1. McDonalds recently negotiated a large purchasing deal for fish, chicken, and beef. They have agreed to purchase 40 million tons of fish, 40 million tons of chicken, and 100 million tons of beef. As such, they want to create an advertising campaign to ensure that consumers eat the correct portion of each meat product.

   After paying to have a commercial produced, McDonalds collects the following data: After watching the commercial once, a person who initially wanted fish now has a 10% chance of buying a fish product, a 60% chance of buying a beef product, and a 30% chance of buying a chicken product; after watching, a person who initially wanted to buy a beef product has a 20% chance of buying a fish product, a 60% chance of buying a beef product, and a 20% chance of buying a chicken product; after watching, a person who initially wanted to by a chicken product has a 40% chance of buying a fish product, a 50% chance of buying a beef product, and a 10% chance of buying a chicken product.

   (a) If the vector $\vec{e}_1$ represents a person who wants to buy a fish product, $\vec{e}_2$ represents a person who wants to buy a beef product, and $\vec{e}_3$ represents a person who wants to buy a chicken product, find a matrix M such that $M\vec{e}_i$ gives the probability of buying fish, beef, or chicken after watching the commercial once.

   (b) Compute the eigenvalues and eigenvectors of M.

   (c) Assume that a fish product takes 50 grams of fish, a beef product takes 50 grams of beef, and a chicken product takes 50 grams of chicken. Further, assume that each time a person watches the commercial it has the same impact (i.e., watching the commercial twice means the likelihood of buying a particular product is given by M^2). If McDonalds ensures that the average customer sees the commercial 3000 times, what are the relative proportions of fish, beef, and chicken McDonalds expects to sell?

   (d) Should McDonalds run the ad? Does the initial population's preferences for fish, beef, or chicken matter? *Explain your reasoning.*

2. Throughout this problem, let P be an $n \times n$ *stochastic matrix.*

   (a) Prove that if $\vec{q}$ is a *probability vector*, then $P\vec{q}$ is also a probability vector.

   (b) Prove that P^k is a stochastic matrix for all $k \geq 0$.

   (c) A *left eigenvector* for P is a non-zero row vector $\vec{w}$ so that $\vec{w}P = \lambda\vec{w}$.
   Does P have a left eigenvector with eigenvalue 1? Why or why not?

   (d) Show that the left and right eigenvectors of P may be different but the left and right eigenvalues of P must be the same. *Hint: you may use facts from linear algebra about determinants and transposes.*

   (e) The *unit n-symplex* is the set of all convex linear combinations of $\vec{e}_1, \dots, \vec{e}_n$. Let S be the unit n symplex, and let $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $\mathcal{P}(\vec{x}) = P\vec{x}$. Show that $\mathcal{P}(S) \subseteq S$. *Hint: start by showing that vectors in S are probability vectors.*

   (f) The Brouwer Fixed-point Theorem states that a continuous map from a simplex into itself has at least one fixed point. Use the Brouwer Fixed-point Theorem to show that P has at least one (right) eigenvector with eigenvalue 1 *which is also a probability vector.*

3. We're going to prove some linear algebra facts because, *just maybe* they'll be useful.

   Let P be an $n \times n$ stochastic matrix and let $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation induced by P (i.e., given by matrix multiplication). Let S be the unit n-symplex.

   (a) Draw S when $n = 1$, 2, and 3.

   (b) Prove that S is equal to the set of all probability vectors in $\mathbb{R}^n$.

   (c) Prove that if $\vec{p}, \vec{q} \in S$, then all convex linear combinations of $\vec{p}$ and $\vec{q}$ are in S.

(d) For the rest of this problem, assume $n \geq 2$.

The *boundary* of S, written ∂S, consists of all vectors in S where at least one coordinate is zero.

Let $\vec{a}, \vec{b} \in S$ be distinct points and let $\ell \subseteq \mathbb{R}^n$ be the line passing through $\vec{a}$ and $\vec{b}$. Prove that ℓ intersects the boundary of S.

(e) Prove that if $V \subseteq \mathbb{R}^n$ is a subspace of *dimension at least two*, then the following holds: if $V \cap S$ is nonempty, then $V \cap \partial S$ is non-empty.

(f) Prove that if $\vec{a}, \vec{b} \in S$ are distinct eigenvectors for P with eigenvalue 1, then there exists a $\vec{d} \in \partial S$ which is an eigenvector for P with eigenvalue 1.

4. Let $\mathcal{M} = (M_0, M_1, \ldots)$ be a stationary Markov chain on a graph $\mathcal{G}$ with n vertices, and let P be the (stochastic) transition matrix for $\mathcal{M}$. Further, suppose $\mathcal{M}$ is modeled by the dynamical system (T, Ω), where Ω is the space of probability distributions on the n vertices.

   (a) Produce examples where $\lim_{k\to\infty} P^k$ exists and does not exist. Can you find conditions on $\mathcal{M}$ and $\mathcal{G}$ so that $\lim_{k\to\infty} P^k$ always exists?

   (b) A *stationary distribution* for $\mathcal{M}$ is defined to be a fixed-point of (T, Ω). Produce examples where $\mathcal{M}$ has exactly 1, 2, and 3 stationary distributions.

   (c) Prove that a convex linear combination of stationary distributions for $\mathcal{M}$ is a stationary distribution for $\mathcal{M}$.

   (d) Prove that $\mathcal{M}$ always has *at least one* stationary distribution.

   (e) A Markov chain is called *primitive* if there exists a $k \in \mathbb{N}$ such that the probability of transitioning from state i to state j in exactly k steps is positive *for every i and j*.

   A distribution $\vec{d} \in \Omega$ is said to have *full support* if none of the entries in $\vec{d}$ are zero.

   Show that if $\mathcal{M}$ is primitive, then every stationary distribution for $\mathcal{M}$ must have full support.

   (f) Prove that if $\mathcal{M}$ is primitive, then $\mathcal{M}$ has a *unique* stationary distribution.

   (g) Show that if $\mathcal{M}$ is primitive and $\lim_{k\to\infty} P^k$ exists, then $P = [\vec{s}|\vec{s}|\cdots|\vec{s}]$, where $\vec{s}$ is the unique stationary distribution for $\mathcal{M}$.

Programming Problems

For the programming problems, please use the Jupyter notebook available at

https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2020-MAT-335-webpage&subPath=homework/homework2-exercises.ipynb

Make sure to comment your code and use "Markdown" style cells to explain what your answers.

1. `np.random.rand()` will generate a random number chosen uniformly from the interval $[0, 1]$. That means, if $x, y \in [0, 1]$ and $x < y$, the probability that the output of `np.random.rand()` lies in $[x, y]$ is $y - x$.

   Using `np.random.rand()`, create a function `pick_random` which inputs a list of the form `[(`⟨*probability*⟩`, `⟨*state*⟩`),...]` and returns one of the states chosen at random (but with the appropriate probability).

2. simulate markov chain and recover stationary distribution and transition probabilities. Do empirical vs. picking at a time.

3. Find the Markov chain with text analysis.