COMS10003 Work Sheet 21

Linear Algebra: Matrices

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1. For the following matrices A, B and C, and vector $\mathbf{v}$

$$A = \begin{bmatrix} 3 & 1 & 5 \\ -4 & 1 & 0 \\ 2 & 4 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 8 & -3 \\ 5 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 \\ 6 & 3 \\ -4 & -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Find:

(a) $A\mathbf{v}$ (c) BA (e) A^TB^T (g) AC (i) A^2
(b) AB (d) $\mathbf{v}\mathbf{v}^T$ (f) $(A+B)\mathbf{v}$ (h) $\mathbf{v}^TC$

2. Determine the rank of the following matrices (by observation):

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 2 & 6 & -4 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence for the following linear systems determine whether a solutions exists or not, and if so, how many (again, by observation)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 2 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

3. Find the matrices which corresponds to the following linear transformations in $\mathbf{R}^2$:

   (a) A projection onto the vector (1,0).

   (b) A counterclockwise rotation through an angle θ followed by a projection onto the vector (1,0).

   (c) Multiplication by a scalar k followed by a counterclockwise rotation through 90^o

   For (b) and (c), are the matrices the same if the order of the two transformations in each case are reversed?

4. Find the matrix representing the linear transformation which projects a vector in $\mathbf{R}^2$ onto the vector $(\cos\theta, \sin\theta)$.

5. Find the 3×3 matrices which correspond to the following linear transformations

   (a) projection of a vector onto the x-y plane

   (b) reflection of a vector through the x-y plane

   (d) counterclockwise rotation of a vector by an angle θ around the y-axis.

6. Prove that $(AB)^T = B^T A^T$. Hint: Let A and B be of size $m \times n$ and $n \times p$, respectively, and represent them by column vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\ldots$, $\mathbf{a}_m$ and $\mathbf{b}_1$, $\mathbf{b}_2$, $\ldots$, $\mathbf{b}_p$, i.e.

$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} \qquad B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \ldots \ \ \mathbf{b}_p]$$

7. Determine the inverse (if it exists) of the following matrices

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Let A and B be invertible matrices of the same size. Show that the product AB is also invertible with inverse $B^{-1}A^{-1}$. Hence by induction show that $(A_1 A_2 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$.

9. Show that: (i) if A has a row consisting of all zeros (a zero row), then the product AB also has a zero row; (ii) if B has a zero column, then AB has a zero column; and (iii) any matrix with a zero row or a zero column is not invertible.

10. For the following matrix A, determine $B = A^n$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$