COMS10003 : Linear Algebra

Matrices

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Introduction

We now move on to the another important component of linear algebra - the matrix. As we shall see, it is really just a data structure that enables us to encode and manipulate linear equations. We'll look at the basic properties of matrices (which are much the same as those for vectors) and see how we can multiply matrices with vectors and other matrices, and how this relates to solving sets of linear equations. As always, I have made use of several textbooks from my bookshelves and these are listed below.

Theory and problems of linear algebra by Seymour Lipschutz, McGraw-Hill,1981.

Linear Algebra and Probability for CS Applications by Ernest Davis, CRC Press, 2012.

Coding the Matrix by Philip N Klein, Newtonian Press, 2013.

Matrices

A **matrix** is simply an array of numbers arranged in rows and columns. We shall use uppercase letters to denote matrices. For example, the matrix A given below has m rows and n columns - it is known as an $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The **components** of the matrix a_{ij} are referenced by the column index j and the row index i, ie a_{ij} is the component in the jth column and the ith row.

Note that the columns and rows of a matrix can also be seen as vectors. In fact, we often represent vectors in matrix form - either as a **column vector** or as a **row vector**, e.g.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

Thus, if we denote the columns of the matrix A by the column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, i.e.

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

then we can write A as a row vector made up of column vectors

$$A = [\,\mathbf{a}_1\,\mathbf{a}_2\,\cdots\,\mathbf{a}_n\,]$$

It follows that we do have to be clear about the respresentation that we are using when we talk about vectors in this form - it is important that we make clear whether a vector is being seen as a column or as a row vector.

Examples - the following are examples of 2×2, 2×3 and 4×4 matrices.

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} -3 & -1 & 4 \\ 1 & 2 & -5 \end{bmatrix} \qquad \begin{bmatrix} 5 & -2 & 8 & -2 \\ 2 & 7 & -2 & 3 \\ 3 & -1 & 0 & 2 \\ -2 & -8 & 1 & 1 \end{bmatrix}$$

It is often useful to consider the **transpose** of a matrix - it is the matrix formed by replacing the rows with the columns and for a matrix A it is denoted A^T, ie

$$A = \begin{bmatrix} 5 & -2 & 8 & -2 \\ 2 & 7 & -2 & 3 \\ 3 & -1 & 0 & 2 \\ -2 & -8 & 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 5 & 2 & 3 & -2 \\ -2 & 7 & -1 & -8 \\ 8 & -2 & 0 & 1 \\ -2 & 3 & 2 & 1 \end{bmatrix}$$

Note therefore that the transpose of a column vector is a row vector and vice versa.

Addition and Scalar Multiplication

The properties described earlier for vectors also applies to matrices. Thus, two matrices are **equal** if they are the same size, ie they have the same number of rows and columns, and if their corresponding components are equal.

If A and B are $m \times n$ matrices then the **sum** $A + B$ is the $m \times n$ matrix whose columns are sums of the corresponding columns in A and B, ie each component of $A + B$ is the sum of the corresponding components in A and B. Note that as for vectors summation is only defined for matrices of the same size.

Example 1

$$\begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & 1 \\ -2 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ 9 & -3 & 10 \\ 2 & -7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 9 \\ 9 & -1 & 11 \\ 0 & -1 & 2 \end{bmatrix}$$

If c is a real number and A is a matrix, then the **scalar multiple** cA is the matrix whose components are c times the corresponding components of A. As with vectors, the negative of the matrix A is $(-1)A$ and thus $A - B = A + (-1)B$. A matrix whose components are all zeros is known as the **zero matrix** and denoted 0, ie $A - A = A + (-1)A = 0$.

Example 2

Given the matrices

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 4 & -2 & -1 \\ 3 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 1 \\ 3 & -6 & 0 \end{bmatrix}$$

then the matrix $C = 4A - 2B$ is given by

$$C = 4 \begin{bmatrix} 2 & -1 & -1 \\ 4 & -2 & -1 \\ 3 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 1 \\ 3 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -4 & -6 \\ 14 & -4 & -6 \\ 6 & 12 & 4 \end{bmatrix}$$

Matrix-Vector Multiplication

We can also multiply matrices together. However we will start by considering a special case - multiplying a vector by a matrix. We shall give the definition first and then consider what it means later. Given a vector $\mathbf{v}$, multiplying by the matrix A is given by

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_i a_{1i}v_i \\ \sum_i a_{2i}v_i \\ . \\ . \\ . \\ \sum_i a_{mi}v_i \end{bmatrix}$$

So it gives another vector, the ith component of which is given by forming the sum of products of corresponding components of the ith row of the matrix and the vector $\mathbf{v}$. In other words, this is the dot product between the ith row and the vector $\mathbf{v}$.

Note that this means we are restricted as to the form of matrices and vectors that we can multiply. Specifically, the number of columns in the matrix must be the same at the number of components in the vector, i.e. n in the above example. Thus, if we have a $m \times n$ matrix, we can multiply a $n \times 1$ column vector with it to give a $m \times 1$ column vector.

From the above we can also express the dot product in a new form. If we have two vectors $\mathbf{u}$ and $\mathbf{v}$ *represented as column vectors*, then

$$\mathbf{u}.\mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 u_2 \ldots u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i$$

We can also write our matrix-vector product in a different form as well. If we consider the matrix A to have columns given by the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, then the product $A\mathbf{v}$ is then

$$A\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots + v_n\mathbf{a}_n = \sum_{i=1}^{n} v_i\mathbf{a}_i$$

i.e. the linear combination of the matrix columns, with weights given by the components of $\mathbf{v}$. Note therefore that the vector $\mathbf{u} = A\mathbf{v}$ must lie within the vector space spanned by the columns of A. This is an important property and we will return to it later.

We can now also introduce another special matrix - the **identity matrix** I - which is a square matrix (same number of rows and columns) and leaves a vector unchanged after multiplication, i.e.

$$I\mathbf{v} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

It is also a diagonal matrix - only it's diagonal components are non-zero.

Example 3

Two examples of multiplying a matrix and a vector are shown below.

$$\begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & 1 \\ -2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -42 \\ 7 \\ 20 \end{bmatrix} \qquad \begin{bmatrix} -3 & -1 & 4 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -15 \end{bmatrix}$$

Linear Systems as Matrix-Vector Multiplication

We now take a look at another interpretation of matrix-vector multiplication. From above, if we have an equation of the form $A\mathbf{x} = \mathbf{b}$, where A is a $m \times n$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ and $m \times 1$ column vectors, respectively, then it corresponds to the following set of simultaneous linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

For example, consider the following set of 3 simultaneous linear equations in 3 unknwons

$$\begin{array}{rcrcrcr} 3x_1 & - & 9x_2 & + & 12x_3 & = & -42 \\ & + & 2x_2 & + & x_3 & = & 7 \\ -2x_1 & + & 6x_2 & & & = & 20 \end{array}$$

where the goal is to determine the values of x_1, x_2 and x_3 which satisfy all three equations. We can write these equations in the form of matrix-vector multiplication, i.e.

$$A\mathbf{x} = \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & 1 \\ -2 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -42 \\ 7 \\ 20 \end{bmatrix} = \mathbf{b}$$

Hence the solution we require is the vector $\mathbf{x}$ which when multiplied by matrix A gives the vector $\mathbf{b}$, i.e. from *Example 3*

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

Note that we can only find a valid $\mathbf{x}$ if the vector $\mathbf{b}$ lies within the vector space spanned by the columns of the matrix A - this is known as the **column space** of A. Moreover, if $m = n$, i.e. for a square matrix A, and the columns of A are linearly independent, then there must always be a unique solution to $A\mathbf{x} = \mathbf{b}$ since $\mathbf{b}$ would always be a member of the set spanned by the columns because it contains all vectors with size equal to size of the columns of A. For example, if

$$A_1 = \begin{bmatrix} -1 & 2 & 0.5 \\ 1 & 1 & -1 \\ 1 & 0 & 0.5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

the columns of A_1 are independent and span $\mathbf{R}^3$ - it must be possible to write $\mathbf{b}$ as a linear combination of the columns and thus $A_1\mathbf{x} = \mathbf{b}$ must have a solution. It is in fact $\mathbf{x} = (2, 1, 2)$.

However, if the columns of A are dependent then $A\mathbf{x} = \mathbf{b}$ may or may not have a solution, dependng on whether $\mathbf{b}$ is a member of the set spanned by the columns. If it is then there will be an infinite number of solutions; whereas if it is not, then there cannot be a solution. For example, with

$$A_2 = \begin{bmatrix} 3 & 2 & 0.5 \\ -1 & 1 & -1 \\ 1 & 0 & 0.5 \end{bmatrix} \qquad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

then $A_2\mathbf{x} = \mathbf{b}_1$ has no solution and $A_2\mathbf{x} = \mathbf{b}_2$ has infinitely many solutions.

The number of linearly independent columns in a matrix is known as the **rank** of the matrix and determines whether the system represented by the matrix has a unique solution or not. An $n \times n$ matrix with n independent columns has a rank of n and there will always be an

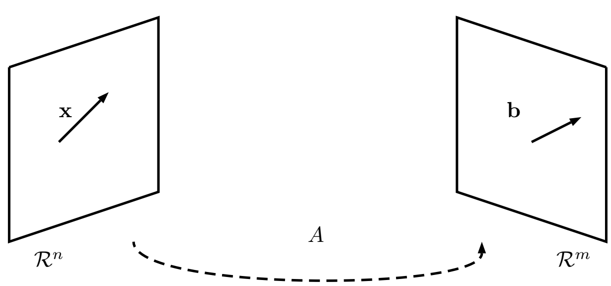
$\mathbf{x}$ and a $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$. Thus, in the above examples A_1 has a rank of 3 and A_2 has a rank of 2.

Linear Transformations

There is also yet another alternative interpretation of matrix-vector multiplication - the idea of **linear transformations**. This is based on the interpretation of the matrix equation $A\mathbf{x} = \mathbf{b}$ as the matrix A **transforming** the vector $\mathbf{x}$ into a new vector $\mathbf{b}$. In other words, we think of A as "acting" (or *operating*) on $\mathbf{x}$ to produce a new vector $\mathbf{b}$.

For example, if A is an $m \times n$ matrix then every vector in $\mathcal{R}^n$ will be transformed to a vector in $\mathcal{R}^m$ - we can think of the whole set of vectors in $\mathcal{R}^n$ being *transformed by the matrix* A into a new set of vectors in $\mathcal{R}^m$ as shown below. When $m = n$ we have vectors in $\mathcal{R}^n$ being transformed to vectors in $\mathbf{R}^n$ and we say that the whole set $\mathcal{R}^n$ is *mapped* into itself. In this case A is often known as a **linear mapping**.

Note that the concept is very much like that of functions which define a rule that transforms one real number into another, eg $y = f(x)$, except that now we are dealing with the transformation of vectors into other vectors, via linear functions in multiple dimensions.



<u>Examples</u>

We can interpret linear mappings in terms of geometric transformations and two examples are given below. There are many more - take a look at the wikipedia page on linear mappings `http://en.wikipedia.org/wiki/Linear_map`.

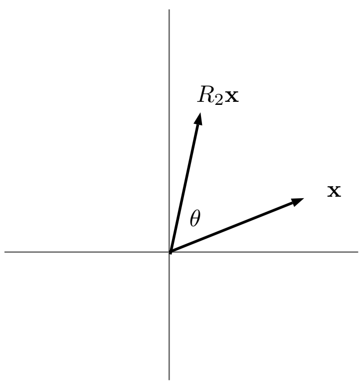
The matrix R_2 defined below corresponds to a rotation by an angle θ counterclockwise about the origin in $\mathcal{R}^2$

$$R_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

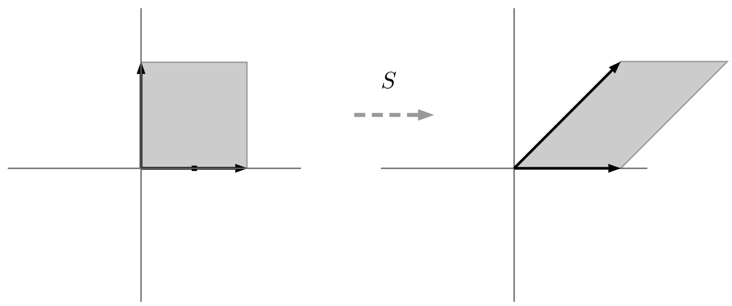
which we prove later and is illustrated below. The following matrix corresponds to a 2-D **shear** mapping, which, for example, converts a square into a parallelogram as also illustrated below.

$$S = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

where c is some positive scalar.



Rotation



Shear

Finding the Matrix of a Transformation

Clearly all matrices represent some form of linear transformation - they will all transform a given vector into another vector. However, if we wish to find the matrix that represents a given transformation, how should we go about it?

Note first of all that a vector $\mathbf{x}$ in $\mathcal{R}^n$ can be written in terms of the vector equation

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + \ldots + x_n\mathbf{e}_n$$

where the vector $\mathbf{e}_i$ corresponds to the ith column of the identity matrix and the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathcal{R}^n$.

If we now consider that we wish to find the matrix A that represents a given transformation, then we have

$$A\mathbf{x} \;=\; A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + \ldots + x_n\mathbf{e}_n) \;=\; x_1A\mathbf{e}_1 + x_2A\mathbf{e}_2 + x_3A\mathbf{e}_3 + \ldots + x_nA\mathbf{e}_n)$$

and hence the columns of A are simply

$$\mathbf{a}_1 = A\mathbf{e}_1 \qquad \mathbf{a}_2 = A\mathbf{e}_2 \qquad \mathbf{a}_3 = A\mathbf{e}_3 \qquad \ldots \qquad \mathbf{a}_n = A\mathbf{e}_n$$

In other words, to find the required matrix all we need do is determine the effect of the transformation on the standard basis $\mathbf{e}_1$, $\mathbf{e}_2$, ., $\mathbf{e}_n$.

<u>Example</u>

To find the matrix which represents a rotation by θ deg counterclockwise about the origin in $\mathbf{R}^2$, note that the standard basis vectors are transformed as follows:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} cos\theta \\ \sin\theta \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

as shown in below. The matrix representation of such a rotation is therefore

$$A \;=\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

i.e. its columns are the result of transforming the standard basis.

Matrix Multiplication

We finish off this section of the notes by considering how we should form the product of two (or more) matrices. In other words, if we have the $n \times n$ matrices A and B and the $n \times 1$ vector $\mathbf{x}$, we seek the matrix C such that

$$AB\mathbf{x} \;=\; (AB)\mathbf{x} \;=\; C\mathbf{x}$$

Note first that

$$B\mathbf{x} \;=\; x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$$

where $\mathbf{b}_i$ is the ith column of B and x_i is the ith component of $\mathbf{x}$, and thus we have

$$\begin{aligned} AB\mathbf{x} &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) \\ &= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_nA\mathbf{b}_n \end{aligned}$$

which follows directly from our definition of the product of a matrix and a vector. Hence, since $C\mathbf{x}$ must be given by

$$C\mathbf{x} \;=\; x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$$

where $\mathbf{c}_i$ is the ith column of C, it must be true that $\mathbf{c}_i \;=\; A\mathbf{b}_i$, or in other words C is the matrix whose columns are the product of A and the corresponding columns of B, ie

$$C \;=\; [\,A\mathbf{b}_1\; A\mathbf{b}_2\; \cdots\; A\mathbf{b}_n\,]$$

Note that C therefore represents the linear transformation corresponding to first applying that defined by B followed by that defined by A. This is sometimes known as a composite transformation.

<u>Example 4</u>

$$\begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & 1 \\ -2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 24 & -18 & -72 \\ -1 & 9 & 2 \\ -8 & 20 & 16 \end{bmatrix}$$

Note that we can multiply any number of matrices in the same way and that we can only multiply matrices that are compatible in terms of row and column size - if A is an $n \times m$ matrix and B is a $p \times q$ matrix then the product AB is only defined if $m = p$ and it results in a $n \times q$ matrix, ie the two 'inner' dimensions must match and the two 'outer' dimensions are then the size of the resulting matrix, eg

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Moreover, from the above definition it is not in general true that $AB = BA$, ie matrix multiplication is **not commutative**. Note also that for an $n \times n$ square matrice A, multiplying by the identity matrix I has no effect, i.e. $IA = A$.

Matrix Inverse

We finish off by considering whether we can undo the effect of applying a matrix to a vector, i.e. is there an inverse linear transformation? It turns out that there isn't always one, but we'll leave those cases until the next section. For now we consider cases when there does exist a **matrix inverse**.

If A is an $n \times n$ matrix and there exists an $n \times n$ matrix B such that

$$BA = I \qquad \text{and} \qquad AB = I$$

where I is the $n \times n$ identity matrix, then we say that the matrix A is **invertible** and that the matrix B is the **inverse** of A. In fact, if an $n \times n$ matrix is invertible, then it has at most one inverse - its inverse is unique. To see this, note that if the matrix A had two inverses B and C, ie so that $CA = AC = BA = AB = I$, then we have

$$B = BI = B(AC) = (BA)C = IC = C$$

ie the matrices B and C must be equal. Since the inverse of a matrix is unique we use a special notation to denote it - A^{-1} - and thus we have

$$A^{-1}A = AA^{-1} = I$$

Note however that A^{-1} **does not** mean '1 over A' - the division of matrices is not defined. Moreover, since the $A^{-1}A = I$ and $AA^{-1} = I$, then the above inverse is only defined for square matrices.

<u>Example</u>

For the two matrices

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \qquad C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

we have

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and thus $C = A^{-1}$, or equivalently $A = C^{-1}$.

Inverse of a 2×2 Matrix

We might ask how to determine the inverse of a matrix. Perhaps not surprisngly this isn't straightforward for large matrices and in the next section we will look at how we can do it. For now, we look at the simple case of 2×2 matrices since for these the inverse is easily defined.

For a 2×2 matrix A its inverse is defined as follows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which we can prove as follows

$$A^{-1}A = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & db - bd \\ ac - ca & ad - cb \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

providing $ad - bc$ is non-zero. In other words, the inverse of a 2×2 matrix corresponds to swapping the two main diagonal components, multiplying the other diagonal components by -1, and dividing throughout by the term $ad - bc$. The latter is known as the **determinant** of the matrix. Unfortunately, similar formula for larger matrices are not so straightforward and rarely, if ever, used in practice. In the next section we shall look at how we can find the inverse of a matrix in a more efficient way using a method based on a technique known as **Gaussian elimination**.
