COMS10003 Workshop Sheet 17 outline solutions.

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Work sheet

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ for

   (a) $f(x,y) = xy\sin xy$

   (b) $f(x,y) = e^{x^2+y^2}$

   (c) $f(x,y) = xe^{xy}$

   (d) $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$

   Solutions: So you just treat the other variables as constants, for brevity $\partial_x f = \partial f/\partial x$ and the same for y. This is a bad notation in every respect except being shorter than all the good notations.

   (a) $\partial_x f = y\sin xy + xy^2\cos xy$ and $\partial_y f = x\sin xy + x^2y\cos xy$

   (b) $\partial_x f = 2xe^{x^2+y^2}$ and $\partial_y f = 2ye^{x^2+y^2}$.

   (c) $\partial_x f = e^{xy} + xye^{xy}$ and $\partial_y f = x^2e^{xy}$.

   (d) $\partial_x f = 3x^2 + 6xy + 3y^2$ and $\partial_y f = 3x^2 + 6xy + 3y^2$

2. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ for

   (a) $f(x,y,z) = xy\ln z$

   (b) $f(x,y,z) = x^2 + y^2 + z^2$

   (c) $f(x,y,z) = x\sin xyz$

   Solutions:

   (a) $\partial_x f = y\ln z$, $\partial_y f = x\ln z$ and $\partial_z f = xy/z$

   (b) $\partial_x f = 2x$ and so on.

   (c) $\partial_x f = \sin xyz + xyz\cos xyz$, $\partial_y f = x^2z\cos xyz$ and $\partial_z f = x^2y\cos xyz$

3. For $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$ work out the directional derivative in the $(2,1)$ direction at $(1,0)$; don't forget to normalize the direction vector.

   Solution: So
   $$\nabla f = (3x^2 + 6xy + 3y^2, 3y^2 + 6xy + 3x^2) \tag{1}$$
   which at $(1,0)$ is $\nabla f = (3,3)$ and since $|(2,1)| = \sqrt{5}$ we have $\mathbf{n} = (2/\sqrt{5}, 1/\sqrt{5})$ giving directional derivative $9/\sqrt{5}$.

4. Find the gradient of $f(x,y) = 3x + y^2 + x^2$ and $f(x,y) = \frac{1}{x^2+y^2}$.

   Solution: $\text{grad}(x + y^2) = (3 + 2x, 2y)$ and

$$\text{grad}\, \frac{1}{x^2+y^2} = \left(\frac{-2x}{(x^2+y^2)^2}, \frac{-2y}{(x^2+y^2)^2}\right) \tag{2}$$

5. Going to three-dimensions in the obvious way, what is the gradient of

$$f(x,y,z) = \sin x + \cos y + \sin z \tag{3}$$

   at $(\pi, 0, \pi)$.

   Solution: First the gradient

$$\nabla f = (\cos x, -\sin y, \cos z) \tag{4}$$

   so at the point specified we have $(-1, 0, -1)$.

6. The divergence is a differential operator acting on a vector field to give a scalar, that's the other way around to the gradient which acts on a scalar to give a vector field. It is defined by

$$\text{div}\, \mathbf{v}(x,y) = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{5}$$

   What is the divergence of (x, y)? What about $(y, -x)$?

   Solution:

$$\text{div}\,(x,y) = 1 + 1 = 2 \tag{6}$$

   and

$$\text{div}\,(y,-x) = 0 \tag{7}$$

7. The Laplacian operator $\Box f = \text{div}(\text{grad} f)$. Write down the formula for $\Box f$ in terms of partial derivatives.

   Solution: This comes from just writing it out, we are looking for the divergence of $(\partial_x f, \partial_y f)$ so

$$\Box f = \partial_x^2 f + \partial_y^2 f \tag{8}$$

   or in better notation

$$\Box f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \tag{9}$$

8. If a surface is given by $f(x,y,z) = c$ where c is a constant, then $\text{grad} f$ is perpendicular to the surface. Examine the two-dimensional version by considering $x^2 + y^2 = 1$. What is the gradient? On $x^2 + y^2 = 1$ we can write $x = \cos\theta$ and $y = \sin\theta$ since these satisfy $x^2 + y^2 = 1$. What does the gradient look like on the surface? Can you find a vector perpendicular to it, and therefore parallel to the surface.

Solution: This is harder to ask than answer. So

$$\nabla(x^2 + y^2) = (2x, 2y) \tag{10}$$

so on the circle we have $\mathbf{v} = (2\cos\theta, 2\sin\theta)$. If we want something perpendicular to that we are looking for $\mathbf{w}$ such that $\mathbf{w} \cdot \mathbf{v} = 0$, or

$$2w_1 \cos\theta + 2w_2 \sin\theta = 0 \tag{11}$$

and staring at that it is clear $(-\sin\theta, \cos\theta)$ works.