

# COMS21103: Stable Matching

## The Algorithm of Happiness

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Based on slides of Raphael Clifford

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# The problem

Imagine a set of students applying for Summer internships.

- ▶ Each student lists employers in order of preference.
- ▶ Each employer lists students in order of preference.
- ▶ Employers extend offers to students.
- ▶ Students accept or decline.

# The problem

Imagine a set of students applying for Summer internships.

- ▶ Each student lists employers in order of preference.
- ▶ Each employer lists students in order of preference.
- ▶ Employers extend offers to students.
- ▶ Students accept or decline.

Can we assign applicants to employers so that for every employer  $E$ , and every applicant  $A$  who is not scheduled to work for  $E$ , at least one of the following is true.

- ▶  $E$  prefers every one of its accepted applicants to  $A$  or
- ▶  $A$  prefers her current situation over working for employer  $E$

If these hold we have a *stable* matching.

# The problem - formulation

Gale and Shapley considered this problem in 1962 but before that the US National Resident Matching Program used something similar to match residents to hospitals in the US. This system is still in use today.

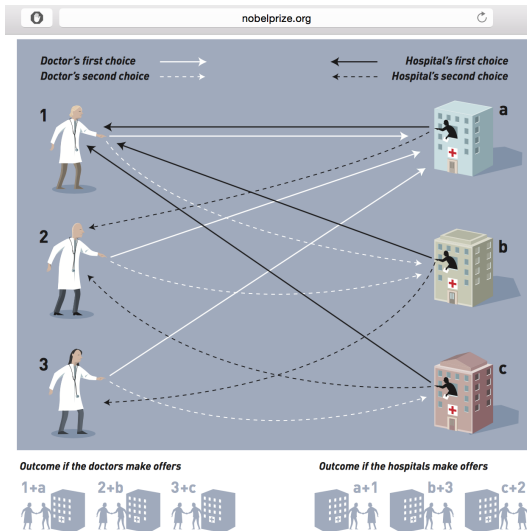
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And... the algorithm has been a blessing to Shapley himself...

<https://www.youtube.com/watch?v=9vOcuRQxc7w>

# Nobel Prize in Economics 2012 - Stable Matching



[http://www.nobelprize.org/nobel\\_prizes/economic-sciences/laureates/2012/popular-economicsciences2012.pdf](http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2012/popular-economicsciences2012.pdf)

# National Resident Matching Program

www.nrmp.org

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**THE MATCH**  
NATIONAL RESIDENT MATCHING PROGRAM®

KEYWORD

SEARCH

ABOUT

NEWS

CONTACT

RESIDENCY

FELLOWSHIP

MATCH PROCESS

POLICIES

MATCH DATA

**THAT'S THE FACE  
OF SOMEONE WHO'S  
MET HER MATCH**

## THE ALGORITHM OF HAPPINESS

THE NRMP MATCHING ALGORITHM PRODUCES A "BEST FIT" FOR APPLICANTS AND PROGRAMS. AND SINCE RESEARCH ON THE ALGORITHM WAS A BASIS FOR AWARDING THE 2012 NOBEL PRIZE IN ECONOMICS, YOU CAN BE CONFIDENT IN ITS RESULTS.

THE MATCH: GETTING IT RIGHT SINCE 1952.

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**RESIDENCY  
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TIMELINE**

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# National Resident Matching Program

www.nrmp.org/match-process/match-algorithm/

Google



KEYWORD SEARCH ABOUT NEWS CONTACT

RESIDENCY FELLOWSHIP MATCH PROCESS POLICIES MATCH DATA

## HOW THE MATCHING ALGORITHM WORKS

The NRMP uses a mathematical algorithm to place applicants into residency and fellowship positions.

### THE MATCHING PROCESS

The process begins with an attempt to match an applicant to the program most preferred on that applicant's rank order list (ROL). If the applicant cannot be matched to that first choice program, an attempt is made to place the applicant into the second choice program, and so on, until the applicant obtains a **tentative** match or all the applicant's choices on the ROL have been exhausted.

A **tentative** match means a program on the applicant's ROL also ranked that applicant and either:

- the program has an unfilled position, in which case there is room in the program to make a tentative match between the applicant and program, or
- the program does not have an unfilled position, but the applicant is more preferred by the program than another applicant who already is tentatively matched to the program. In that case, the applicant who is less preferred by the program is removed to make room for a tentative match with the more preferred applicant.

Matches are "**tentative**" because an applicant who is matched to a program may be removed from that program to make room for an applicant more preferred by the program. When an applicant is removed from a tentative match, an attempt is made to re-match that applicant, starting from the top of the applicant's ROL. This process is carried out for all applicants until each applicant has either been tentatively matched to the most preferred choice possible or all choices submitted by the applicant have been exhausted.

When the Match is complete, all tentative matches become final. **For an example of how the matching algorithm works, select Run A Match from the Match Resources options to the right.**

### MATCH RESOURCES

RESIDENCY TIMELINE

FELLOWSHIP TIMELINE

ABOUT ROL

NOBEL PRIZE

MATCH FEES

RUN A MATCH

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COMS21103: Stable Matching

# The problem - formulation

We will use Gale and Shapley's original problem of date matching.

Consider a society of  $n$  "women" ( $w_1, \dots, w_n$ ) and  $n$  "men" ( $m_1, \dots, m_n$ ).

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We will use Gale and Shapley's original problem of date matching.

Consider a society of  $n$  "women" ( $w_1, \dots, w_n$ ) and  $n$  "men" ( $m_1, \dots, m_n$ ).

- ▶ A matching is a one to one correspondence between the men and the women (we assume a purely monogamous, heterosexual society).
- ▶ Each female has a (strict) preference list for the males and each male has a (strict) preference list for the females.
- ▶ The matching is unstable if there exists  $w_i$  and  $m_j$  such that
  - ▶  $w_i$  and  $m_j$  are not matched to each other.
  - ▶  $w_i$  prefers  $m_j$  to her match.
  - ▶  $m_j$  prefers  $w_i$  to his match.
- ▶ If matching is not unstable, it is said to be stable.
- ▶ In other words, a matching is stable when there does not exist any alternative pairing ( $m_i, w_j$ ) in which both  $m_i$  and  $w_j$  are individually better off than they would be with the person to which they are currently matched.

Does a stable matching always exist? How quickly can we find one?

# The Gale-Shapley “Proposal” Algorithm

- Let  $i$  be the smallest value such that  $m_i$  is unmatched.

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- ▶  $m_i$  proposes to the most desirable woman (according to his list) that hasn't already rejected him.
- ▶ She accepts if either a) she is currently unmatched, or b) she finds  $m_i$  more desirable than her current match (in which case her current match becomes unmatched.)

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- ▶ Repeat until there are no unmatched men left.

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- ▶ Repeat until there are no unmatched men left.

Does the algorithm terminate? Is the resulting matching stable? The following example is taken from <http://bit.ly/26MnHW>.



# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Victor proposes to Bertha.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>rd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Victor proposes to Bertha.

- Bertha accepts since previously unmatched.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>rd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
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Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Wyatt proposes to Diane.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>rd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Wyatt proposes to Diane.

- Diane accepts since previously unmatched.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Xavier proposes to Bertha.

*Women's Preference List*

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Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Xavier proposes to Bertha.

- Bertha dumps Victor and accepts Xavier.

*Women's Preference List*

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Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

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Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Victor proposes to Amy.

*Women's Preference List*

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Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
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Erika	Yancey	Wyatt	Zeus	Xavier	Victor



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Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Victor proposes to Amy.

- Amy accepts since previously unmatched.

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Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Yancey proposes to Amy.

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Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Amy.

- Amy rejects since she prefers Victor.

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Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Yancey proposes to Diane.

*Women's Preference List*

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Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Diane.

- Diane dumps Wyatt and accepts Yancey.

# Stable matching example

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Xavier	Bertha	Erika	Clare	Diane	Amy
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Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

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Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Bertha.

# Stable matching example

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Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
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Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Bertha.

- Bertha rejects since she prefers Xavier.

# Stable matching example

*Men's Preference List*

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Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Amy.



# Stable matching example

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Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Amy.

- Amy rejects since she prefers Victor.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Wyatt proposes to Clare.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Clare.

- Clare accepts since previously unmatched.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Zeus proposes to Bertha.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Bertha.

- Bertha rejects since she prefers Xavier.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Zeus proposes to Diane.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Diane.

- Diane rejects Yancey and accepts Zeus.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Yancey proposes to Clare.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor



# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Clare.

- Clare rejects since she prefers Wyatt.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Yancey proposes to Bertha.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
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Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Bertha.

- Bertha rejects since she prefers Xavier.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
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Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Erika.

# Stable matching example

*Men's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Yancey proposes to Erika.

- Erika accepts since previously unmatched.

*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

# Stable matching example

*Men's Preference List*

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*Women's Preference List*

	0 <sup>th</sup>	1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>
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Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

STOP

- Everyone matched.
- Stable matching!

# Stable matching - perfection

## Lemma

*All men and women are matched when G-S terminates.*

# Stable matching - perfection

## Lemma

*All men and women are matched when G-S terminates.*

**Proof:** (By contradiction)

Suppose, for sake of contradiction, that Zeus is not matched on termination of algorithm.

- ▶ Some woman, say Amy, must therefore also not be matched upon termination.
- ▶ But every woman remains matched once they have been matched the first time, so no one must have proposed to Amy.
- ▶ But, Zeus proposes to everyone since he ends up unmatched. This gives the contradiction.



# Stable matching - stability

## Lemma

*Any matching found by G-S is stable.*

# Stable matching - stability

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*Any matching found by G-S is stable.*

**Proof:** (By contradiction) Suppose the matching has at least one unstable pair. Say Amy prefers Victor but is engaged to another man (e.g. Wyatt) and Victor prefers Amy but is engaged to another woman (e.g. Diane). Victor must have proposed to Amy earlier than Diane.

- ▶ Case 1: Amy was free when Victor proposed to her. Amy would not have swapped Victor for Wyatt..
- ▶ Case 2: When Victor proposed to Amy, she rejected him because she was already engaged to another man she preferred. Then the only way Amy can be matched with Wyatt is if she preferred him to the previous man. Then Wyatt must be preferred over Victor also, giving the contradiction.

# Stable matching - running time

## Theorem

*The G-S algorithm terminates after at most  $n^2$  proposals.*

# Stable matching - running time

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*The G-S algorithm terminates after at most  $n^2$  proposals.*

**Proof:** Let  $P(t)$  be the number of pairs  $(m_j, w_i)$  such that  $m_j$  has proposed to  $w_i$  by the end of proposal number  $t$ .

- ▶  $P(t + 1) > P(t)$  as each proposal that is made involves a new pair.
- ▶  $P(\cdot) \leq n^2$  as there are no more than  $n^2$  pairs of men and women in total.
- ▶ Therefore G-S terminates after at most  $n^2$  proposals.

# Stable matching - random starts

## Fact

*All executions of G-S result in the same set of matched pairs.*

# Stable matching - random starts

## Fact

*All executions of G-S result in the same set of matched pairs.*

Order of proposers (i.e. men) does not change the outcome of the algorithm

# Stable matchings are not unique

but... there can be many stable matchings for a given set of preferences.  
Consider the following:

- ▶ Bob prefers Alice to Sarah
- ▶ Charlie prefers Sarah to Alice
- ▶ Alice prefers Charlie to Bob
- ▶ Sarah prefers Bob to Charlie

# Stable matchings are not unique

but... there can be many stable matchings for a given set of preferences.  
Consider the following:

- ▶ Bob prefers Alice to Sarah
- ▶ Charlie prefers Sarah to Alice
- ▶ Alice prefers Charlie to Bob
- ▶ Sarah prefers Bob to Charlie

G-S will make Bob engaged to Alice and Charlie engaged to Sarah (maybe in the other order) and stop. However, pairing Charlie with Alice and Bob with Sarah is also stable and preferable for the women.



# Fairness

## Theorem

*Every execution of G-S results gives the best possible valid partner for each man.*

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A woman is a *valid partner* of a man if there is stable matching that contains that pairing.

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## Theorem

*Every execution of G-S results gives the best possible valid partner for each man.*

A woman is a *valid partner* of a man if there is stable matching that contains that pairing.

**Proof:** (By contradiction). Assume some man is paired with a woman who is not his best valid partner. Consider the first such case and the first rejection this man  $m$  receives from his best valid partner  $w$ .

- ▶ When  $m$  is rejected,  $w$  is matched with a man, say  $Z$ , whom she prefers to  $m$ .
- ▶  $Z$  only proposes to  $w$  if she's his best valid partner - higher preferences must have rejected him.
- ▶ Thus the pairing  $(w, m)$  is unstable.

# Fairness

How could we get the best solution for women?

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If women propose instead of men.

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How could we get the best solution for women?

If women propose instead of men.

Similarly... G-S results in the best solution for hospitals in the national residence matching program

# G-S Algorithm - Github

<https://github.com/xocoatzin/algorithms>

# Further reading



# Further reading

- ▶ **Algorithm Design**

J. Kleinberg and E. Tardos.

Pearson Education, ISBN: 0321372913

- ▶ Section 1.1 – A First Problem: Stable Matching

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[3,985 views - Nov 2015]

- ▶ S Sawhney <https://www.youtube.com/watch?v=pc5WSJkFk24>  
[11,217 views - Nov 2015]