COMS10003 Workshop Sheet 18.

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Introduction

This worksheet is about Fourier series.

Useful facts

- **Trigonometric identities**: adding angles

$$\begin{aligned} \sin A \pm B &= \sin A \cos B \pm \cos A \sin B \\ \cos A \pm B &= \cos A \cos B \mp \sin A \sin B. \end{aligned} \tag{1}$$

- **Trigonometric identities**: products

$$\begin{aligned} \cos A \cos B &= \frac{1}{2}[\cos{(A-B)} + \cos{(A+B)}] \\ \sin A \sin B &= \frac{1}{2}[\cos{(A-B)} - \cos{(A+B)}] \\ \sin A \cos B &= \frac{1}{2}[\sin{(A+B)} + \sin{(A-B)}]. \end{aligned} \tag{2}$$

- **Trigonometric identities**: double angles

$$\begin{aligned} \sin 2A &= 2 \sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A. \end{aligned} \tag{3}$$

- The integral of a sine or a cosine over its whole period is zero, the negative and positive balance.

- **The Kronecker delta**: δ_{nm} is zero when $n \neq m$ and one when $n = m$.

- **Periodic**: A function is periodic if for some constant L, $f(t+L) = f(t)$. The smallest such L is called the *period*.

- **Periodic, even and odd** A function $f(t)$ has period L if $f(t+L) = f(t)$, it is odd if $f(-t) = -f(t)$ and even if $f(-t) = f(t)$.

- A function with period L has the **Fourier series expansion**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{L}\right).$$

where

$$\begin{aligned} a_0 &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)dt \\ a_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt \\ b_n &= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\sin\left(\frac{2\pi nt}{L}\right)dt \end{aligned}$$

- In the lectures we considered only $L = 2\pi$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt).$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \end{aligned}$$

Work sheet

1. Establish that

$$\int_{-\pi}^{\pi} \sin mt \ \cos ntdt = 0 \tag{4}$$

for integers n and m.

2. What is $\sum_{n=0}^{\infty} a_n \delta_{n3}$?

3. Show by checking whether $f(t) = -f(-t)$ for odd, $f(t) = f(-t)$ for even and neither for neither which of the following are odd, even or neither: $\sin t$, $t^3 + t$, $t^3 + 2t^2$ and $|t|$.

4. Consider an odd function $f(t)$. By doing a change of variable $t' = -t$ show

$$\int_{-1}^{1} f(t)dt = 0 \tag{5}$$

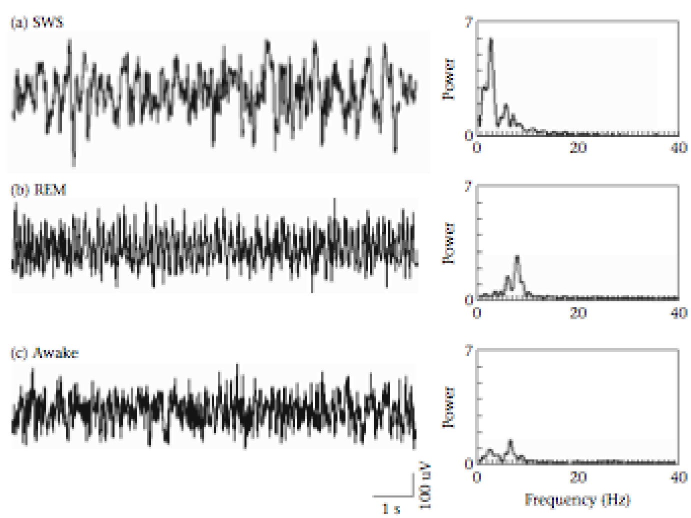
5. If $f(t)$ is even and $g(t)$ is odd, what is $f(t)g(t)$?

6. What is the Fourier series for

$$f(t) = \begin{cases} -1 & t \in (-\pi, -\pi/2) \\ 1 & t \in (-\pi/2, \pi/2) \\ -1 & t \in (\pi/2, \pi) \end{cases} \tag{6}$$

with $f(t+2\pi) = f(t)$. Try to use odd and even arguments to avoid doing the integral for b_n.

7. There are, roughly speaking, two different types of sleep, slow wave sleep (SWS) when the brain is largely in active and the body is relaxed, and rapid eye movement (REM) sleep when we dream, the brain has a similar activity pattern to waking and the body is paralysed. This was discovered using electroencephalogram (EEG), the recording of the electrical activity in the brain using electrodes placed on the scalp. Here we see, on the left, EEG traces for a mouse during REM, SWS and waking, on the right we see a measure of the size of the Fourier components, roughly the a_n and b_n for different frequencies: $\sin(nt)$ has period $2\pi/n$ and so has frequency $n/2\pi$. Although the brain is largely inactive during SWS what little activity there is is synchronized and is characterized by delta waves and theta waves. Can you guess what frequency delta waves and theta waves are at? [Picture taken from Lima SL, Rattenborg NC, Lesku JA, Amlaner CJ. (2005) *Sleeping under the risk of predation.* Animal Behaviour 70:723-736.]



Exercise sheet

The difference between the work sheet and the exercise sheet is that the solutions to the exercise sheet won't be given and the problems are designed to be more suited to working on on your own, though you are free to discuss them in the work shop if you finish the work sheet problems. Selected problems from the exercise sheet will be requested as part of the continual assessment portfolio.

1. Establish that
$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt \ \cos ntdt = \delta_{mn} \tag{7}$$
for integers n and m.

2. Read the wikipedia page for the Fast Fourier Transform.

3. Using `gnuplot` or whatever plot the function, written here in a Fourier series form
$$f(t) = \sin(t) + 0.1\sin(10t) + 0.2\cos(11t) \tag{8}$$
and compare it to $g(t) = \sin(t)$. The original $f(t)$ has details at two scales, around 2π and around $2\pi/10$. $g(t)$ is obtained from $f(t)$ by dropping the higher frequency Fourier modes. Changing a function by removing some of the Fourier modes is called *band pass filtering*.

4. Find the Fourier series for $\sin^3 t$; a quick way to do this is to regard it as a trigonometry problem, rather than a Fourier series problem, that is use the trigonometric identities to express it in terms of sines and cosines, rather than doing all the integrals: so start by writing $\sin^3 t = \sin^2 t \sin t$ and then write $\sin^2 t$ in terms of $\cos 2t$.