1 Intersection of regular languages ($\star$)

Let $L = \{w \mid w$ has an even number of '$\texttt{a}$'s and one or two '$\texttt{b}$'s$\}$ over the alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$. L is the intersection of two simpler languages. Give DFAs for these two languages and use the generic construction for the intersection of regular languages to construct a DFA for L.

2 Constructing DFAs ($\star$)

Construct DFAs for the following languages over the alphabet $\{\texttt{0}, \texttt{1}\}$.

1. $\{\varepsilon, \texttt{0}\}$
2. $\{w \mid w$ begins with a $\texttt{1}$ and ends with a $\texttt{0}\}$
3. $\{w \mid w$ contains the substring $\texttt{0101}$, i.e. $w = u\texttt{0101}v$ for some $u, v\}$
4. $\{w \mid w$ has length at least 3 and the third symbol is a $\texttt{0}\}$
5. $\{w \mid w$ is any string except $\texttt{11}$ or $\texttt{111}\}$
6. $\{w \mid$ at every odd position of w there is a $\texttt{1}\}$

3 Constructing NFAs ($\star$)

Give NFAs for the following languages with the number of states specified over the alphabet $\Sigma = \{a, b\}$.

1. $\{w \mid w$ ends with $\texttt{aa}\}$ with 3 states
2. $\{w \mid w$ contains the substring $\texttt{abab}\}$ with 5 states
3. $\{w \mid w$ contains an even number of '$\texttt{a}$'s or exactly two '$\texttt{b}$'s$\}$ with 6 states
4. $\{\texttt{a}\}$ with 2 states

5. ($\star\star$) Can you construct DFAs for the above languages with the same number of states? Why not?

4 NFAs need only one accept state

For every NFA there is an equivalent NFA with exactly one accept state.

1. ($\star$) Give a construction to turn an NFA into one with exactly one accept state. (Note: don't forget the case when your original NFA has no accept states at all.)
2. ($\star\star$, optional) Formally prove that your construction works.

5 k-fold repetition is regular ($\star$)

Let $B_n := \{a^k \mid k \text{ is a multiple of } n\}$. Show that for every $n \geq 1$ the language B_n is regular. (Note: The aim of this problem is to describe a DFA or NFA for a given n abstractly (but not necessarily formally) since you can't draw a single diagram that covers all cases of n.)

6 Odd one out ($\star\star$)

Let L be the language of all strings over the alphabet $\{\mathtt{0}, \mathtt{1}\}$ that do not contain a pair of `1`s that are separated by an odd number of symbols. Construct a DFA for L with exactly 5 states.

Hints:

- How many `1`s can a word in L have maximally?
- You may find it helpful to construct an NFA with 4 states for the complement of L.

7 Reversing is regular ($\star\star$, optional)

For a word w, let $w^{\mathcal{R}}$ be the reverse of w, i.e. the reverse of $w = w_1 w_2 \dots w_{n-1} w_n$ is $w_n w_{n-1} \dots w_2 w_1$. For a language L, let $L^{\mathcal{R}} := \{w^{\mathcal{R}} \mid w \in L\}$.

Show that if L is a regular language then so is $L^{\mathcal{R}}$. For an extra challenge you can try and prove this formally.