COMS10003 : Workshop on Logic

Propositional Logic

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Introduction

For this workshop you should read up on *Propositional Logic*, covering the following topics:

- Introduction to Propositional Logic, including syntax and semantics;
- Truth tables for compound propositions;
- Normal forms.

Reading will help you find solutions to the tasks in this worksheet.

Note, this worksheet contains tasks on several topics related to Propositional Logic in order of increasing difficulty for each topic. Schedule your work so that you find an answer to those parts of the worksheet that enable you to solve the rest of the questions alone.

For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Preparation

Design your own syntax/semantics reference card. It will help you remember the symbols used for the different connectives. If necessary, add the truth tables to this card for those connectives that you find hard to remember.

Before you start, compare the reference cards within your group. See whether you can improve your card based on what you have seen.

Task 1: Formalization

Task 1.1: Express the following statements in propositional logic, using:

p : It is raining.
q : It is snowing.
r : It is freezing.

1. It is raining.
2. It is not snowing.
3. It is either raining or snowing.
4. It is raining, but it is not snowing.
5. It is not both raining and snowing.
6. If it is freezing then it is snowing.
7. When it is snowing, then it is freezing.
8. If it is not freezing then it is raining.
9. If it is freezing or snowing then it is not raining.
10. Either it is not raining or, if it is not raining, then it is snowing.
11. Either it is not raining or, if it is not raining but freezing, then it is snowing.
12. It is neither raining nor snowing.
13. It is not the case that, if it is snowing, then it is not snowing.
14. It is raining if and only if it is not snowing.
15. If it is both raining and freezing, then it is snowing.

Task 1.2: Express the following propositions in clear English, using:

p : It is sunny.
q : It is raining.
r : I play tennis.
s : I go swimming.

1. $p \Rightarrow r$
2. $q \Rightarrow \neg r$
3. $p \Leftrightarrow q$
4. $p \Rightarrow (r \vee s)$
5. $(p \vee q) \Rightarrow s$

Task 1.3: State the *converse* and the *contrapositive* of each of the following propositions:

1. If we have frost tonight, then I won't cycle in tomorrow.
2. My cat comes in whenever it is hungry.
3. When you hear the fire alarm, you must vacate the building.
4. If it is hot tomorrow, then we will go swimming.
5. People who don't pay their tax by the deadline will be fined.

Task 2: Syntax

Task 2.1: Eliminate as many brackets as possible:

1. $(((p \vee (\neg q)) \vee s) \Leftrightarrow ((\neg(s \wedge (\neg r)))))$
2. $((((\neg p) \vee s) \wedge q) \vee (\neg(\neg(\neg s))))$

Task 2.2: Based on the syntax of *Propositional Logic* as presented in the lecture, develop a method to decide whether or not a given string of symbols from the alphabet is a syntactically correct proposition.

Hint: You may want to define a function that takes a string as argument, identifies the top-level connective and recursively traverses the argument(s) of that connective until either a propositional variable or a nullary connective is reached, or a syntactic error is encountered.

Task 3: Truth Tables

Task 3.1: How can you determine the truth table for a compound proposition? Explain in simple steps what you need to do.

Task 3.2: Give the truth table for each of the following propositions:

1. $p \Leftrightarrow \neg p$
2. $p \lor (q \land r)$ and $(p \lor q) \land r$
3. $(p \lor \neg q) \Rightarrow (r \land p)$
4. $(p \Leftrightarrow \neg q) \lor (q \Rightarrow p)$
5. $\neg(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$
6. $(p \Leftrightarrow (p \land \neg p)) \Leftrightarrow \neg p$
7. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

Task 3.3

1. For two propositonal variables, p and q, how many assignments of the truth values true (T) and false (F) exist?

2. Let $V = \{T, F\}$ be the set of truth values.
   A function from V^n, the set $\{(v_1, v_2, \ldots, v_n) \mid v_i \in V,\ \mathit{where}\ 1 \leq i \leq n\}$, to V is called a function of degree n.

   For instance, $\wedge$ is a function from V^2, the set $\{(T, T), (T, F), (F, T), (F, F)\}$, to $\{T, F\}$, such that $\wedge(T, T) = T$, $\wedge(T, F) = F$, $\wedge(F, T) = F$ and $\wedge(F, F) = F$.

   How many different functions of degree 2 exist?

3. Give a truth table that shows all these functions of degree 2 in some natural order. Explain your approach.

   We have discussed the truth tables for negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), xor ($\oplus$), implication ($\Rightarrow$) and equivalence ($\Leftrightarrow$). Identify the columns in your table that correspond to the semantics of these connectives.

   Which other important connectives can you find in your truth table? List these and discuss their semantics.

4. In general, for a compound proposition with n propositional variables, how many assignments of truth values exist? In general, how many different functions of degree n exist? Discuss.

Task 4: Normal Forms

Task 4.1: Propositions can be expressed in what is called *Normal Form*. We distinguish two types of normal form, the *Conjunctive Normal Form* and the *Disjunctive Normal Form*.

1. Give a definition for *Conjunctive Normal Form*.
2. Give a definition for *Disjunctive Normal Form*.
3. Based on a truth table for a given compound proposition, state how to derive the *Disjunctive Normal Form* of that proposition.

Task 4.2: Write the following propositions in Disjunctive Normal Form:

1. $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$
2. $(p \vee \neg q) \Rightarrow (r \wedge p)$
3. $(p \Leftrightarrow \neg q) \vee (q \Rightarrow p)$
4. $\neg(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$
5. $(p \Leftrightarrow (p \wedge \neg p)) \Leftrightarrow \neg p$
6. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

Task 4.3: A set of connectives is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition that uses connectives only from that set.

Identify the set of connectives used in Disjunctive Normal Form. Is this set *functionally complete*? Is this the smallest set of functionally complete connectives?