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Language Engineering

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Structural Induction: p11

- The use of compositional definitions permits the use of structural induction proofs (which are often simpler than normal proofs by induction)

- Proofs by structural induction have two parts:

1. Base Cases:

   prove that the property holds for each basis element

2. Inductive Cases:

   assume property holds for all constituents of a composite element and prove the property holds for the composite element itself

Example - Binary Numerals: p10

Syntax: $n \; ::= \; 0 \mid 1 \mid n\,0 \mid n\,1$

Semantics:

$\mathcal{N}\,[[\, 0 \,]] = 0$ // basis elements

$\mathcal{N}\,[[\, 1 \,]] = 1$

$\mathcal{N}\,[[\, n\,0 \,]] = 2 * \mathcal{N}\,[[\, n \,]]$ // compound elements

$\mathcal{N}\,[[\, n\,1 \,]] = 2 * \mathcal{N}\,[[\, n \,]] + 1$

Proof of Functionality

- Theorem: $\mathcal{N}$ is a function: $\forall n \in \text{Num}.\forall a,b \in Z.\mathcal{N}[[n]]=a \wedge \mathcal{N}[[n]]=b \rightarrow a=b$
- Proof: by induction in the structure of the numeral n
- Base cases
  - if n=0 then $\mathcal{N}[[n]]=a \wedge \mathcal{N}[[n]]=b \rightarrow a=b=0$ as only first rule applies
  - if n=1 then $\mathcal{N}[[n]]=a \wedge \mathcal{N}[[n]]=b \rightarrow a=b=1$ as only second rule applies
- Induction step
  - if n=n' 0 then $\mathcal{N}[[n']]=i \wedge \mathcal{N}[[n']]=j \rightarrow i=j$ by ind. hyp.

    $\therefore \mathcal{N}[[n]]=a \wedge \mathcal{N}[[n]]=b \rightarrow a=b=2i$ as only third rule applies
  - if n=n' 1 then $\mathcal{N}[[n']]=i \wedge \mathcal{N}[[n']]=j \rightarrow i=j$ by ind. hyp.

    $\therefore \mathcal{N}[[n]]=a \wedge \mathcal{N}[[n]]=b \rightarrow a=b=2i+1$ as only fourth rule applies
  - (using the fact that integer addition and multiplication are functions)

Proof of Totality

- Theorem: $\mathcal{N}$ is a function: $\forall n \in Num. \exists a \in Z. \mathcal{N}[[n]]=a$
- Proof: by induction in the structure of the numeral n
- Base cases
  - if n=0 then $\mathcal{N}[[n]]=0=a$ as first rule applies
  - if n=1 then $\mathcal{N}[[n]]=1=a$ as second rule applies
- Induction step
  - if n=n' 0 then $\mathcal{N}[[n']]=i$ by ind. hyp.

    $\therefore \mathcal{N}[[n]]=2i=a$ as third rule applies
  - if n=n' 1 then $\mathcal{N}[[n']]=i$ by ind. hyp.

    $\therefore \mathcal{N}[[n]]=2i+1=a$ as fourth rule applies
  - (using the fact that addition of 1 and multiplication by 2 are total)

Combined Proof of Total Functionality

- Theorem: $\mathcal{N}$ is a total function: $\forall n \in Num. \exists! a \in Z. \mathcal{N}[[n]]=a$ where $\exists! x. P(x)$ means "there is exactly one x such that P" and is defined $\exists x. \forall y. P(y) \leftrightarrow x=y$
- i.e. we need to show $\forall n \in Num. \exists a \in Z. \forall b \in Z. \mathcal{N}[[n]]=b \leftrightarrow a=b$ in FOL
- Proof: by induction in the structure of the numeral n
- Base cases
  - if n=0 then $\forall b.\ \mathcal{N}[[n]]=b \leftrightarrow b=0$ so a=0 as only first rule applies
  - if n=1 then $\forall b.\ \mathcal{N}[[n]]=b \leftrightarrow b=1$ so a=1 as only second rule applies
- Induction step
  - if n=n' 0 then $\exists! b \in Z. \mathcal{N}[[n']]=b$ by ind. hyp.

    $\therefore\ \forall y.\ \mathcal{N}[[n]]=y \leftrightarrow a=y=2b$ as only third rule applies
  - if n=n' 1 then $\exists! b \in Z. \mathcal{N}[[n']]=b$ by ind. hyp.

    $\therefore\ \forall y.\ \mathcal{N}[[n]]=y \leftrightarrow a=y=2b+1$ as only fourth rule applies