COMS21103: Order Statistics

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Based on slides of Raphael Clifford

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Order Statistics

- In this lecture we look at order statistics.
  - Select the ith smallest of n elements (the element of rank i).
  - If $i = 1$ we have the minimum
  - If $i = n$, we have the maximum
  - If $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$, then we have the median
- A simple solution is just to sort the input and choose the ith element of the sorted array.
- Worst case is $\Theta(n \log n)$ time using Merge Sort or Heap Sort for example.
- We can do better than that!

Partitioning revisited

- Recall... for Quicksort we used a random partitioning that was able to take a pivot x and rearrange the input array A so that all values $\leq x$ are to the left of x and those $\geq x$ are to right of x.

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- Assume RAND-PARTITION(A, p, q) partitions $A[p, \ldots, q]$ according to a randomly chosen pivot and returns r, the index of the pivot.
- First we have to choose a random pivot and then call PARTITION with the pivot placed at end of the input array.

Input: A, p, q
$i \leftarrow$ RANDOM(p, q);
swap $A[q]$ and $A[i]$;
return PARTITION(A, p, q);

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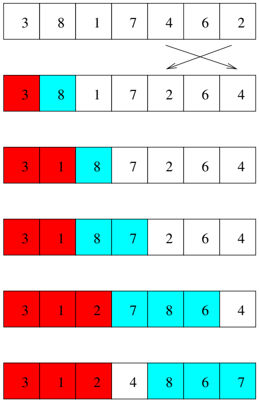
PARTITION(A, p, q) returns the index of the pivot element after partitioning.

Input : A, p, q
Output: Index of pivot
$x \leftarrow A[q]$ $\triangleright$ x is the pivot;
$i \leftarrow p - 1$;
for $j \leftarrow p$ **to** $q - 1$ **do**
  if $A[j] \leq x$ **then**
    $i \leftarrow i + 1$;
    swap $A[i]$ and $A[j]$;
  end
end
swap $A[i + 1]$ and $A[q]$;
return $i + 1$;



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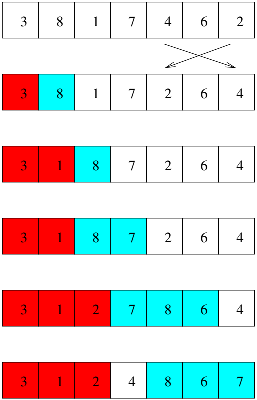


- At the start of every iteration, the following *invariants* are true
  - For $p \leq k \leq i$, $A[k] \leq x$
  - For $i + 1 \leq k < j$, $A[k] > x$
  - $A[q] = x$

Randomised algorithm

We can use the partition function repeatedly to find the *i*th smallest element.

- Pick a pivot at random
- Partition according to the pivot
- Now we only need to look in one of the two "halves" after the partitioning
- Recurse until either we happen to choose the *i*th smallest element as a pivot or the half we need to look in only has one element in it

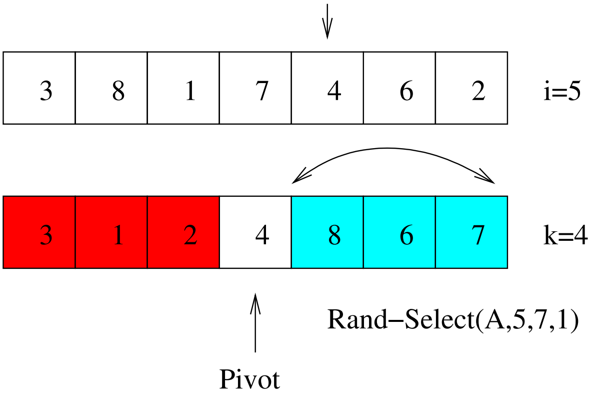
Randomised algorithm

RAND-SELECT(A, p, q, i) returns the ith smallest element of $A[p, \ldots, q]$

Input : A, p, q and i
Output: The ith smallest value in $A[p, \ldots, q]$
if $p = q$ **then**
    return $A[p]$;
end
$r \leftarrow$ RAND-PARTITION(A, p, q);
$k \leftarrow r - p + 1$ $\quad\triangleright$ pivot index in $A[p, \ldots, q]$;
if $i = k$ **then**
    return $A[r]$;
end
if $i < k$ **then**
    return RAND-SELECT$(A, p, r - 1, i)$;
else
    return RAND-SELECT$(A, r + 1, q, i - k)$;
end

Order Statistics Example

In this example we want to find the 5th smallest number in the array
3, 8, 1, 7, 4, 6, 2.



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Master Theorem - Case 3
Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function and $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) \tag{1}$$

If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

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  - In fact, the average case is also linear time and the method is very fast in practice.

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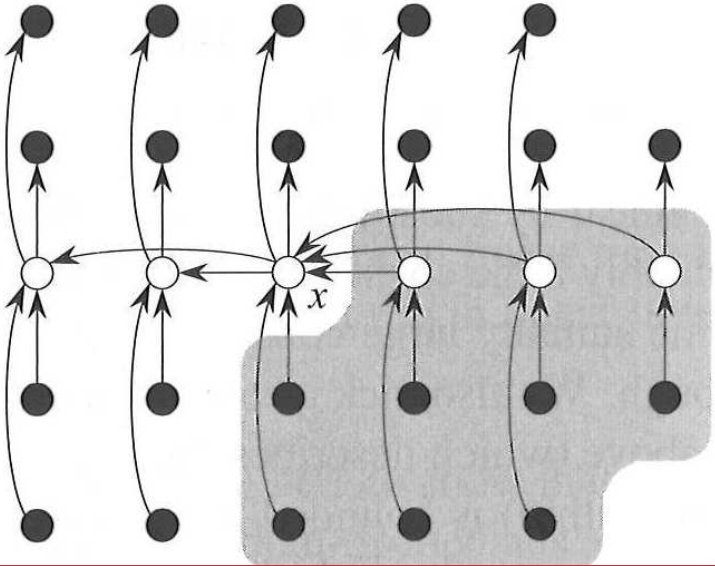
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
3. Partition $A[p, \ldots, q]$ using the pivot x. $k = \text{rank}(x)$
4. if $i = k$ then return x
5. elsif $i < k$
    - then recursively SELECT the ith smallest element in the lower part
    - else recursively SELECT the $i - k$th smallest element in the higher part

BFPRT Pivot Method



BFPRT Pivot Method - Example

A = [2, 3, 1, 7, 2, 8, 3, 9, 4, 12, 15, 3, 5, 7, 1]

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  [2, 3, 1, 7, 2] $\rightarrow$ Median 2
  [8, 3, 9, 4, 12] $\rightarrow$ Median 8
  [15, 3, 5, 7, 1] $\rightarrow$ Median 5

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- Partition using median and re-order groups
  [2, 1, 2, 7, 3]
  [1, 3, 5, 7, 15]
  [3, 4, 8, 9, 12]

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- Partition using median and re-order groups
  [2, 1, 2, 7, 3]
  [1, 3, 5, 7, 15]
  [3, 4, 8, 9, 12]
- Median chosen is 5

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4. $T(\lfloor n/5 \rfloor)$ time to find median of group medians
5. $\Theta(n)$ time to partition n elements around the pivot
6. Time to recursively call SELECT on one "half" of the input

- There are $\lfloor n/5 \rfloor$ group medians, at least half of which are $\leq x$, the pivot by definition. This is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians.
- Therefore at least $3\lfloor n/10 \rfloor$ elements are $\leq x$ in total
- Similary at least $3\lfloor n/10 \rfloor$ elements are $\geq x$ in total

Summary

- RAND-SELECT is linear time on average and simple to implement but harder to analyse.
- However, RAND-SELECT runs in $\Theta(n^2)$ in the worst case.
- SELECT runs in linear time in the worst case.
- SELECT is slightly harder to implement but simpler to analyse.

Further Reading

- Introduction to Algorithms
  T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein.
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 9 – Order statistics
- Algorithms
  S. Dasgupta, C.H. Papadimitriou and U.V. Vazirani
  `http://www.cse.ucsd.edu/users/dasgupta/mcgrawhill/`
  - Chapter 2, Section 2.4 – Medians