COMS10003 : Workshop on Logic

Introduction to Propositional Logic

Kerstin Eder

October 15, 2014

Introduction

For this workshop you should read up on *Propositional Logic*, covering the following topics:

- Introduction to Propositional Logic, including syntax and semantics;
- Truth tables for compound propositions;
- Normal forms.

Reading will help you find solutions to the tasks in this worksheet.

Note, this worksheet contains tasks on several topics related to Propositional Logic in order of increasing difficulty for each topic. Schedule your work so that you find an answer to those parts of the worksheet that enable you to solve the rest of the questions alone.

For each task and subtask, use the whiteboard to present and to discuss your solutions; take turns. Active participation is required from all group members.

Preparation

Design your own syntax/semantics reference card. It will help you remember the symbols used for the different connectives. If necessary, add the truth tables to this card for those connectives that you find hard to remember.

Before you start, compare the reference cards within your group. See whether you can improve your card based on what you have seen.

Task 1: Formalization

Task 1.1: Express the following statements in propositional logic, using:

p : It is raining.
q : It is snowing.
r : It is freezing.

Answer: *It is important that you discuss why you have chosen to map the English language statement to a certain formal statement. In particular, you may find that opinions differ within the group, e.g. whether you should use disjunction (normal "or") or exclusive or ("xor").*

1. It is raining.
   Answer: p
2. It is not snowing.
   Answer: $\neg q$
3. It is either raining or snowing.
   Answer: $p \vee q$
4. It is raining, but it is not snowing.
   Answer: $p \wedge \neg q$
5. It is not both raining and snowing.
   Answer: $\neg(p \wedge q) = \neg p \vee \neg q$
6. If it is freezing then it is snowing.
   Answer: $r \Rightarrow p$
7. When it is snowing, then it is freezing.
   Answer: $q \Leftrightarrow r$
8. If it is not freezing then it is raining.
   Answer: $(\neg r \Rightarrow p) = (\neg p \Rightarrow r)$
9. If it is freezing or snowing then it is not raining.
   Answer: $(r \vee q) \Rightarrow \neg p$
10. Either it is not raining or, if it is not raining, then it is snowing.
    Answer: $\neg p \vee (\neg p \Rightarrow q) = \neg p \vee (p \vee q) = TRUE,$ *a tautology*
11. Either it is not raining or, if it is not raining but freezing, then it is snowing.
12. It is neither raining nor snowing.
13. It is not the case that, if it is snowing, then it is not snowing.
14. It is raining if and only if it is not snowing.
15. If it is both raining and freezing, then it is snowing.

Task 1.2: Express the following propositions in clear English, using:

p : It is sunny.
q : It is raining.
r : I play tennis.
s : I go swimming.

1. $p \Rightarrow r$
   Answer: *If it is sunny, I play tennis.*
2. $q \Rightarrow \neg r$
   Answer: *If it is raining, I don't play tennis.*
3. $p \Leftrightarrow q$
   Answer: *It is sunny if and only if it is raining.*
4. $p \Rightarrow (r \vee s)$
   Answer: *If it is sunny, I either play tennis or go swimming.*
5. $(p \vee q) \Rightarrow s$
   Answer: *If it is either sunny or raining, I go swimming.*

Task 1.3: State the *converse*, *inverse* and the *contrapositive* of each of the following propositions:

Answer:

Statement:	$p \Rightarrow q$
Converse:	$q \Rightarrow p$
Inverse:	$\neg p \Rightarrow \neg q$
Contrapositive:	$\neg q \Rightarrow \neg p$

1. If we have frost tonight, then I won't cycle in tomorrow.
   Answer: *Converse: If I don't cycle tomorrow, then we will have frost tonight.*
   Inverse: If we don't have frost tonight, then I will cycle in tomorrow.
   Contrapositive: If I cycle tomorrow, we won't have frost tonight.
2. My cat comes in whenever it is hungry.
   Answer: *Note that this statement has the "if" part last, i.e. it could be rephrased to: "If it is hungry, then my cat comes in."*
   Converse: Whenever my cat comes in, it is hungry.
   Inverse: Whenever my cat does not come in, it is not hungry.
   Contrapositive: Whenever my cat doesn't come in, it is not hungry.
3. When you hear the fire alarm, you need to vacate the building.
4. If it is hot tomorrow, then we will go swimming.
5. People who don't pay their tax by the deadline will be fined.

Task 2: Syntax

Task 2.1: Eliminate as many brackets as possible:

1. $(((p \vee (\neg q)) \vee s) \Leftrightarrow ((\neg(s \wedge (\neg r)))))$

   Answer: $(((p \vee (\neg q)) \vee s) \Leftrightarrow ((\neg(s \wedge (\neg r)))))$
   $= (p \vee \neg q \vee s) \Leftrightarrow \neg(s \wedge \neg r)$
   $= p \vee \neg q \vee s \Leftrightarrow \neg(s \wedge \neg r)$
   (Note that this can now be simplfied, but the brackets in the rhs expression are important.)

2. $((((\neg p) \vee s) \wedge q) \vee (\neg(\neg(\neg s))))$

   Answer: $((((\neg p) \vee s) \wedge q) \vee (\neg(\neg(\neg s))))$
   $= ((\neg p \vee s) \wedge q) \vee \neg\neg\neg s$
   (Again, this can now be simplified.)

Task 2.2: Based on the syntax of *Propositional Logic* as presented in the lecture, develop a systematic method to decide whether or not a given string of symbols from the alphabet (as given in the definition in the lecture) is a syntactically correct proposition.

Hint: You may want to define a function that takes a string of symbols from our alphabet as argument, identifies (matches on) the top-level connective symbol and recursively traverses the argument(s) of that connective until either a propositional variable or a truth value is reached, or a syntactic error is encountered.

Answer: *(Solution Hint)*

In lectures, we gave a formal definition of propositional logic and concluded it consists of two parts, the alphabet and grammar. Based on this we defined what constitutes a well formed formula. Deciding whether a string (which can be thought of a sequence of symbols from the alphabet) is part of the language (i.e. syntactically correct), follows the same pattern as the computation of its truth-value.

Given an input string like $(\neg(p \vee q) \Rightarrow (r \wedge p))$*, a recursive function that determines the truth value of a proposition could work like this, given that* $v(p) = t$*,* $v(q) = f$ *and* $v(r) = t$*, where* v *is a function from a set of propositional variables to the set* $\{t, f\}$*:*

decide($(\neg(p \vee q) \Rightarrow (r \wedge p))$*) depends on*

decide($\neg(p \vee q)$*) which depends on*

decide($(p \vee q)$*) which depends on*

decide(p)=true

and decide(q)=true

so decide($(p \vee q)$*)=true*

so decide($\neg(p \vee q)$*)=*$false$

and decide($(r \wedge p)$*) which depends on*

decide(r)=true

and decide(p)=true

so decide($(r \wedge p)$*)=true*

so decide($(\neg(p \vee q) \Rightarrow (r \wedge p))$*)=true*

Use the above to write a similar method to determine whether or not a given string of symbols is syntactically correct.

Task 3: Truth Tables

Task 3.1: How can you determine the truth table for a compound proposition? Explain in simple steps what you need to do.

Answer: *The key thing is to start with the innermost part of the proposition first. Each propositional variable is represented as one column in the truth table, and then the inner statements consisting of these variables and connectives need to be evaluated, which can be done by putting each inner statement in a new column of the truth table. We can then regard the inner clauses as propositional variables, which make up the outer clauses and can continue as described above. This way we gradually build up the entire compound proposition, which will go in the final column on the right hand side of the table.*

Task 3.2: Give the truth table for each of the following propositions:

1. $p \Leftrightarrow \neg p$

   Answer:

p	$\neg p$	$p \Leftrightarrow \neg p$
T	F	F
F	T	F

2. $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$
3. $(p \vee \neg q) \Rightarrow (r \wedge p)$
4. $(p \Leftrightarrow \neg q) \vee (q \Rightarrow p)$

   Answer:

p	p	$(p \Leftrightarrow \neg q) \vee (q \Rightarrow p)$
T	T	T
T	F	T
F	T	T
F	F	T

5. $\neg(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$

   Answer:

p	q	$\neg(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$
T	T	F
T	F	F
F	T	F
F	F	F

6. $(p \Leftrightarrow (p \wedge \neg p)) \Leftrightarrow \neg p$

   Answer:

p	$\neg p$	$p \wedge \neg p$	$p \Leftrightarrow (p \wedge \neg p)$	$(p \Leftrightarrow (p \wedge \neg p)) \Leftrightarrow \neg p$
T	F	F	F	T
F	T	F	T	T

7. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

Task 3.3

1. For two propositonal variables, p and q, how many assignments of the truth values true (T) and false (F) exist?

   Answer: *Each variable can either be true or false, and the total number of assignments is therefore the sum of all possible combinations of assignments of the individual propositional variables. Another way to think about it: If we represent p and q as bits, where 1 means true and 0 means $false$, then we have a 2-bit string. One bit can represent two numbers, 1 and 0. A 2-bit string can therefore represent $2 * 2 = 2^2 = 4$ numbers, which is also the total number of assignments for p and q.*

p	q	*Possible bit representation*
T	T	11
T	F	10
F	T	01
F	F	00

2. Let $V = \{T, F\}$ be the set of truth values.
   A function from V^n, the set $\{(v_1, v_2, \ldots, v_n) \mid v_i \in V,\ where\ 1 \leq i \leq n\}$, to V is called a function of degree n.

   For instance, $\wedge$ is a function from V^2, the set $\{(T,T), (T,F), (F,T), (F,F)\}$, to $\{T, F\}$, such that $\wedge(T,T) = T$, $\wedge(T,F) = F$, $\wedge(F,T) = F$ and $\wedge(F,F) = F$.

   How many different functions of degree 2 exist?

   Answer: *One should note that the function V^n to V is essentially a way to represent truth tables. n is the number of propositional variables, each of which is represented as a column in the truth table. The final column represents the result of some logical operation on all possible combination of values of the propositional variables, and the values of its rows correspond to the set V. A function of degree 2 operates on two propositional "input" variables, one such example would be $\wedge$. The question is now how many different assignments are there for the final column. The answer is 16, as there are 4 rows, each of which can be either true or $false$. We can think of each row of the result column as one bit in a 4-bit string, and we know that a 4-bit can represent $2 * 2 * 2 * 2 = 2^4 = 16$ different numbers.*

3. Give a truth table that shows all these functions of degree 2 in some natural order. Explain your approach.

   We have discussed the truth tables for negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), xor ($\oplus$), implication ($\Rightarrow$) and equivalence ($\Leftrightarrow$). Identify the columns in your table that correspond to the semantics of these connectives.

   Which other important connectives can you find in your truth table? List these and discuss their semantics.

Answer:

p	q	$\bot$	$\downarrow$	$\not\Leftarrow$	$\neg p$	$\not\Rightarrow$	$\neg q$	$\oplus$	$\uparrow$	$\wedge$	$\Leftrightarrow$	q	$\Rightarrow$	p	$\Leftarrow$	$\vee$	$\top$
T	T	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
T	F	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
F	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
F	F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T

Without a systematic approach it is very easy to miss out one of the columns. By thinking about it in terms of a binary number, one could easily see each column as a 4-bit string. Counting up, from 0 to 15, and substituting T and F for 0 *and* 1 *one, one would come up with the table above. The other connectives include projection (p and q), which evaluate to true, whenever p and q are true respectively.* $\top$ *and* $\bot$ *denote tautology and contradiction,* $\Leftarrow$ *is converse implication,* $\downarrow$ *and* $\uparrow$ *are* NOR *and* $NAND$, $\not\Rightarrow$ *represents material nonimplication (or abjunction), and* $\not\Leftarrow$ *is converse nonimplication.*

4. In general, for a compound proposition with n propositional variables, how many assignments of truth values exist? In general, how many different functions of degree n exist? Discuss.

   Answer: *For two propositional variables, there are* $2^2 = 4$ *different combinations of their individual truth values, and as we saw above, there are* 16 *functions of degree* 2 *or in other words, there are 16 possible assignments of truth values for a compound proposition with 2 variables. With 3 propositional variables, there are are* $2^3 = 8$ *different combinations of their truth values, leading to a total of* 2^{2^3} *possible assignments for a compound proposition with* 3 *variables. For n propositional variables, there are* 2^n *possible combinations of their truth values, and* 2^{2^n} *functions* V^n *to* V, *or* 2^{2^n} *possible assignment of truth values.*

Task 4: Normal Forms

Task 4.1: Propositions can be expressed in what is called *Normal Form*. We distinguish two types of normal form, the *Conjunctive Normal Form* and the *Disjunctive Normal Form*.

1. Give a definition for *Conjunctive Normal Form*.

   Answer: *A formula in Conjunctive Normal Form is a conjunction of disjunctions of atomic propositions or negations of atomic propositions, i.e. it is an AND of ORs.*
   Example: $\neg A \wedge (B \vee C)$

2. Give a definition for *Disjunctive Normal Form*.

   Answer: *A formula in Disjunctive Normal Form is a disjunction of conjunctions of atomic propositions or negations of atomic propositions, i.e. it is an OR of ANDs.*
   Example $(A \wedge B) \vee C$

   Note that these definitions allow degenerate *conjunctions and disjunctions (i.e. conjunctions with* one *conjunct and disjunctions with* one *disjunct). For instance,* A *is already in CNF and it also is in DNF. Likewise,* $A \wedge B$ *is in CNF and also in DNF.*

   Note further that, if we allow repetitions of atomic propositions and their negations, then a contradiction, such as $A \wedge \neg A$*, already is in DNF.*

3. Based on a truth table for a given compound proposition, state how to derive the *Disjunctive Normal Form* of that proposition.

   Answer: *Given a compound proposition like* $\neg(r \vee (q \wedge (\neg r \Rightarrow \neg p)))$*, to derive its DNF we begin by giving the truth table for the statement:*

p	q	r	$\neg(r \vee (q \wedge (\neg r \Rightarrow \neg p)))$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

   For each of the three true cases, we form the conjunction that gives a T *value only in that case:*

   (a) $p \wedge q \wedge \neg r$

   (b) $p \wedge \neg q \wedge \neg r$

(c) $\neg p \wedge \neg q \wedge \neg r$

Then we form the disjunction of these three conjunctions to obtain the answer:
$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$

Task 4.2: Write the following propositions in Disjunctive Normal Form:

1. $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$
   Answer: $p \vee (q \wedge r)$ *and* $(p \wedge r) \vee (q \wedge r)$
2. $(p \vee \neg q) \Rightarrow (r \wedge p)$
3. $(p \Leftrightarrow \neg q) \vee (q \Rightarrow p)$
4. $\neg(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$
5. $(p \Leftrightarrow (p \wedge \neg p)) \Leftrightarrow \neg p$
6. $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

Task 4.3: A set of connectives is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition that uses connectives only from that set.

Identify the set of connectives used in Disjunctive Normal Form. Is this set *functionally complete*? Is this the smallest set of functionally complete connectives?

Answer: *The set of connectives used in Disjunctive Normal Form is functionally complete, but it is not the smallest possible such set. A set consisting only of $NAND$ or NOR connectives is also functionally complete, and contains only a single logical connective.*