Example 0.1. *Consider a binary symmetric channel, with probability matrix*

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

Find the capacity of the channel.

Solution:

We know, from the matrix P that the probability of a correct transmission is

$$\Pr(Y=0|X=0) = P(Y=1|X=1) = 1-p$$

and the probability of incorrect transmission for each symbol is given by

$$\Pr(Y=0|X=1) = \Pr(Y=1|X=0) = p.$$

We can choose the input distribution as: $\Pr(X=0)=\alpha$ and $\Pr(X=1)=1-\alpha$, $\alpha \in [0,1]$.

We need $H(Y)$ and $H(Y|X)$ in order to derive $I(X;Y) = H(Y) - H(Y|X)$.

To compute $H(Y) = \sum_{y\in\{0,1\}} \Pr(Y=y)\log_2(\Pr(Y=y))$, we need $\Pr(Y=0)$ and $\Pr(Y=1)$.

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$$\begin{aligned}\Pr(Y=0) = &\sum_{x\in\{0,1\}} \Pr(X=x)\cdot\Pr(Y=0|X=x)\\ =&\Pr(X=0)\Pr(Y=0|X=0)+\Pr(X=1)\Pr(Y=0|X=1)\\ =&\alpha\cdot(1-p)+(1-\alpha)\cdot p = p+\alpha\cdot(1-2p) = D.\end{aligned}$$

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$$\begin{aligned}\Pr(Y=1) = &\sum_{x\in\{0,1\}} \Pr(X=x)\cdot P(Y=1|X=x)\\ =&\Pr(X=0)\Pr(Y=1|X=0)+\Pr(X=1)\Pr(Y=1|X=1)\\ =&\alpha\cdot p+(1-\alpha)\cdot(1-p) = 1-p-\alpha\cdot(1-2p) = 1-D.\end{aligned}$$

Therefore we have:

$$\begin{aligned}H(Y) &= -\sum_{y\in\{0,1\}} \Pr(Y=y)\cdot\log_2(\Pr(Y=y))\\ &= -\big[\Pr(Y=0)\cdot\log_2(\Pr(Y=0))+\Pr(Y=1)\cdot\log_2(\Pr(Y=1))\big]\\ &= -\big[D\cdot\log_2(D)+(1-D)\cdot\log_2(1-D)\big]\\ &= H(D) \quad \text{(the binary entropy function)}\end{aligned}$$

Next we compute the conditional entropy:

$$H(Y|X) = \sum_{x\in\{0,1\}} \Pr(X=x)H(Y|X=x)$$
$$=\Pr(X=0)\cdot H(Y|X=0)+\Pr(X=1)\cdot H(Y|X=1)$$
$$=-\Pr(X=0)\cdot\big[\Pr(Y=0|X=0)\log_2(\Pr(Y=0|X=0))+\Pr(Y=1|X=0)\log_2(\Pr(Y=1|X=0))\big]$$
$$-\Pr(X=1)\cdot\big[\Pr(Y=0|X=1)\log_2(\Pr(Y=0|X=1))+\Pr(Y=1|X=1)\log_2(\Pr(Y=1|X=1))\big]$$
$$=-\alpha\cdot\big[(1-p)\log_2(1-p)+p\log_2 p\big]-(1-\alpha)\cdot\big[p\log_2(p)+(1-p)\log_2(1-p)\big]$$
$$=\alpha\cdot H(p)+H(p)-\alpha\cdot H(p) = H(p).$$

Thus we have:

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(p + \alpha \cdot (1-2p)) - H(p) \end{aligned}$$

We have that $C = \max_X I(X;Y)$. Note that $H(Y|X) = H(p)$ does not depend on α, therefore to compute the maximum of $I(X;Y)$, we only need to find the value of α which maximizes $H(p+(1-2p)\cdot\alpha)$. Further, we know that the binary entropy function $H(x)$ takes its maximum value when $x = 1/2$ and that when $\alpha = 1/2$, the value $p + \alpha \cdot (1-2p) = 1/2$. Hence $I(X;Y)$ reaches its maximum value when $\alpha = 1/2$. Then

$$\begin{aligned} C &= \max_\alpha \big(H(p+\alpha\cdot(1-2p)) - H(p) \big) \\ &= H(1/2) - H(p) = 1 - H(p) \end{aligned}$$

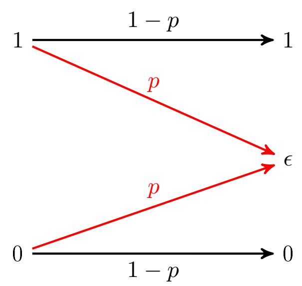
Note that when the crossover probability is $p = 1/2$, then $C = 0$, i.e. we have no information about the transmitted bit from the received bit.

Example 0.2. *Given a Binary Erasure Channel with probability matrix*

$$P = \begin{pmatrix} 1-p & 0 \\ p & p \\ 0 & 1-p \end{pmatrix}$$

Find the capacity of the channel.

<u>Solution</u>: We can represent the BEC as follows:



We choose an input distribution $\Pr(X=0) = \alpha$ and $\Pr(X=1) = 1-\alpha$.
As before we have to compute $H(Y) - H(Y|X)$.

$$\begin{aligned} H(Y|X) &= \sum_{x\in\{0,1\}} \Pr(X=x) H(Y|X=x) \\ &= \Pr(X=0)\cdot H(Y|X=0) + \Pr(X=1)\cdot H(Y|X=1) \end{aligned}$$

Now we have that

$$\begin{aligned} H(Y|X=0) = -\big[& \Pr(Y=0|X=0)\log_2(\Pr(Y=0|X=0)) + \\ & \Pr(Y=\epsilon|X=0)\log_2(\Pr(Y=\epsilon|X=0)) + \\ & \Pr(Y=1|X=0)\log_2(\Pr(Y=1|X=0)) \big] \\ = -\big[& (1-p)\cdot\log_2(1-p) + p\log_2 p \big] = H(p) \end{aligned}$$

and in the same way $H(Y|X=1) = H(p)$. So $H(Y|X) = \alpha H(p) + (1-\alpha)H(p) = H(p)$. Then $C = \max_\alpha(H(Y) - H(Y|X)) = \max_\alpha(H(Y) - H(p))$. We know that $H(Y) \leq \log_2 3$ (because in general $H(Y) \leq \log_2 m$ and in this case $m = 3$), but we cannot achieve this by any choice of the input distribution. So we have to work a bit harder, and compute $H(Y)$.

- $\Pr(Y=0) = \Pr(X=0)\Pr(Y=0|X=0) + \Pr(X=1)\Pr(Y=0|X=1) = (1-\alpha)(1-p) + 0 = (1-\alpha)(1-p)$.
- $\Pr(Y=\epsilon) = \Pr(X=0)\Pr(Y=\epsilon|X=0) + \Pr(X=1)\Pr(Y=\epsilon|X=1) = p$
- $\Pr(Y=1) = \Pr(X=0)\Pr(Y=1|X=0) + \Pr(X=1)\Pr(Y=1|X=1) = 0 + \alpha(1-p) = (\alpha)(1-p)$.

Then using these values to compute $H(Y)$, we get

$$H(Y) = H(p) + (1-p)H(\alpha)$$

and

$$C = \max_\alpha H(p) + (1-p)H(\alpha) - H(p) = \max_\alpha (1-p)H(\alpha).$$

The max of $H(\alpha)$ is reached for $\alpha = 1/2$ and $H(1/2) = 1$, so $C = 1 - p$.