Introduction to coding theory

CoCoNut, 2016
Emmanuela Orsini

References

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What does she say?

“Wel*ome to t*is c*ass!” $\longrightarrow$

What does she say?

"Wel*ome to t*is c*ass!" $\longrightarrow$ "Welcome to this class!"

Why is this example working?

- English has in built redundancy, so that it can tolerate *errors*.

Coding theory I

More in general, consider the following applications of *data storage* or *transmission*:

- CDs and DVDs
- Satellite/Digital Television
- Deep space probes
- Internet communications
- Mobile phones
- Computer hard disks/memory/floppy etc

In all of these the data can become corrupted.

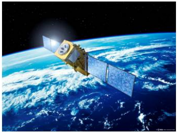
- It is prone to errors

However they still work

- How?

Coding theory - Applications

- Internet
- Mobile phones
- Satellite broadcast
  - TV

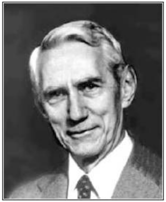


- Deep space telecommunications
  - Mars Rover
- Data storage

Codes are all around us!

Coding theory - The birth

"The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point"
(Claude Shannon, 1948)

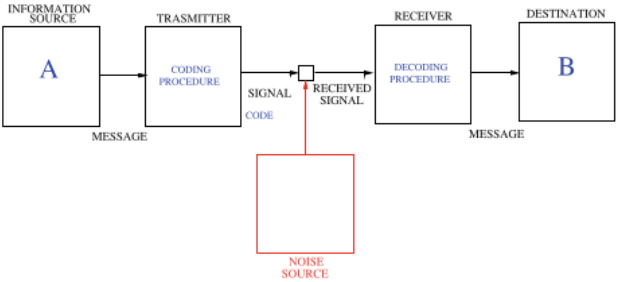


- In 1948, Claude E. Shannon wrote *"A Mathematical Theory of Communication"*, which marked the beginning of both Information and Coding Theory

- In 1950, Richard W. Hamming wrote *"Error Detecting and Error Correcting Codes"*, which was the first paper explicitly introducing error-correcting codes

Coding theory II

The general idea is that of adding some kind of redundancy to the message that we want to send over a communication channel



Digital Data

Digital data is sent as a series of ones and zeros.

- 1111010111110101010001101010101l1

Sometimes an error occurs:

- 11110111111101010100011010101011

We would like to be able to either detect or correct such errors.

Detection

- Good if we can request a resend of the data

Correction

- Needed if data cannot be resent (e.g. CD/DVD) or too costly to resend (e.g. deep space probe)

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Simple Error Detection

Most data is first bundled up into a group of bits before sending

- e.g. 4, 8, 32 or 64 bits at a time

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A simple detection trick is to add a parity bit

Suppose we wish to transmit 4 bits

- 0110

We add in an extra bit which signals whether the original data

has an even or odd number of ones

- The extra bit denotes the parity of the original bits

$$\begin{array}{ccc} 0110 & \longrightarrow & 0110\mathbf{0} \\ 1111 & \longrightarrow & 1111\mathbf{0} \\ 1000 & \longrightarrow & 1000\mathbf{1} \\ 1011 & \longrightarrow & 1011\mathbf{1} \end{array}$$

Simple Error Detection

The previous example can be described mathematically as follows.

Simple Error Detection

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We wish to send four message bits

$$m_1 m_2 m_3 m_4 \in \{0,1\}^4$$

To do this we add a fifth bit equal to

$$m_5 = m_1 \oplus m_2 \oplus m_3 \oplus m_4$$

where

$$x \oplus y = x + y \pmod 2$$

The resulting five bits is called a codeword

Simple Error Detection

We can now detect whether a single error has occurred.

Suppose you receive the following data using the previous example:

- 10101 Errors
- 01110 Errors
- 11101 No errors
- 11111 Errors
- 00000 No errors
- 00001 Errors

Trouble is we do not know where the errors occurred

Detecting errors - Hamming code I

Again sticking to four bits of message

$$m_1 m_2 m_3 m_4$$

The idea is to use multiple parity-check bits.

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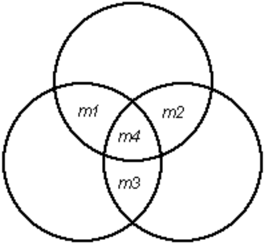
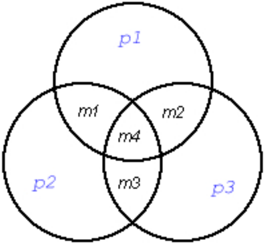
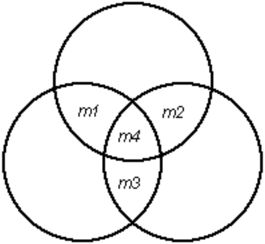
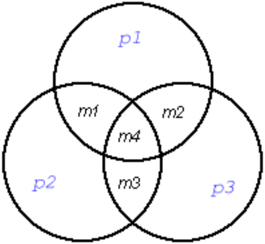
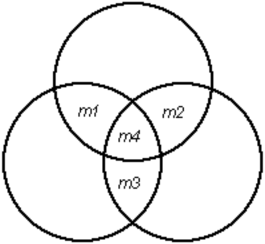
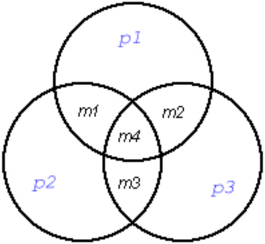




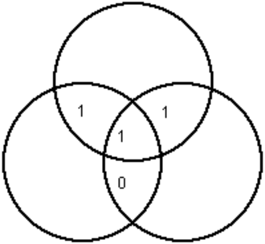

m1
m2
m4
m3

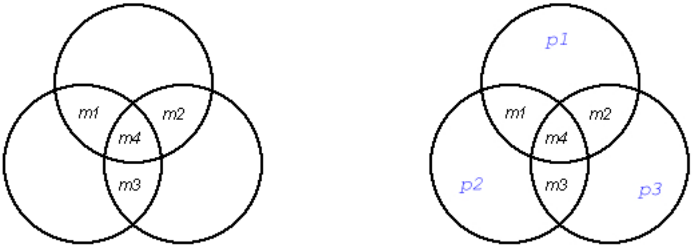

p1
m1
m2
m4
p2
m3
p3





Suppose $\mathbf{m} = 1101$ is the message





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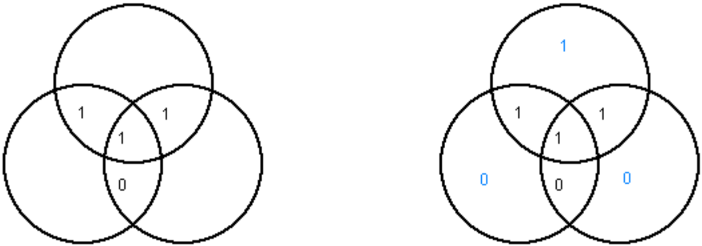
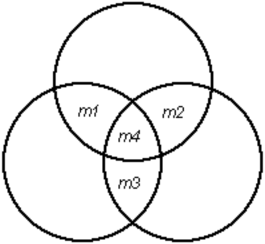
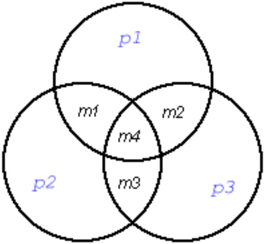
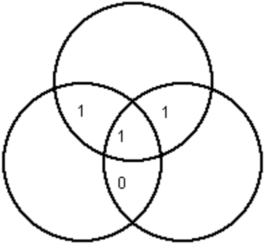
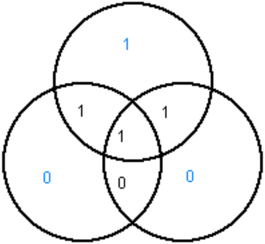






Suppose $\mathbf{m} = 1101$ is the message $\rightarrow 1101100$ is the codeword





Detecting errors - Hamming code III

Suppose that after transmission one symbol is flipped and $\mathbf{r} = 1001100$ is received.

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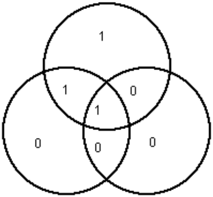

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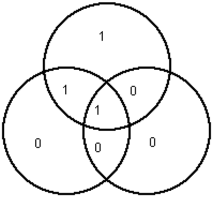

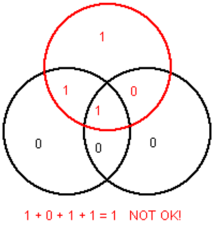

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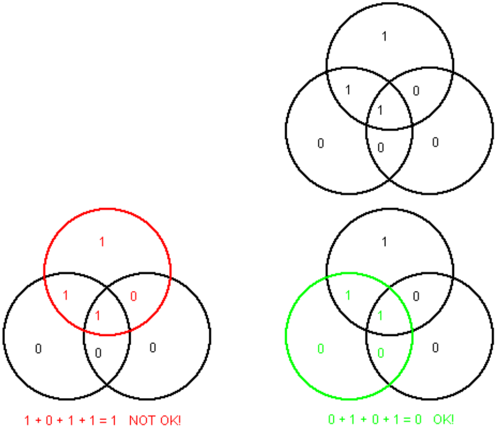

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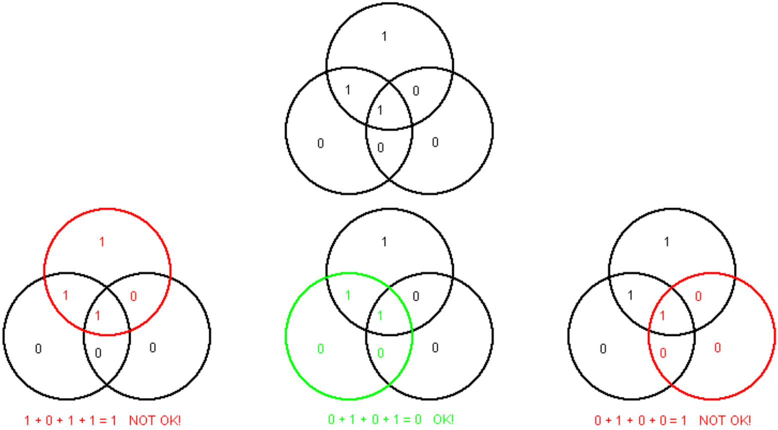
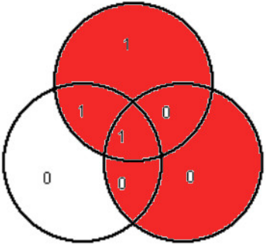


Correcting errors - Hamming code IV

$\mathbf{c} = 1101100$ and $\mathbf{r} = 1001100$



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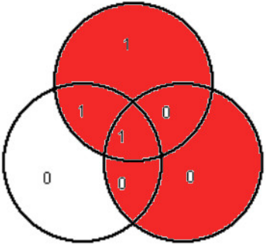

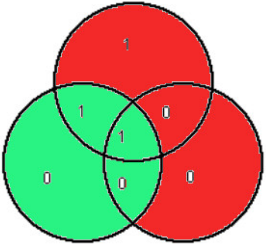

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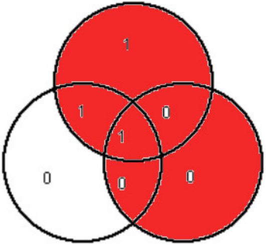

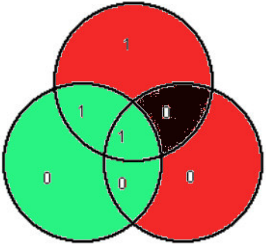

$\rightarrow$ the error is at m_2

Hamming code - Basic idea

- Use multiple parity bits, each covering a subset of the message bits

$$
\begin{aligned}
m_1 + m_4 + m_2 + p_1 = 0 \quad &\longrightarrow \quad \{m_1, m_4, m_2, p_1\} = C_1 \\
m_1 + m_3 + m_4 + p_2 = 0 \quad &\longrightarrow \quad \{m_1, m_3, m_4, p_2\} = C_2 \\
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- No two message bits belong to exactly the same subsets. In this way it is possible to correct one error and to detect two errors.

Hamming code - Basic idea

- Use multiple parity bits, each covering a subset of the message bits

$$m_1 + m_4 + m_2 + p_1 = 0 \longrightarrow \{m_1, m_4, m_2, p_1\} = C_1 \; OK$$
$$m_1 + m_3 + m_4 + p_2 = 0 \longrightarrow \{m_1, m_3, m_4, p_2\} = C_2$$
$$m_2 + m_4 + m_3 + p_3 = 0 \longrightarrow \{m_2, m_4, m_3, p_3\} = C_3 \; ERRORS$$

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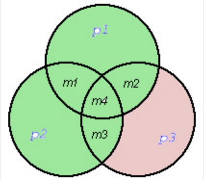
- No two message bits belong to exactly the same subsets. In this way it is possible to correct one error and to detect two errors.

Example

Suppose two errors occurred, at m_2 and p_1. Then:

- The code detects that some errors occurred
- The code concludes the error is at p_3, introducing an extra error



Hamming code

Return

- Enc : $\{0,1\}^4 \to \{0,1\}^7$ that maps the 2^4 strings of 4 bits $\mathbf{m}$ into a codeword $\mathbf{c}$
- We can write down all the codewords:

Information bits	Codeword	Information bits	Codeword
0000	0000000	1000	1000110
0001	0001111	1001	1001001
0100	0010101	1010	1010101
0011	0011100	1011	1011010
0010	0010011	1100	1100011
0101	0101010	1101	1101100
0110	0110110	1110	1110000
0111	0111001	1111	1111111

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0010	0010011	1100	1100011
0101	0101010	1101	1101100
0110	0110110	1110	1110000
0111	0111001	1111	1111111

- C contains 16 codewords of length 7

Block codes - Notation I

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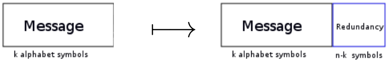
- We consider codes C over $\mathcal{A}$. If $q = 2$ the code is called binary
- Let Enc be an injective map:

$$\text{Enc} : \mathcal{A}^k \longrightarrow \mathcal{A}^n \qquad C \text{ is the image of Enc}$$



the entire block is called codeword

- n is the length of a codeword

Definition

A block code is a code with fixed length n, i.e. a non-empty subset of $\mathcal{A}^n$

- If a block code $C \subseteq \mathcal{A}^n$ contains $M = q^k$ codewords, then M is the size of C

Block codes - Notation II

A block code of length n and size M is denoted by (n, M)-code

- $k = \log_q(M)$ message length
- $n - \log_q(M)$ redundancy
- $R = \frac{\log_q(M)}{n}$ information rate
  Average amount of real information in each block of n symbols transmitted over a channel

Hamming

Example

The Hamming code we have seen before is a binary $(7, 16)$ block code with information rate $4/7$.

Definition (Hamming distance)

Given two strings $\mathbf{x}$ and $\mathbf{y} \in \mathcal{A}^n$, the **Hamming distance** between $\mathbf{x}$ and $\mathbf{y}$ is

$$d(\mathbf{x}, \mathbf{y}) = |\{i | x_i \neq y_i\}|.$$

Example

$\mathbf{v}_1 = 01011$
$\mathbf{v}_2 = 11110$

$d(\mathbf{v}_1, \mathbf{v}_2) = 3$

$\mathbf{w}_1 = 3211$
$\mathbf{w}_2 = 0213$

$d(\mathbf{w}_1, \mathbf{w}_2) = 2$

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Definition (Code distance)

The (Hamming) **minimum distance of a code** C is given by

$$d(C) = min\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}.$$

Definition (Hamming weight)

The **Hamming weight** of a string $\mathbf{x}$, $\mathrm{wt}(\mathbf{x})$, is defined as the number of non-zero symbols in the string.

Decoding problem

Why is the distance of a code important?

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Let C be an (n, M) code and suppose that a codeword $\mathbf{c}$ is sent over a noisy channel:

$\mathbf{c}$

$\mathbf{r}$

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1. if $\mathbf{r} \in C$ then no correction is needed 

Decoding problem

Why is the distance of a code important?

Let C be an (n, M) code and suppose that a codeword $\mathbf{c}$ is sent over a noisy channel:

1. if $\mathbf{r} \in C$ then no correction is needed 
2. if $\mathbf{r} \notin C$, then some errors occurred 

Decoding problem - Why is $d(C)$ important?

If $\mathbf{r} \notin C$: the decoder has to find the codeword c that has been sent

A possible strategy is the Maximum Likelihood Decoding (MLD): find the most likely codeword transmitted, i.e. the codeword $\mathbf{c}$ which maximizes the probability that $\mathbf{r}$ is the received word given that $\mathbf{c}$ has been sent.

We will see that for some types of channel MLD is equivalent to finding the coderword $\mathbf{c}$ closest to $\mathbf{r}$ in the Hamming distance (Nearest neighbour decoding):

$$\min_{\mathbf{c} \in C} d(\mathbf{r}, \mathbf{c})$$

From now on we will assume a type of channel such that we can use the minimum distance decoding to perform MLD

Decoding problem - Why is $d(C)$ important?

Let $\mathbf{x} \in \mathcal{A}^n$ and $t \in \mathbb{N}$, define

$$\mathcal{B}_t(\mathbf{x}) = \{\mathbf{y} \in \mathcal{A}^n \mid d(\mathbf{x}, \mathbf{y}) \leq t\}$$

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$$\mathcal{B}_t(\mathbf{x}) = \{\mathbf{y} \in \mathcal{A}^n \mid d(\mathbf{x}, \mathbf{y}) \leq t\}$$

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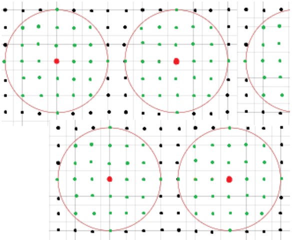

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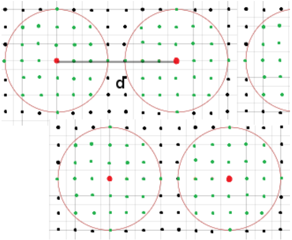

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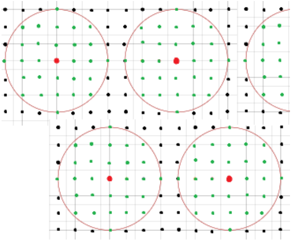

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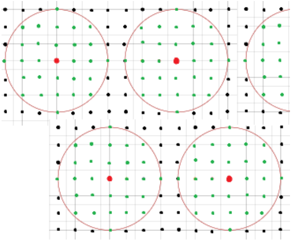

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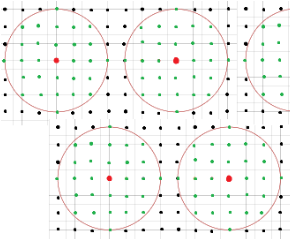

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- if $\mathbf{r}$ = red word (codeword), then no errors
- if $\mathbf{r}$ = green word, we should correct it to the red coderword that is the center of the ball it lies in
- if $\mathbf{r}$ = black word, then we are not able to correct because, if we increase the radii, balls would overlap

Error correction and error detection capability

More formally, we have the following definition:

- The error detection capability of a code C is the number e of errors that the code can detect. A *e-error detecting* code has minimum distance $d = e + 1$.

- The error correction capability of a code C is the number of errors that the code can correct. A *t-error detecting* code has minimum distance d such that $d = \lfloor \frac{t-1}{2} \rfloor$.

Erasures

In a similar way we can define the erasure correction capability of a code. An **erasure** occurs when a transmitted symbol is unreadable and at its place an extra symbol ϵ is introduced.

Example

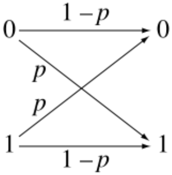
$\mathbf{c} = 1001100 \longrightarrow \mathbf{r} = 100\epsilon100$

- A code can correct s erasures if $s < d$
- The condition for simultaneous correction of t errors and s erasures is

$$d \geq 2t + s + 1.$$

Binary Symmetric Channel (BSC)

$\mathcal{X} = \{0, 1\}$ input alphabet and $\mathcal{Y} = \{0, 1\}$ output alphabet.
A BSC is parametrized by the probability p, $0 \leq p < 1/2$, that an input bit is flipped. p depends on the noise level and is called *crossover probability*.



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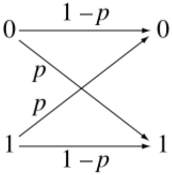

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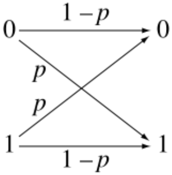

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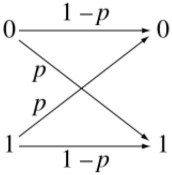

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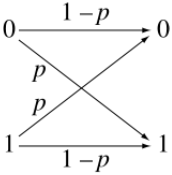

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- $\Pr(\mathbf{c}|\mathbf{c}) = (1 - p)^5$
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Suppose $\mathbf{c}$ is transmitted codeword and $\mathbf{r}$ is received word $\rightarrow \mathbf{c} = \mathbf{r} + \mathbf{e}$

Binary Symmetric Channel

Suppose $\mathbf{c}$ is transmitted codeword and $\mathbf{r}$ is received word $\rightarrow \mathbf{c} = \mathbf{r} + \mathbf{e}$
Given two codewords $\mathbf{c}_1, \mathbf{c}_2$, then

$$\begin{aligned} \Pr(\mathbf{r}|\mathbf{c}_1) \leq \Pr(\mathbf{r}|\mathbf{c}_2) &\iff d(\mathbf{r}, \mathbf{c}_1) \geq d(\mathbf{r}, \mathbf{c}_2) \\ &\iff \mathrm{wt}(\mathbf{r} + \mathbf{c}_1) \geq \mathrm{wt}(\mathbf{r} + \mathbf{c}_2) \\ &\iff \mathrm{wt}(\mathbf{e}_1) \geq \mathrm{wt}(\mathbf{e}_2) \end{aligned}$$

The most likely codeword sent is the one corresponding to the error of smallest weight

Do we need more structure?

Binary Hamming code $(7, 16)$: $\text{Enc} : \{0,1\}^4 \to \{0,1\}^7$

Information bits	Codeword	Information bits	Codeword
0000	0000000	1000	1000110
0001	0001111	1001	1001001
0100	0010101	1010	1010101
0011	0011100	1011	1011010
0010	0010011	1100	1100011
0101	0101010	1101	1101100
0110	0110110	1110	1110000
0111	0111001	1111	1111111

We need $n \cdot 2^k$ bits to store a binary code $\text{Enc} : \{0,1\}^k \to \{0,1\}^n$

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We need extra structure that would facilitate a succinct representation of the code

Can we do better?

Mathematically we can describe the $(7,16)_2$ Hamming code by a matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

so that, if we represent a message by the vector $\mathbf{m} = (m_1\ m_2\ m_3\ m_4)$, we can encode by computing

$$\mathbf{c} = \mathbf{m} \cdot G$$

Suppose we wish to transmit $\mathbf{m} = (1\ 0\ 1\ 0)$, we then compute

$$(1\,0\,1\,0) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} = (1\,0\,1\,0\,1\,0\,1)$$

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Linear codes - Definition

The previous example is an example of **linear code**.

Definition (Linear code)

Let q be a prime power. Then $C \subseteq \{0, 1, \ldots, q-1\}^n = \mathbb{F}_q^n$ is a linear code if it is a linear subspace of $\mathbb{F}_q^n$. If C has dimension k and distance d then it will be referred to as an $[n, k, d]_q$ or just an $[n, k]_q$ code.

- $\mathbb{F}_q^n$ denote the vector space of all n-tuples over the finite field $\mathbb{F}_q$.