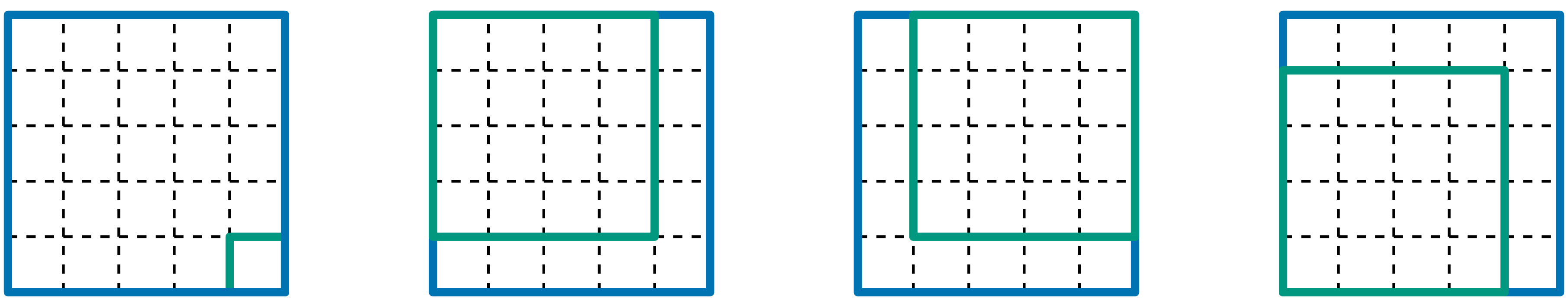

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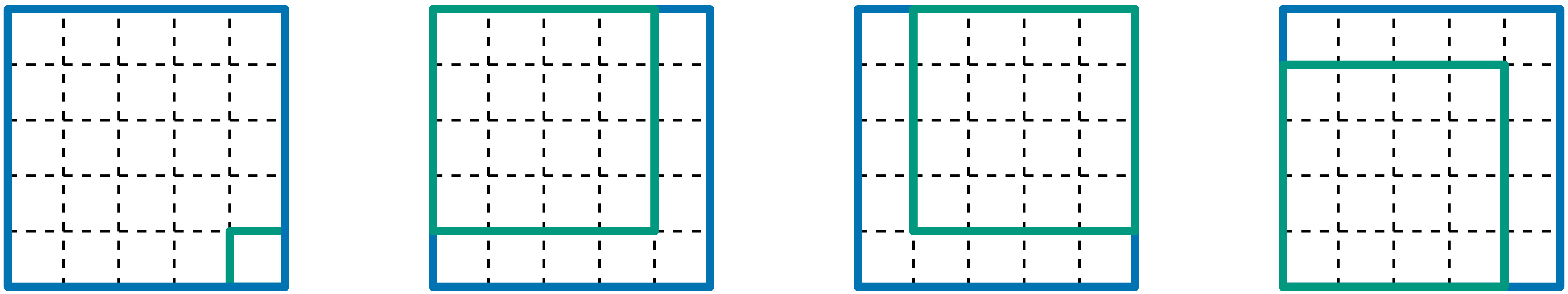

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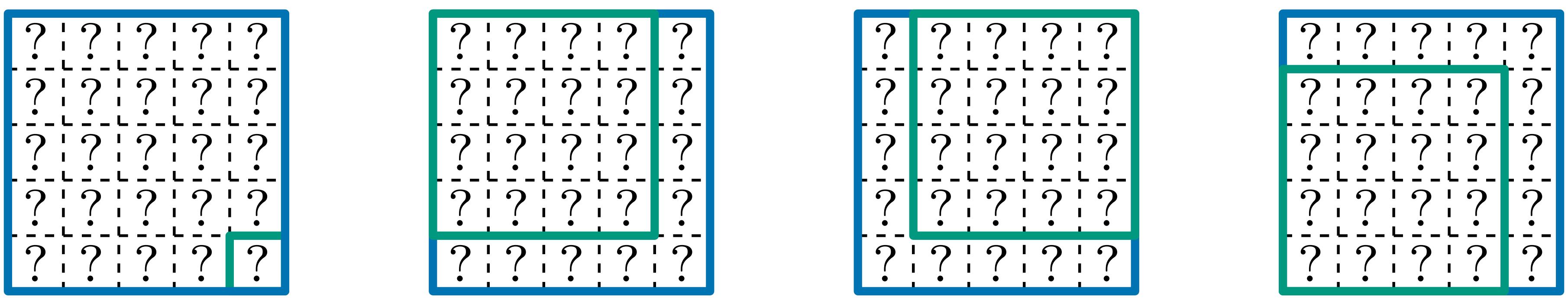

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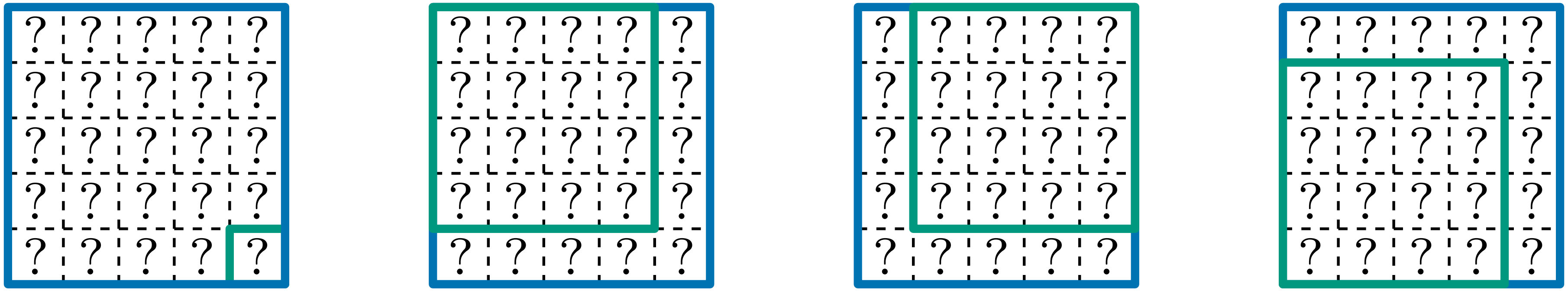

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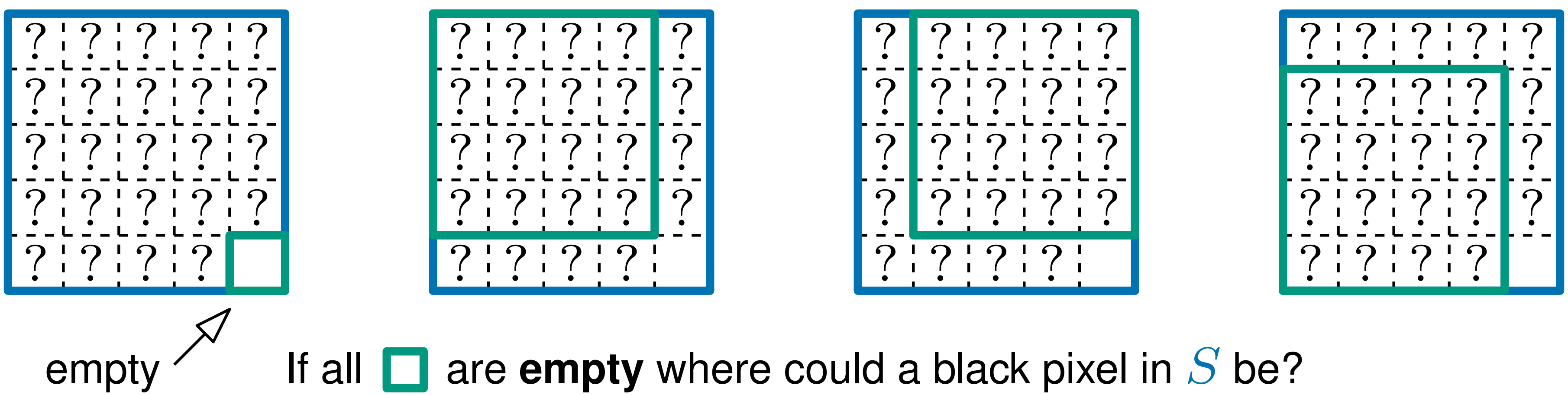

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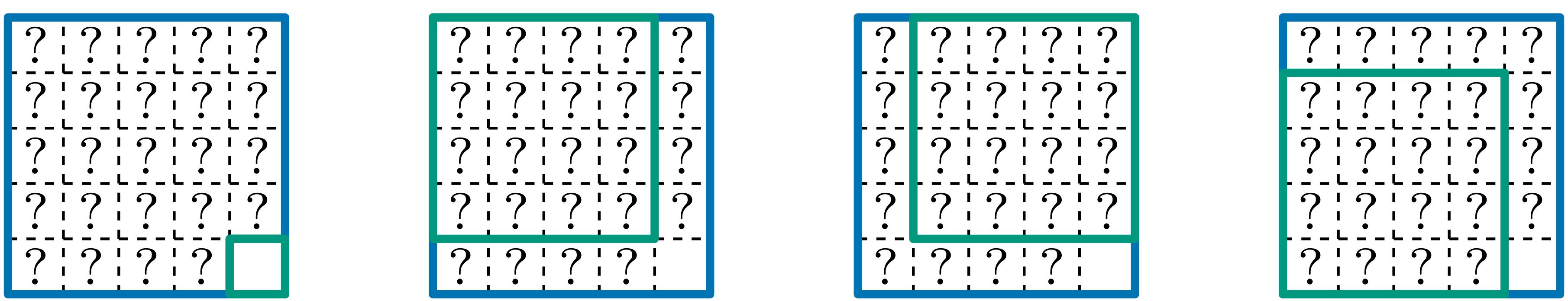

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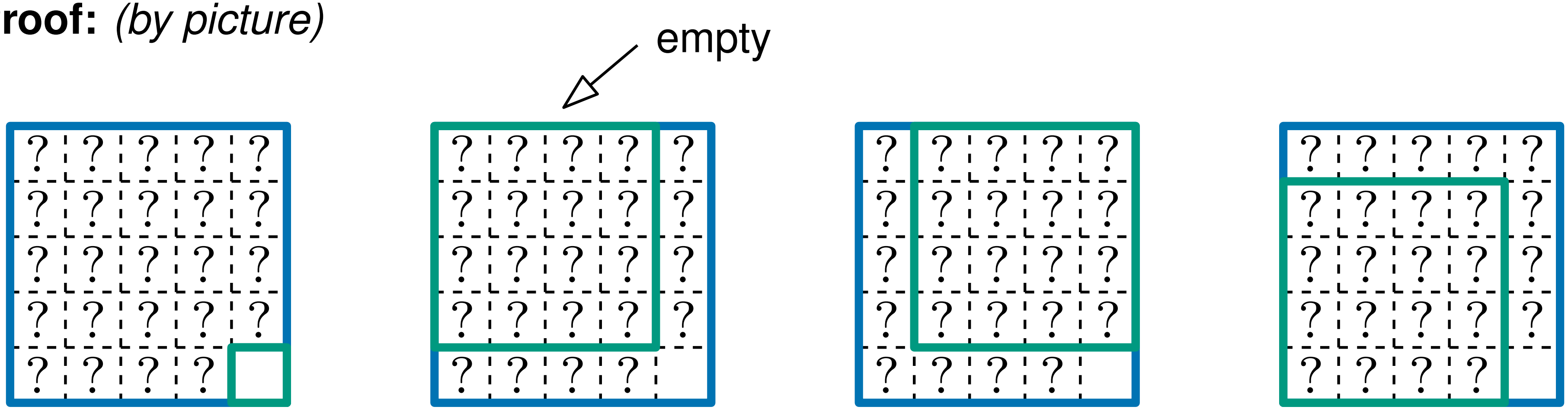

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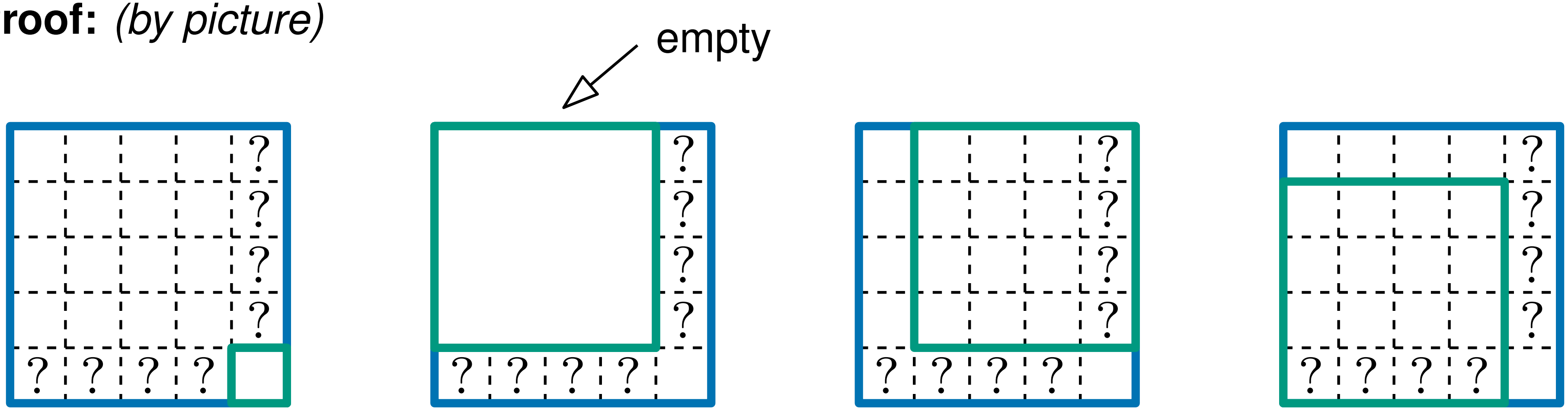

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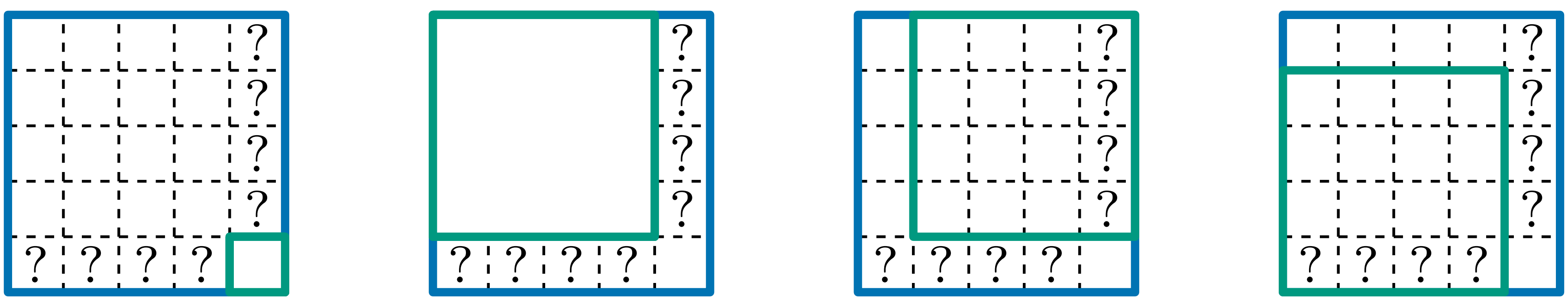

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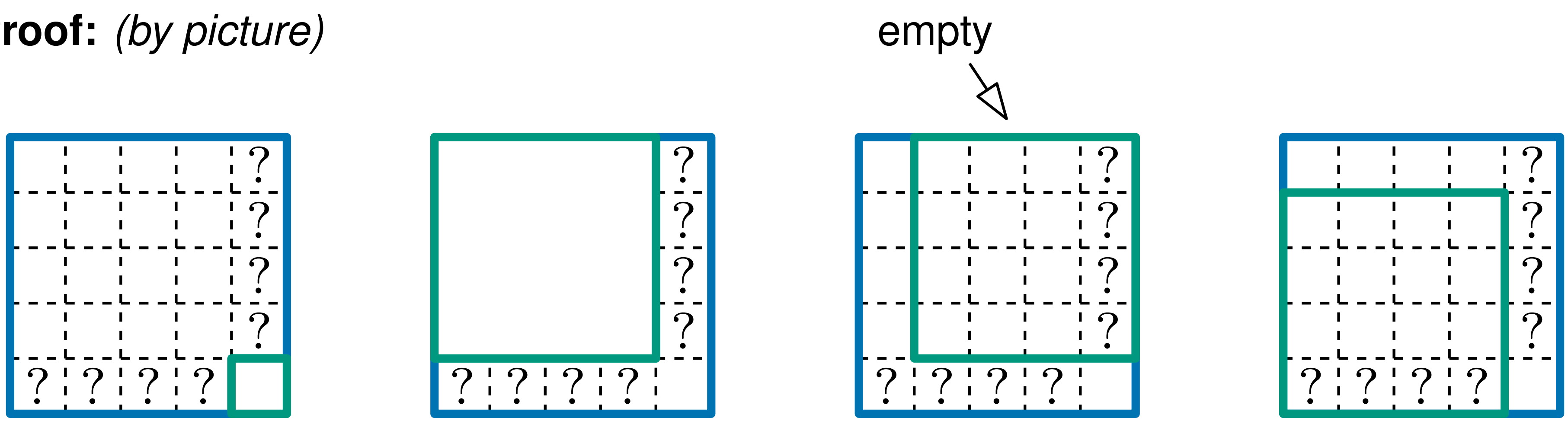

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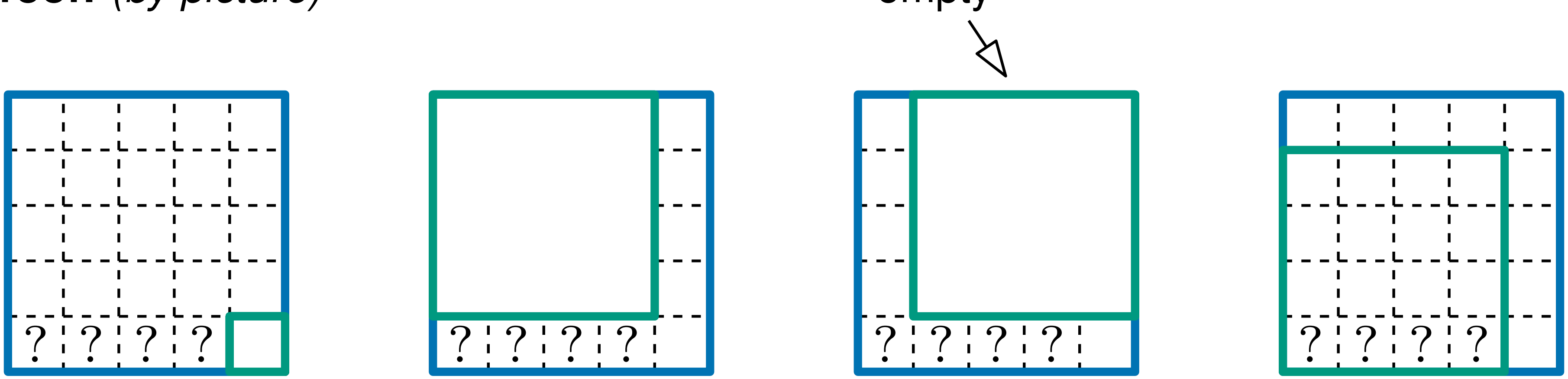

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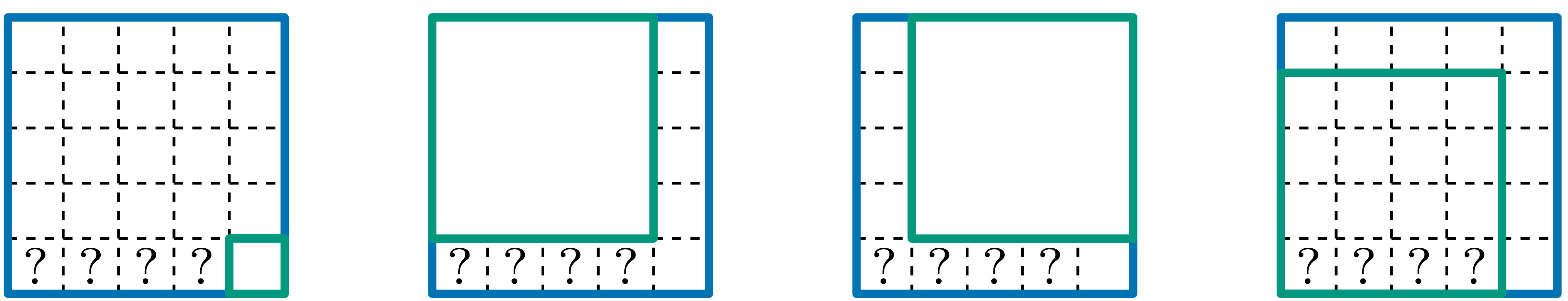

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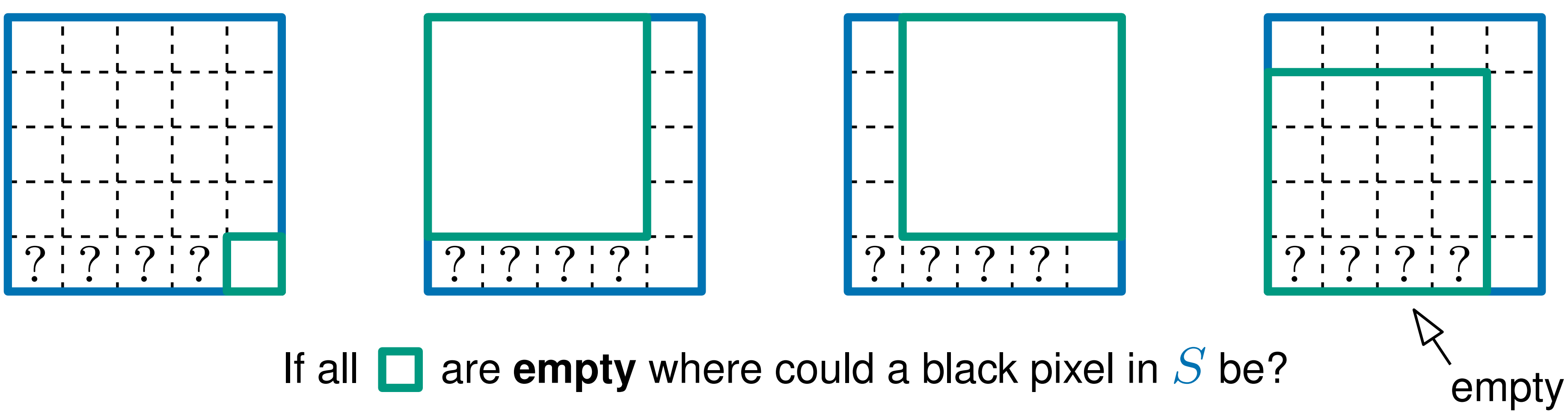

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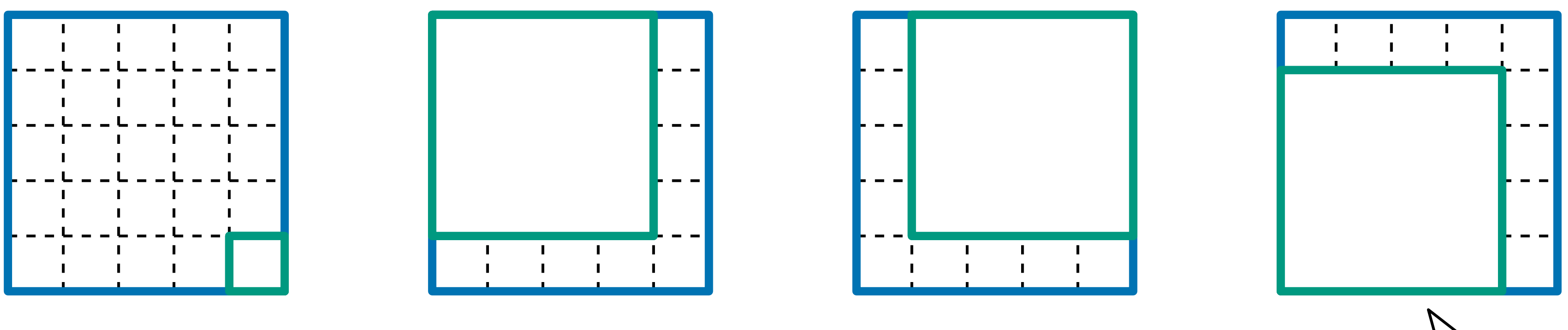

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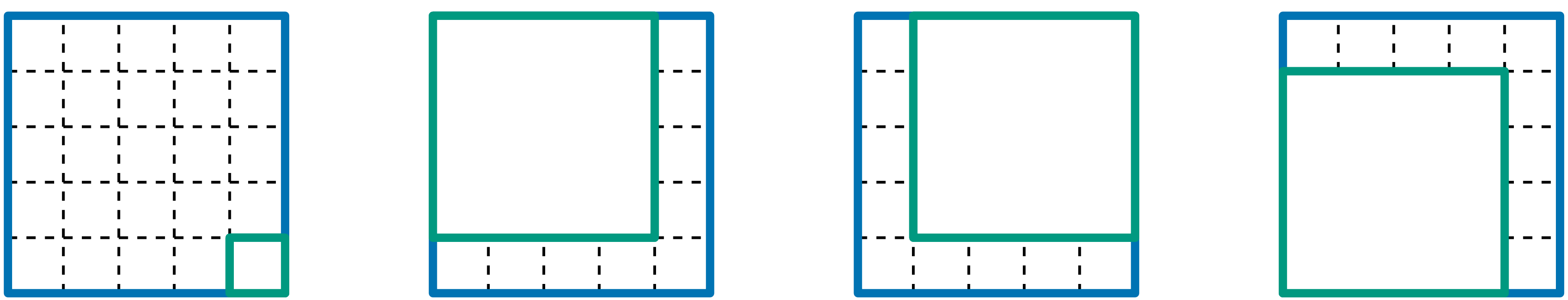

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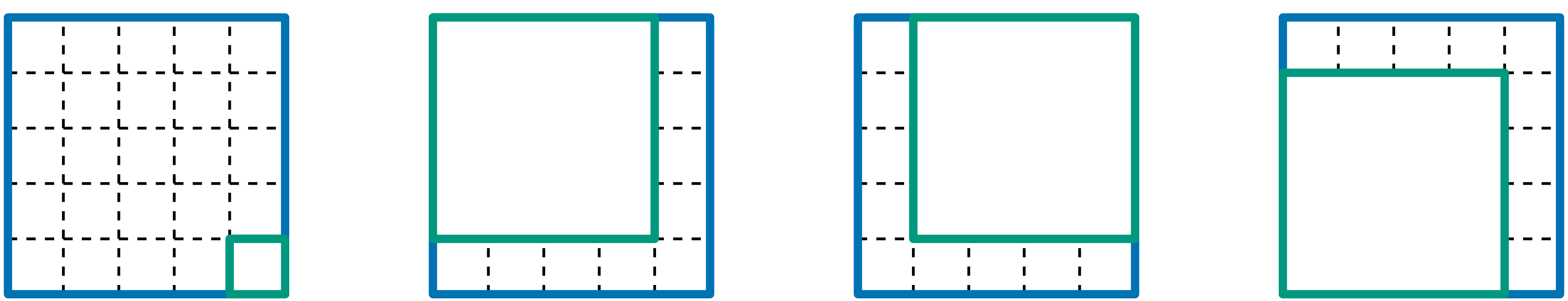

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If all $\square$ are **empty** then S is **empty**

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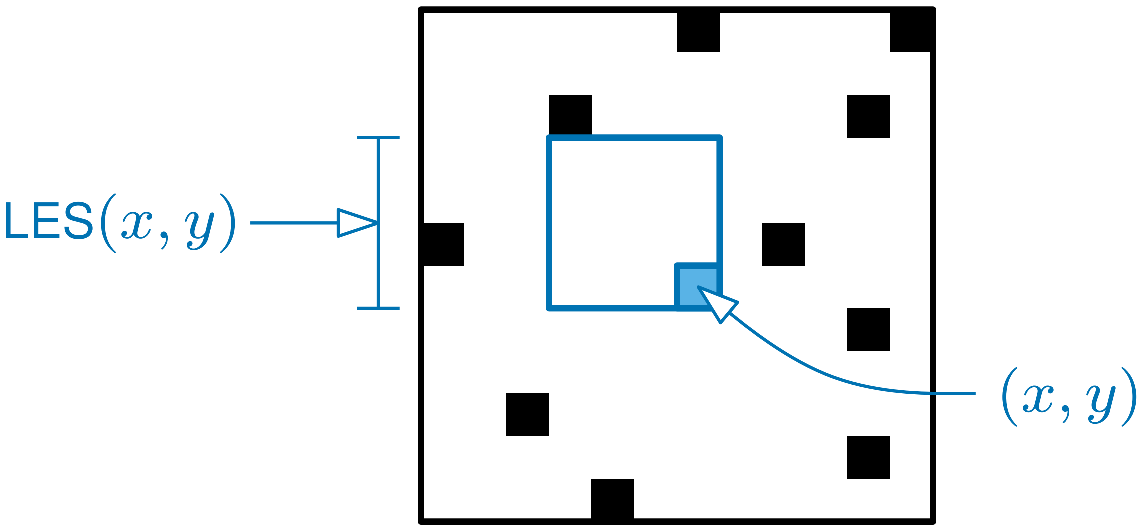
Let LES(x, y) be the size (i.e. side length) of the largest empty square whose bottom right is at (x, y)



Then:

If the pixel (x, y) is not empty then LES$(x, y) = 0$.

If (x, y) is empty and in the first row or column,

$$\text{LES}(x, y) = 1.$$

If (x, y) is empty and not in the first row or column,

$$\text{LES}(x, y) = \min(\text{LES}(x-1, y-1), \text{LES}(x-1, y), \text{LES}(x, y-1)) + 1.$$

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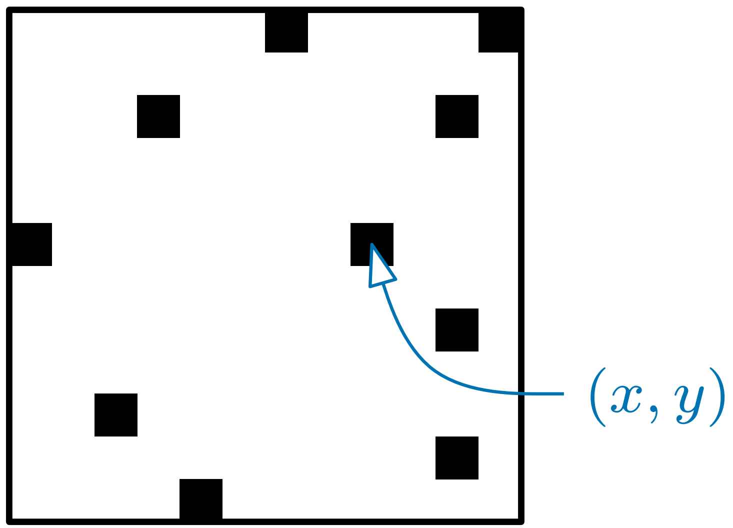

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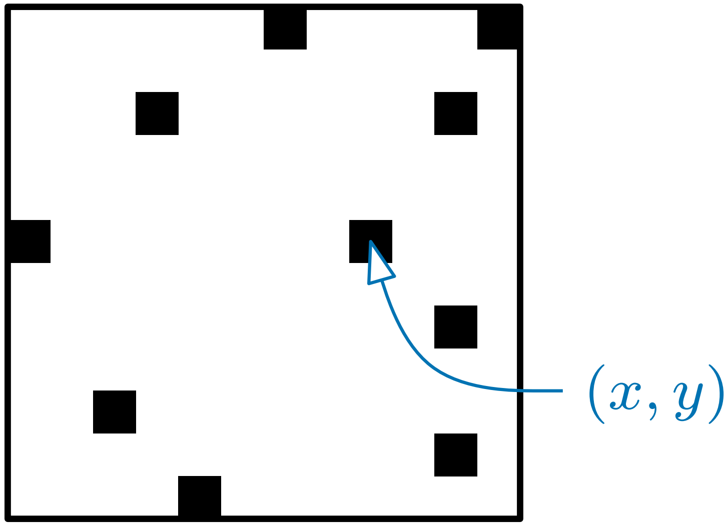

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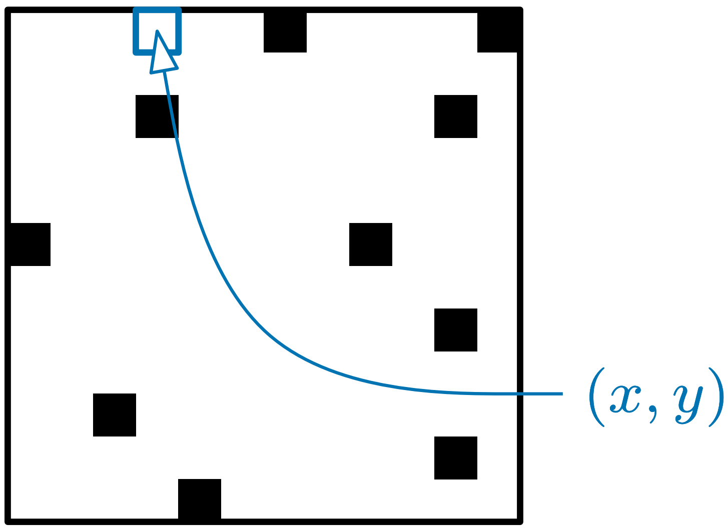

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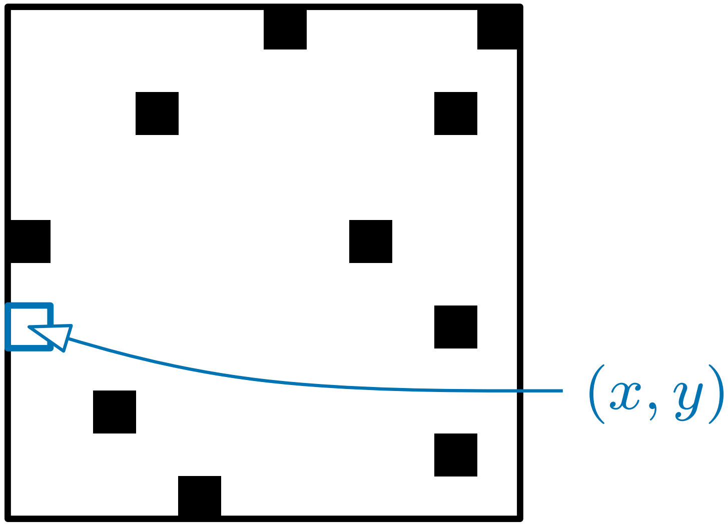

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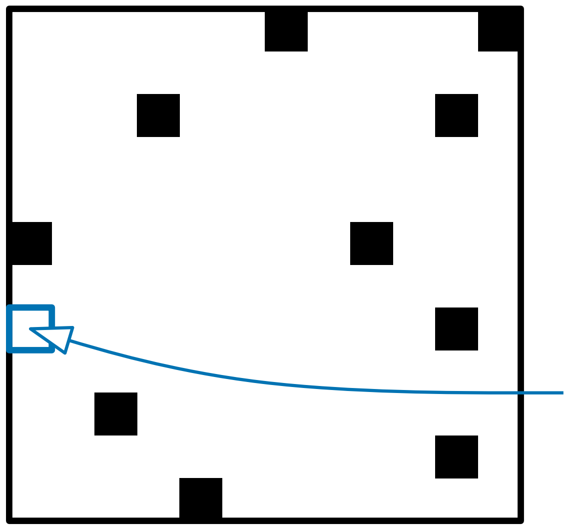

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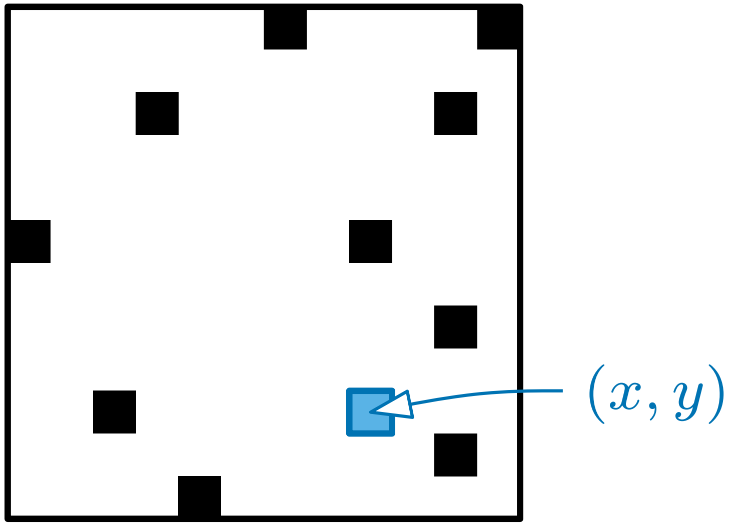

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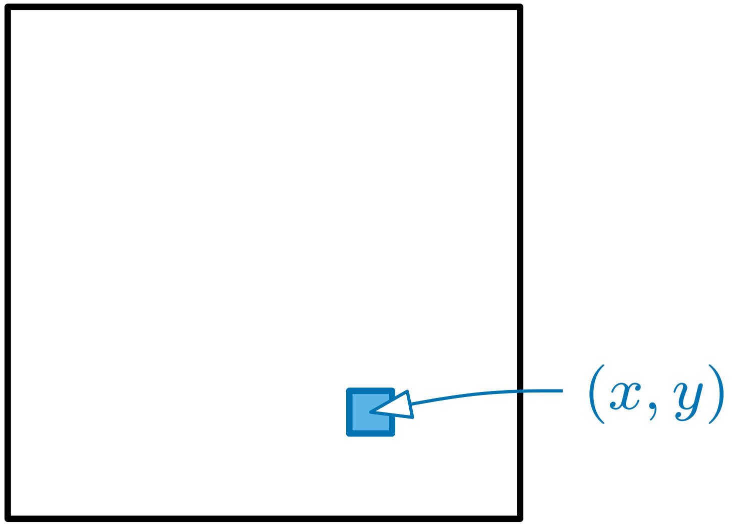

Then:

If the pixel (x, y) is not empty then LES$(x, y) = 0$. ✓

If (x, y) is empty and in the first row or column,

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If (x, y) is empty and not in the first row or column,

$$\text{LES}(x, y) = \min(\text{LES}(x-1, y-1), \text{LES}(x-1, y), \text{LES}(x, y-1)) + 1.$$

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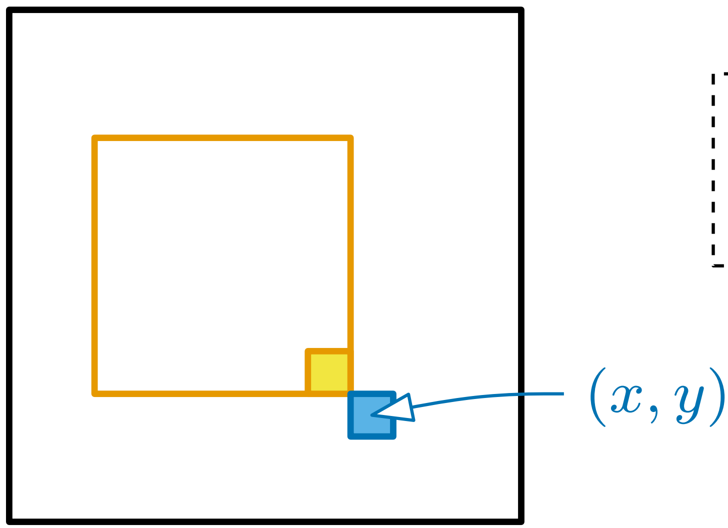

$$\mathsf{LES}(x, y) \leqslant \mathsf{LES}(x-1, y-1) + 1$$

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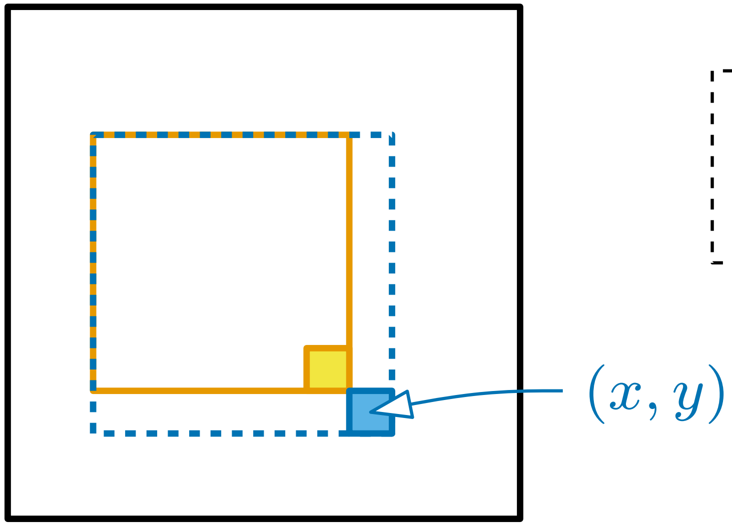

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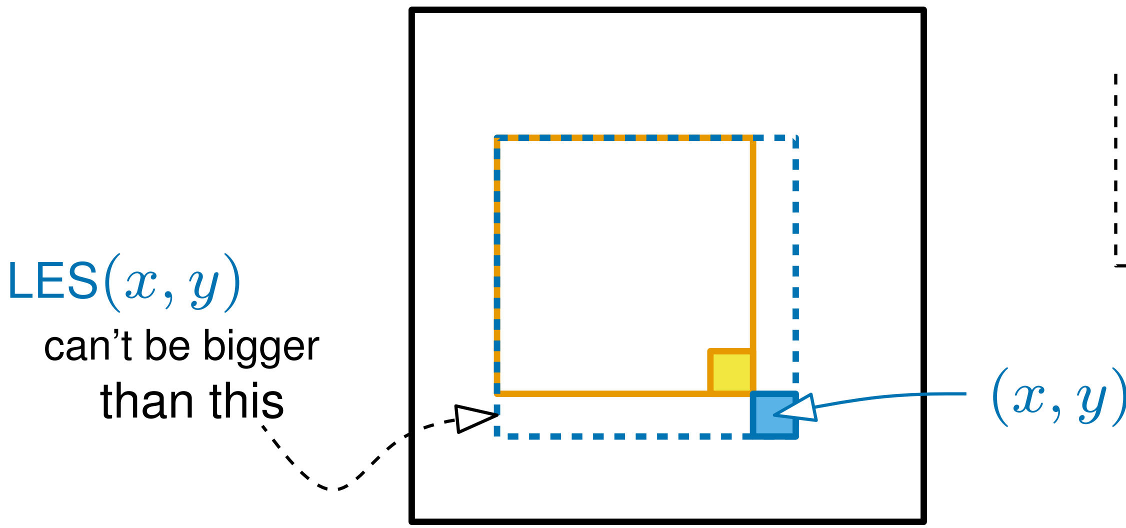

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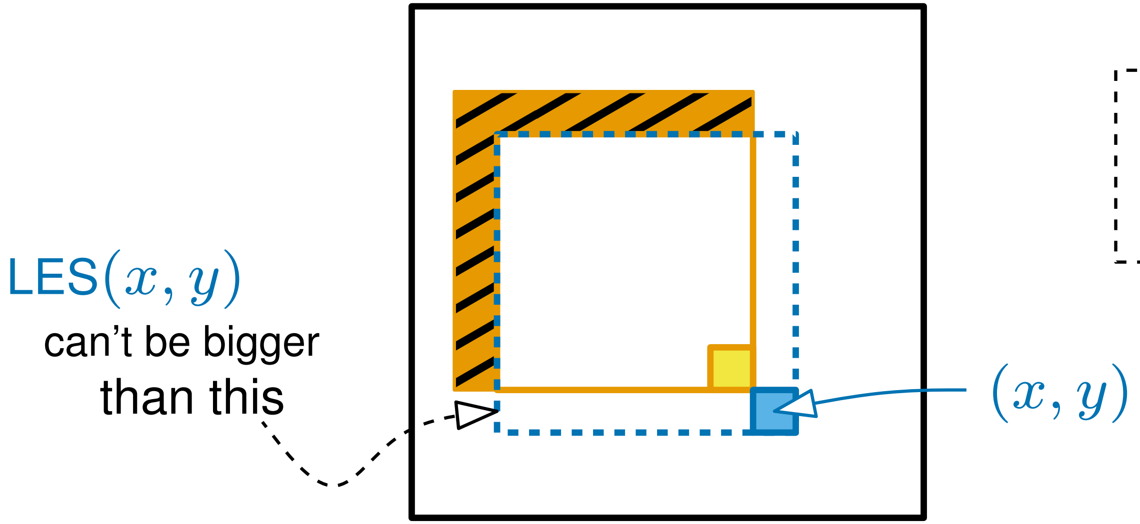

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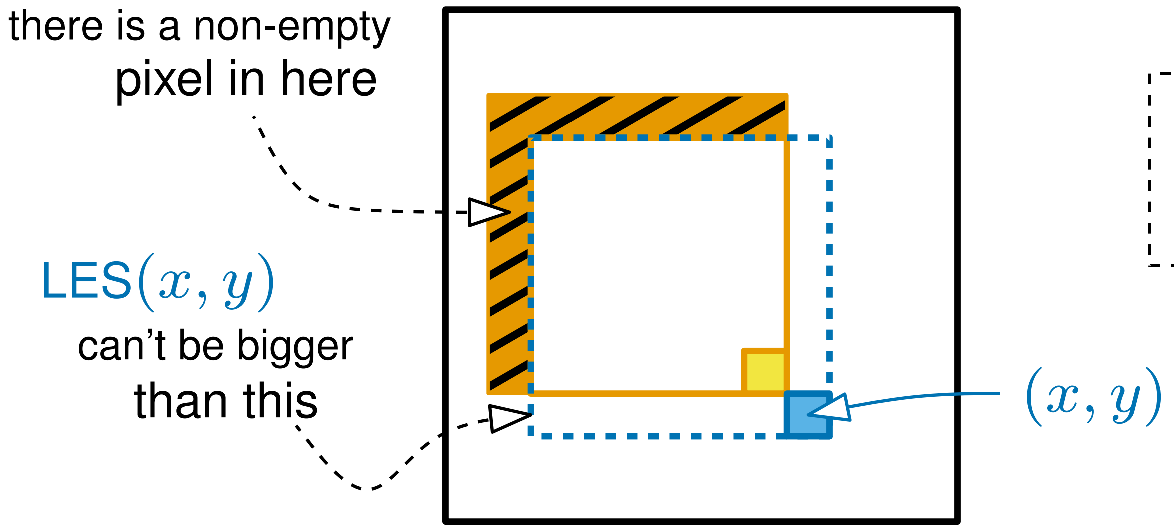

LES$(x, y) \leqslant$ LES$(x - 1, y - 1) + 1$

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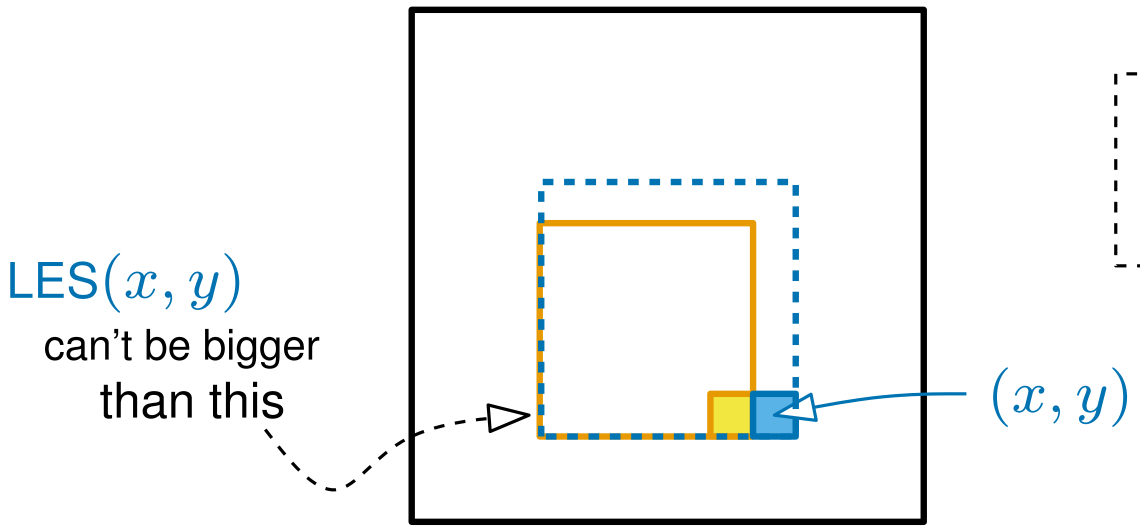

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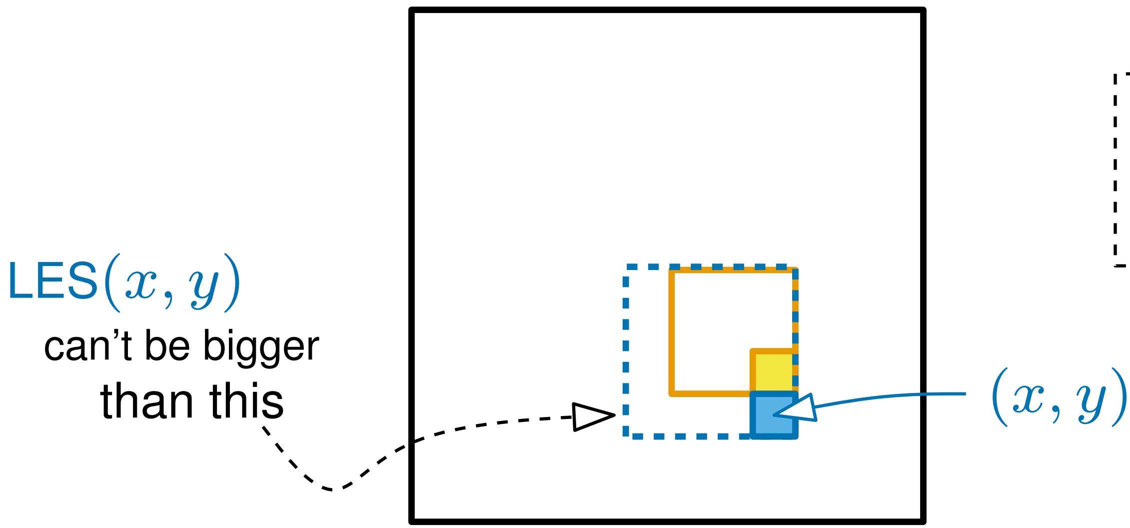

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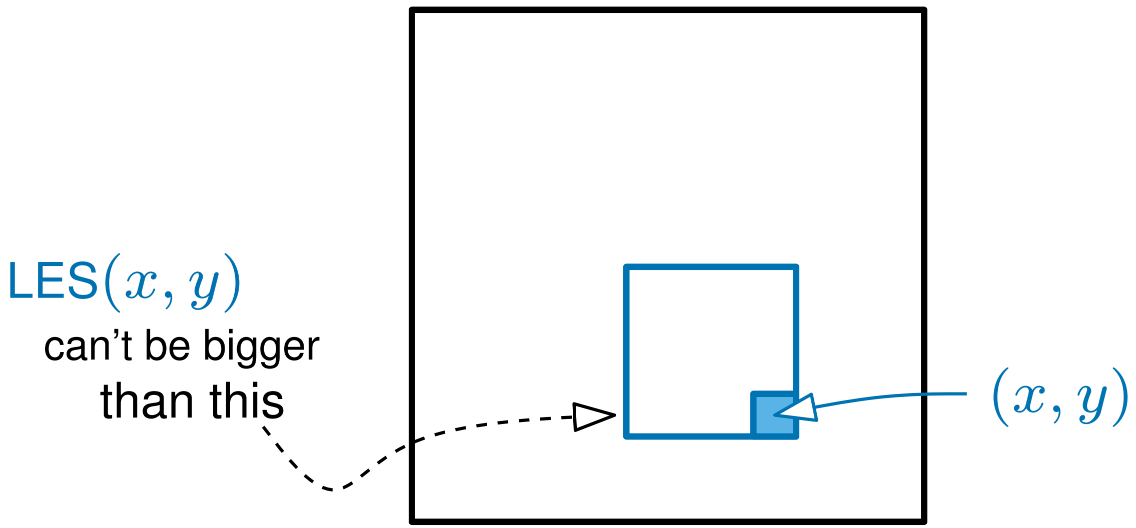

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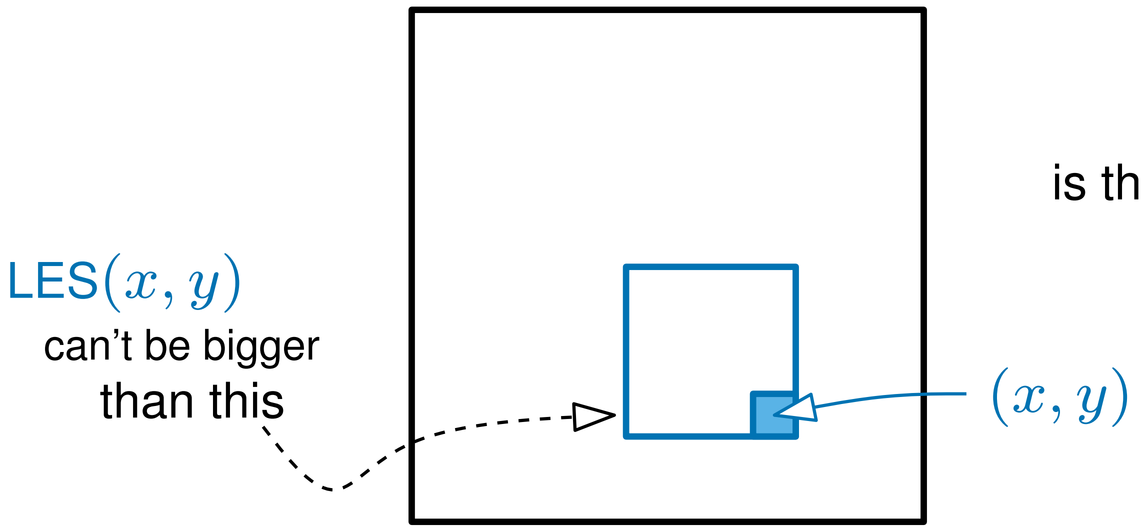

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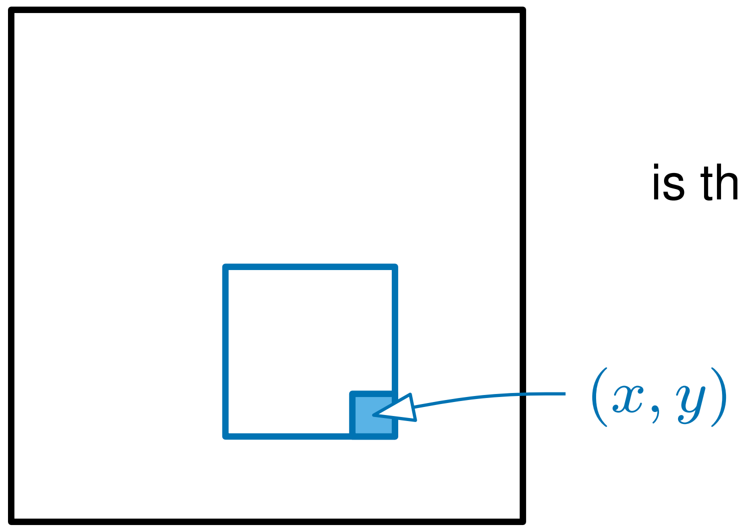

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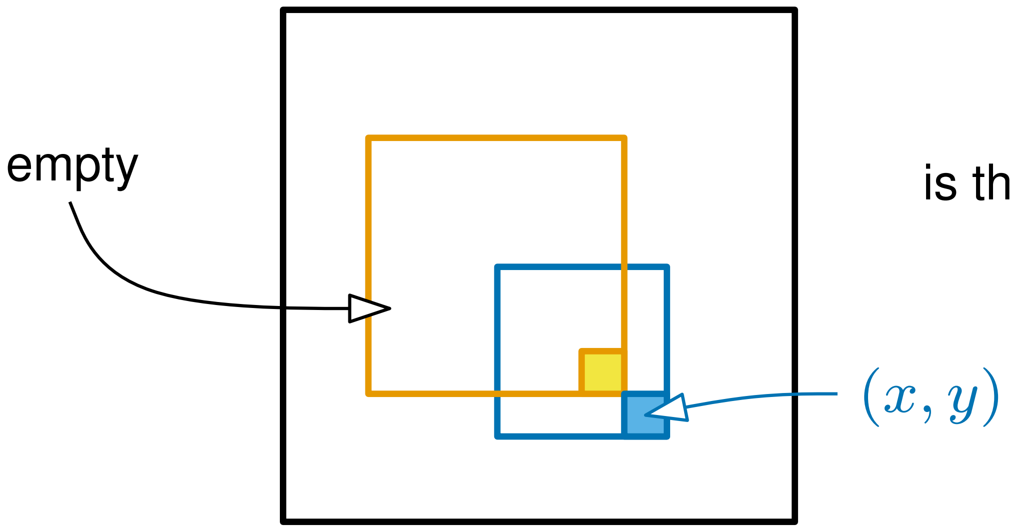

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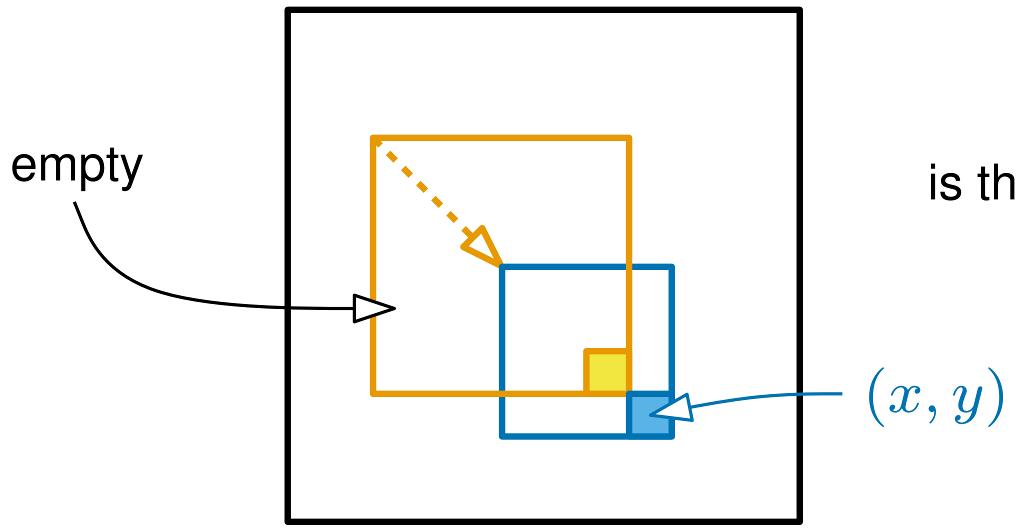

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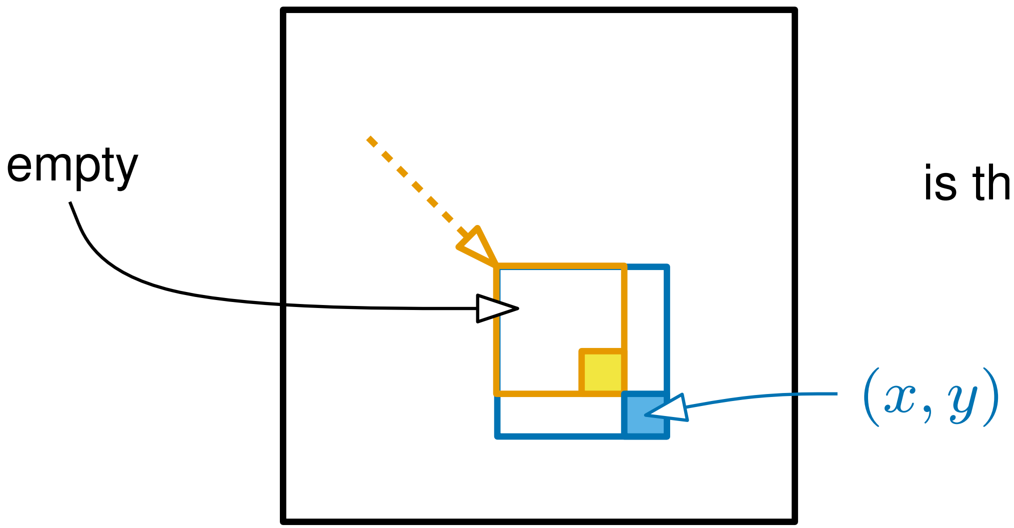

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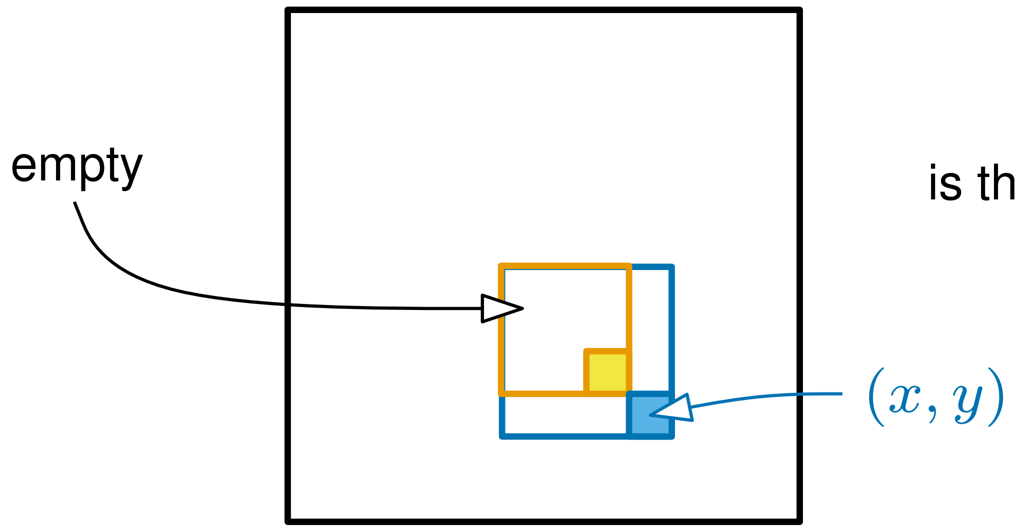

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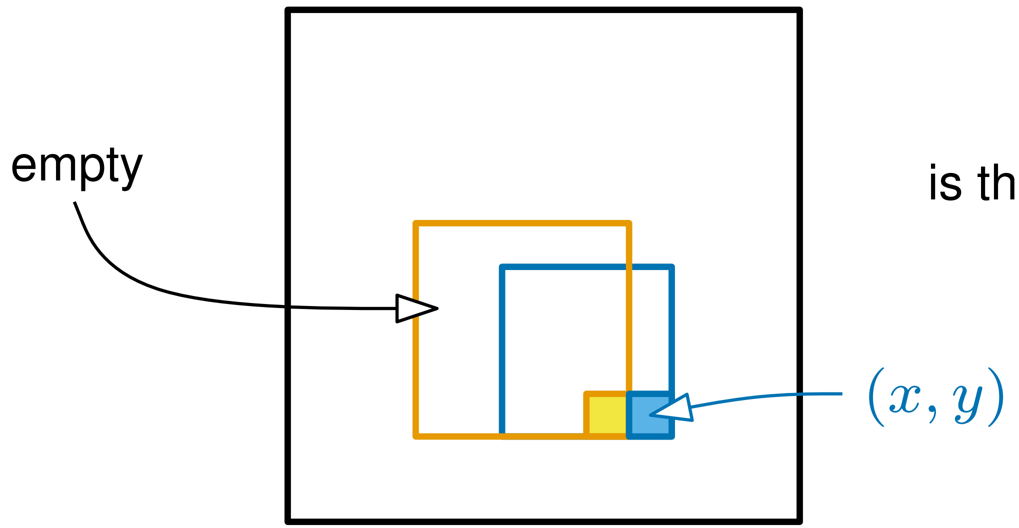

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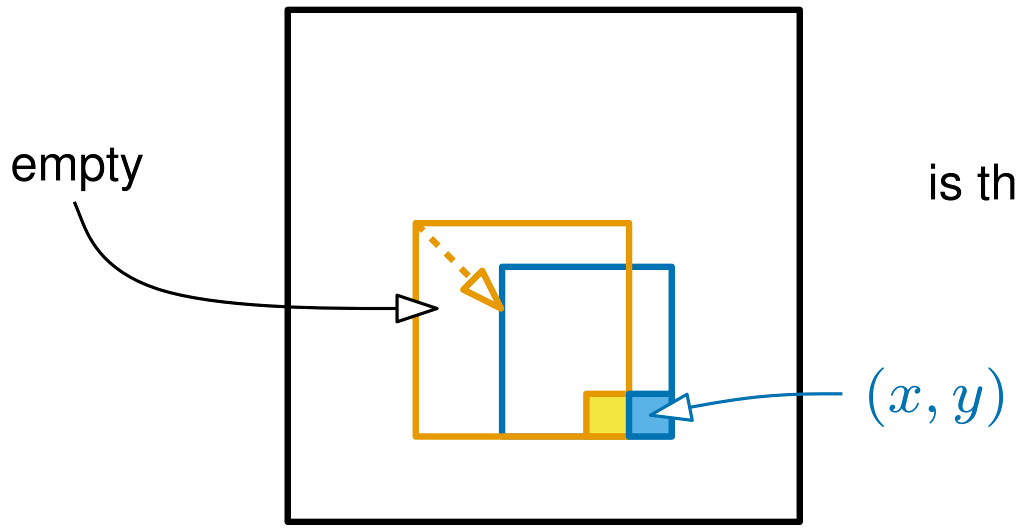

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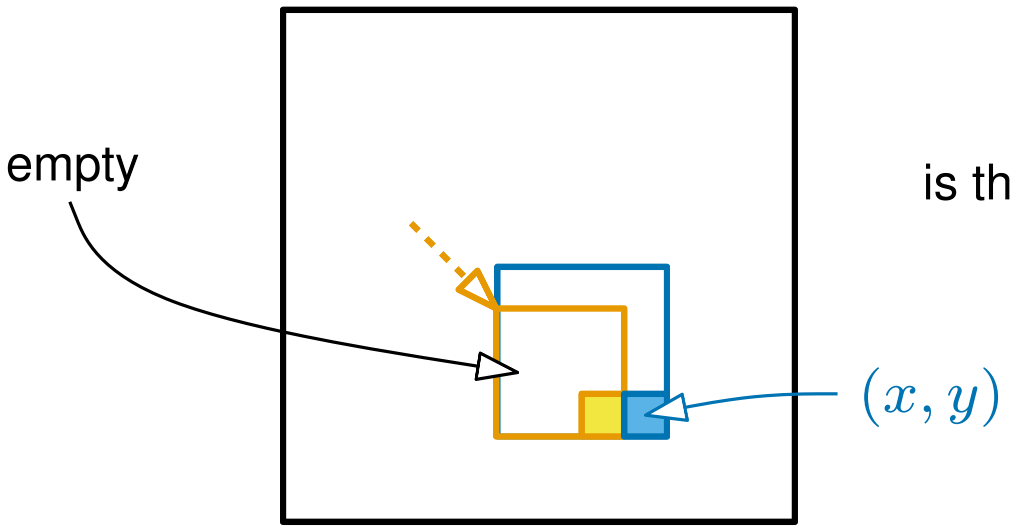

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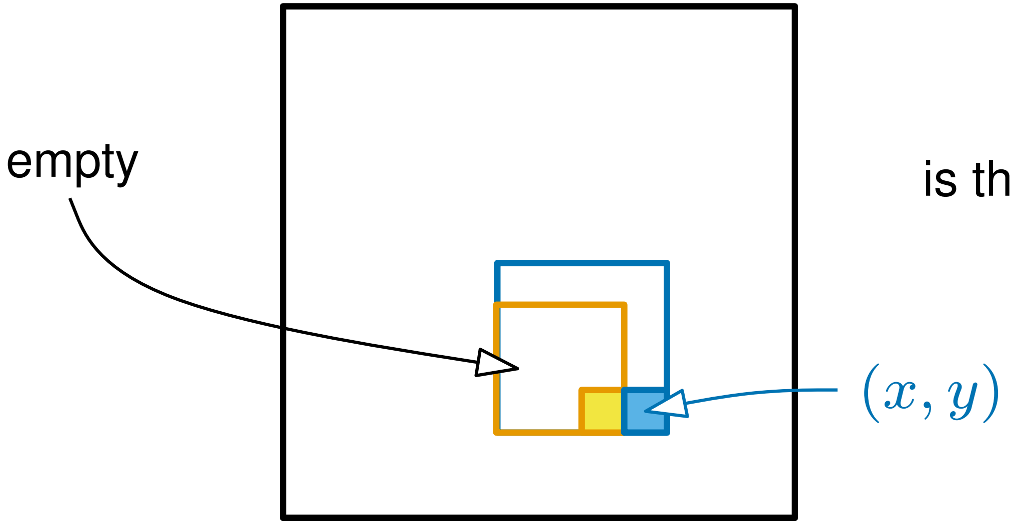

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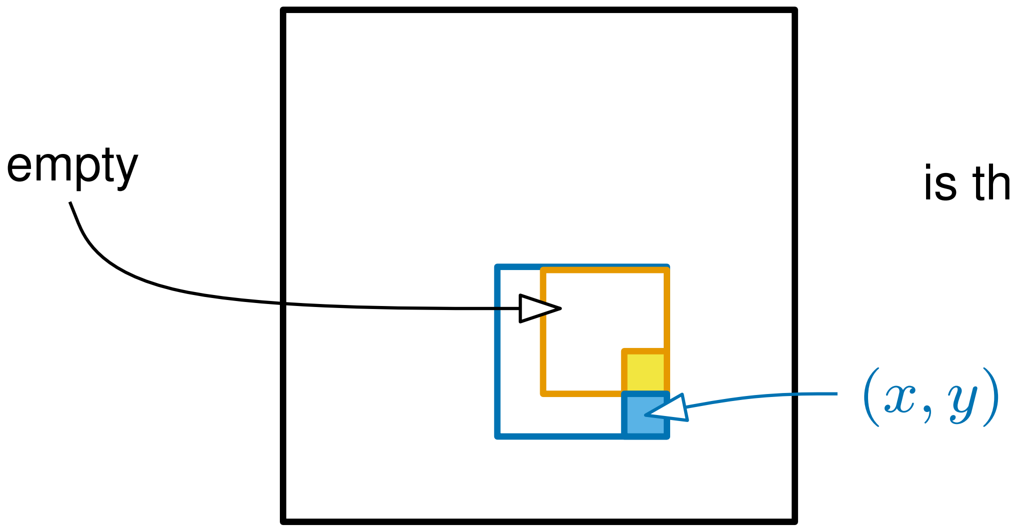

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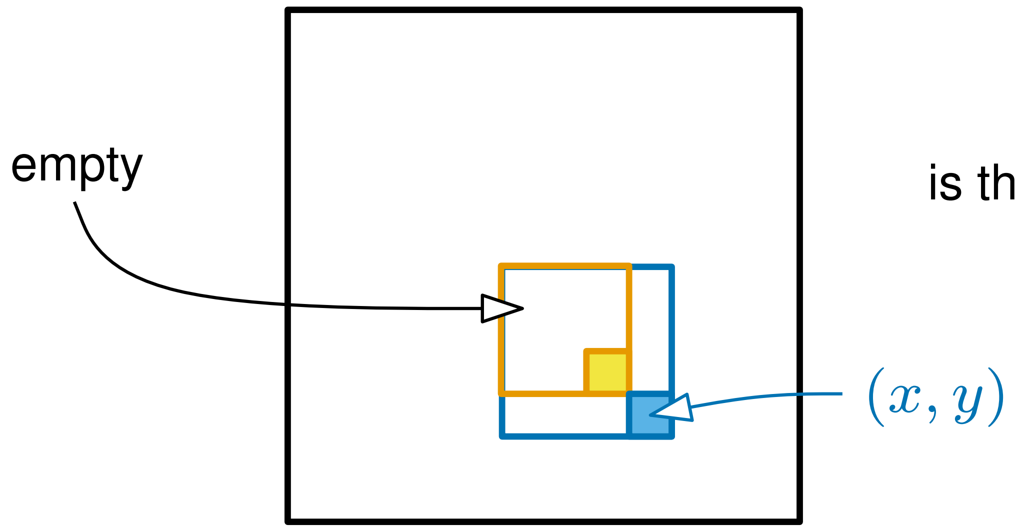

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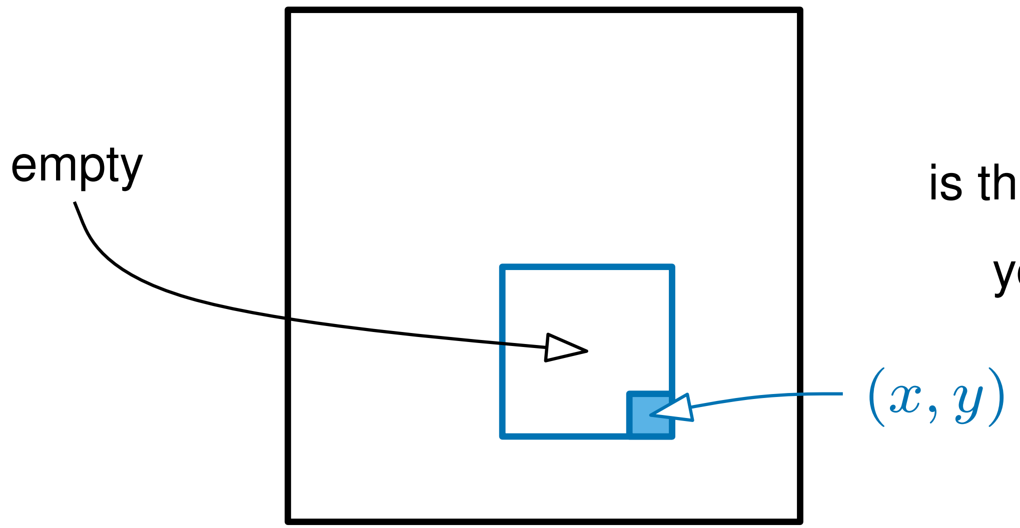

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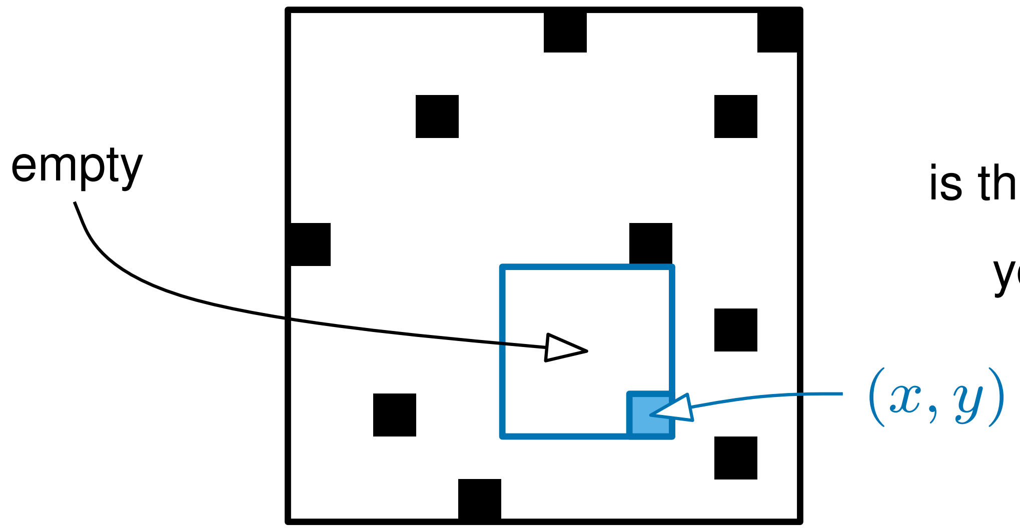

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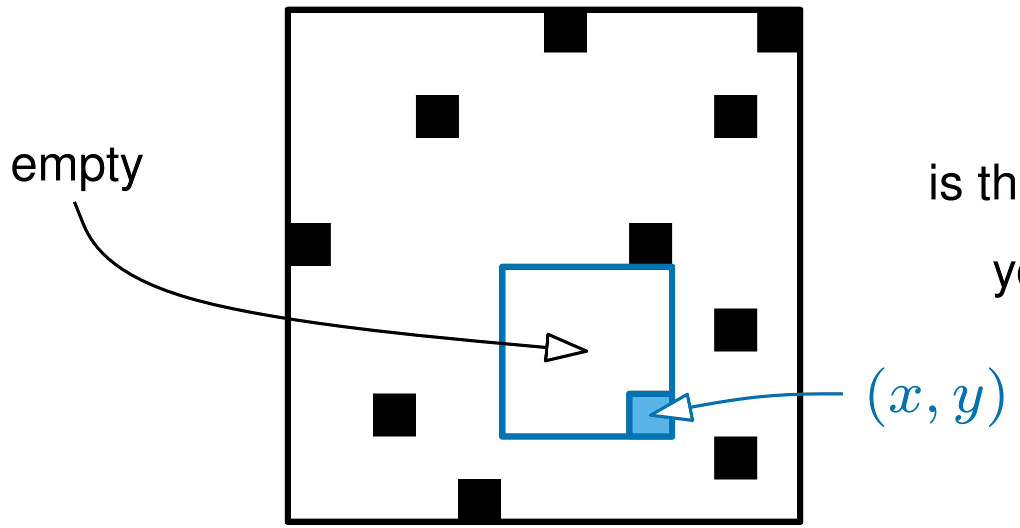

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2. Write down a recursive algorithm

We can use the recursive formula to get a recursive algorithm...

LES(x, y)

```
If pixel (x,y) is not empty
  Return 0
If (x = 1) or (y = 1)
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Return min(LES(x-1, y-1), LES(x-1, y), LES(x, y-1)) + 1
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LES(x, y) computes the size of the largest empty square whose bottom right is at (x, y)

Therefore, the maximum of LES(x, y) over all x and y gives the size of the largest empty square in the whole image

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What is the time complexity of this algorithm?

How efficient is the recursive algorithm?

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Let's compute LES$(4, 4)$...

How efficient is the recursive algorithm?

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Let's compute LES$(4, 4)$... *(and consider the recursive calls)*

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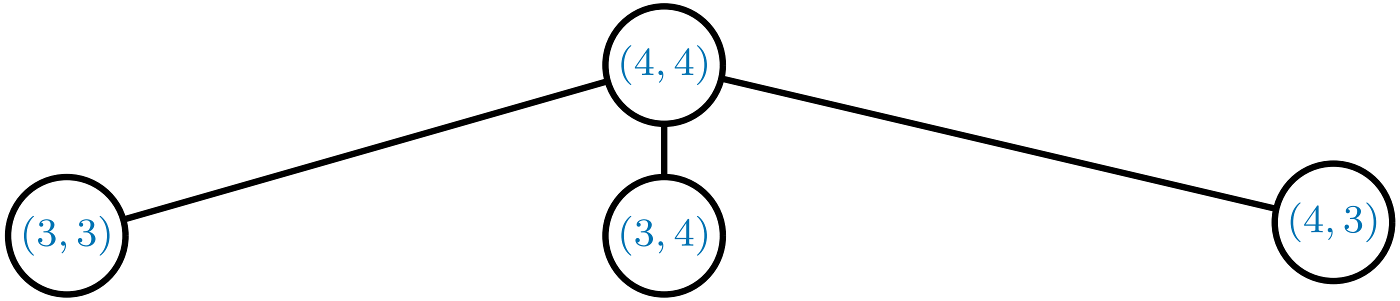

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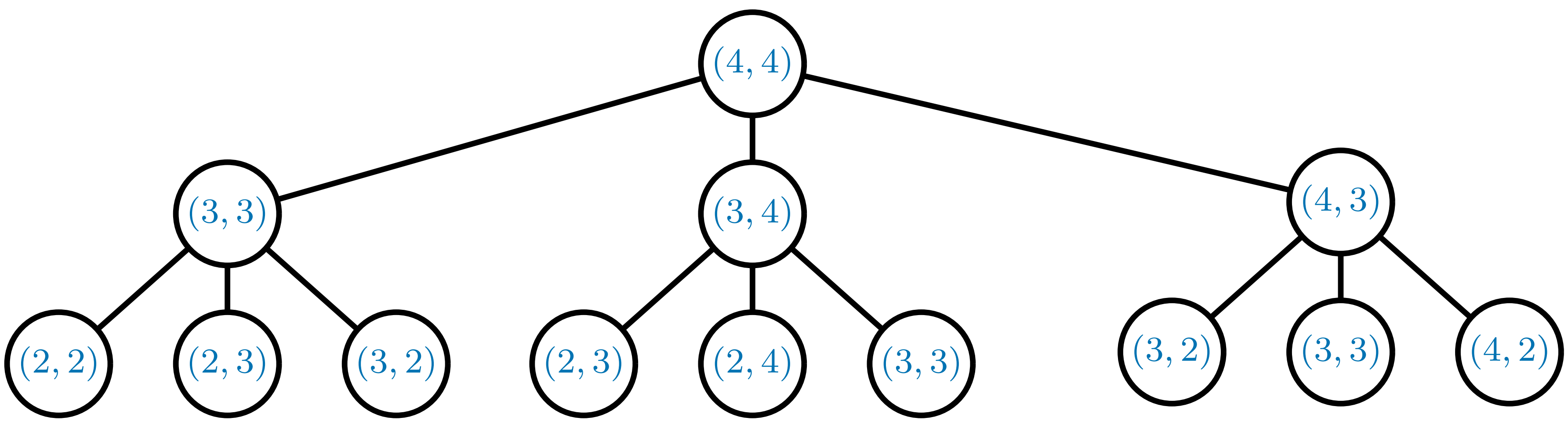

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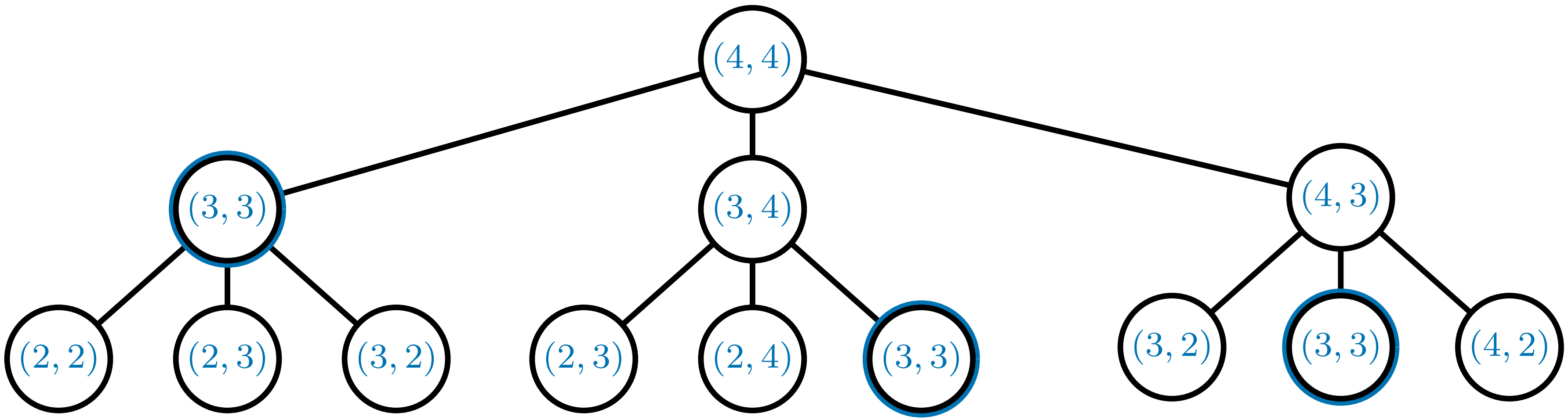

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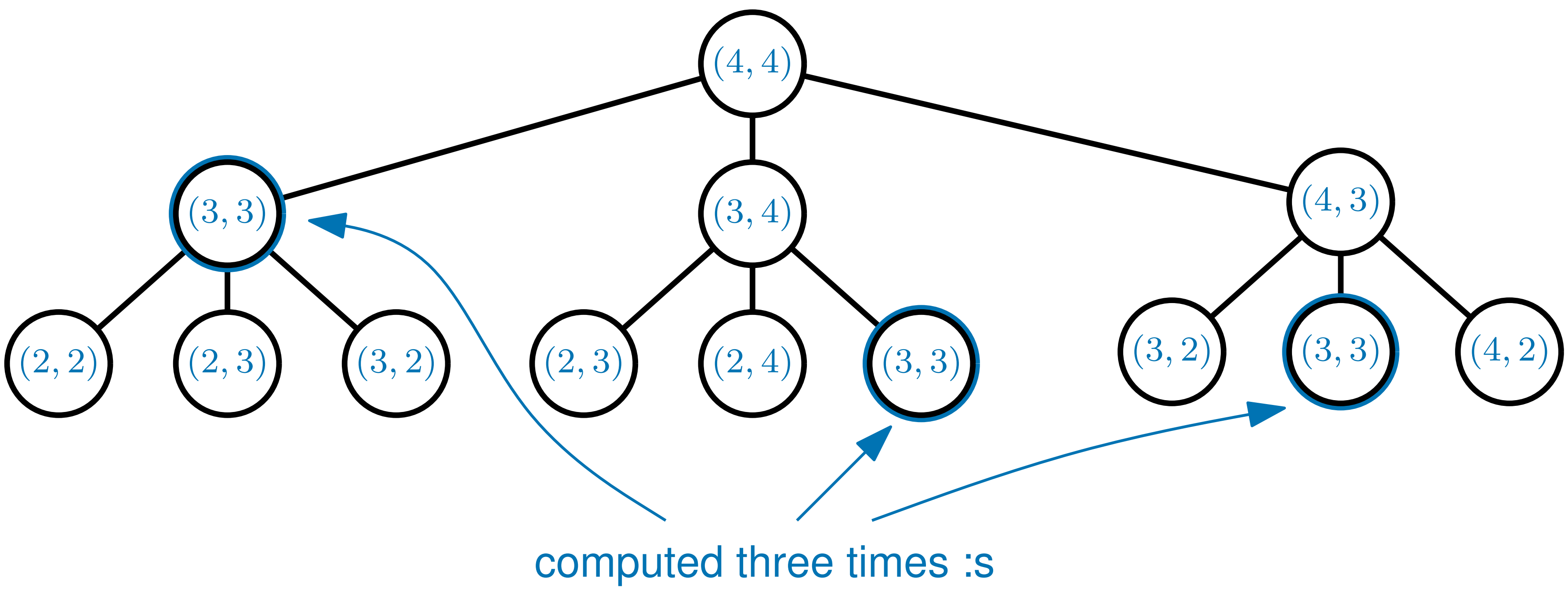

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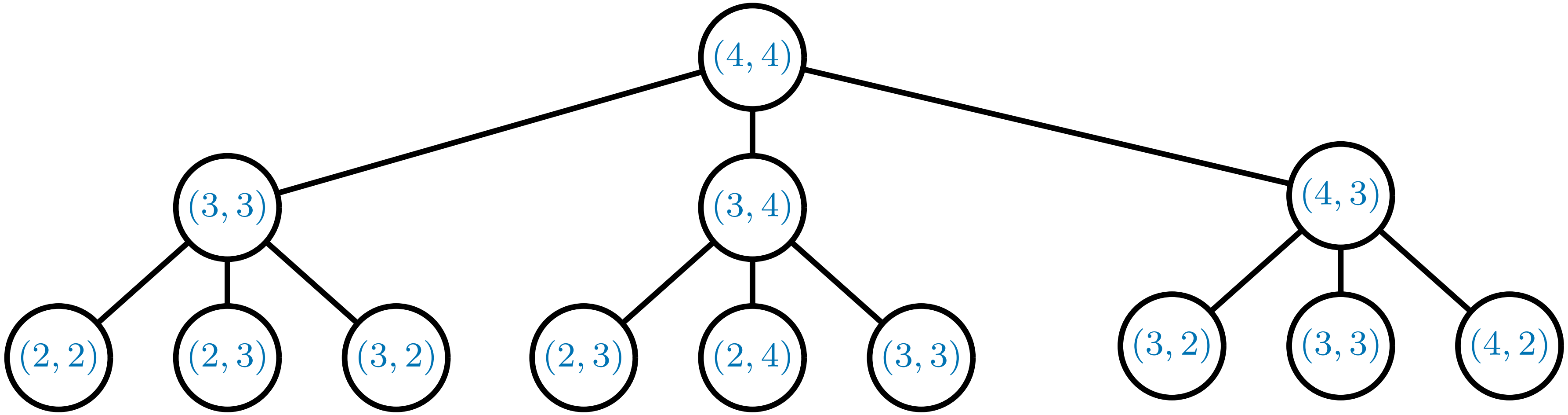

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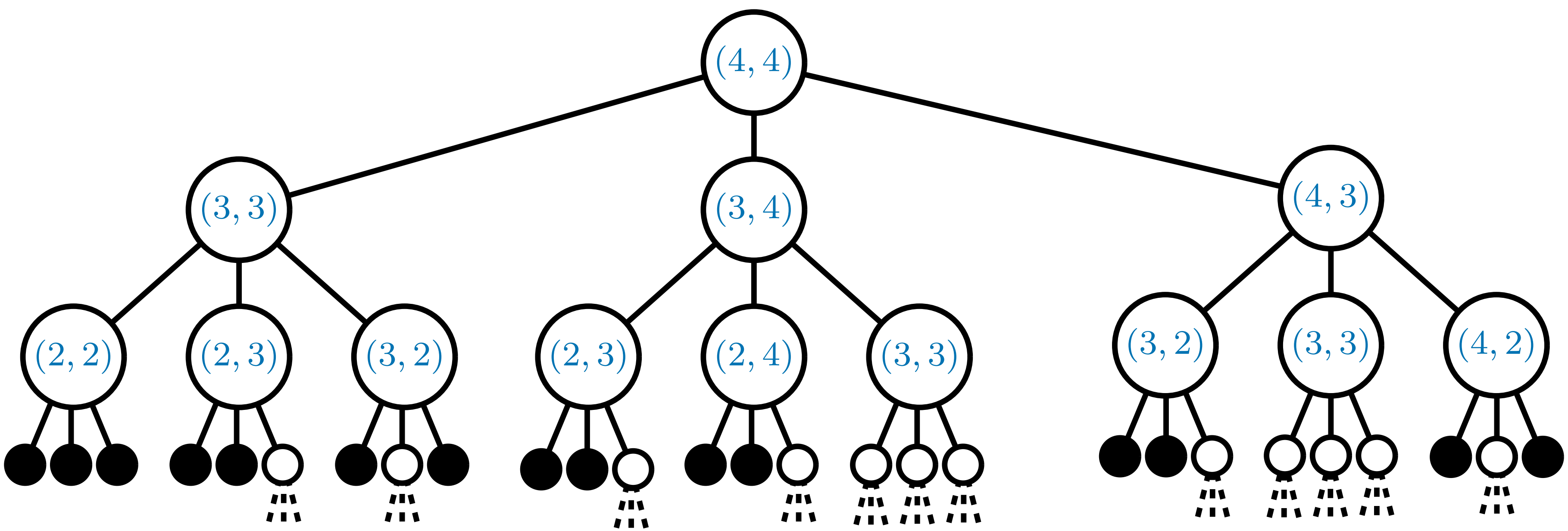

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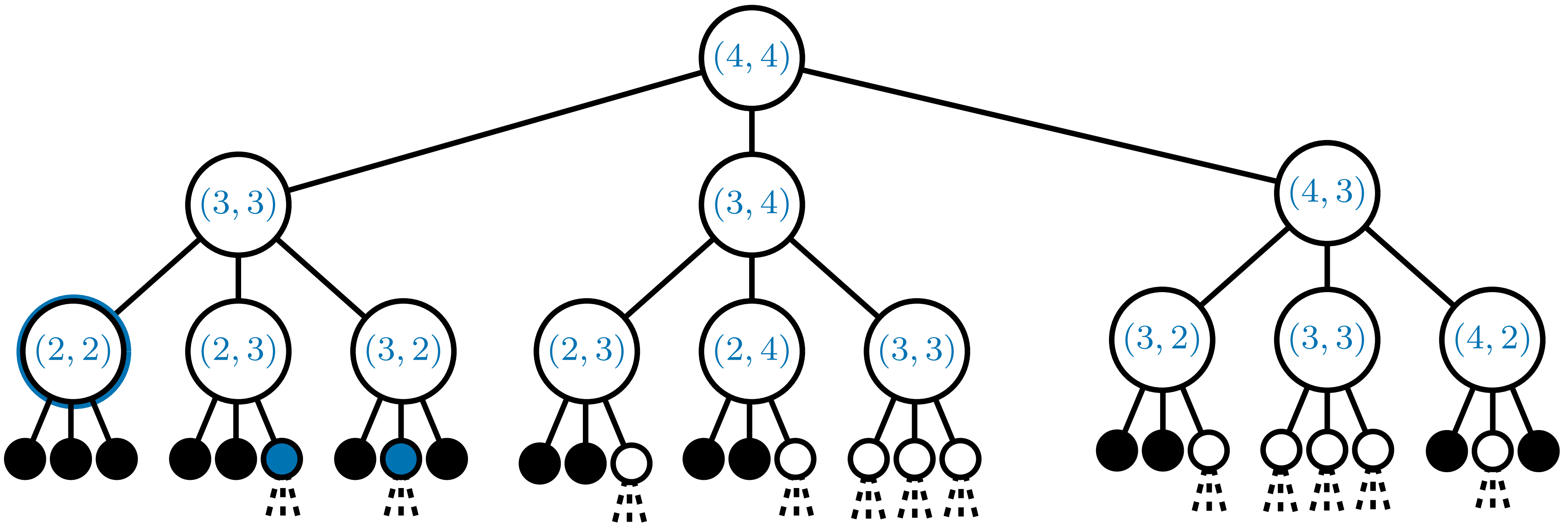

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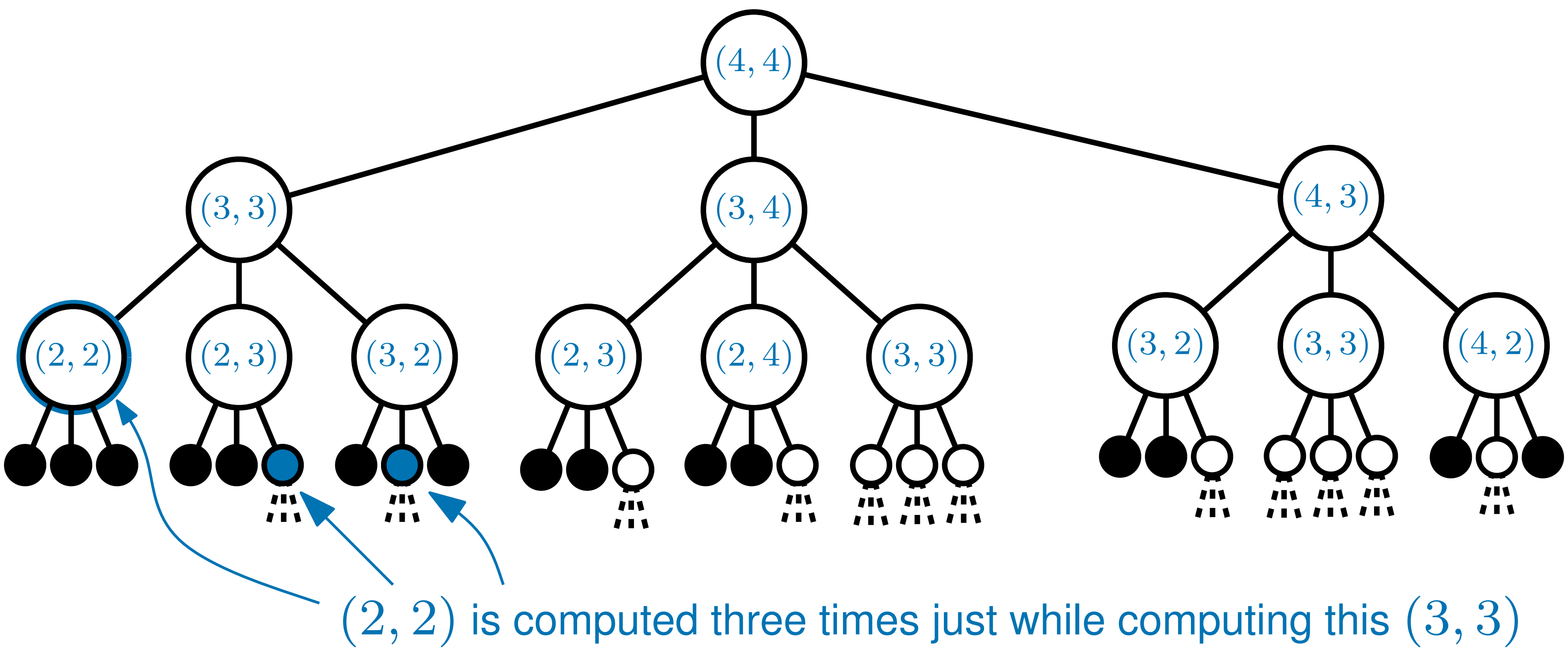

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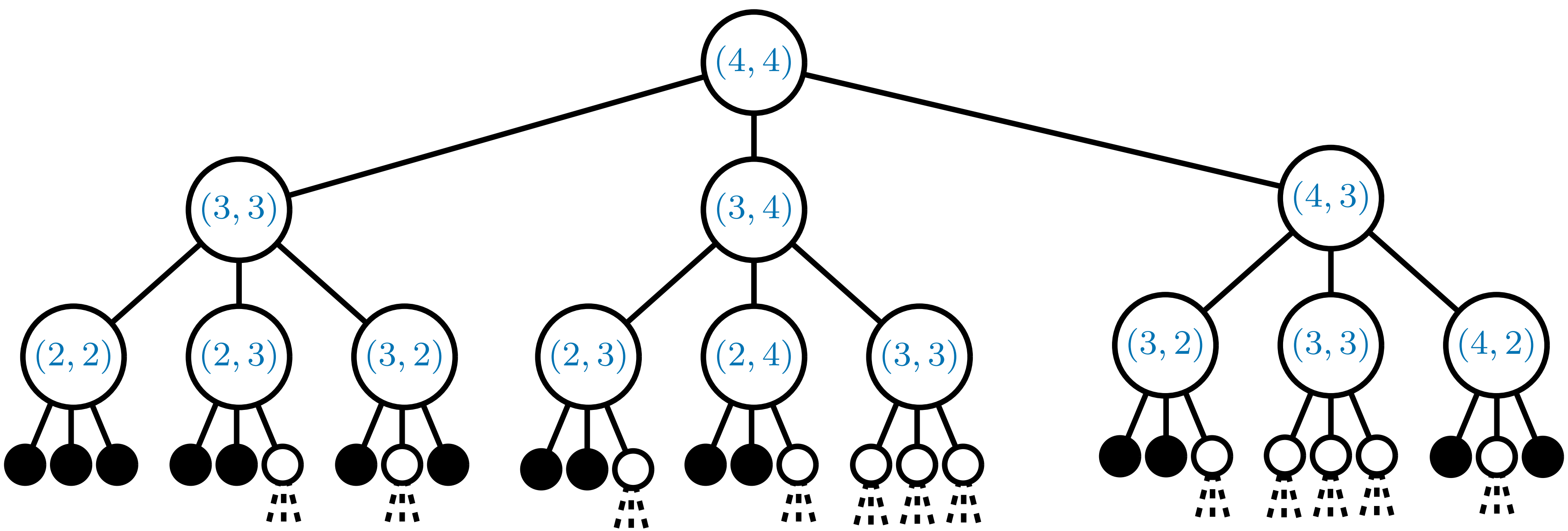

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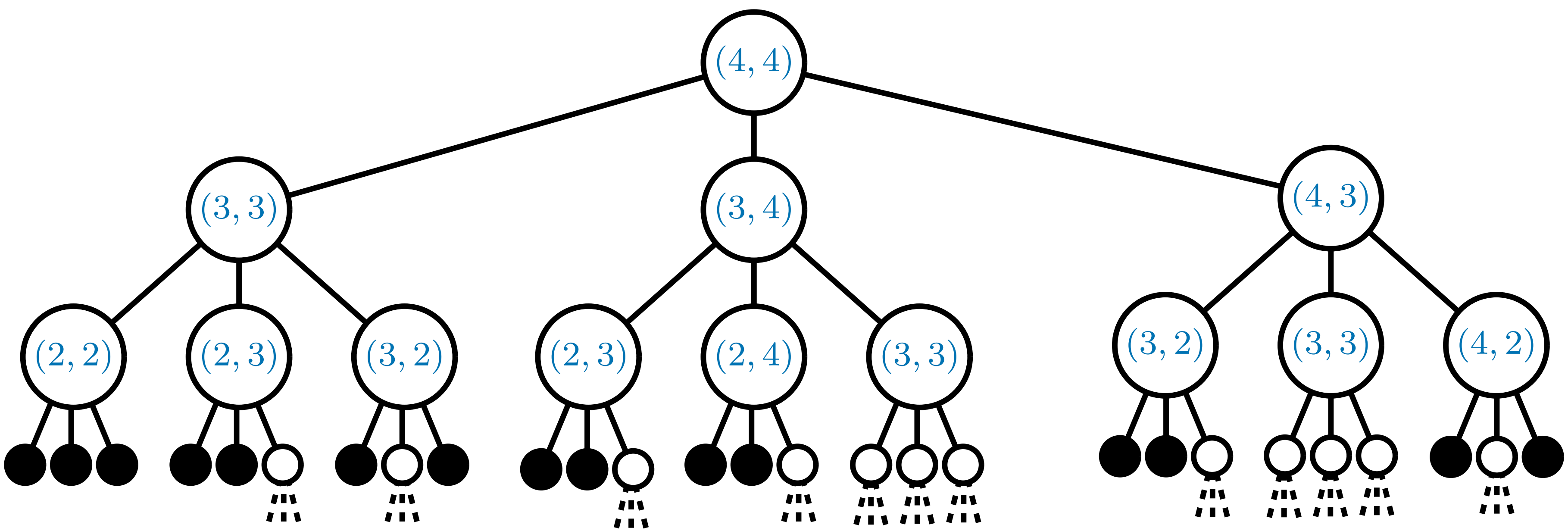

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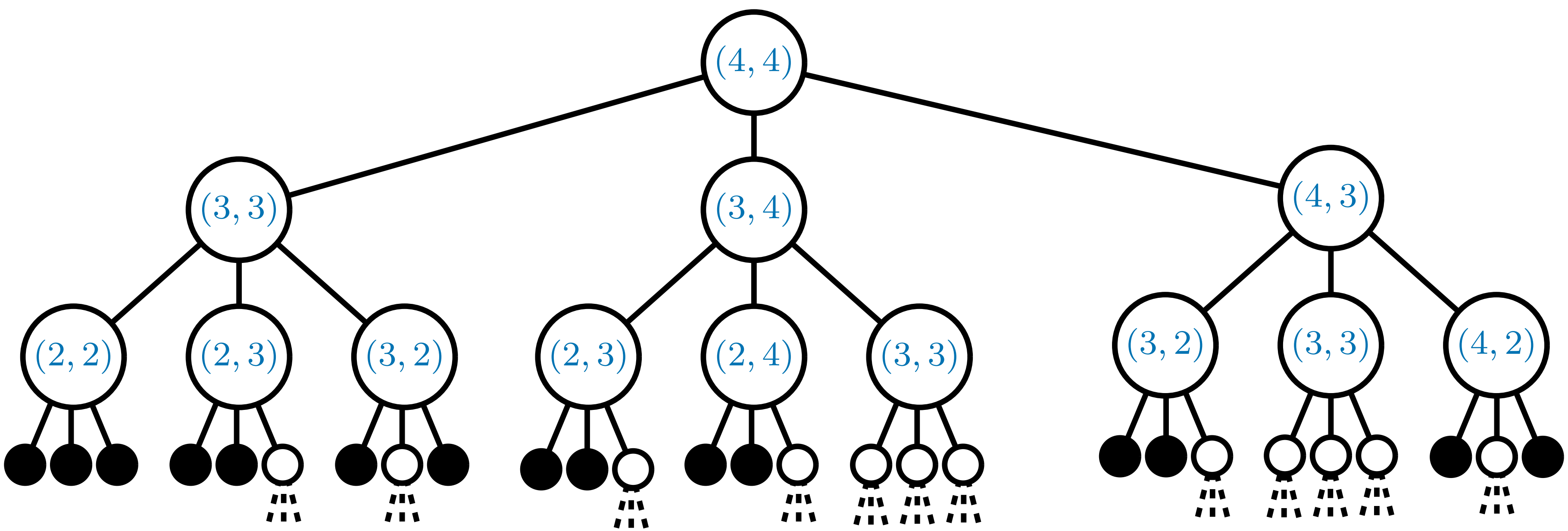

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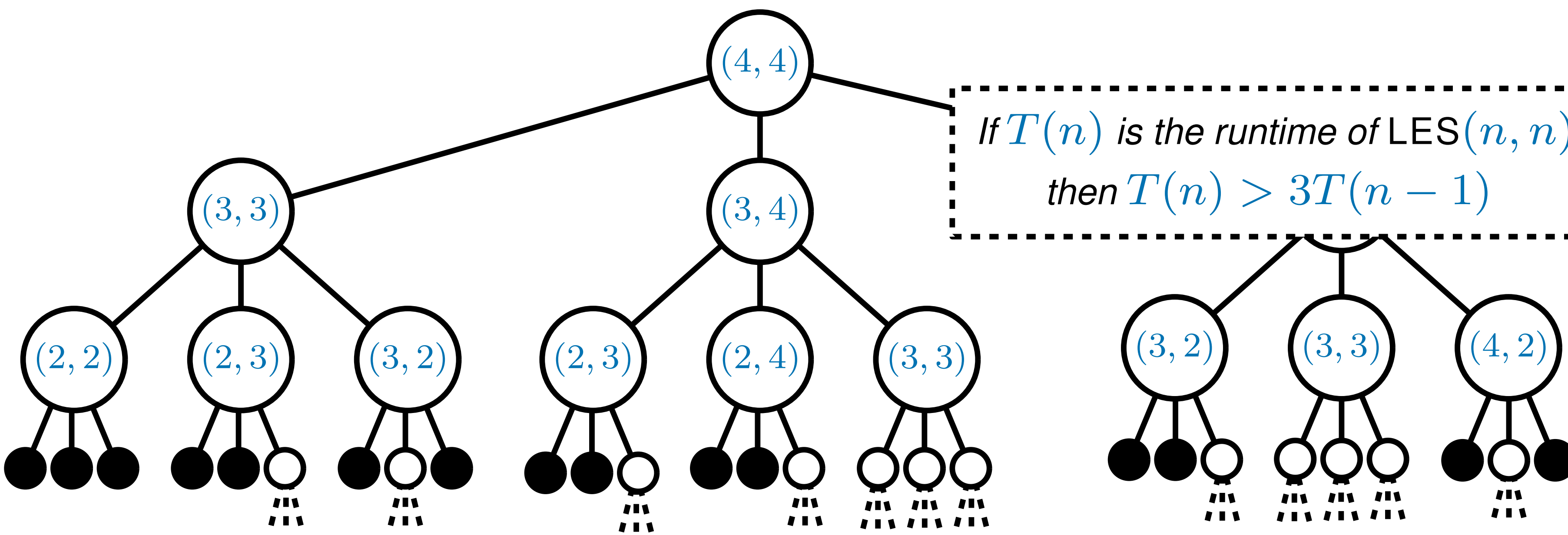

If $T(n)$ *is the runtime of* LES(n,n) *then* $T(n) > 3T(n-1)$

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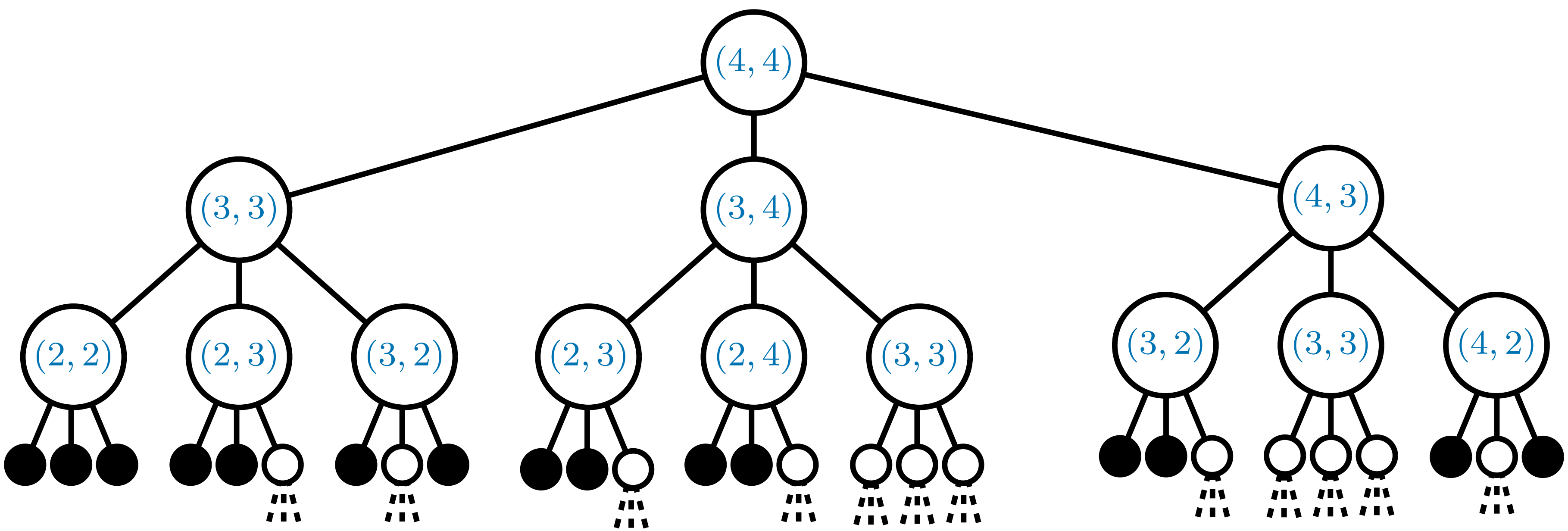

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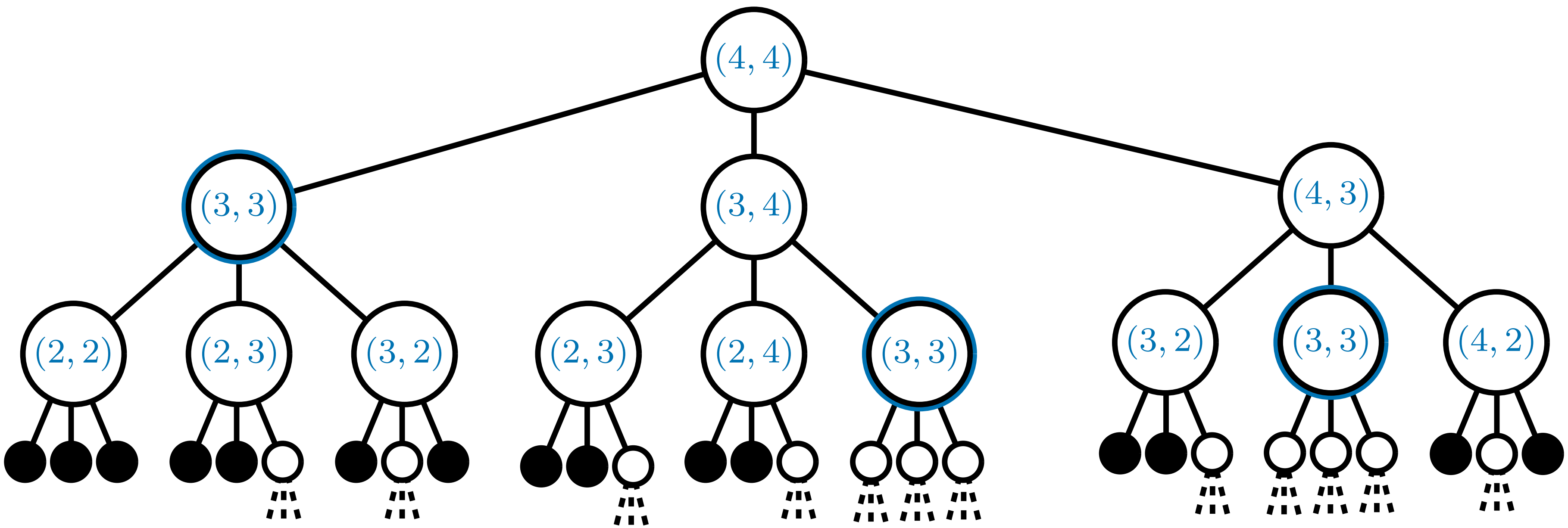

What should we do about all this repeated computation?

3. Store the solutions to subproblems

$\textsc{MemLES}(x, y)$

```
If pixel (x, y) is not empty
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If LES[x, y] undefined
  LES[x, y] = min (MemLES(x − 1, y − 1), MemLES(x − 1, y), MemLES(x, y − 1)) + 1
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In the $\textsc{MemLES}$ version of the algorithm
we store solutions to previously computed subproblems
in an $(n \times n)$ 2D array called LES

3. Store the solutions to subproblems

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This is called memoization *(not memorization)*

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The time complexity of computing $\mathrm{MEMLES}(n, n)$ is now $O(n^2)$

(in fact, computing $\max_{x,y} \mathrm{MEMLES}(x, y)$ takes $O(n^2)$ time too)

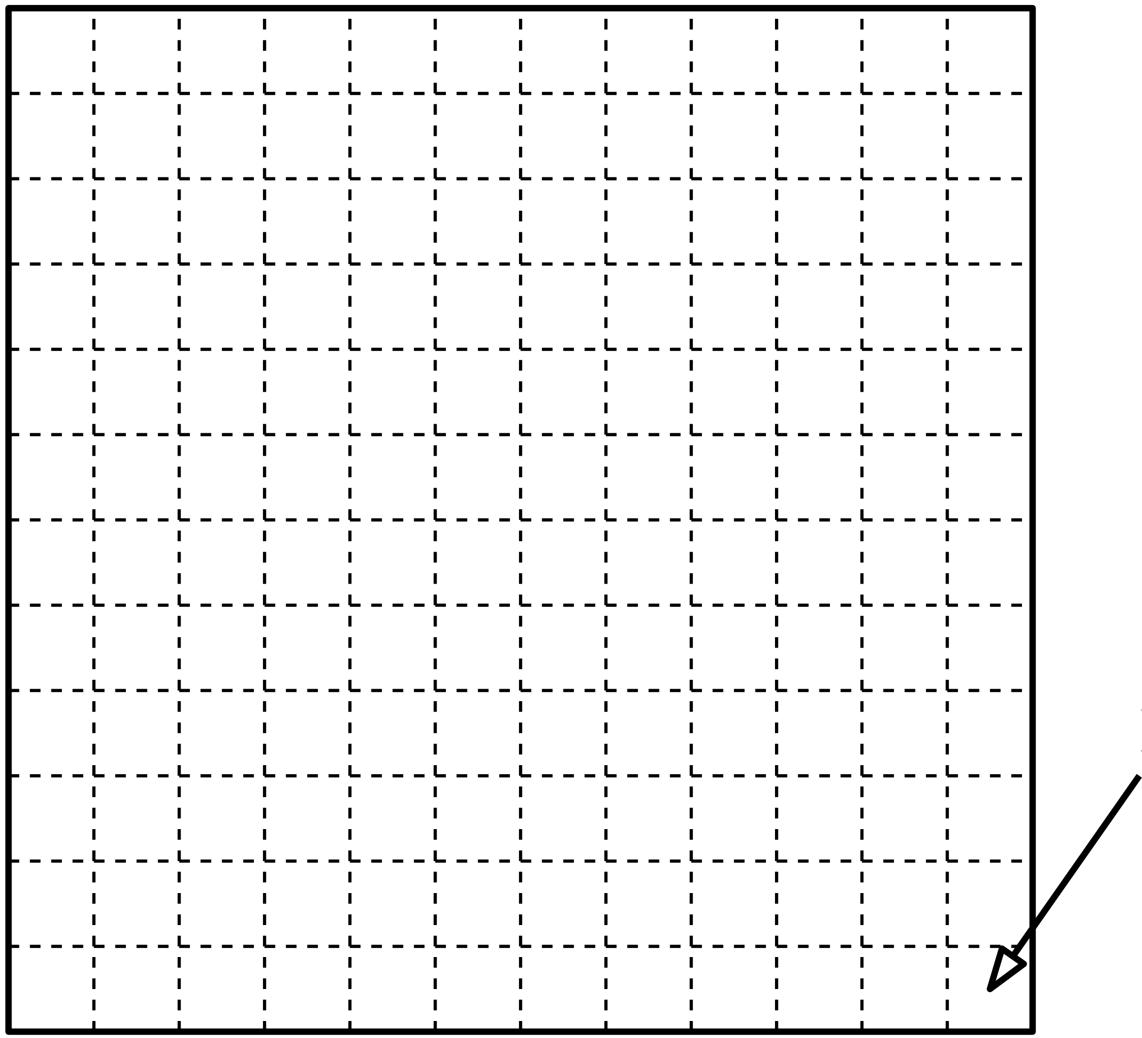
The dependency graph

$$\mathrm{LES}[x,y] = \min\left(\textsc{MemLES}(x-1,y-1), \textsc{MemLES}(x-1,y), \textsc{MemLES}(x,y-1)\right)+1$$

(for $x, y > 1$ and (x, y) non empty)

The 2D array

$\mathrm{LES}:$



$\mathrm{LES}[n,n]$

What information do we need to compute $\mathrm{LES}[n,n]$?

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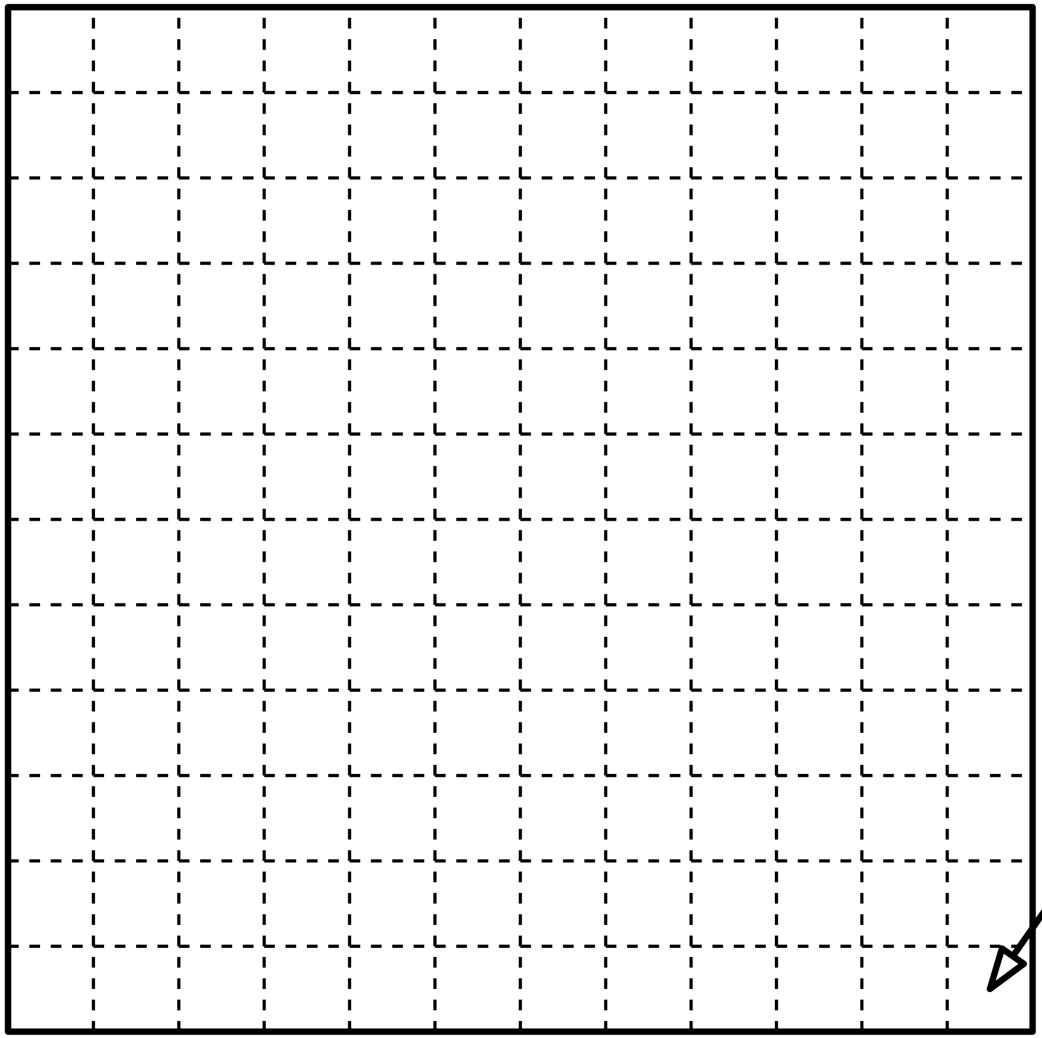

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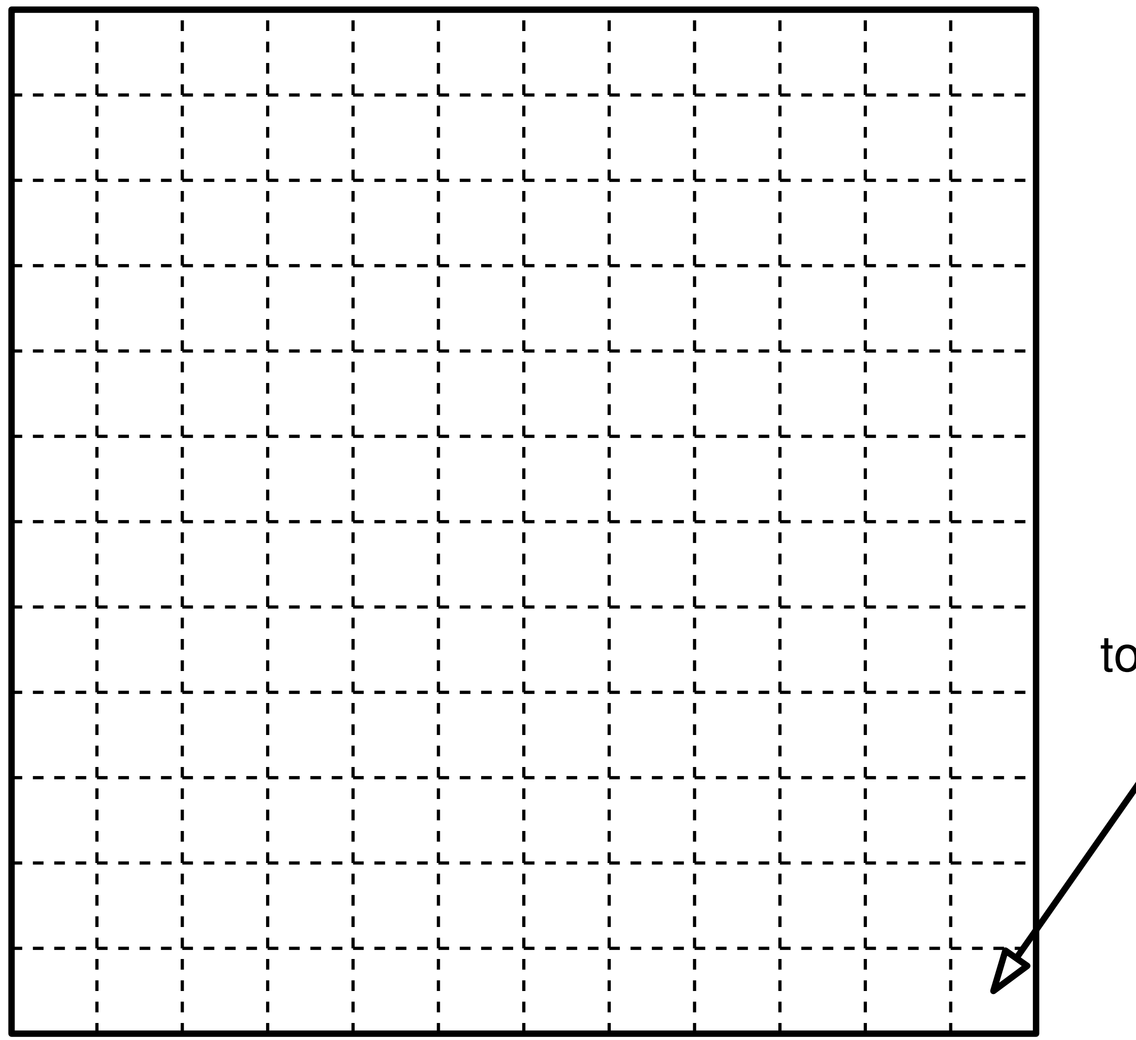

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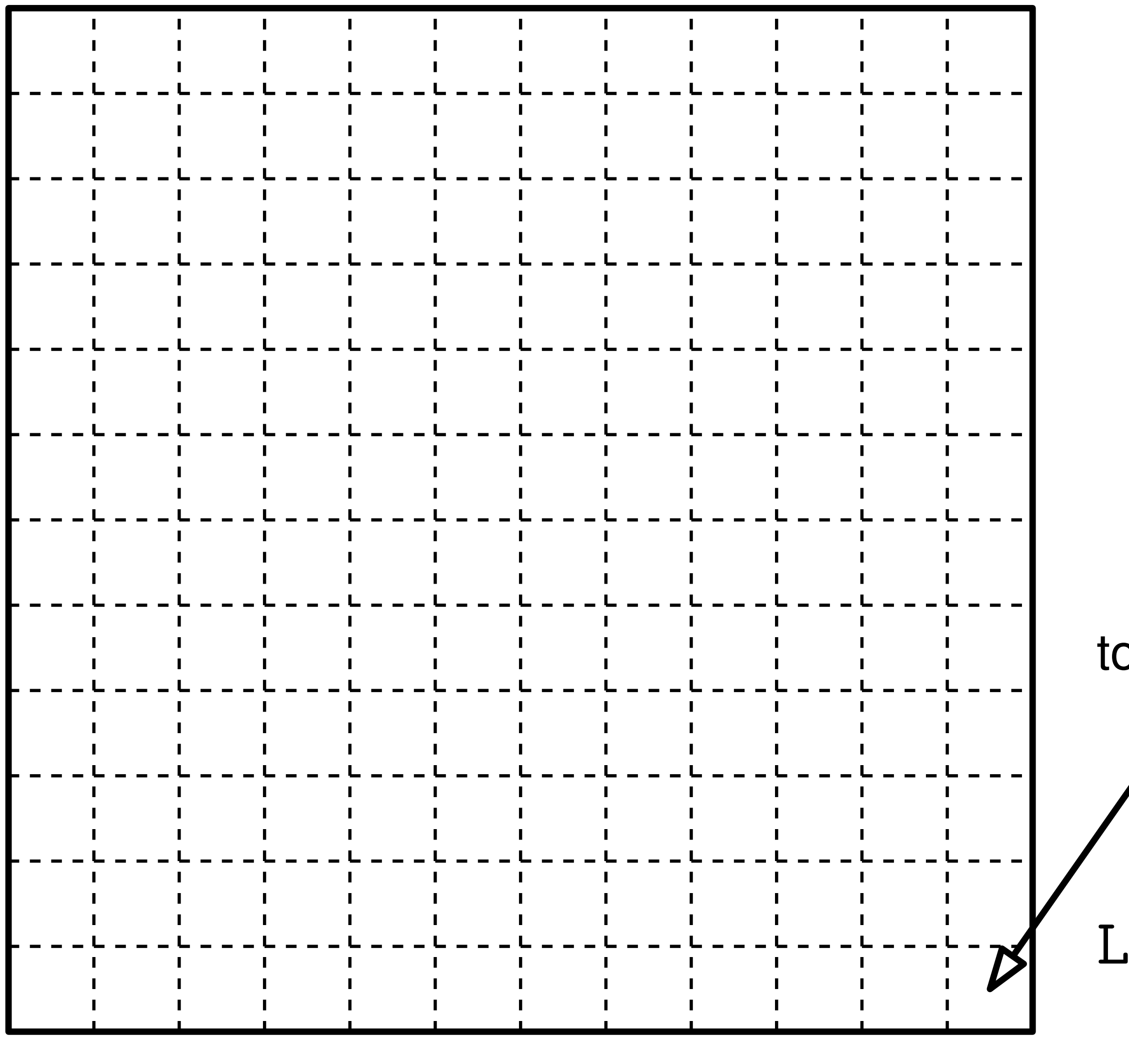

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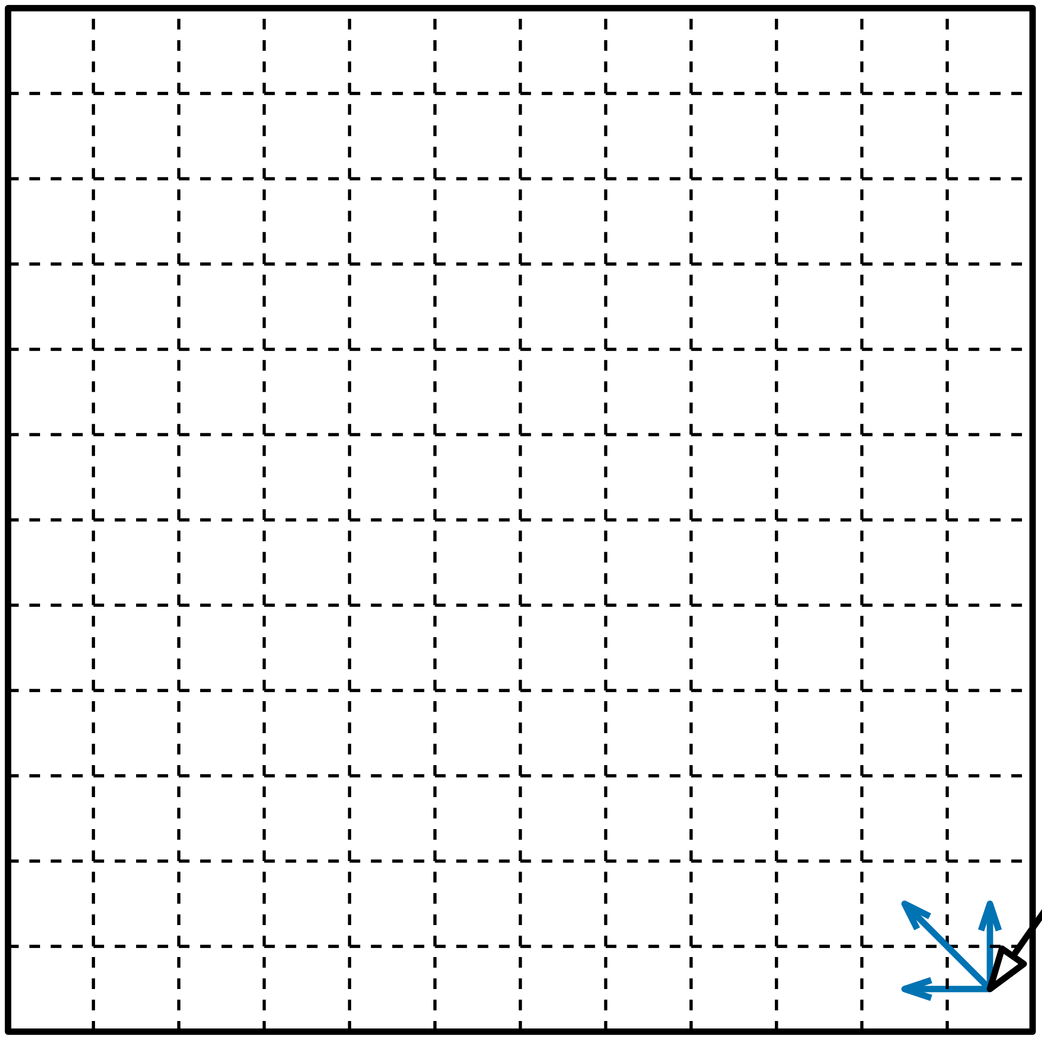

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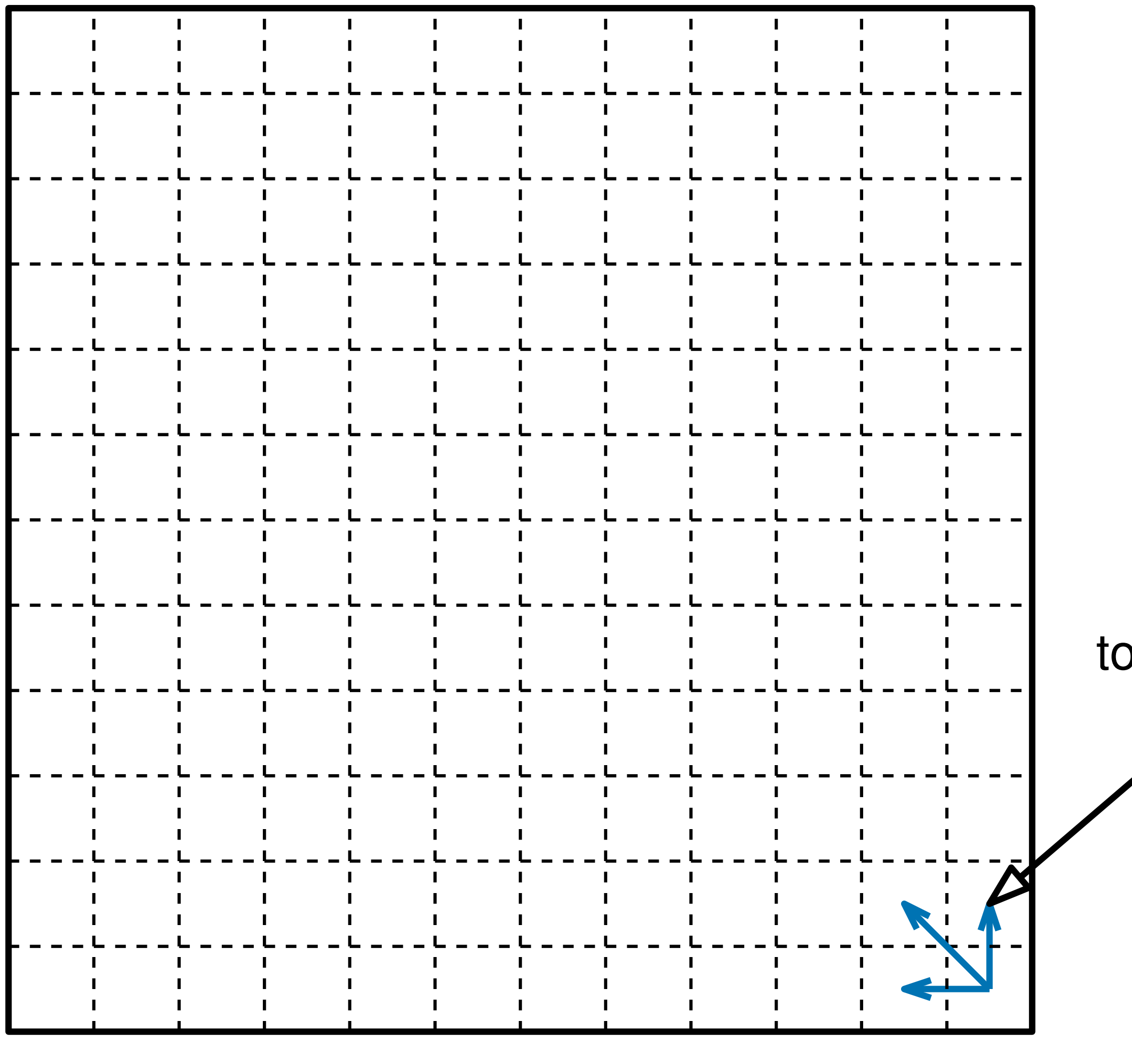

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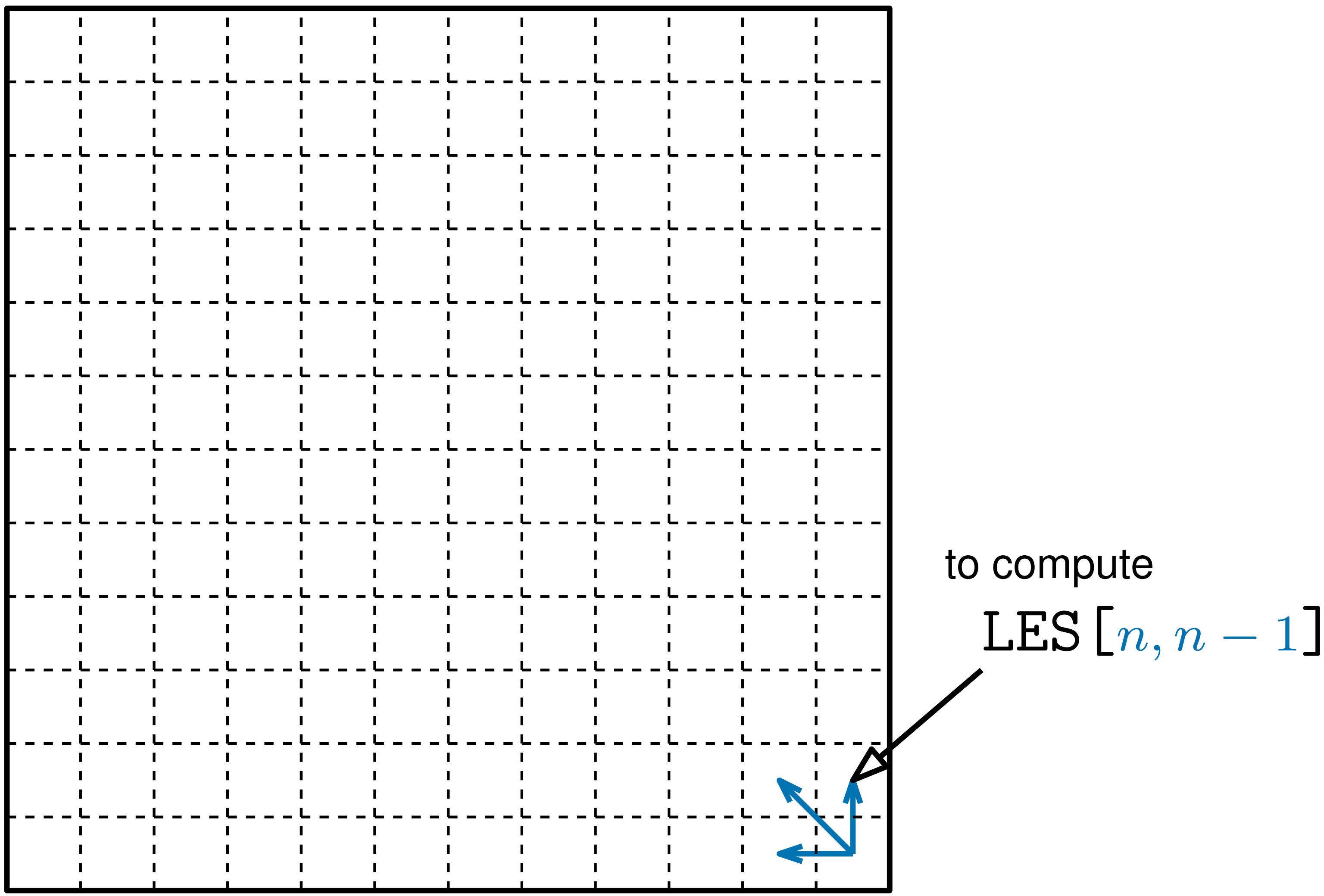

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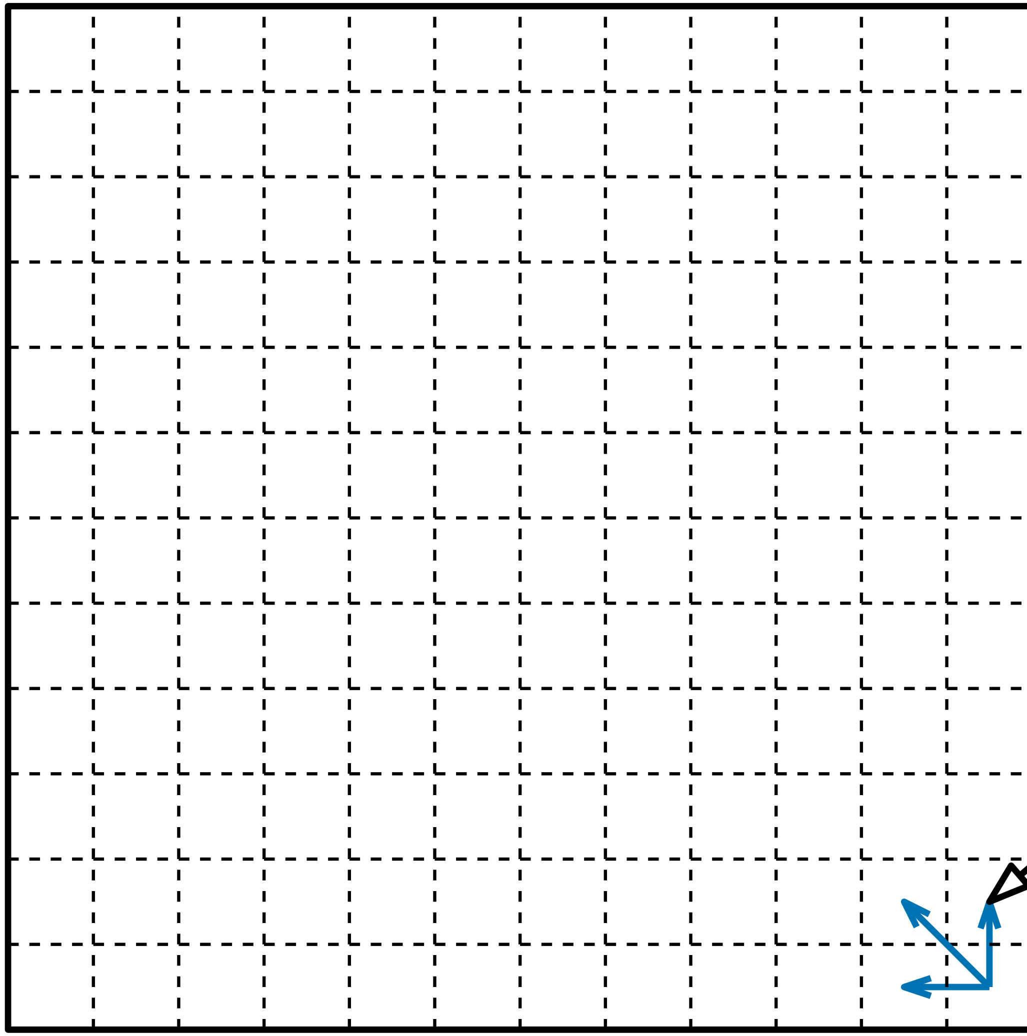

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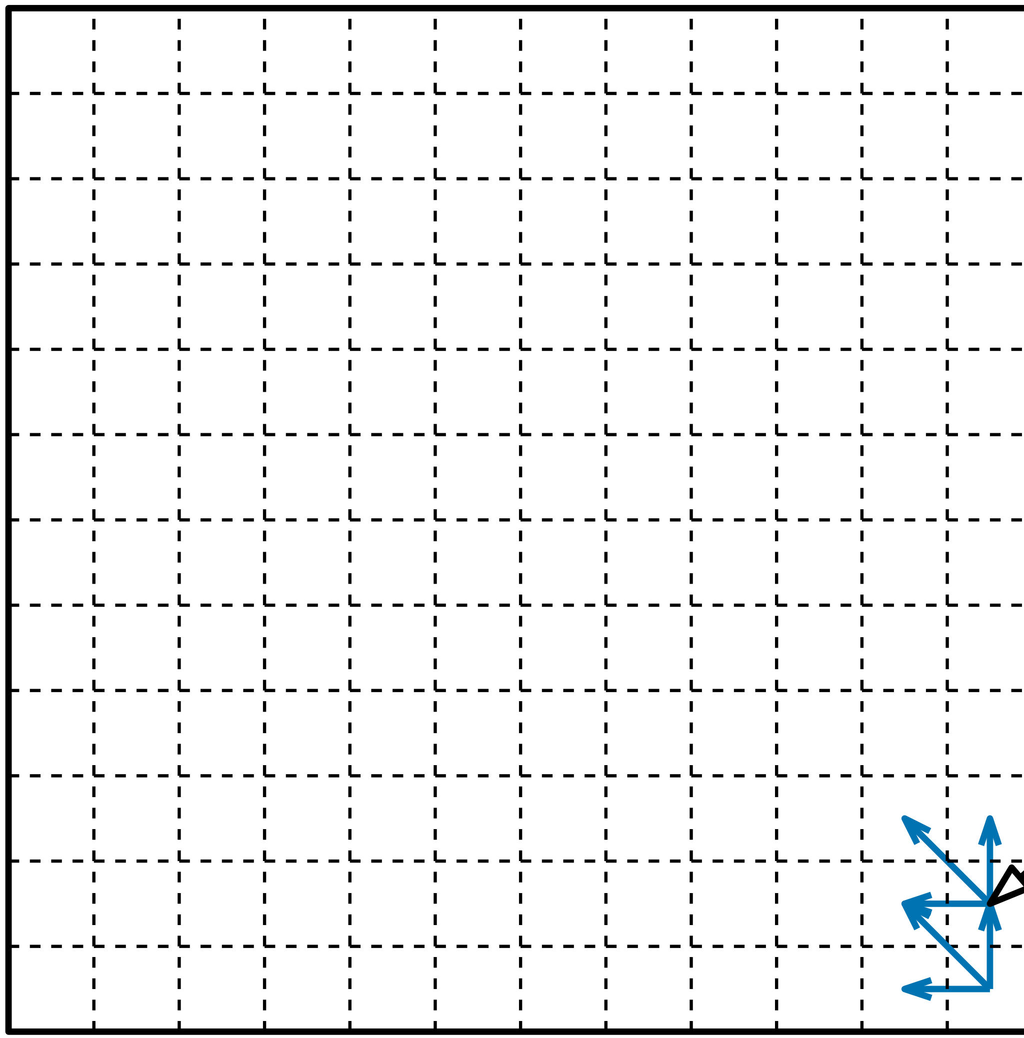

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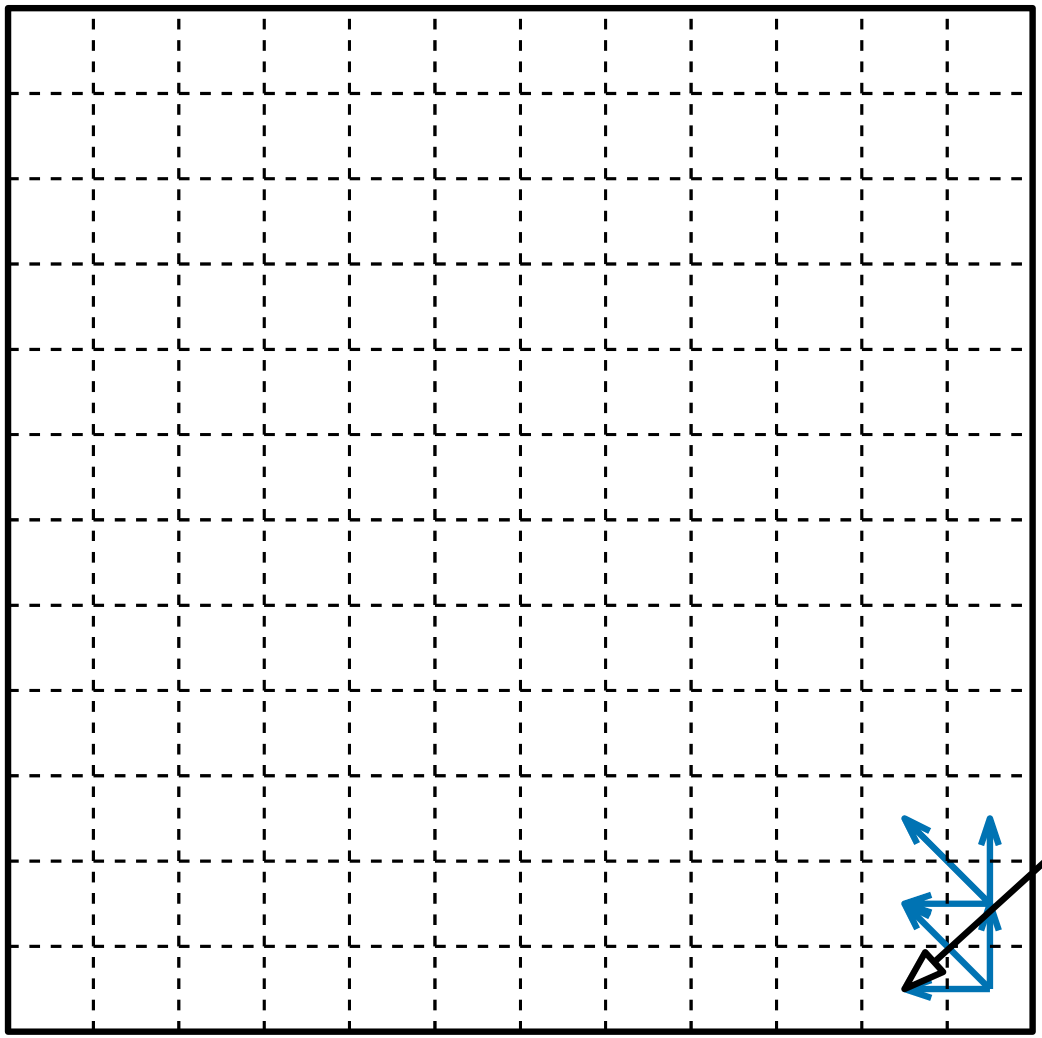

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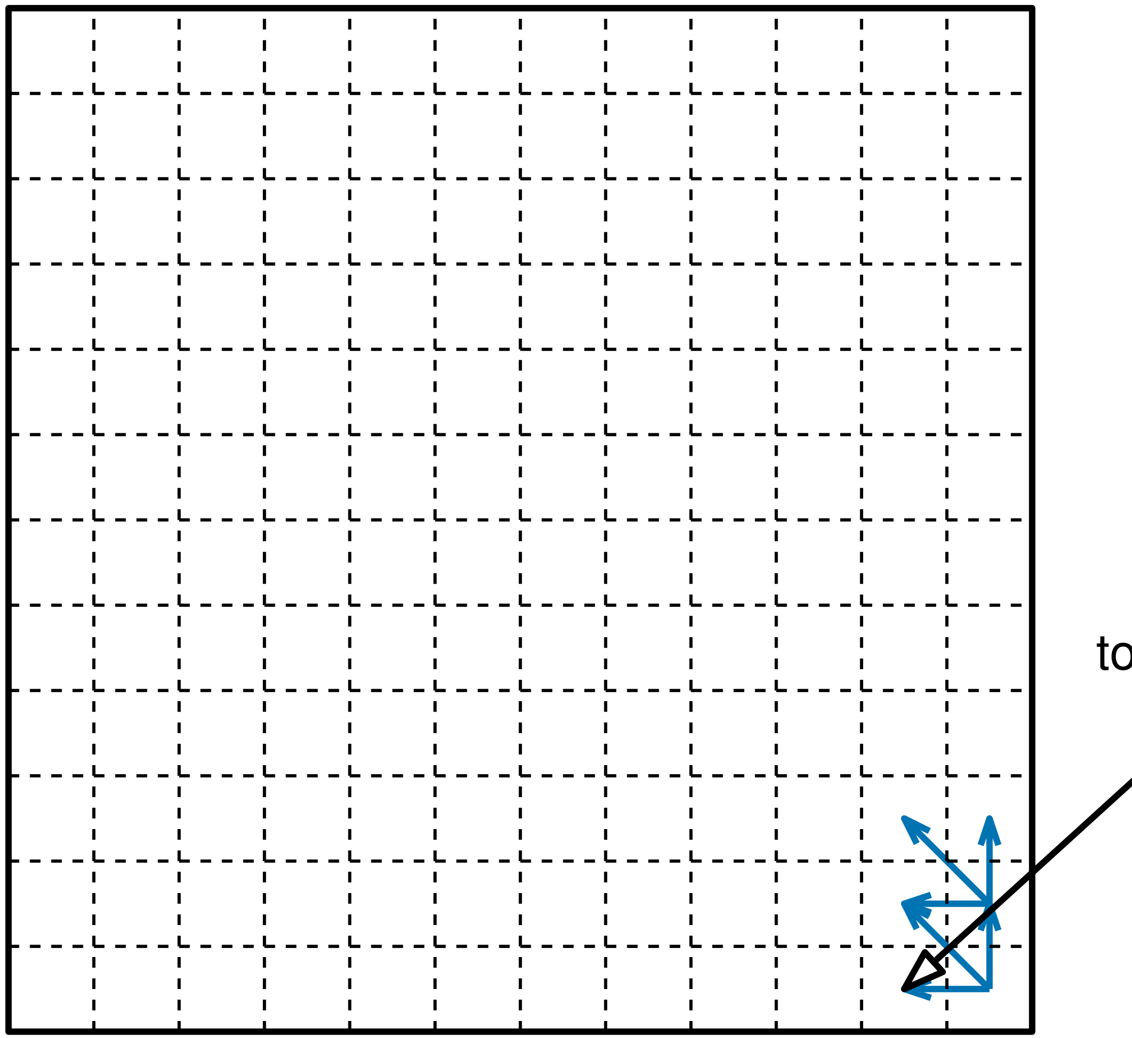

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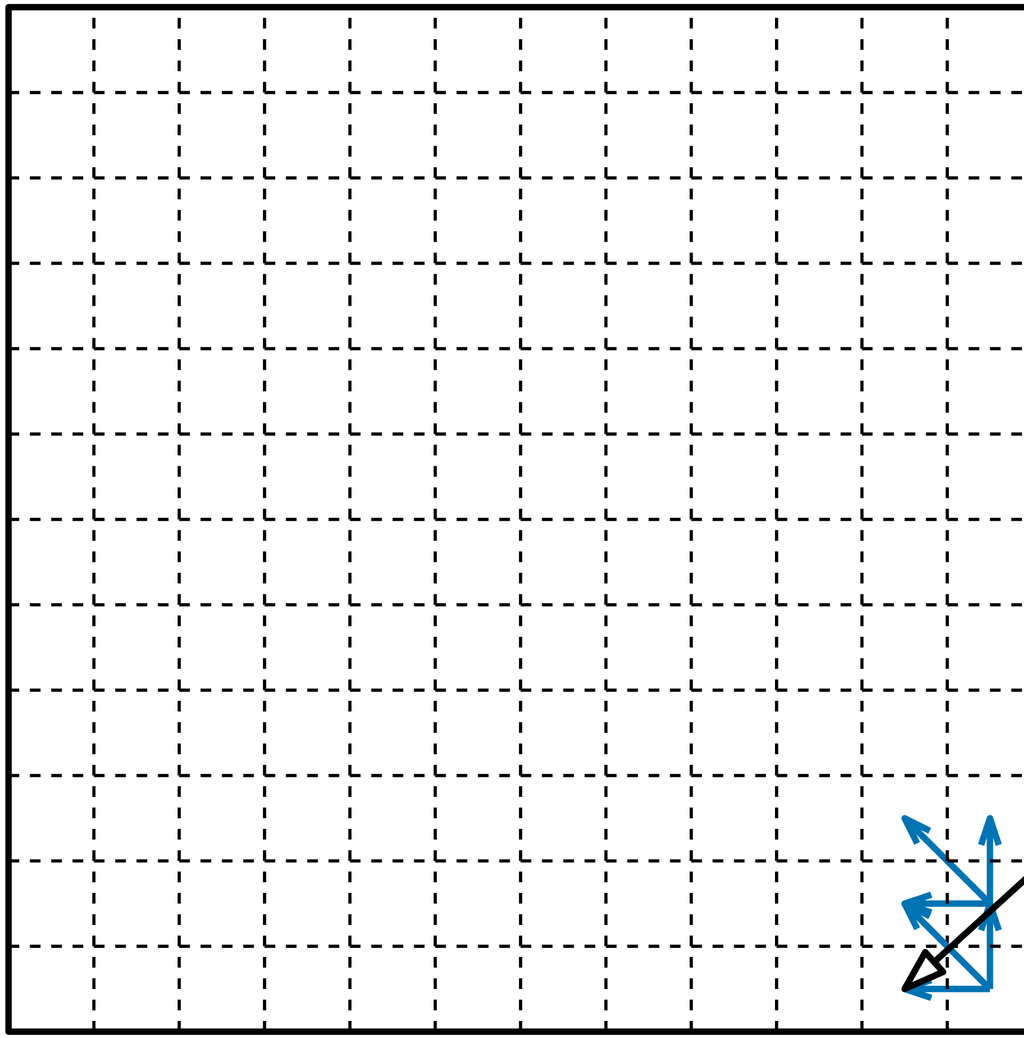

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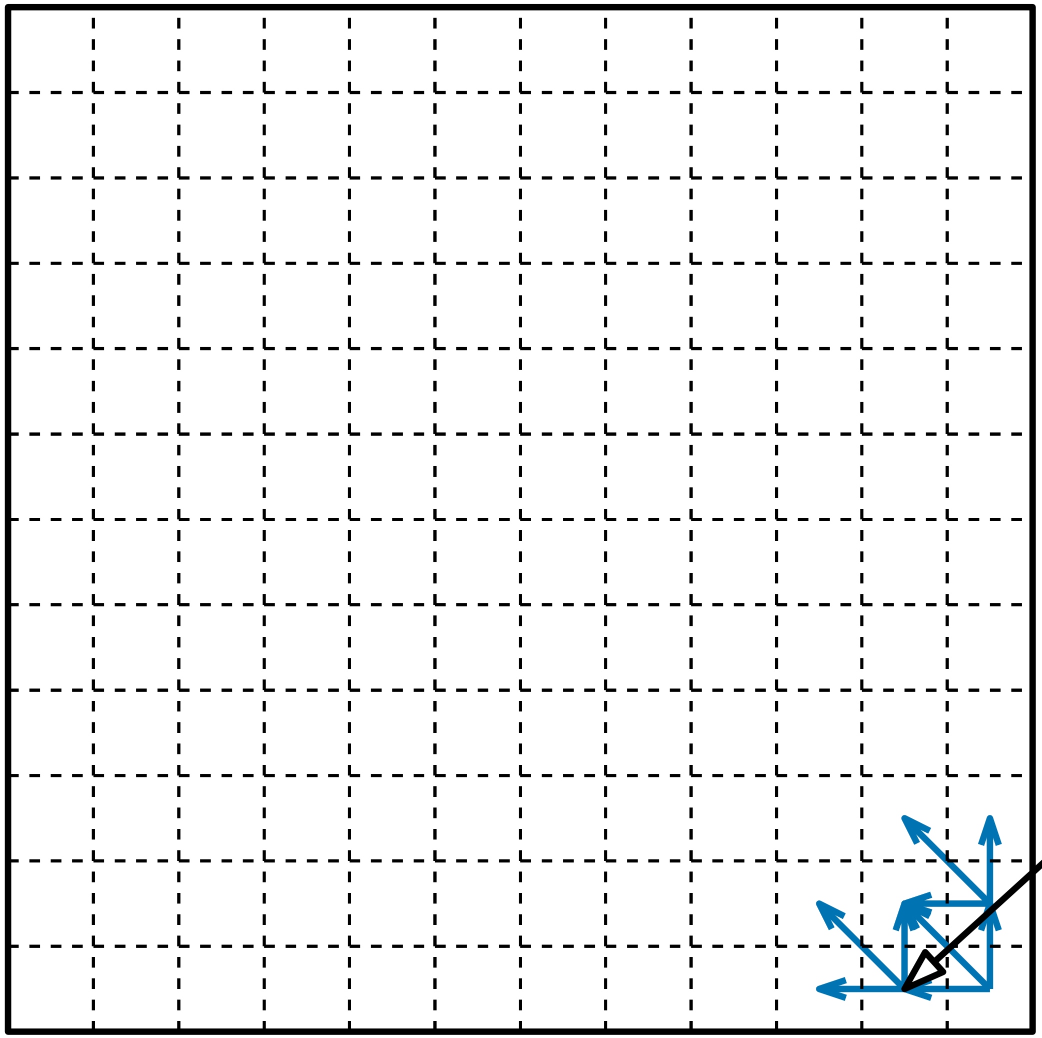

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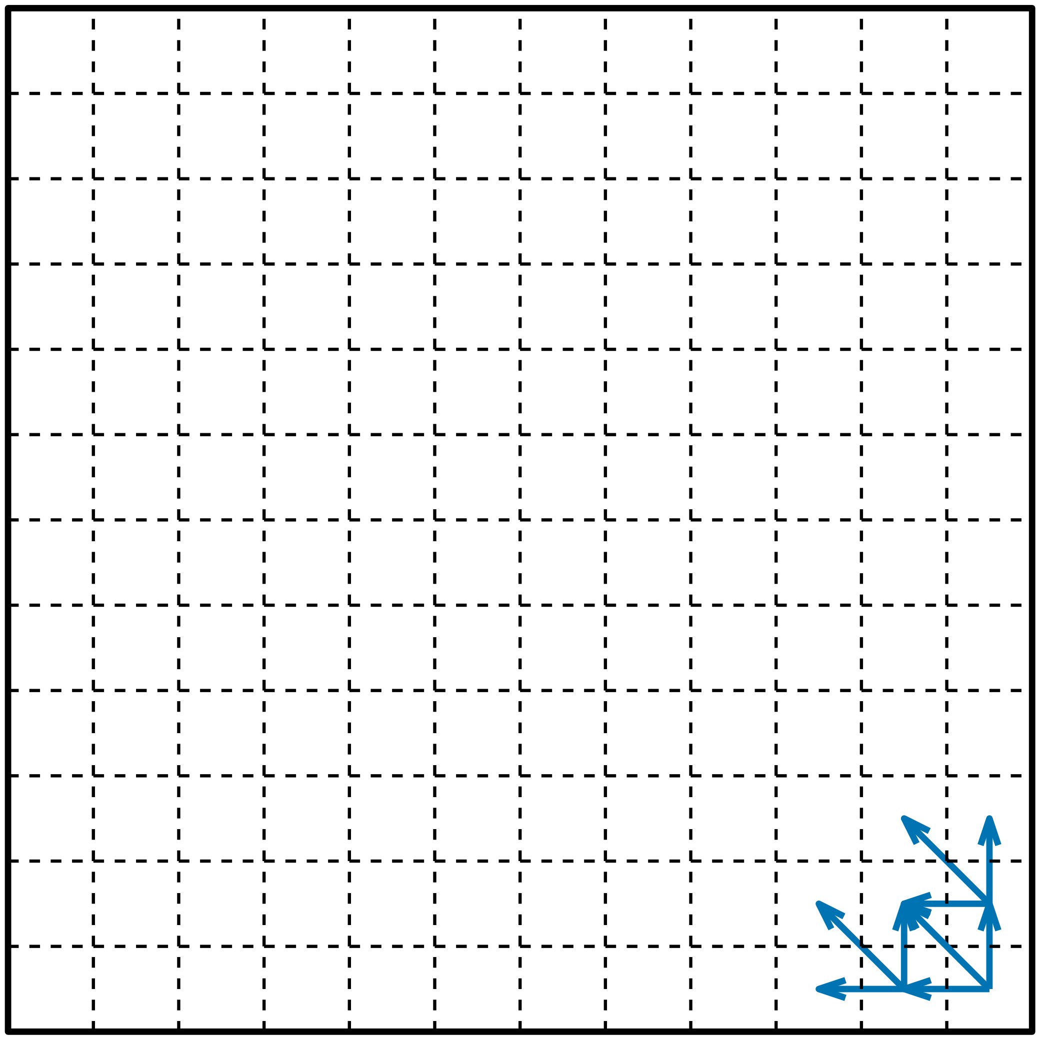

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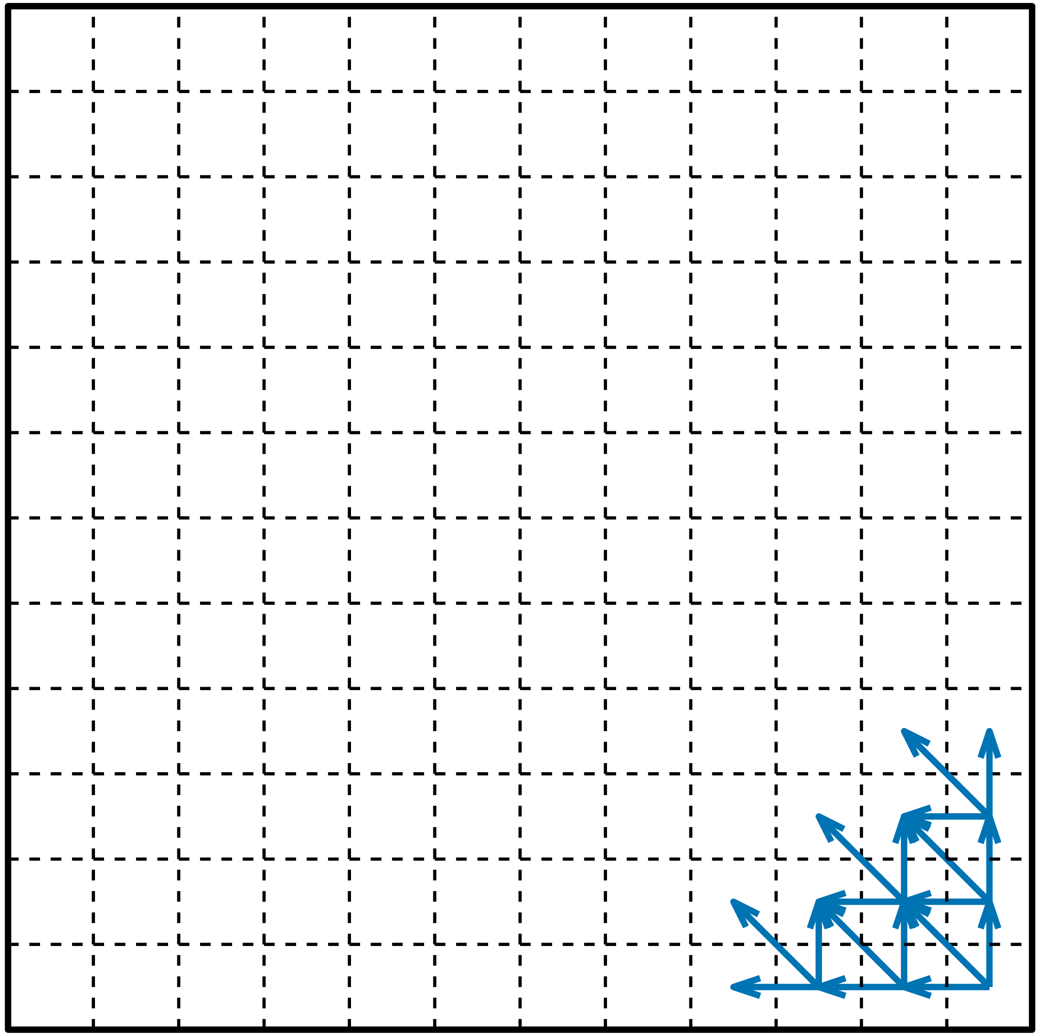

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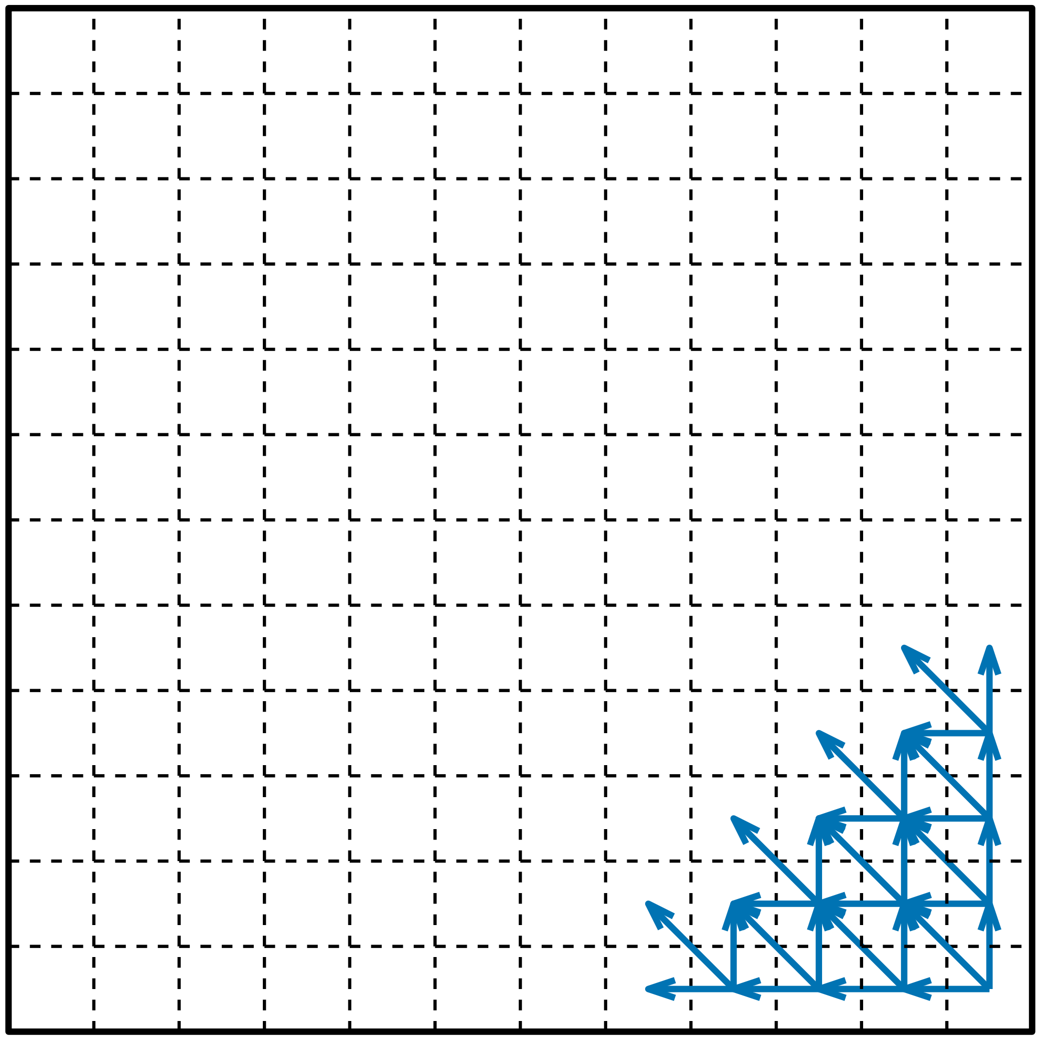

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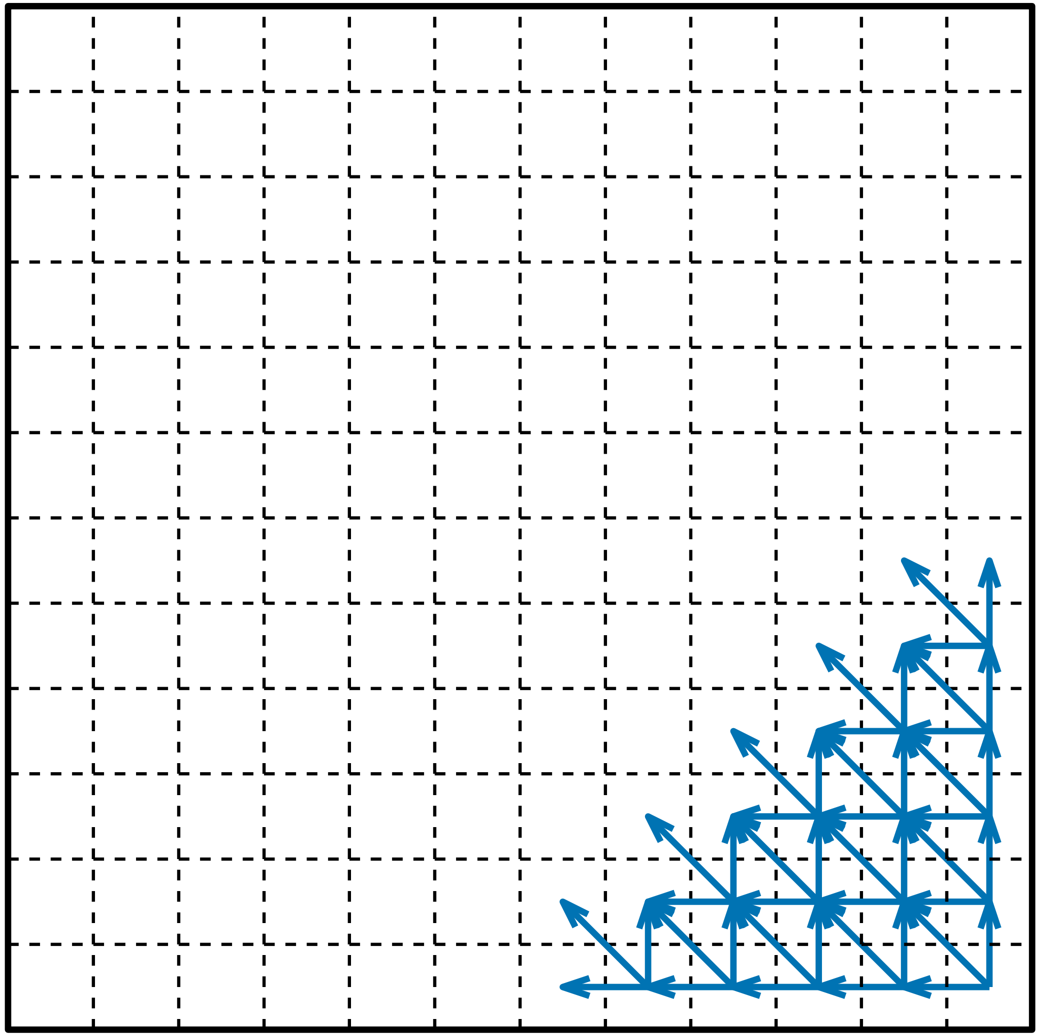

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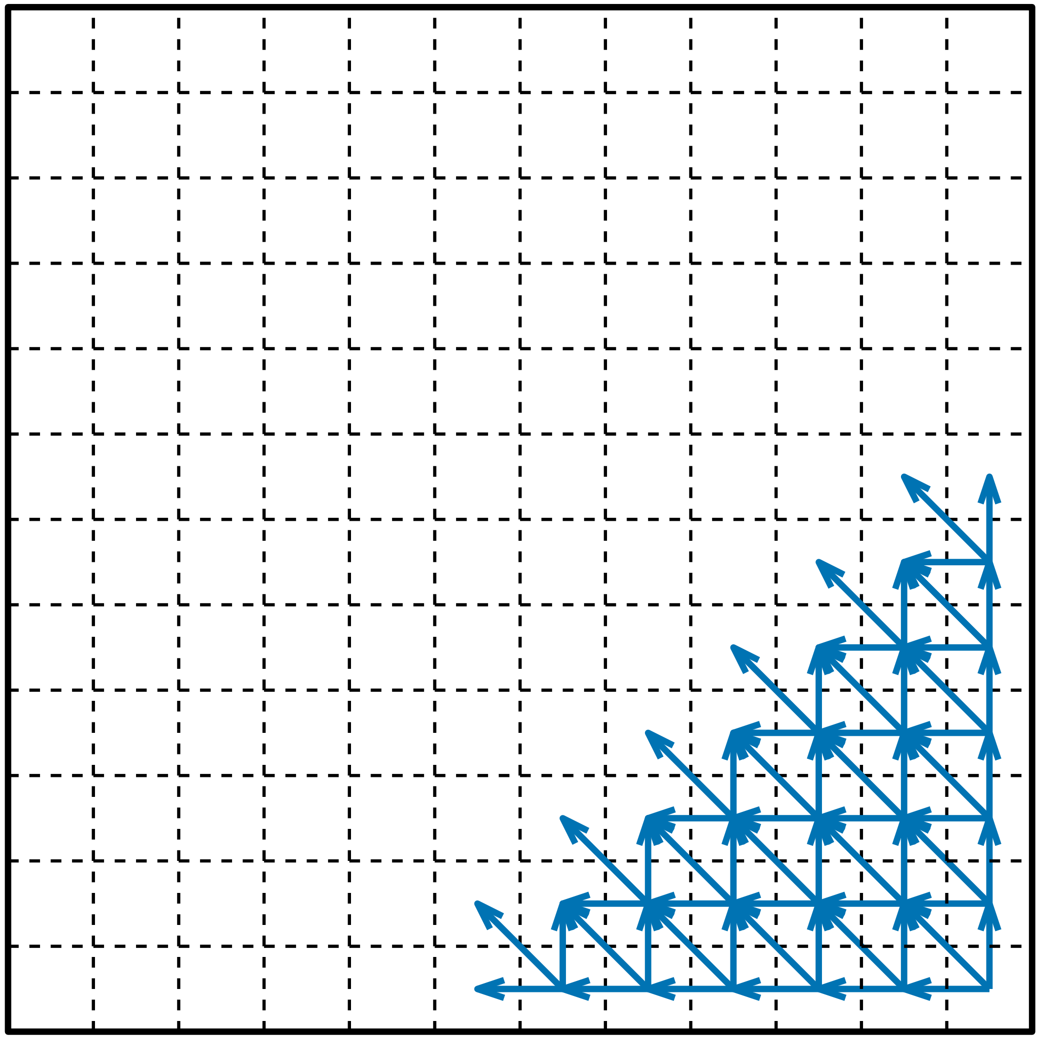

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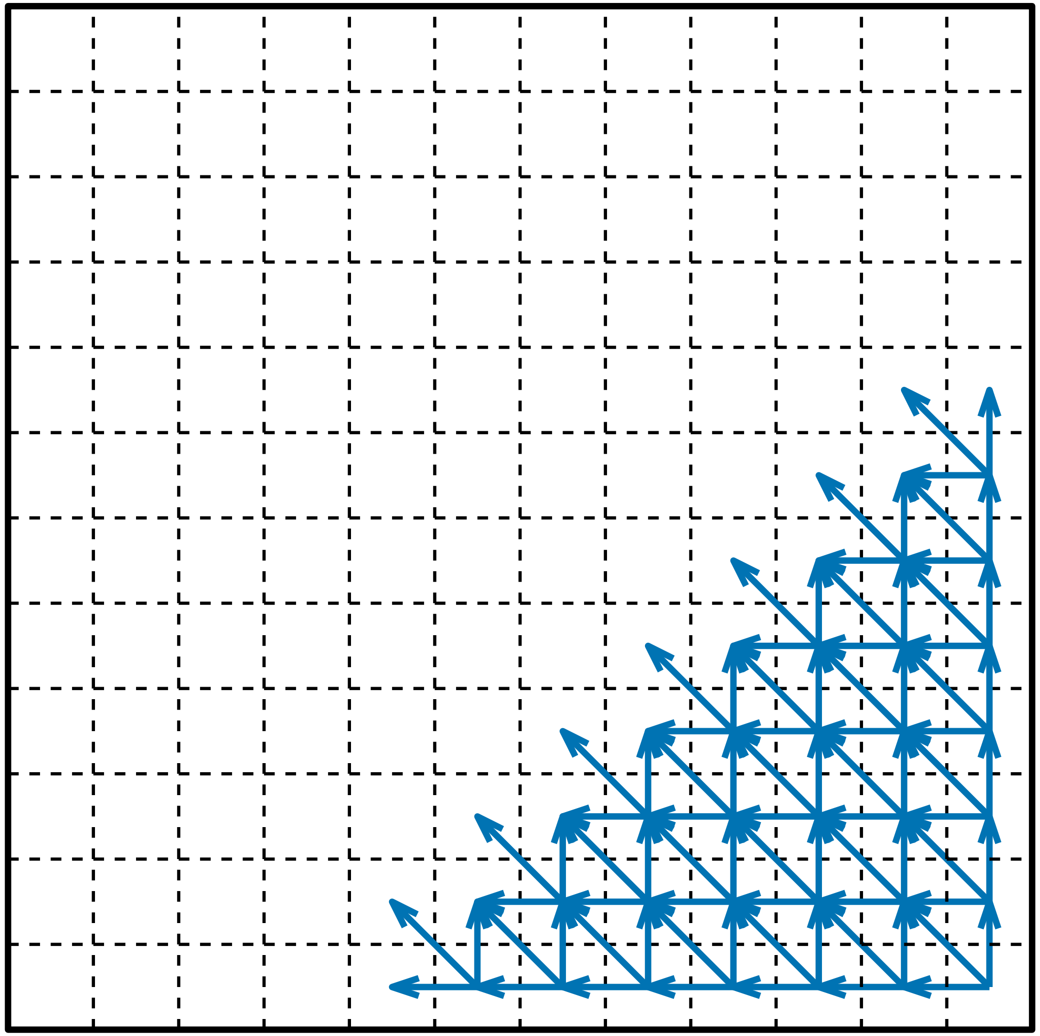

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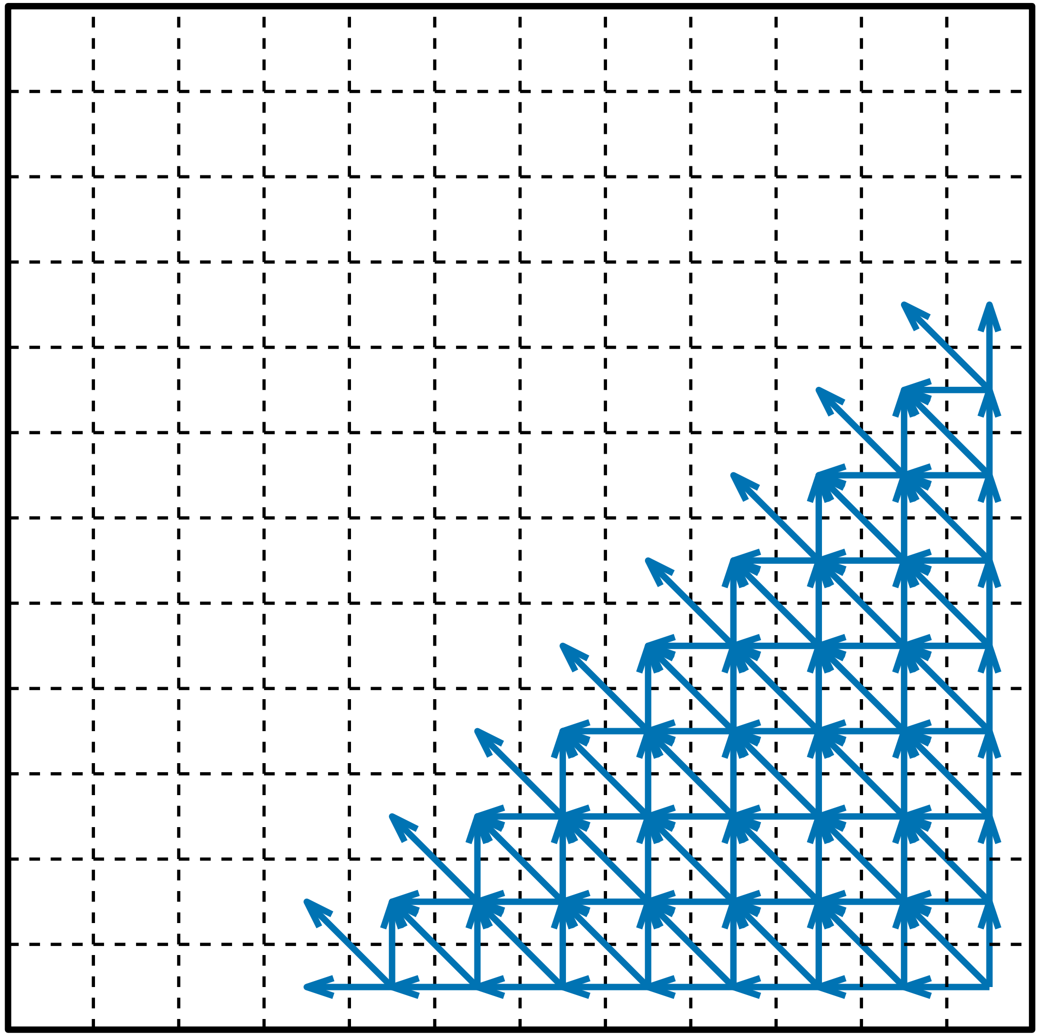

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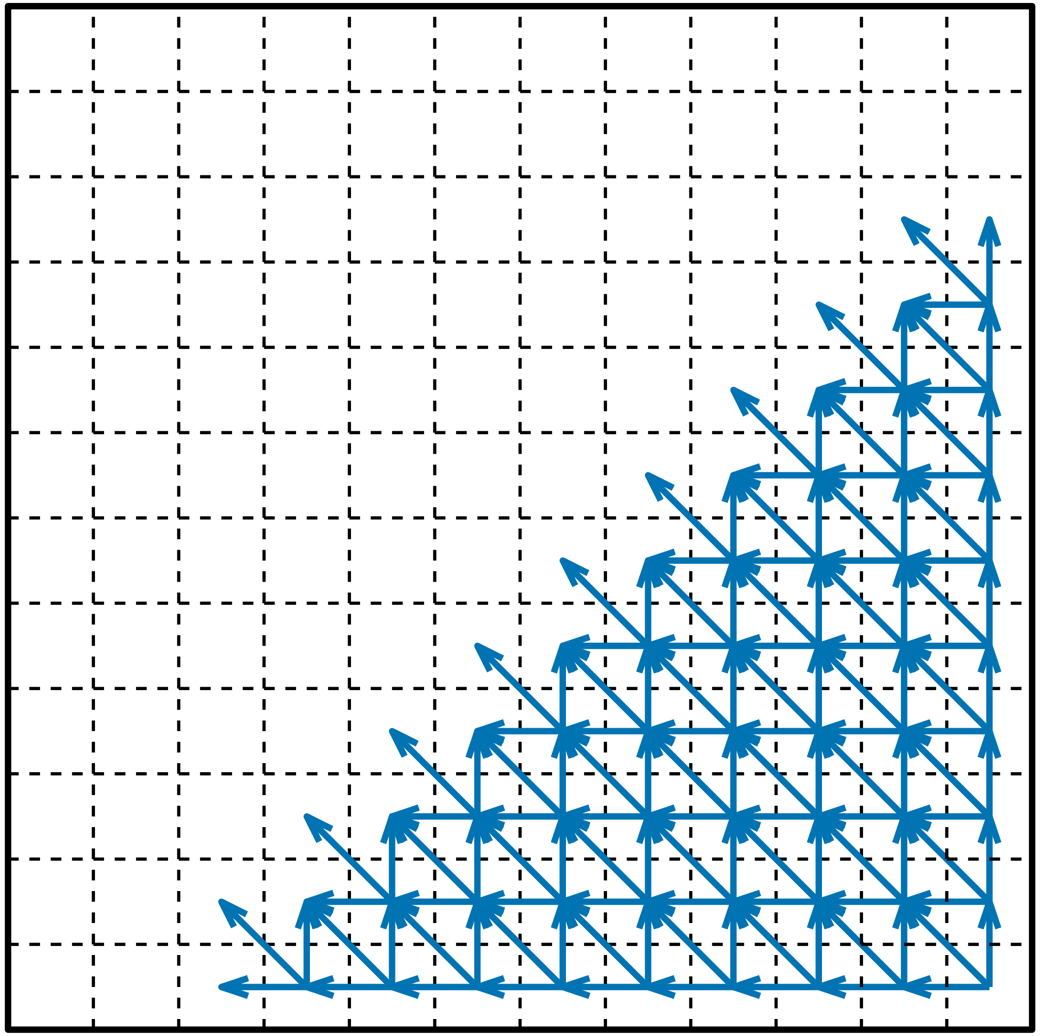

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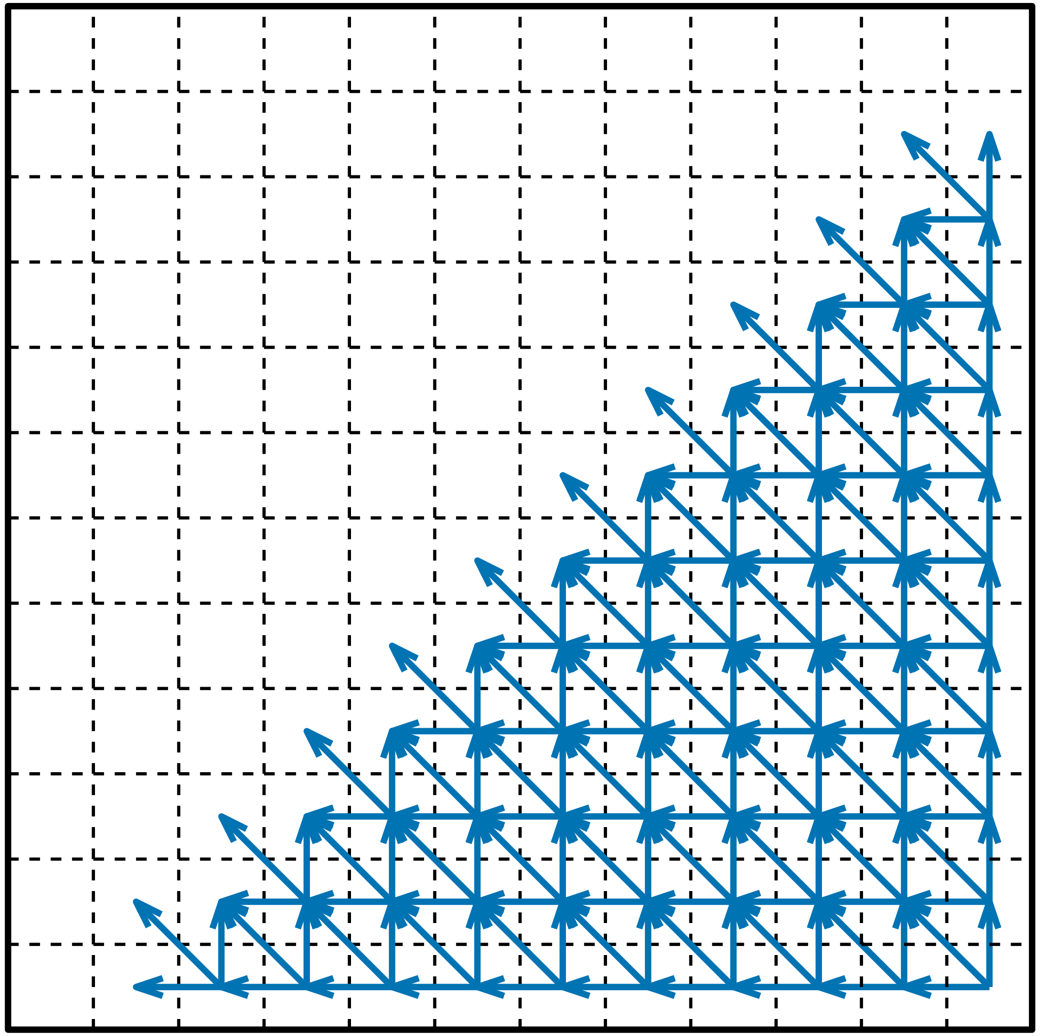

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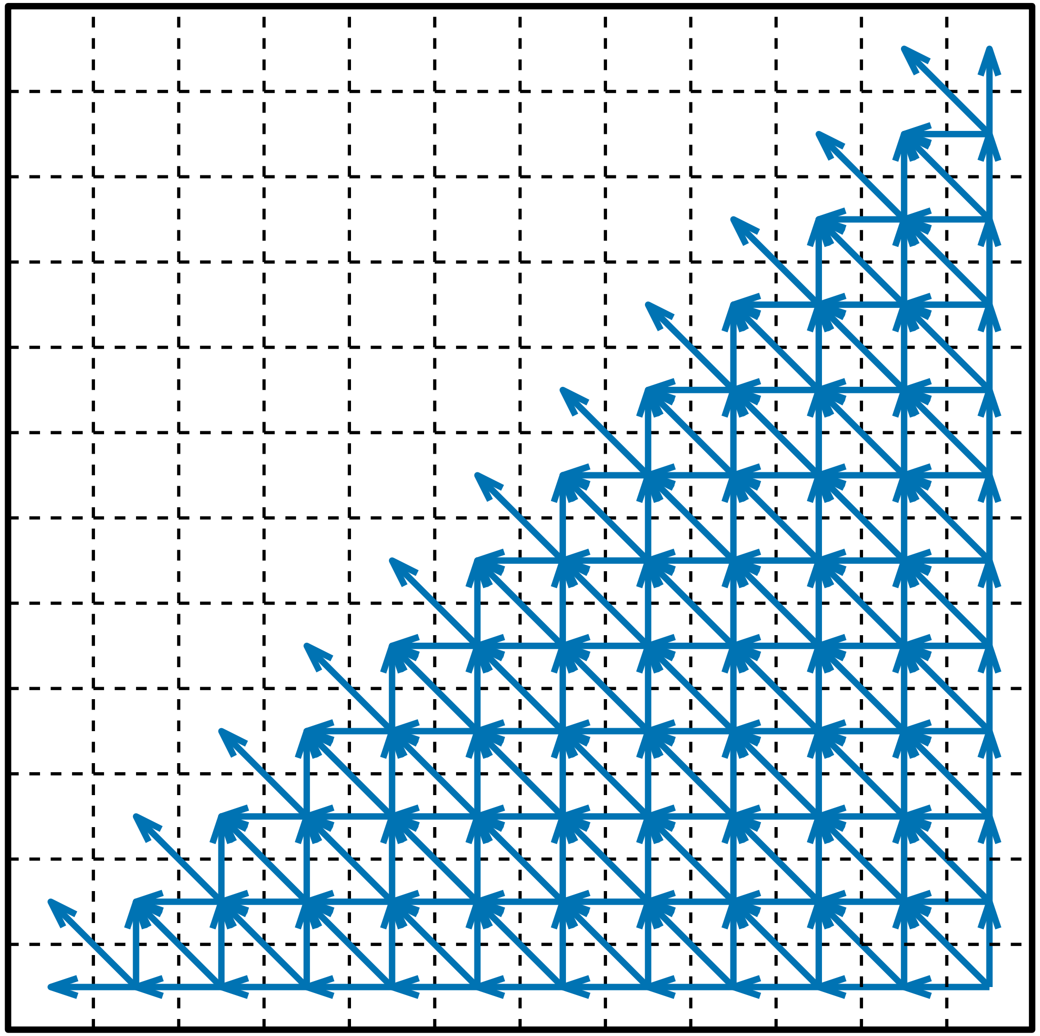

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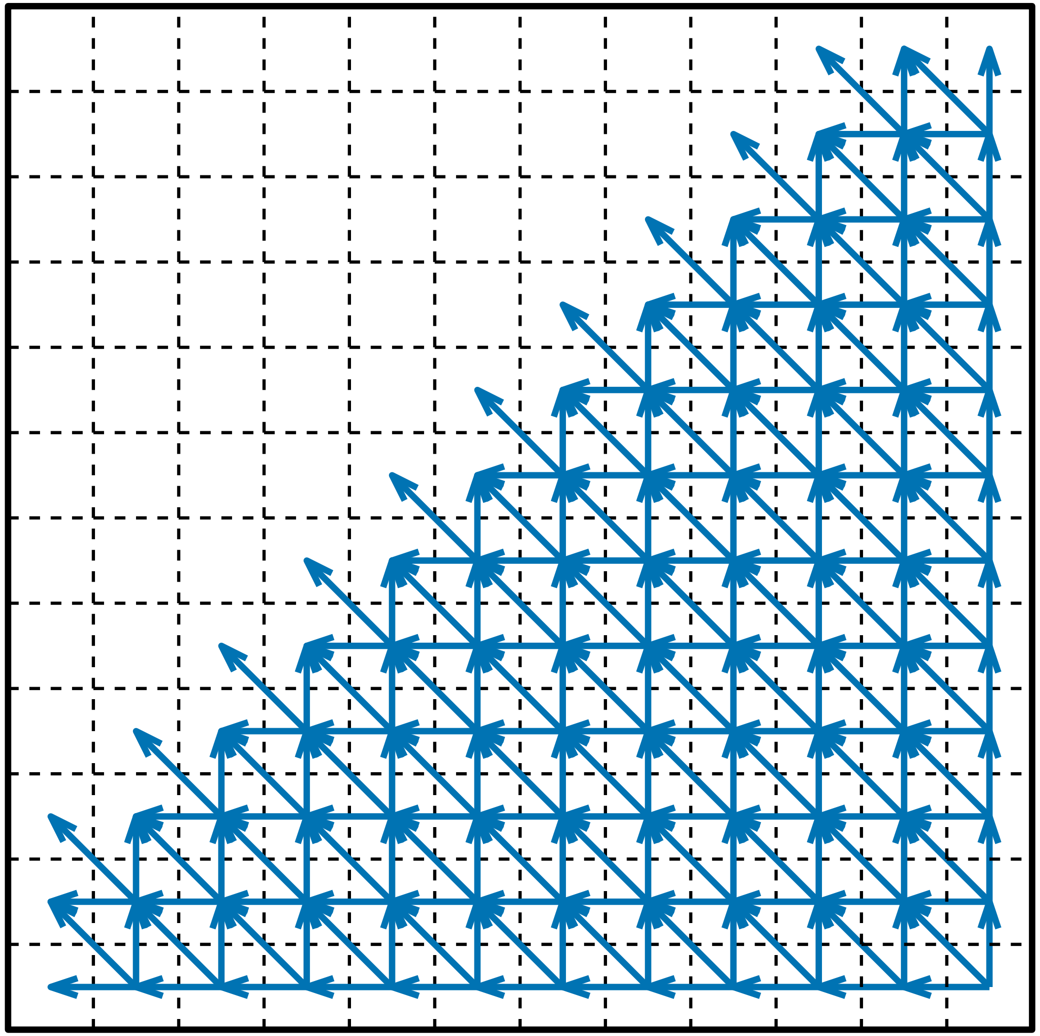

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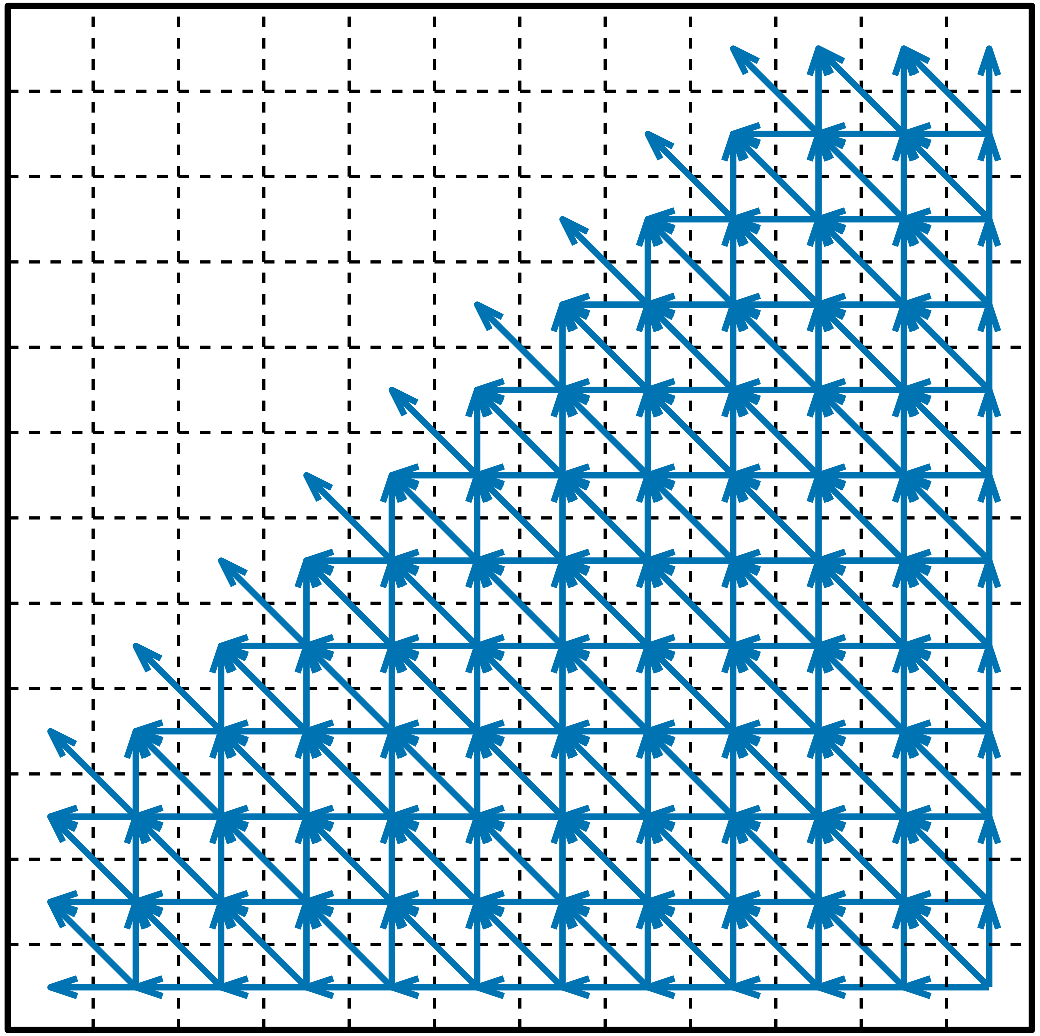

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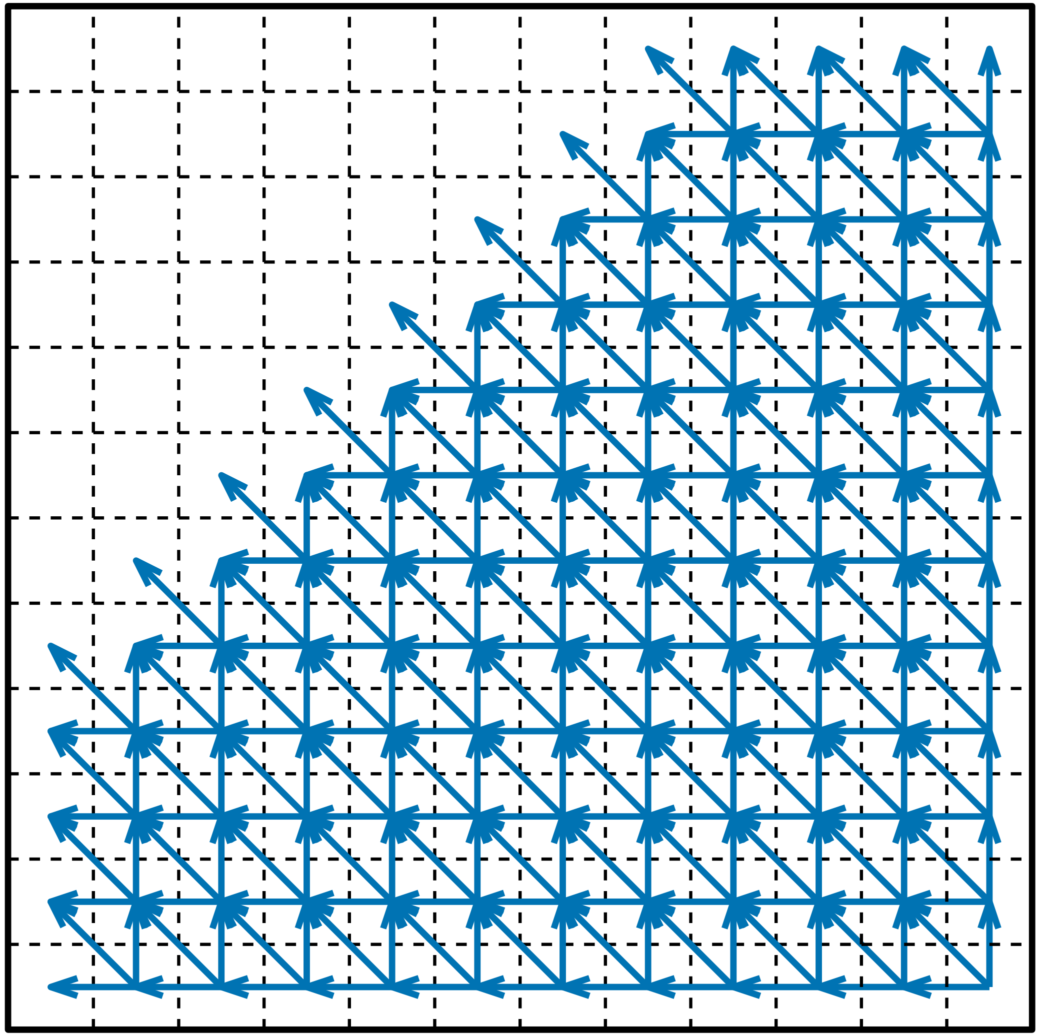

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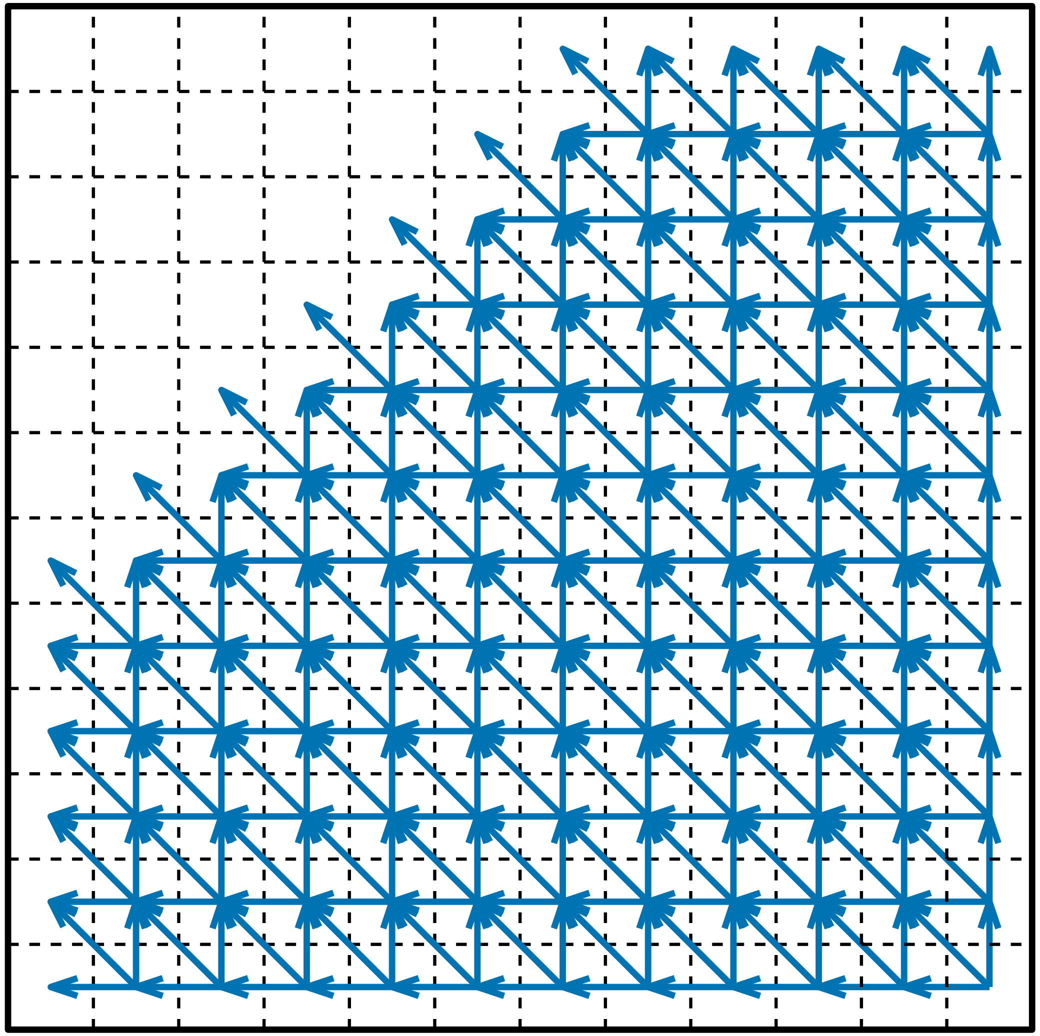

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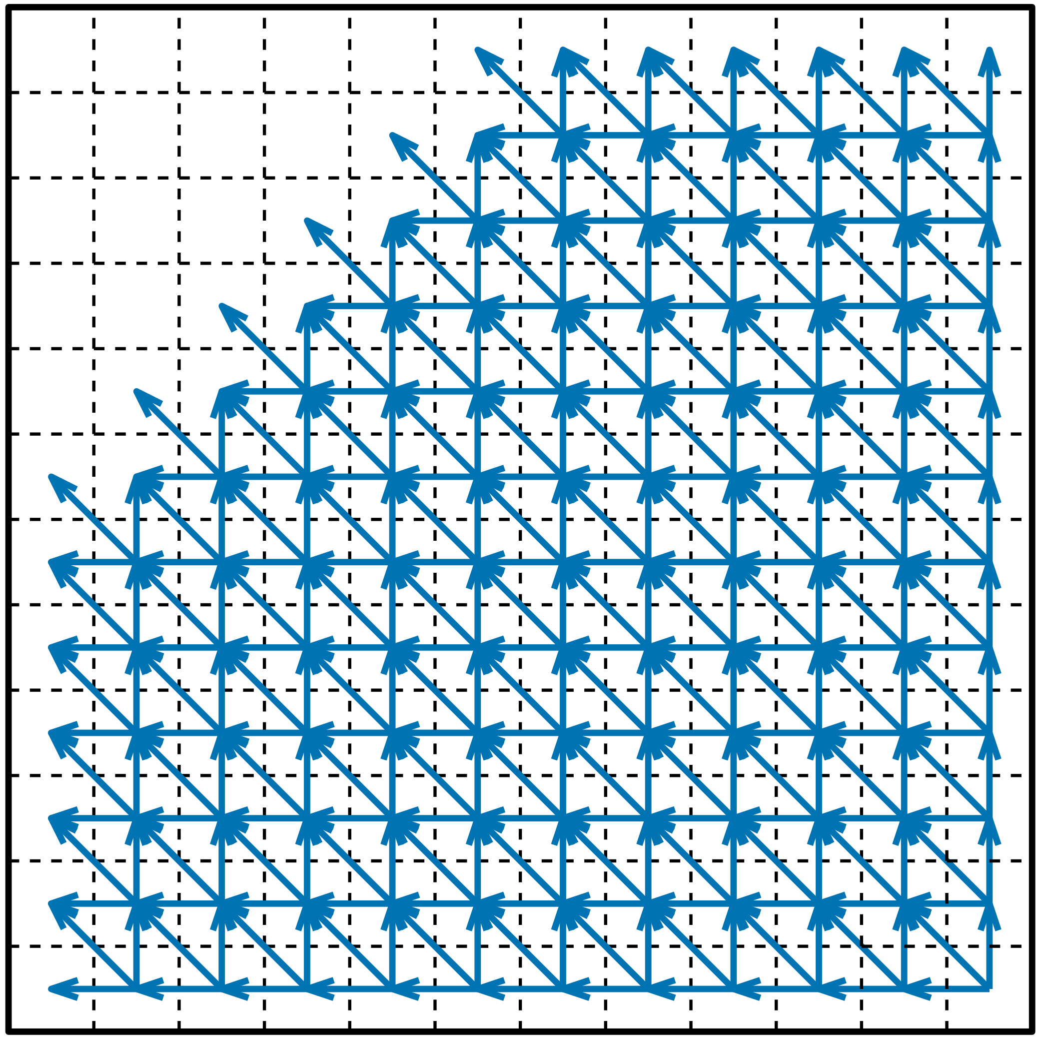

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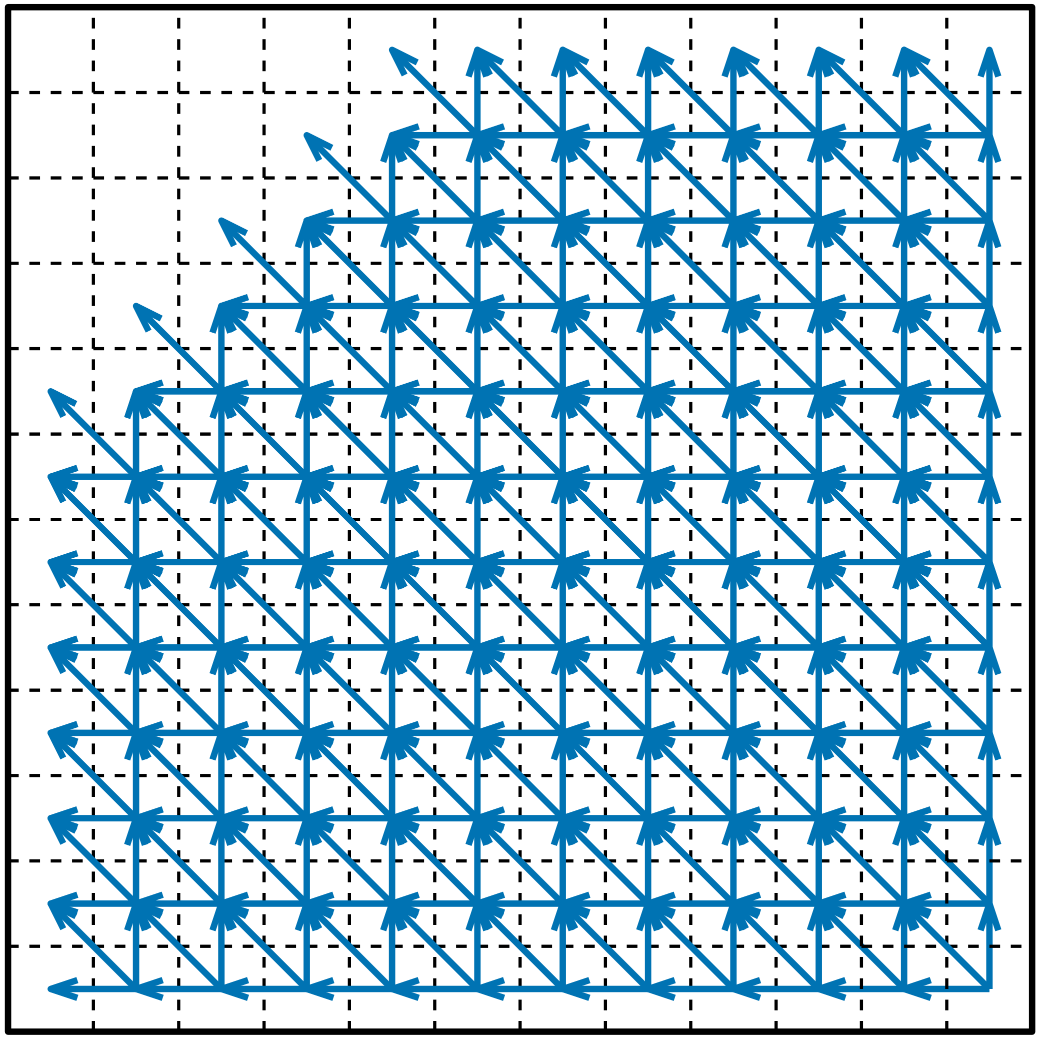

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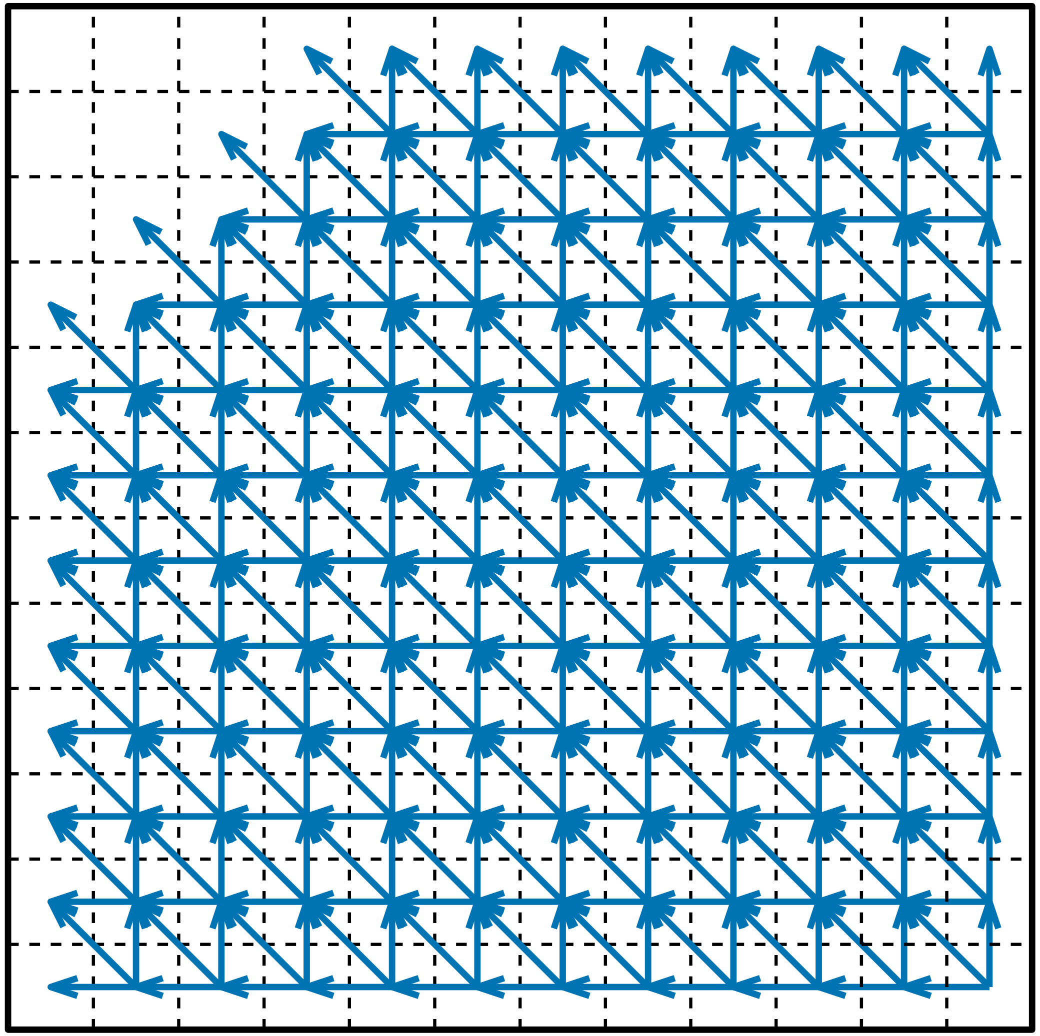

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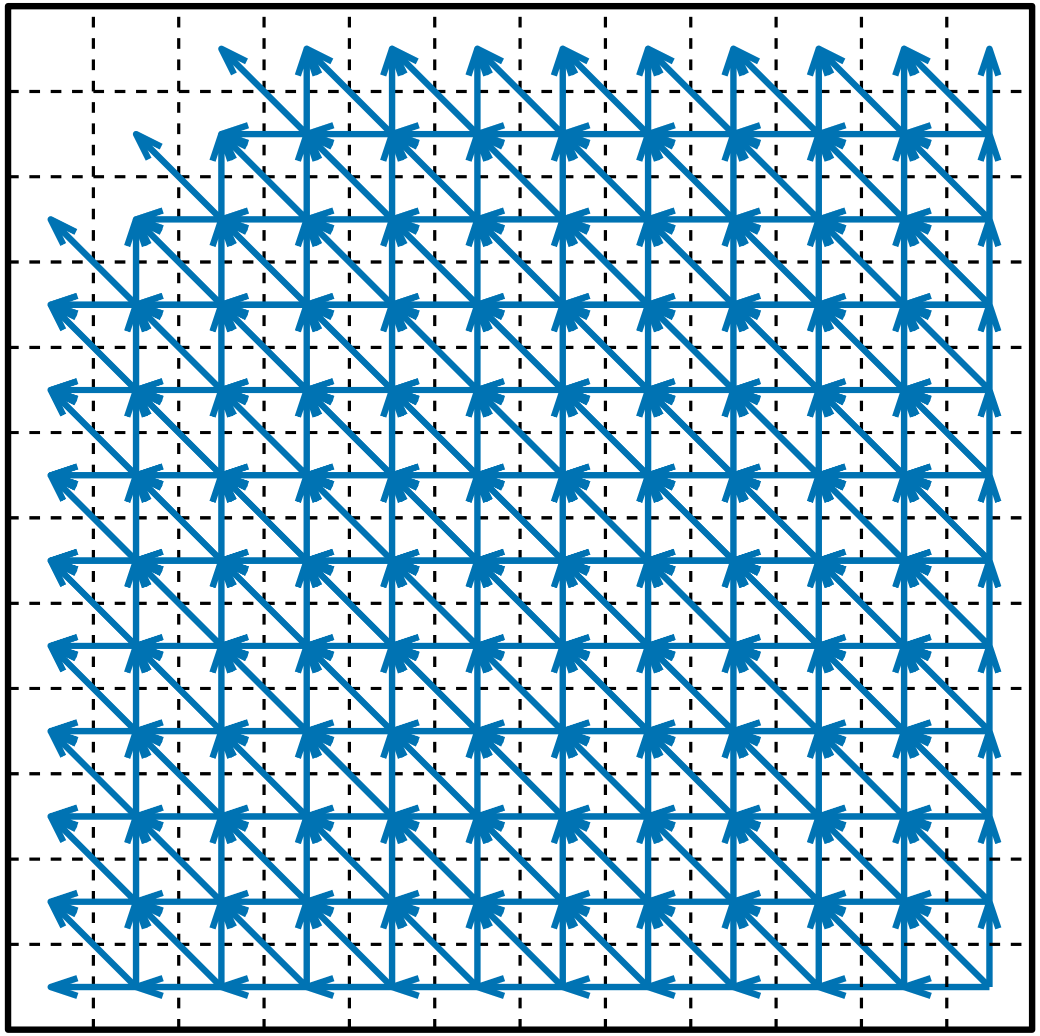

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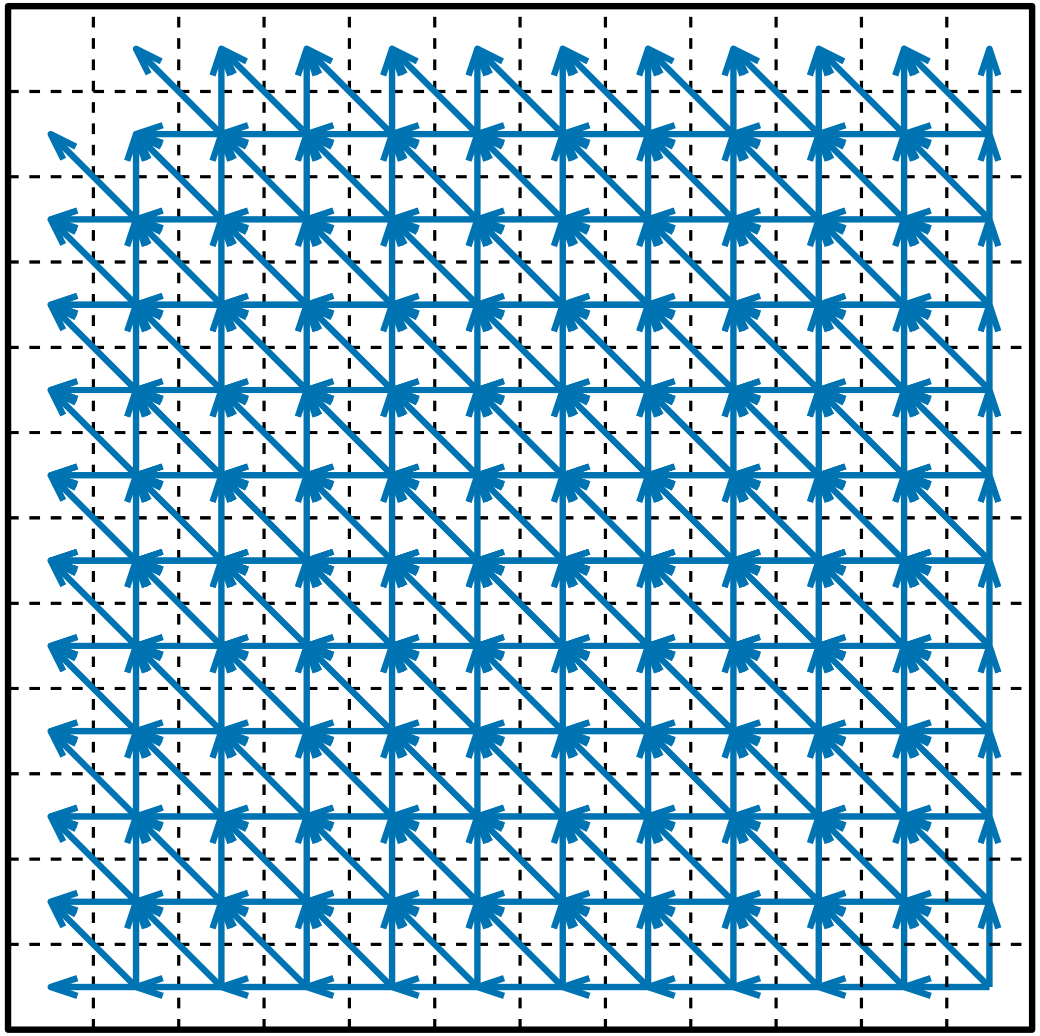

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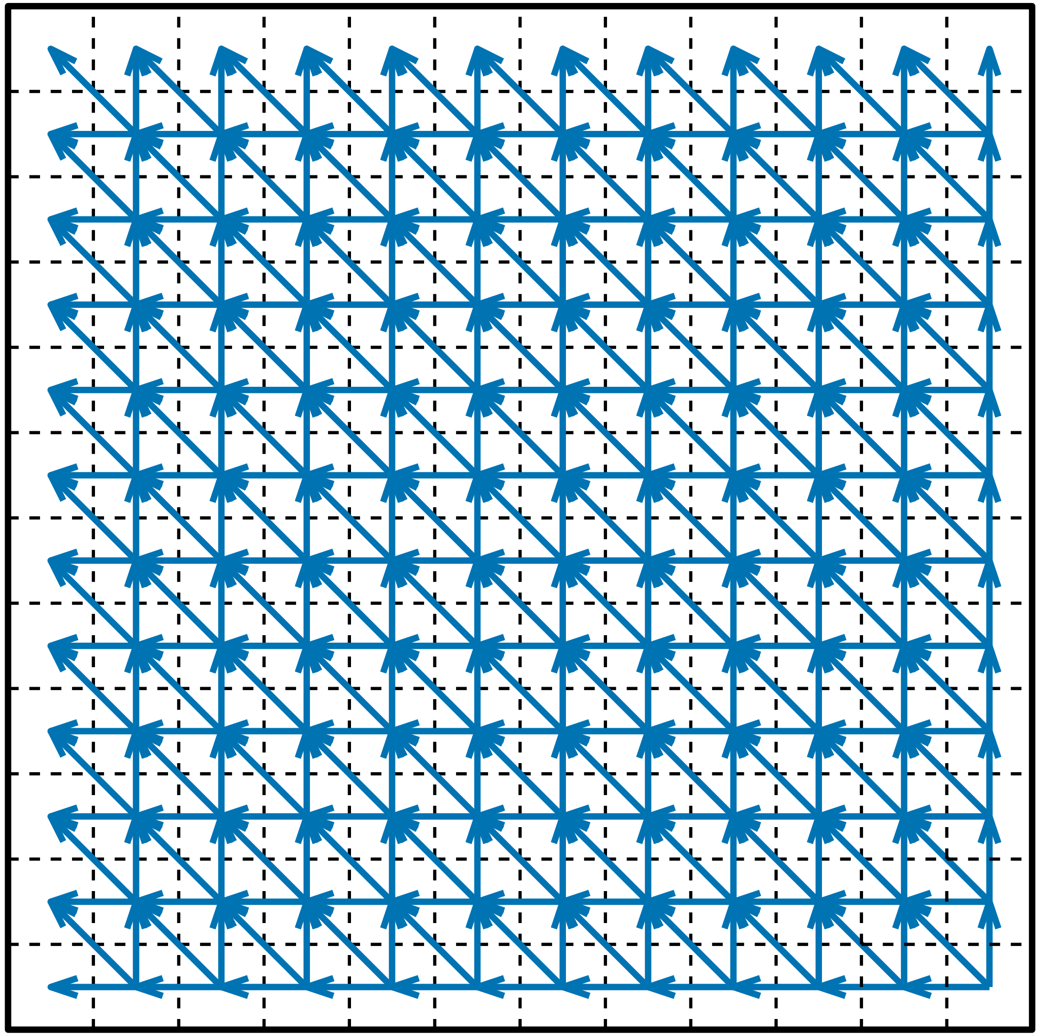

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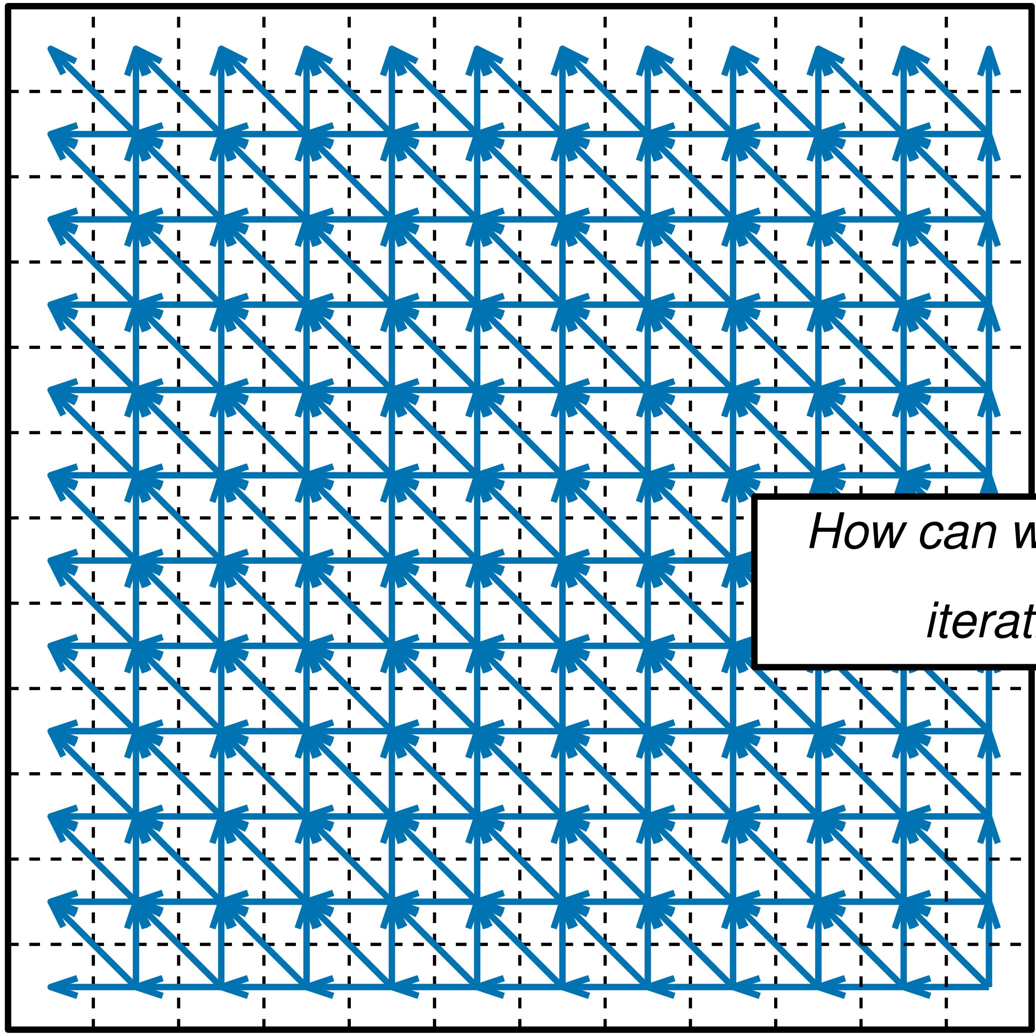

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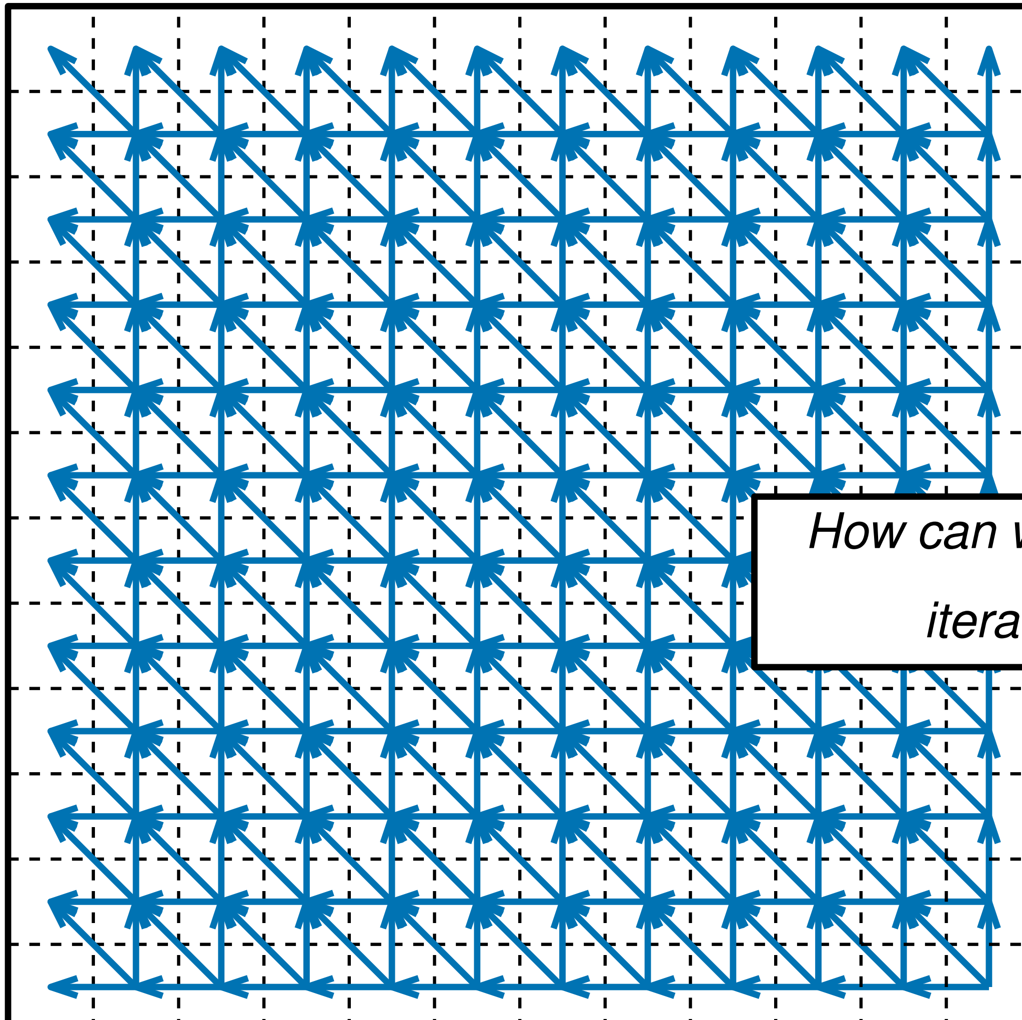

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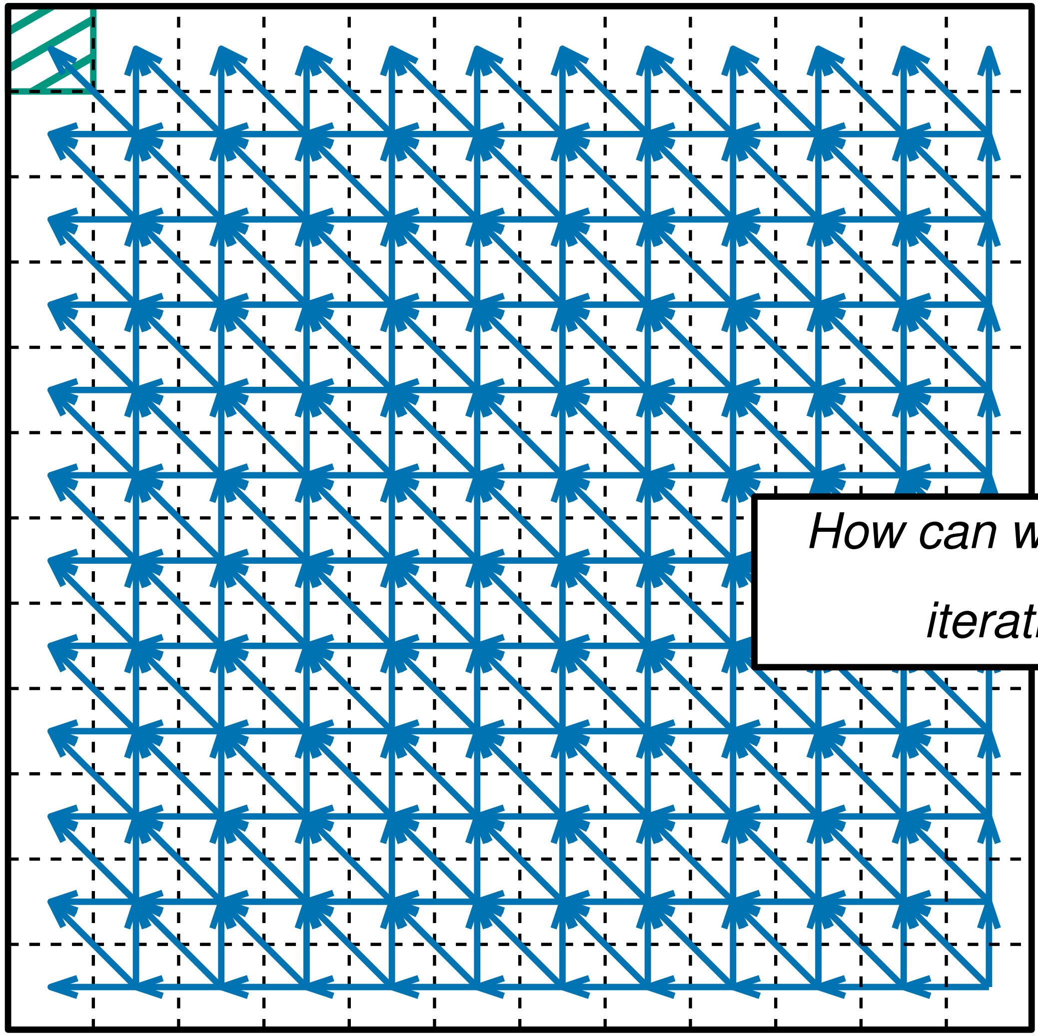

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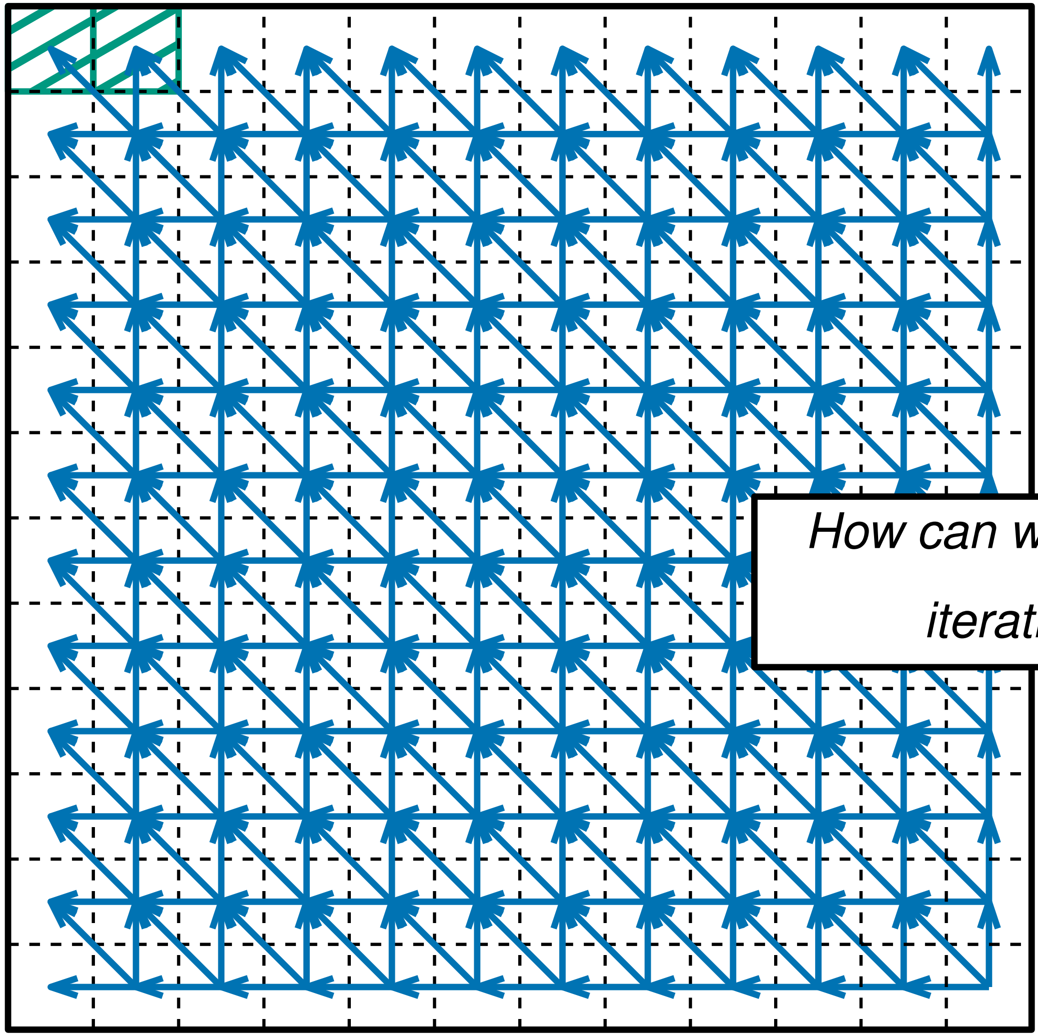

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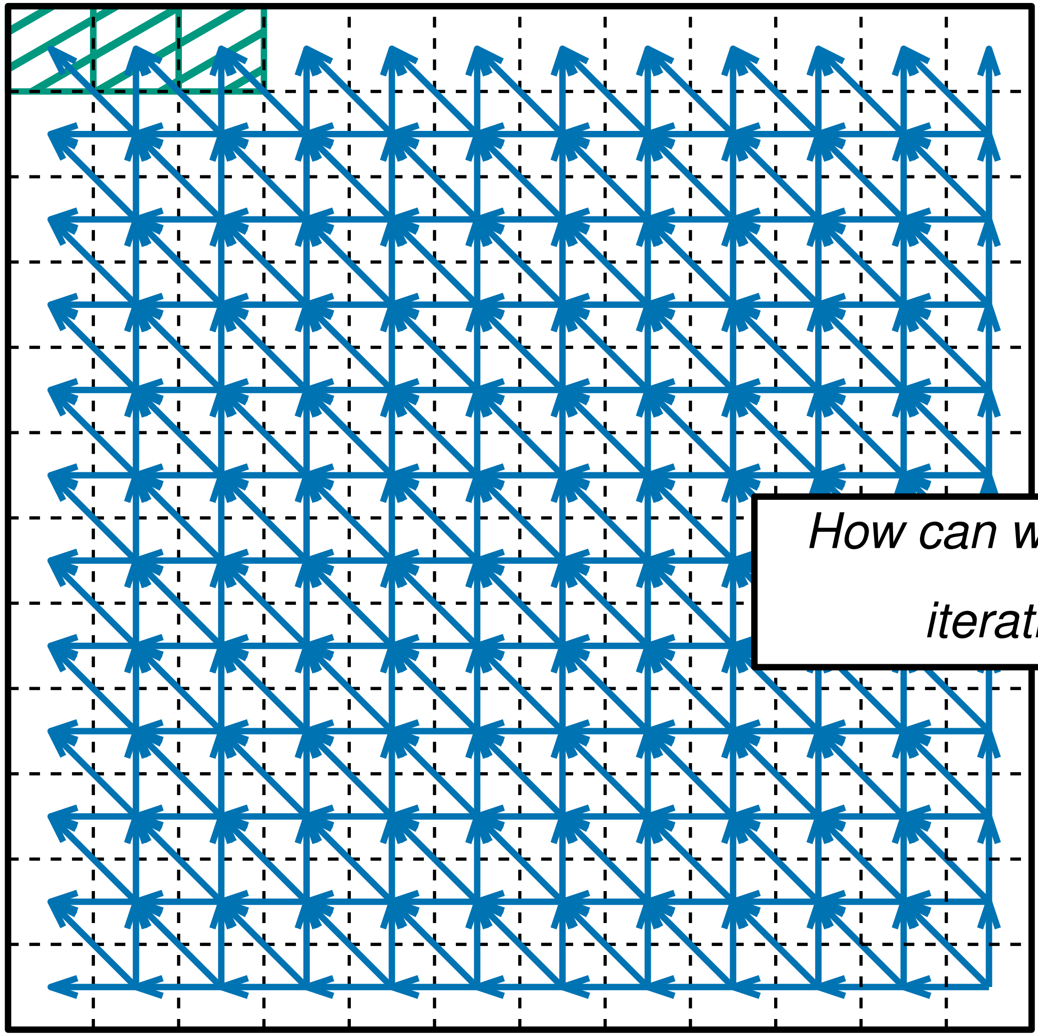

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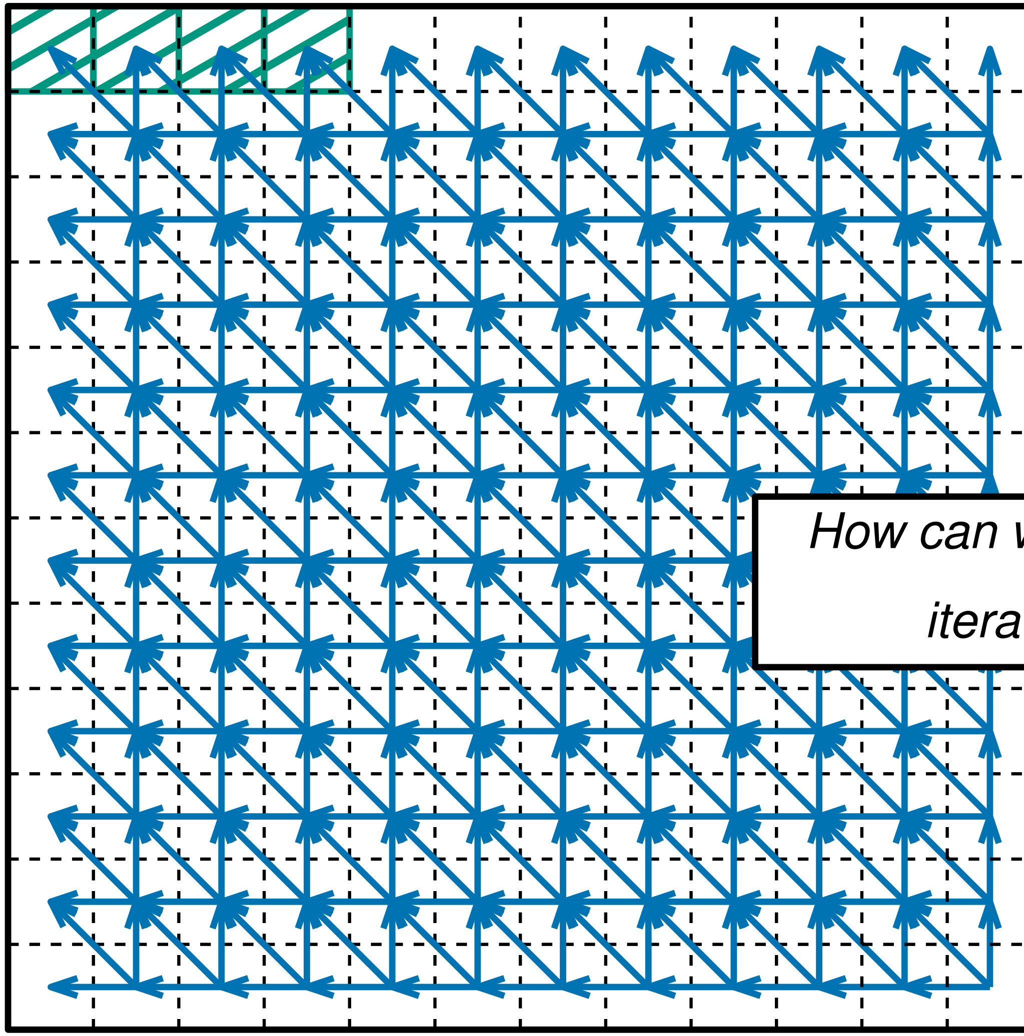

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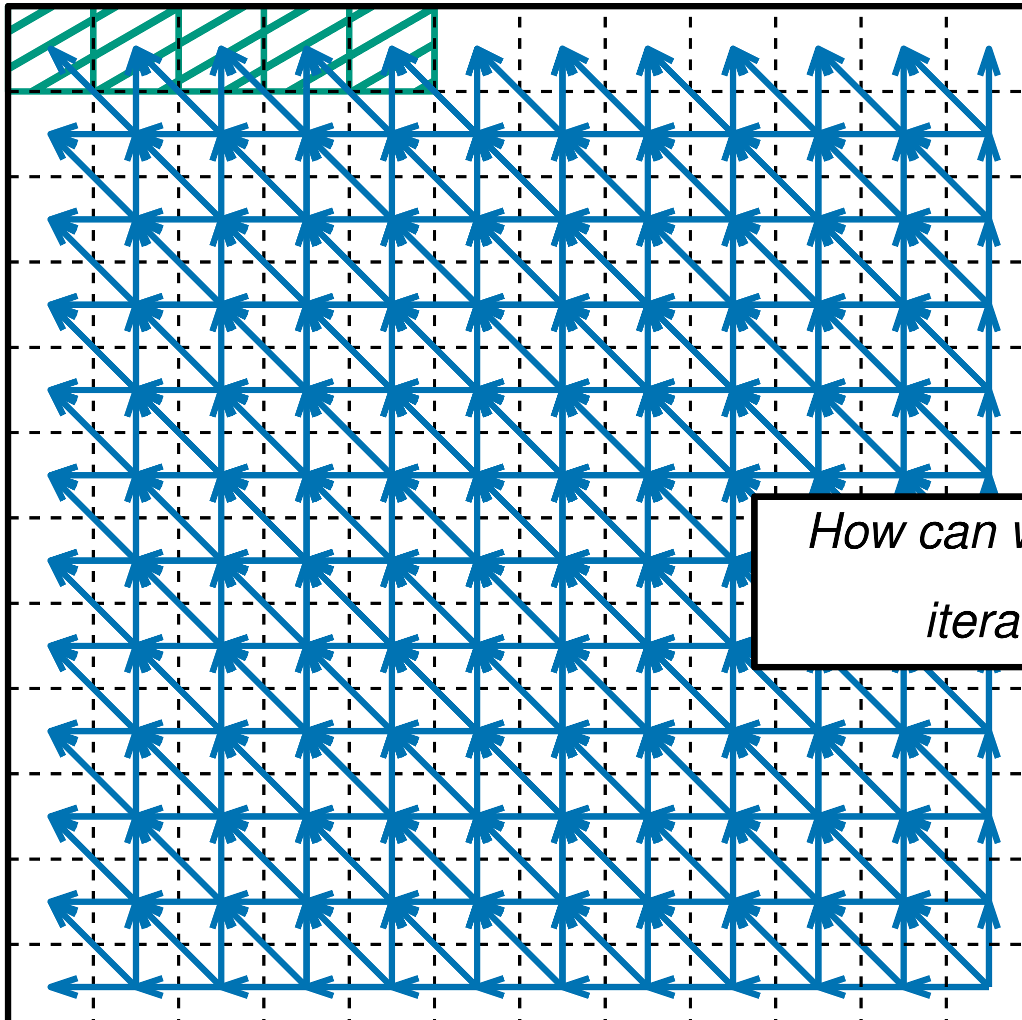

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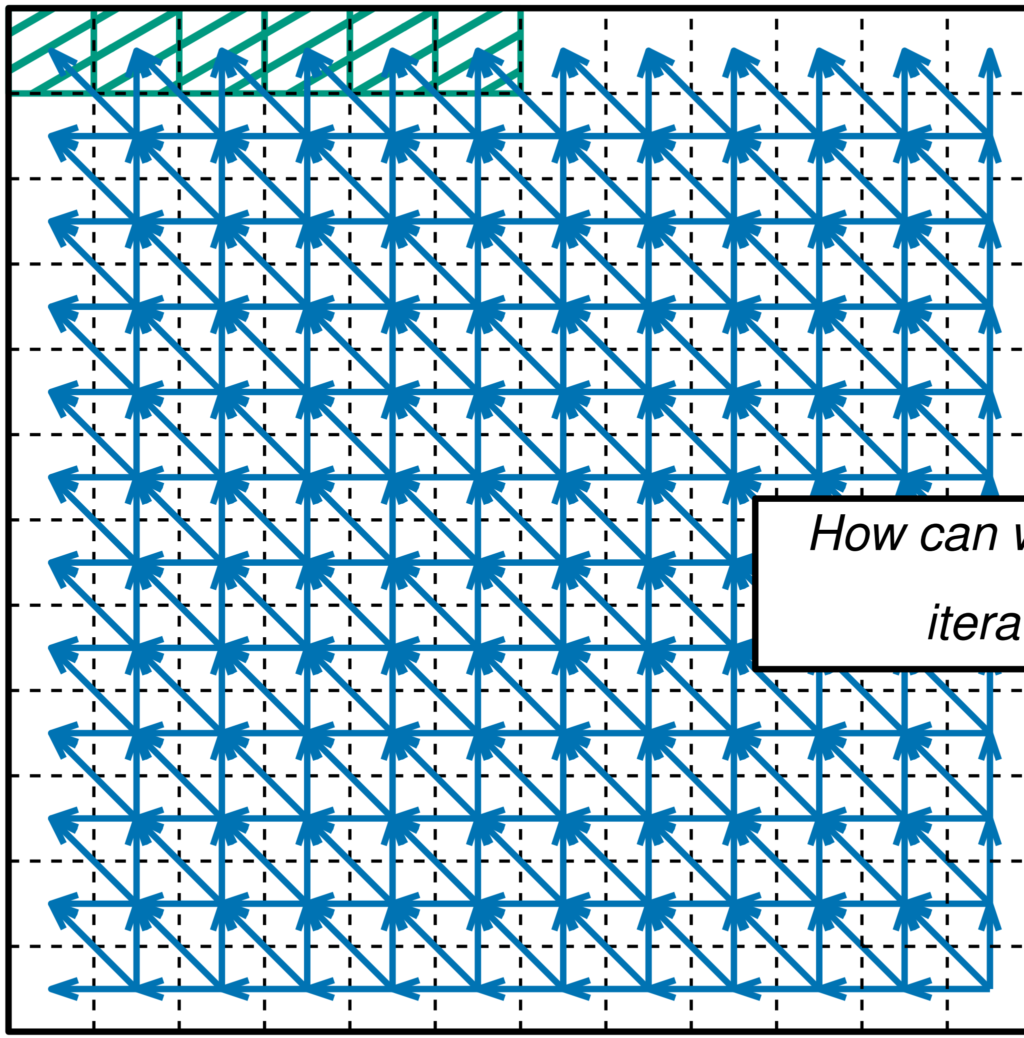

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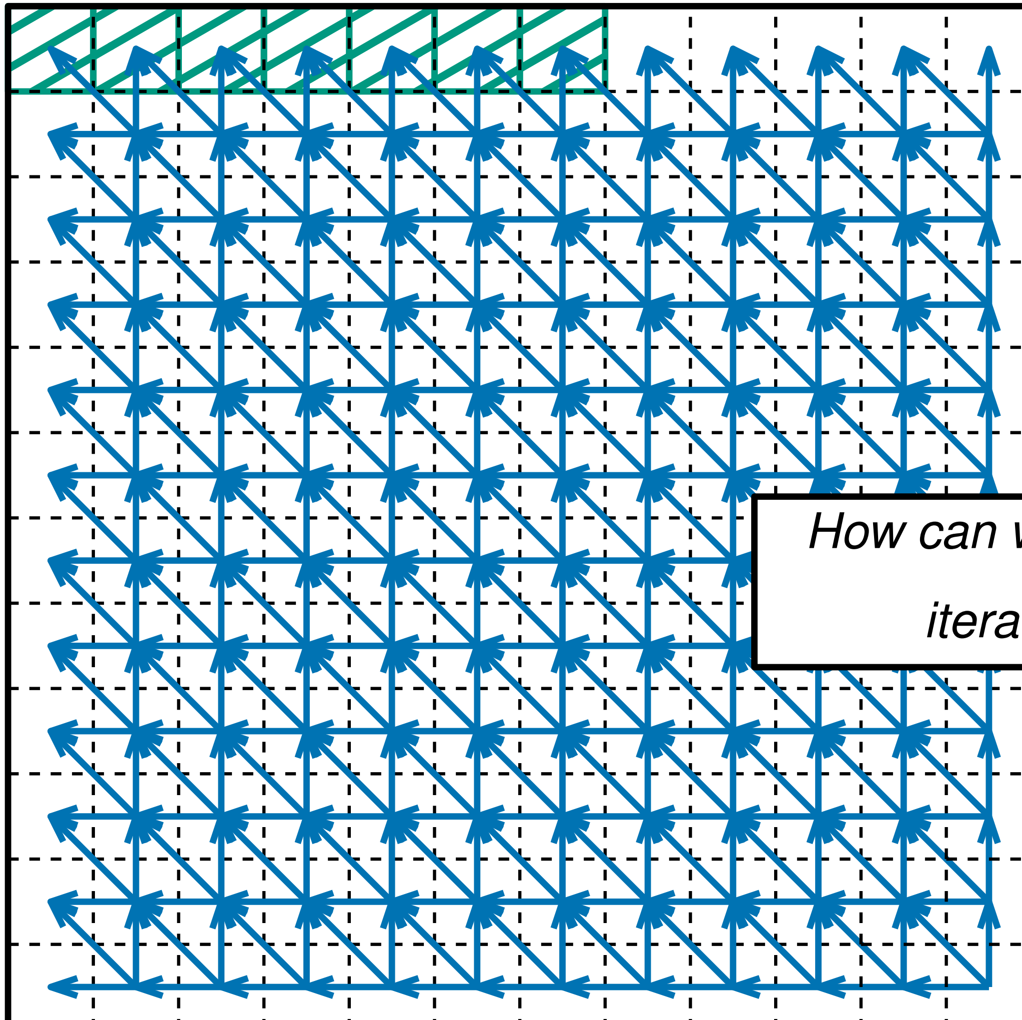

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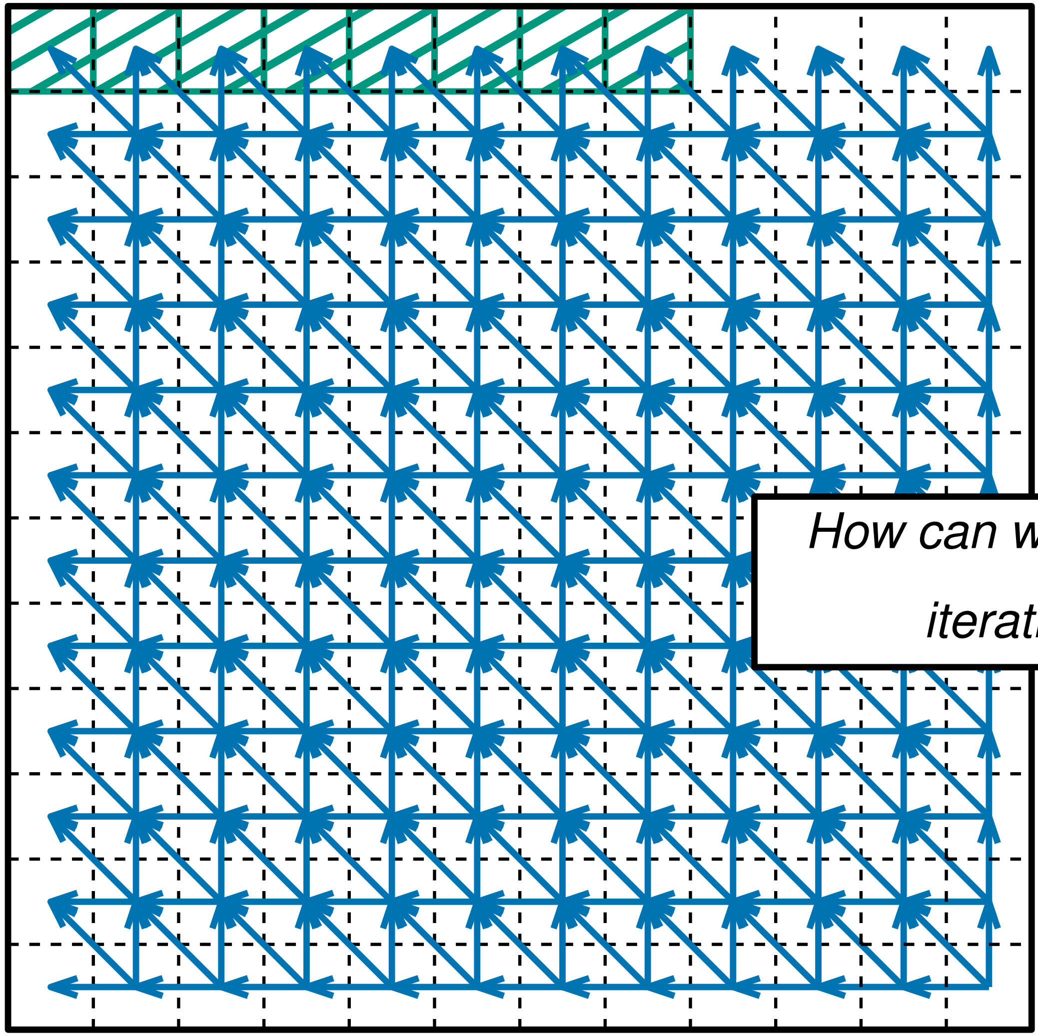

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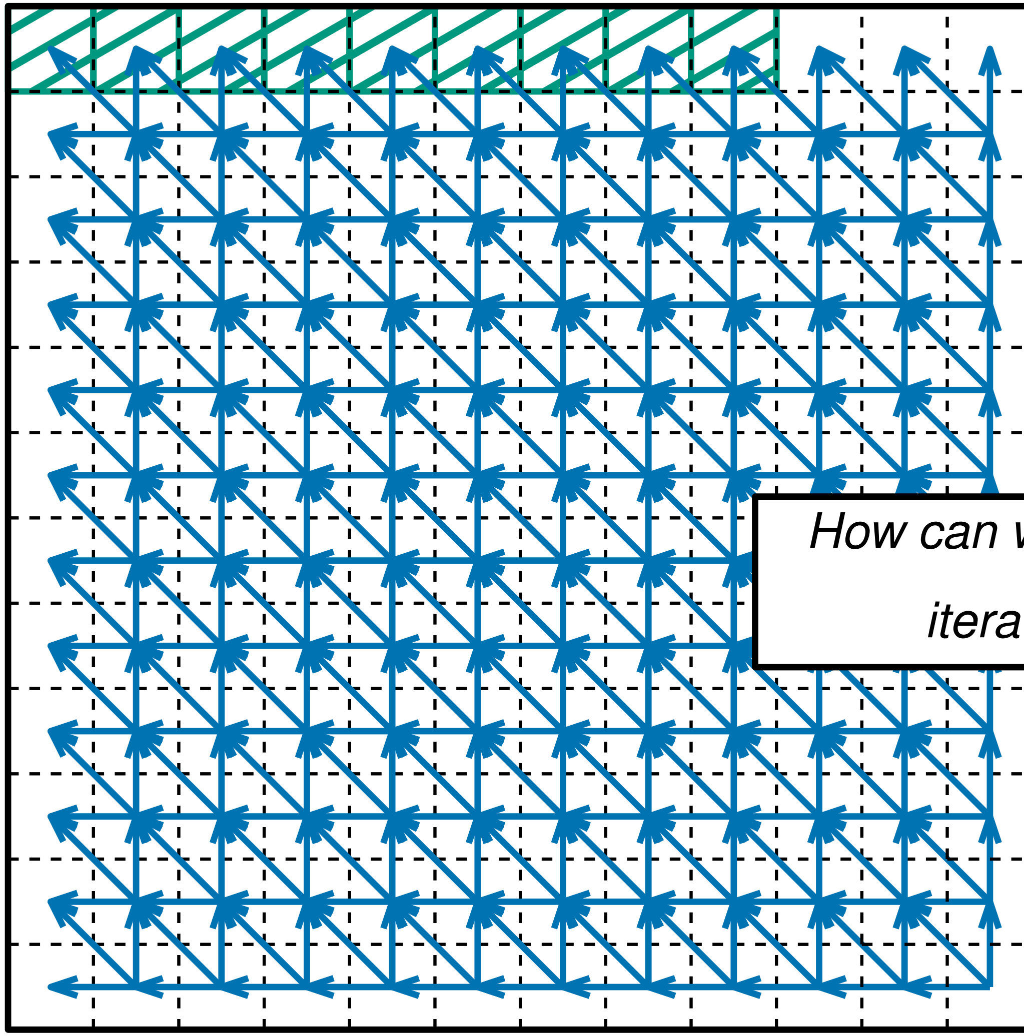

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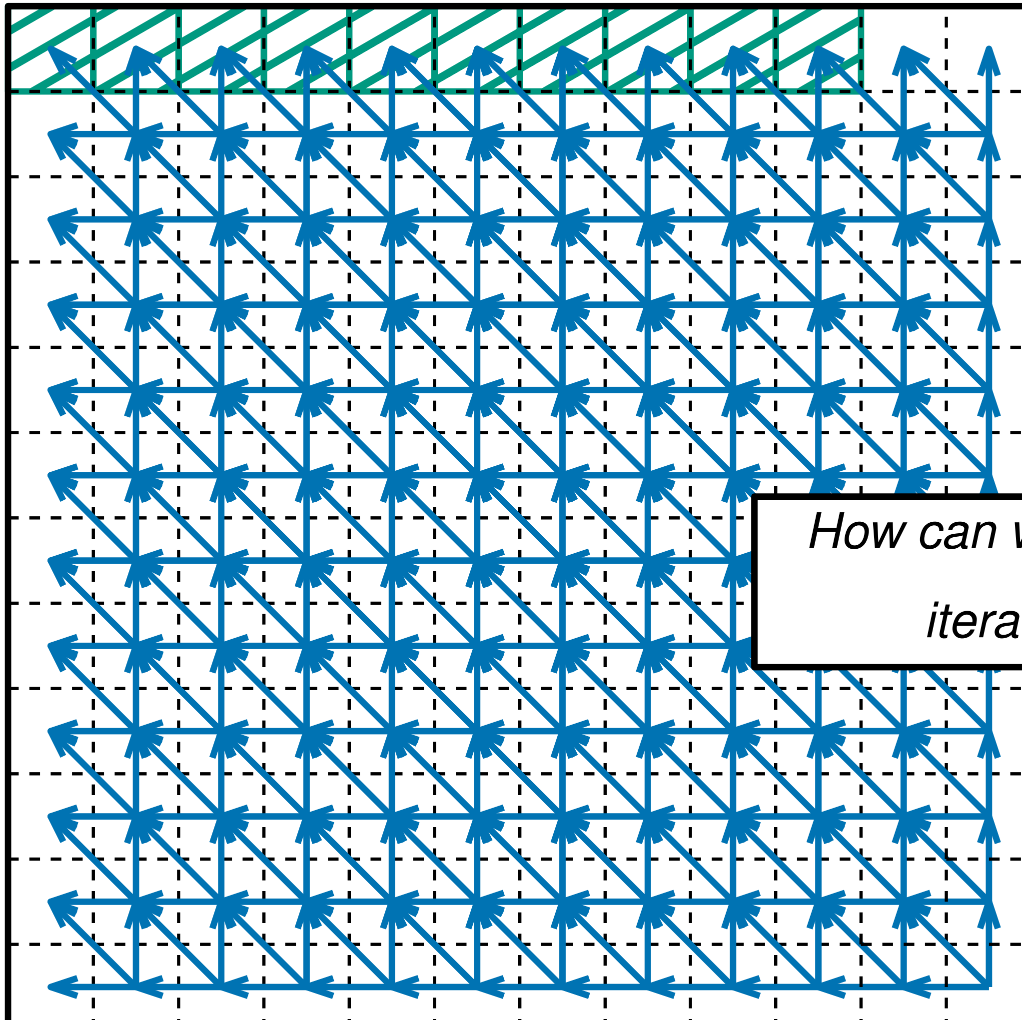

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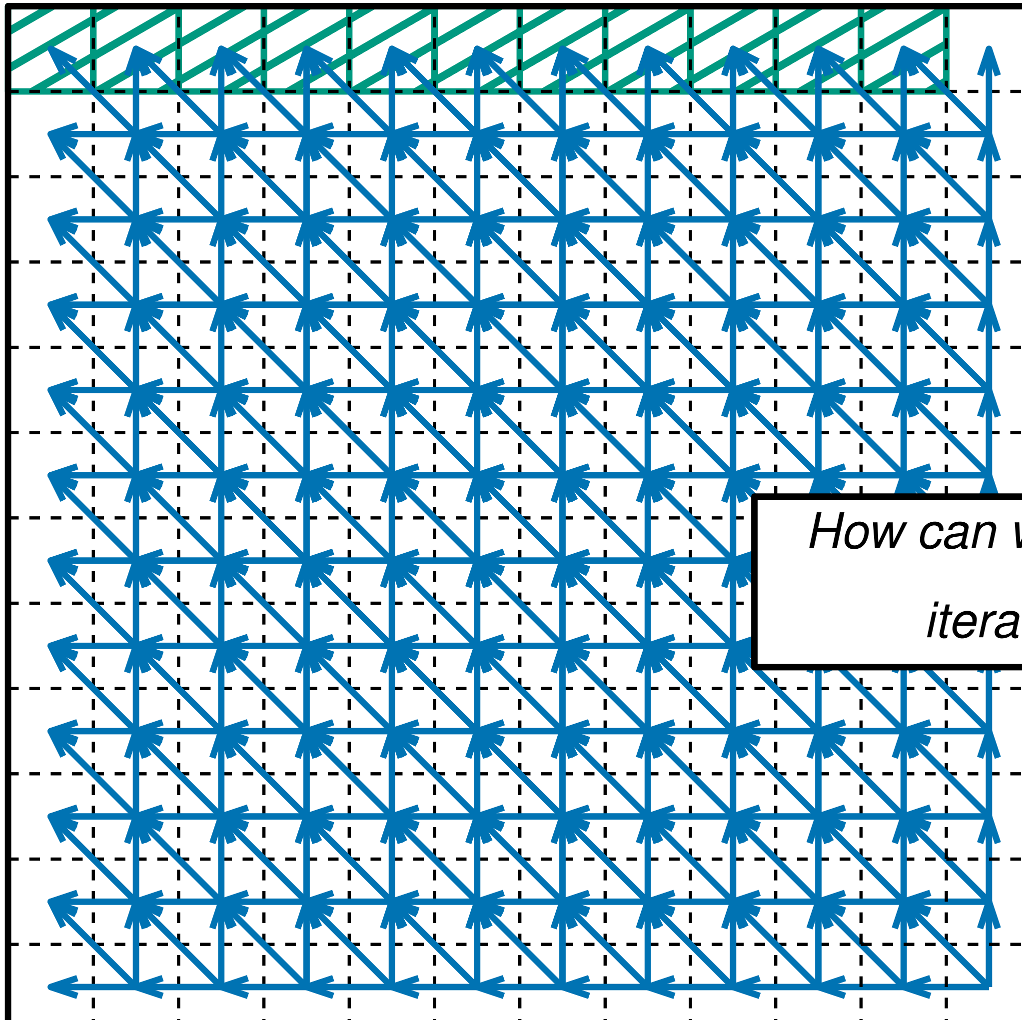

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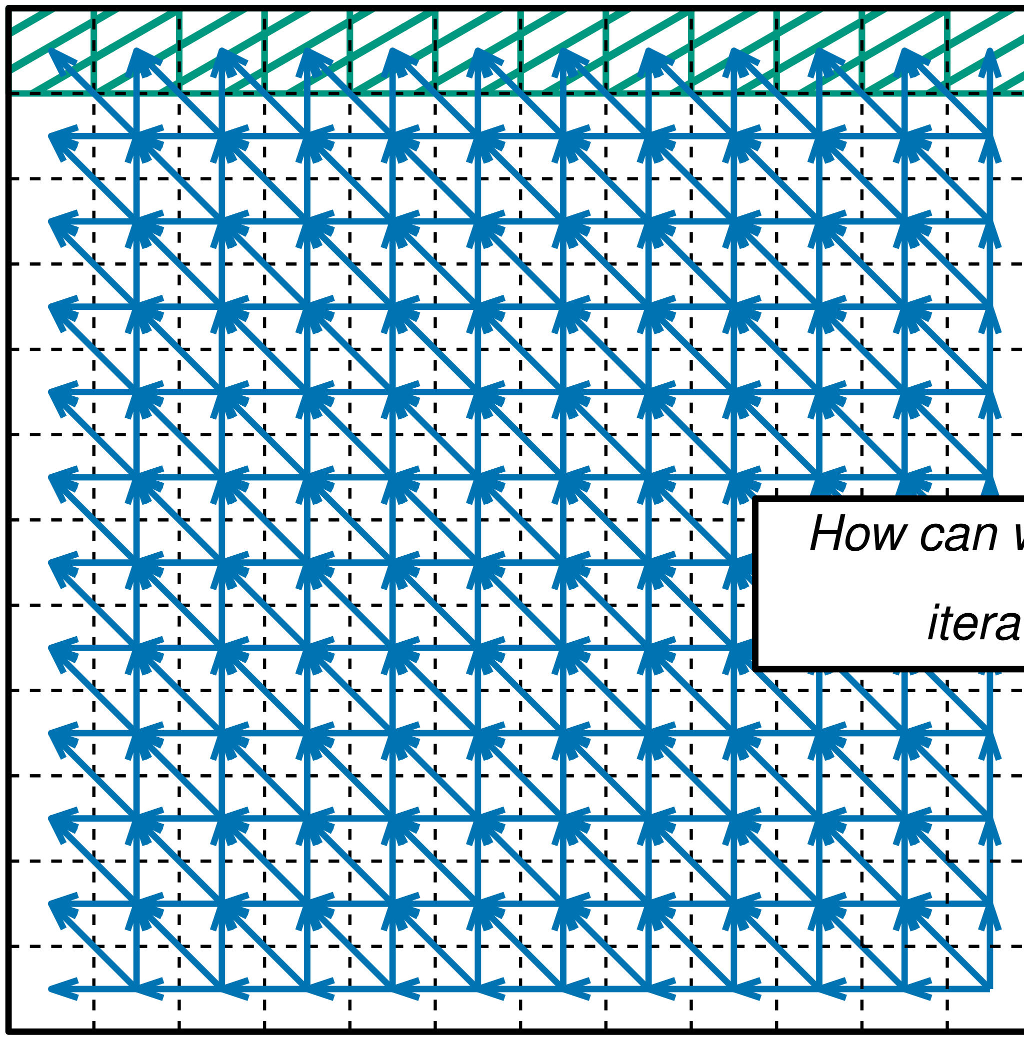

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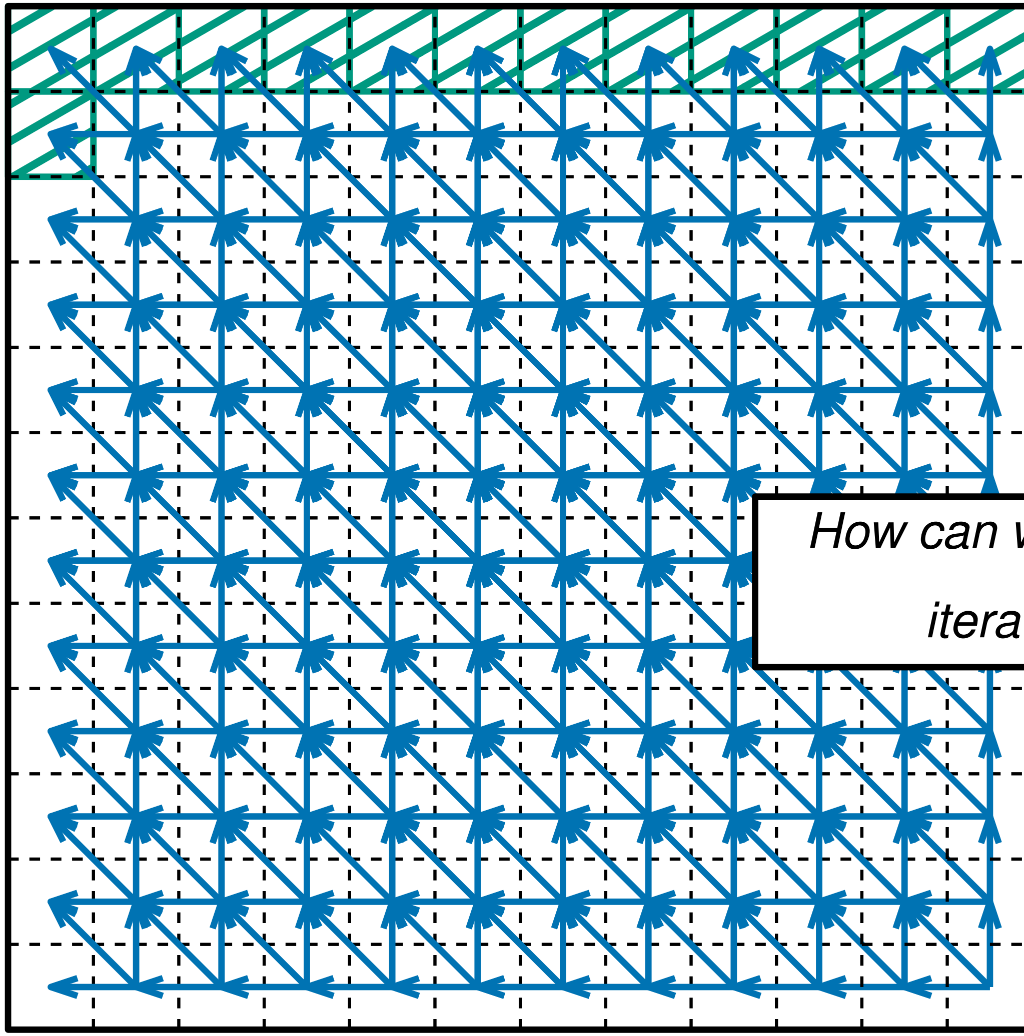

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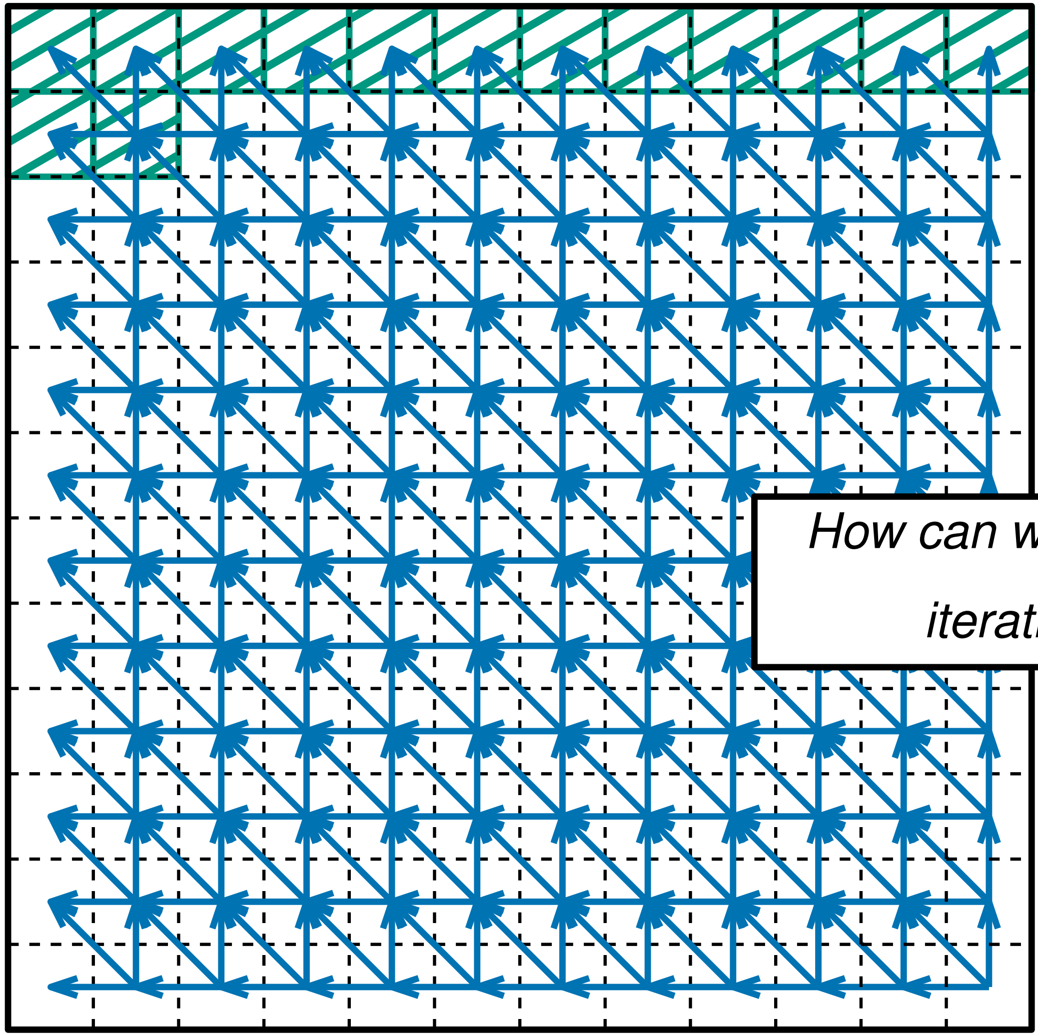

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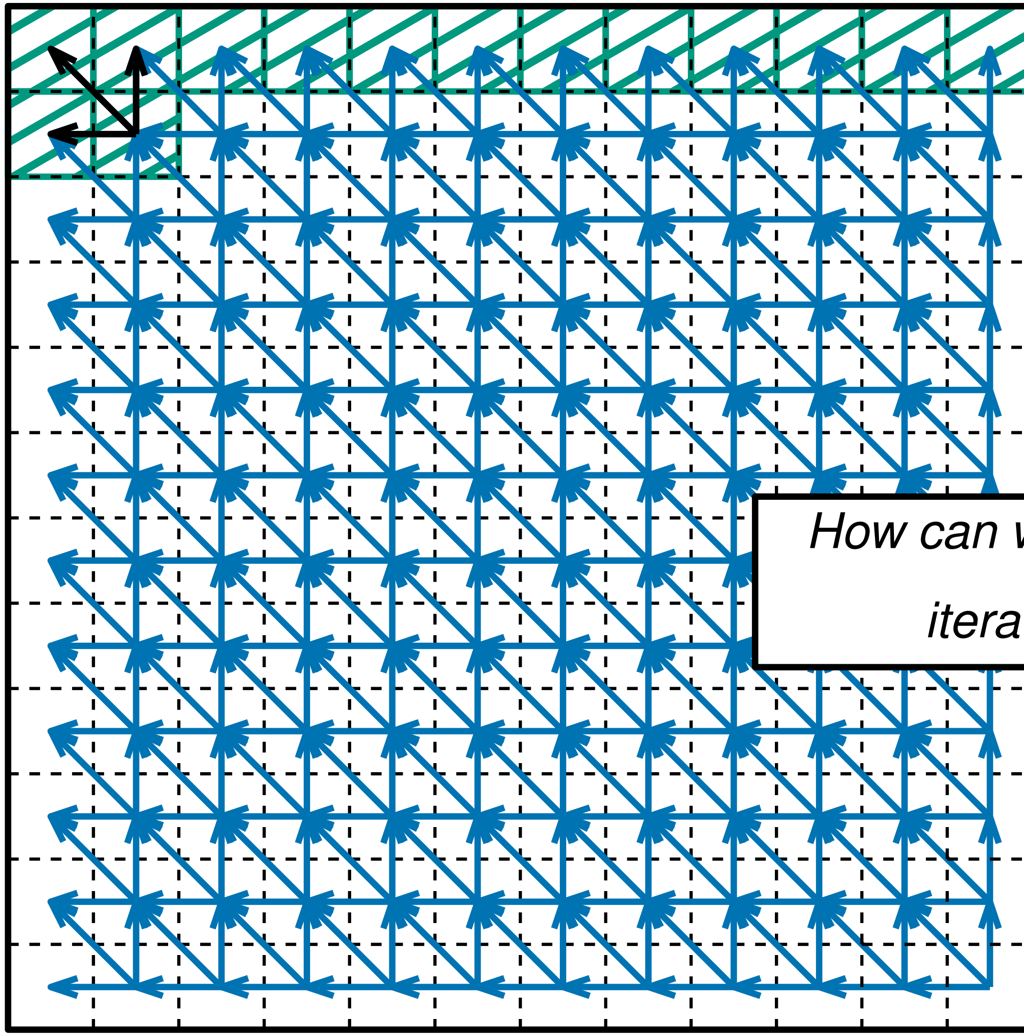

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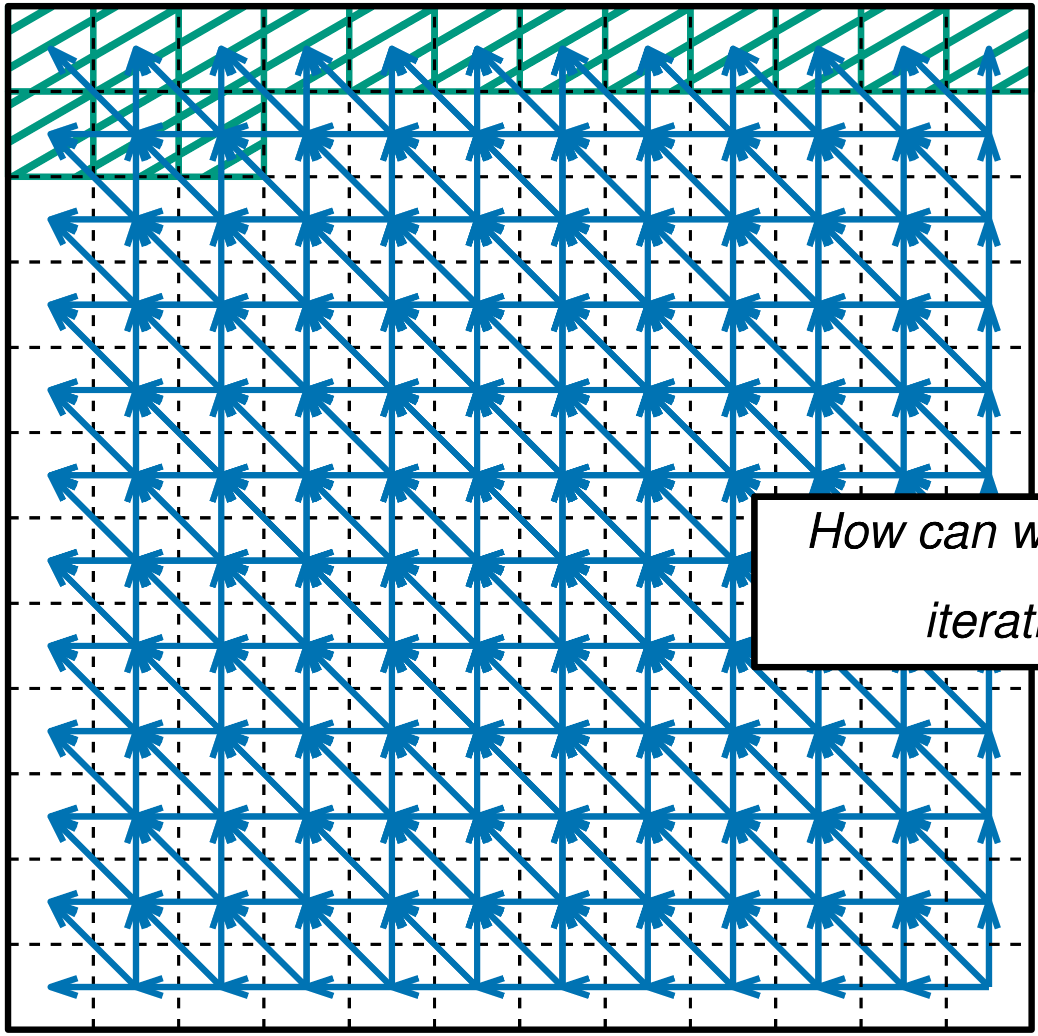

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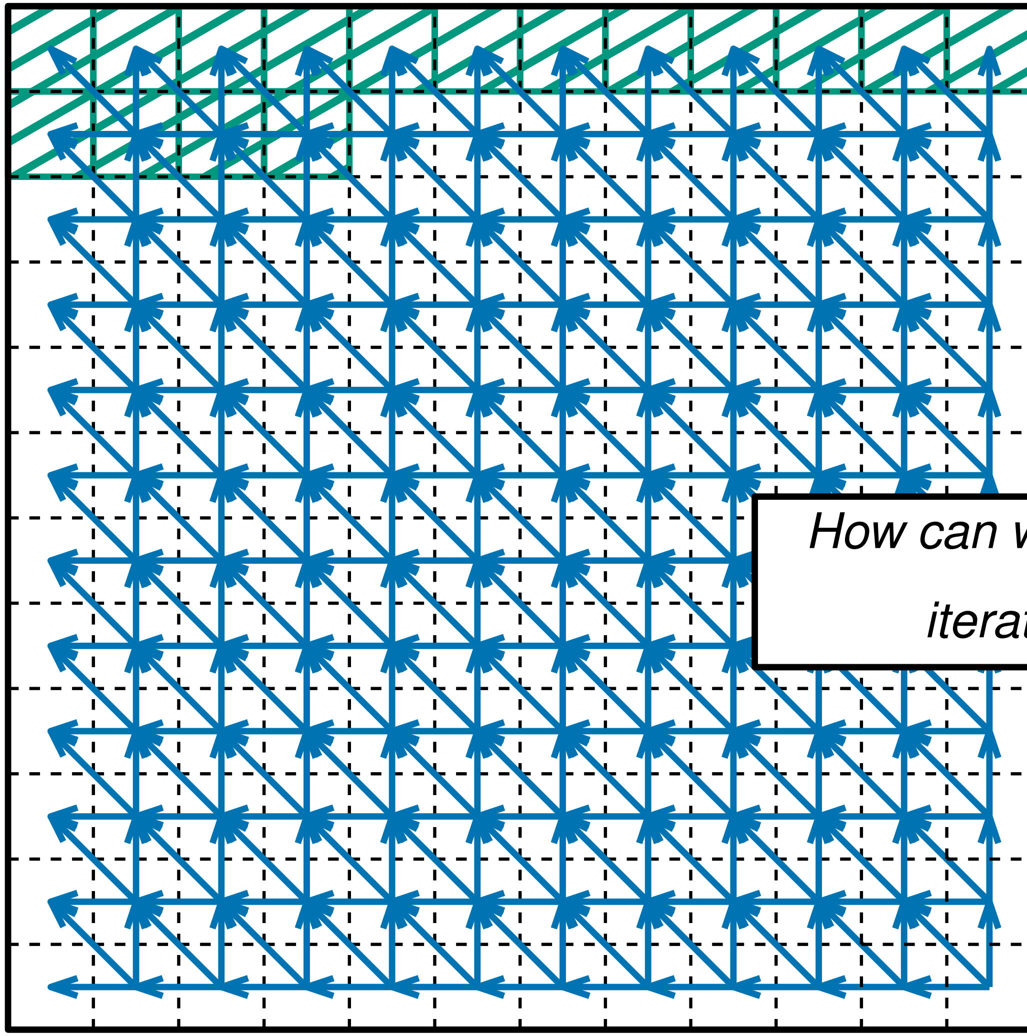

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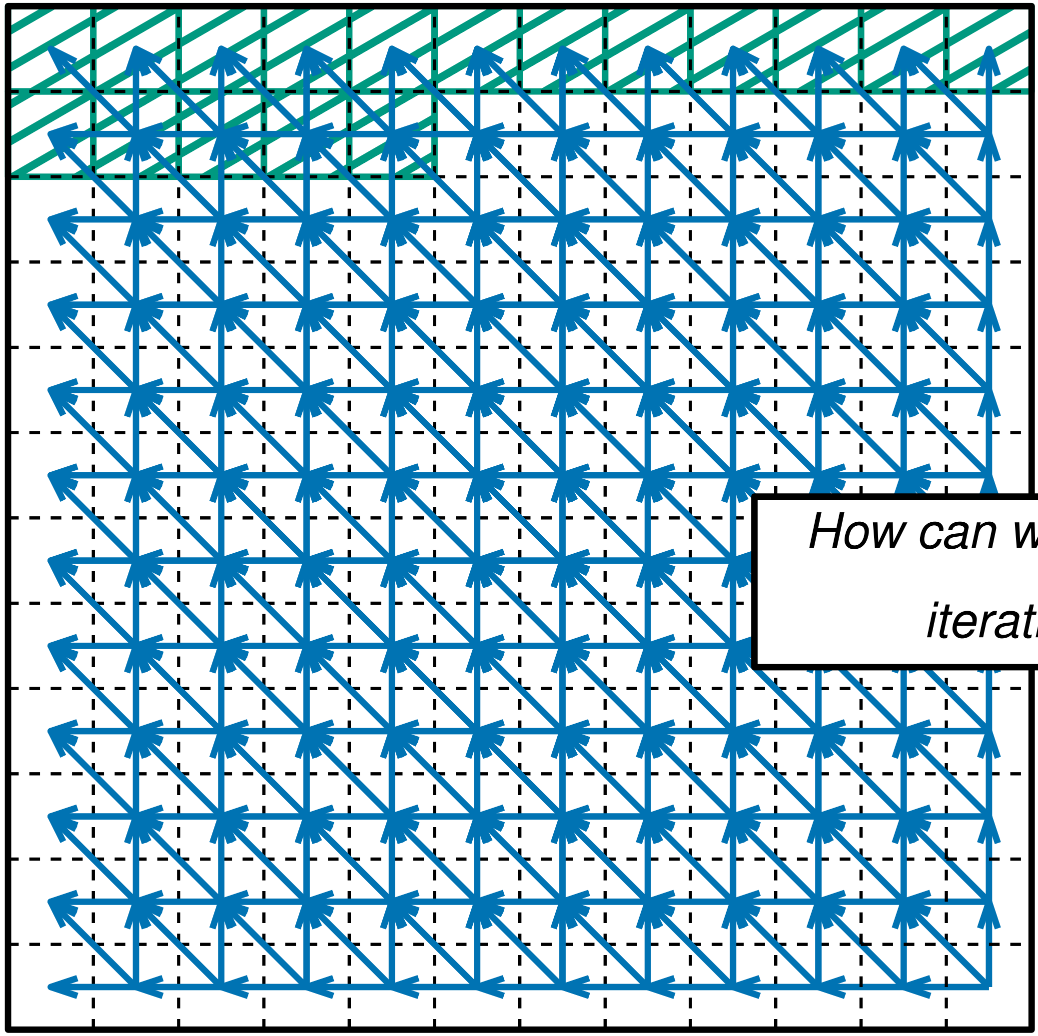

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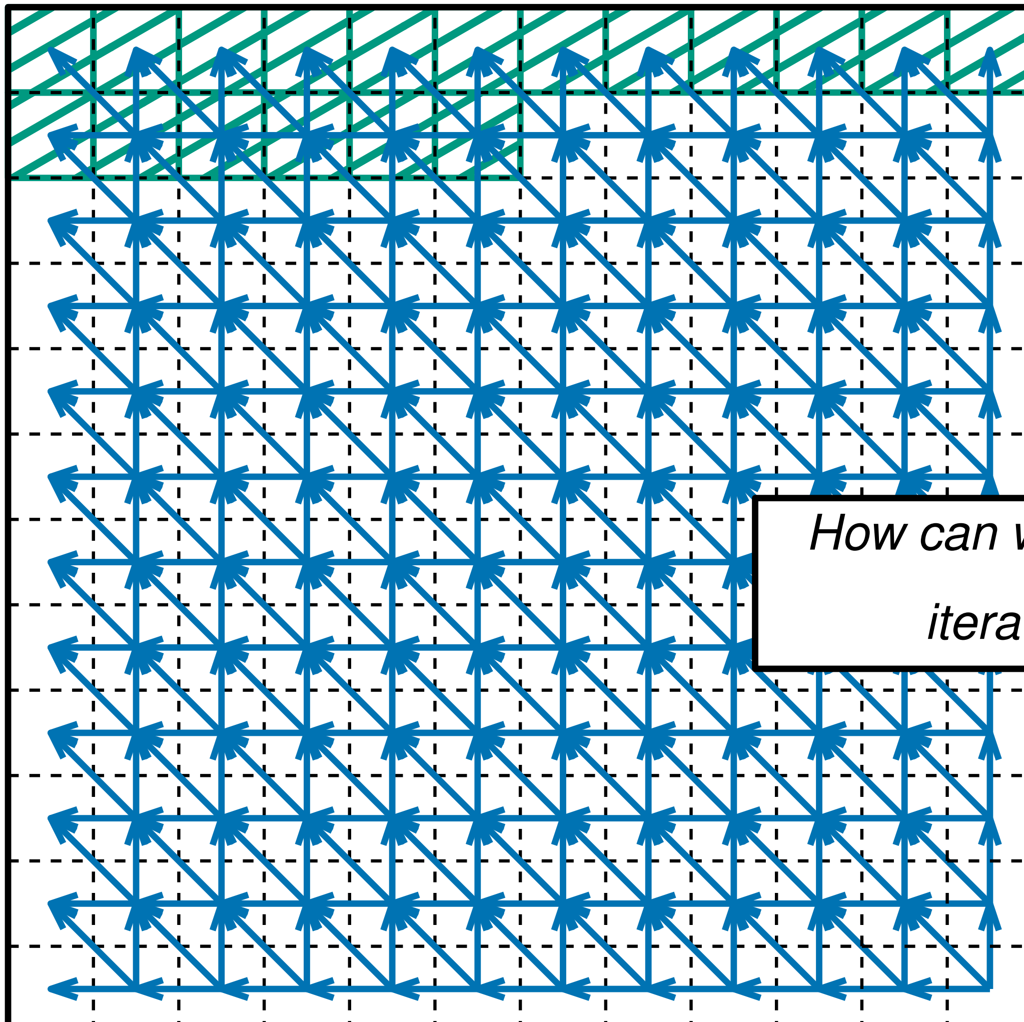

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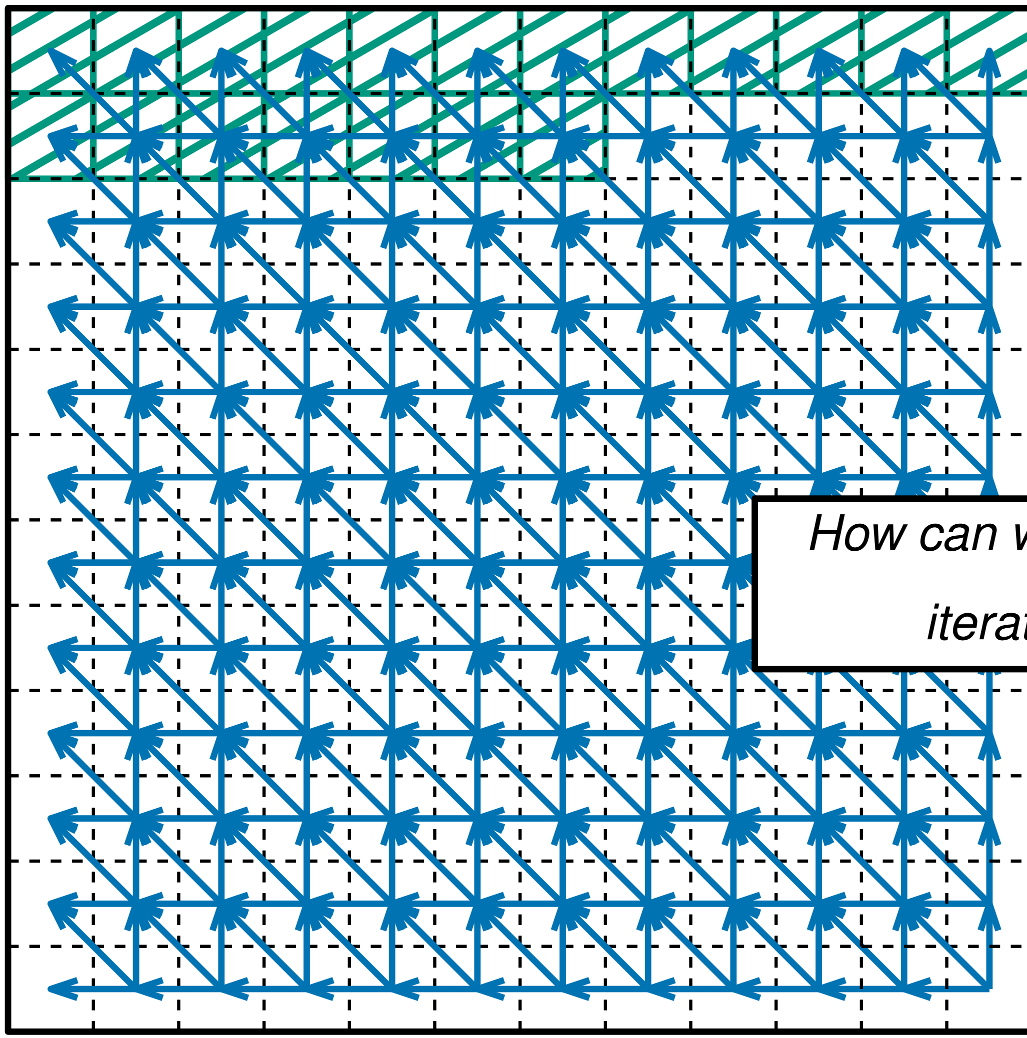

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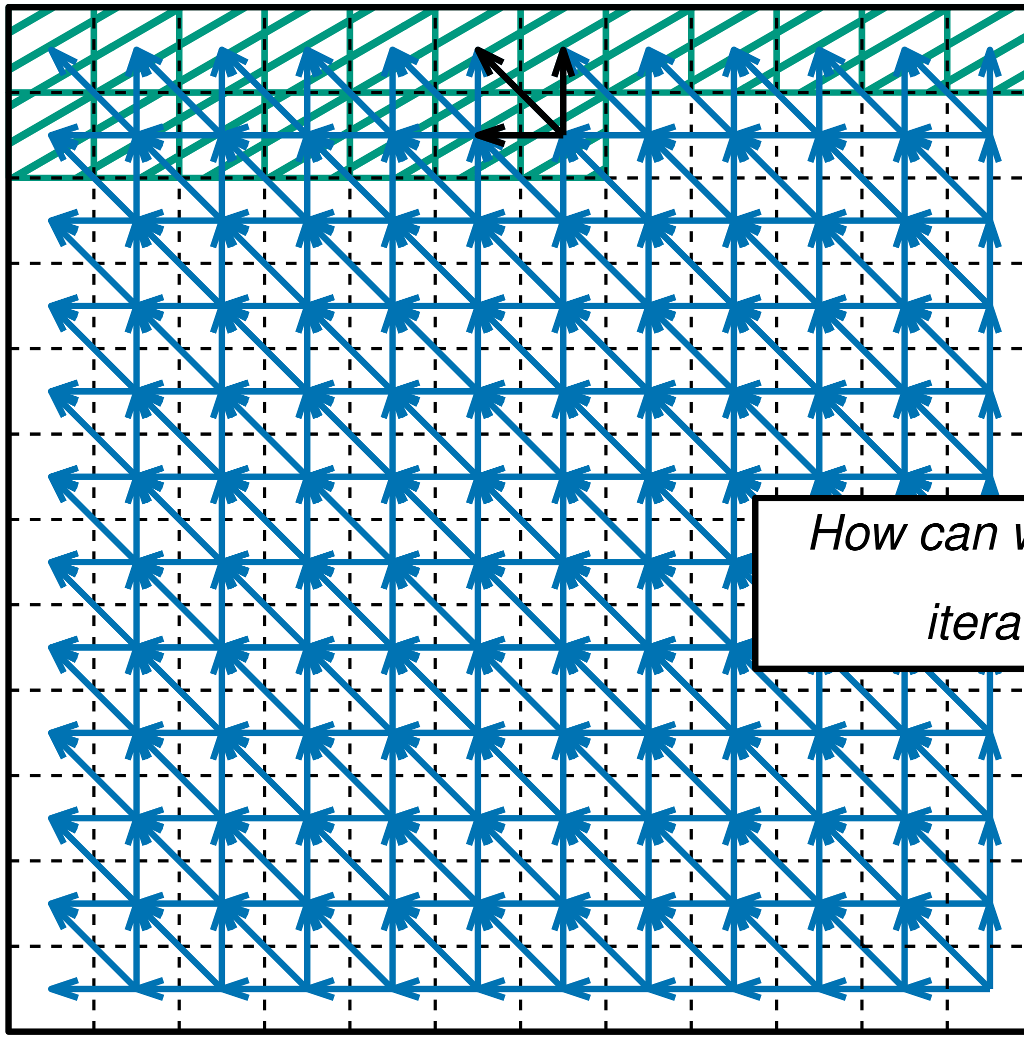

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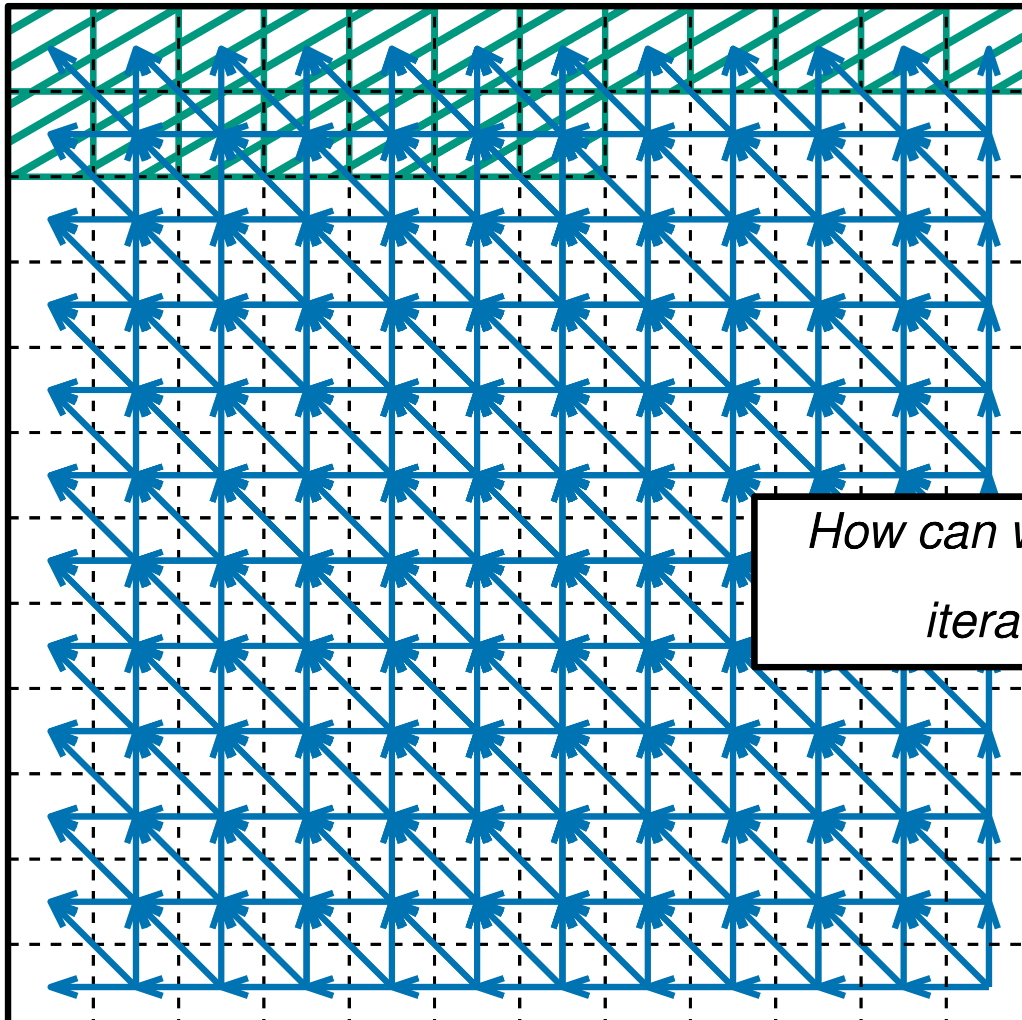

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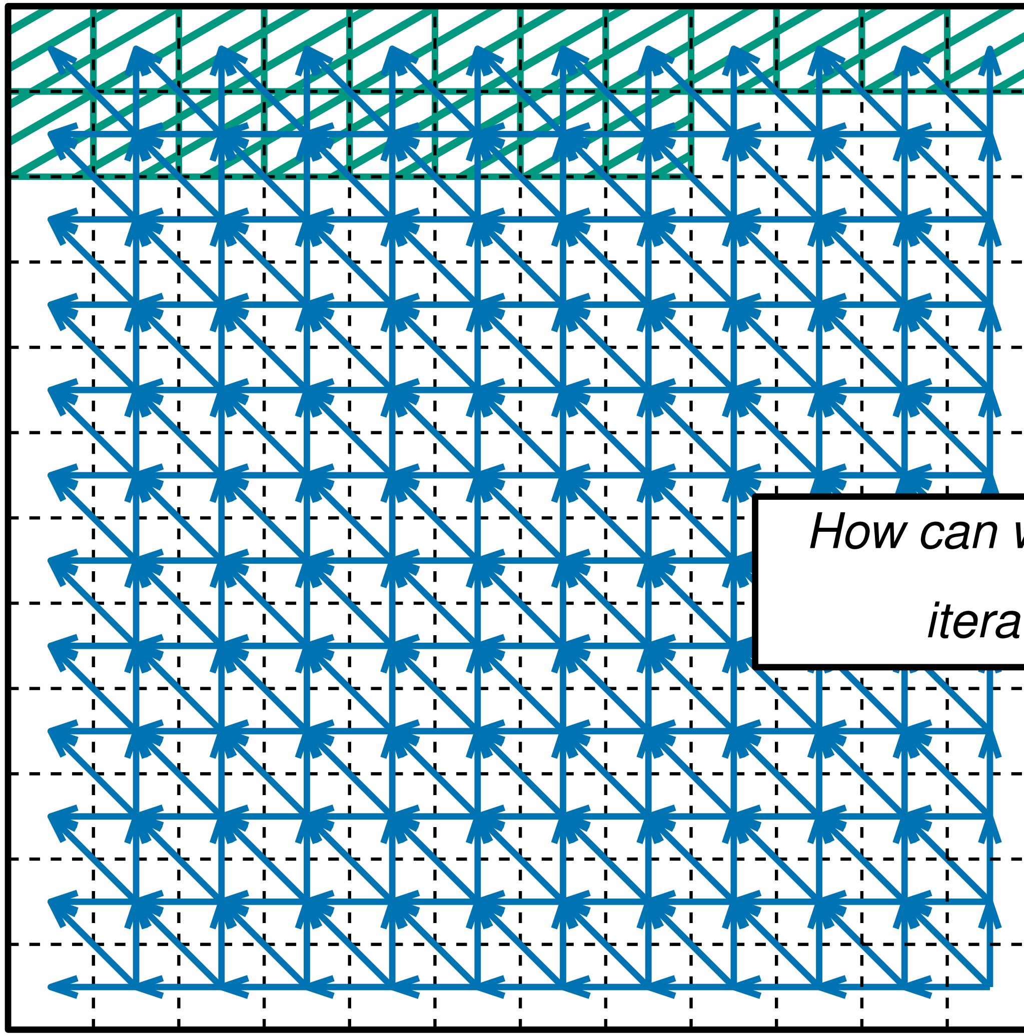

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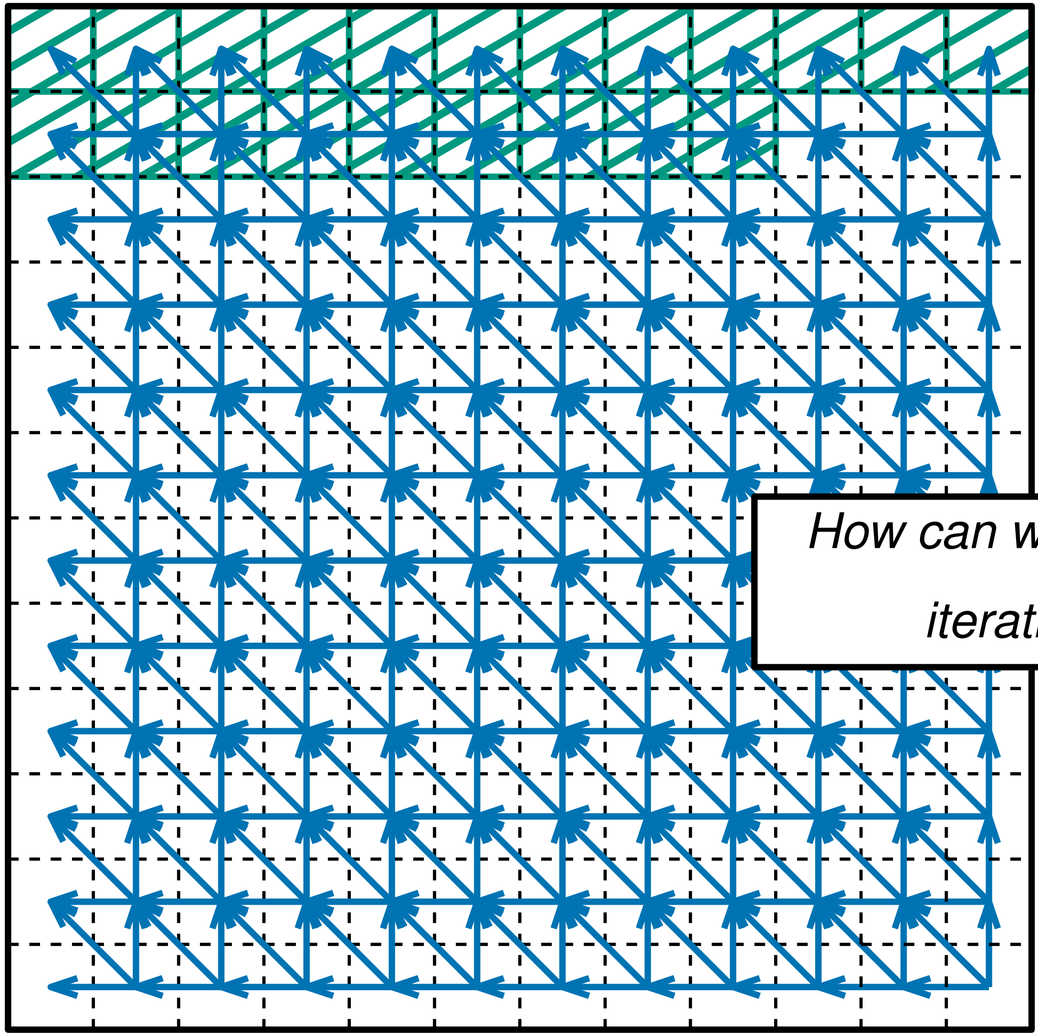

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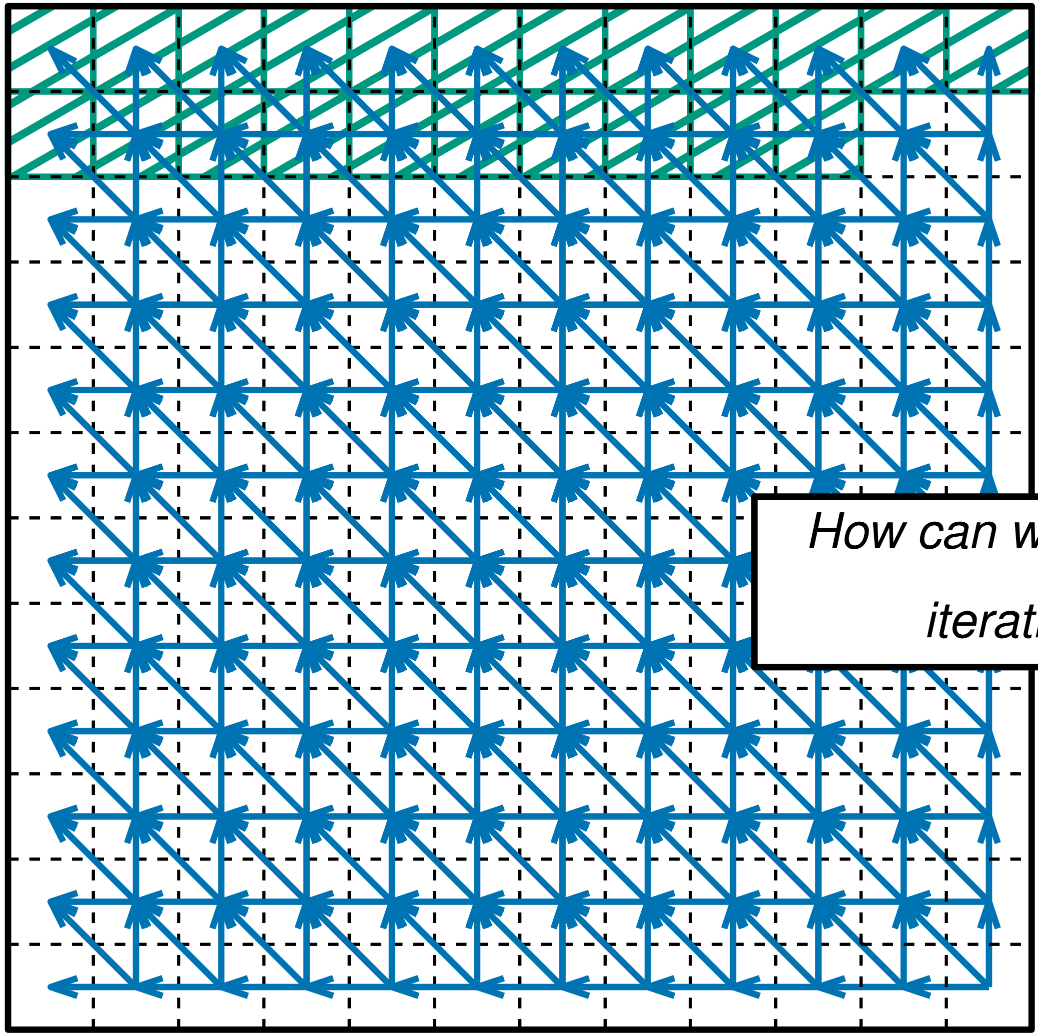

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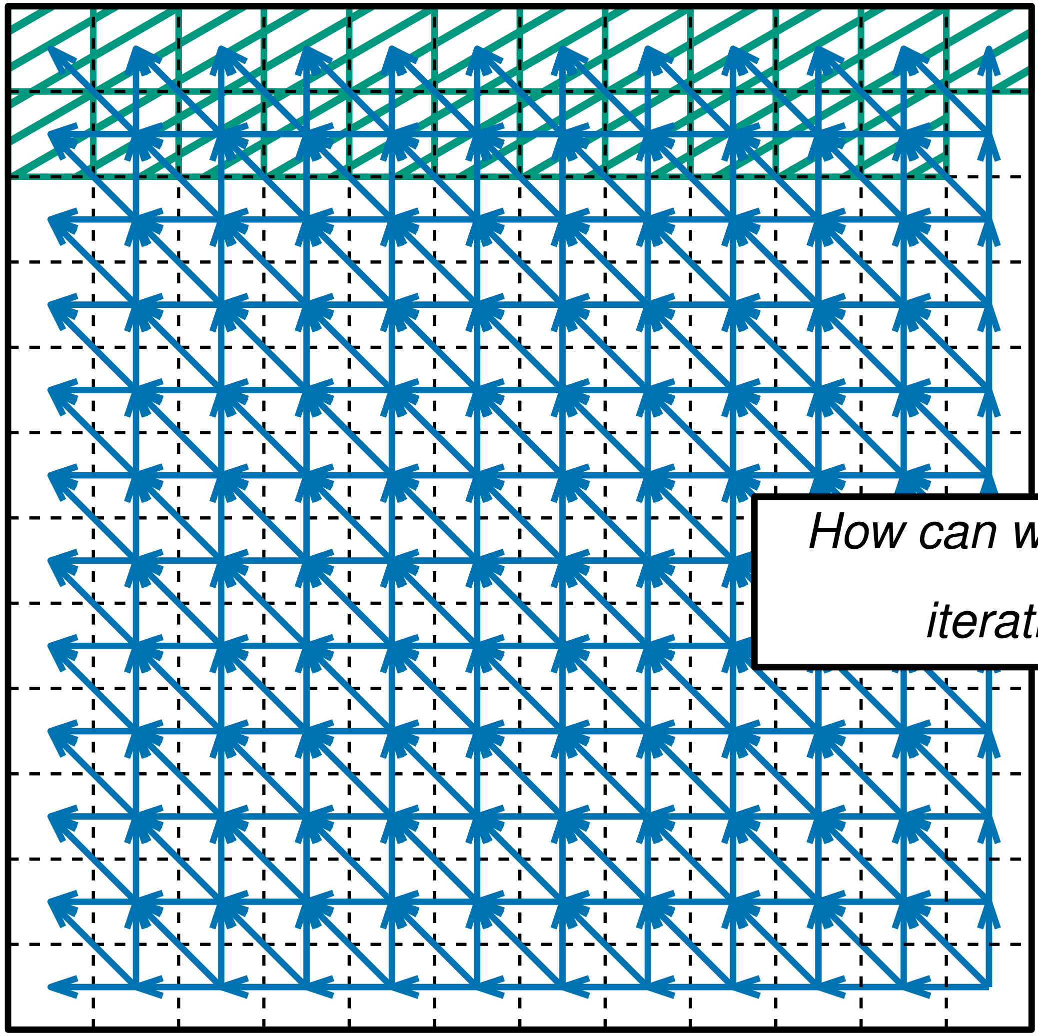

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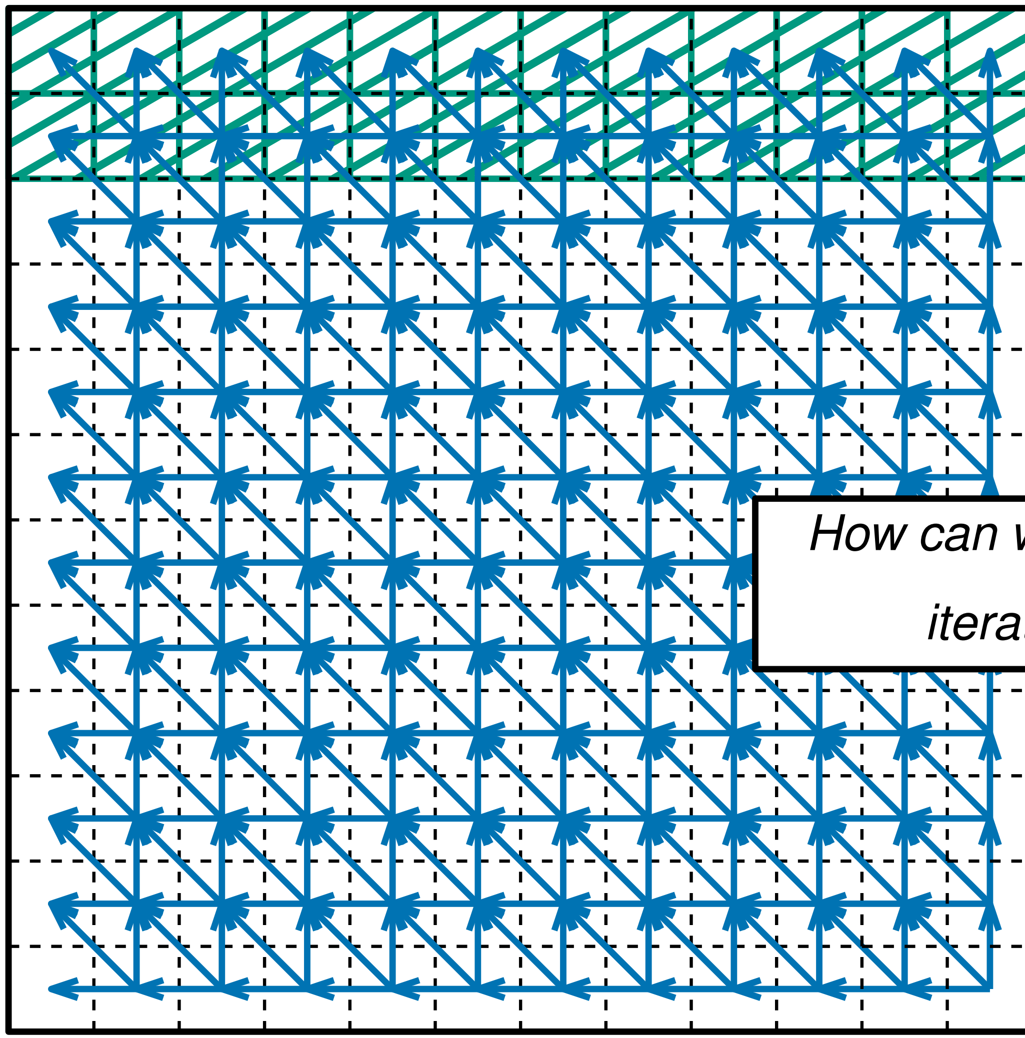

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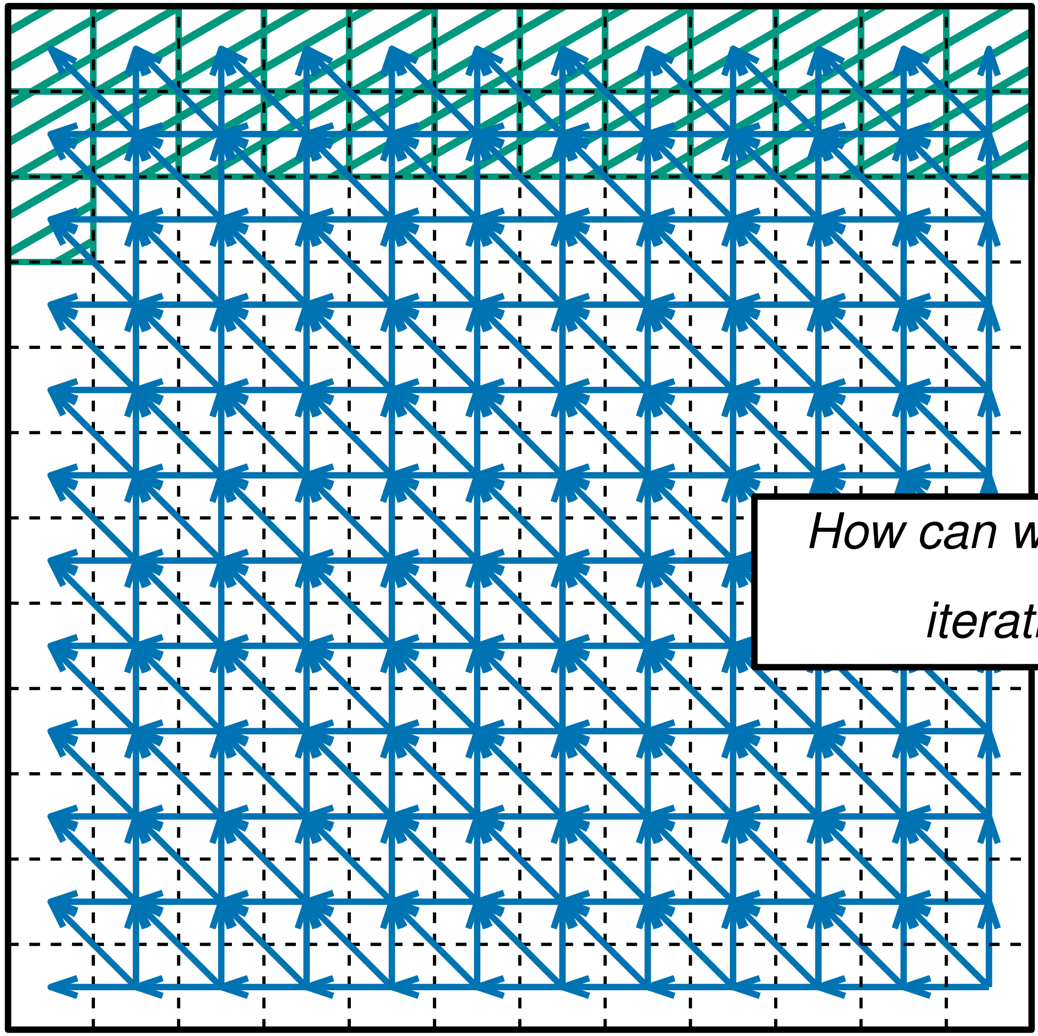

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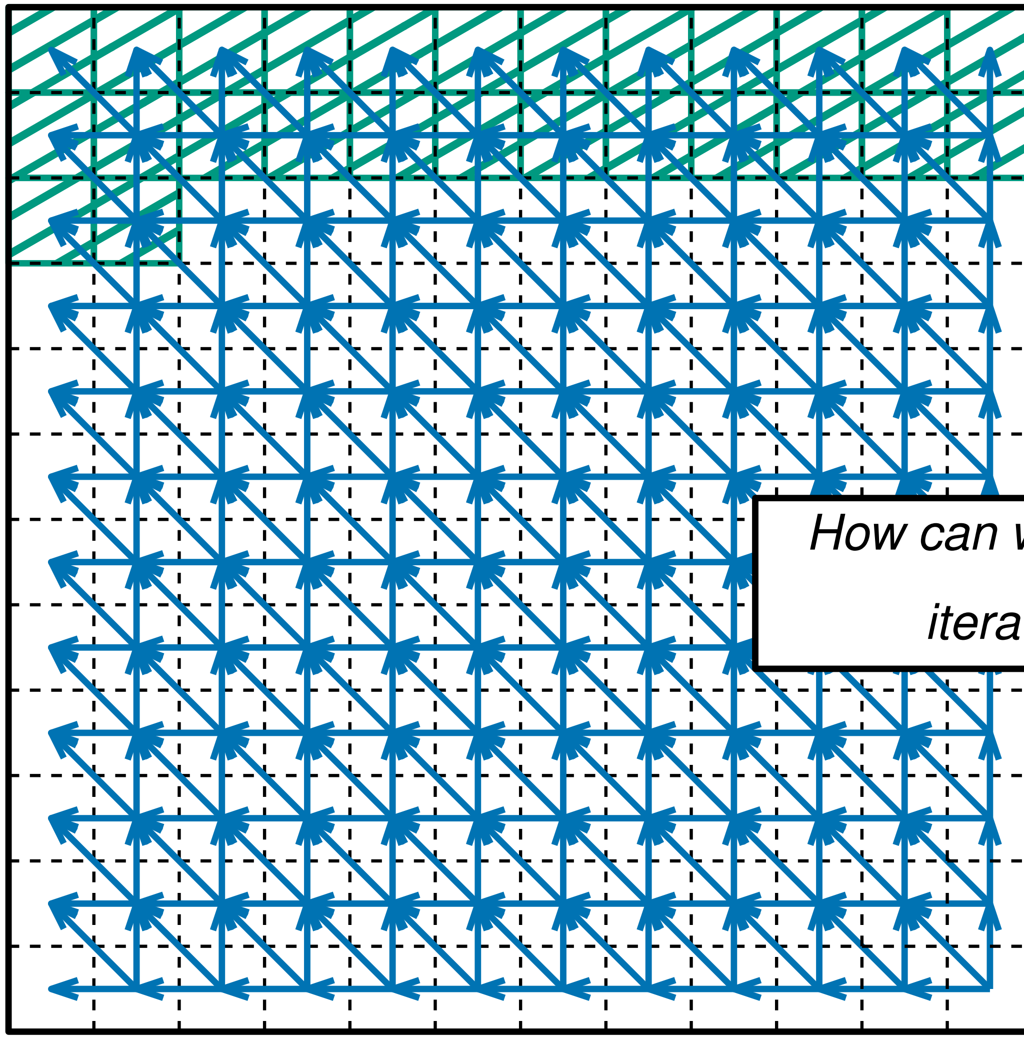

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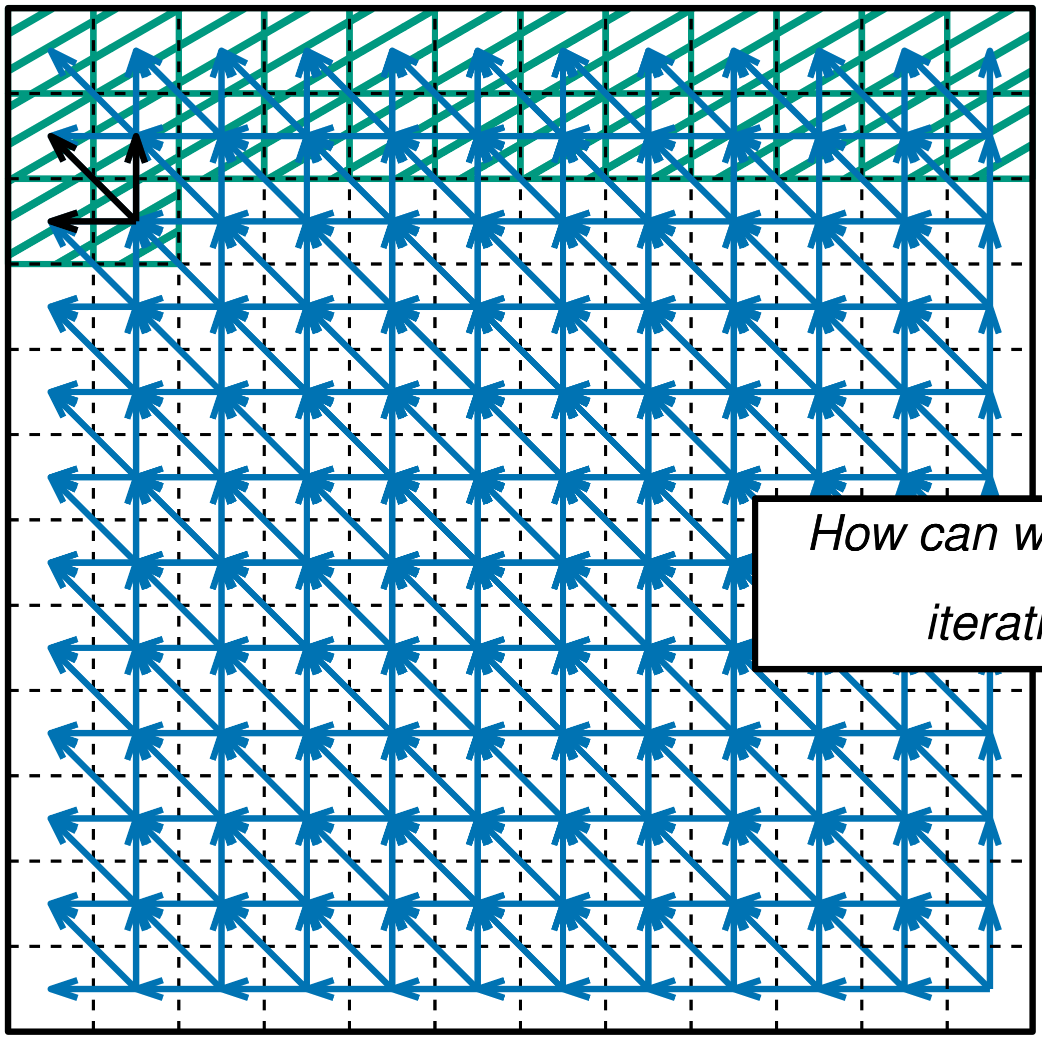

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4. Derive an iterative algorithm

```
ItLES(n)
```

```
For y = 1 to n
  For x = 1 to n
    If pixel (x, y) is not empty
      LES[x, y] = 0
    Else If (x = 1) or (y = 1)
      Return 1
    Else
      LES[x, y] = min(LES[x − 1, y − 1], LES[x − 1, y], LES[x, y − 1]) + 1
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This iterative version of the algorithm

runs in $O(n^2)$ time

and avoids making any recursive calls

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Maximum of $\mathrm{LES}[x, y]$ over all x and y

gives the size of the largest empty square in the whole image

this also takes $O(n^2)$ time

~~Introduction~~ Summary

Dynamic programming is a technique for finding efficient algorithms for problems which can be broken down into simpler, overlapping subproblems.

The basic idea:

1. Find a recursive formula for the problem
   - in terms of answers to subproblems.

   (typically this is the hard bit)

2. Write down a naive recursive algorithm

   (typically this algorithm will take exponential time)

3. Speed it up by storing the solutions to subproblems (memoization)

   (to avoid recomputing the same thing over and over)

4. Derive an iterative algorithm by solving the subproblems in a good order

   (iterative algorithms are often better in practice, easier to analyse and prettier)

in other words. . .

Dynamic programming is *recursion without repetition*

End of part one

Part two

Weighted Interval Scheduling

Dynamic programming is a technique for finding efficient algorithms for problems which can be broken down into simpler, overlapping subproblems.

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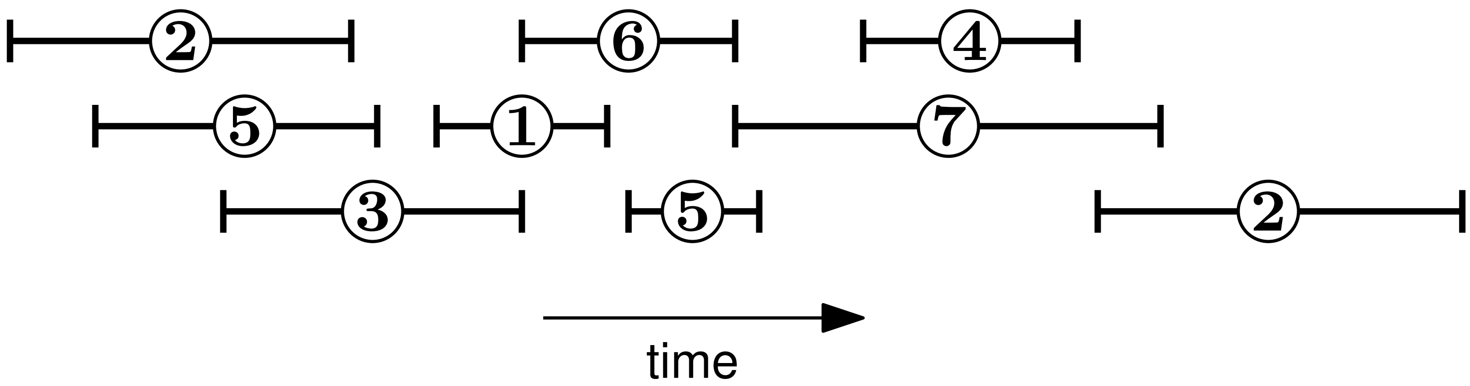
find the *schedule* with largest total weight



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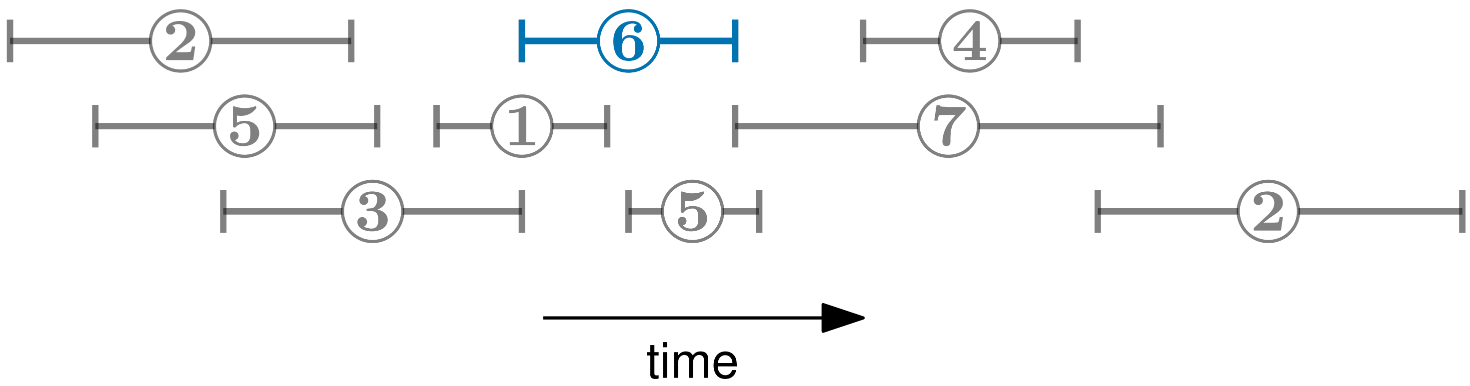

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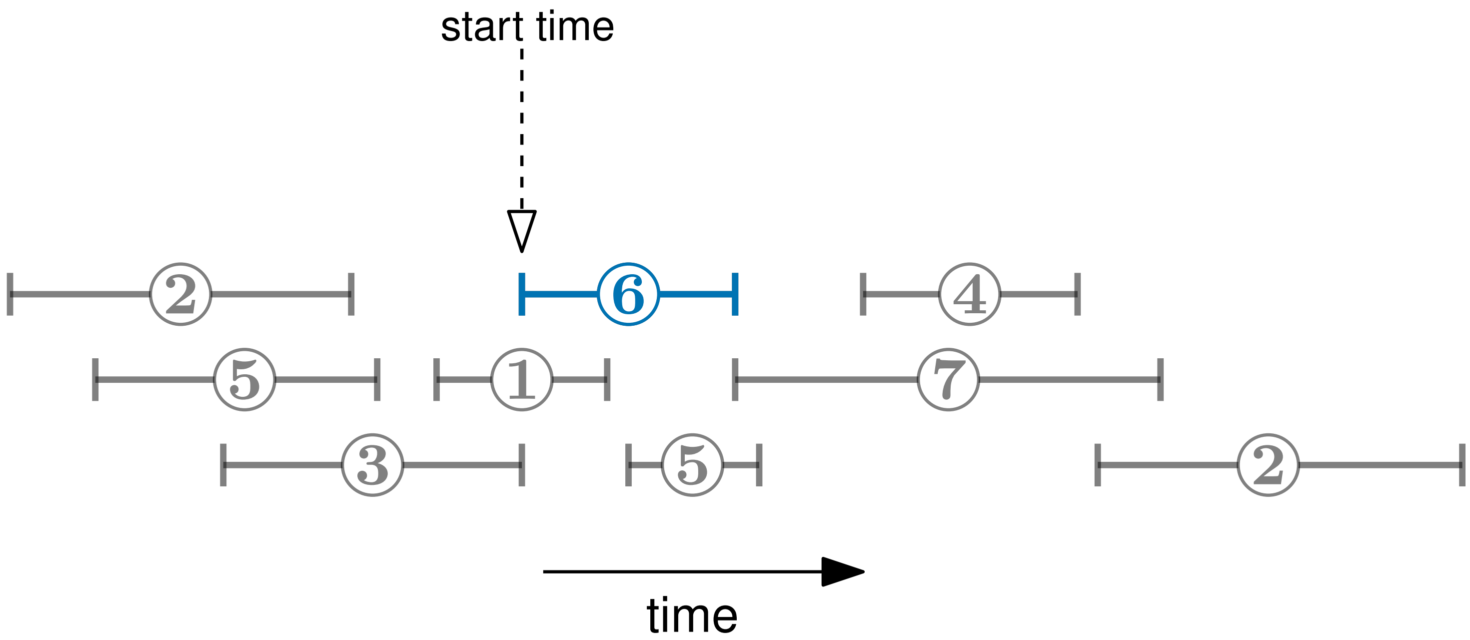

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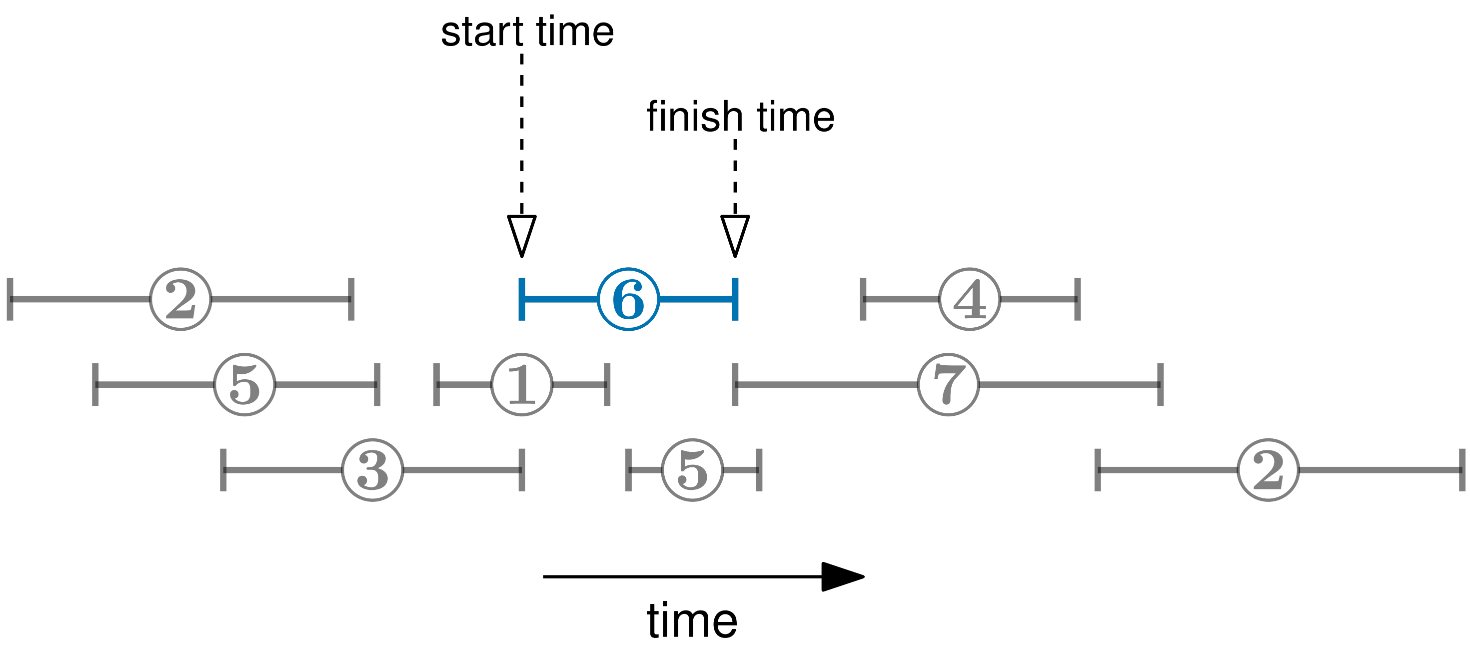

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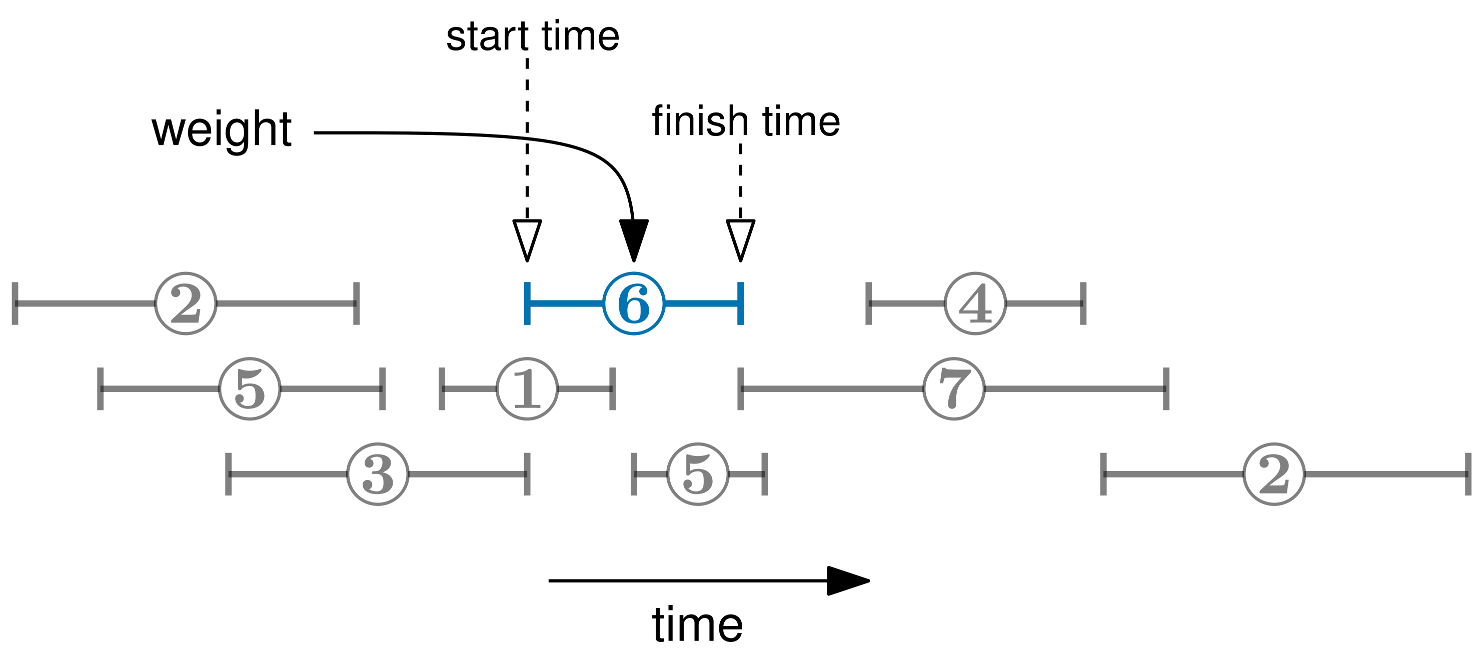

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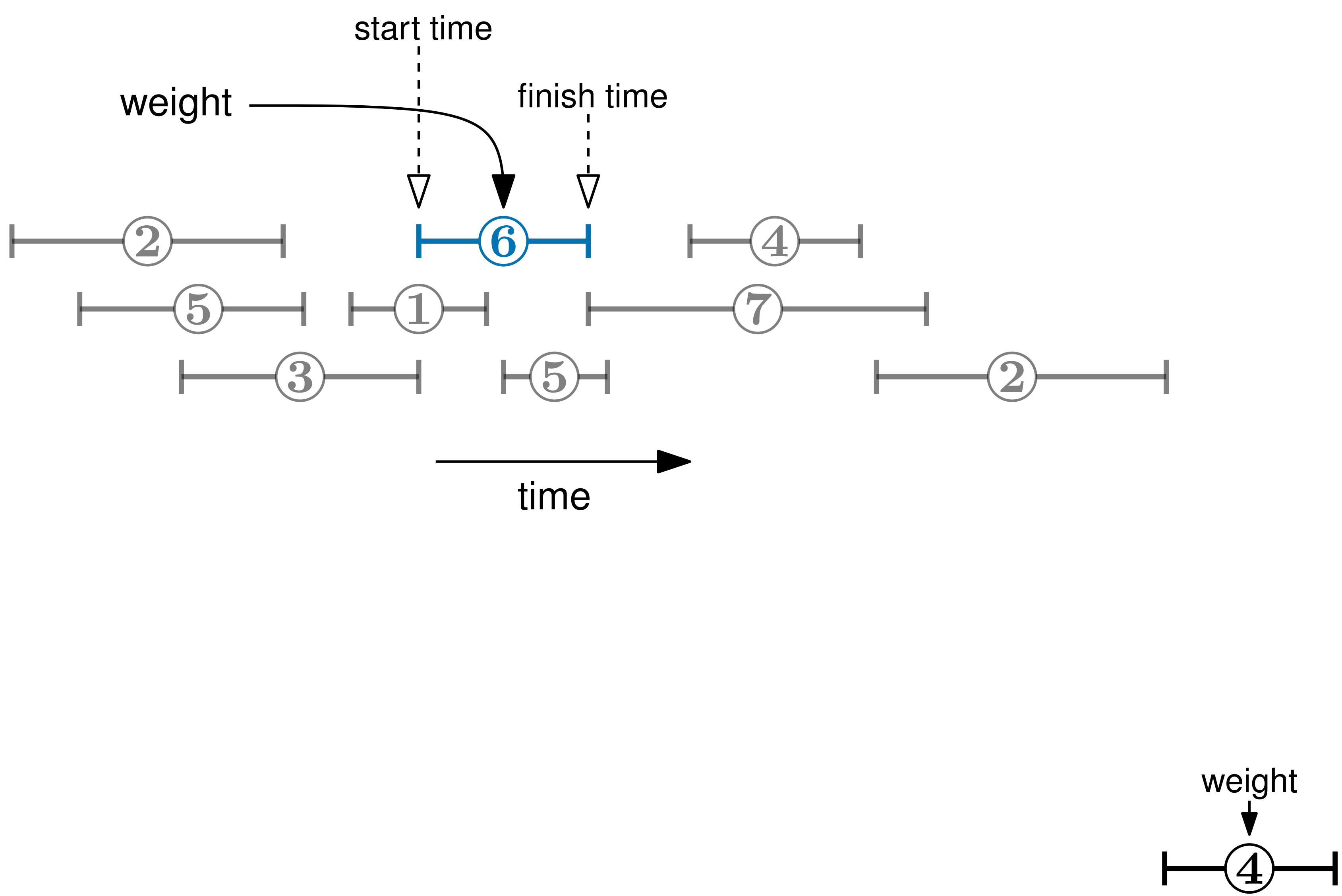

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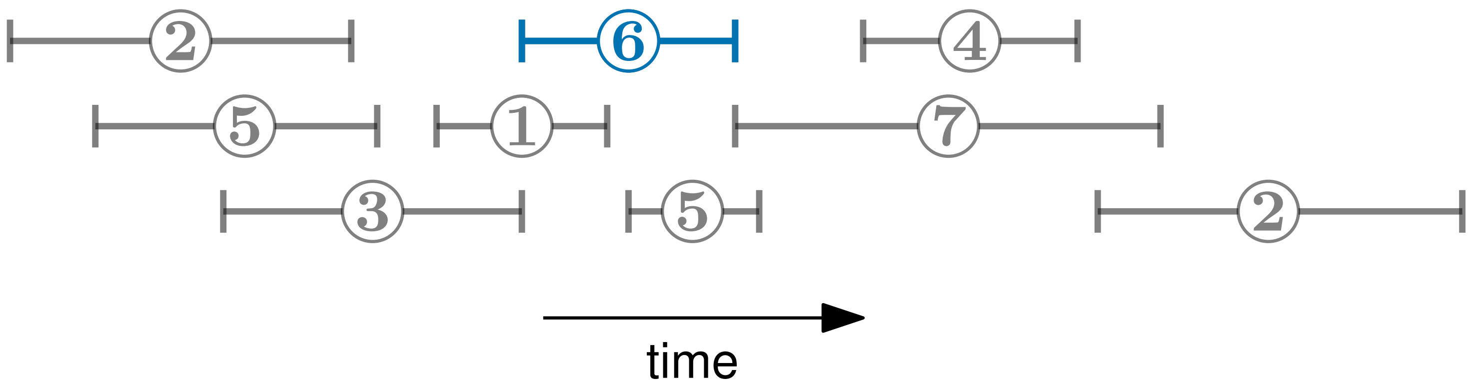

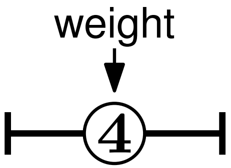

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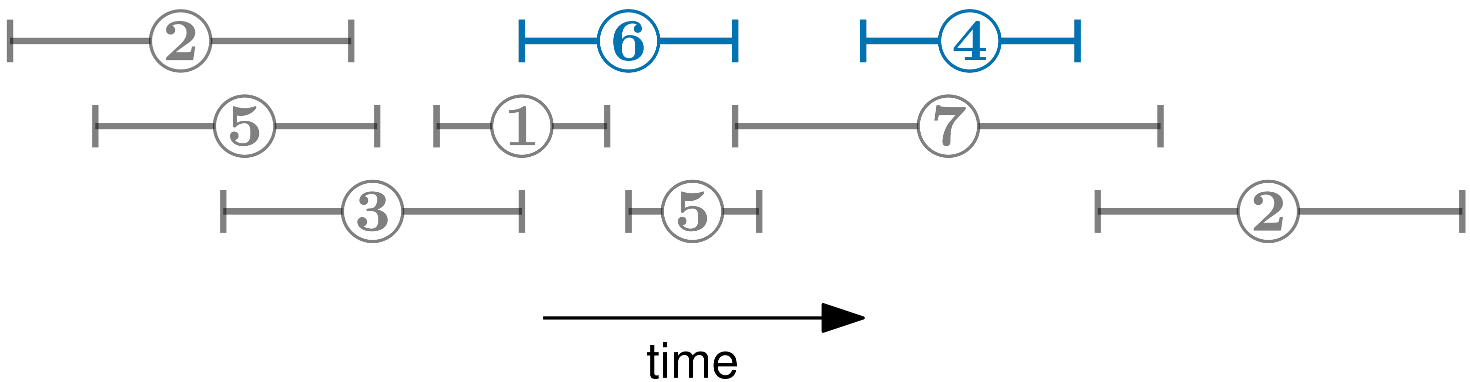

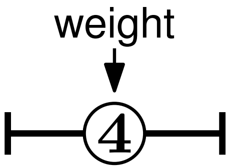

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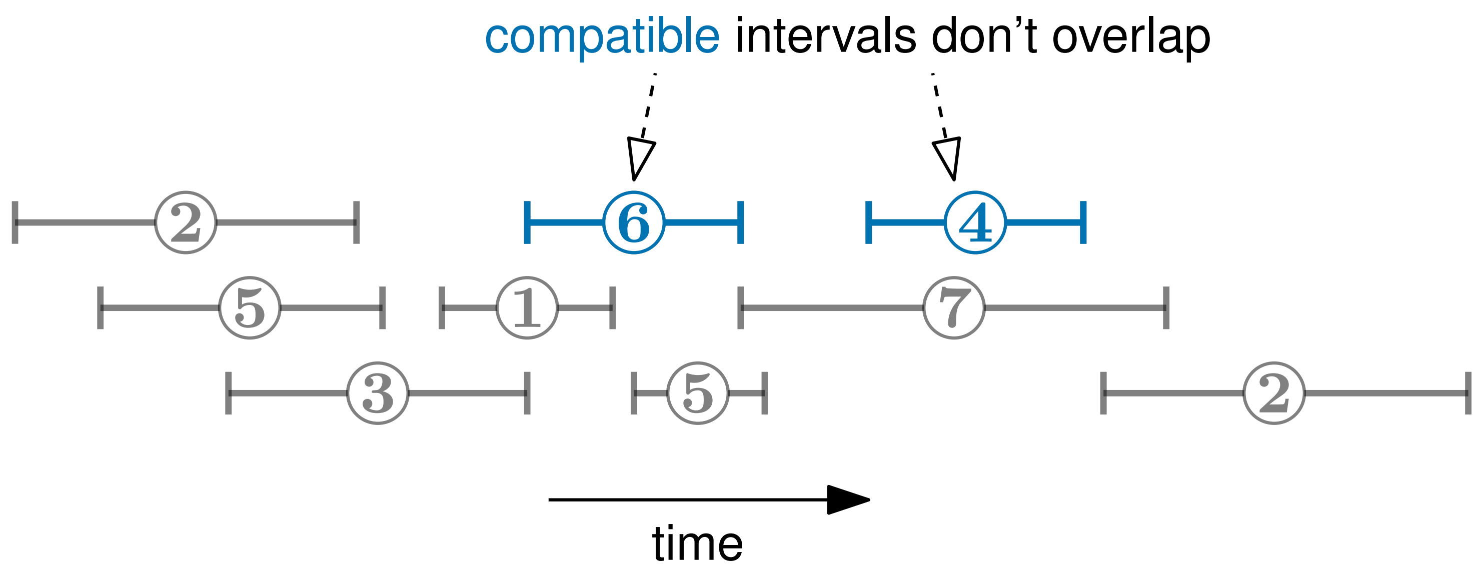

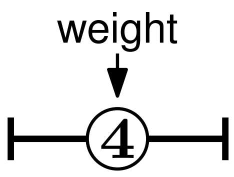

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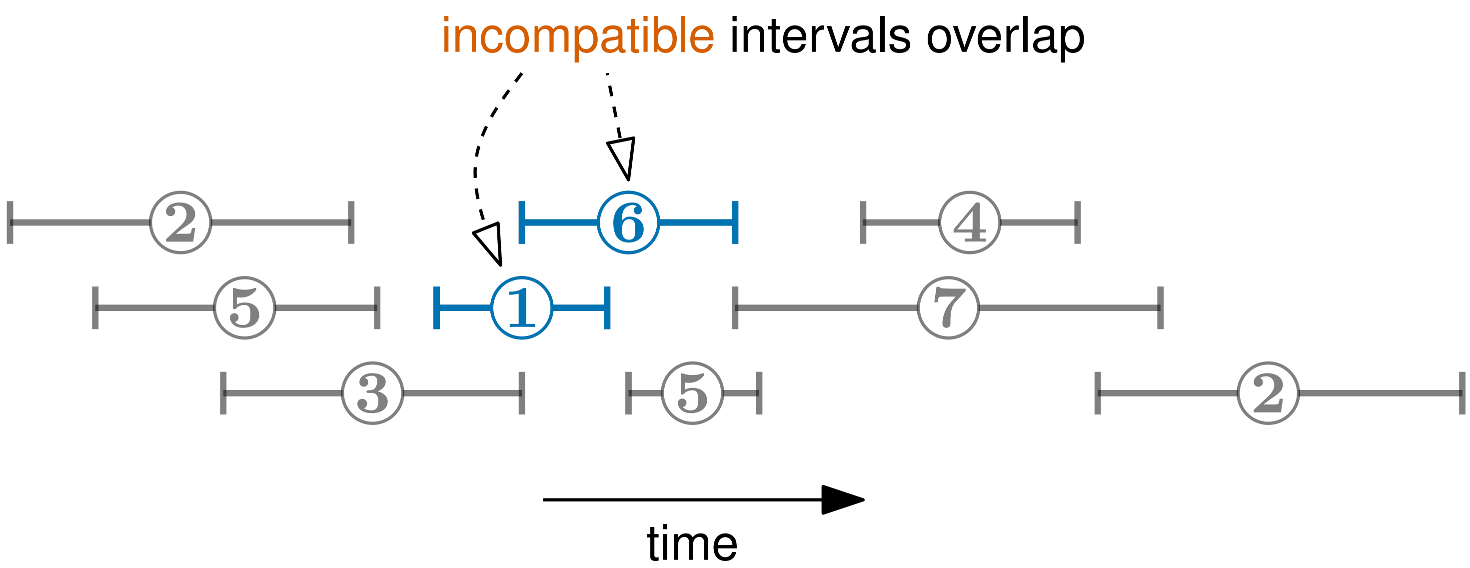

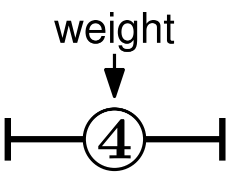

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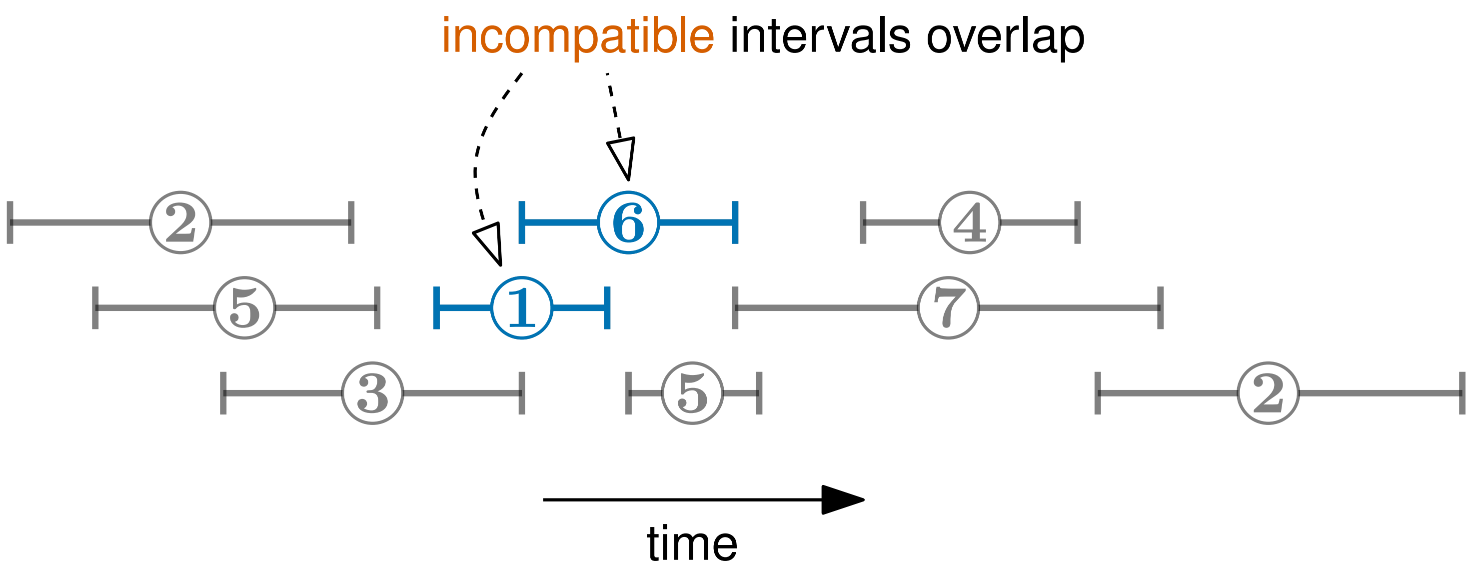

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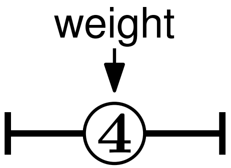

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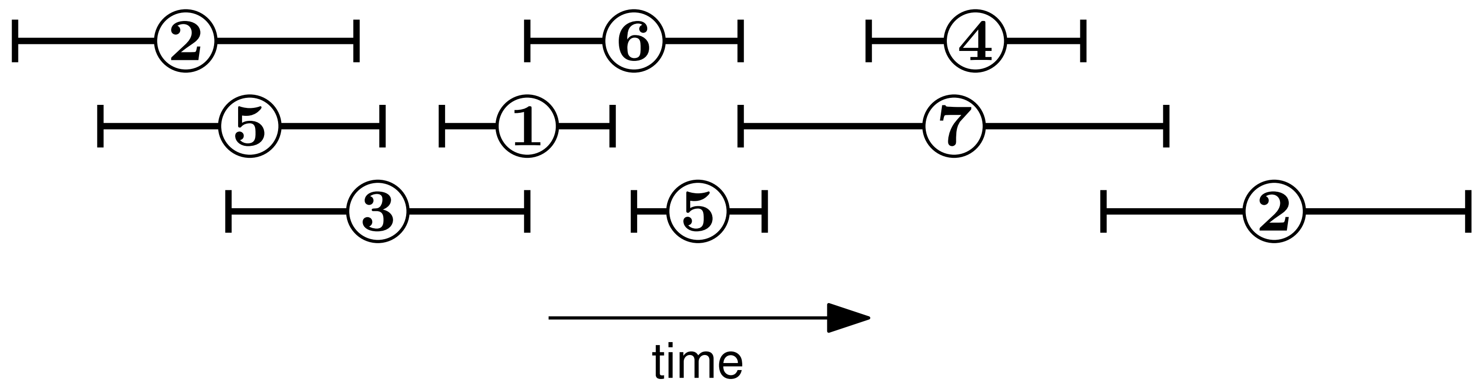

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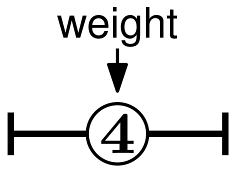

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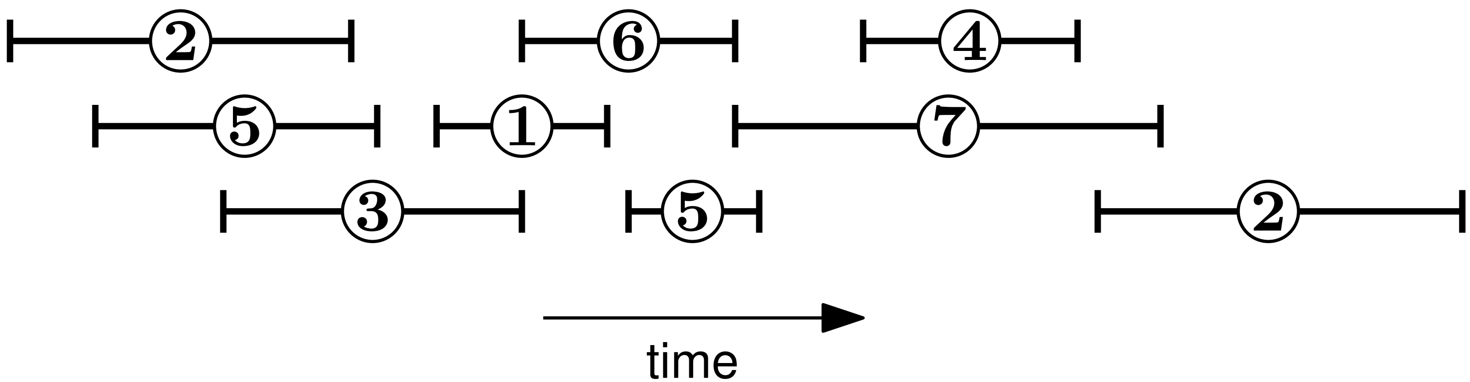

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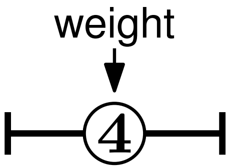

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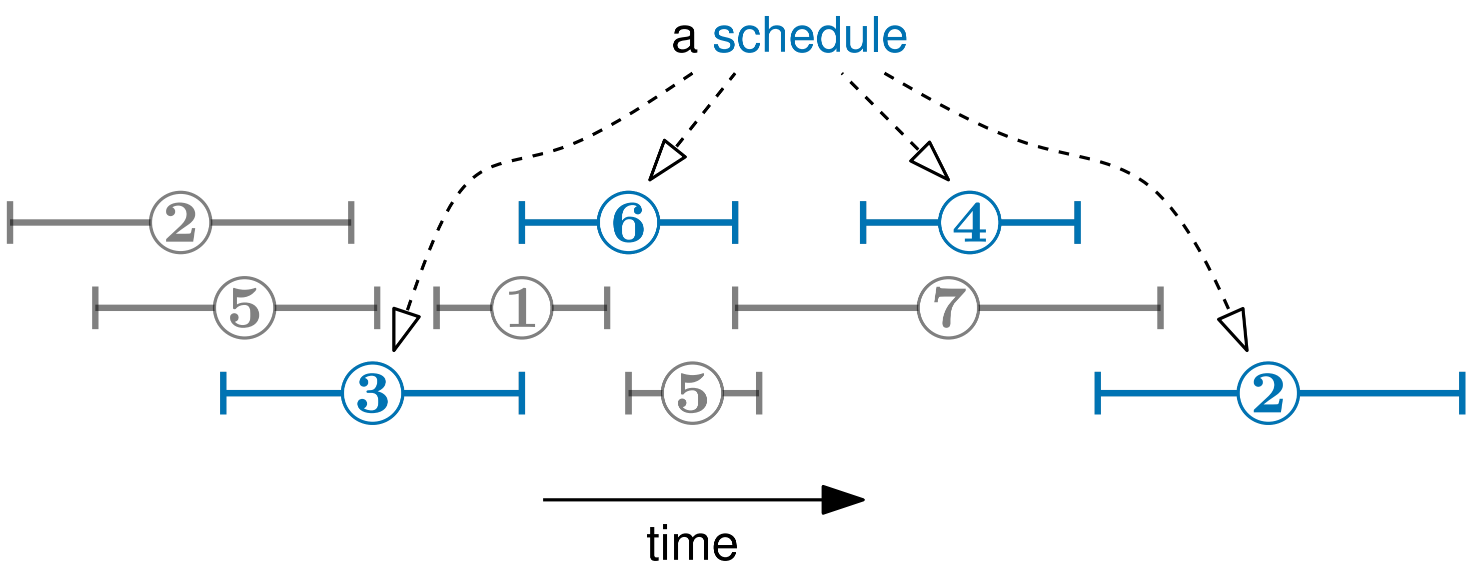

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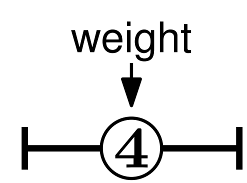

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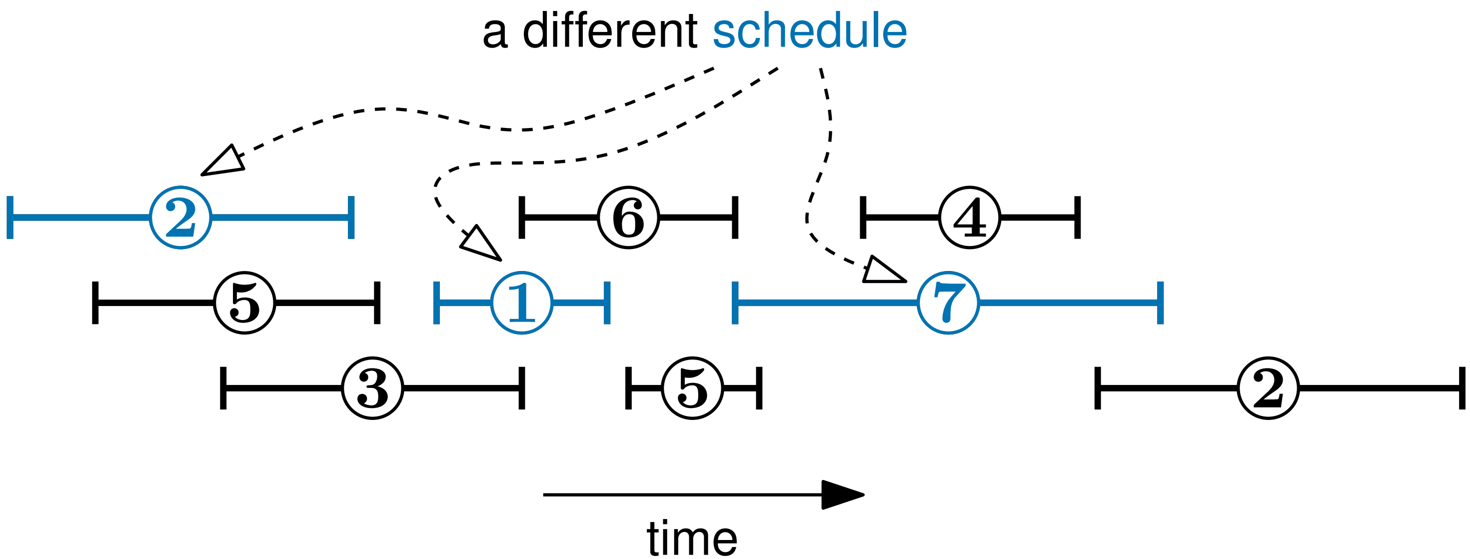

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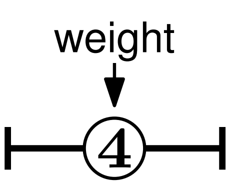

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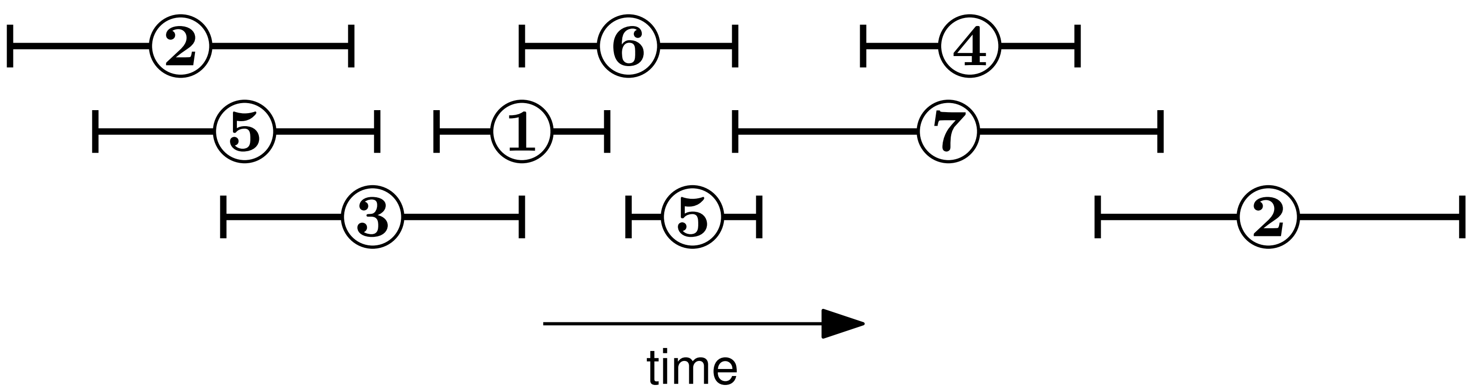

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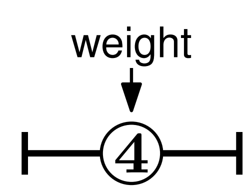

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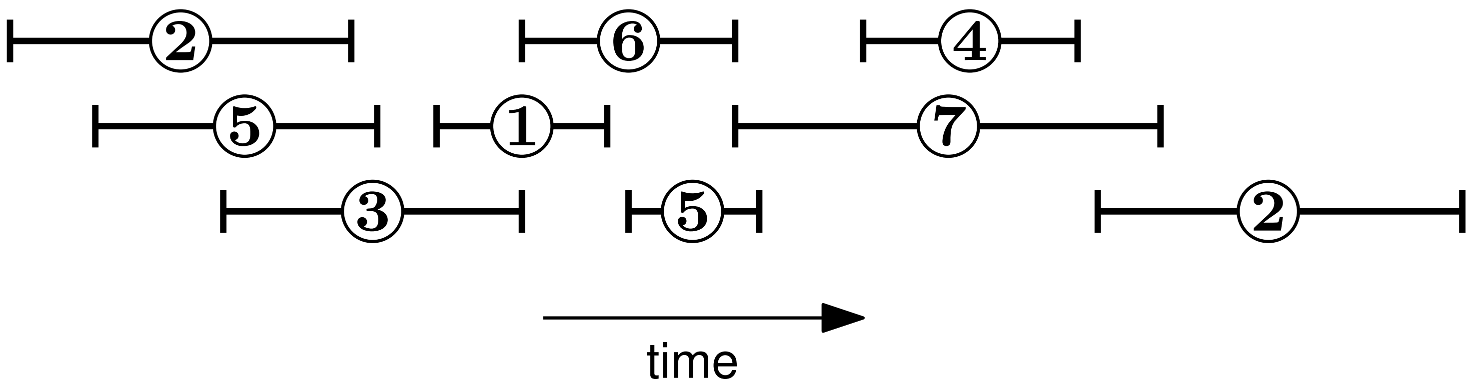

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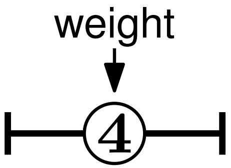

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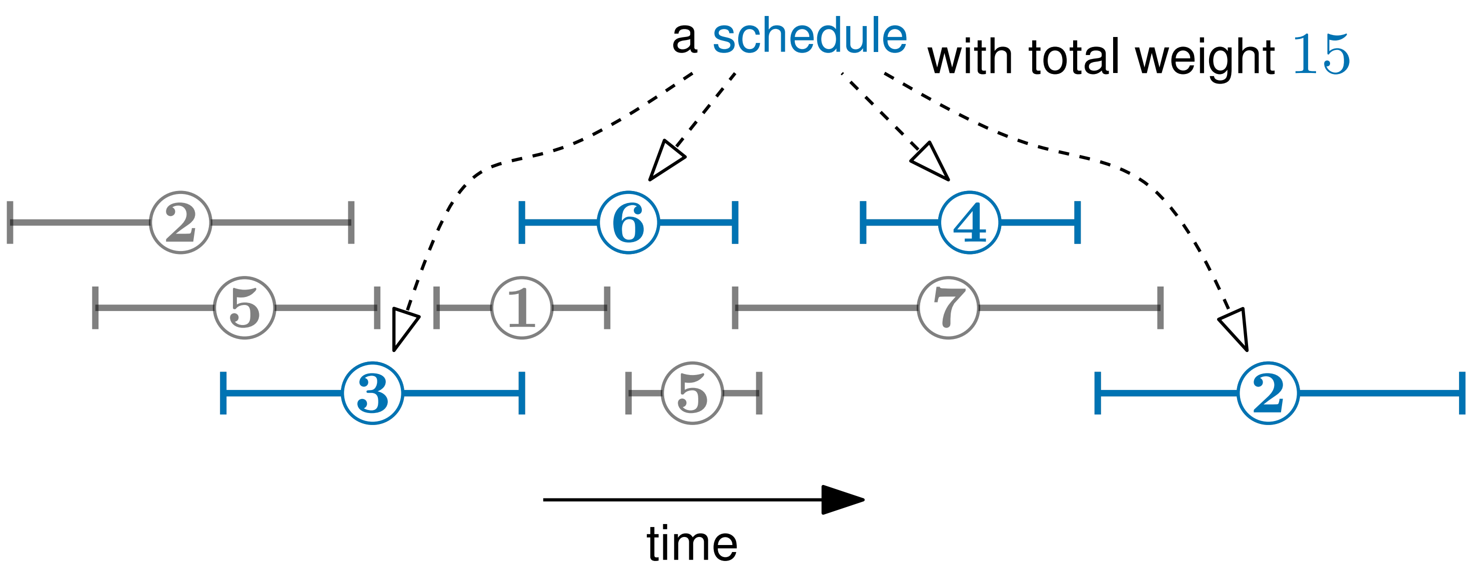

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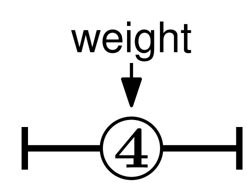

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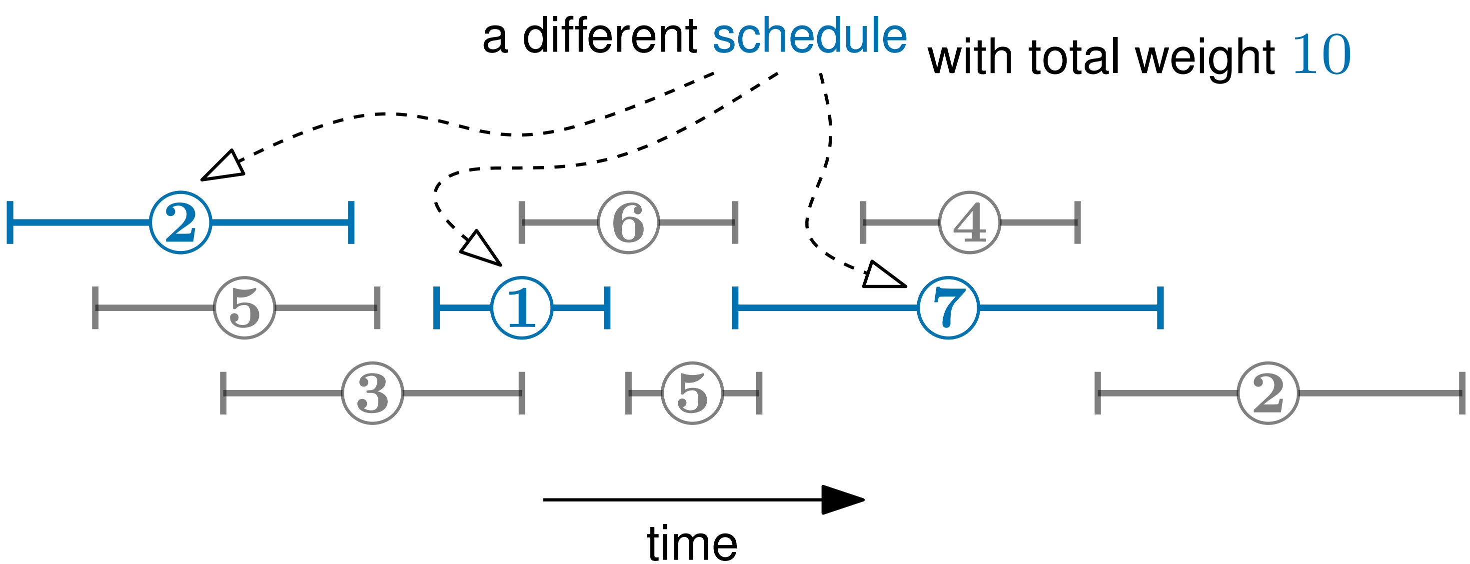

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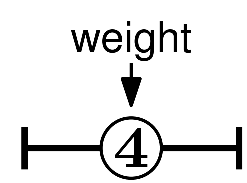

Weighted Interval Scheduling

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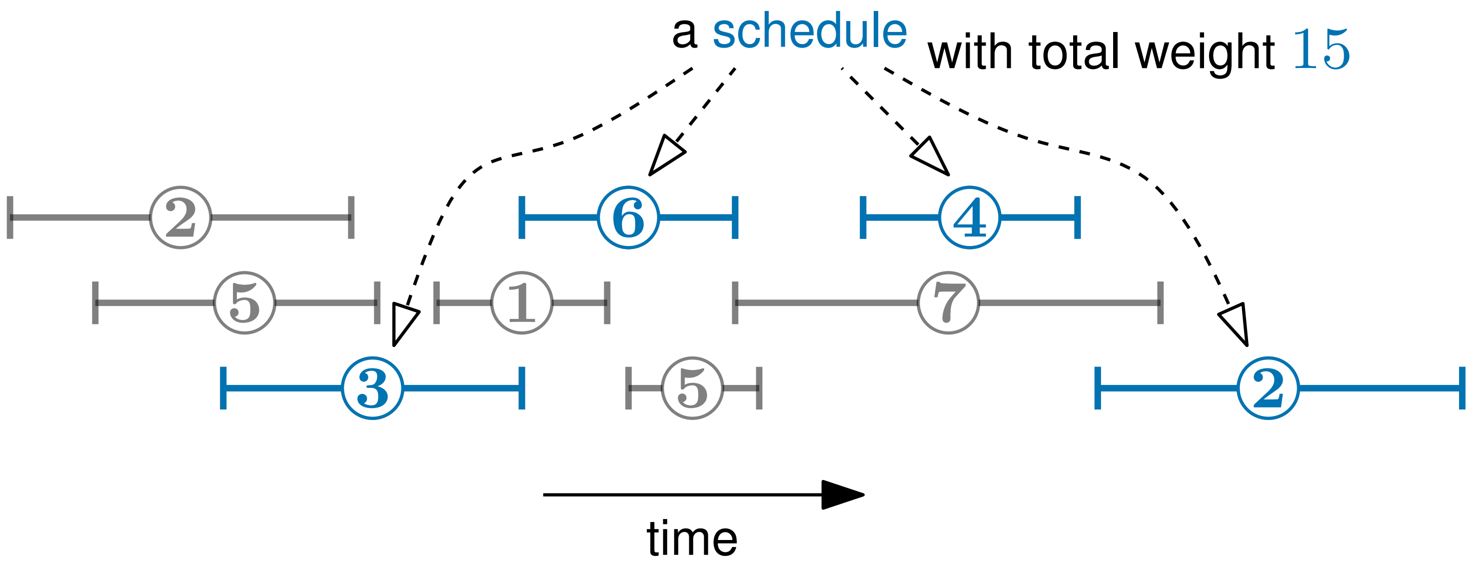

Two intervals are compatible if they don't overlap

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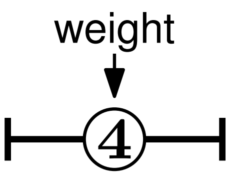

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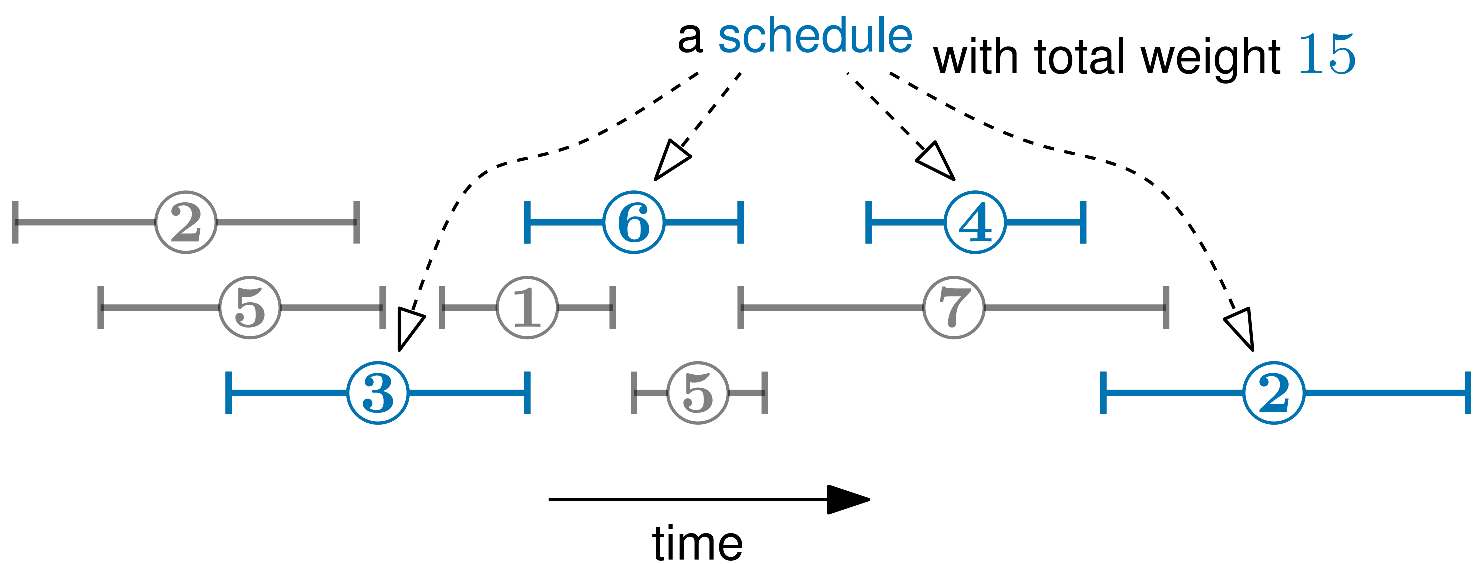

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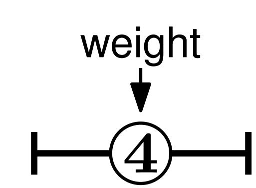

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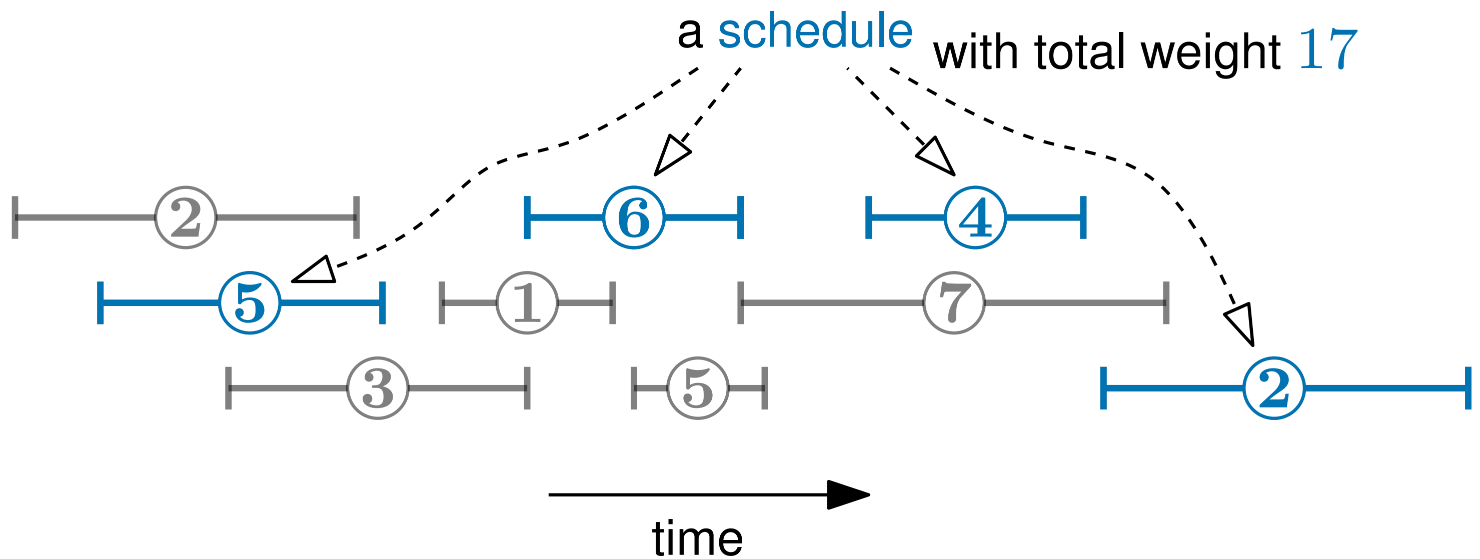

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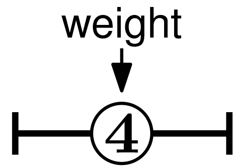

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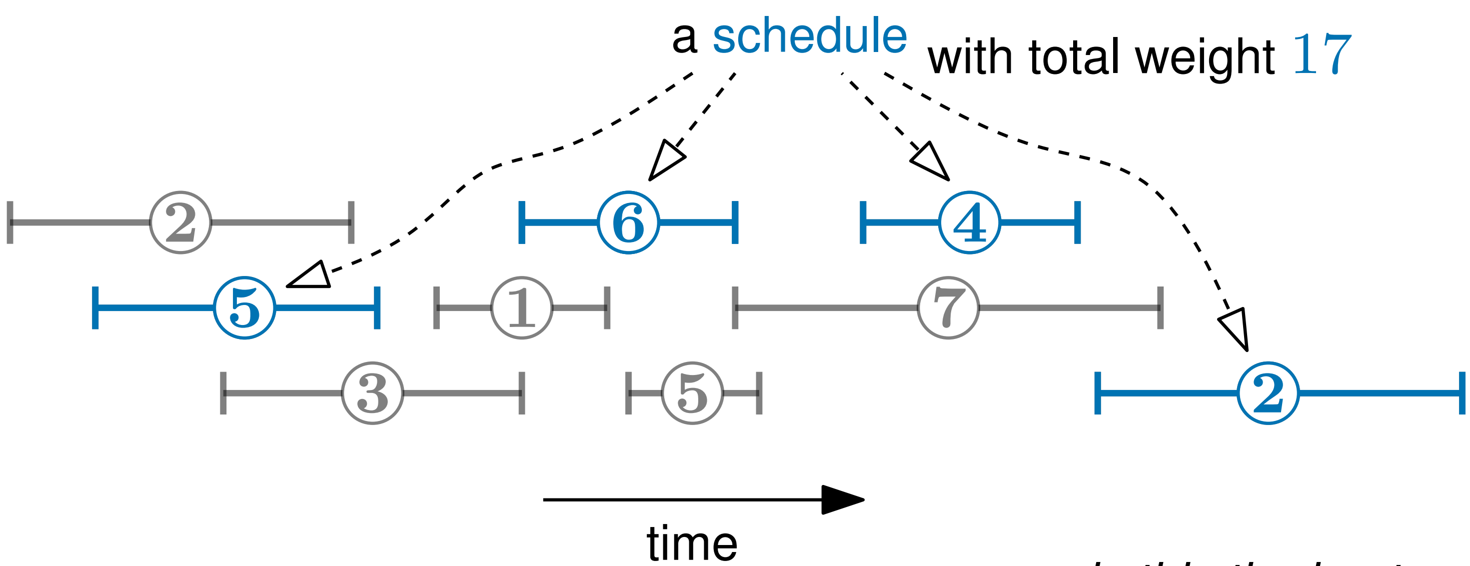

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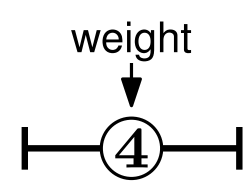

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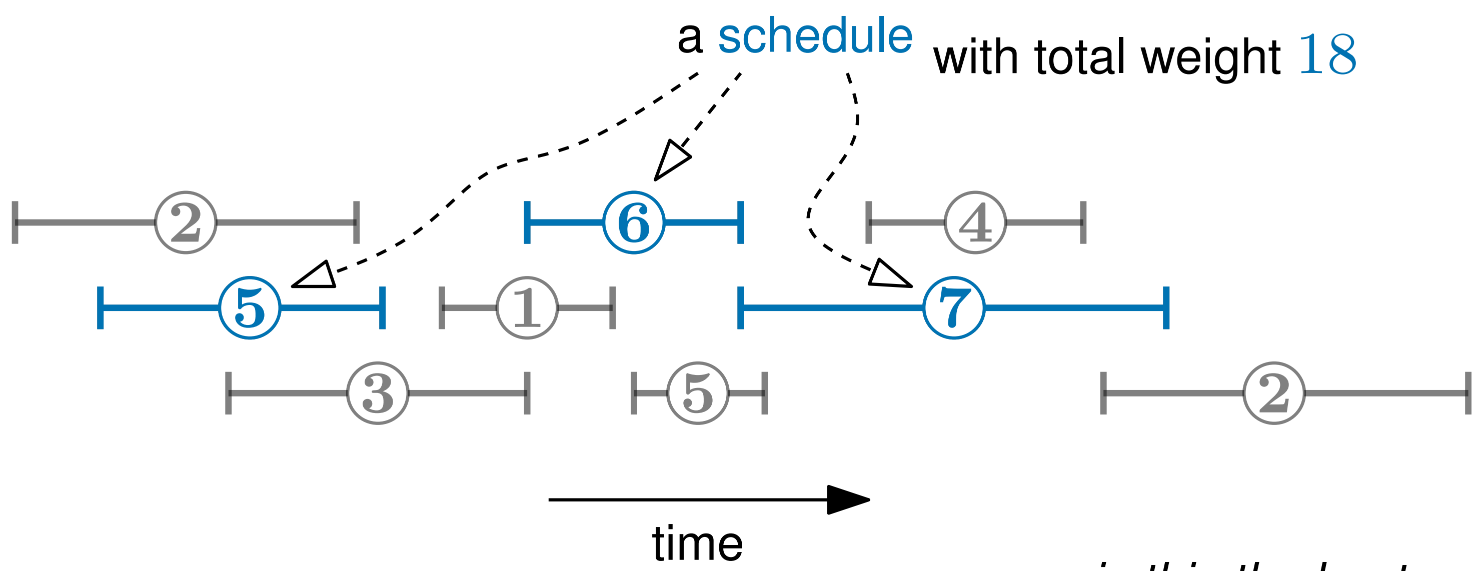

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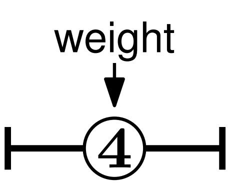

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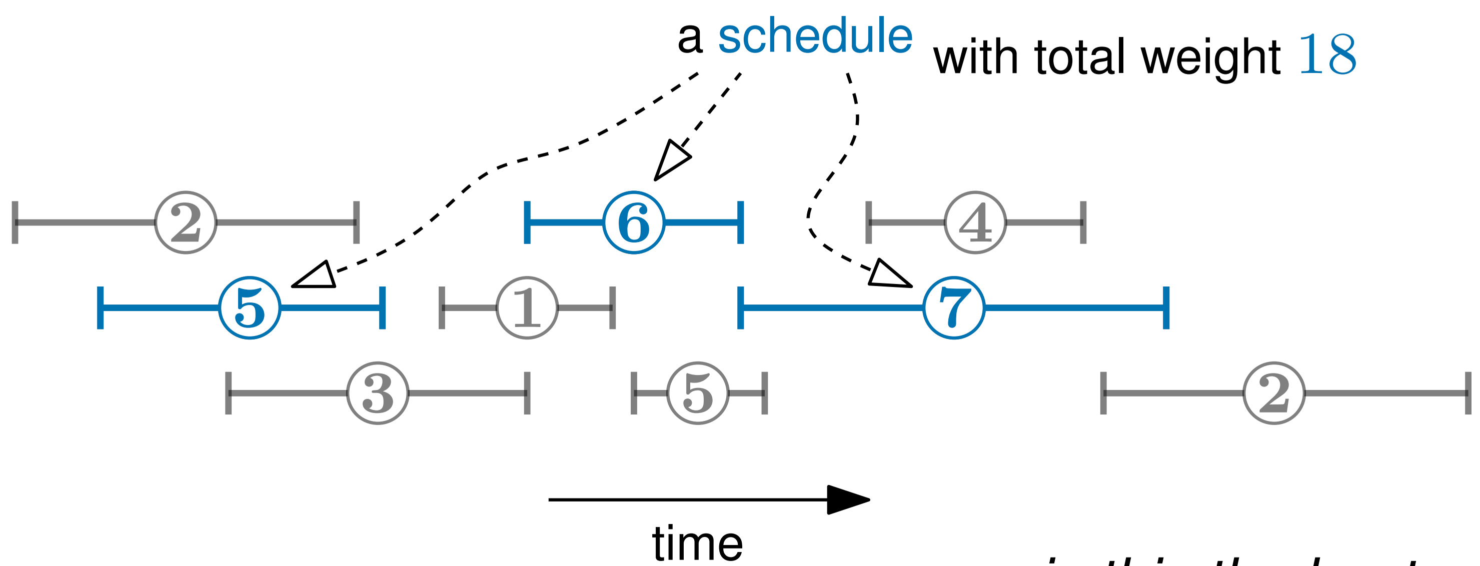

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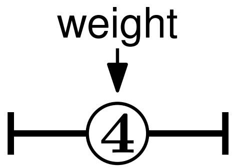

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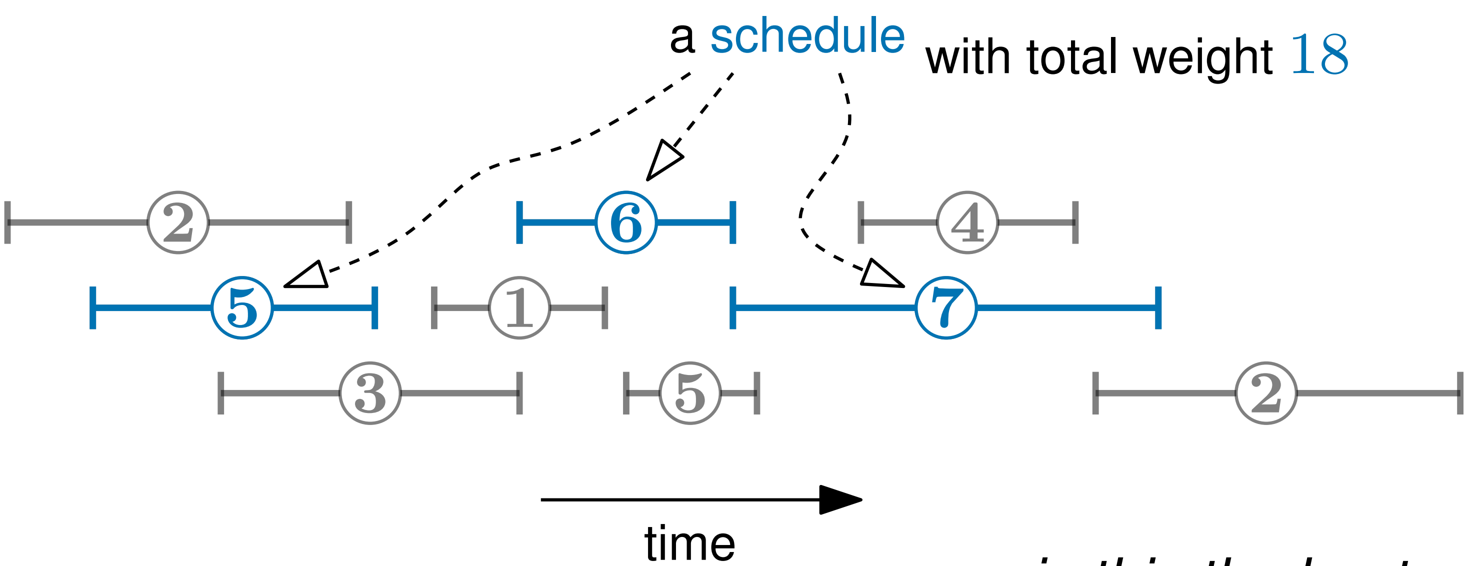

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yes

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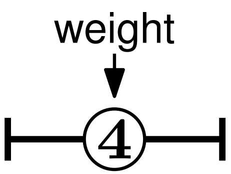

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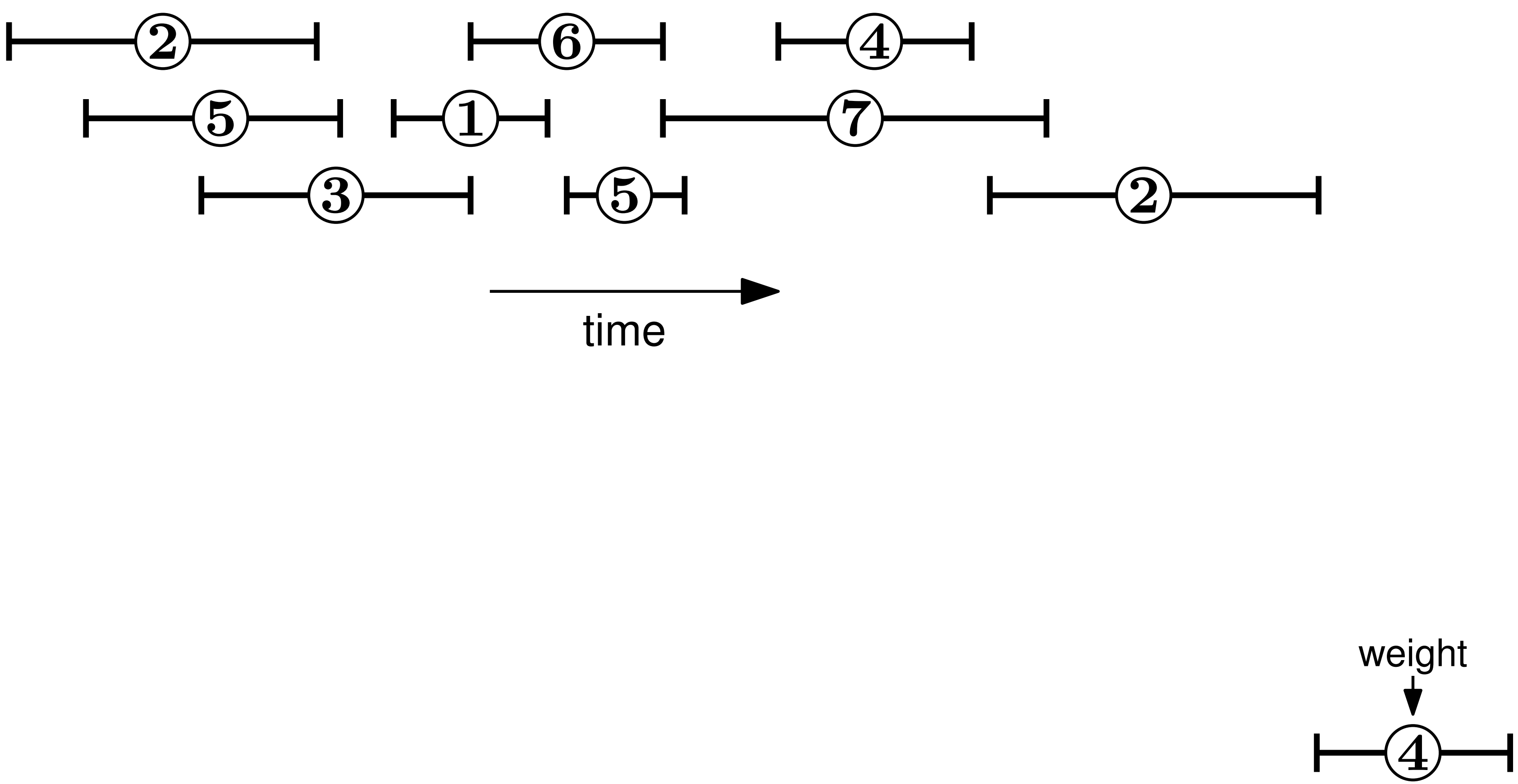

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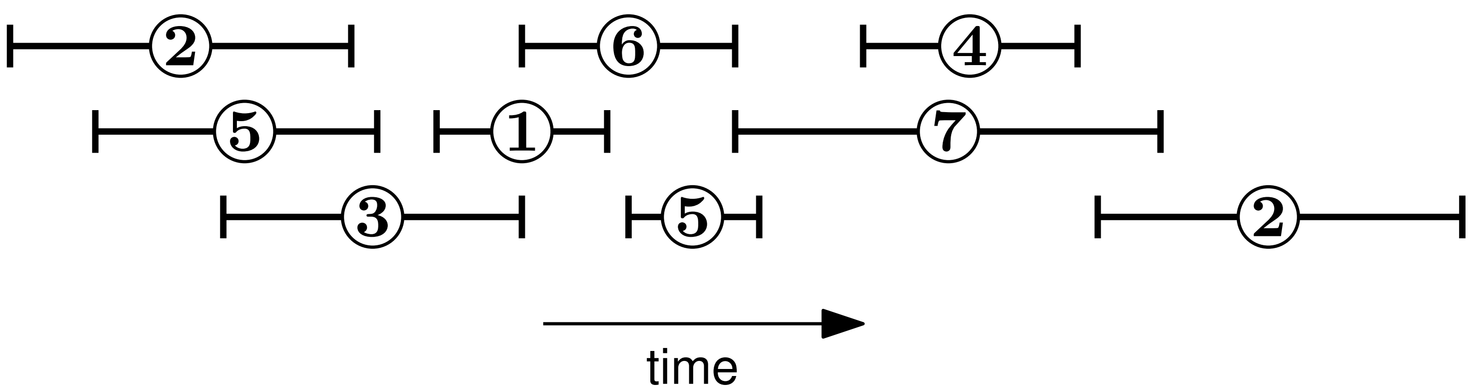

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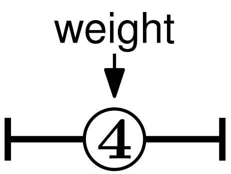

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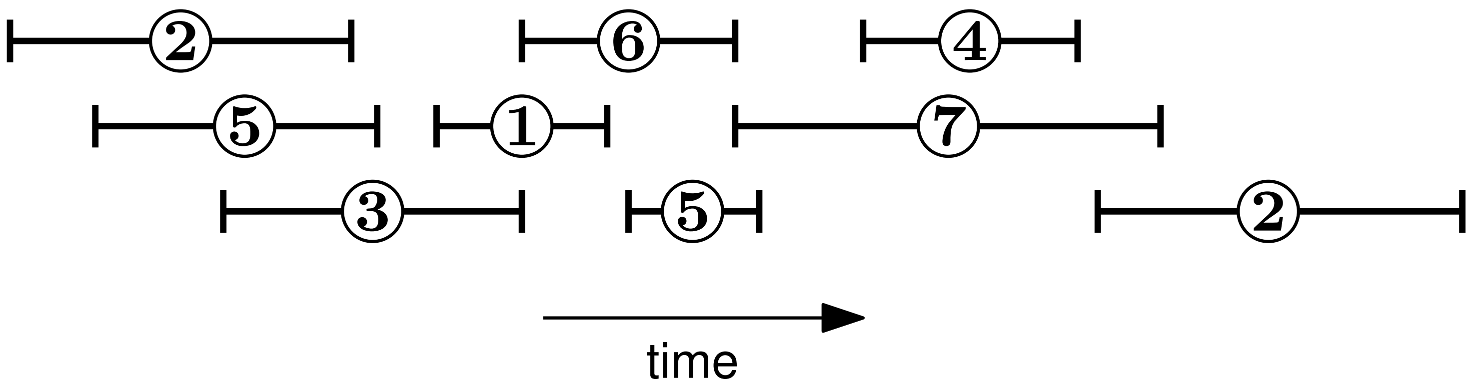

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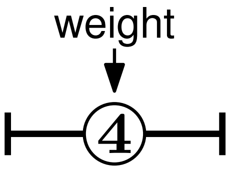

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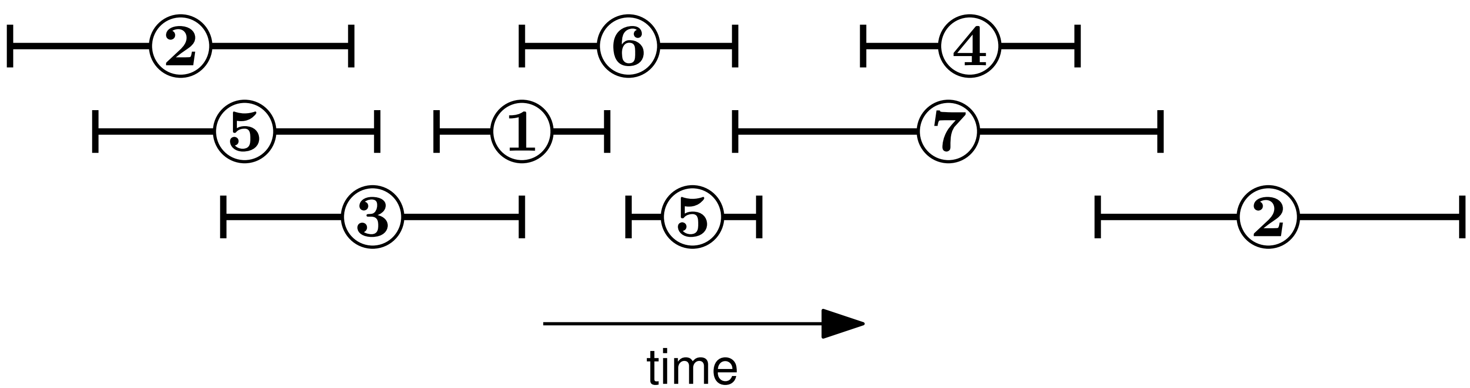

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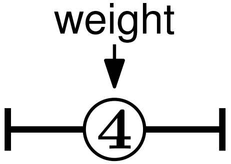

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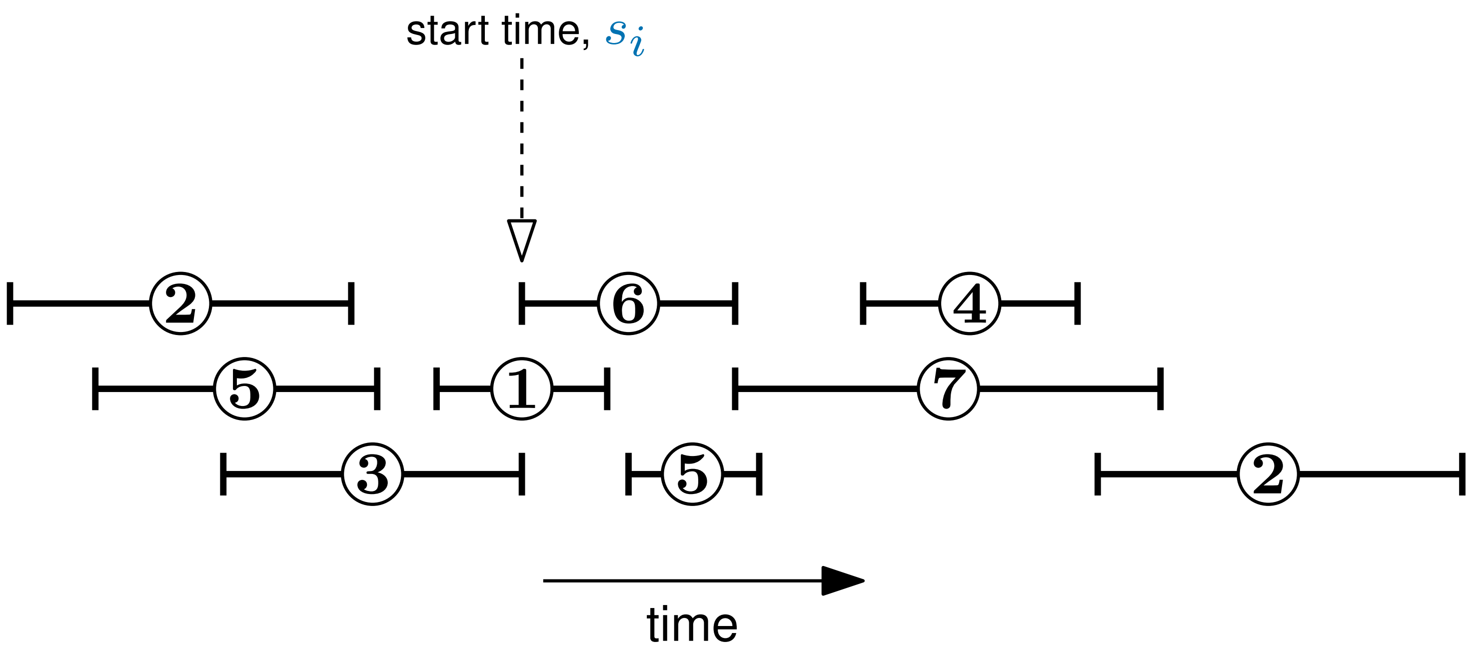

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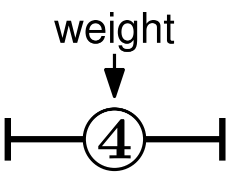

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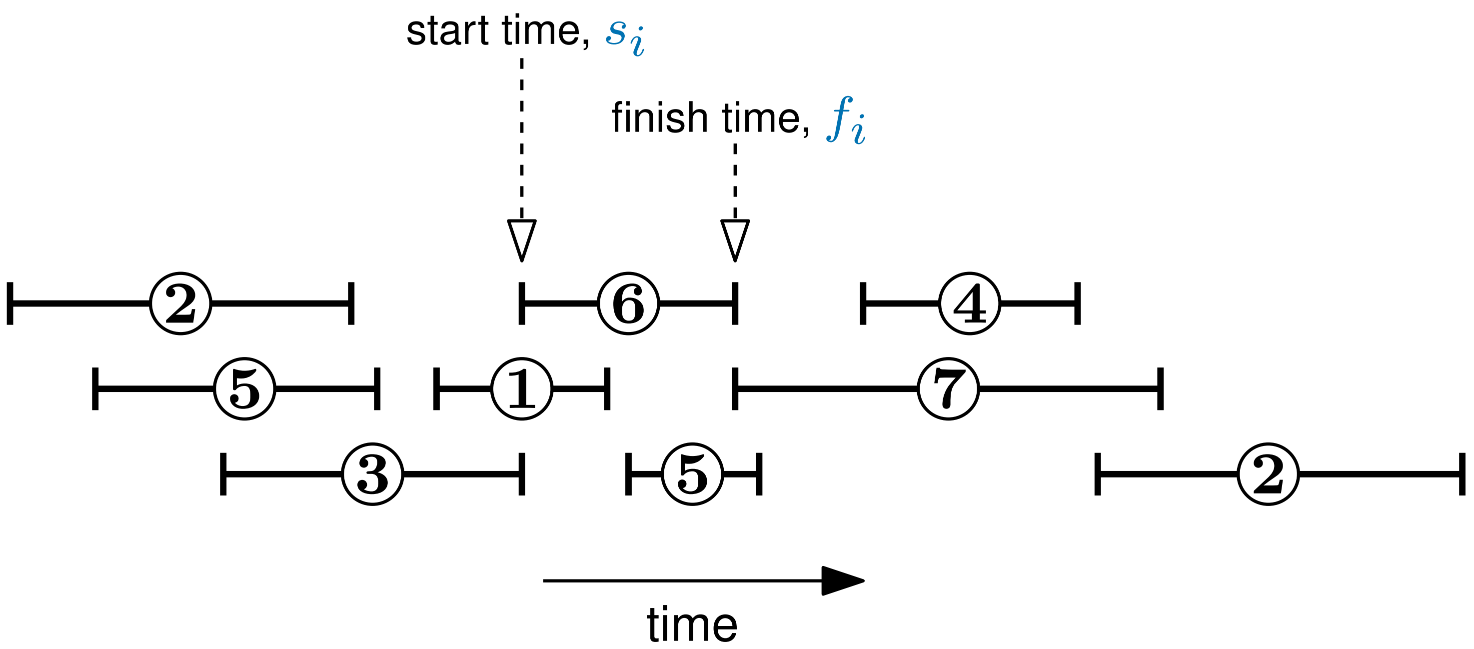

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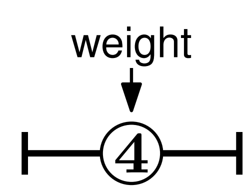

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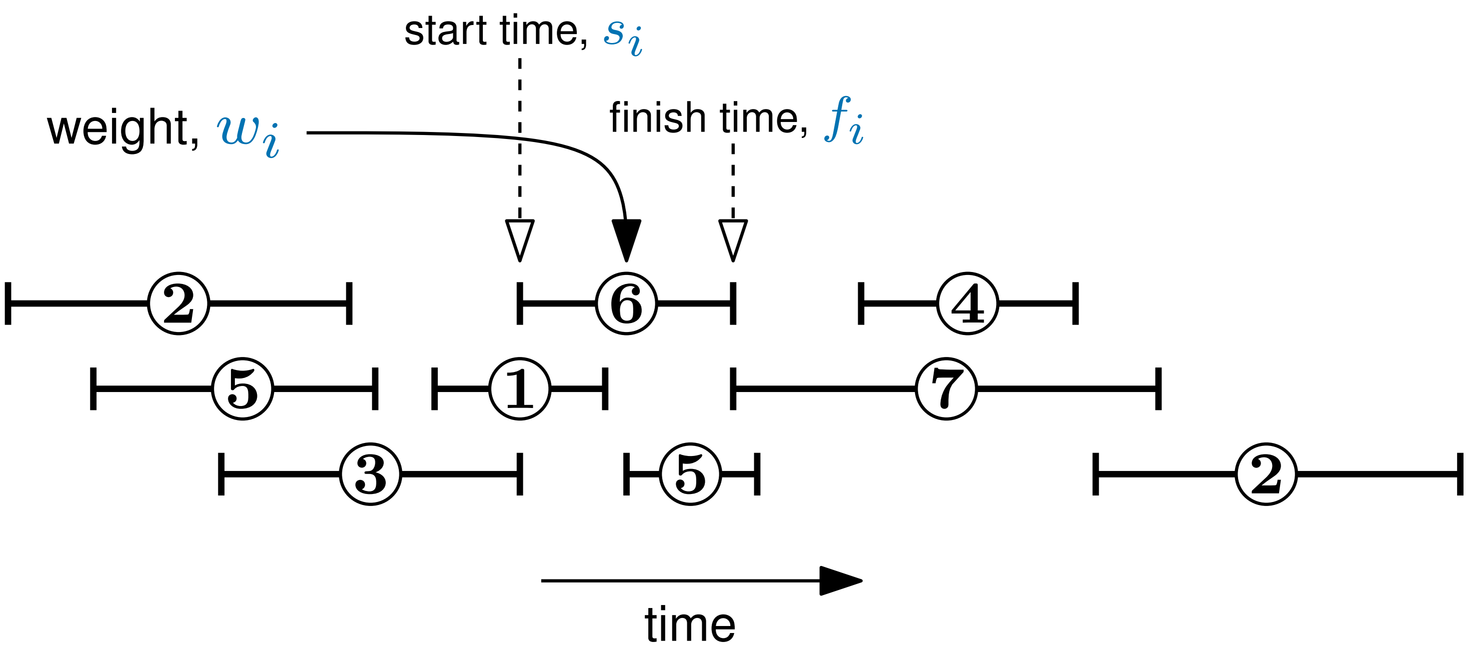

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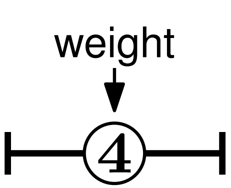

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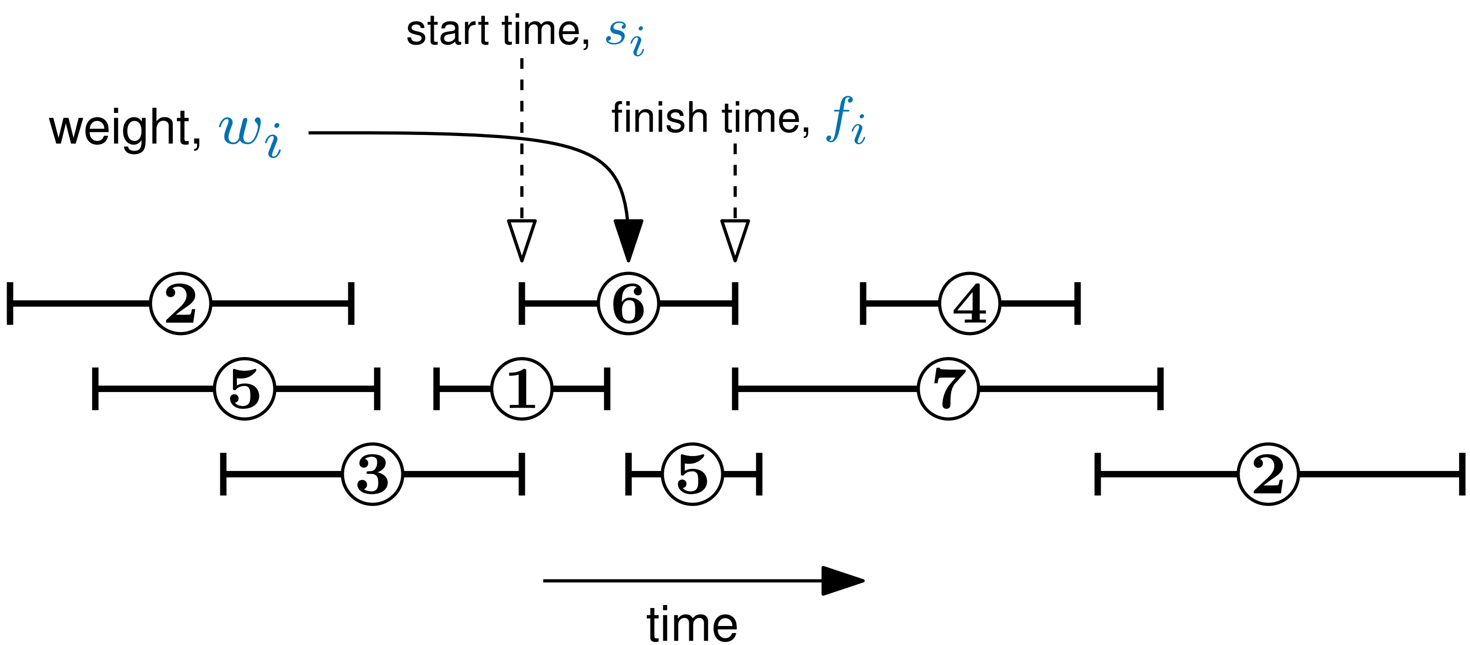

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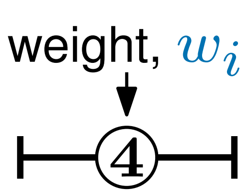

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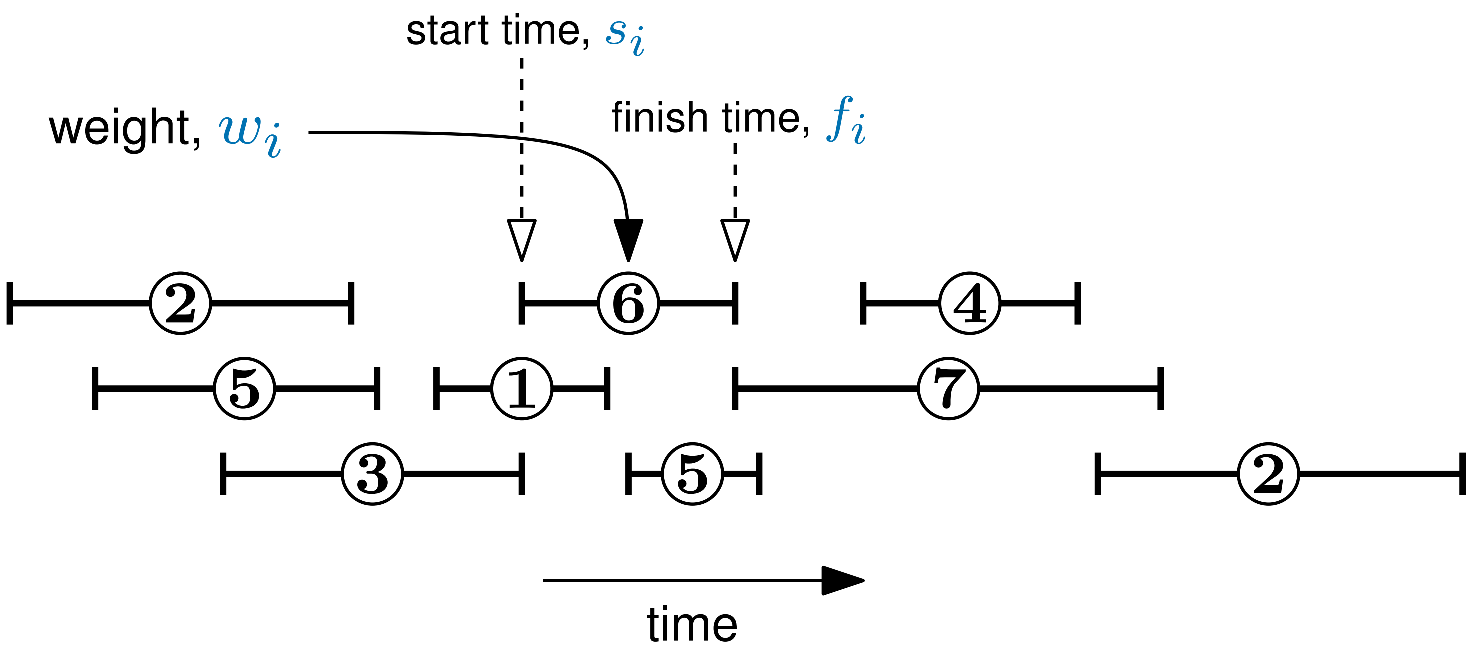

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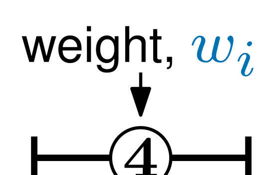

The intervals are sorted by *finish time* i.e. $f_i \leqslant f_{i+1}$

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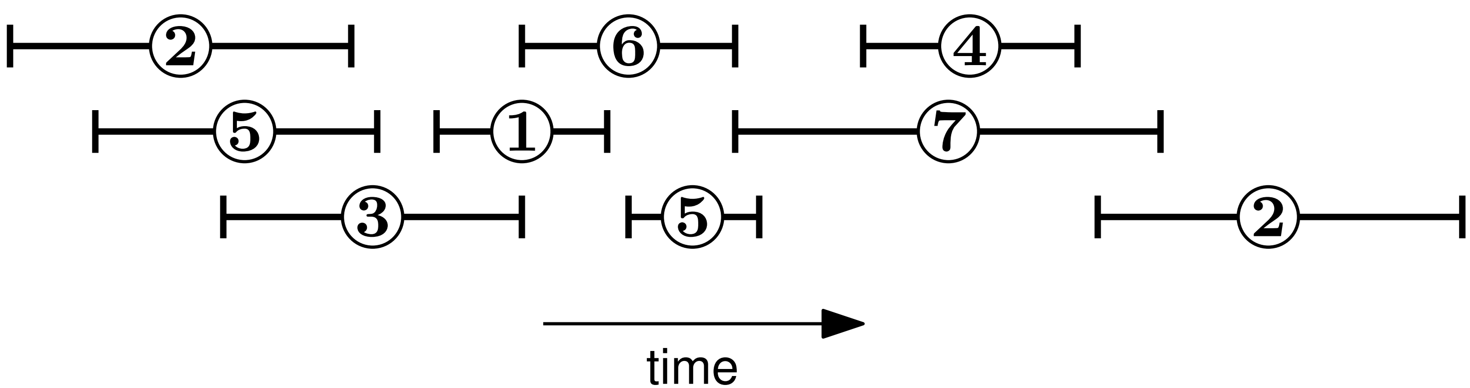

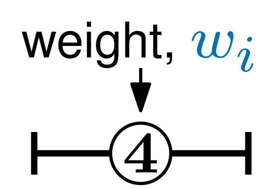

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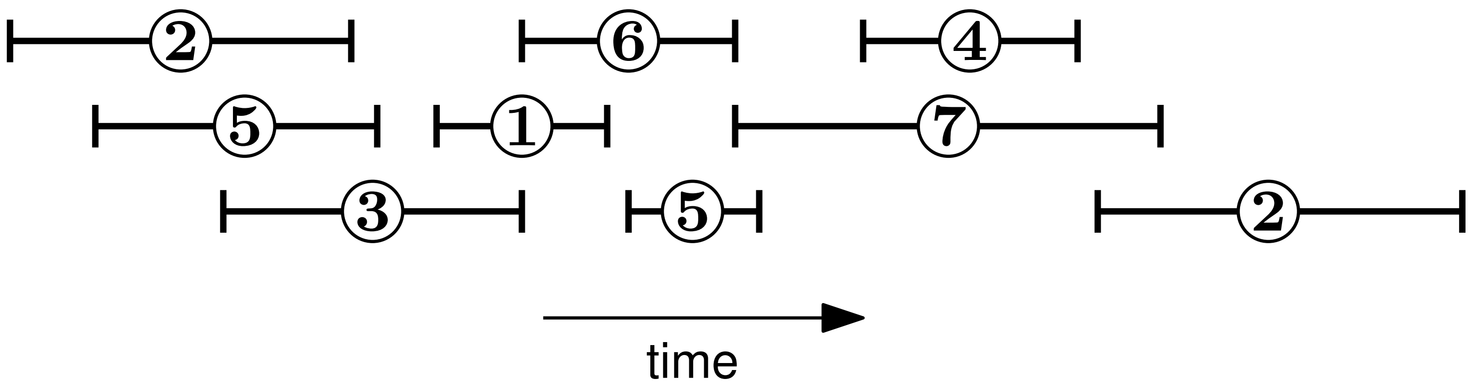
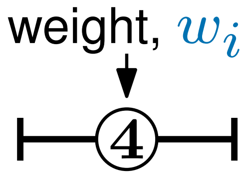


The intervals in the input are sorted by *finish time*

interval i finishes before interval $i+1$ finishes



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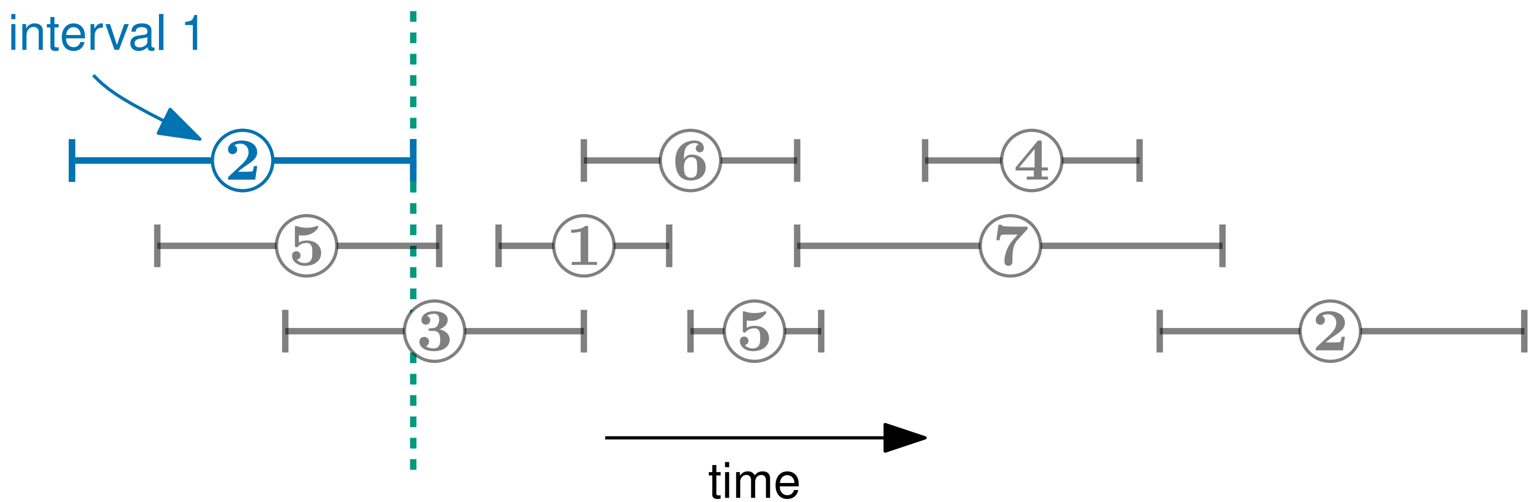

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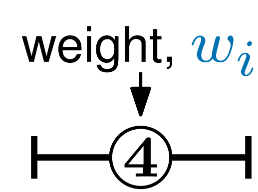

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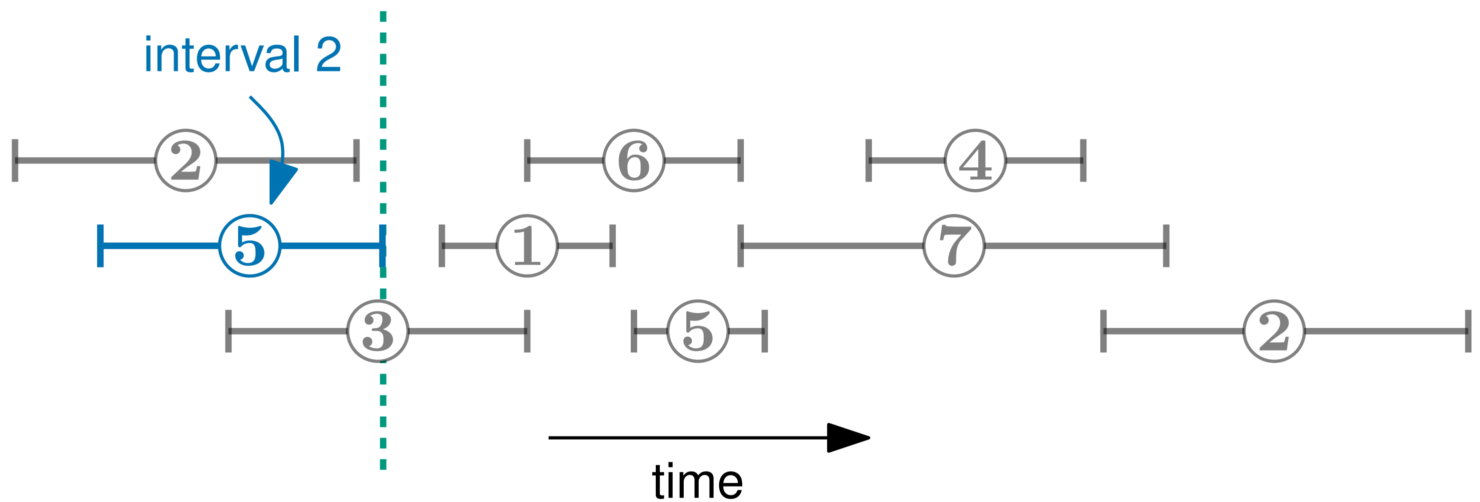

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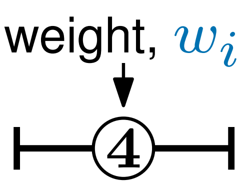

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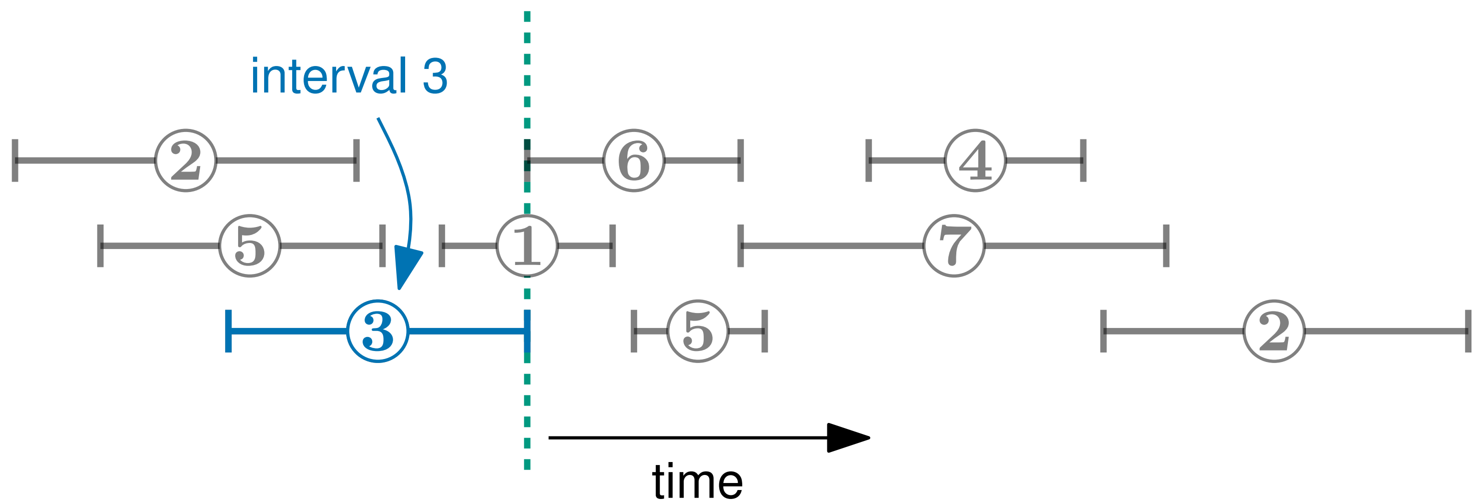

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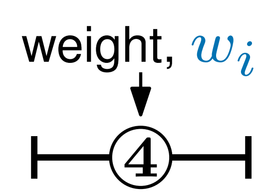

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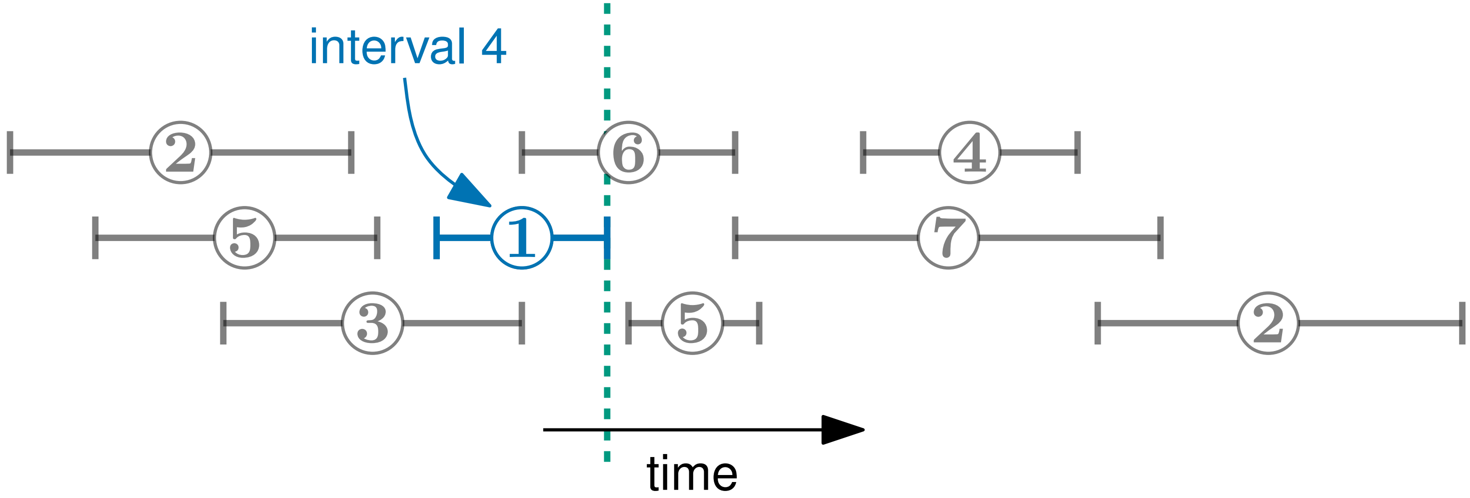

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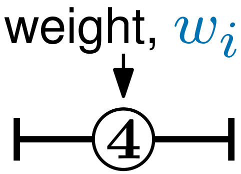

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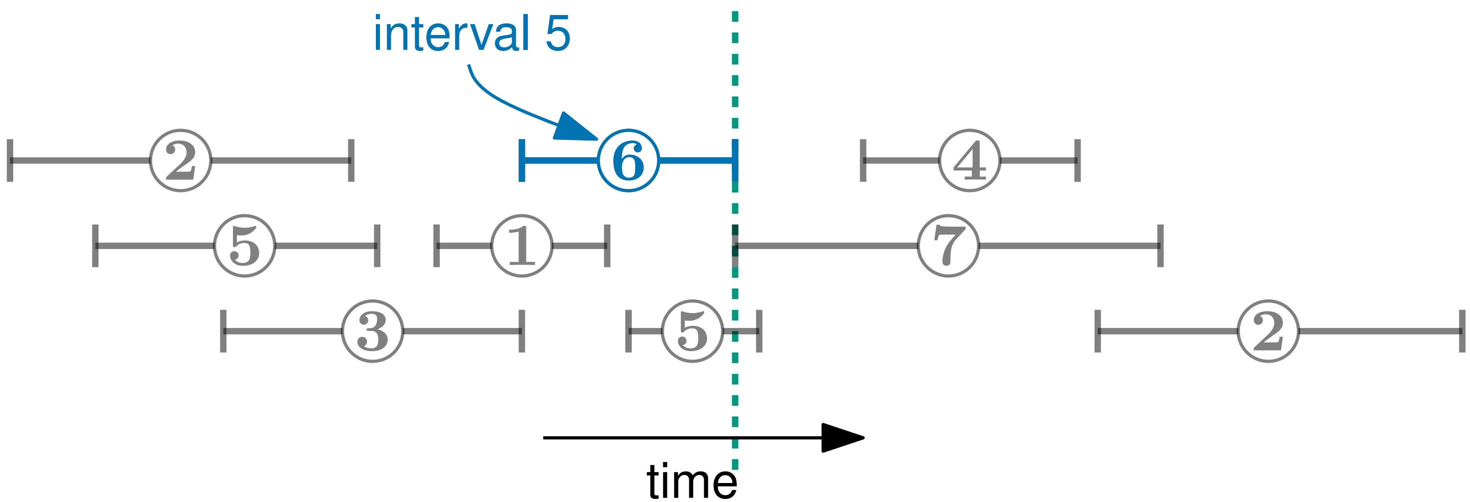

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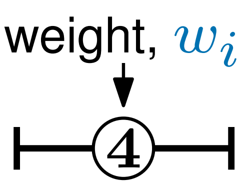

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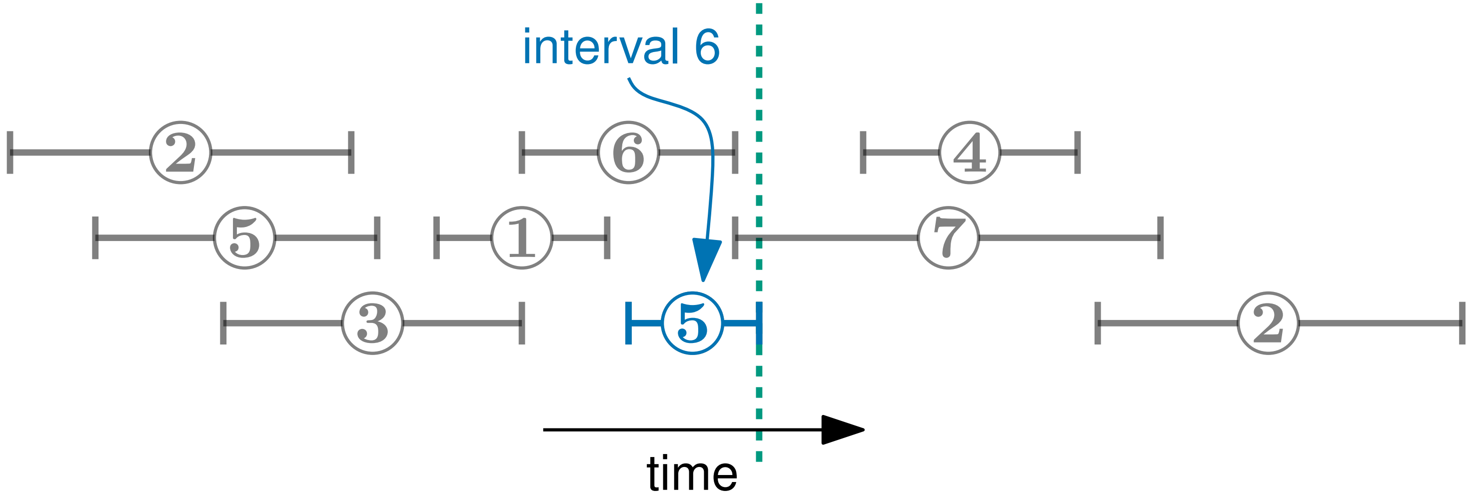

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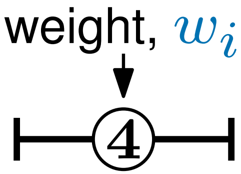

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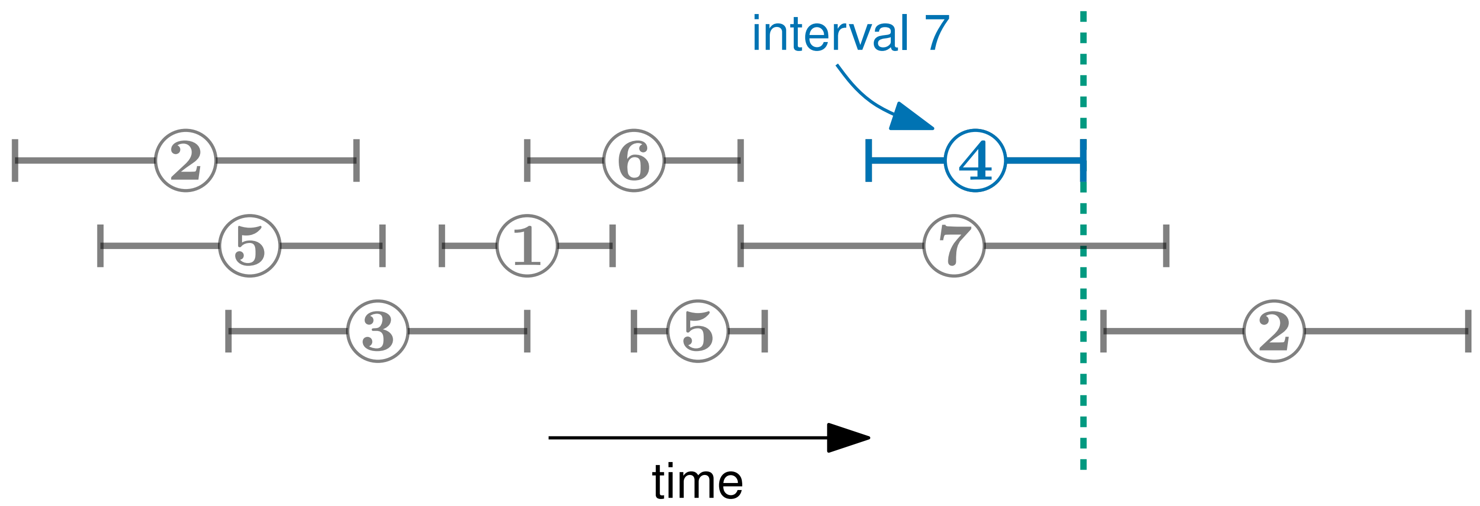

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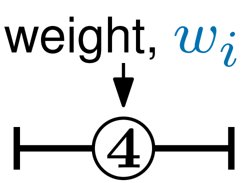

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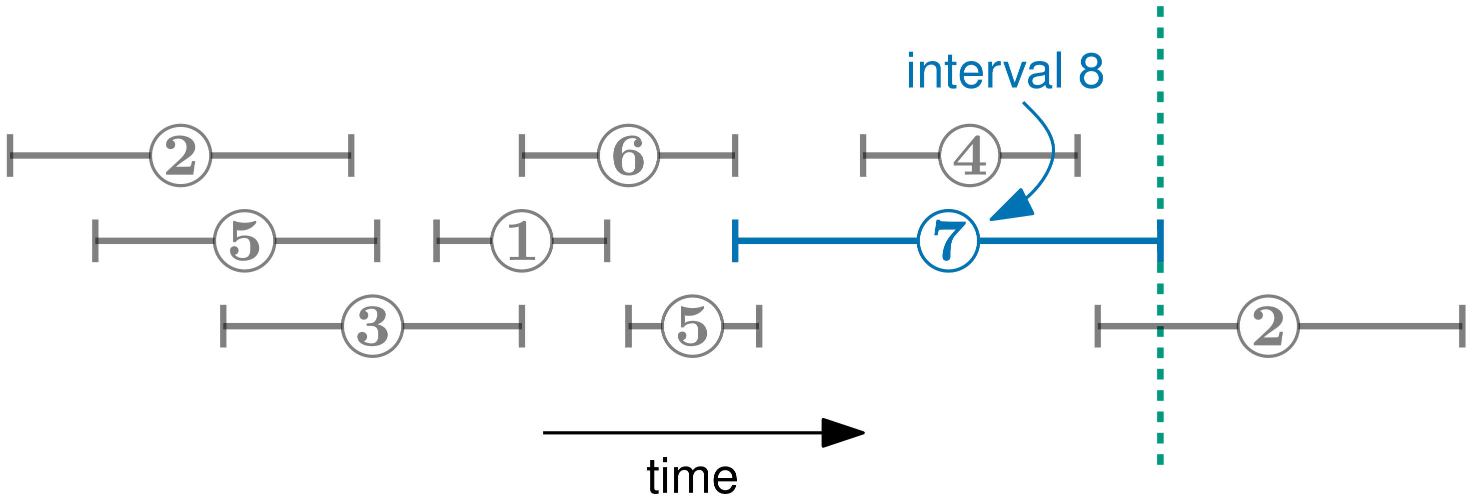

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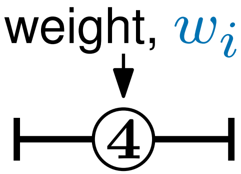

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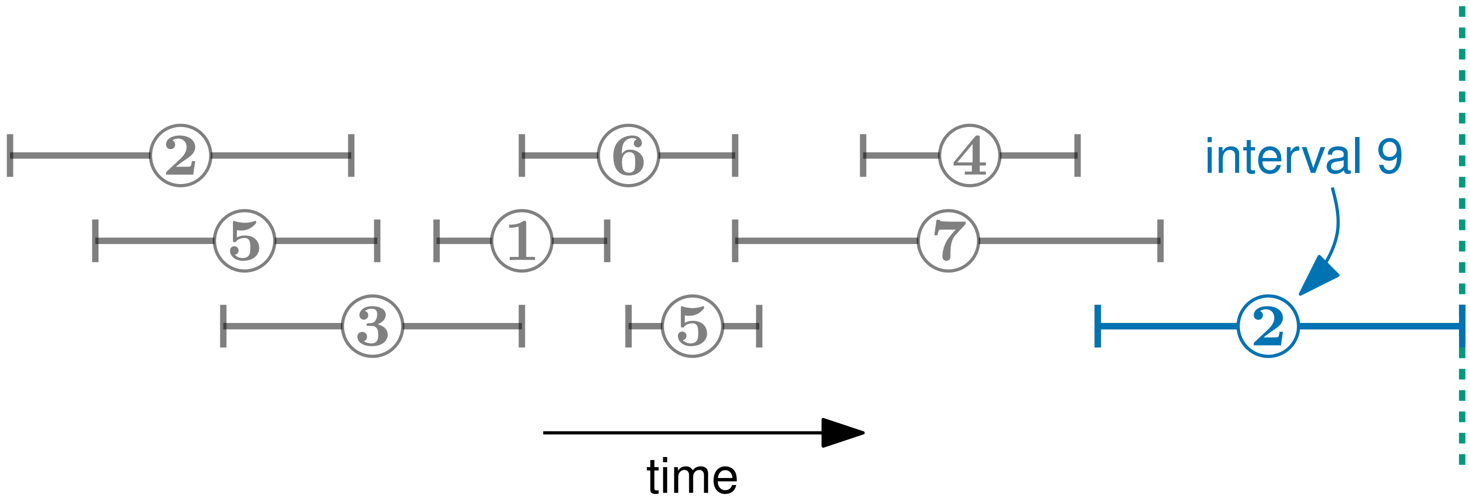

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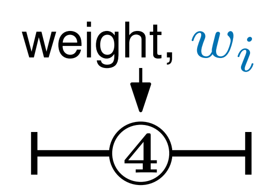

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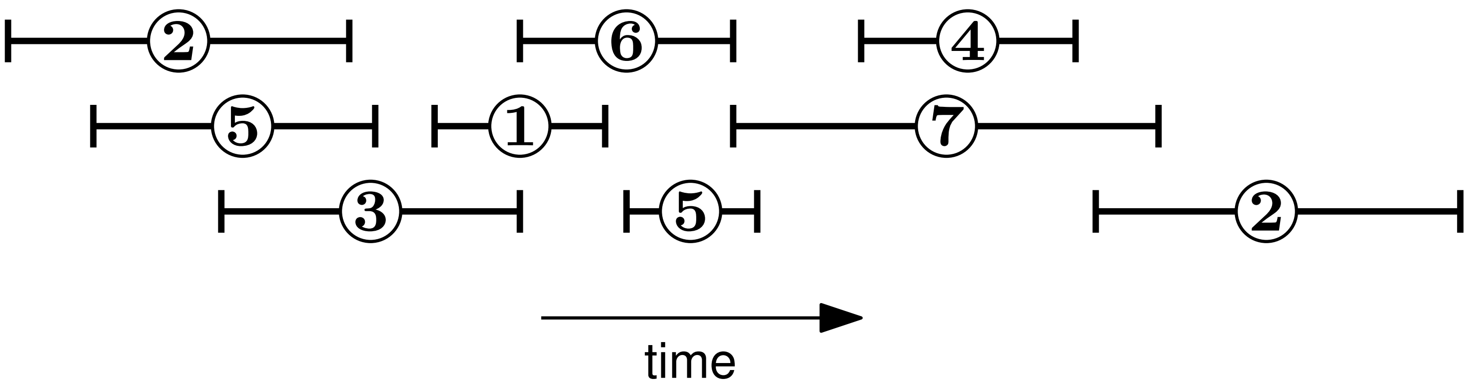

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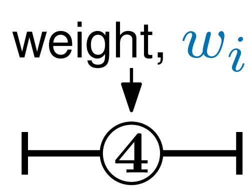
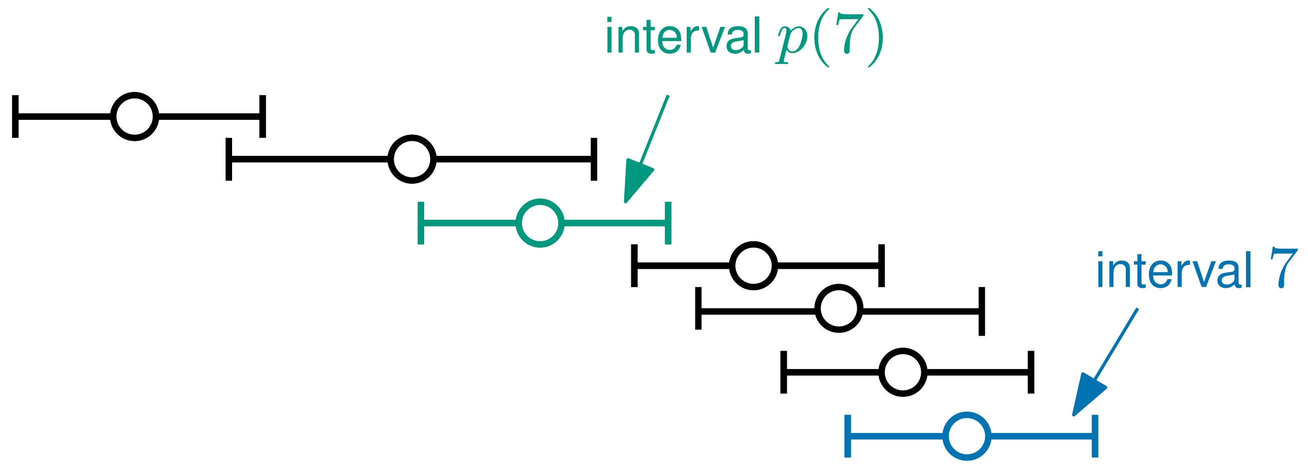


Compatible Intervals



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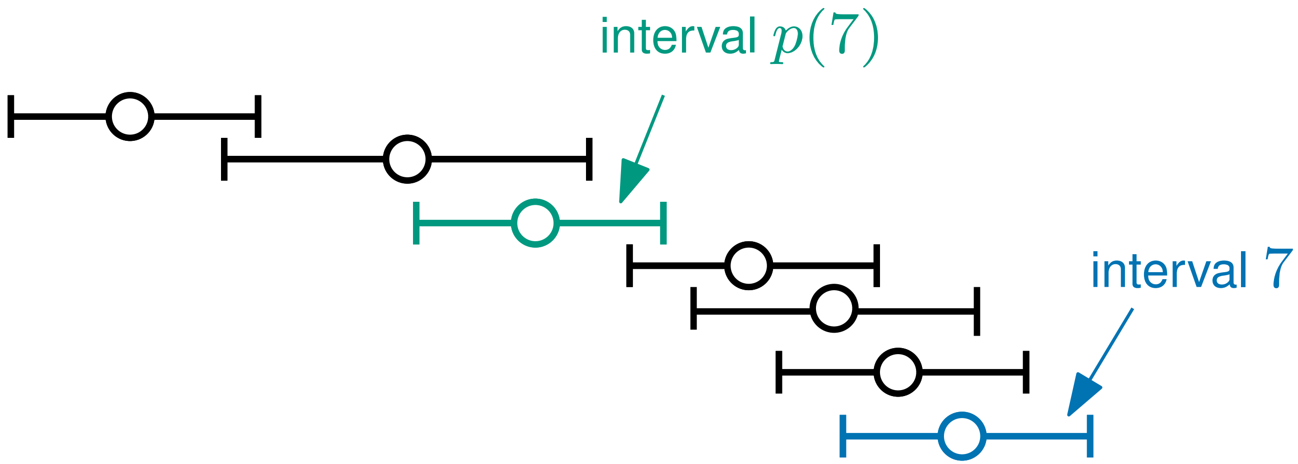
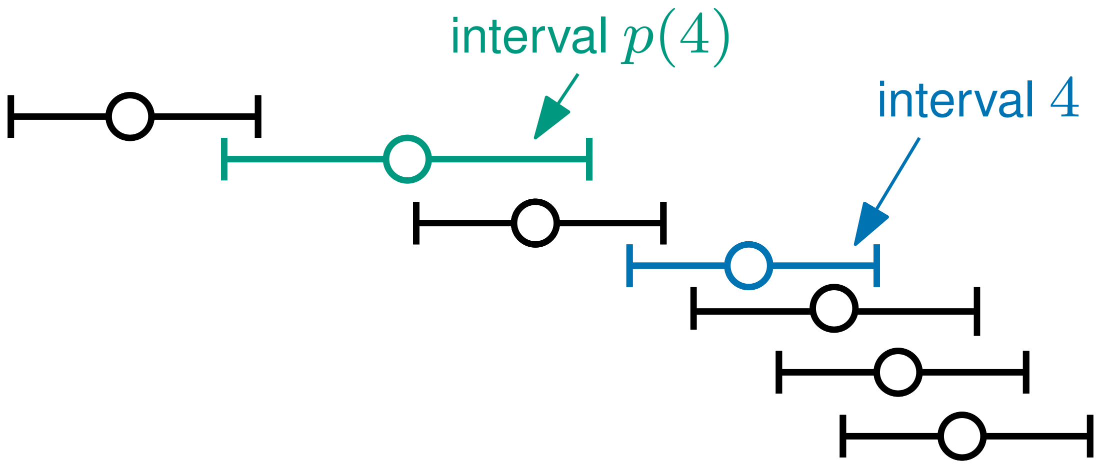


For all i,

Let $p(i)$ be the rightmost interval (in order of finish time) which finishes before the i-th interval but doesn't overlap it

Compatible Intervals



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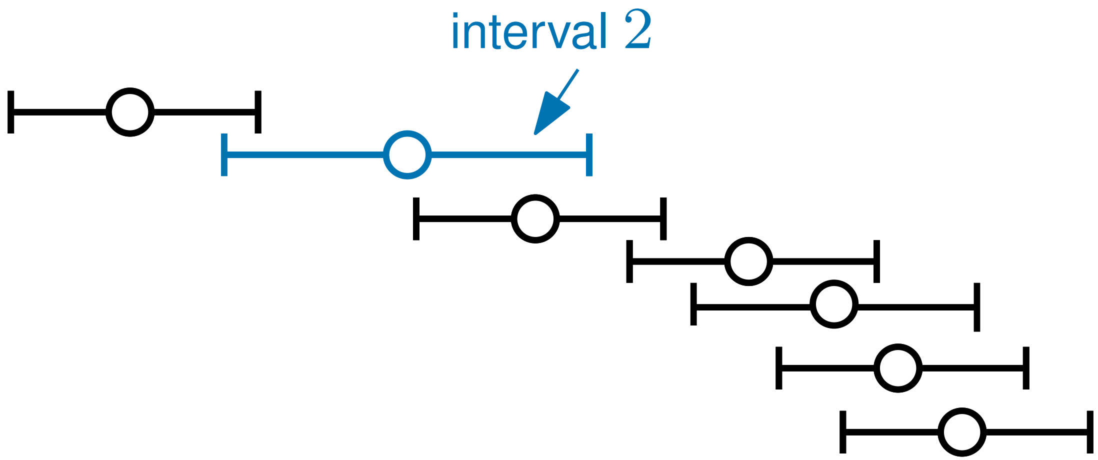

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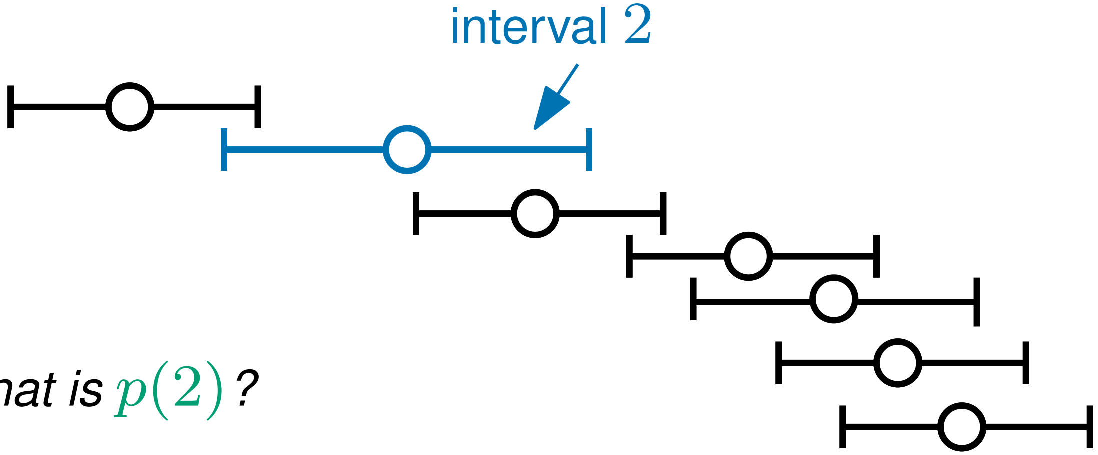

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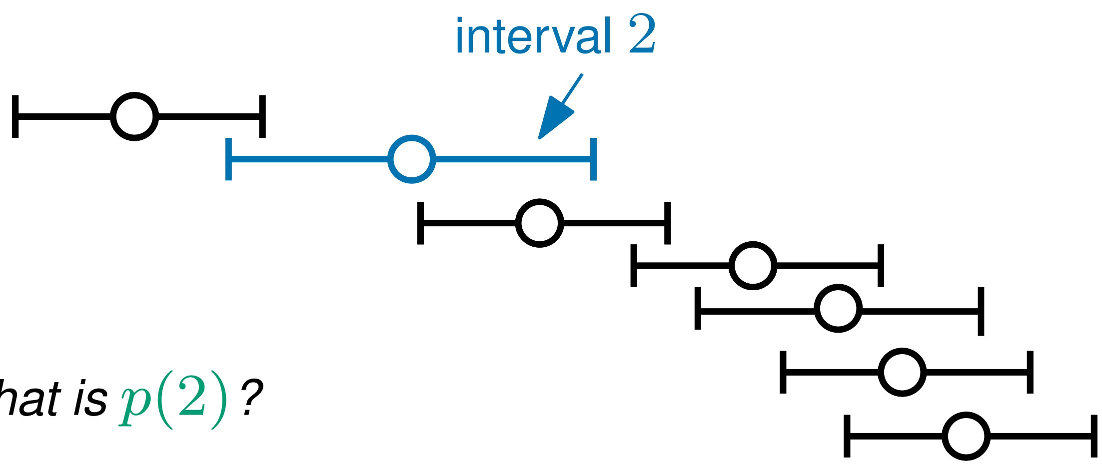

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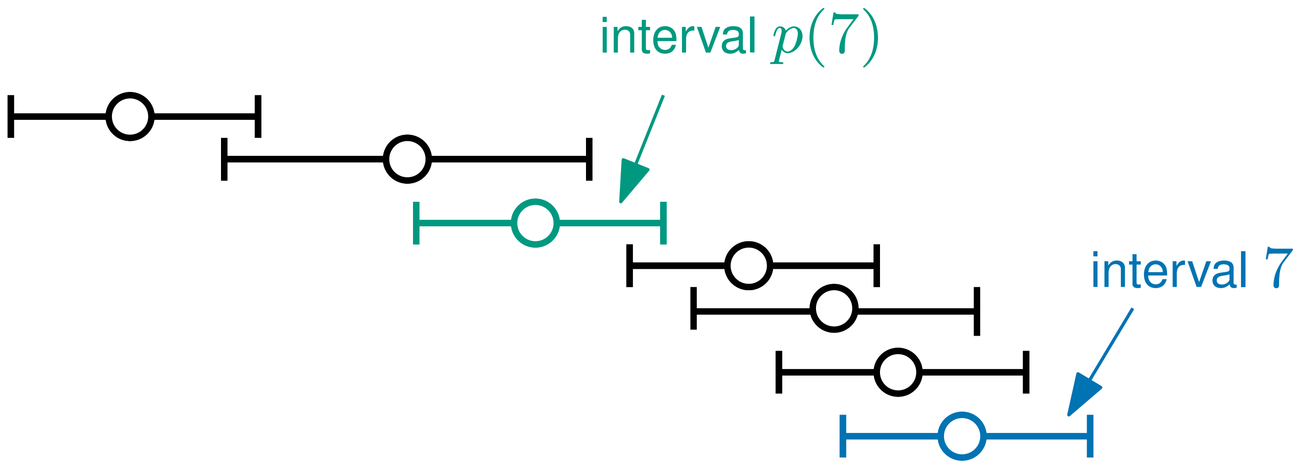

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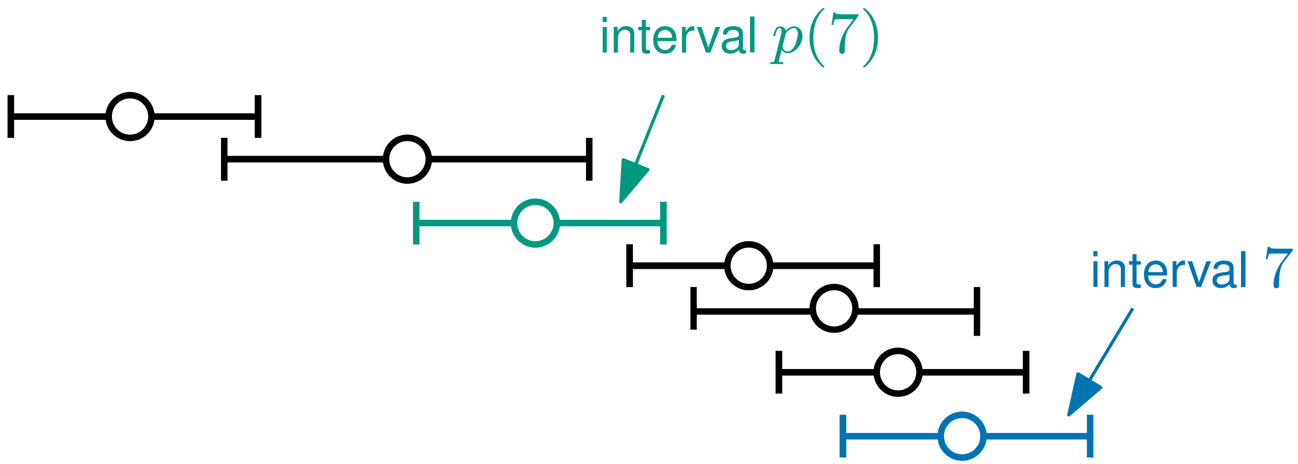

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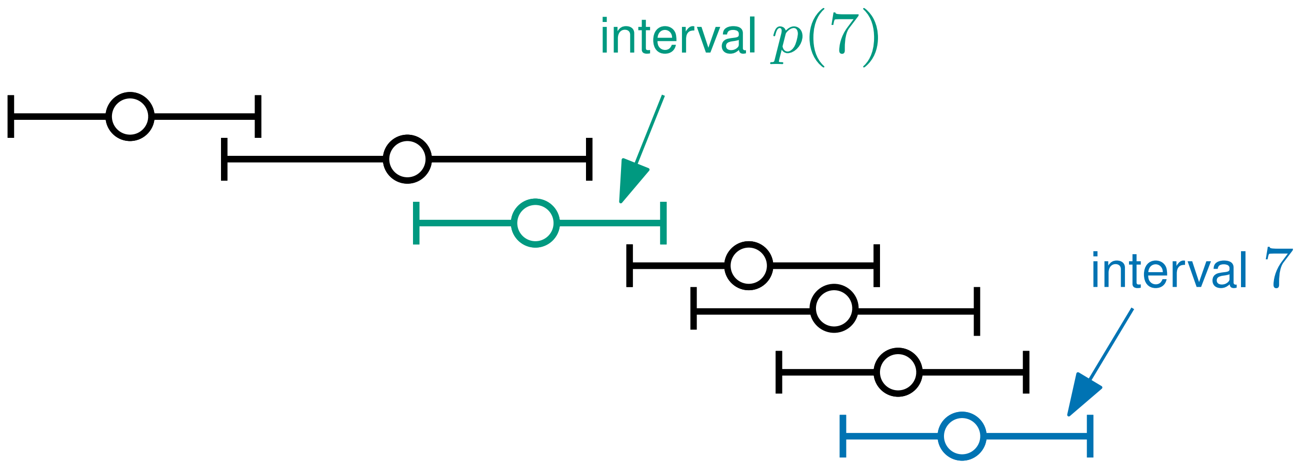

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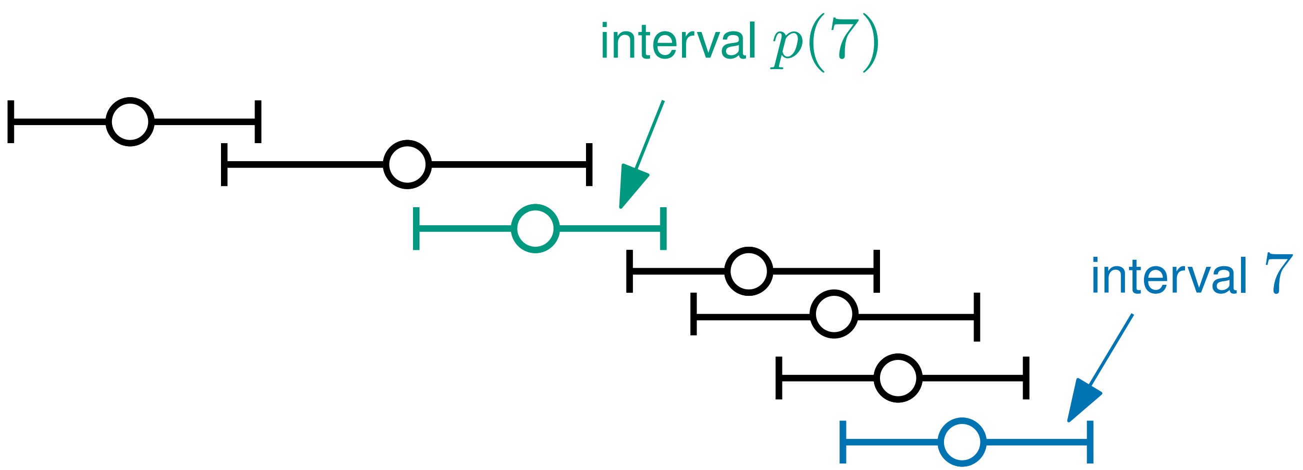

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- we'll come back to this at the end

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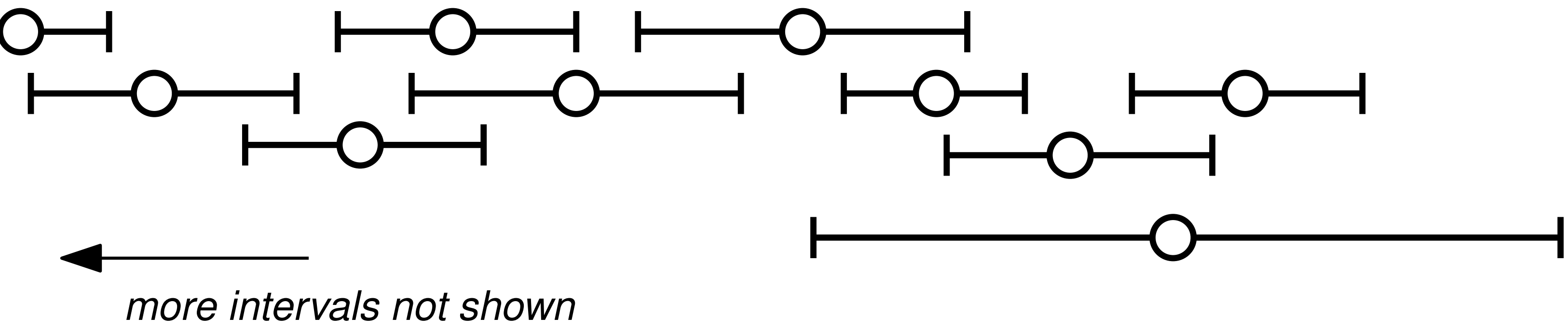
Consider some optimal schedule $\mathcal{O}$ for intervals $\{1, 2, 3 \ldots, n\}$ with weight OPT...



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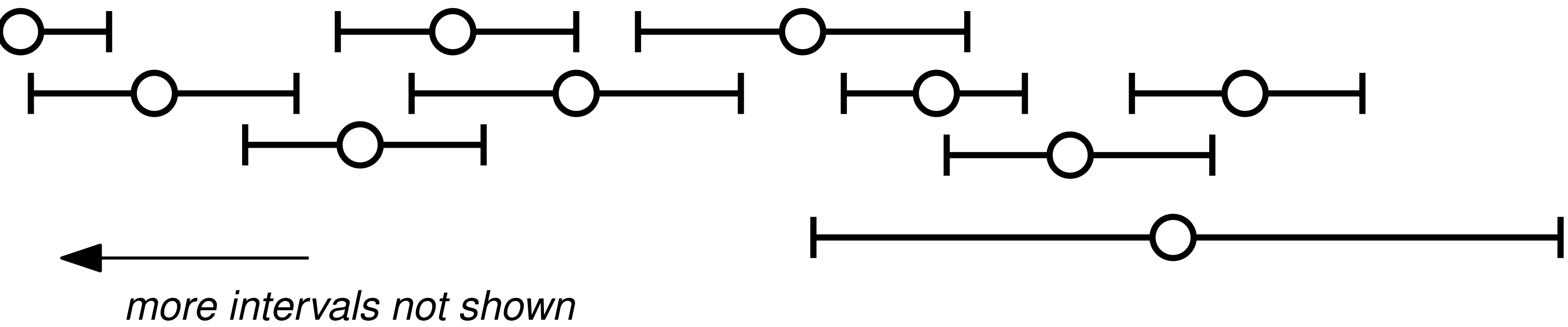

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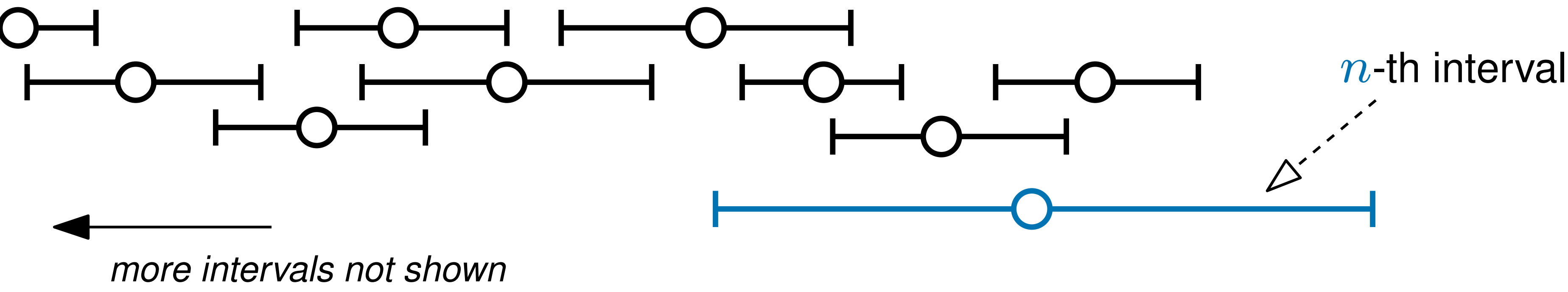

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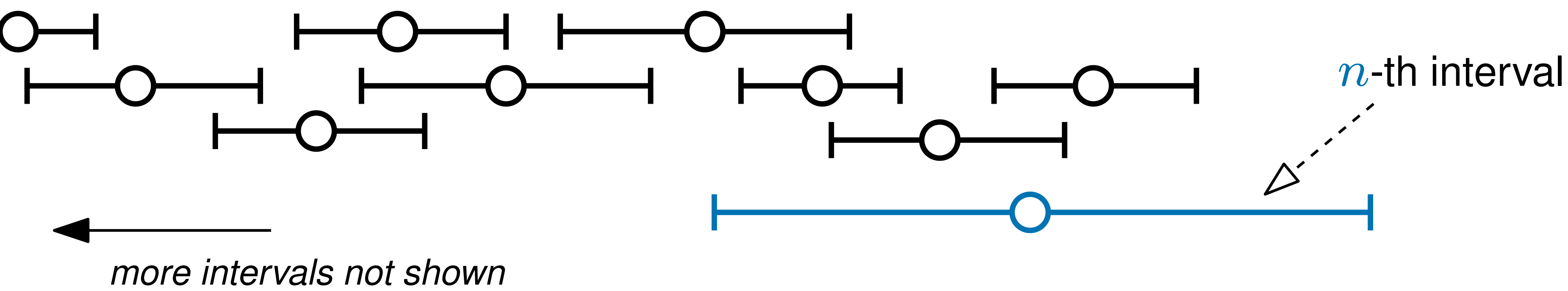

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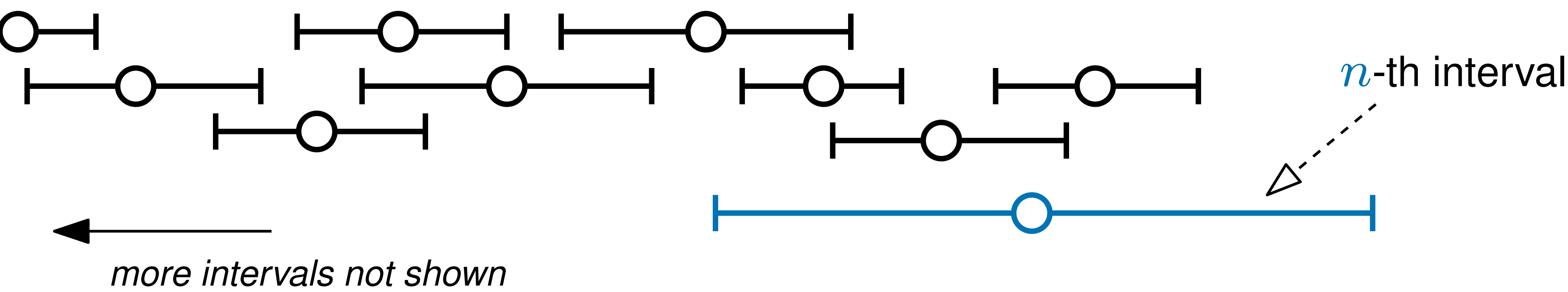

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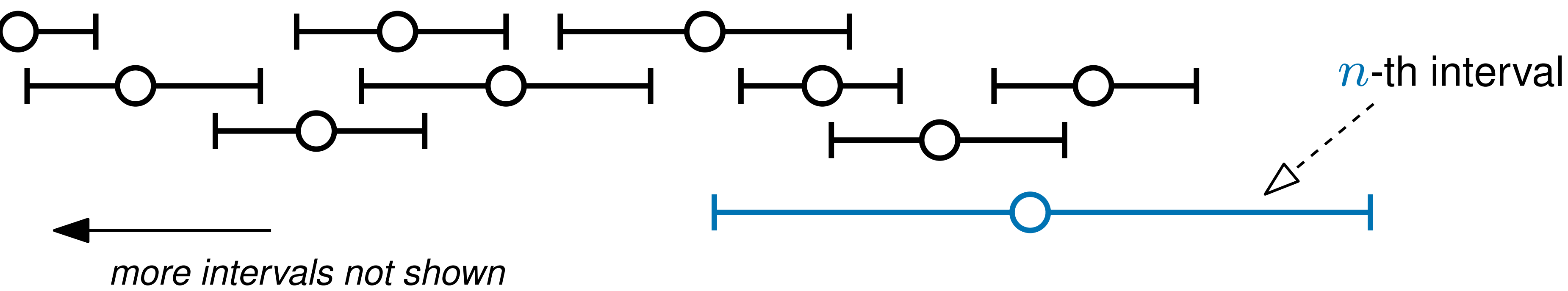

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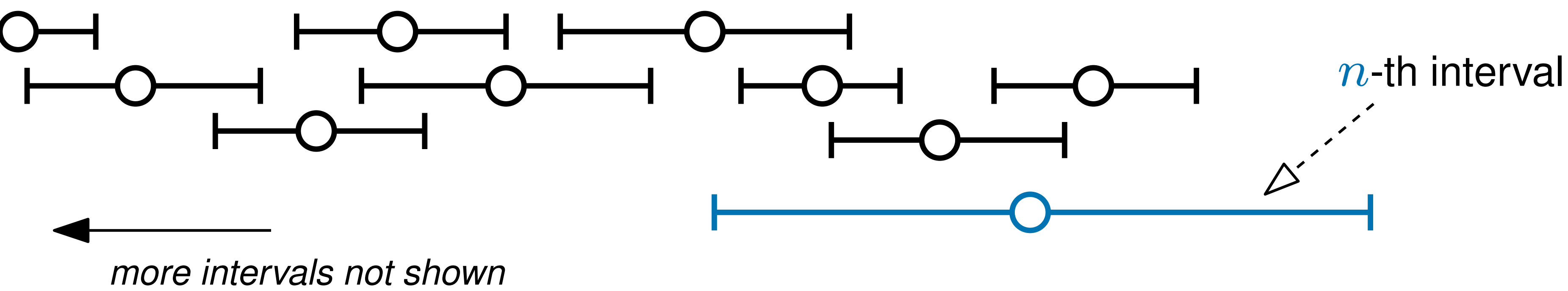

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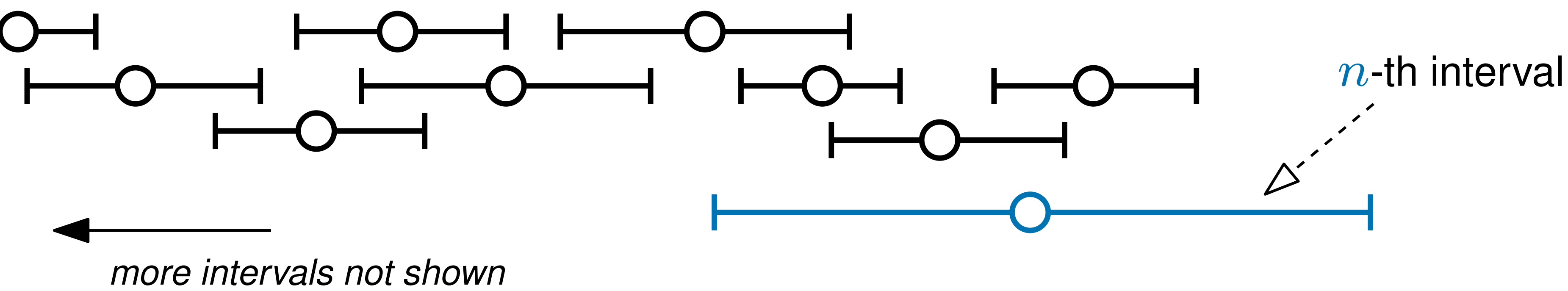

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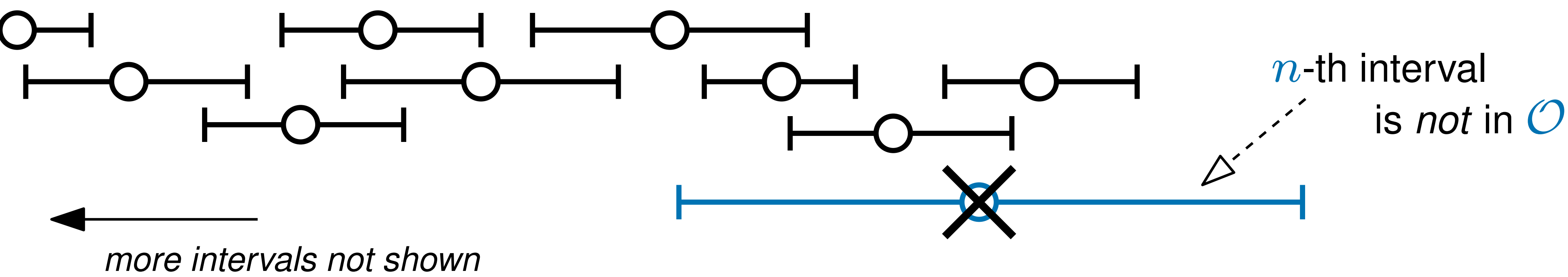

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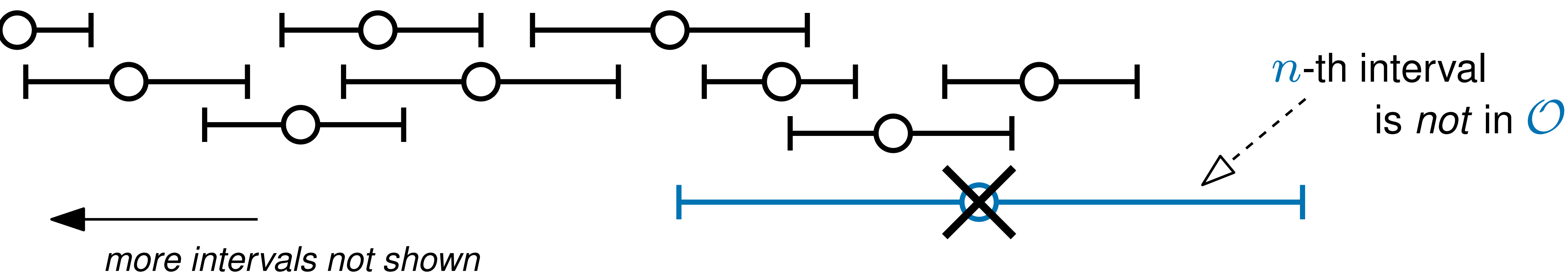

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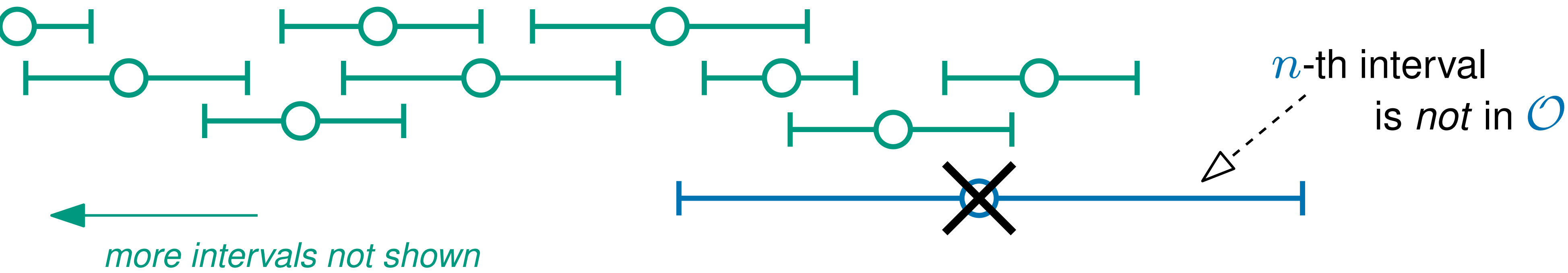

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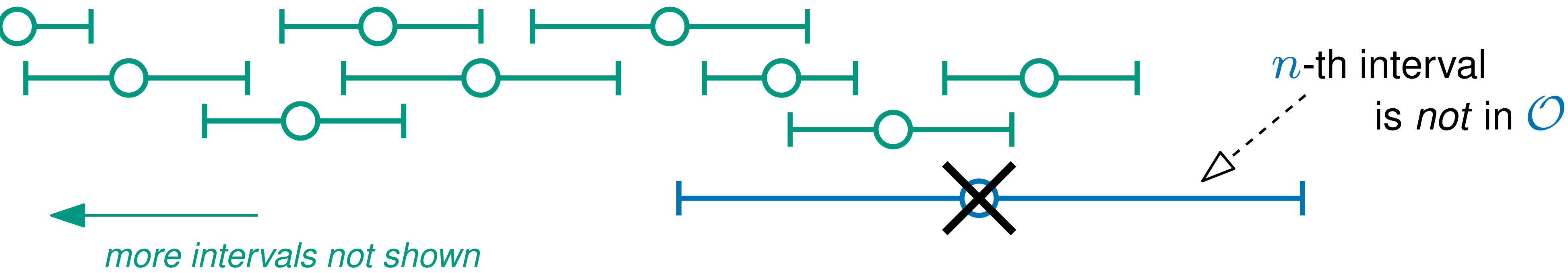

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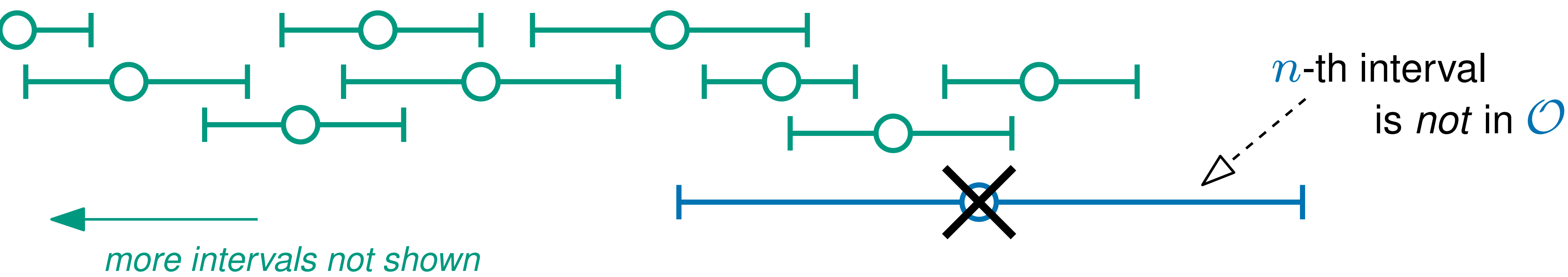

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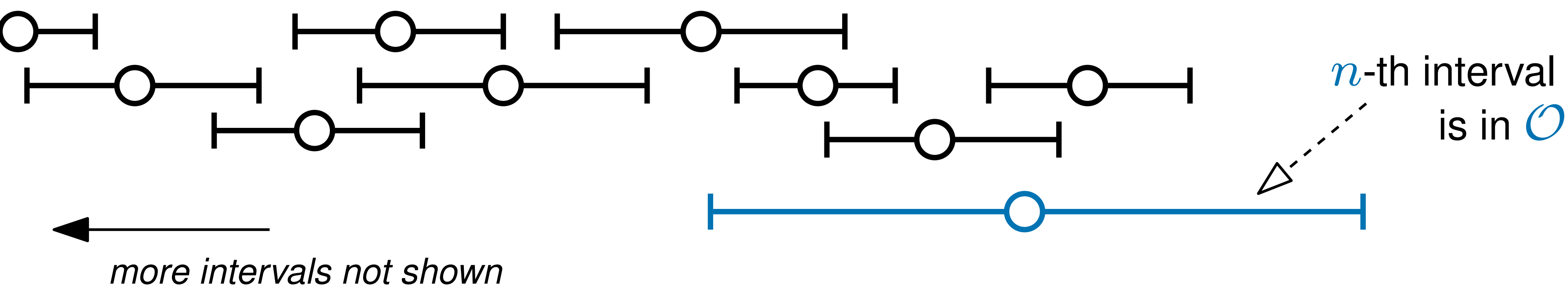

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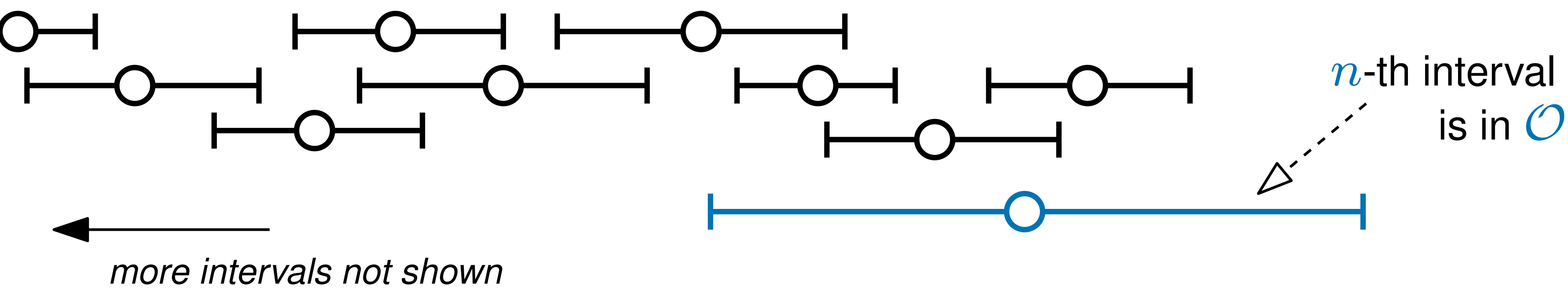

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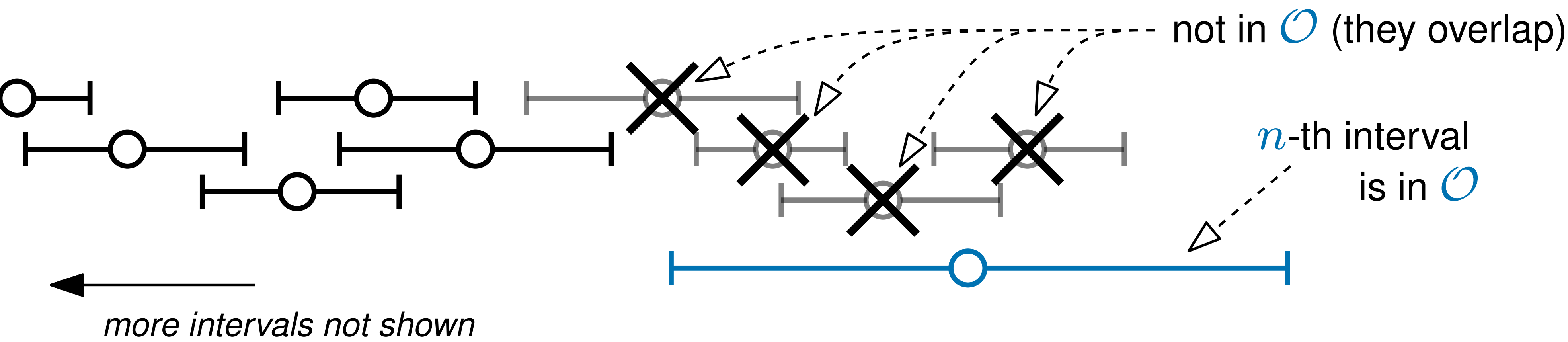

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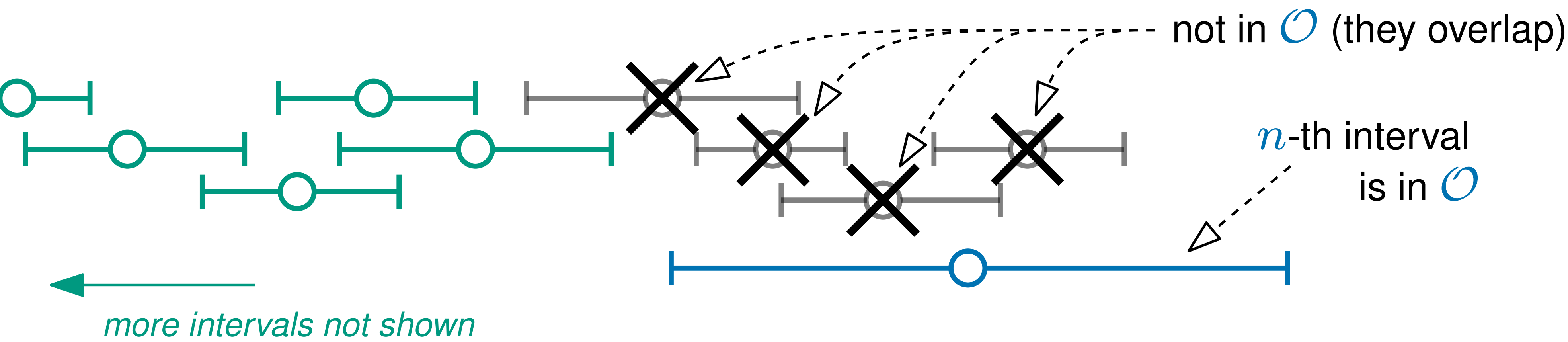

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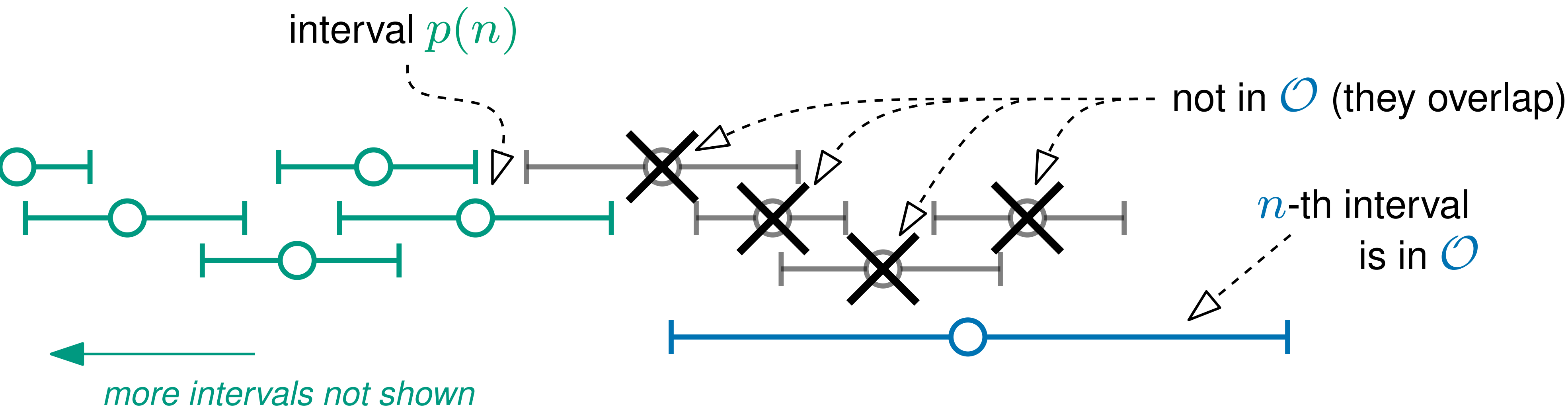

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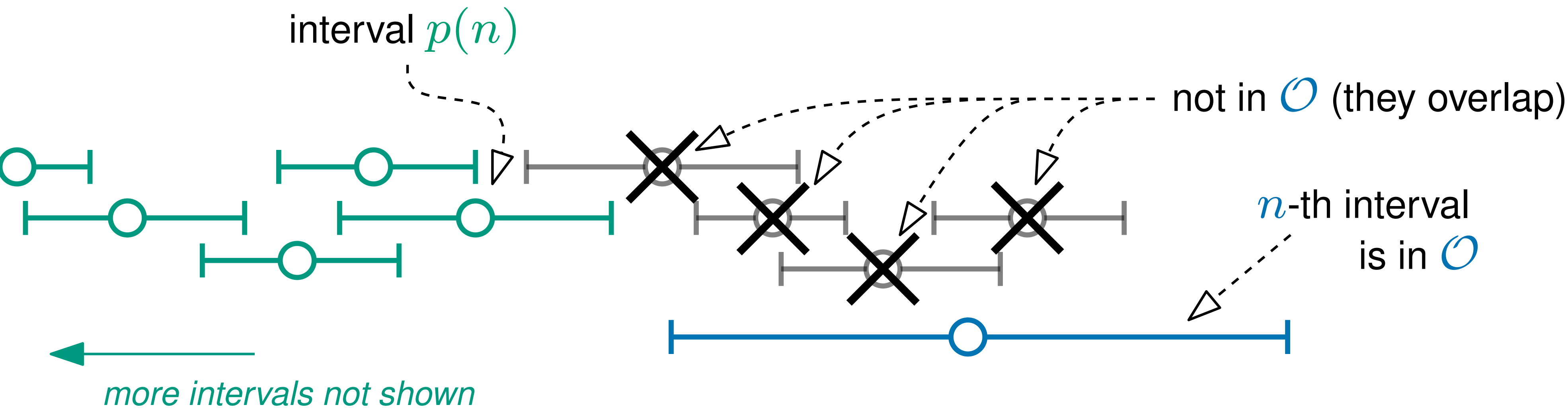

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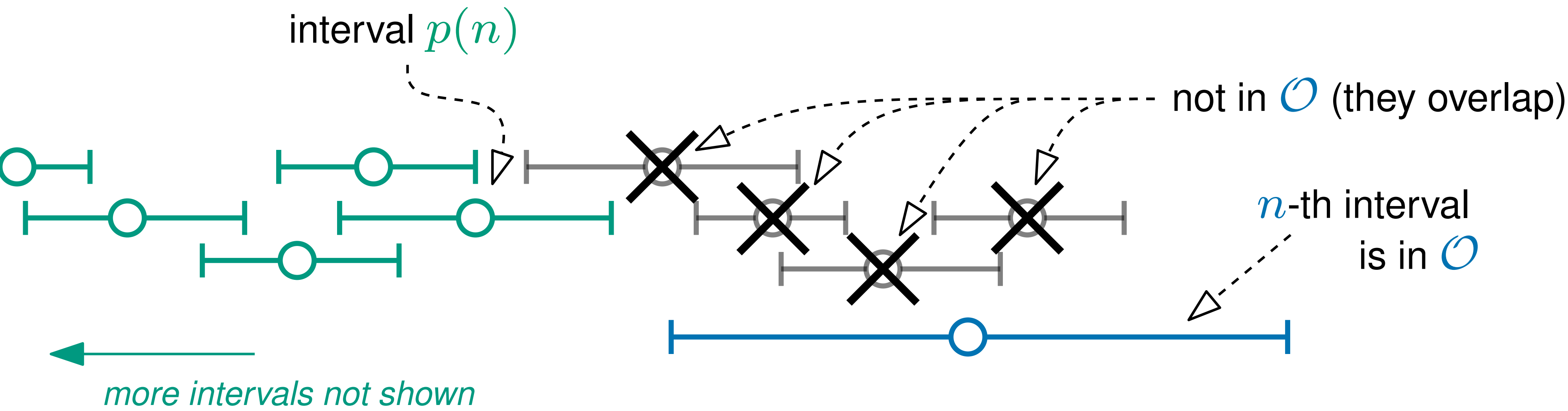

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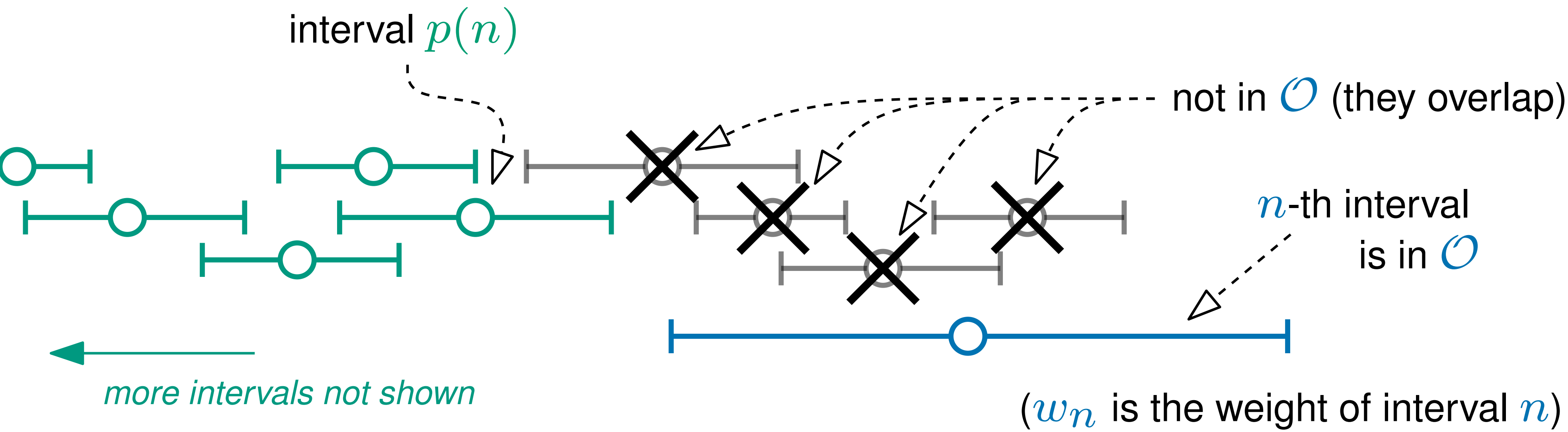

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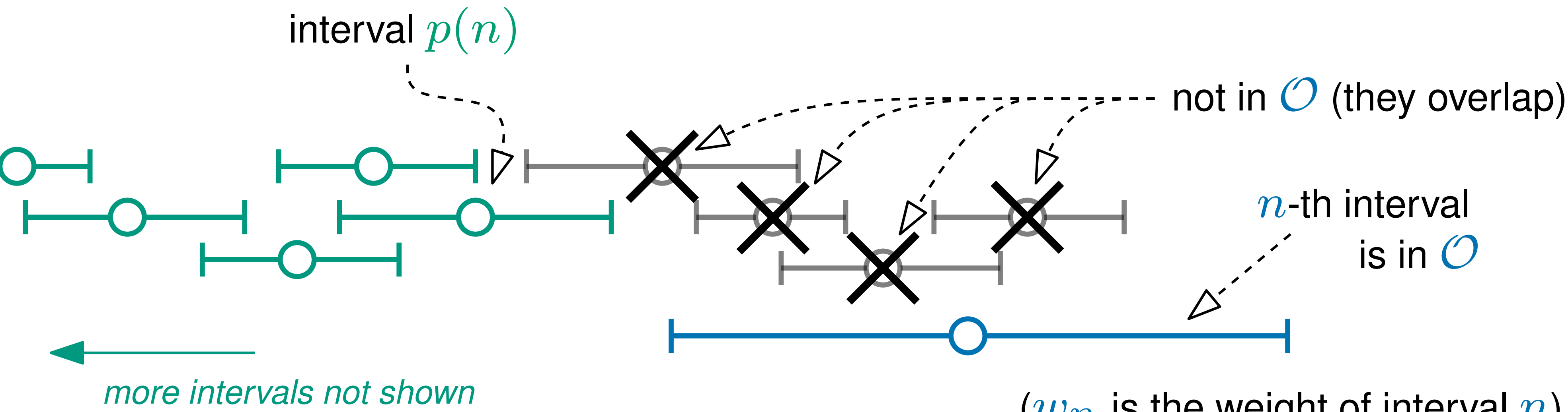

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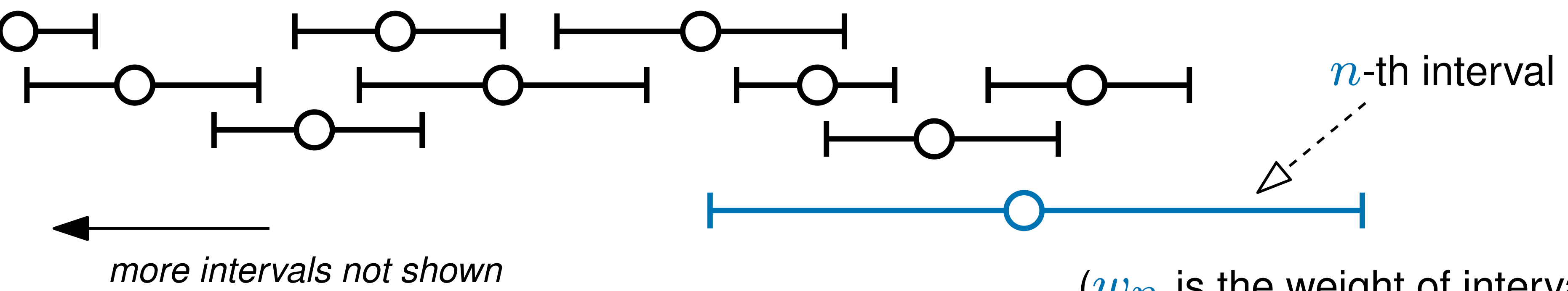

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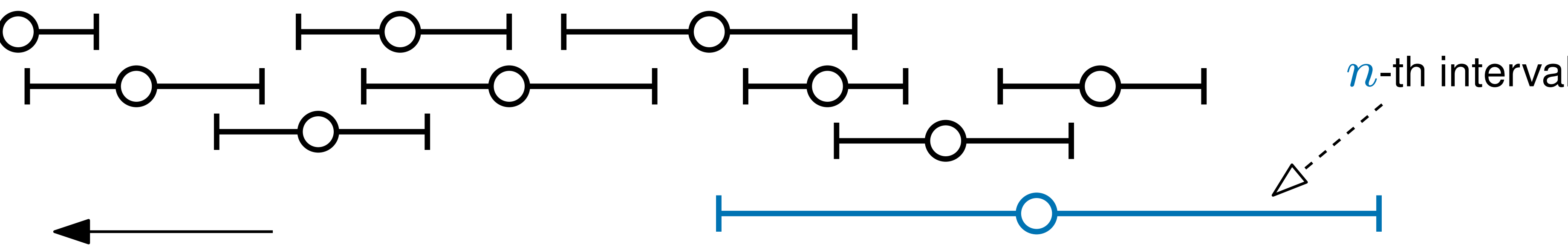

more intervals not shown

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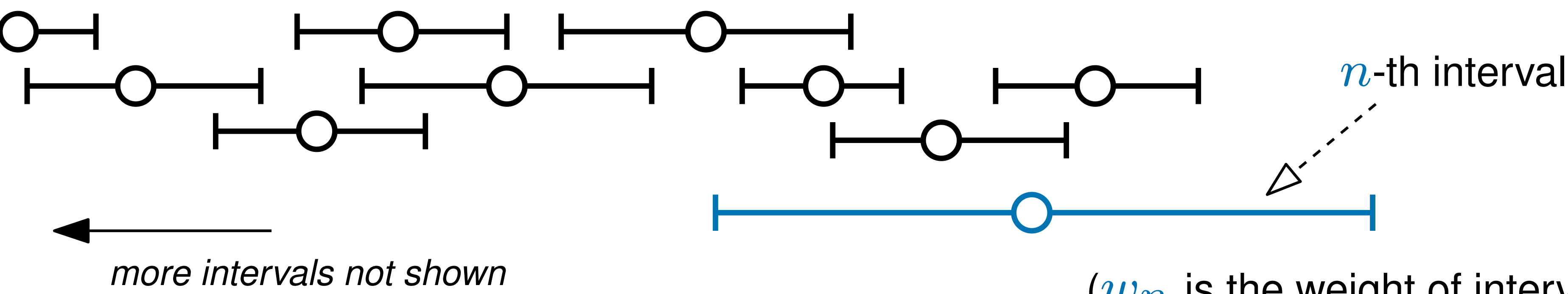

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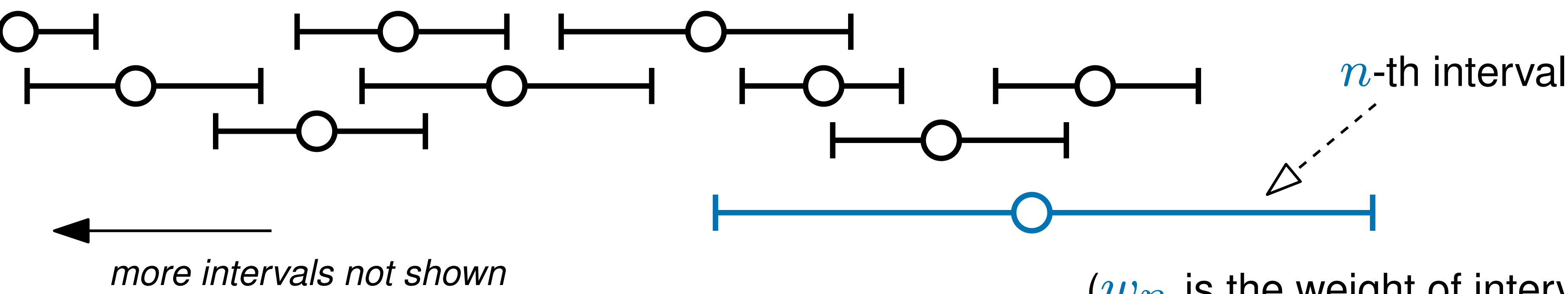

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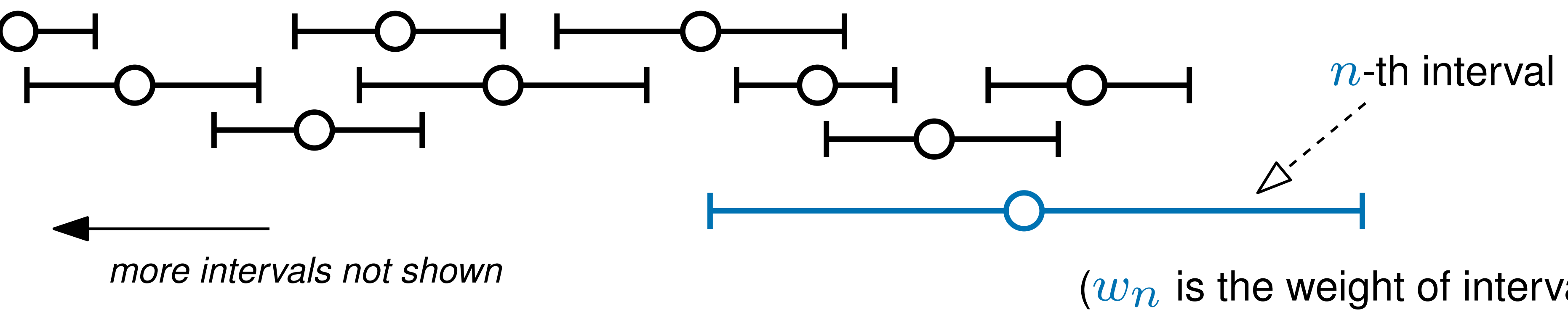

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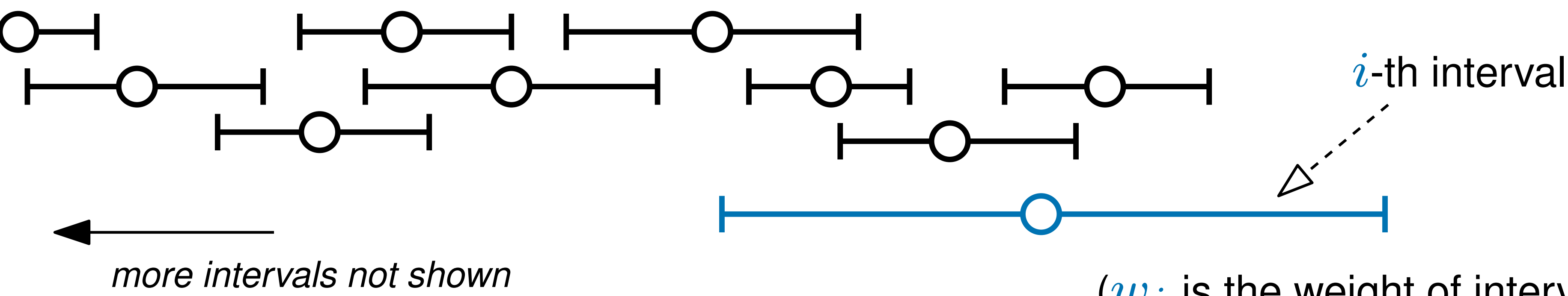

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Once again, we can use the recursive formula to get a recursive algorithm...

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WIS(i)
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If (i = 0)
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Return max(WIS(i - 1), WIS(p(i)) + w_i)
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$\texttt{WIS}(i)$ computes the weight of an optimal schedule for intervals $\{1, 2, 3, \ldots, i\}$

Therefore, $\texttt{WIS}(n)$ gives the weight of the optimal schedule (for the full problem)

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What is the time complexity of this algorithm?

How efficient is the recursive algorithm?

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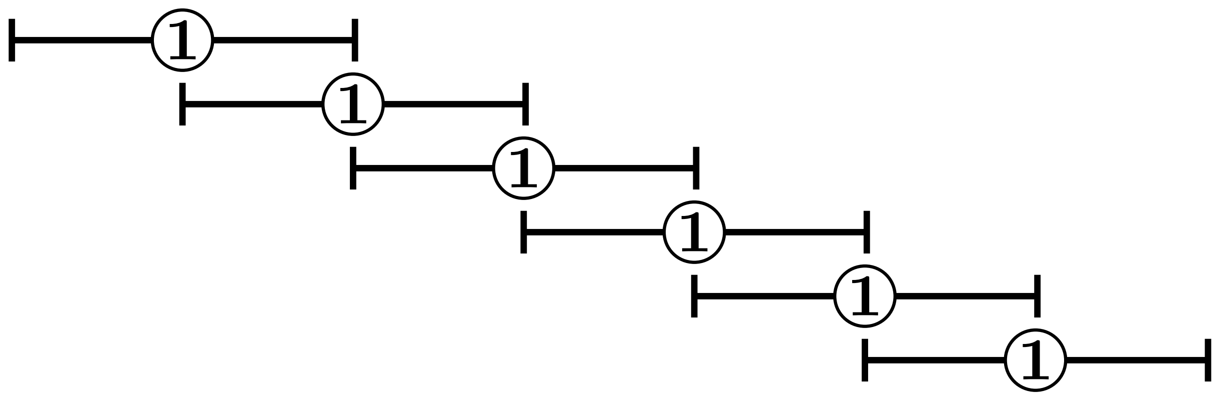
consider this simple input with $n = 6$



How efficient is the recursive algorithm?

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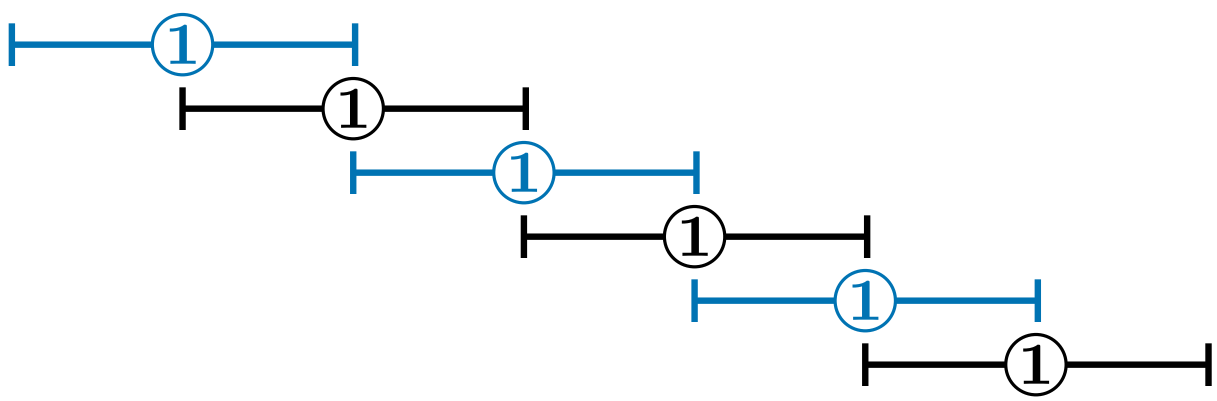

(the best schedule has weight 3)

How efficient is the recursive algorithm?

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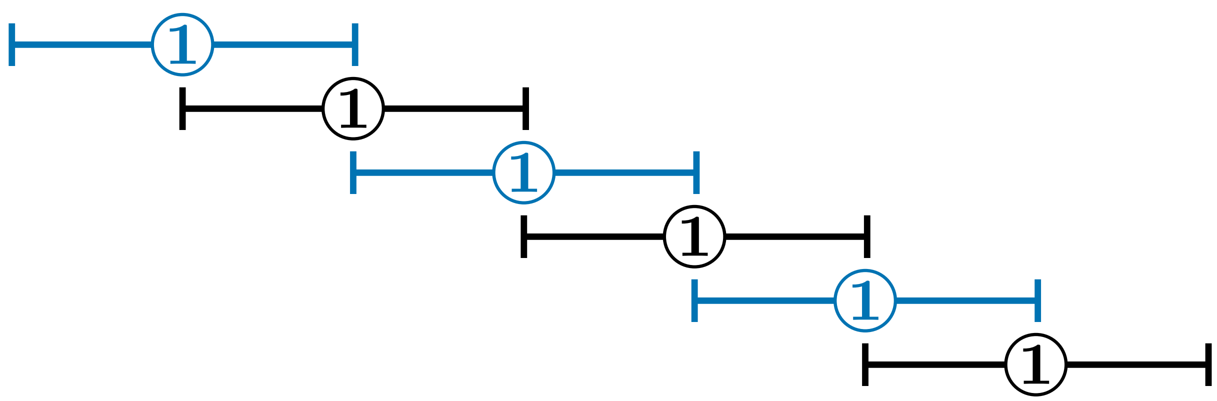

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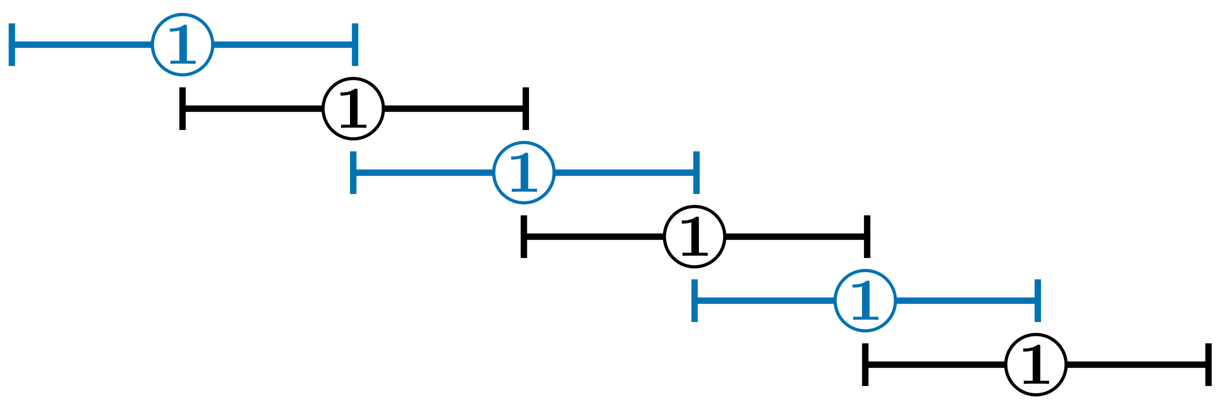

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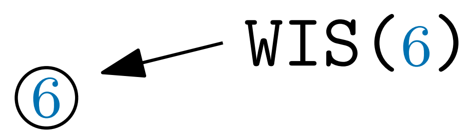

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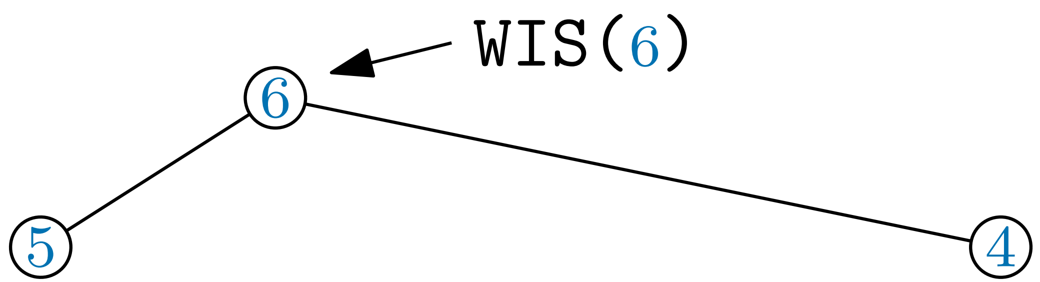

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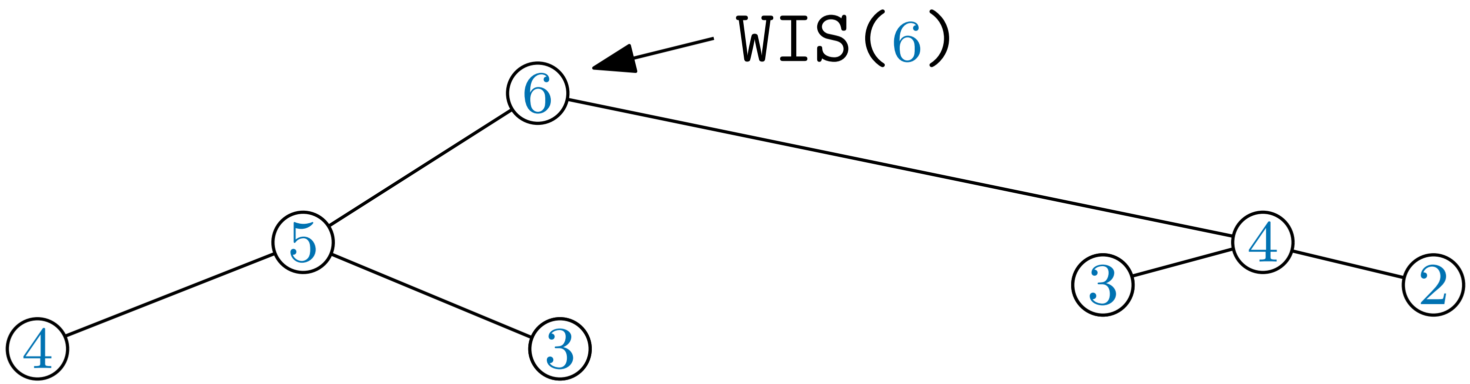

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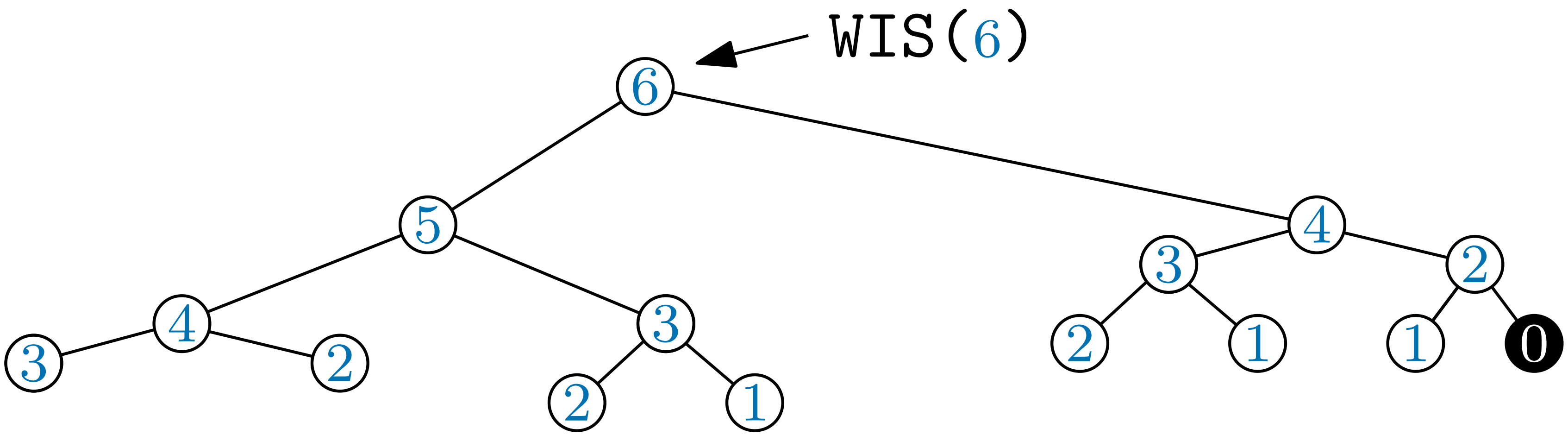

so WIS(i) makes recursive calls to WIS($i-1$) and WIS($i-2$)

How efficient is the recursive algorithm?

`WIS(`i`)`
`If` $(i = 0)$
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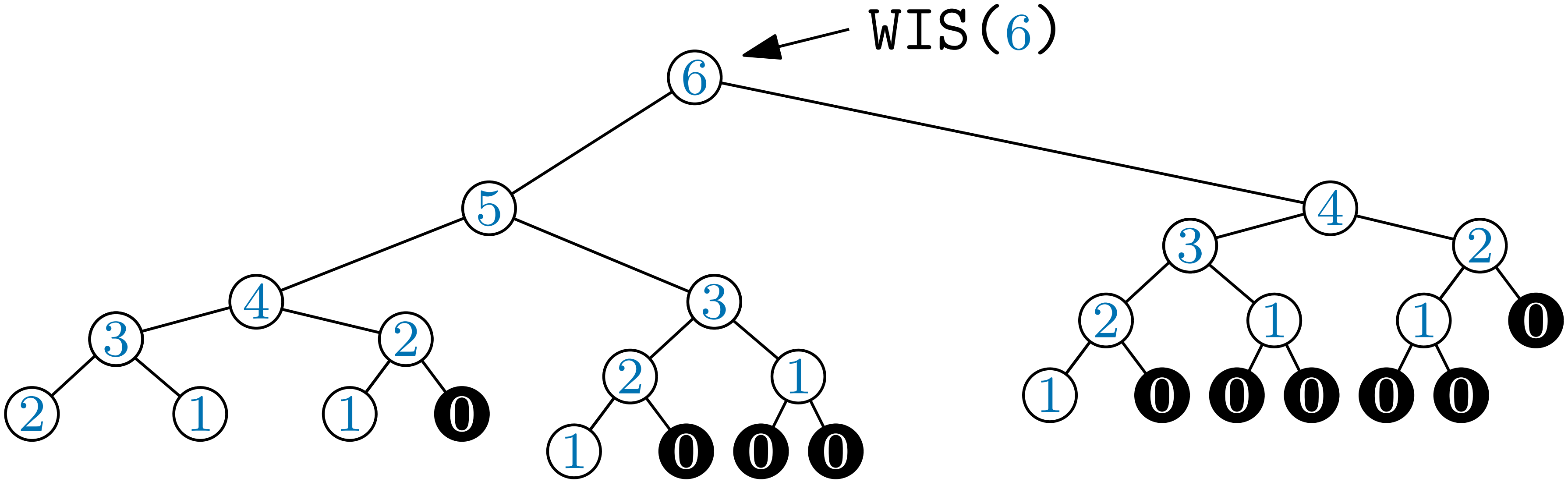

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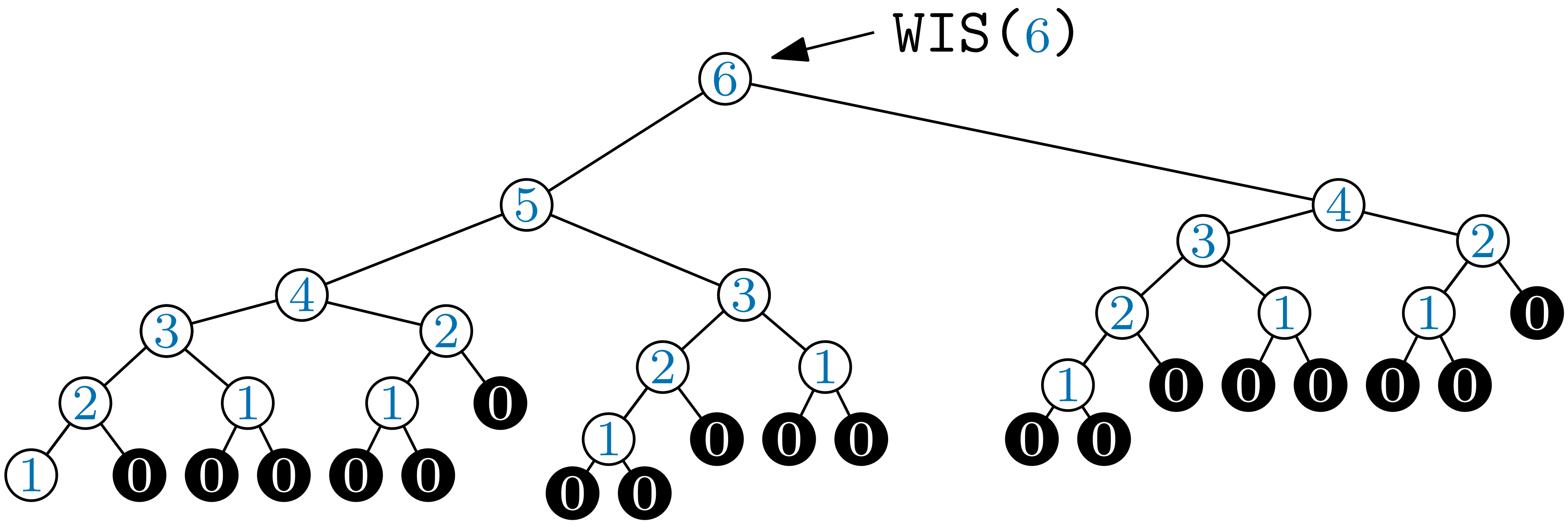

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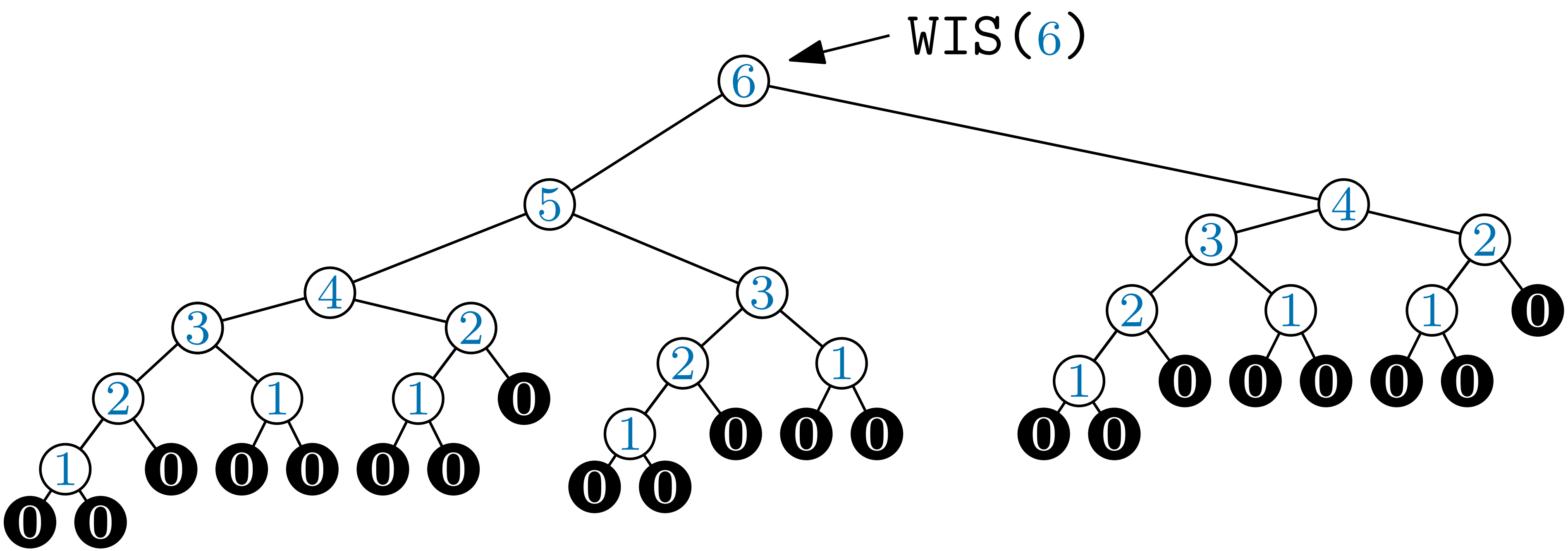

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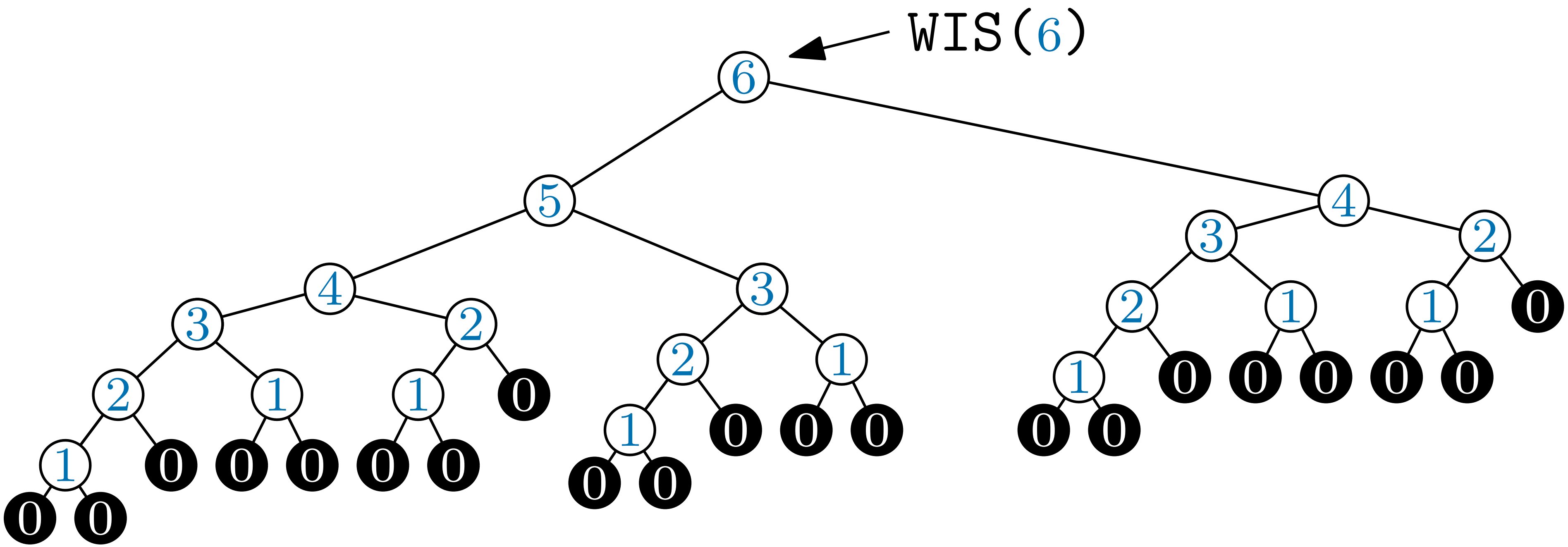

This doesn't look good (but it does look familiar)

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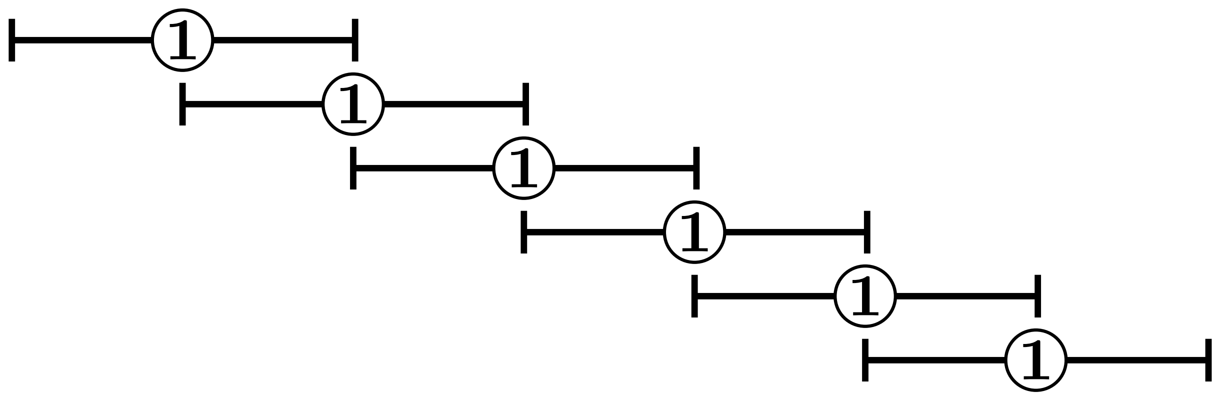

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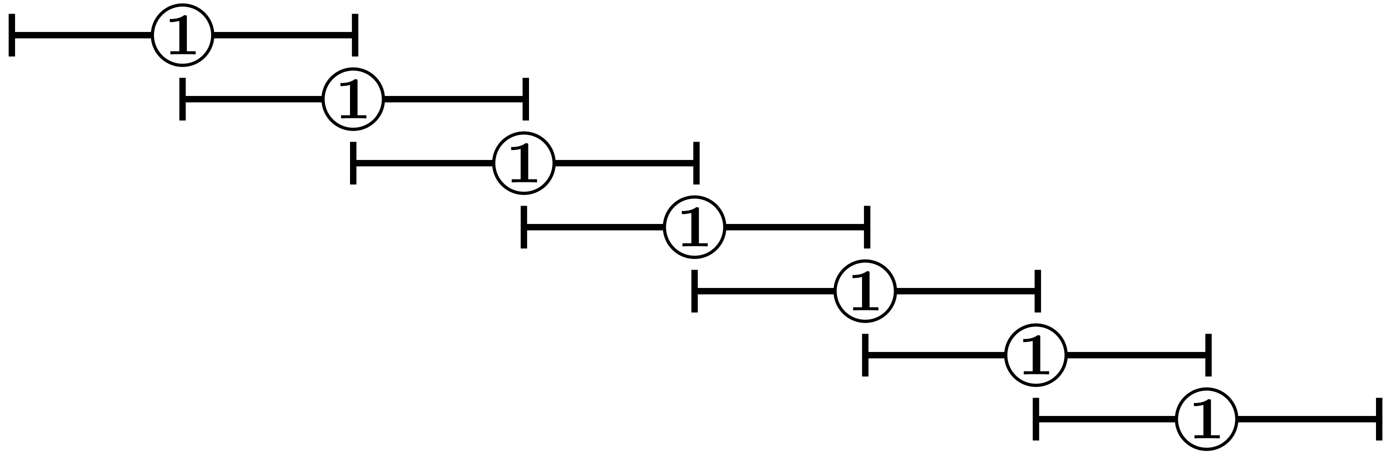

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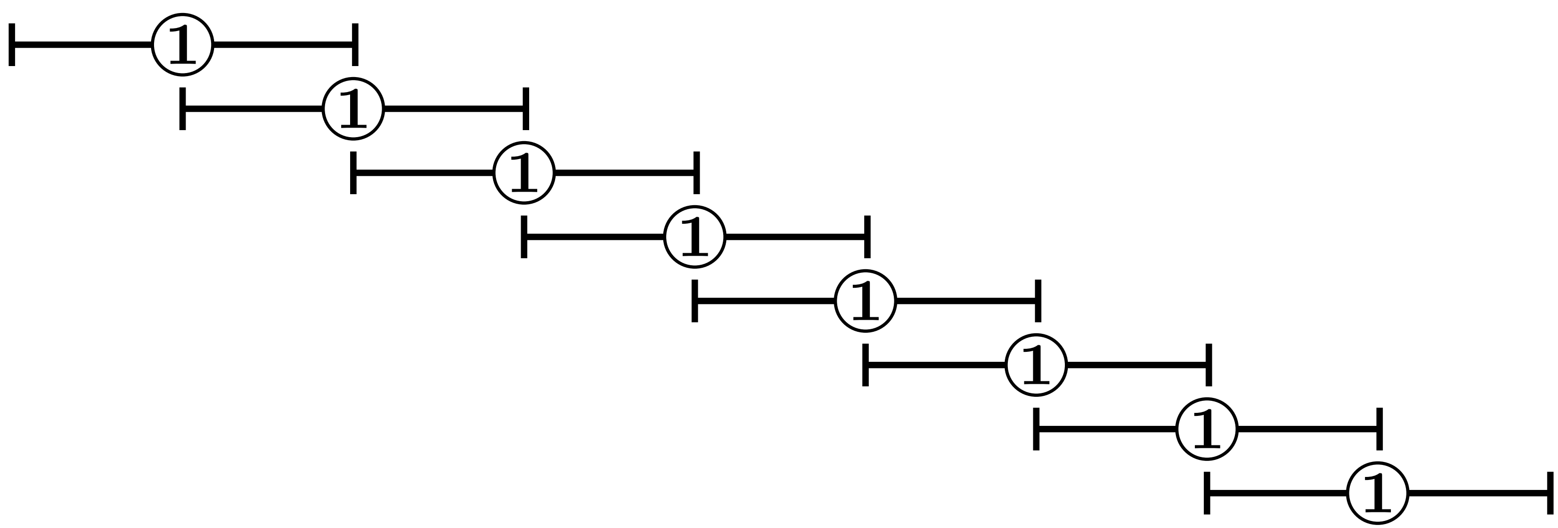

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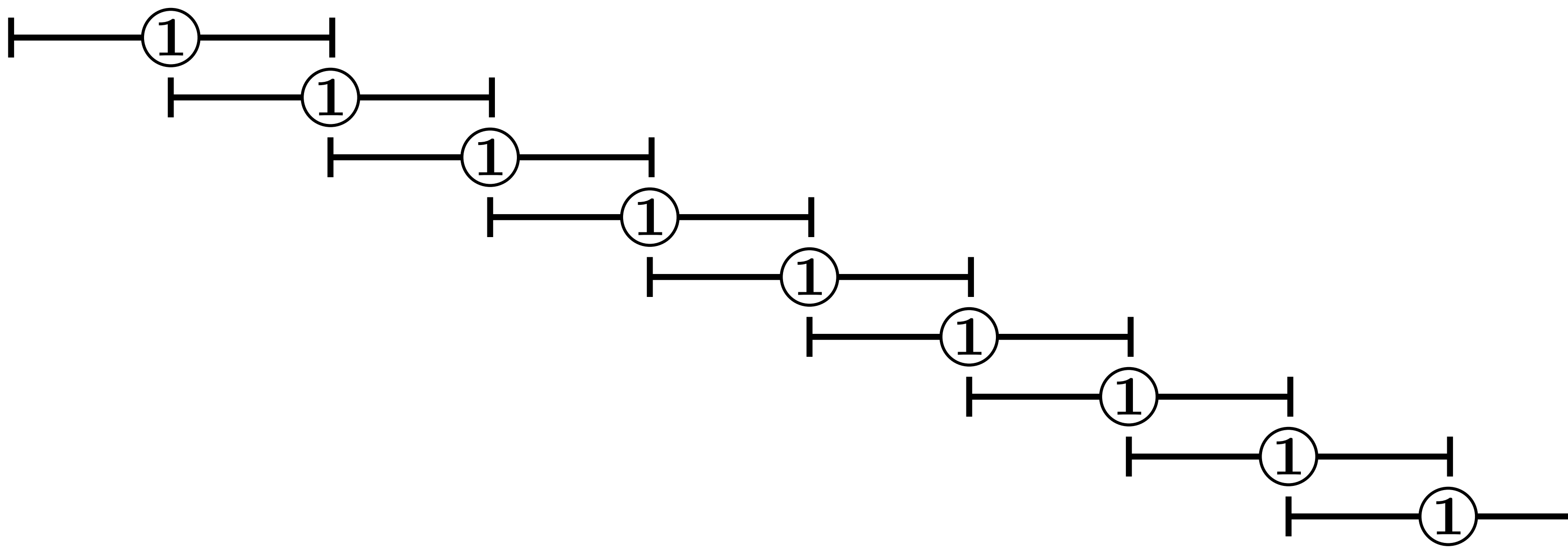

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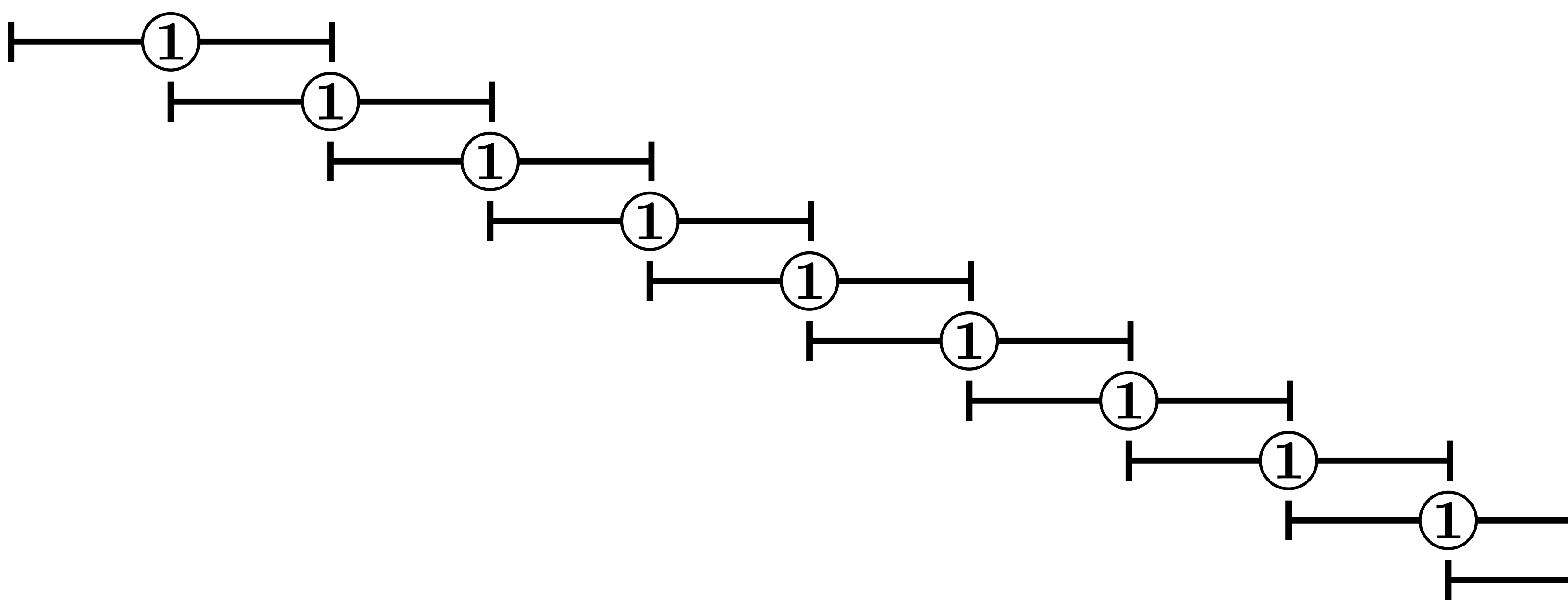

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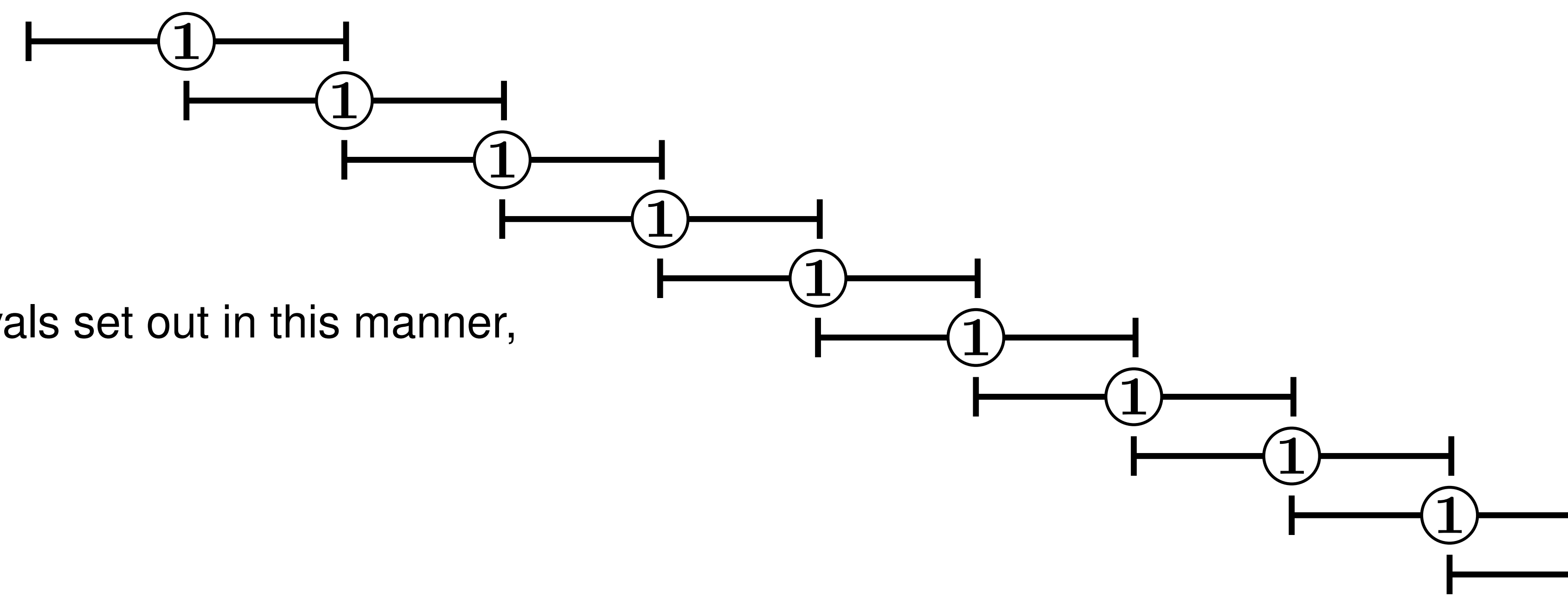

Given n intervals set out in this manner,

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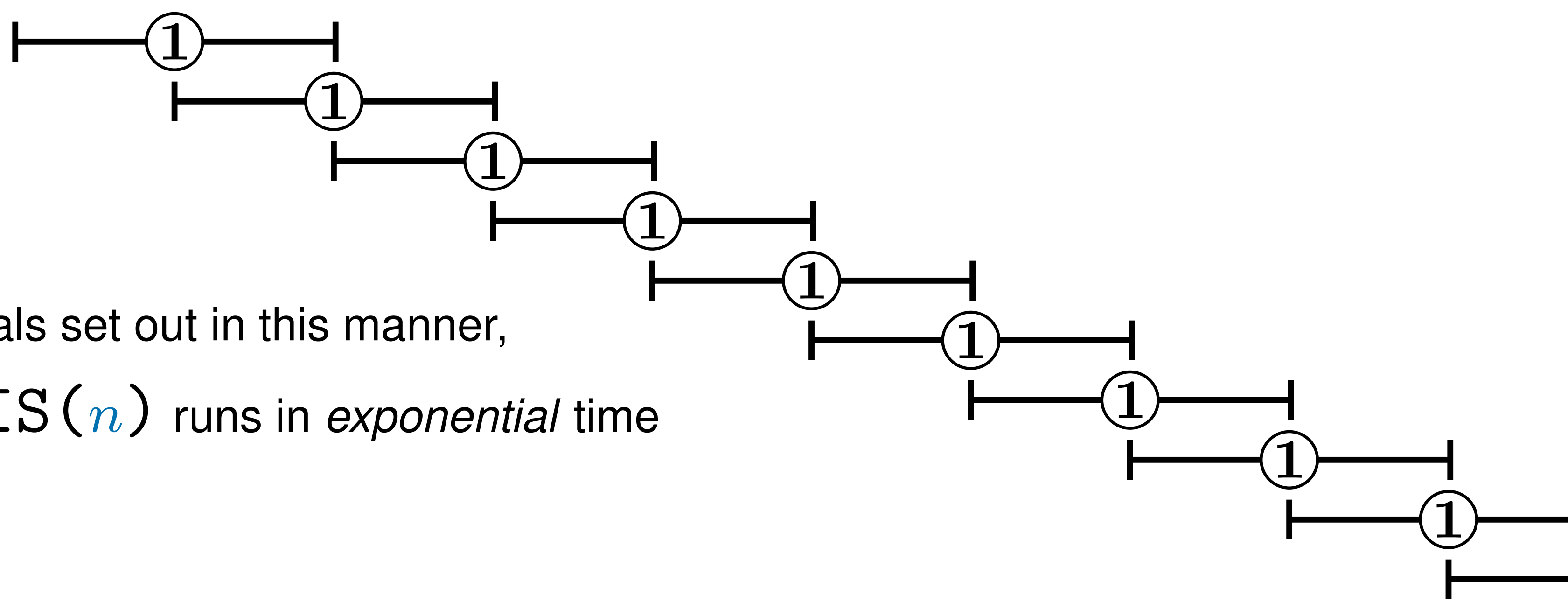

Given n intervals set out in this manner,

WIS(n) runs in *exponential* time

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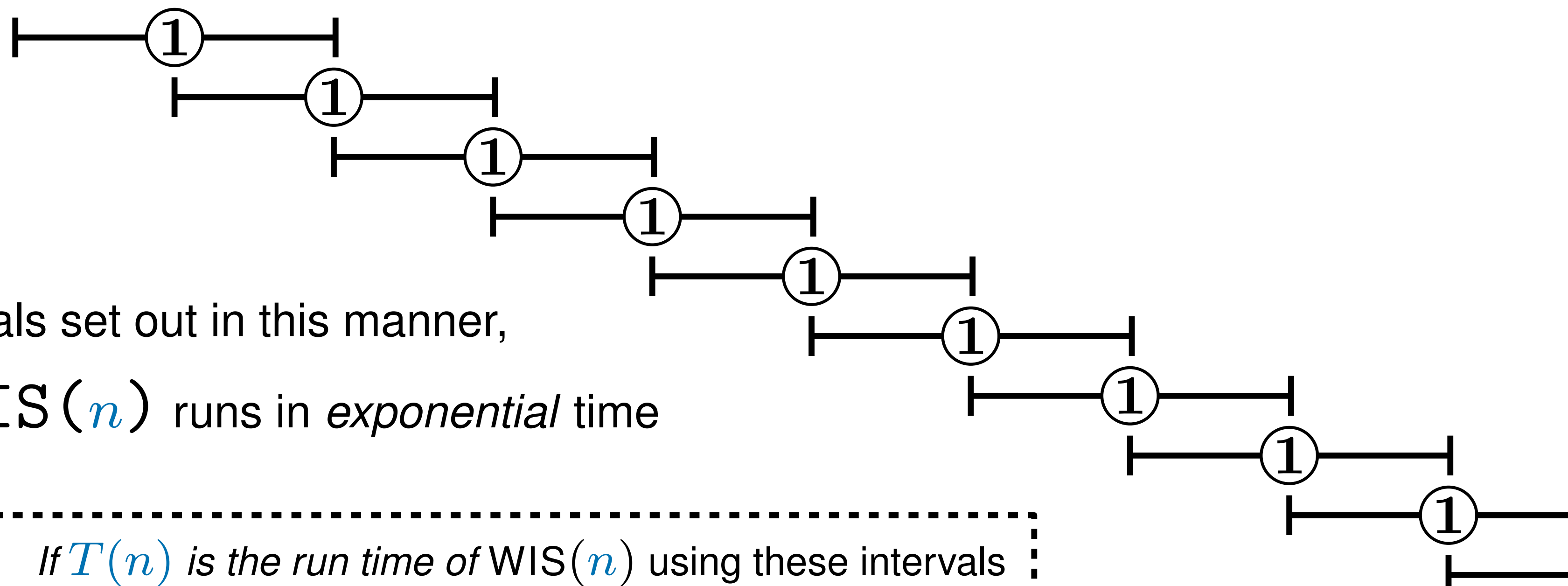

Given n intervals set out in this manner,

$\texttt{WIS}(n)$ runs in *exponential* time

If $T(n)$ *is the run time of* WIS(n) using these intervals
then $T(n) > 2T(n-2)$

3. Store the solutions to subproblems

```
MemWIS(i)
```

```
If (i = 0)
  Return 0
If WIS[i] undefined
  WIS[i] = max (MemWIS(i − 1), MemWIS(p(i)) + w_i)
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In the MEMWIS version of the algorithm we store solutions to previously computed subproblems in an n length array called WIS

3. Store the solutions to subproblems

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The time complexity of computing $\textsc{MemWIS}(n)$ is now $O(n)$

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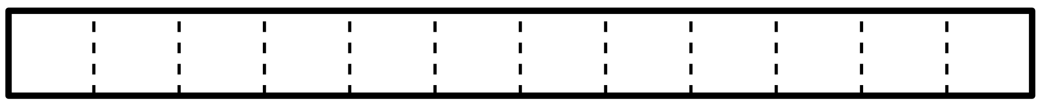
Each entry WIS$[i]$ is only computed *once*

The time complexity of computing MemWIS(n) is now $O(n)$

because every recursion causes an unfilled entry to be filled in the array

The dependency graph

The array

WIS: 

What information do we need to compute $\texttt{WIS}[i]$?

The dependency graph

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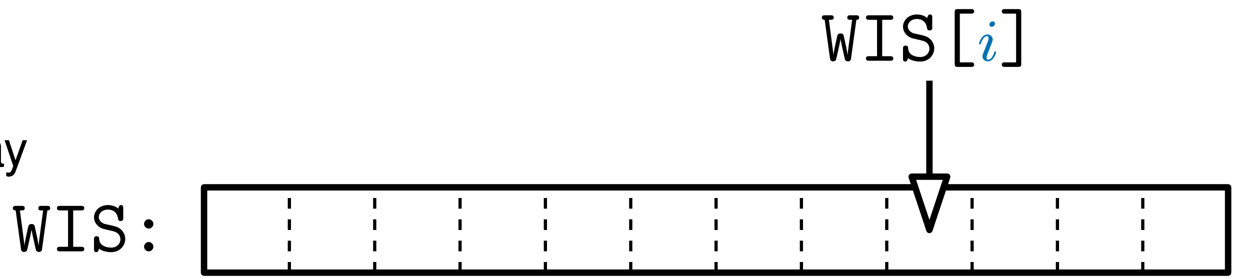

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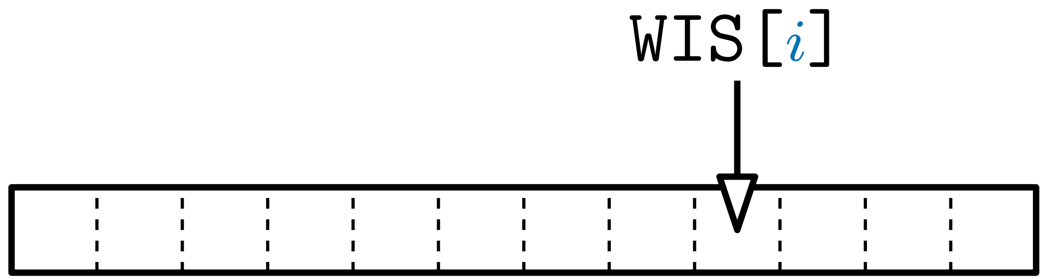

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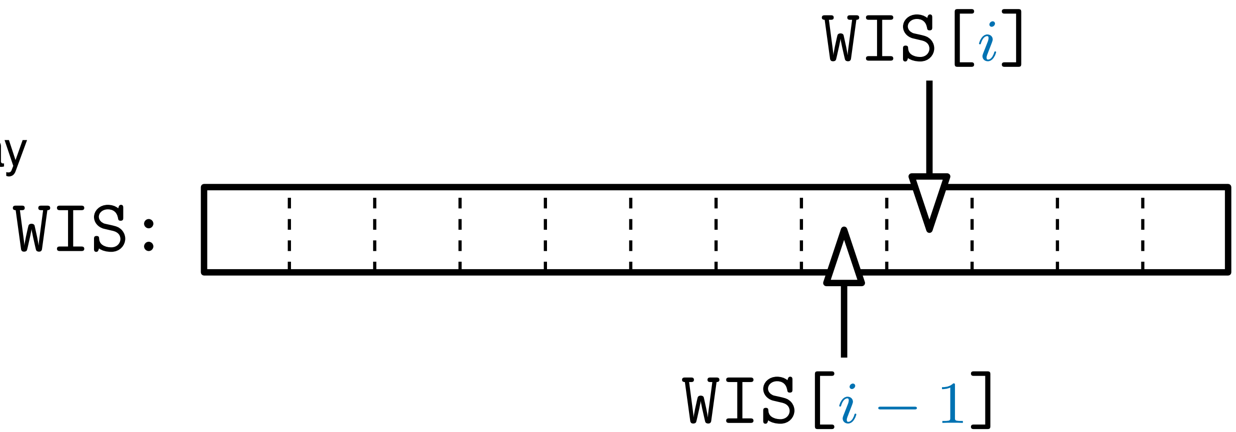

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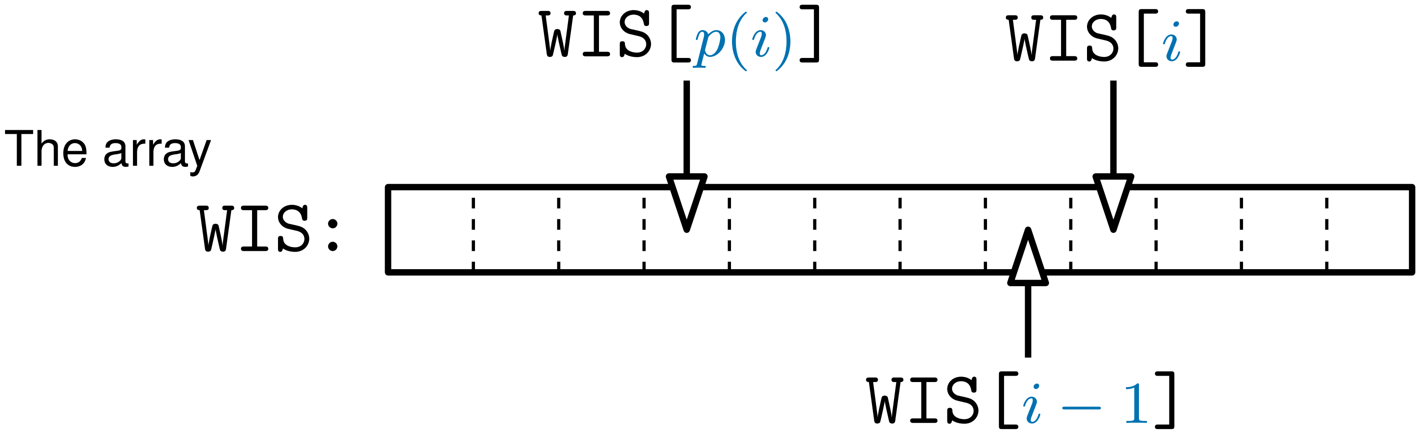

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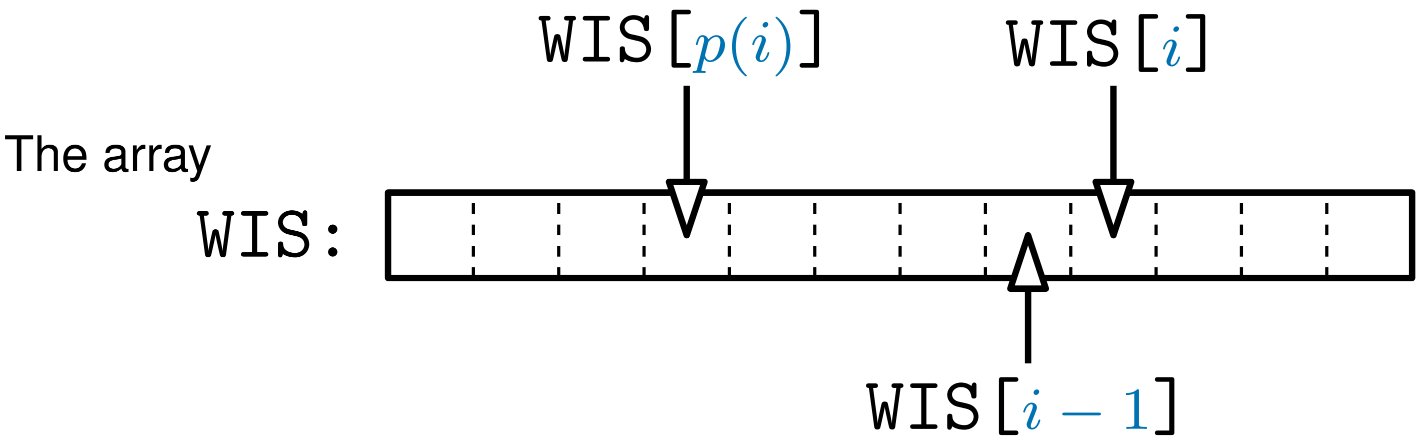

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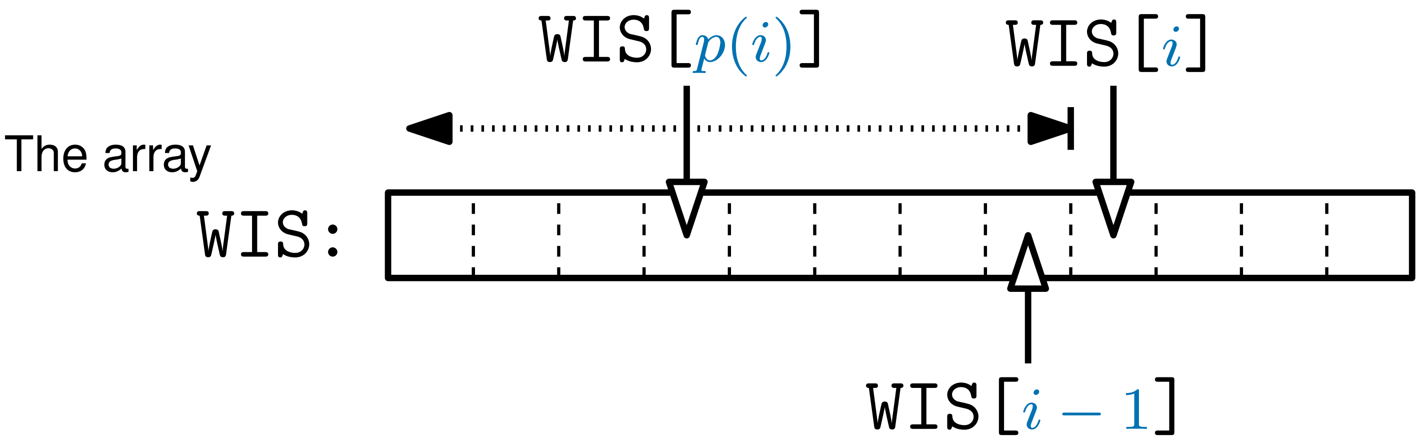

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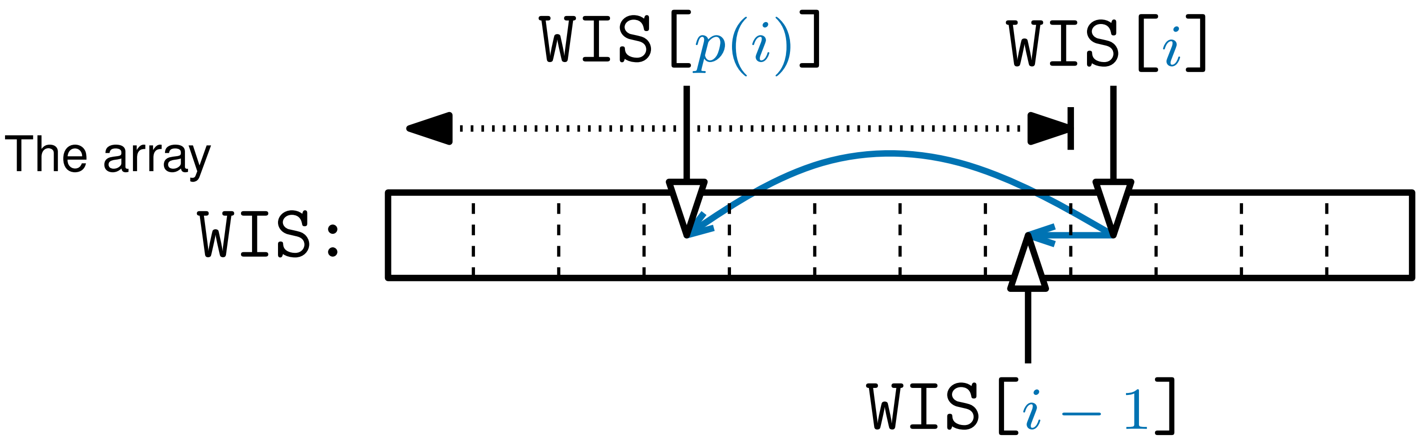

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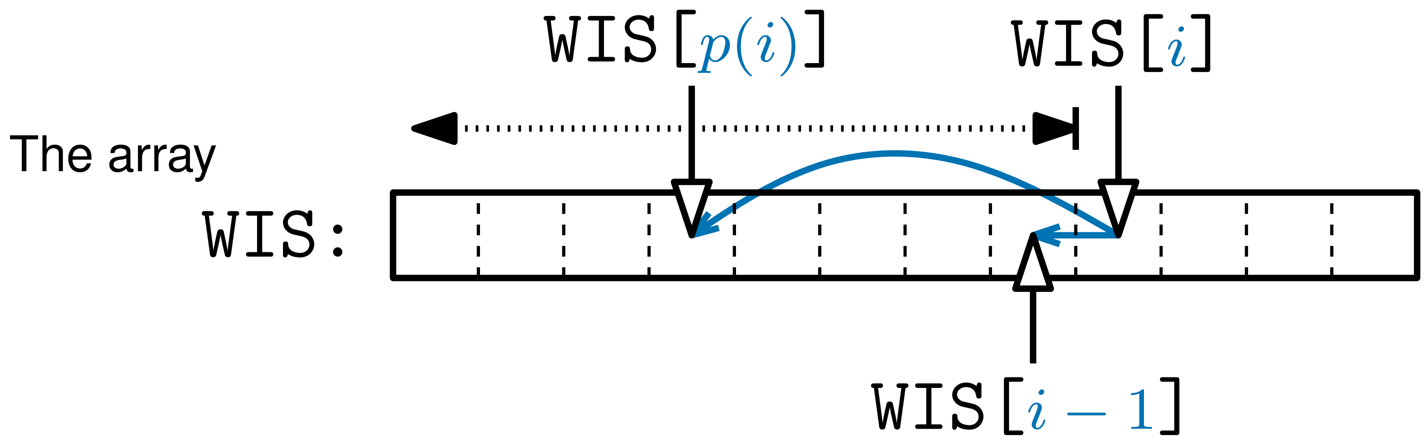

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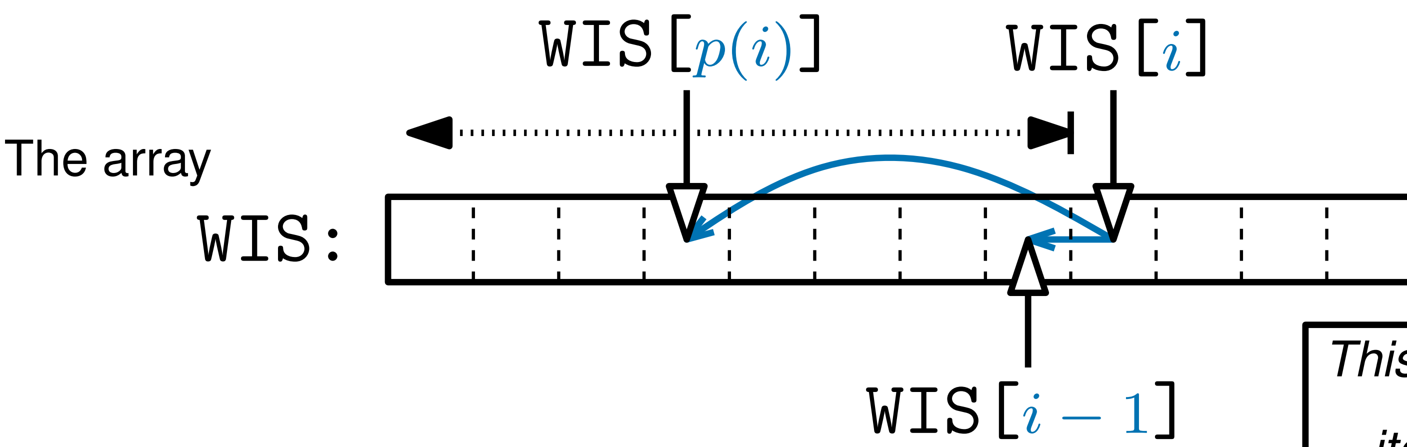

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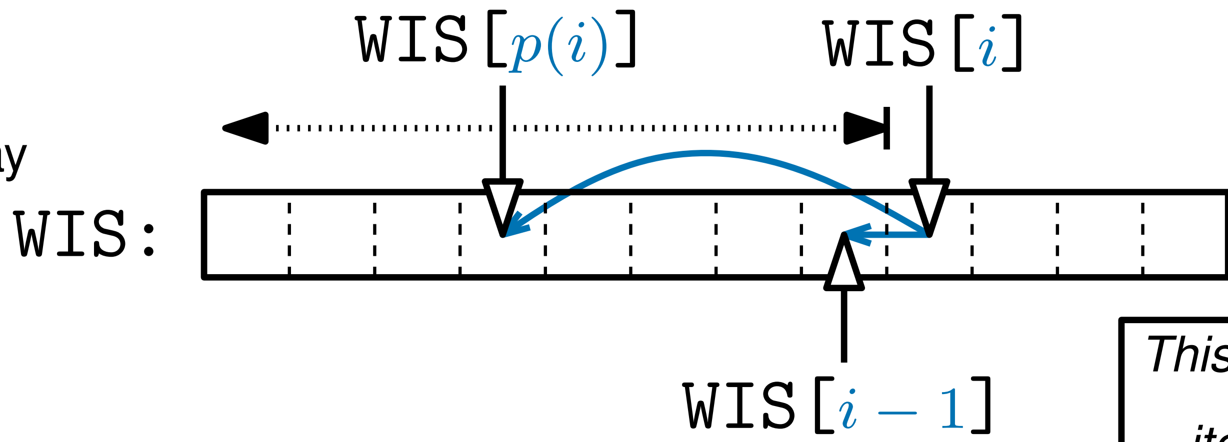

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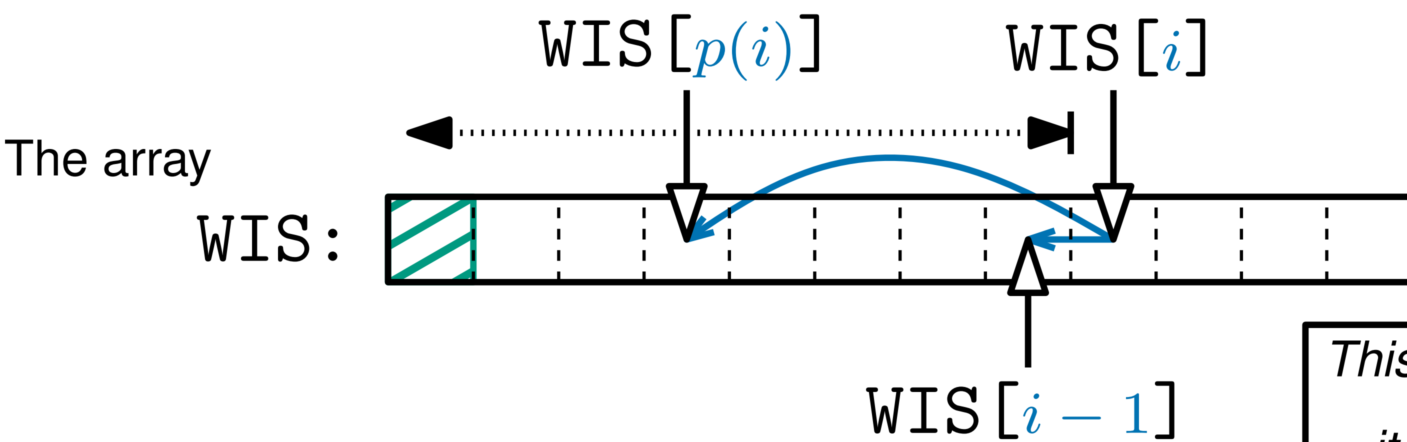

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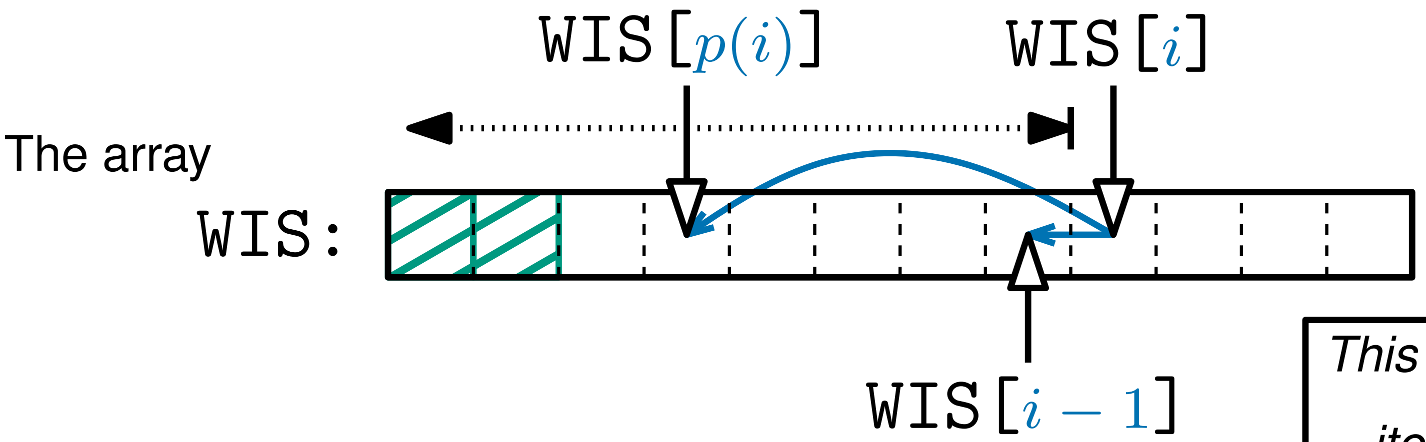

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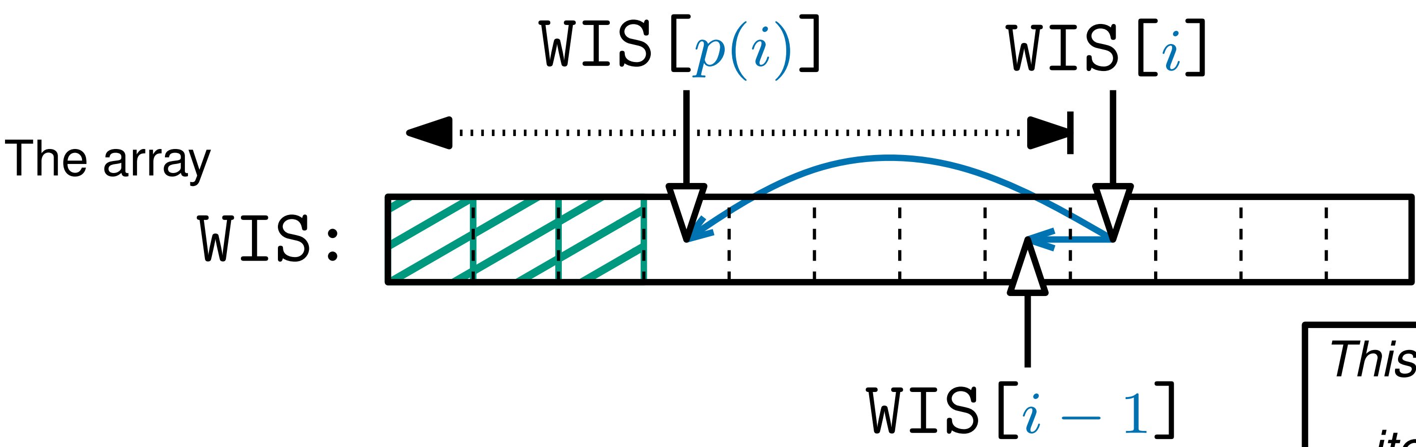

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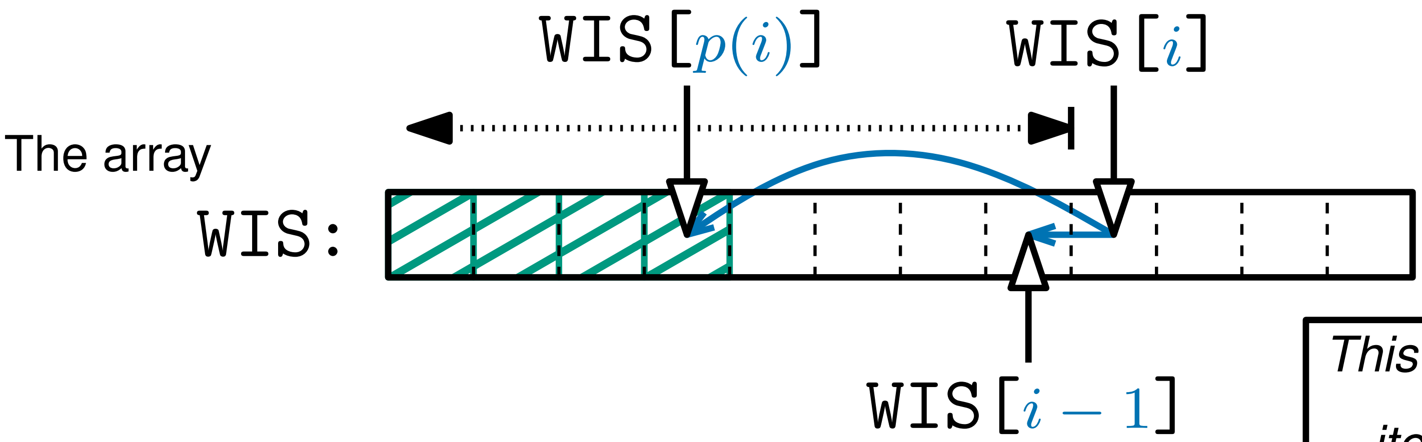

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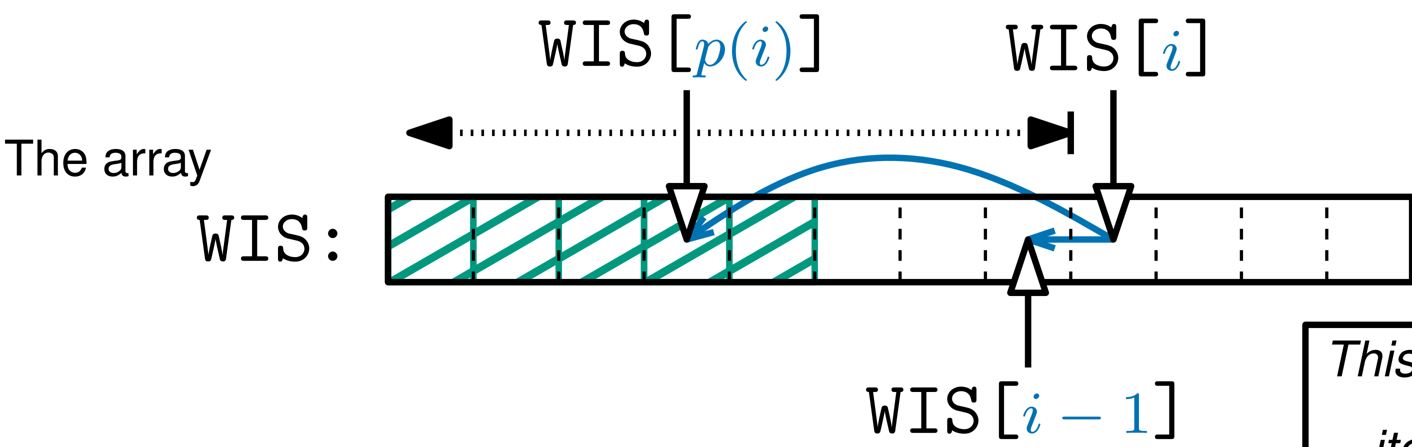

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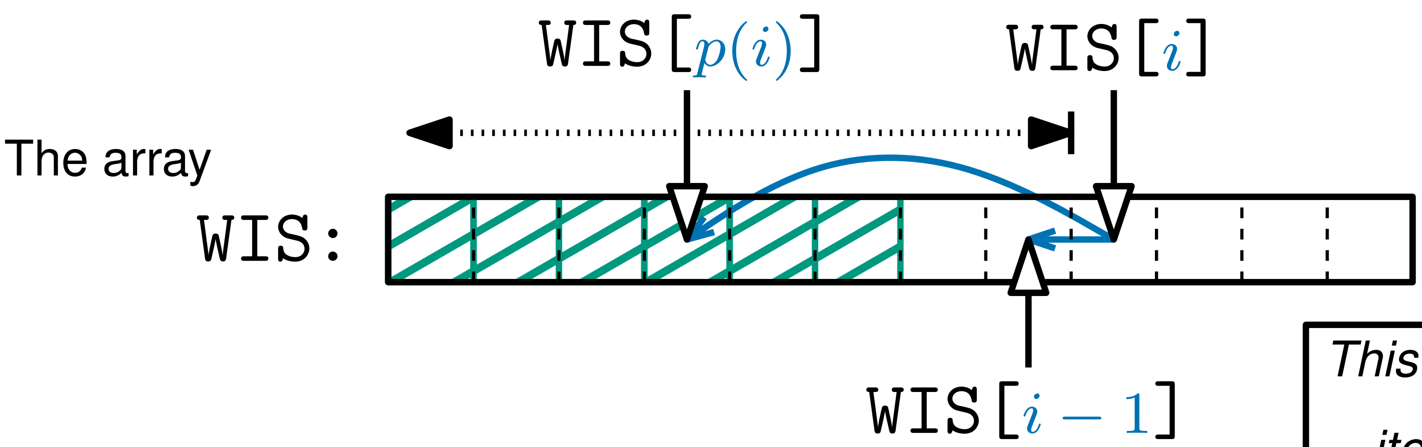

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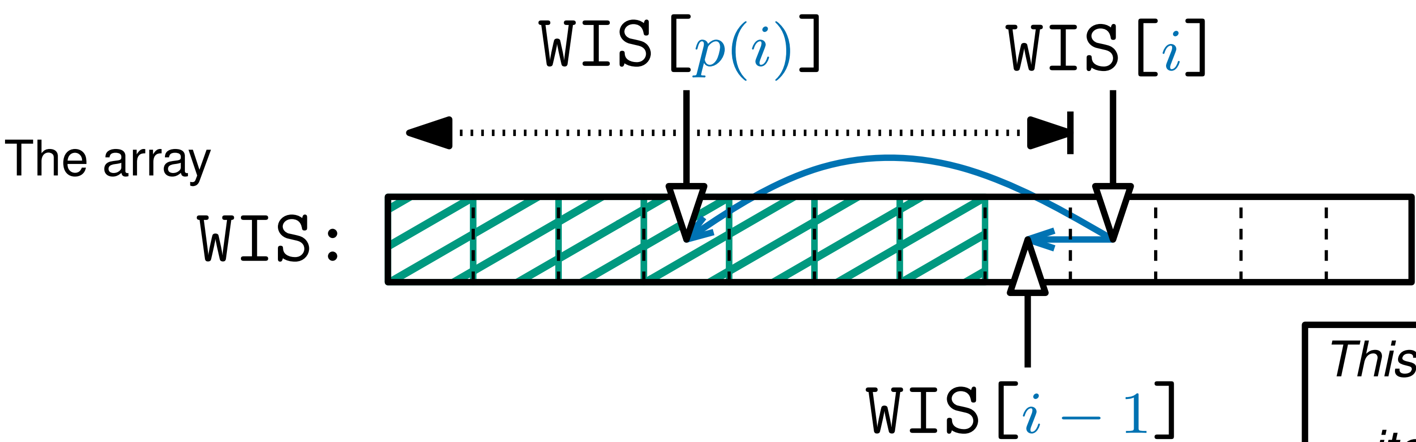

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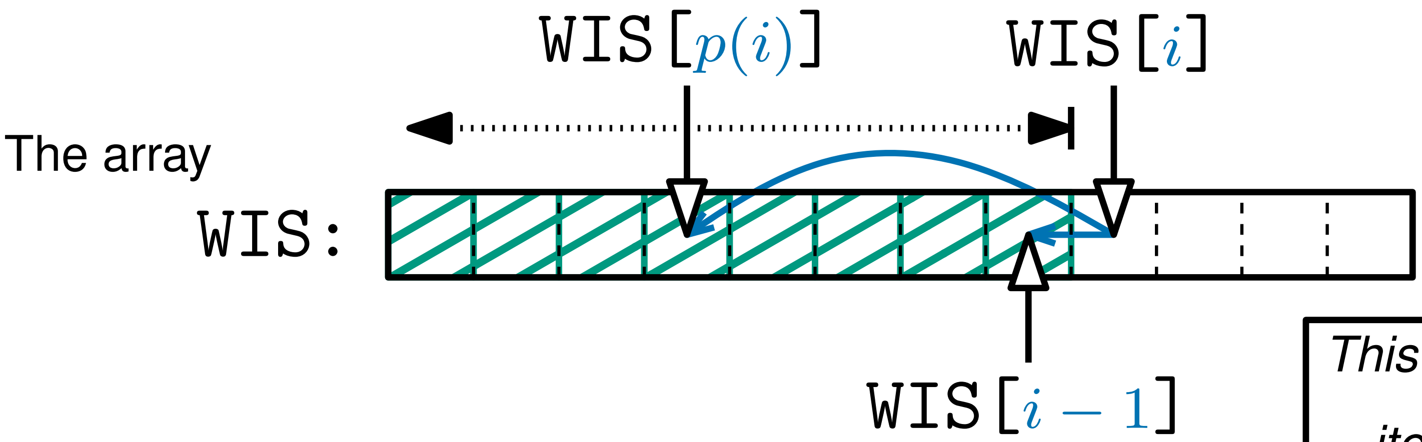

This suggests another iterative algorithm

Fill in the array from the left again

What information do we need to compute WIS[i] ?

to compute WIS[i] we need WIS[$i-1$] and WIS[$p(i)$]

both of which are to the *left* of WIS[i]

(somewhere)

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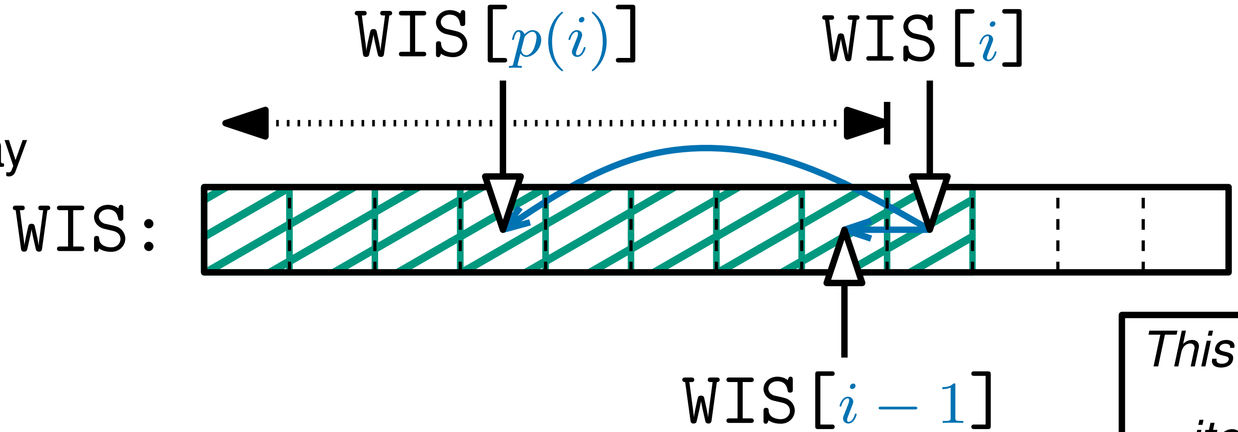
all of the dependencies go left. . .

The array



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4. Derive an iterative algorithm

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ItWIS(n)
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This is an iterative dynamic programming algorithm
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it runs in $O(n)$ time

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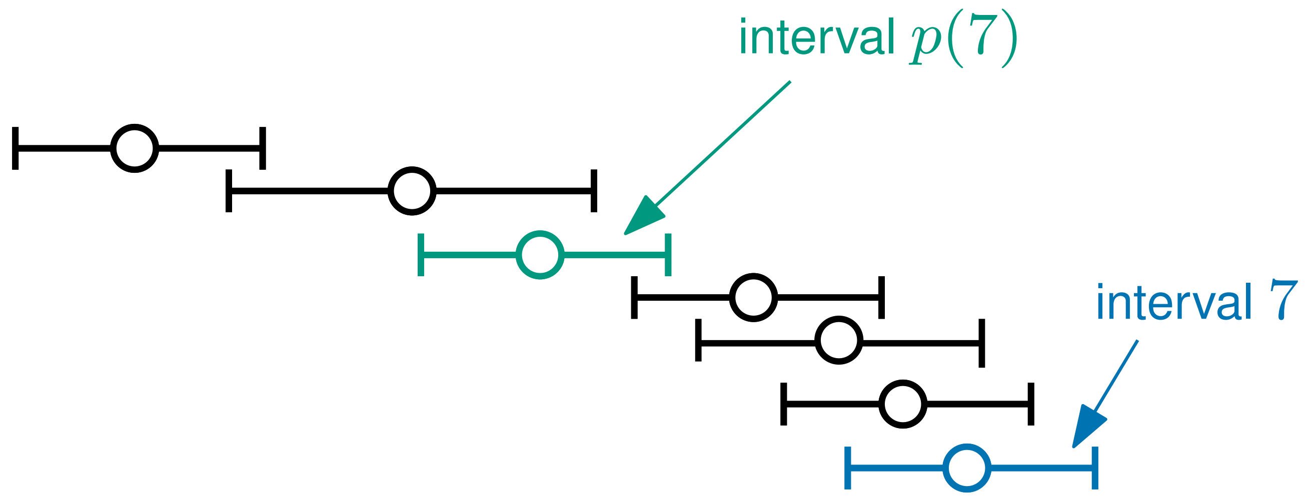
…but it requires than you precomputed all the $p(i)$ values

How do you find all those $p(i)$ values?



Revised Claim: We can precompute any $p(i)$ in $O(\log n)$ time

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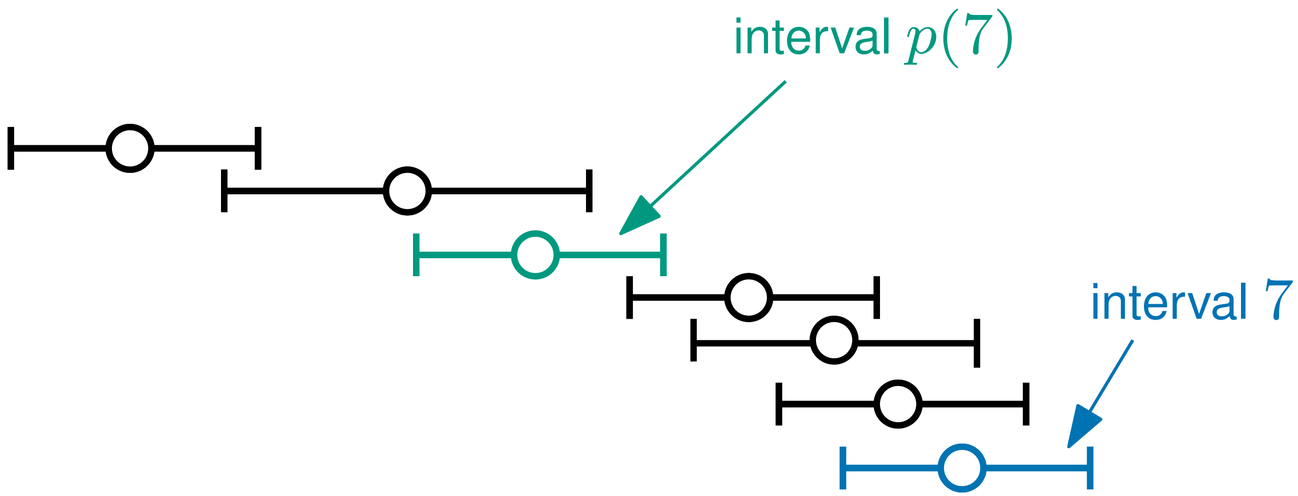

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Recall that s_i is the start time of interval i

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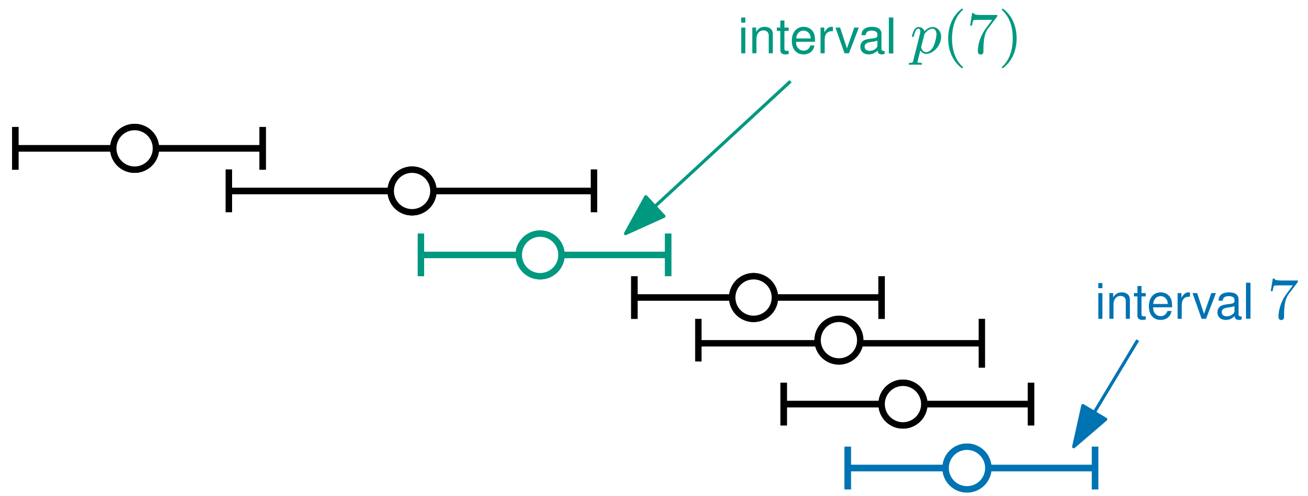

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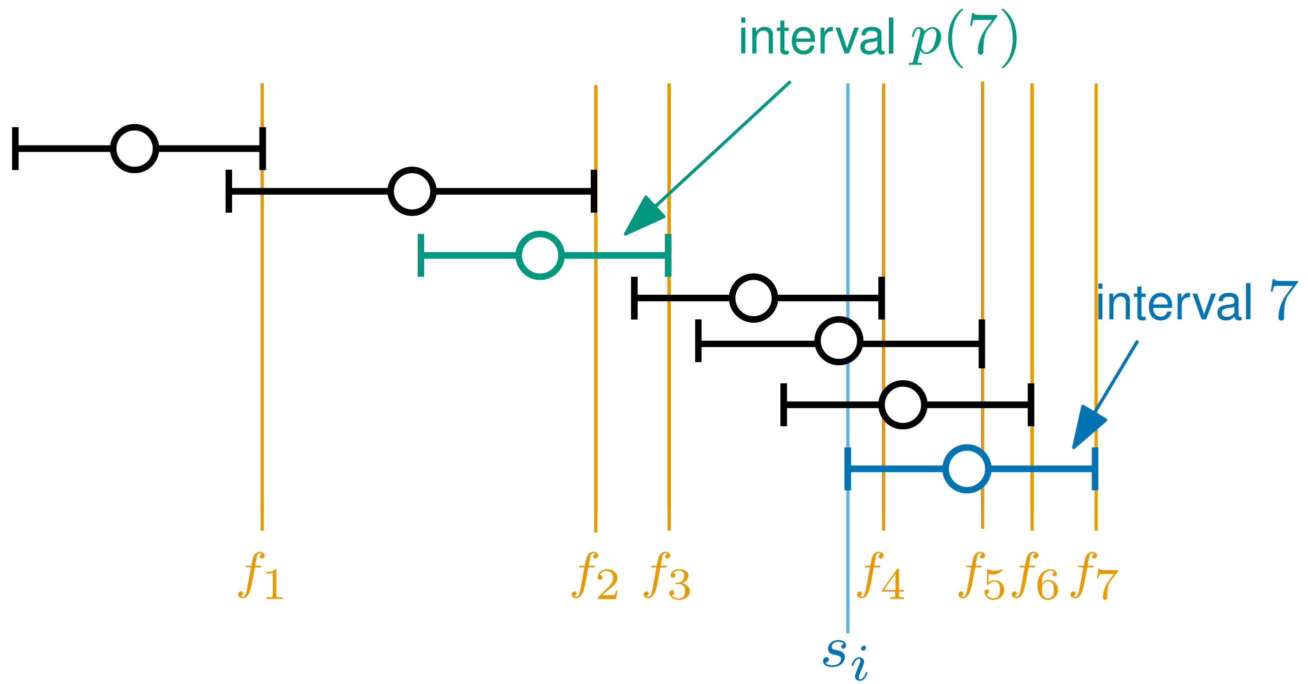

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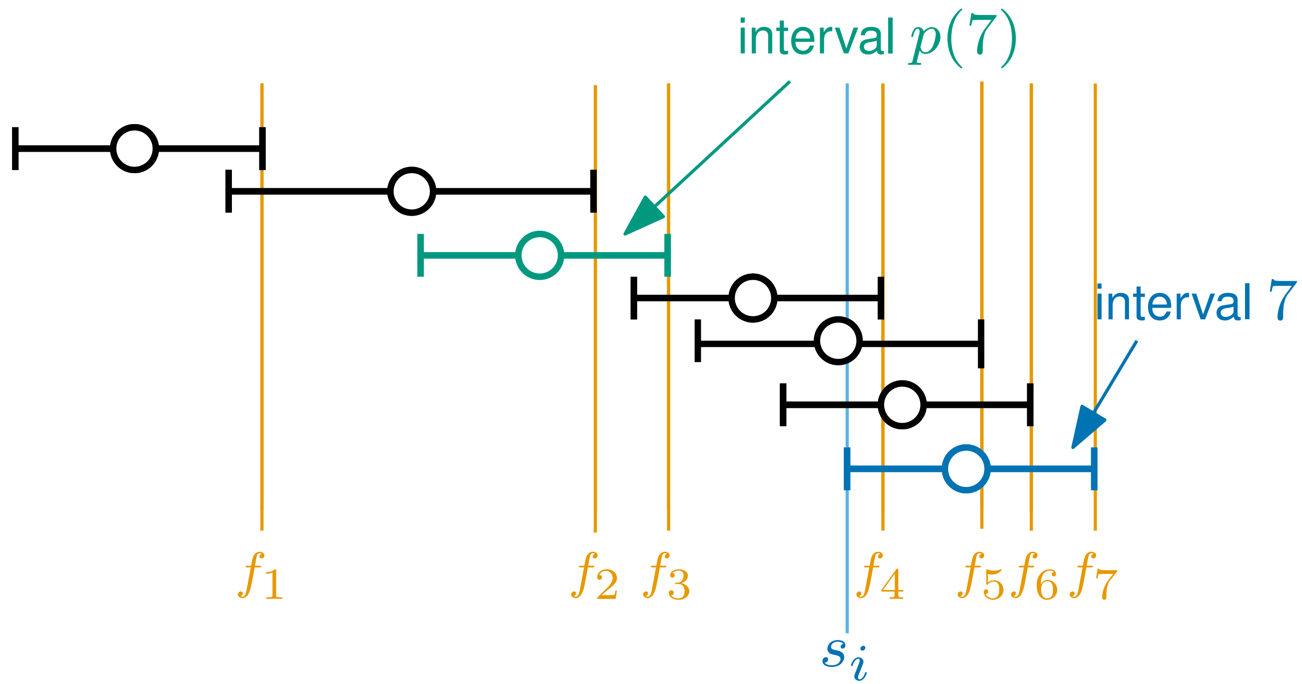

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As the input is sorted by finish times, we can find j by binary search in $O(\log n)$ time

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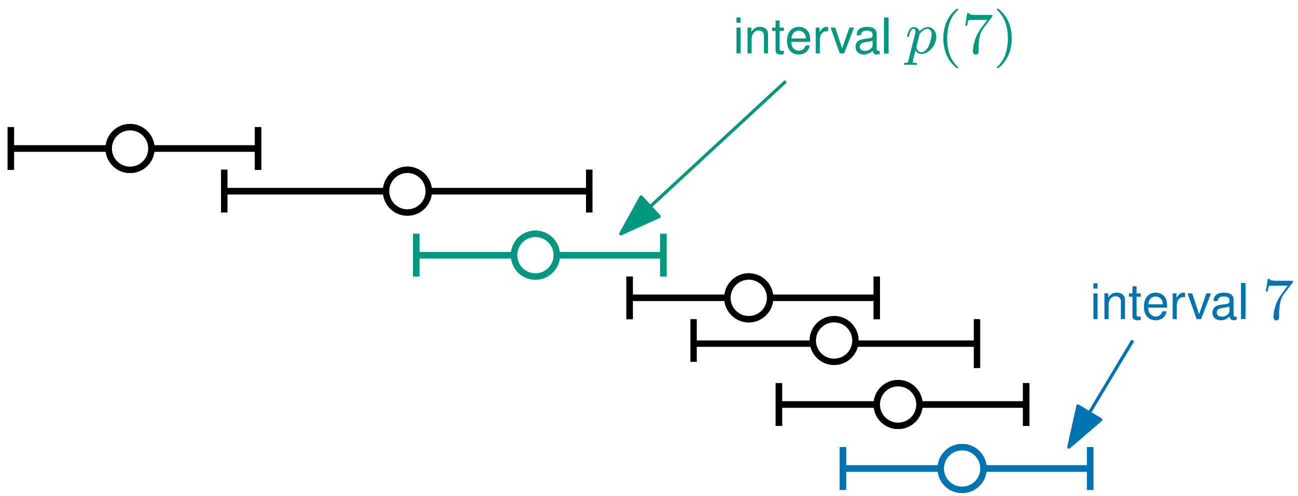

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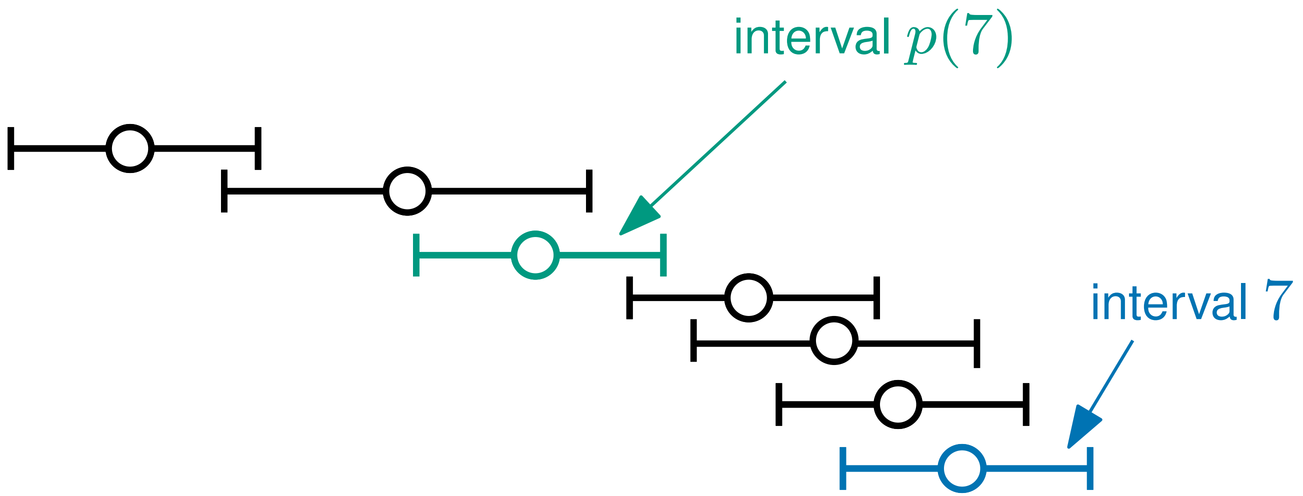

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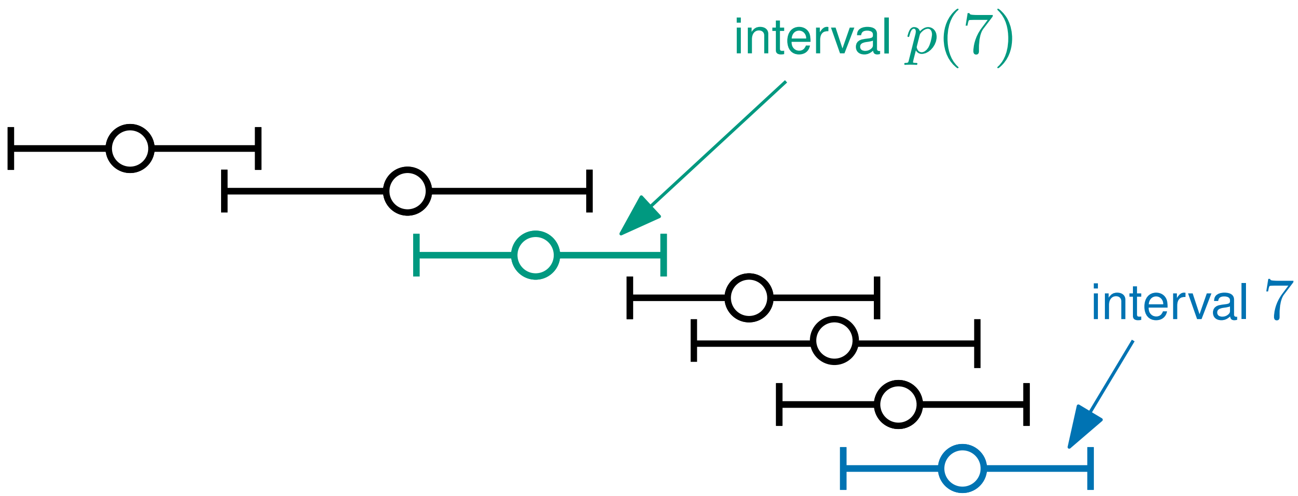

Revised Claim: We can precompute any $p(i)$ in $O(\log n)$ time

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(by using the revised claim n times)

Wait, did you want the actual schedule?

$\texttt{ItWIS}(n)$ finds the weight of the optimal schedule
but doesn't find the actual schedule

$\texttt{ItWIS}(n)$

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There is an optimal schedule for $\{1, 2, \ldots, i\}$ containing
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(by the argument we saw earlier)

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$\texttt{ItWIS}(n)$ finds the weight of the optimal schedule
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$\texttt{ItWIS}(n)$	$\textsc{FindWIS}(i)$

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	If $(i = 0)$
	  Return nothing
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The final algorithm

ITWIS(n) finds the weight of the optimal schedule
and FINDWIS(n) finds the actual schedule

ITWIS(n)

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FINDWIS(i)

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If (i = 0)
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If WIS[i − 1] ⩽ WIS[p(i)] + w_i
  Return FindWIS(p(i)) then i
Return FindWIS(i − 1)
```

The final algorithm:

Step 1: Find all the $p(i)$ values

Step 2: Run ITWIS(n) to find the optimal weight

Step 3: Run FINDWIS(n) to find the schedule

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The final algorithm:

$O(n \log n)$ time

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The final algorithm:

Step 1: Find all the $p(i)$ values ← $O(n \log n)$ time

Step 2: Run $\texttt{ItWIS}(n)$ to find the optimal weight ← $O(n)$ time

Step 3: Run $\textsc{FindWIS}(n)$ to find the schedule ← $O(n)$ time

The final algorithm

$\texttt{ItWIS}(n)$ finds the weight of the optimal schedule
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$O(n)$ time

Step 3: Run $\texttt{FindWIS}(n)$ to find the schedule

Overall this takes $O(n \log n)$ time

[Dynamic programming] is a technique for finding efficient algorithms for problems which can be broken down into simpler, overlapping subproblems.

The basic idea:

1. Find a recursive formula for the problem
- in terms of answers to subproblems.

(typically this is the hard bit)

2. Write down a naive recursive algorithm

(typically this algorithm will take exponential time)

3. Speed it up by storing the solutions to subproblems (memoization)

(to avoid recomputing the same thing over and over)

4. Derive an iterative algorithm by solving the subproblems in a good order

(iterative algorithms are often better in practice, easier to analyse and prettier)

in other words. . .

Dynamic programming is *recursion without repetition*