COMS10003 : Linear Algebra

Solving Linear Equations and Inverting Matrices

Andrew Calway

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Introduction

In this section we are going to look at two things. One, how to solve sets of linear equations; and two, how to find the inverse of a matrix (if it exists). Both of these are used throughgout computer science and again it is important that you have a good understanding of the fundamentals and the language we use to describe them. In particular, we are going to focus on two algorithms - Gaussian Elimination to solve sets of linear equations and the Gauss-Jordan Method to find the inverse of a matrix. These are in fact closely related as we shall see. As always, I have made use of several textbooks from my bookshelves and these are listed below.

Theory and problems of linear algebra by Seymour Lipschutz, McGraw-Hill,1981.

Linear Algebra and Probability for CS Applications by Ernest Davis, CRC Press, 2012.

Coding the Matrix by Philip N Klein, Newtonian Press, 2013.

Linear Equations and Linear Systems

Just to recap from last time, a **system of linear equations**, or simply a **linear system**, is a set of such equations involving the same set of variables x_1, x_2, ..., x_n. The term 'set of simultaneous linear equations' is also often used. Examples are

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 6 \\ 3x_1 & - & x_2 & = & 4 \end{array} \qquad \begin{array}{rcrcrcl} x_1 & + & x_2 & - & x_3 & = & 2 \\ 2x_1 & - & x_2 & - & 5x_3 & = & 1 \\ 3x_1 & - & 4x_2 & + & x_3 & = & 6 \end{array}$$

A solution to such a system is a set of values for x_1, x_2, ..., x_n which **simultaneously** satisfies all the equations in the system. For example, $x_1 = 2$ and $x_2 = 2$ is a solution to the first system above since $(2) + 2(2) = 6$ and $3(2) - (2) = 4$. The set of all possible solutions to a system are known as the **solution set** and two systems are said to be **equivalent** if they have the same solution set.

Example

We can solve the following system of two equations in two unknowns x_1 and x_2 (a '2×2 system') by the method of substitution

$$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 4 \end{array}$$

From the first equation we obtain $x_1 = 2x_2 + 1$ and substituting for x_1 in the second equation we obtain $2x_2 + 2 = 4$, giving $x_2 = 1$ and $x_1 = 3$.

Geometric Interpretation

It is useful to consider the above example in geometric terms. Each linear equation defines a straight line in the 2-D plane with axes x_1 and x_2 as shown in Fig. 1a, ie

$$x_2 = x_1/2 - 1/2 \qquad\qquad x_2 = x_1 - 2$$

and the solution corresponds to where the two lines intersect, $x_1 = 3$ and $x_2 = 1$. In other words, all points along each are solutions to their respective equations and thus where they intersect must be a solution to both equations. Similar geometric interpretation of linear problems will be useful to us throughout the course.

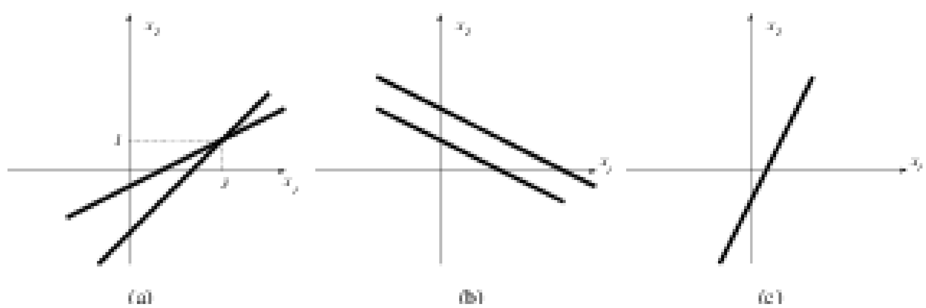

Figure 1: Geometric interpretation of solving a 2×2 linear system: (a) single unique solution - lines intersect at single point; (b) no solution - parallel lines; (c) infinitely many solutions - lines coincide.

Existence and Uniqueness of Solutions

We were able to find solutions to the above examples. However, this is not always the case. Consider the following 2×2 system

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 4 \\ 2x_1 & + & 4x_2 & = & 4 \end{array}$$

Using the first equation to substitute for x_1 in the left hand side (lhs) of the second equation then yields $2(4 - 2x_2) + 4x_2 = 8$ and hence to the conclusion that $8 = 4$! In this case the two equations are **inconsistent** - if we multiply both sides of the first equation by 2, then one equation is saying that $2x_1 + 4x_2$ equals 8 and the other that it equals 4. They cannot both be correct. We can see clearly what is happening by considering the geometry. As shown in Fig. 1b the inconsistency corresponds to the two equations defining straight lines which are parallel and never intersect - there does not exist a common solution and hence the system does not have a solution.

Now consider the following 2×2 system

$$\begin{array}{rcrcl} 2x_1 & - & x_2 & = & 1 \\ 4x_1 & - & 2x_2 & = & 2 \end{array}$$

In this case the two equations are saying the same thing - multiplying both sides of the first equation by two gives the second equation - and thus any values of x_1 and x_2 which satisfy $x_2 = 2x_1 - 1$ will be a solution of the system. Clearly there are infinite number of such solutions, ie for any x_1 we can find an x_2 which satisfies $x_2 = 2x_1 - 1$. As illustrated in Fig. 1c, the geometric interpretation is that the two equations correspond to the same straight line, ie $x_2 = 2x_1 - 1$, and all points along the line are 'intersections' - there are an infinite number of solutions.

It is also clear from the geometry that the above three possibilities for the existence of a solution to a given 2×2 system are all that there exists - two straight lines in the 2-d plane either: intersect at a single point; are parallel and do not intersect; or coincide and hence intersect at every point. We can therefore conclude that such a system either has a single solution, no solution or an infinite number of solutions.

In fact, the above is a **general principle of all linear systems** and we can state that

> *A given system of linear equations has either:*
>
> 1. *a single unique solution; or*
> 2. *no solution; or*
> 3. *an infinite number of solutions.*

A system which has a unique solution or an infinite number of solutions is said to be **consistent**, whilst a system having no solution is said to be **inconsistent**.

3×3 Linear Systems

(a) Consider the 3×3 system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & x_3 & = & 2 \\ 2x_1 & - & x_2 & - & 5x_3 & = & 1 \\ 3x_1 & - & 4x_2 & + & x_3 & = & 6 \end{array}$$

From the first equation we have $x_3 = x_1 + x_2 - 2$ and substituting this into the second equation yields

$$2x_1 - x_2 - 5(x_1 + x_2 - 2) = 1 \quad \Rightarrow \quad 3x_1 + 6x_2 = 9$$

giving $3x_1 = 9 - 6x_2$ and thus substituting into the third equation

$$(9 - 6x_2) - 4x_2 + ((3 - 2x_2) + x_2 - 2) = 6 \quad \Rightarrow \quad x_2 = 4/11$$

from which $x_1 = 25/11$ and $x_3 = 7/11$. For a 3×3 system each equation defines a **plane** in **3-d space** with axes x_1, x_2, x_3 and in the above case the three planes intersect at a single point corresponding to the unique solution for the system. This geometric interpretation for such a 3×3 system is illustrated in Fig. 2a.

(b) The 3×3 system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & 2 \\ 2x_1 & - & x_2 & - & 5x_3 & = & 1 \\ 2x_1 & + & 4x_2 & - & 6x_3 & = & 6 \end{array}$$

does not have a solution since the first and third equations are inconsistent, ie the first says that $x_1 + 2x_2 - 3x_3$ equals 6, whilst the third says that it equals 3. In this case two of the planes in 3-d space are parallel and thus there is no common intersection of all three planes. An example of this is shown in Fig. 2b. Note that this is only one example of inconsistency in a 3×3 system - inconsistency exists whenever there is no common intersection point of the three planes, whether they are parallel or not, as shown in Figs. 2c and 2d.

(c) In the 3×3 system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & 2 \\ 2x_1 & - & x_2 & - & 5x_3 & = & 1 \\ 3x_1 & + & x_2 & - & 8x_3 & = & 3 \end{array}$$

the third equation is equal to the sum of the first and second equations and thus any set of values for x_1,x_2,x_3 which satisfies both the first two equations will also satisfy the third equation. Since there are an infinite number of these, ie common solutions to two equations correspond to the line in 3-d space where the two respective planes intersect, the system must also have an infinite number of solutions. Such a case is illustrated in Fig. 2e - the three planes intersect in a line. Note that a system would also have an infinite number of solutions whenever two or all three planes coincide.

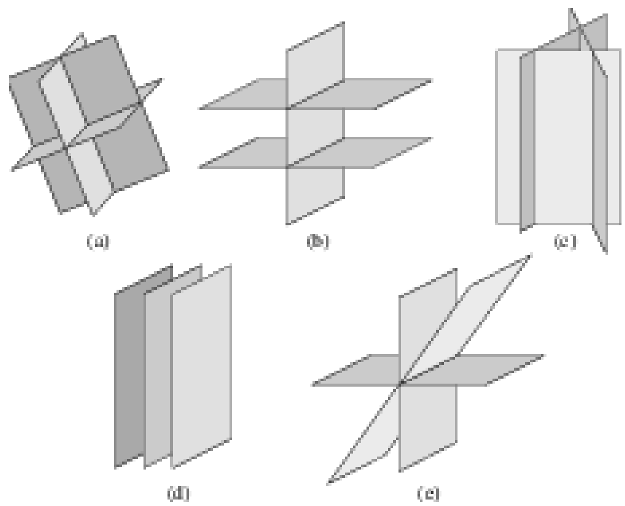

Figure 2: Geometric interpretation of solving a 3×3 system: (a) single unique solution - planes intersect at single point; (b) no solution - two parallel planes; (c) no solution - no common intersection; (d) no solution - all planes are parallel; (e) infinitely many solutions - planes intersect along a line.

The General Case - n $\times$ m Linear Systems

In general we are interested in solving systems consisting of n equations in m unknowns - an $n \times m$ system with m unknowns x_1, x_2, ..., x_m:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1m}x_m & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2m}x_m & = & b_2 \\ \cdots & & \cdots & & \cdots & & \cdots & & . \\ \cdots & & \cdots & & \cdots & & \cdots & & . \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nm}x_m & = & b_n \end{array}$$

where a_{ij} are the coefficients and b_i are the constants. In this course we shall only consider systems in which there are the same number of unknowns as equations , ie in which $n = m$.

In the above examples we were able to solve a 2×2 without difficulty using simple substitution. We applied a similar approach to the 3×3 systems, although the method was clearly less straightforward. In fact when dealing with larger systems, ie with $n > 3$, the approach is not practical and we require a more systematic and efficient method. One such scheme and the most useful is that of **Gaussian Elimination**. We shall consider it in the next section.

Three Elementary Operations

Gaussian elimination is a systematic procedure for solving linear systems. It is based on the principle of replacing a system with an equivalent system which is easier to solve. An equivalent system is derived from the system of interest by applying a sequence of **elementary operations**, ie application of an elementary operation to a system yields a system with the same solution set.

The following three elementary operations are used in Gaussian elimination:

1. Replace one equation by the sum of itself and a multiple of another equation.
2. Multiply all terms in one equation by a non-zero constant.
3. Interchange any two equations.

Convince yourself that applying any one of these operations will not change the solution set of a given system.

Gaussian Elimination - An Example

The best way to describe Gaussian elimination is by way of an example. Consider the following 3×3 linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 0 \\ x_1 & + & 5x_2 & + & 6x_3 & = & -1 \end{array} \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

The elimination algorithm proceeds as follows:

(i) We first eliminate x_1 in eqns (2) and (3) by subtracting eqn (1) from each equation, ie

$$
\begin{array}{rcrcrcrl}
x_1 & + & x_2 & + & x_3 & = & 2 & \quad (1) \\
 & & 2x_2 & + & 2x_3 & = & -2 & \quad (2) \ \leftarrow \ (2)-(1) \\
 & & 4x_2 & + & 5x_3 & = & -3 & \quad (3) \ \leftarrow \ (3)-(1)
\end{array}
$$

Note that this corresponds to two elementary operations and thus the resulting system must have the same solution set.

(ii) We then eliminate x_2 from eqn (3) by subtracting twice eqn (2) from it, ie

$$
\begin{array}{rcrcrcrl}
x_1 & + & x_2 & + & x_3 & = & 2 & \quad (1) \\
 & & 2x_2 & + & 2x_3 & = & -2 & \quad (2) \\
 & & & & x_3 & = & 1 & \quad (3) \ \leftarrow \ (3)-2\,(2)
\end{array}
$$

The system is now said to be in **triangular form** and from eqn (3) we can deduce that $x_3 = 1$ in the solution for this and the original system.

(iii) We can then find values for x_1 and x_2 using **back substitution** - recursively substitute known values into eqns (1) and (2), ie from eqn (2) with $x_3 = 1$

$$2x_2 + 2 \ = \ -2 \quad \Rightarrow \quad x_2 \ = \ -2$$

and from eqn (1) with $x_2 = -2$ and $x_3 = 1$

$$x_1 - 2 + 1 \ = \ 2 \quad \Rightarrow \quad x_1 \ = \ 3$$

Hence the (unique) solution to the triangular and the original system is $x_1 = 3$, $x_2 = -2$ and $x_3 = 1$. Convince yourself that it is a solution to the original system by substituting the values for x_1, x_2, and x_3 into the original equations (you should do this whenever you implement Gaussian elimination by hand in case you've made an arithmetic error).

Gaussian elimination therefore consists of two distinct stages:

1. Reduce the set of equations to **triangular form** using **forward elimination** - use each equation in turn to eliminate the left-most term from the equations below;

2. Find the required solution using **back substitution** - solve for each unknown from bottom to top by substituting known values into each successive equation.

At each step in the forward elimination process, the equation used to eliminate terms in the equations below is known as the **pivot equation** and the coefficient of its left-most term is known as the **pivot**, eg the first pivot in the above example is 1 and the second pivot is 2. To eliminate a given term in an equation below, the current pivot is divided into the coefficient to give a **multiplier** which defines the multiple of the pivot equation to subtract from the equation. For example, in step (ii) above, eqn (2) is the pivot equation and the multiplier required to eliminate the x_2 term from eqn (3) is the coefficient 4 divided by the pivot 2, ie 4/2=2.

From the above description it is clear that the elimination process can only succeed if the pivot at each step is non-zero - if a zero pivot occurs at any given step it will be impossible to eliminate terms from the equations below. It turns out that in some cases we can overcome the problem and still find a solution, whilst in others it will indicate that the original system either has no solution, ie it's equations are inconsistent, or it has infinitely many solutions. We consider these cases below.

Equation Interchange

Consider the 3×3 system

$$\begin{array}{rcrcrcrl} x_1 & + & x_2 & + & x_3 & = & -2 & \qquad (1) \\ 3x_1 & + & 3x_2 & - & x_3 & = & 6 & \qquad (2) \\ x_1 & - & x_2 & + & 2x_3 & = & -1 & \qquad (3) \end{array}$$

The first pivot is 1 and thus the multipliers required to eliminate the x_1 term from eqns (2) and (3) are 3 and 1 respectively, ie

$$\begin{array}{rcrcrcrl} x_1 & + & x_2 & + & x_3 & = & -2 & \qquad (1) \\ & & & - & 4x_3 & = & 12 & \qquad (2) \leftarrow (2) - 3(1) \\ & - & 2x_2 & + & x_3 & = & 1 & \qquad (3) \leftarrow (3) - (1) \end{array}$$

However, at this point the pivot in eqn (2) is zero, ie the coefficient of x_2, and hence the elimination procedure cannot proceed. Of course, the way out of this difficulty is simple - rearrange the equations by interchanging eqns (2) and (3)

$$\begin{array}{rcrcrcrl} x_1 & + & x_2 & + & x_3 & = & -2 & \qquad (1) \\ & - & 2x_2 & + & x_3 & = & 1 & \qquad (2) \\ & & & - & 4x_3 & = & 12 & \qquad (3) \end{array}$$

giving a triangular system which is equivalent to the original system and from which back substitution gives

$$x_3 = -3 \qquad x_2 = -2 \qquad x_1 = 3$$

Thus whenever a zero pivot occurs we should always seek to interchange the equations in order obtain a non-zero pivot. However, as we shall find in the next section, this is not always possible.

Breakdown of Elimination

In some cases the forward elimination process will break down and cannot be rectified by interchanging equations. This occurs if the system has no solution or if it has infinitely many solutions. Consider the 3×3 system:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & -2 \\ 3x_1 & + & 3x_2 & - & x_3 & = & 6 \\ x_1 & + & x_2 & + & x_3 & = & -1 \end{array}$$

After applying the first step of forward elimination we obtain

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & -2 \\ & & & - & 4x_3 & = & 12 \\ & & & & 0 & = & 1 \end{array}$$

However, the second step cannot take place since we have a zero pivot in the second equation. Moreover, it is not solved by interchanging the second and third equations - neither has a non-zero x_2 term. We also note that the system contains a contradiction - the 3rd equation can never be true and thus indicates that the equations in the original system are inconsistent (as is obvious from the first and third equations in the original system). Now consider the system:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & -2 \\ 3x_1 & + & 3x_2 & - & x_3 & = & 6 \\ 2x_1 & + & 2x_2 & + & 2x_3 & = & -4 \end{array}$$

In this case forward elimination yields

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & -2 \\ & & & - & 4x_3 & = & 12 \\ & & & & 0 & = & 0 \end{array}$$

which again has zero second pivot and thus elimination cannot proceed. However, in this case the equations are consistent, the only difficulty is that we have only 2 equations in 3 unknowns and thus the system has infinitely many solutions, ie any solution with $x_3 = -3$ and $x_1 = 1 - x_2$.

Gaussian elimination therefore breaks down if we encounter a zero pivot which cannot be rectified by interchanging equations. In such cases, the system will either be inconsistent and have no solution, or will have infinitely many solutions.

Matrix Notation

Since the elimination process depends only on the coefficients and constants of a system, it is useful to adopt a notation which avoids having to write out the unknowns at each step. All we have to do is ensure that the order of the coefficients and constants is maintained.

We can do this using matrices. We can represent the coefficients of a system by its **coefficient matrix**, eg a 3×3 system and its coefficient matrix are shown below

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & 5 \\ 4x_1 & - & 6x_2 & & & = & -2 \\ -2x_1 & + & 7x_2 & + & 2x_3 & = & 9 \end{array} \qquad \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

note that the second equation has no x_3 term and thus the corresponding matrix entry is zero. The **augmented matrix** of a system is the coefficient matrix with an added right-most column containing the constants, eg for the above system the augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

We can apply the elimination process to the augmented matrix in exactly the same way as we did with the equation form of the system using **row operations**, eg to solve the above system:

(i) Subtract twice row 1 from row 2 and add row 1 to row 3 to give

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

Note that this gives the augmented matrix of an equivalent system since the row operations correspond the elementary operations used before.

(ii) Replace row 3 by the sum of itself and row 2 to give

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \equiv \qquad \begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & 5 \\ & - & 8x_2 & - & 2x_3 & = & -12 \\ & & & & x_3 & = & 2 \end{array}$$

from which $x_3 = 2$. The augmented matrix is now said to be in **echelon** ("steplike") **form**.

(iii) Use back substitution as before to give the required solution, ie

$$-8x_2 = -8 \;\;\rightarrow\;\; x_2 = 1 \qquad\qquad 2x_1 = 2 \;\;\rightarrow\;\; x_1 = 1$$

Note: don't forget to check that this is a solution to the original system by inserting the values for the unknowns in the original equations.

The General Case

The advantage of Gaussian elimination is that it is a **systematic** procedure which involves applying the same series of steps to any system of linear equations in order to determine a solution. Thus, in general, given a linear system having n equations in n unknowns with coefficients a_{ij} and constants b_i, ie

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array} \qquad \begin{array}{c} (1) \\ (2) \\ \\ (\mathrm{n}) \end{array}$$

elimination proceeds by taking each pivot equation (i), $1 \leq i < n$, and updating eqns $(i+1)$ to (n) such that (assuming that no equation interchange is required)

$$a_{jk} \;=\; a_{jk} - \frac{a_{ji}}{a_{ii}} a_{ik} \;=\; a_{jk} - m_{ij} a_{ik} \qquad b_j \;=\; b_j - m_{ij} b_i \qquad \begin{array}{ccccc} i+1 & \leq & j & \leq & n \\ i+1 & \leq & k & \leq & n \end{array}$$

where j is the updated equation index, k is the coefficient index, and m_{ij} is the multiplier used to eliminate the x_ith term from the jth equation. This gives the equivalent triangular system

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 & & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & & & \cdots & + & \vdots & = & \vdots \\
 & & & & a_{n-1n-1}x_{n-1} & + & a_{n-1n}x_n & = & b_{n-1} \\
 & & & & & & a_{nn}x_n & = & b_n
\end{array}
$$

which can then be solved using back substitution to find the unknowns x_i, ie

$$
x_i \;=\; \frac{1}{a_{ii}}\left(b_i \;-\; \sum_{j=i+1}^{n} a_{ij}x_j\right)
$$

Finding the Inverse of a Matrix - the Gauss-Jordan Method

If we wish to find the inverse of an $n \times n$ matrix A, then we seek the $n \times n$ matrix A^{-1} such that $AA^{-1} = I$. If we denote the columns of A^{-1} by $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$, then using the column interpretation of matrix multiplication we have

$$
A\,[\mathbf{x}_1\;\mathbf{x}_2\;\ldots\;\mathbf{x}_n\,] \;=\; [A\mathbf{x}_1\;A\mathbf{x}_2\;\ldots\;A\mathbf{x}_n\,] \;=\; [\,\mathbf{e}_1\;\mathbf{e}_2\;\ldots\;\mathbf{e}_n\,]
$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_n$ are the columns of the $n \times n$ identity matrix, ie

$$
\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}
$$

In other words, this corresponds to trying to solve the n linear systems

$$
A\mathbf{x}_1 = \mathbf{e}_1 \qquad A\mathbf{x}_2 = \mathbf{e}_2 \qquad \cdots \qquad A\mathbf{x}_n = \mathbf{e}_n
$$

which all have the same coefficient matrix A. Obviously we could solve each of these using Gaussian elimination as before. However, since they all have the same coefficient matrix we can actually solve them all at the same time by carrying out Gaussian elimination on all the systems simultaneously. An example will provide the best description of the method. It is known as the **Gauss-Jordan Method**.

Example

If we wish to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

then we seek to solve

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which represents three linear systems. We can solve these by applying Gaussian elimination simultaneously using matrix notation, ie we form the 3×6 augmented matrix

$$[A\ I] = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

and apply row operations to reduce the matrix to echelon form, ie first subtract twice row 1 from row 2 and add row 1 to row 3 to give

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$$

and then add row 2 to row 3 to give

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

Note that in doing so we have effectively applied the forward elimination stage to each of the three systems simultaneously. Moreover, we could now proceed to solve each system separately by extracting the relevant coefficients and constants and applying back substitution, hence obtaining the columns of the inverse matrix. However, there is a better and quicker way of doing this simultaneously on all three systems.

First, we add twice row 3 to row 2 and subtract row 3 from row 1 to give

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

and then add 1/8 of row 2 to row 1

$$\begin{bmatrix} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

and finally multiply row 1 by 1/2 and row 2 by $-1/8$ to give

$$\begin{bmatrix} 1 & 0 & 0 & \frac{6}{8} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

If we now consider each of the three systems represented by this reduced matrix we find that the solutions are simply the three columns on the right, ie for first system we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} \frac{6}{8} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

and hence $\mathbf{x}_1$ is equal to the vector on the right. The required inverse matrix is therefore

$$A^{-1} = \begin{bmatrix} \frac{6}{8} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

which is confirmed by

$$AA^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} \frac{6}{8} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrices Having No Inverse

What happens to the Gauss-Jordan method if a matrix does not have an inverse? Clearly if a matrix does not have an inverse, then at least one of the linear systems we need to solve will not have a solution and hence when we perform Gaussian elimination we shall encounter a row in the coefficient matrix having all zeros. Since the Gauss-Jordan method is based on Gaussian elimination this also what happens when we try to find the inverse of a matrix which is not invertible - we encounter an all zero row in the left half of the augmented matrix. The following example illustrates the effect.

Example

We wish to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

To do so we apply forward elimination using row operations to the augmented matrix

$$[\,A\;I\,] \quad = \quad \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix}$$

by adding row 1 to row 2 and subtracting five times row 1 from row 3 to give

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix}$$

and then subtracting twice row 2 from row 3 to give

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}$$

from which we cannot solve any of the three linear systems and hence cannot determine the columns of the inverse of the matrix A - it does not exist.

Note that in the above case, none of the three linear systems could have had a solution since the columns of A are linearly dependent, ie

$$7\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix} - 5\begin{bmatrix} -2 \\ 5 \\ -4 \end{bmatrix}$$

and none of the columns of the identify lie in the column space of the matrix. Hence the matrix does not have an inverse. In fact, this is generally true:

> *An $n \times n$ square matrix is invertible if and only if it has n linearly independent columns.*