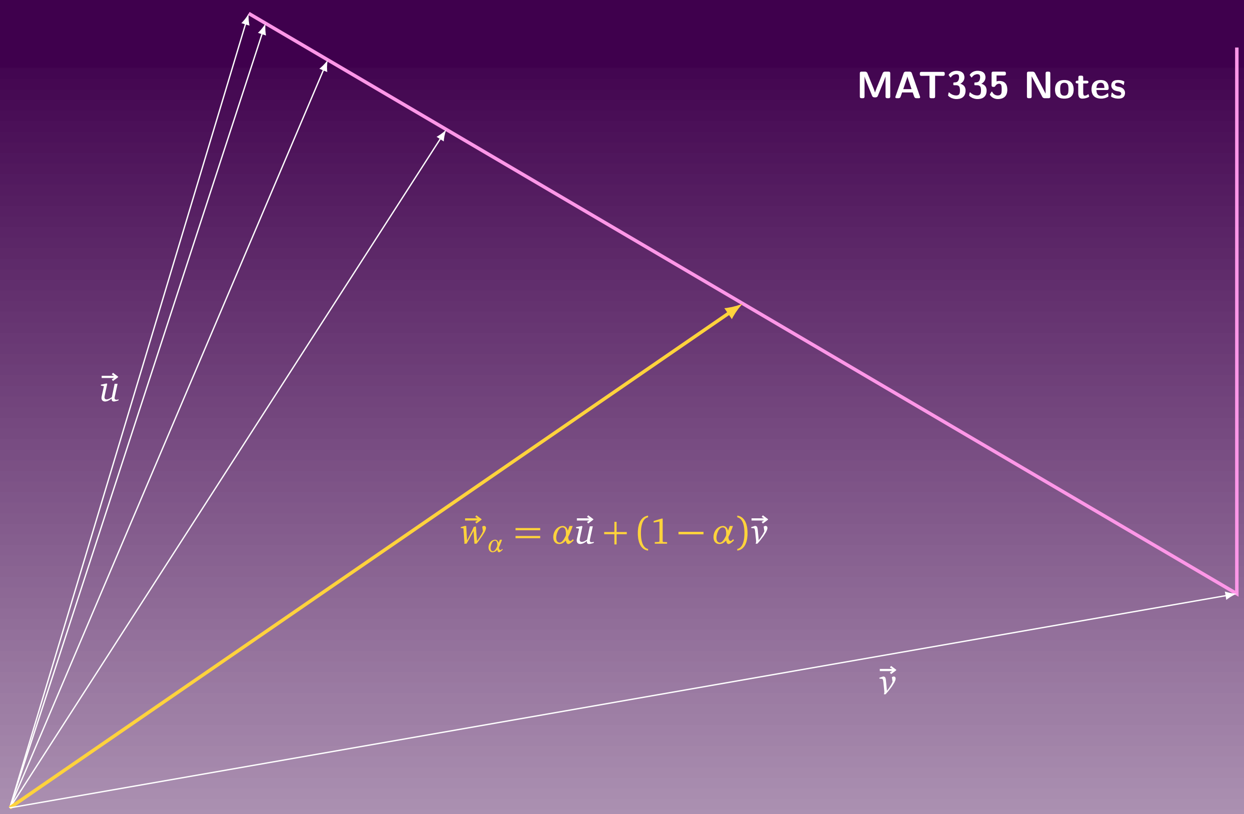

MAT335 Notes

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Dynamical Systems

Dynamical System

DEF A pair (T,X) is called a ***dynamical system*** if X is a set and $T : X \to X$ is a function. In the context of a dynamical system, T is often called a ***transformation***.

This definition is very general, and most things you encounter could be considered a dynamical system. For example, if $X = \{\text{air molecules and their positions on earth}\}$ and $T : X \to X$ is the result of the wind blowing for one second, then (T,X) is a dynamical system. Alternatively, we could take the state of your computer's memory (RAM) to be a set and your processor executing a single instruction to be a transformation.

It's hard to say much about general dynamical systems. However, throughout the course, we will find ways to classify dynamical systems. Once we "narrow the field", we'll be able to say lot's of interesting things.

Newton's Method

Newton's method is a way of using tangent-line approximations to functions to estimate their roots. It is an iterative procedure.[1]

---

1 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $T_f : \{\text{guesses}\} \to \{\text{guesses}\}$ be a single application of Newton's method.

1.1 Find a general formula for T_f.

1.2 Let $f(x) = x(x-2)(x-3)$. Compute $T_f^n(4)$ for $n = 0, 1, 2, 3$.

1.3 Do you think

$$\lim_{n\to\infty} T_f^n(4)$$

converges? If so, what does it converge to? Can you prove your answer?

Fixed Point

DEF Let (T,X) be a dynamical system. A point $a \in X$ is called a ***fixed point*** if $T(a) = a$.

Basin of Attraction

DEF Let (T,X) be a dynamical system and let $x \in X$. The ***basin of attraction*** of x is the set

$$A_x = \{y \in X : \lim_{n\to\infty} T^n y = x\}.$$

Eventually, we will talk about more general *basins of attraction*, but for now we will limit ourselves to that of a single point.

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2 Let $f(x) = x(x-2)(x-3)$ and let T_f be the function that applies a single iteration of Newton's method (as before).

2.1 Is $[3,4) \subseteq A_3$ for T_f? What about $[100, 1000]$? $(2,3)$?

2.2 Describe A_3.

2.3 Is A_3 connected?

[1] Whenever something is iterative, you should think dynamics!

Inverse Image

DEF

Let $f : A \to B$ be a function and let $X \subseteq B$. The ***inverse image*** of X under f, denoted $f^{-1}(X)$, is

$$f^{-1}(X) = \{x \in A : f(x) \in X\}.$$

Note: a function *need not* be invertible to have inverse images. In fact, the idea of inverse images applies to *every* function.

3

3.1 Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Find $g^{-1}(\{1\})$, $g^{-1}(\{4\})$, $g^{-1}(\{0\})$, $g^{-1}(\{-1\})$, and $g^{-1}([3,4])$.

3.2 Let f and T_f be as before. (Recall, $f(x) = x(x-2)(x-3)$). Find $T_f^{-1}([3,4))$.

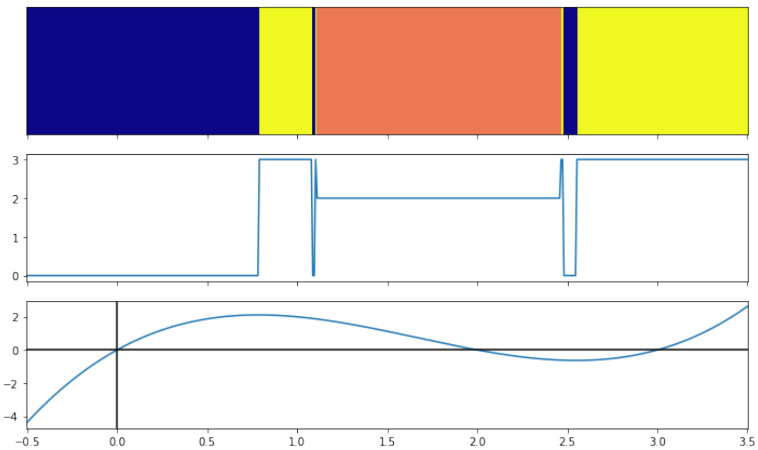
3.3 Define

$$Q = \bigcup_{n \geq 0} T_f^{-n}([3, 3.1))$$

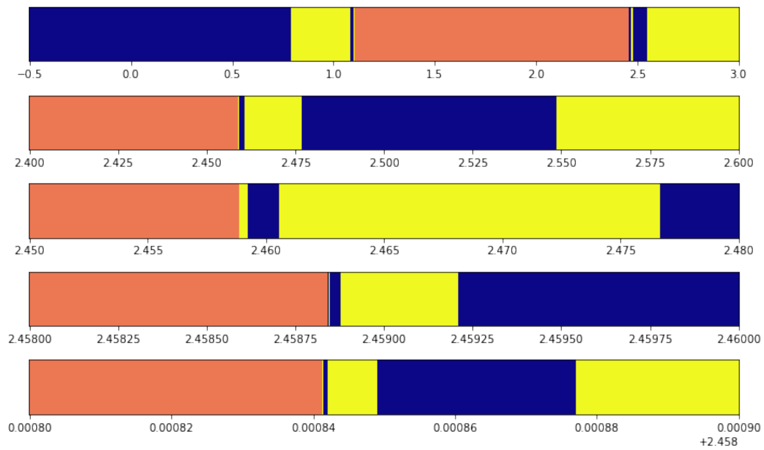
where T_f^0 is the identity function.

Is $Q = A_3$? Why or why not?

Using a computer, we can graph A_0, A_2, and A_3.



Zooming in around $x \approx 2.4588$:



We've just seen our first **fractal**! For now, we will define a *fractal* as a set with repeated patterns at all scales.

Fractals

Let's construct some famous fractals.

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4 Let K_0 be an equilateral triangle with sides of length 1. Let K_1 be the result of applying to each side of K_0. Repeat this process to get K_2 from K_1, etc. and define

$$K_\infty = \lim_{n\to\infty} K_n.$$

4.1 Draw K_0, K_1, and K_2.

4.2 Find the perimeter of K_0, K_1, and K_2. Find a general formula for the perimeter of K_n.

4.3 What is the perimeter of K_∞? What is the area enclosed by K_∞?

K_∞ is called the *Koch Snowflake*.

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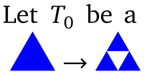
5 Let T_0 be a filled-in equilateral triangle. To get T_1, T_2, etc., apply the substitution rule  to each (sub)triangle of T_0, T_1, etc.. Define T_∞ to be the limit of this process.

5.1 Draw T_0, T_1, and T_2.

5.2 Find a formula for the area of T_n.

5.3 Compute the area of T_∞.

5.4 Is T_∞ the empty set? Why or why not?

T_∞ is called *Sierpinski's Triangle*.

---

6 Let $C_0 = [0,1]$ be the unit interval. Recursively define C_i by the substitution rule , which removes the middle $1/3$ of every interval. Define C_∞ to be the limit of this process.

6.1 Compute the length of C_n.

6.2 Compute the length of C_∞.

C_∞ is called the *Cantor set*.

Our normal sense of measurement fails when it comes to these fractals. We need a new idea: *similarity dimension*.

Dimension

Dimension can be thought of as a relationship between scale and content.[2]

7

7.1 Let $\ell = [0,1)$, $2\ell = [0,2)$, $3\ell = [0,3)$, etc.. How many disjoint copies of ℓ does it take to cover $n\ell$?

7.2 Let $S = [0,1)^2$, $2S = [0,2)^2$, etc.. How many disjoint copies of S does it take to cover nS?

7.3 Let $C = [0,1)^3$, $2C = [0,2)^3$, etc.. How many disjoint copies of C does it take to cover nC?

7.4 Based on the patterns you see, describe an algorithm that can be used to find the dimensions of ℓ, S, and C.

7.5 Let T be the filled-in equilateral triangle. Apply your algorithm to $2T$.

7.6 Let T_∞ be the Sierpinski triangle. Apply your algorithm to $2T_\infty$. Does the number you get make sense?

Similarity Dimension

DEFINITION

A set $Q \subseteq \mathbb{R}^n$ has ***similarity dimension*** d if there exists a $c \in \mathbb{Z}$ and $s \in \mathbb{R}_{\geq 1}$ satisfying

$$d = \log_s(c)$$

and sQ (Q scaled up by a factor of s) is covered by c copies of Q (at most overlapping on their boundaries).

8

8.1 Compute the similarity dimension of a (a) a line segment, (b) the Cantor set, and (c) Sierpinski's triangle.

8.2 Compute the similarity dimension of the Koch snowflake.

What about sets that aren't self-similar?

9

Let K'_∞ be the "Koch snowflake" obtained with the substitution rule $\to$.

9.1 Find the perimeter and dimension of K'_∞.

9.2 Let K_{strange} be the "Koch snowflake" obtained by the rule $\to$ or chosen randomly at each stage. What should the dimension of K_{strange} be? Can you compute it's similarity dimension?

[2]Here "content" refers to the "stuff inside" of an object.

We need a way to define dimension for shapes that aren't self-similar. Let's again work from sets whose dimension we know: cubes.

Box Covering

DEFINITION

A d-dimensional ***box covering*** of $X \subseteq \mathbb{R}^k$ is a collection $C = \{B_i\}$ of d-dimensional cubes which satisfy

1. if $i \neq j$, B_i and B_j intersect at most on their boundaries;
2. $B_i \cap X \neq \{\}$ for all i;
3. $X \subseteq \bigcup_i B_i$.

Outer Measure

DEFINITION

The ***d-dimensional outer measure*** of $X \subseteq \mathbb{R}^k$ is

$$\lim_{n\to\infty} \inf_{C_n} \text{volume}(C_n)$$

where C_n is a d-dimensional box covering of X with cubes of side-length $1/n$.

You should think of $\inf_{C_n}$ volume(C_n) as the "smallest possible box covering of size $1/n$ that still covers the set".

---

10 Let $\ell \subseteq \mathbb{R}^3$ be the line segment from $\vec{0}$ to $(1,0,0)$.

10.1 Find the 1, 2, and 3-dimensional outer measures of ℓ.

10.2 Does ℓ have a 0-dimensional outer measure?

10.3 Let $T \subseteq \mathbb{R}^3$ be the filled in triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$. Find the 1, 2, and 3-dimensional outer measures of T.

10.4 Find the 1-dimensional outer measure of the Cantor set.

What would it mean to have a fractional-dimensional outer measure? Let B be a d-dimensional box with side lengths k. Its volume is k^d. Divide the box in half along every dimension and each sub-box has volume $(k/2)^d = (1/2)^d k^d$, and so there must be 2^d sub-boxes.

What if there were fewer "sub-boxes"?

---

11 Let C_α be the Cantor-like set obtained by removing the middle α of each subinterval. (I.e., the standard Cantor set is $C_{1/3}$.)

11.1 Find the number of boxes in a 1-dimensional box-covering of C_0 (the interval) and $C_{1/3}$ where the width of each box is 1/3, 1/9, 1/27, etc..

11.2 Based on what you know about how many width-k boxes it takes to fill d-dimensional space, find a formula relating d, the number of boxes, and the width of the boxes.

11.3 Use your formula to estimate d for $C_{1/3}$. How does this compare to the similarity-dimension of $C_{1/3}$?

Box-counting Dimension

DEFINITION

Let $X \subseteq \mathbb{R}^m$ and let $B \subseteq \mathbb{R}^m$ and let B be a minimal-dimensional, minimally-sized box such that $X \subseteq B$. The ***box-counting dimension*** of X is

$$d = \lim_{n\to\infty} \frac{\log(\text{\# of sub-boxes of } B_n \text{ that intersect } X)}{\log n},$$

where B_n is B "cut" along each axis into n equally-spaced slices.

11.4 Find the box-counting dimension of $C_{1/3}$.

11.5 Find the box-counting dimension of the unit simplex in $\mathbb{R}^2$. I.e. $\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2 \text{ for some } \alpha, \beta \geq 0 \text{ satisfying } \alpha + \beta \leq 1\}$

11.6 Intuitively, what should $\lim_{\alpha\to 0} \dim(C_\alpha)$ be? Find the box-counting and similarity dimension of C_α and verify.

Box-counting dimension is difficult to compute exactly, but it's useful for approximations. Computers are pretty good at counting boxes!

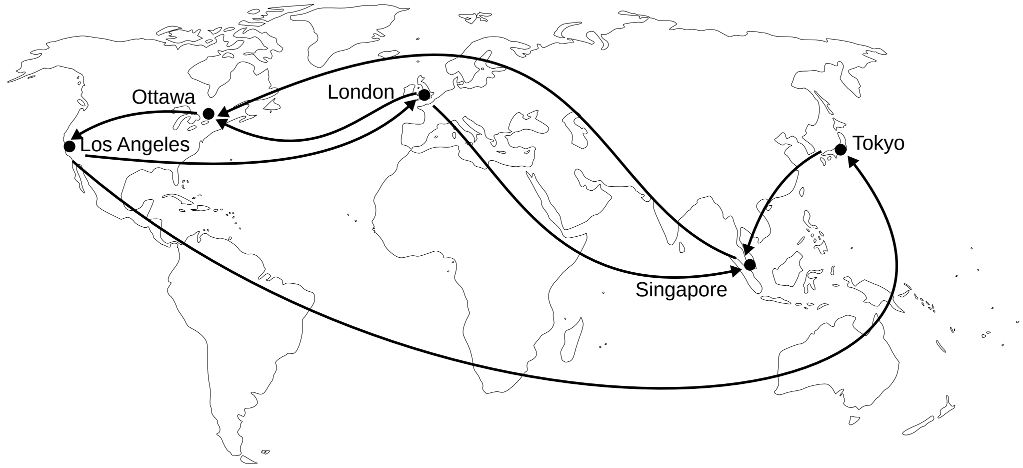
Transition Matrix

DEFINITION

Let $\mathcal{G}$ be a directed graph with vertices $\{1,\ldots,n\}$. A ***transition matrix*** for $\mathcal{G}$ is an $n\times n$ matrix $A=[a_{ij}]$ where

$$a_{ij}=\begin{cases}1 & \text{if there is an edge from } j \text{ to } i\\ 0 & \text{otherwise}\end{cases}.$$

12 The map shows the direct, one-way flights offered by the Pacific Rim air shipping company.



We can think of their flight map as a graph (in the graph-theory sense).

12.1 Write down a *transition matrix* A for the above graph.

(a) What do the diagonal entries tell you about the available flights?

(b) Should $a_{ij}=a_{ji}$? Explain.

12.2 Write down a matrix $B=[b_{ij}]$ where the entry b_{ij} indicates the number of ways to take exactly two flights from city i to city j.

(a) Compute A^2 and compare with B.

(b) What information does the 1st row of A give you about flights?

(c) What information does the 2nd column of A give you about flights?

(d) Based upon your last two answers what does the $1,2$ entry of A^2 tell you about flights?

12.3 Compute A^3. What does it tell you about shipping routes?

12.4 A package with a lost tracking number is getting kicked around from route to route! Each time it lands, it is randomly (and with equal probability) put on another flight. After several weeks (and 100s of flights), the package is finally noticed. Where is it most likely to be?

Markov Chain

DEF

Given a graph G, a ***stationary Markov chain on*** G, is a random process, denoted $X_1, X_2, \ldots$ where X_i indicates the location on the graph at the ith step and the probability distribution of X_i is *completely determined* by the value of X_{i-1}.

Some people call stationary Markov chains "memoryless" processes because what happened more than one step prior has no affect on what happens in the next step.

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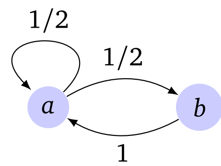
13 Consider the following directed graph with transition probabilities labeled.



13.1 Find a transition matrix, T, for the graph.

13.2 Find a matrix P_1 whose i, j entry represents the probability of transitioning from state j to state i in exactly one step.

13.3 Find a matrix P_2 whose i, j entry represents the probability of transitioning from state j to state i in exactly two steps.

13.4 How do P_2 and P_1^2 relate?

13.5 Does $\lim_{n\to\infty} P_1^n$ exist? If so, what is it?

Stochastic Matrix

DEF

A vector $\vec{v} \in \mathbb{R}^n$ is called a ***probability vector*** if its entries are non-negative and sum to one. A matrix is called a ***stochastic matrix*** if its columns are probability vectors.

In the context of Markov chains, we also refer to stochastic matrices as *transition* matrices.[3]

---

14 You're a picky eater. You have meals of healthy food and dessert, but you never have healthy food twice in a row. Each time you eat dessert, you flip a weighted coin to decide what meal to eat next. Let p represent the weight of the coin.

14.1 Are your eating habits described by a Markov chain? Why or why not?

14.2 Draw a graph representing this situation.

14.3 Your friend visits you for New Years and sees your having a healthy dinner. You lose touch after that, but bump into each other at a restaurant six years later. What type of meal are you most likely to be eating? Does this depend on p?

---

15 15.1 Is a Markov chain a dynamical system? Why or why not?

15.2 Consider the Markov chain with states $\{a, b\}$ given by the transition matrix $\begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$. Given that you start in state a, give probability vectors indicating the probability of being in a particular state after 0, 1, 2, and 3 steps along the Markov chain.

15.3 Can this Markov chain be *modeled* by a dynamical system? If so, describe such a model.

[3] Yes, this word is doing double duty!

Realization of a Markov Chain

DEFINITION

Given a Markov Chain $\mathcal{M} = (X_1, X_2, \ldots)$ with state space S, a ***realization*** of $\mathcal{M}$ is a sequence of states whose transitions are allowed by the Markov chain. I.e., an allowed element of $S^{\mathbb{N}}$.

A realization $\vec{r} \in S^{\mathbb{N}}$ is called ***generic*** for $\mathcal{M}$ if the transition probabilities for $\mathcal{M}$ can be recovered from $\vec{r}$ by averaging.

---

16 Consider the Markov chain $\mathcal{M}$ with state space $[0,1]$ and the transition rule

$$X_{i+1} = \begin{cases} X_i/3 & \text{with probability } 1/2 \\ X_i/3 + 2/3 & \text{with probability } 1/2 \end{cases}$$

16.1 Using a random number generator, write down the initial segment (up to 4 transitions) of two different realizations of $\mathcal{M}$.

16.2 Let $\vec{r} = (r_0, r_1, \ldots)$ be a realization of $\mathcal{M}$. Is it possible that $\lim_{i\to\infty} r_i$ exists? Why or why not?

16.3 Let $\vec{s} = (s_0, s_1, \ldots)$ be a realization for $\mathcal{M}$ that is generic. Is it possible that $\lim_{i\to\infty} s_i$ exists? Why or why not?

16.4 Can $\mathcal{M}$ be modeled by a dynamical system? If so, describe the model.

16.5 Suppose we start off with a uniform distribution on $[0,1]$. Draw the resulting distribution after 1, 2, and 3 steps along $\mathcal{M}$.

DEFINITION

Iterated Function System (IFS)

An ***iterated function system*** with functions $(f_1,\dots,f_n)$ and transition probabilities $(p_1,\dots,p_n)$ is a stationary Markov chain where transitions are given by the rule

$$X_{i+1} = \begin{cases} f_1(X_i) & \text{with probability } p_1 \\ f_2(X_i) & \text{with probability } p_2 \\ & \vdots \end{cases}$$

DEFINITION

Invariant Set

Let $\mathcal{I}$ be an iterated function system with functions $(f_1,\dots,f_n)$ and non-zero transition probabilities $(p_1,\dots,p_n)$. The set X is called an ***invariant set*** for $\mathcal{I}$ if

$$X = \bigcup_i f_i(X).$$

---

17 Let $\mathcal{I}$ be the iterated function system with transition probabilities $(1/2,1/2)$ and functions $(f_1:[0,1]\to[0,1], f_2:[0,1]\to[0,1])$ given by $f_1(x)=x/3$ and $f_2(x)=x/3+2/3$.

17.1 Find an invariant set for $\mathcal{I}$.

17.2 Is the Cantor set an invariant set for $\mathcal{I}$?

17.3 Is there a larger invariant set for $\mathcal{I}$ than the Cantor set? Why or why not?

17.4 If the probabilities for f_1 and f_2 change, will that affect the invariant sets?

---

18 Define $f_{\vec{a}}:[0,1]^2\to[0,1]^2$ by $f_{\vec{a}}(\vec{x})=\vec{x}/2+\vec{a}$. Let $\mathcal{F}$ be the iterated function system with functions $(f_{\vec{0}}, f_{(1/2,0)}, f_{(0,1/2)})$ and equal probabilities.

18.1 Draw a maximal invariant set for $\mathcal{F}$.

18.2 Can you create an iterated function system so that the Sierpinski triangle is an invariant set? If so, do it!

Continuous-time Dynamical Systems

The dynamical systems we've seen so far involve taking one "step" at a time. However, as we experience life, it seems like time advances continuously, not discretely. Can we capture that in a dynamical system?

Continuous-time Dynamical System

DEF Let X be a set and let $(T^a : X \to X)_{a\in\mathbb{R}}$ be a family of functions satisfying $T^a \circ T^b = T^{a+b}$. Then, (T^a, X) is called a ***continuous time dynamical system***.

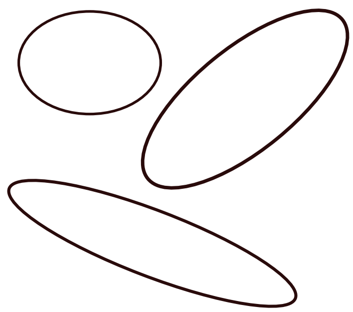
19

19.1 Let $S^a : \mathbb{R} \to \mathbb{R}$ be defined by $S^a(x) = ax$. Is $(S^a, \mathbb{R})$ a continuous time dynamical system?

19.2 Let $T^a : \mathbb{R} \to \mathbb{R}$ be defined by $T^a(x) = 2^a x$. Is $(T^a, \mathbb{R})$ a continuous time dynamical system?

19.3 Let $\mathcal{F} = \{\text{functions from } \mathbb{R} \text{ to } \mathbb{R}\}$ and define $E^t : \mathcal{F} \to \mathcal{F}$ by $E^t(f(x)) = f(x+t)$. Is $(E^t, \mathcal{F})$ a continuous time dynamical system?

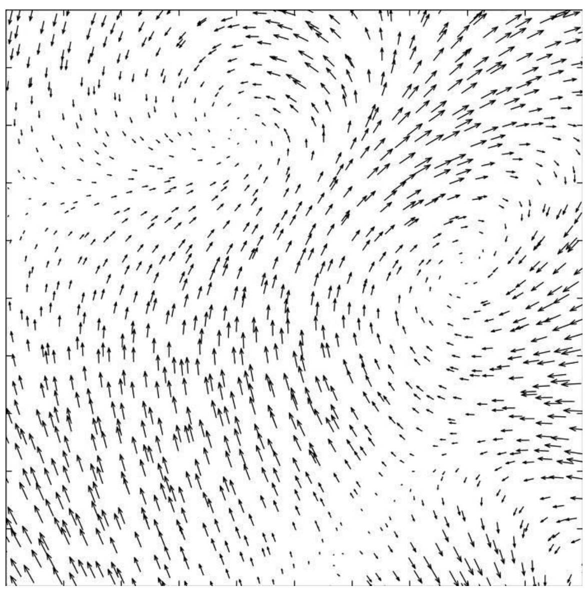
20.1 Consider the following garden paths.



Let P represent the set of points on the paths and define $W^t : P \to P$ to be the result of walking at unit speed counter-clockwise around your path for t seconds.

Is (W^t, P) a continuous time dynamical system?

20.2 You're watching the surface of a lake and carefully map out the velocity of the water at each point. You produce the following picture of velocities.



Let S be the points on the surface of the lake and let $\Phi^t : S \to S$ be the result of "flowing" along the surface for t seconds. Is (Φ^t, S) a continuous time dynamical system?

Systems like (Φ^t, S) come up often because they are described by *local* rules. That is, recorded in the picture is information about where you flow after tiny time increments—finding where you end up after bigger time increments takes work.

Vector Field (DEF)

Given a set X, a ***n-dimensional vector field on X*** is a function $F : X \to \mathbb{R}^n$ which assigns a vector to each point in X.

Vector fields show up a lot in physics because *velocities* and *accelerations* of particles in a fluid naturally produce vector fields.

---

21 Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field and let $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which "flows" points along the vector field with velocity at a given point equal to the vector at that point.

21.1 Find a V so that all points flow vertically at unit speed.

21.2 Find a V so that all points flow radially outward and increase in speed.

21.3 Find a V so that all points flow around in a circle.

21.4 Find a V so that all points flow around in a circle at the same speed.

21.5 Find a V so that all points flow around in a circle with the same period (that is, it takes the same amount of time for any given point to go around the circle).

Flow & Orbit (DEFINITION)

Let (W^t, X) be a continuous time dynamical system and let $x \in X$. The ***flow of W^t starting at x*** is the function $f : \mathbb{R} \to X$ defined by

$$f(t) = W^t(x).$$

The ***orbit of x under W^t*** is the set $\{f(t) : t \in \mathbb{R}\}$, where f is the flow of W^t starting at x. We call the set $\{f(t) : t \in \mathbb{R}^+\}$ the ***forward orbit*** of x.

We will often notate the flow of W^t starting at x by $\phi_t(x)$. As a flow, we think of t as the variable.

---

22 Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field and let $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that flows points along the vector field.

22.1 If possible, find a V so that some orbits are infinitely long.

22.2 If possible, find a V so that some orbits are not infinitely long.

22.3 If possible, find a V so that some orbits are infinitely long and others are not.

22.4 If possible, find a V so that all orbits are finite line segments.

22.5 If possible, find a V so that the forward orbit of every point ends up at $\vec{e}_1$.

22.6 If possible, find a V so that the forward orbit of every point ends up at $\vec{0}$ or $\vec{e}_1$ (and at least one orbit heads towards $\vec{0}$ and one towards $\vec{e}_1$).

DEFINITION

Stable & Unstable Points

Let $(W^t, \mathbb{R}^n)$ be a continuous time dynamical system and let $\phi_t(\vec{w})$ represent the flow of W^t starting at $\vec{w} \in \mathbb{R}^n$. We call the point $\vec{x} \in \mathbb{R}^n$ ***stable*** if for all $\varepsilon > 0$, there exists a $\delta > 0$ so that for all $\vec{y} \in \mathbb{R}^n$,

$$\|\vec{x} - \vec{y}\| < \delta \qquad \text{implies} \qquad \|\phi_t(x) - \phi_t(y)\| < \varepsilon \qquad \text{for all} \qquad t > 0.$$

Otherwise, $\vec{x}$ is called ***unstable***.

---

23 Given a 2×2 matrix M, define a vector field $V_M : \mathbb{R}^2 \to \mathbb{R}^2$ by $V_M(\vec{x}) = M\vec{x}$. Consider the following matrices and their associated vector fields:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

23.1 Explain in plain language what it means for x to be a *stable* point of a continuous dynamical system.

23.2 Use an online plotter (for example `https://www.desmos.com/calculator/eijhparfmd` or `https://anvaka.github.io/fieldplay`) to plot V_X for $X \in \{I, A, B, C, D, E\}$ and determine which points are stable/unstable for each.

23.3 Compute the eigenvalues for each of the matrices. Can you relate the eigenvalues to stable/unstable points?

Differential Equations

To really get a handle on what's going on, let's think about some differential equations!

24

24.1 Let a be a constant. Find a non-trivial solution to the differential equation $f'(t) = af(t)$.

24.2 Consider the (boring) vector field $V : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $V(t) = at$. Find a non-trivial flow for the corresponding dynamical system. Is V given by matrix multiplication? If so, what's the matrix?

24.3 Let $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and define the vector field $W : \mathbb{R}^2 \to \mathbb{R}^2$ by $W(\vec{x}) = D\vec{x}$. Find a non-trivial flow for the corresponding dynamical system.

Can you give conditions on a and b so that $\vec{0}$ is stable/unstable with respect to the corresponding dynamical system? If a and b are complex numbers, does that change your answer?

24.4 Suppose $(W^t, \mathbb{R}^n)$ is a continuous dynamical system. Define what the *derivative of W^t with respect to time* should mean. How does this relate to vector fields and flows?

24.5 Prove that if $f : \mathbb{R} \to \mathbb{R}^n$ is a function and M is a matrix, then $(Mf)' = M(f')$.

24.6 Suppose A is a matrix and $A = PDP^{-1}$ where D is diagonal. Further, suppose f is a solution to $f'(t) = Df(t)$. Find a solution to the differential equation $\Phi'(t) = A\Phi(t)$.

24.7 Let M be a matrix and consider the continuous dynamical system coming from the vector field $\vec{x} \mapsto M\vec{x}$. Classify the behaviour near $\vec{0}$ based on the eigenvalues of M.

25

Consider the continuous dynamical systems $(W_1^t, \mathbb{R}^2)$, $(W_2^t, \mathbb{R}^2)$, $(W_3^t, \mathbb{R}^2)$, and $(W_4^t, \mathbb{R}^2)$ given by flows along the vector fields

$$V_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \qquad V_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y^2 \\ -y \end{bmatrix} \qquad V_3\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x + y^2 \end{bmatrix} \qquad V_4\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + y^2 \\ -y \end{bmatrix}$$

25.1 Classify $(0, 0)$ as stable or unstable for each system.

25.2 Classify $(0.01, 0.01)$ as stable or unstable for each system.

25.3 Conjecture: under what conditions will a first-order approximation about at a fixed point tell you about stability *near* that fixed point.

DEFINITION

Structural Stability

A continuous dynamical system (W^t, X) is called ***structurally stable at $\vec{x} \in X$ with respect to fixed points/stable points/periodic points/etc.*** if the topology of the set of fixed points/stable points/periodic points/etc. near $\vec{x}$ remains invariant with respect to small perturbations of W^t.

26

26.1 What should a *small perturbation* of $(W^t, \mathbb{R}^n)$ mean?

26.2 What should it mean for the *topology* of a set to be *invariant*?

26.3 Out of the examples in 25, which are structurally stable at $\vec{0}$?

26.4 Revisit your conjecture from 25.3. Can you rephrase it in terms of structural stability?

Time-1 Map

DEFINITION

Let (W^t, X) be a continuous dynamical system. The ***time-1 map*** associated with (W^t, X) is the dynamical system (T, X) where

$$T(x) = W^1(x)$$

for all $x \in X$.

---

27 Let S be a circle of circumference 1 and let (W_k^t, S) be the continuous dynamical system that flows points counter-clockwise along S at speed k.

27.1 For which k is the time-1 map associated with (W_k^t, S) trivial?

27.2 For which k does the time-1 map associated with (W_k^t, S) have periodic points?

Circle Maps

Let S be a circle of circumference 1. We can associate S with the unit interval $[0,1]$ provided we "glue" the endpoints together. We write $[0,1]/0 \sim 1$ to notate the set $[0,1]$ where 0 and 1 are considered the same point. That is, if you move to the right starting at 0.99, you'll wrap around and end up at 0.01.

Given a function $f : [0,1) \to \mathbb{R}$, we can create a function $g : [0,1) \to [0,1)$ via the formula

$$g(x) = f(x) \mod 1.$$

That is, compute the value $f(x)$ with "wrap around", and that's what $g(x)$ is.

Symbolic Coding

DEFINITION

Let (T, X) be a discrete dynamical system and let $\mathcal{P} = \{P_a, P_b, \ldots\}$ be a partition of X. A ***symbolic coding*** of $x \in X$ relative to the partition $\mathcal{P}$ is the sequence

$$\mathbb{C}(x) = (c_0, c_1, \ldots)$$

where

$$c_i = \begin{cases} a & \text{if} \quad T^i x \in P_a \\ b & \text{if} \quad T^i x \in P_b \\ \vdots & \end{cases}$$

---

28 Let (W_k^t, S) be as in 27. We can describe this dynamical system by

$$W_k^t(x) = x + kt \mod 1.$$

Let $P_a = [0, 1/2)$ and $P_b = [1/2, 1)$ be a partition of $[0,1]/0 \sim 1$, and let (T_k, S) be the time-1 map for (W_k^t, S).

28.1 Let $k = 0.25$. Find $\mathbb{C}(0)$ and $\mathbb{C}(1/3)$ for the T_k.

28.2 Suppose $\mathbb{C}(x) = (a, a, b, b, a, a, b, b, \ldots)$. What can you say about x and k?

28.3 Is $(a, b, a, a, b, b, a, a, a, b, b, b, \ldots)$ the symbolic coding of any point? Why or why not?

28.4 For which k can x be *exactly* recovered from $\mathbb{C}(x)$?

28.5 Let $\{a, b\}^{\mathbb{N}}$ be the set of sequences of a's and b's. We can think of $\mathbb{C} : S \to \{a, b\}^{\mathbb{N}}$ as a function. For which k is $\mathbb{C}$ one-to-one? Onto?

To really understand this situation, we need to know some things about *probability theory*. A central object of study in probability theory are *random variables*.

DEF

Random Variable

Let Q be a set. A ***random variable with state space*** Q is a function $X : [0,1] \to Q$.

We think of a random variable $X : [0,1] \to Q$ as a "to be determined quantity". Of course, if $\omega \in [0,1]$, then $X(\omega) \in Q$ is an actual element of the state space. It would be incorrect to write "$X \in Q$".

DEFINITION

Probability Function

The ***probability function*** on $[0,1]$, is a function

$$\mathbb{P} : \{\text{measureable subsets of } [0,1]\} \to [0,1]$$

that assigns every measurable subset of $[0,1]$ a number corresponding to the "percentage" of $[0,1]$ it occupies.

We often call elements of the domain of $\mathbb{P}$ *events* instead of calling them sets.

29 Let X and Y be random variables with state space $\{H, T\}$ defined by

$$X(\omega) = \begin{cases} H & \text{if } \omega > 1/2 \\ T & \text{otherwise} \end{cases} \qquad Y(\omega) = \begin{cases} H & \text{if } \omega > 2/3 \\ T & \text{otherwise} \end{cases}$$

29.1 By writing $X = H$, we mean the event (set) $X^{-1}(H) = \{\omega \in [0,1] : X(\omega) = H\}$.

What is $\mathbb{P}(X = H)$ and $\mathbb{P}(X = T)$? If you were given a fair coin, what would the probability of getting heads or tails be?