Lab 03
Outbreak!

MATH130 (Calculus I)
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An experimental subject has escaped from the lab, and the dangerous *Smurfus verdigans* bacterium is spreading like wildfire through the local Smurf population. Before we can put control measures into place, we need to understand how the disease is progressing. This situation calls for some math, and that's where you come in.

The facts on the ground

Let's get you up to speed. There's fifty thousand Smurfs in the greater Geneva area. *S. verdigans* was eliminated in these parts over a hundred years ago, so none of the locals have ever been infected; accordingly, they're all **susceptible**, save a handful with hereditary immunity. If a susceptible Smurf is exposed to an **infected** one, there's a chance of transmission. Once a Smurf catches *S. verdigans*, he[1] will be sick for an average of fourteen days, during which he can transmit the disease to others. Afterwards, the disease dies down, although the unfortunate Smurf is left with permanent disfiguring greenness; we classify these Smurfs as **recovered**. Recovered Smurfs are immune to further infection with *S. verdigans*.

Setting up a model

At any given time, there are three populations of Smurfs: the susceptible, the infected, and the recovered. Let's let t denote the number of days since the outbreak began; then $S(t)$, $I(t)$, and $R(t)$ can represent the numbers of susceptible, infected, and recovered Smurfs on day t.

To make sense of how things change over time, we're going to need to think about the *rates of change* of each of these quantities. Let $S'(t)$, $I'(t)$, and $R'(t)$ denote the rates of change[2] of the susceptible, infected, and recovered populations in Smurfs ("Sm") per day ("d"). (For example, if $I'(4) = 200$, that means that the infected population is growing by 200 Sm/d.)

Now that we have some names for things, it's time to set up a model. How many Smurfs recover from *S. verdigans* on a given day? Of course, it depends on how many are infected! As an approximation, we'll say that one-fourteenth of the infected recover on any given day, so that[3]

$$R' = \frac{I}{14}\,\mathrm{Sm/d}. \tag{1}$$

Mathematicians call equation (1) a *rate equation* or *differential equation*; we don't bother writing the "(t)" bits when it's clear from context that we mean that R' and I are functions of t.

How many Smurfs become infected with *S. verdigans* on a given day? Naturally, this depends on the value of S, since a Smurf can only become infected if he is susceptible. Since *S. verdigans* is transmitted from infected Smurfs, however, it also depends on I; in particular, if no Smurfs had the disease, none could ever

[1] Almost all Smurfs are male. No one knows why.
[2] Mathematicians call these functions the *derivatives*, and we'll be spending quite a bit of time with them in this course.
[3] Think about this line carefully before you proceed!

catch it! It turns out that the *product* of S and I is the relevant value, since it measures how much contact there can be between infected and susceptible Smurfs. Thus, we expect that $S'(t) = -aS(t)I(t)$ for some positive constant a (since infection causes Smurfs to *leave* the susceptible population). Our experimental data show that $a \approx 0.00001$. Thus,

$$S' = -0.00001 \cdot SI \,\text{Sm/d}. \tag{2}$$

What about the infected population? There are *two* influences on the value of I': the infection of susceptible Smurfs and the recovery of infected Smurfs. In fact, $I' = -S' - R'$, since every Smurf entering I came from S and every Smurf leaving S goes to R. Combining this with equations (1) and (2), we obtain

$$I' = 0.00001 \cdot SI - \frac{I}{14} \text{Sm/d}. \tag{3}$$

Problem 1. Use equations (1) to (3) to calculate $S' + I' + R'$. What does this value mean? Interpret your result in a sentence or two.

Using the model

Whew! That's a lot of math. Of course, it boils down to just equations (1) to (3), which describe how R, S, and I change over time. If we use the equations to calculate the rate of change of some quantity (say, S', the rate of change of the susceptible population), we can then use this to estimate how large that population will be the next day. For example, if $S = 40000$ on some day and $S' = -5000$, we estimate that the next day we will find $S = 40000 - 5000 = 35000$ susceptible Smurfs remaining.

Problem 2. As we mentioned above, there are fifty thousand Smurfs in the local population. On the first day of the outbreak ($t = 1$), there were 2100 infected smurfs and 2500 who are already immune for genetic reasons. Use equations (1) to (3) to estimate how many susceptible, infected, and recovered Smurfs there will be tomorrow ($t = 2$). Repeat this process to estimate S, I, and R for $t = 3$ (using your values for $t = 2$) and for $t = 4$ (using your values for $t = 3$). Write your results in the table below.

t	S	I	R	S'	I'	R'
1	45400	2100	2500			
2						
3						
4						

Table 1: Estimates for the first four days

Further analysis

The process used in problem 2 would allow us to keep estimating populations far into the future. However, there are two problems with this: it's tedious and it's error-prone. The first problem could easily be solved with the use of a computer program; for the second, we'll need a more sophisticated theory of solving differential equations, which we'll revisit later in the term.

However, there are some useful things we can say about this system with purely analytic methods. Here's an example. According to equation (3), the rate at which the infected population grows is given by $I' = 0.00001 \cdot SI - \frac{I}{14}$. You should have found in Table 1 that $I'(1) \approx 803.4\,\text{Sm/d}$. This number is positive, so

I is increasing initially. In fact, I will increase as long as I' is positive. If I' becomes negative, I will start to shrink—good news, since we're trying to control the outbreak!

Problem 3. Factor out I from equation (3). Use your factorization to describe the conditions under which $I' = 0$ (in terms of values of I and S). When is I' positive? When is it negative? (Note that S, I, and R are never negative, since they are numbers of Smurfs—a decidedly cheery and unfailingly positive species.)

Suppose we are able to develop a vaccine which makes Smurfs immune to *S. verdigans*; how many should we immunize to ensure that the disease doesn't spread?

Problem 4. As you saw in problem 3, a vaccine-only campaign is going to be difficult. Let's look at the transmission part of this system and consider how a quarantine policy might help. The value $a = 0.00001$ in equations (2) and (3) is called the *transmission constant* of *S. verdigans*, and it determines how efficiently the disease passes from one Smurf to another. Suppose the scientists are able to quarantine infected Smurfs, reducing a by half (to 0.000005). Rewrite equations (1) to (3) to incorporate this new constant.

In problem 3, you should have found that whenever there were fewer than 7143 susceptible Smurfs, the number of infected would decline instead of grow; this number is called the *outbreak threshold* for this disease and population. Repeat that calculation using your new equations (1) to (3). How does the outbreak threshold change? Under the conditions on the ground (2100 initial infected, 45400 initial susceptible), will an outbreak occur, or will the number of infected simply fall towards 0?

Problem 5. In problem 4, you should have found that the new transmission coefficient was not low enough to prevent an outbreak. Find a (positive) value of the transmission coefficient which does prevent the outbreak. What is the largest value that succeeds at preventing outbreak?