University of Bristol
COMS21103: Data Structures and Algorithms
Problem Set 2 with Answers

Remark: All the problems are from the textbook, and Problems with $\star$ are more challenging. However, we will mainly focus on Problem 1 to 4 during the problem class.

Problem 1: Problems from class. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers. *(These recurrences are from our Monday's class.)*

1. $T(n) = T(\lceil n/2 \rceil) + 1$
2. $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$
3. $T(n) = 2T(\sqrt{n}) + 1$

Solutions:

(1) Without loss of generality, we drop the ceiling function, and look at $T(n) = T(n/2) + 1$. We use the master theorem: We have that $a = 1, b = 2$, and $f(n) = 1$. Hence $n^{\log_b a} = n^0 = 1 = f(n)$. By the case 2 of the master theorem, we have that $f(n) = \Theta(\lg n)$.

(2) We drop the flooring function, and look at $T(n) = 2T(n/2 + 17) + n$. Our guess is that $T(n) = \Theta(n \log n)$, and we use the substitution method to prove it. We first show that $T(n) = O(n \lg n)$, i.e. $T(n) \leq cn \cdot \lg n - dn$ for some constant $c > 0$, $d > 0$, and any $n \geq n_0$ for some n_0. We assume that the base case holds trivially, and the statement holds for any problem of size less than n. Then, it holds for large enough n that

$$\begin{aligned} T(n) &\leq 2\left(c(n/2 + 17) \cdot \lg(n/2 + 17) - d(n/2 + 17)\right) + n \\ &\leq (cn + 34c)\lg(2n/3) - 2d(n/2 + 17) + n \\ &\leq cn\lg(2n/3) + 34c\log n - dn - 34d + n \\ &\leq cn\lg n - dn, \end{aligned}$$

where the last inequality holds by choosing $c = 1$. Now, we prove that $T(n) = \Omega(n \log n)$, i.e. $T(n) \geq cn\lg n$ for some constant $c > 0$. Substituting our guess in the recurrence, we have that

$$\begin{aligned} T(n) &\geq 2c(n/2 + 17)\lg(n/2 + 17) + n \\ &= cn\lg(n/2 + 17) + 34c\lg(n/2 + 17) + n \\ &\geq cn\lg(n/2) + 34c\lg(n/2) + n \\ &= cn\lg n - cn + 34c\lg n - 34c + n \\ &= cn\lg n + (1 - c)n + 34c(\lg n - 1) \\ &\geq cn\lg n, \end{aligned}$$

where the last inequality holds by if $c = 1/2$. In summary, we obtain that $T(n) = \Theta(n \lg n)$.

(3) We ignore the ceiling function, and study $T(n) = 2T(\sqrt{n}) + 1$. By defining $n = 2^m$, we have that

$$T(2^m) = 2T(2^{m/2}) + 1.$$

We further define $S(m) = T(2^m)$, and rewrite the equation above as

$$S(m) = 2S(m/2) + 1.$$

By case 1 of the master theorem, we have that $S(m) = \Theta(m)$. Hence

$$T(n) = T(2^m) = S(m) = \Theta(m) = \Theta(\lg n).$$

Problem 2: Recurrence examples. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

1. $T(n) = 2T(n/2) + n^3$
2. $T(n) = T(9n/10) + n$
3. $T(n) = 16T(n/4) + n^2$
4. $T(n) = 7T(n/3) + n^2$
5. $T(n) = 7T(n/2) + n^2$
6. $T(n) = 2T(n/4) + \sqrt{n}$
7. $T(n) = T(n-1) + n$
8. $T(n) = T(\sqrt{n}) + 1$

Solutions:

(1) We apply the master theorem, and have that $a = 2$, $b = 2$, and $f(n) = n^3$. Since $n^{\log_b a} = n^{\log_2 2} = n$, and $f(n) = n^3 = \Omega(n^{1+\varepsilon})$ for $\varepsilon = 2$, we need to verify the regularity condition of the master theorem, i.e. $af(n/b) \le cf(n)$ for some constant $c < 1$. Notice that

$$af(n/b) = 2(n/2)^3 = n^3/4 \le cf(n)$$

for $c = 1/4 < 1$, hence the regularity condition is satisfied, and we have that $T(n) = \Theta(n^3)$.

(2) We apply the master theorem, and have that $a = 1$,$b = 10/9$, and $f(n) = n$. Since $n^{\log_b a} = n^0 = 1$ and $f(n) = n = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon = 1$, we need to check the regularity condition from case 3. Notice that

$$af(n/b) = 9n/10 \le cf(n),$$

where $c = 0.99$. Hence, case 3 of the master theorem applies, and we have that $T(n) = \Theta(n)$.

(3) We apply the master theorem, and have that $a = 16, b = 4$, and $f(n) = n^2$. Since $n^{\log_b a} = n^2 = \Theta(f(n))$, case 2 of the master theorem applies, and we have that $T(n) = \Theta(n^2 \lg n)$.

(4) We apply the master theorem, and have that $a = 7$, $b = 3$, and $f(n) = n^2$. Since $n^{\log_b a} = n^{\log_3 7} \approx n^{1.77}$, and $f(n) = \Omega(n^{1.77+\varepsilon})$ for $\varepsilon = 0.2$, we need to verify the regularity condition of Case 3 in the master theorem: We have that

$$af(n/b) = 7(n/3)^2 = 7n^2/9 \le cf(n)$$

for $c = 0.9$. Hence we can apply case 3 of the master theorem, and obtain that $T(n) = \Theta(n^2)$.

(5) We apply the master theorem, and have that $a = 7$, $b = 2$, and $f(n) = n^2$. Since $n^{\log_b a} \approx n^{2.8}$ and $f(n) = O(n^{2.8-\varepsilon})$ for $\varepsilon = 0.6$, case 1 of the master theorem applies, and we have that $T(n) = \Theta(n^{\lg 7})$.

(6) We apply the master theorem, and have that $a = 2$, $b = 4$, and $f(n) = n^{1/2}$. Since $n^{\log_b a} \approx n^{1/2} = \Theta(f(n))$, case 2 of the master theorem applies, and we have that $T(n) = \Theta(\sqrt{n} \lg n)$.

(7) We can simply expand this recursion formula, and obtain that $T(n) = \Theta(n^2)$.

(8) We ignore the ceiling function, and study $T(n) = T(\sqrt{n}) + 1$. By defining $n = 2^m$, we have that

$$T(2^m) = T(2^{m/2}) + 1.$$

We further define $S(m) = T(2^m)$, and rewrite the equation above as

$$S(m) = S(m/2) + 1.$$

By case 2 of the master theorem, we have that $S(m) = \Theta(\lg m)$. Hence

$$T(n) = T(2^m) = S(m) = \Theta(\lg m) = \Theta(\lg \lg n).$$

Problem 3: More recurrence examples. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficient small n. Make your bunds as tight as possible, and justify your answers.

1. $T(n) = 3T(n/2) + n \lg n$.
2. $T(n) = 5T(n/5) + n/\lg n$.
3. $T(n) = 4T(n/2) + n^2\sqrt{n}$
4. $T(n) = 3T(n/3 + 5) + n/2$
5. $T(n) = 2T(n/2) + n/\lg n$
6. $T(n) = T(n/2) + T(n/4) + T(n/8) + n$
7. $T(n) = T(n-1) + 1/n$
8. $T(n) = T(n-1) + \lg n$
9. $T(n) = T(n-2) + 2\lg n$
10. $T(n) = \sqrt{n}T(\sqrt{n}) + n$

Solutions:

(1) We apply the master theorem, and have that $a = 3, b = 2$, and $f(n) = n \lg n$. Since $n^{\log_b a} = n^{\lg 3} \approx n^{1.58}$, and $f(n) = n \lg n = O(n^{1.58-\varepsilon})$ for $\varepsilon = 0.2$, case 1 of the master theorem applies and we have that $T(n) = \Theta(n^{\lg 3})$.

(2) We need to use the recursion tree method, and write $T(n)$ as

$$\begin{aligned} T(n) &= \sum_{i=1}^{\log_5 n} 5^i \cdot \frac{n/5^i}{\lg(n/5^i)} \\ &= \sum_{i=1}^{\log_5 n} 5^i \cdot \frac{n/5^i}{\log_5(n/5^i)} \cdot \log_5 2 \\ &= \sum_{i=1}^{\log_5 n} \frac{n \cdot \log_5 2}{\log_5 n - i} \\ &= \sum_{i=1}^{\log_5 n} \frac{n \cdot \log_5 2}{i} \\ &= (\log_5 2) \cdot n \cdot H_{\log_5 n}, \end{aligned}$$

where $H_{\log_5 n}$ is the $\log_5 n$-th harmonic number, and of the order $\Theta(\lg \lg n)$. Hence, we have that $T(n) = \Theta(n \lg \lg n)$.

(3) We apply the master theorem. We have that $a = 4$, $b = 2$, and $f(n) = n^{2.5}$. Notice that $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for $\varepsilon = 0.5$, hence we need to verify the regularity condition. We have that $af(n/b) = 4(n/2)^{2.5} \leq 0.8n^{2.5} = 0.8f(n)$. By case 3 of the master theorem, we have have $T(n) = \Theta(n^{2.5})$.

(4) We guess that $T(n) = \Theta(n \lg n)$. We first prove that $T(n) = O(n \lg n)$, i.e. $T(n) \leq cn \lg n$ for some constant c. By induction, we have for a large enough n that

$$\begin{aligned} T(n) &\leq 3c(n/3 + 5)\lg(n/3 + 5) + n/2 \\ &\leq (cn + 15c)\lg(n/2) + n/2 \\ &\leq cn \lg n - cn + 15c \lg(n/2) + n/2 \\ &\leq cn \lg n \end{aligned}$$

for $c = 10$. Next, we prove that $T(n) = \Omega(n \lg n)$, i.e. $T(n) \geq cn \lg n$ for some constant c. By induction, we have that

$$\begin{aligned} T(n) &\geq 3c(n/3+5)\lg(n/3+5) + n/2 \\ &\geq cn\lg(n/3) + n/2 \\ &\geq cn\lg n - cn\log 3 + n/2 \\ &\geq cn\lg n, \end{aligned}$$

where the last inequality holds if $c = 10$. This finishes the proof of $T(n) = \Omega(n \lg n)$. Combing these two steps together, we have that $T(n) = \Theta(n \lg n)$.

(5) We need to use the recursion tree method, and write $T(n)$ as

$$\begin{aligned} T(n) &= \sum_{i=1}^{\lg n} 2^i \cdot \frac{n/2^i}{\lg(n/2^i)} \\ &= \sum_{i=1}^{\lg n} 2^i \cdot \frac{n/2^i}{\lg(n/2^i)} \\ &= \sum_{i=1}^{\lg n} \frac{n}{\lg n - i} \\ &= \sum_{i=1}^{\lg n} \frac{n}{i} \\ &= n \cdot H_{\lg n}, \end{aligned}$$

where $H_{\lg n}$ is the $\lg n$-th harmonic number, and of the order $\Theta(\lg \lg n)$. Hence, we have that $T(n) = \Theta(n \lg \lg n)$.

(6) We guess that $T(n) = \Theta(n)$. We first prove that $T(n) \leq cn$ for some constant $c > 0$. Then we have that

$$\begin{aligned} T(n) &= T(n/2) + T(n/4) + T(n/8) + n \\ &\leq cn/2 + cn/4 + cn/8 + n \\ &= (7c/8 + 1)n \\ &\leq cn \end{aligned}$$

for $c = 80$. On the other hand, $T(n) = \Omega(n)$ due to the term n in the expression of $T(n)$. In summary, $T(n) = \Theta(n)$.

(7) By the definition of $T(n)$ we have that $T(n) = \sum_{i=1}^{n} 1/i = \Theta(\lg n)$.

(8) By definition, we have that

$$T(n) = \sum_{i=2}^{n} \lg i = \lg(n!) = \Theta(n \lg n),$$

where the last equality is from our Problem Set 1.

(9) The analysis is similar with (8), and we have that $T(n) = \Theta(n \lg n)$.

(10) We first change the variables, and let $n = 2^{2^k}$. Then,

$$T(n) = T\left(2^{2^k}\right) = 2^{2^{k-1}} T\left(2^{2^{k-1}}\right) + 2^{2^k}.$$

For simplicity, let $S(k) = T\left(2^{2^k}\right)$

$$S(k) = T\left(2^{2^k}\right) = 2^{2^{k-1}} S\left(k-1\right) + 2^{2^k}.$$

Then, we have that

$$S(k) = 2^{2^{k-1}} \cdot \left(2^{2^{k-2}} S(k-2) + 2^{2^{k-1}}\right) + 2^{2^k}$$
$$= 2^{2^{k-1}+2^{k-2}} S(k-2) + 2^{2^k} + 2^{2^k},$$

and can express $S(k)$ by

$$S(k) = 2^{\sum_{i=1}^{k-1} 2^i} + k \cdot 2^{2^k} = 2^{2^k - 2} + k \cdot 2^{2^k}.$$

Therefore

$$T(n) = S(k) = \Theta\left(k \cdot 2^{2^k}\right) = \Theta(n \cdot \lg\lg n).$$

Problem 4: Use a recursion tree to give an asymptotically tight solution to the recurrence $T(n) = T(\alpha n) + T((1-\alpha)n) + cn$, where α is a constant in the range $0 < \alpha < 1$ and $c > 0$ is also a constant.

Solution: $T(n) = \Theta(n \lg n)$

$\star$ **Problem 5:** Consider the regularity condition $af(n/b) \leq cf(n)$ for some constant $c < 1$, which is part of case 3 of the master theorem. Give an example of constants $a \geq 1$ and $b > 1$ and a function $f(n)$ that satisfies all the conditions in case 3 of the master theorem except the regularity condition.

Solution: Let $a = 1, b = 2$. Let $f(n) = n^2$ if n is an odd number, and $f(n) = n$ if n is an even number.

$\star$ **Problem 6:** Show that case 3 of the master theorem is overstated, in the sense that the regularity condition $af(n/b) \leq cf(n)$ for some constant $c < 1$ implies that there exists a constant $\varepsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \varepsilon})$.

Solution: Assuming that $af(n/b) \leq cf(n)$ for some constant $c < 1$, we have that

$$f(n) \geq \left(\frac{a}{c}\right) f\left(\frac{n}{b}\right) \geq \left(\frac{a}{c}\right)^2 f\left(\frac{n}{b^2}\right) \geq \cdots \geq \left(\frac{a}{c}\right)^{\log_b n} \Omega(1).$$

Therefore,

$$f(n) = \Omega\left(\left(\frac{a}{c}\right)^{\log_b n}\right) = \Omega\left(n^{\log_b (a/c)}\right) = \Omega\left(n^{\log_b a + \log_b (1/c)}\right).$$

Since $c < 1$, it holds that $\log_b(1/c) > 0$, and we can simply set $\varepsilon = \log_b(1/c)$. This finishes the proof.