Introduction to Mathematical Logic

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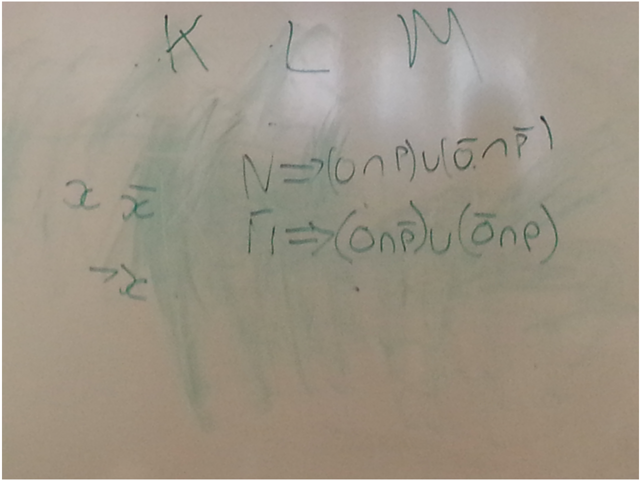
Introduction



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- We've explored different ways of *informal* knowledge representation.

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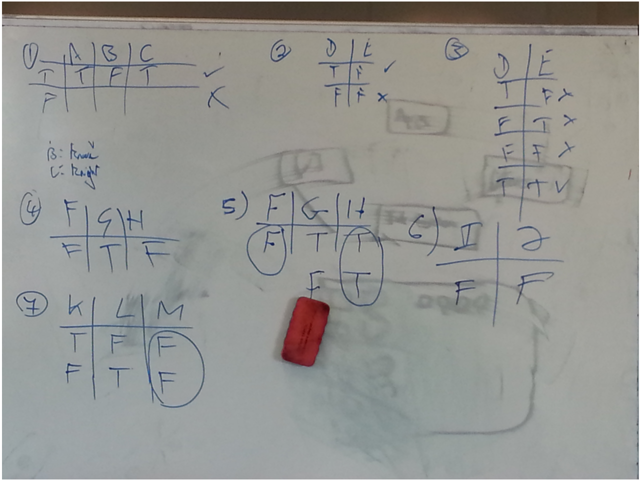

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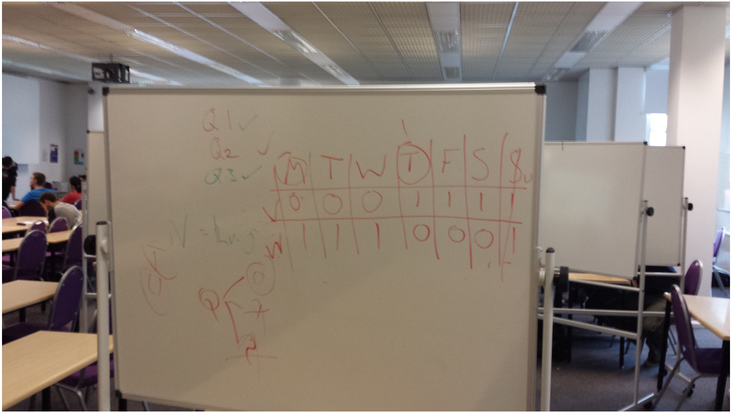

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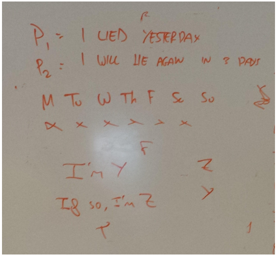

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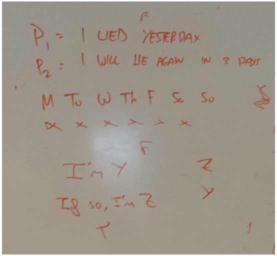

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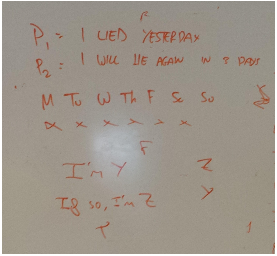

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- This lecture introduces you to logic.

Some Feedback from last year

'Logic' chapter - very boring, tedious and at times quite confusing because of all the formalities.

Everything is taught formally, rather than intuitively.

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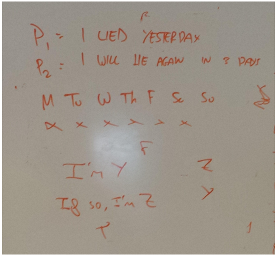

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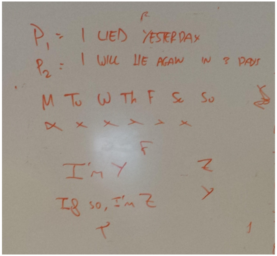

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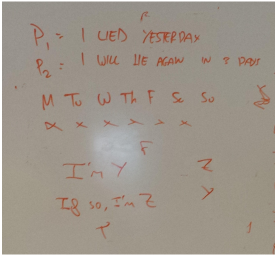

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- We will focus on *propositional logic*.

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- The *proof theory* is concerned with manipulating formulae according to certain rules.

Propositional Logic

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- It is possible to determine whether any given statement is a proposition by prefixing it with

  It is true that . . .

  and seeing whether the result makes sense.

Propositions

Which of these are propositions?

1. Good morning!
2. London is the capital of the UK.
3. Grass is blue.
4. I am hungry.
5. Who is speaking?
6. 'Logic' has five letters.
7. 1+1
8. $1 + 1 = 2$
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What about *“This statement is false.”*

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- Alternatively we could write something like $light\_is\_on$ so that the meaning of the propositional variable becomes obvious.

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$\Leftrightarrow$	if and only if	equivalence, iff	($\leftrightarrow$)

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p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
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p	It is Monday.
q	It is sunny.
$p \wedge q$	It is Monday and it is sunny.
	It is Monday but it is sunny.
	It is Monday. It is sunny.

The word *both* is often useful eg *it is both Monday and sunny*.

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Alternative short version

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p	q	$p \oplus q$
T	T	F
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p	It is raining.
q	It is sunny.
$p \oplus q$	It is raining or it is sunny, but not both.

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- Conjunction and disjunction are both *binary* connectives.

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$p \Rightarrow q$ If I study hard then I get rich.
Whenever I study hard, I get rich.
That I study hard implies I get rich.
I get rich, if I study hard.

More About Implication

Let p be *"I study hard."* and let q be *"I get rich."*

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- However if I've studied hard but failed to become rich then the proposition is clearly false.
- Note that $(p \Rightarrow q)$ is equivalent to $(\neg q \Rightarrow \neg p)$.

Even More About Implication

- **Notation** and **Terminology**

p	$\Rightarrow$	q
antecedent		**consequent**
premise		conclusion
hypothesis		
sufficient condition		necessary condition

Examples

$p \Rightarrow q$	If I study hard then I get rich.
$q \Rightarrow p$	If I get rich then I study hard. (the **converse.**)
$\neg p \Rightarrow \neg q$	If I don't study hard then I don't get rich. (the **inverse.**)
$\neg q \Rightarrow \neg p$	If I don't get rich then I don't study hard. (the **contrapositive.**)

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    - nullary connectives **true** and **false**;
    - unary connective $\neg$

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- The language of propositional logic is based on two components, an **alphabet** and a **grammar**.
- The **alphabet** consist of the following sets:
  - A set *PROP*, of propositional variables, p, q, r, s etc.
  - A set of propositional connectives:-
    - nullary connectives **true** and **false**;
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Based on the **alphabet**, we can build up compound propositions, or formulae. But what constitutes a well-formed formula?
The **grammar** defines this.

Language of Propositional Logic: Well-Formed Formulae

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This is an *inductive* definition. It can be used to determine whether a given formula is a wff, e.g. is $(p \Rightarrow q(\neg r))$ or $((\neg(p \vee q)) \wedge (\neg r))$ a wff?

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- Solution: $(p \Leftrightarrow (((\neg q) \vee r) \Rightarrow p))$

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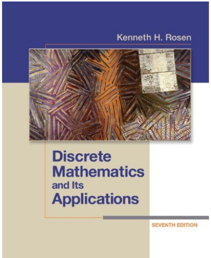
- Workshop sheet will be online this evening.

Suggested Reading

A good textbook on Discrete Mathematics is the one by Rosen:



Kenneth H. Rosen

Discrete Mathematics and Its Applications (7th Edition)

Read the parts on *Logic* to advance your understanding of the material in this lecture. Solve the exercises in the book to practice problem solving.

Note: There are many books on *Logic*. The best level for you would be an *Introduction to Logic*. The Library in QB offers a large variety of textbooks covering this subject.