MAT2310/6Sept01

Elementary Matrices and LU Decomposition

1 Elementary Matrices

An $n \times n$ **elementary matrix** is a matrix obtained by performing one elementary row operation on I_n, the $n \times n$ identity matrix. Three types of elementary matrices correspond to the three types of elementary row operations. For example,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are 3×3 elementary matrices. E_1 corresponds to the elementary row operation $\underline{\mathbf{2}} \longleftrightarrow \underline{\mathbf{3}}$, E_2 corresponds to $\underline{\mathbf{2}} \times 3$, and E_3 $\underline{\mathbf{2}} + 2 \times \underline{\mathbf{1}}$.

Elementary matrices are invertible. An elementary matrix and its inverse are of the same type. For E_1, E_2 and E_3 above, one can easily verify that

$$E_1^{-1} = E_1,\ E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Performing an elementary row operation on a matrix is the same as premultiplying the matrix by the elementary matrix corresponding to the elementary row operation. For example, let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & 1 & 2 \end{pmatrix}.$$

Applying $\underline{\mathbf{2}} + 2 \times \underline{\mathbf{1}}$ to A, or premultiplying it by E_3 give the same matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 7 & 10 & 13 & 16 \\ 9 & 0 & 1 & 2 \end{pmatrix}.$$

Matrices A and B are **row equivalent** if there are elementary matrices $E_1, \ldots, E_n$ such that $A = E_n \ldots E_1 B$. Alternatively they are row equivalent if one is the result of applying a sequence of elementary row operations to another.

Theorem (Theorem 1.4.3) Let A be an $n \times n$ matrix. The followings are equivalent:

(a) A is nonsingular;

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution;

(c) A is row equivalent to I_n.

Proof: (a) $\Rightarrow$ (b) Premultiply both sides of $A\mathbf{x} = \mathbf{0}$ with A^{-1}, we get $x = \mathbf{0}$ as the only solution.

(c)$\Rightarrow$ (a) If A is row equivalent to I_n, there are elementary matrices E_i, $i = 1, \ldots, k$, such that $E_k \cdots E_1 A = I$. Premultiplying both sides by E_k^{-1}, E_{k-1}^{-1} and so on, we get $A = E_1^{-1} \cdots E_k^{-1}$. Recall that a product of invertible matrices is invertible, and the inverse is the product of the inverses of the matrices in the reverse order. So $A^{-1} = E_k \cdots E_1$.

(b) $\Rightarrow$ (c) We will prove that if A is not row equivalent to I_n, $A\mathbf{x} = \mathbf{0}$ will have infinitely many solutions. Let U be an REF of A. If A and hence U is not row equivalent to I, some of the diagonal entries of U must be zero. U must have a zero row. So $A\mathbf{x} = \mathbf{0}$ is equivalent to a system with more variables than equations. So it has infinitely many solutions. QED

We now have an efficient way to find the inverse of a nonsingular matrix. If A is row equivalent to I_n, then there are elementary matrices E_i, $i = 1, \ldots, k$, such that $E_k \ldots E_1 A = I_n$. Premultiplying both sides with E_k^{-1}, E_{k-1}^{-1} and so on, one gets $A = E_1^{-1} \ldots E_k^{-1}$. As products of nonsingular matrices are nonsingular, A is nonsingular. Also $A^{-1} = E_k \ldots E_1$.

Examples and practices

(1) Find the inverse of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

We put A and I_2 side by side. Apply the same elementary row operations to both, and reduce A to the identity, as follows.

$$\begin{array}{cc}
 & \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \\
\begin{array}{c} \underline{\mathbf{2}} - 3 \times \underline{\mathbf{1}} \\ \longrightarrow \end{array} & \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \\
\begin{array}{c} \underline{\mathbf{2}}/(-2) \\ \longrightarrow \end{array} & \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)
\end{array}$$

$$\xrightarrow{\underline{\mathbf{1}} - 2\times\underline{\mathbf{2}}} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

A^{-1} is the matrix on the right:

$$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$

Writing down the elemenary matrices corresponding to the elementary row operations, A^{-1} can be written as a product of elementary matrices:

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

Also, A can be written as

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(2) Find the inverse of

$$\begin{pmatrix} 4 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}.$$

2 LU Decomposition

Let A be an $n \times n$ matrix. Let a_{ij} be its entry in the ith row and the jth column. a_{ii}, $i = 1, \ldots, n$, are called the diagonal entries. A is said to be **lower triangular** if the entries below the diagonal are zero: $a_{ij} = 0$ for $i > j$. It is **upper triangular** if the entries above the diagonal are zero: $a_{ij} = 0$ for $i < j$. Its is called **diagonal** if it is both upper and lower triangular, or the off diagonal entries are zero.

If A can be reduced to an upper triangular matrix via type III elementary row operations, then A admits an LU decomposition: $A = LU$ where L and U are lower and upper triangular respectively. We will illustrate this by examples later.

Consider a system $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x}^T = (x_1, \ldots, x_n)$ is a vector of variables, $\mathbf{b}^T = (b_1, \ldots, b_n)$ is a vector of constants. Suppose $A = LU$ is an LU decomposition. Then it can be solved easily by forward and backward substitutions, as follows. The system can be written as $LU\mathbf{x} = \mathbf{b}$. Let

$U\mathbf{x} = \mathbf{y}$. Then $L\mathbf{y} = \mathbf{b}$. We solve the later system for $\mathbf{y}$ and then the former system for $\mathbf{x}$. As L and U are triangular matrices, only forward and backward substitutions are involved.

We illustrate these by an example.

Examples and practices

(1) Find an LU decomposition for

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix}.$$

We perform the following sequence of elementary row operations:

$$\begin{array}{cc} & \begin{pmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{pmatrix} \\ \begin{array}{c} \underline{\mathbf{2}} - (1/2) \times \underline{\mathbf{1}} \\ \longrightarrow \\ \underline{\mathbf{3}} - 2 \times \underline{\mathbf{1}} \end{array} & \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{pmatrix} \\ \begin{array}{c} \underline{\mathbf{3}} + 3 \times \underline{\mathbf{2}} \\ \longrightarrow \end{array} & \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}. \end{array}$$

In terms of elementary matrices, we have $E_3E_2E_1A = U$, where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \; E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix}.$$

So $A = E_1^{-1}E_2^{-1}E_3^{-1}U$. Let $L = E_1^{-1}E_2^{-1}E_3^{-1} =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix}.$$

Then we have $A = LU$.

(2) With A as in the last example, solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}^T = (1, 3/2, -1)$, and $\mathbf{x}^T = (x_1, x_2, x_3)$ is a vector of variables.

Write the system as $LU\mathbf{x} = \mathbf{b}$. Let $U\mathbf{x} = \mathbf{y}$, where $\mathbf{y}^T = (y_1, y_2, y_3)$ is a vector of variables. Then $L\mathbf{y} = \mathbf{b}$.

From the last example and the given data, $L\mathbf{y} = \mathbf{b}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \\ -1 \end{pmatrix}.$$

The first equation gives $y_1 = 1$. Substitute this into the second equation gives $y_2 = 1$. And then the third equation gives $y_3 = 0$. This is a simple process of forward substitution.

Now that $\mathbf{y}$ is known, we can solve for $\mathbf{x}$ from $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now use backward substitution: from the last equation, $x_3 = 0$. Substitute this into the second equation, we get $x_2 = 1/3$. Then the first equation gives $x_1 = -1/6$.

(3) For each of the following matrices, either find an LU decomposition, or prove that such a decomposition doesn't exist:

$$\begin{pmatrix} 3 & 1 \\ 9 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(4) Solve the system

$$\begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

by first finding an LU decomposition for the coefficient matrix, and then solve two systems of equations.