Problems for Lecture 3

Group #2 of Cohort #2

Group member: Georgios Terzakis, Yue Yu, Xiaoqing Lu, Chong Cao

Problem 1

(a) The 3-year bond with coupon rate 10% and par value $100 is calculated as

$$\text{Price} = \frac{10}{1+5\%} + \frac{10}{(1+5.5\%)^2} + \frac{110}{(1+6.5\%)^3} = 109.57$$

Assuming that the bond yield is *r*, we have

$$\text{Price} = \frac{10}{1+r} + \frac{10}{(1+r)^2} + \frac{110}{(1+r)^3} = 109.57$$

Solving numerically, we get the YTM as r=6.4%.

(b) Forward rates

$$(1+r_1)(1+{}_1f_1) = (1+r_2)^2 \Rightarrow {}_1f_1 = \frac{(1+r_2)^2}{1+r_1} - 1 = 6.0\%$$

$$(1+r_1)(1+{}_1f_2)^2 = (1+r_3)^3 \Rightarrow {}_1f_2 = \left[\frac{(1+r_3)^3}{1+r_1}\right]^{1/2} - 1 = 7.2\%$$

$$(1+r_2)(1+{}_2f_1) = (1+r_3)^3 \Rightarrow {}_2f_1 = \frac{(1+r_3)^3}{1+r_2} - 1 = 8.5\%$$

(c) In order to guarantee 3-year return on the coupon bond, the coupon payments can be re-invested at forward rates. The total value of coupon bond at the end of third year can be obtained as

$$\text{FV} = 10(1+{}_1f_2)^2 + 10(1+{}_2f_1) + 10 + 100 = 132.36$$

The 3-year total return of the coupon bond can be obtained as

$$\text{Return} = \text{FV - Price} = 132.36 - 109.57 = 22.79$$

Also, the annual return rate (APR) is calculated as

$$R = \left(\frac{132.36}{109.57}\right)^{1/3} - 1 = 6.5\%$$

Problem 2

The forward rate can be derived from spot rates as

$$(1+r_1)(1+{}_1f_2)^2=(1+r_3)^3 \Rightarrow {}_1f_2=\left[\frac{(1+r_3)^3}{1+r_1}\right]^{1/2}-1$$

Given that r_1=6% and r_3=7%, the forward rate is calculated as 7.5%.

IF we invest \$1M one year from now, the cash flow at year three would be

$$1\text{M}\times(1+{}_1f_2)^2=1.15\text{M}$$

Therefore, the profit would be around 0.15M at the end of year three.

In order to lock in the forward rate, we can short 1-year ZCB and long 3-year ZCB from the bond dealer. Assuming that the par value of ZCB is \$100, the cash flows are as follows

	T=0 Year	**T=1 Year**	**T=3 Year**
1-year ZCB	Short 10000 units 1-year ZCB, receive cash: \$943396.2	Receive 1M dollars; Close short position of 1-year ZCB	
3-year ZCB	With cash \$943396.2, long 11557.01 units of 3-year ZCB		Payoff from 11557.01 units of 3-year ZCB is \$1155701
	Net cash = **0**	Net cash = 1M – 1M = **0**	Net cash =11557.01 – 1M = **155701**

With the bond investment strategy described above, we can get the same interest income as that of 1M investment by forward rate.

Problem 3

The price of coupon bonds with par value \$100 can be calculated as

$$\text{Price}=\frac{c}{1+r_1}+\frac{c}{(1+r_2)^2}+\cdots+\frac{100+c}{(1+r_T)^T}$$

where c is the coupon, T represents the maturation and r_i denotes the spot rates.

The current price for bonds A, B, C and D can be computed as

$$\text{Price}_A=88.62,\ \text{Price}_B=79.06,\ \text{Price}_C=102.27,\ \text{Price}_D=104.19$$

Suppose that the yield increases to 3.8% from 3.5% (Δr = 0.3%), the new prices and changes will be

$$\text{Price}_A=87.36,\ \text{Price}_B=76.88,\ \text{Price}_C=100.90,\ \text{Price}_D=101.65$$

$$\Delta_A = -1.43\%,\ \Delta_B = -2.75\%,\ \Delta_C = -1.34\%,\ \Delta_D = -2.43\%$$

Suppose that the yield decreases to 3.2% from 3.5% (Δr = -0.3%), the new prices and changes will be

$$\text{Price}_A = 89.91,\ \text{Price}_B = 81.30,\ \text{Price}_C = 103.67,\ \text{Price}_D = 106.80$$

$$\Delta_A = 1.45\%,\ \Delta_B = 2.83\%,\ \Delta_C = 1.36\%,\ \Delta_D = 2.51\%$$

The durations of each bond can be calculated as

$$D_A = 4.88,\ D_B = 9.47,\ D_C = 4.59,\ D_D = 8.38$$

The approximate estimate for price changes can be obtained from duration and yield change as

$$\frac{\Delta P}{P} = -D\Delta r$$

For yield change of Δr = 0.3%, the price changes will be

$$\Delta_A = -1.46\%,\ \Delta_B = -2.84\%,\ \Delta_C = -1.38\%,\ \Delta_D = -2.51\%$$

For yield change of Δr = -0.3%, the price changes will be

$$\Delta_A = 1.46\%,\ \Delta_B = 2.84\%,\ \Delta_C = 1.38\%,\ \Delta_D = 2.51\%$$

Conclusions:

For the same yield change, the longer the duration, the bigger change in bond price (Δ_B, $\Delta_D > \Delta_A$, Δ_C).

For the same yield change and duration, the higher the coupon rate, the smaller change in bond price ($\Delta_B > \Delta_D$ and $\Delta_A > \Delta_C$).

Problem 4

(a) Suppose that the hedging portfolio consists of *x* units of bond A, *y* units of bond B, and *z* units of bond C. In order to synthetically replicate the liability, the hedging portfolio should have the same cash flow with the 31-year custom-made bond. The cash flows are shown as follows (par value = $100)

Year	**Bond A**	**Bond B**	**Bond C**	**Portfolio**	**Custom bond**
1~29	0	4	6	$4y+6z$	0
30	0	104	106	$104y+106z$	1M
31	100	0	0	$100x$	2M

We can solve the system of equations as

$$\begin{cases} 4y+6z=0 \\ 104y+106z=1\text{M} \\ 100x=2\text{M} \end{cases} \Rightarrow \begin{cases} x=20000 \\ y=30000 \\ z=-20000 \end{cases}$$

Therefore, we should buy 20000 units of bond A and 30000 units of bond B and sell 20000 units of bond C to hedge the liability.

(b) Suppose that the hedging portfolio consists of *x* units of 10-year zero-coupon bond, *y* units of 15-year zero-coupon bond. Each ZCB has a duration of 10 and 15 year, respectively.
Assuming that the term structure is currently flat at 6%, the PV of ZCB can be obtained as

$$PV_{10\text{yr}} = \frac{100}{(1.06)^{10}} = 55.84\,,\ PV_{15\text{yr}} = \frac{100}{(1.06)^{15}} = 41.73$$

The market value of the replicating portfolio is

$$C = x\bullet PV_{10\text{yr}} + y\bullet PV_{15\text{yr}} = 55.84x + 41.73y$$

The duration of the replicating portfolio is also obtained as

$$D_C = 10x\frac{PV_{10\text{yr}}}{C} + 15y\frac{PV_{15\text{yr}}}{C} = \frac{1}{C}(558.4x + 625.9y)$$

The PV of liability can be computed as

$$L = \frac{1\text{M}}{(1.06)^{30}} + \frac{2\text{M}}{(1.06)^{31}} = 0.50\text{M}$$

The duration of liability can also be computed as

$$D_L = 30\times\frac{1\text{M}}{(1.06)^{30}L} + 31\times\frac{2\text{M}}{(1.06)^{31}L} = 30\times 0.346 + 31\times 0.654 = 30.65$$

Under the duration-based hedging, the hedging portfolio should have the same market value as well as same interest-rate sensitivity (measured using duration model) as the liability.

$$\begin{cases} C = L \\ D_C = D_L \end{cases} \Rightarrow \begin{cases} 55.84x + 41.73y = 0.50\text{M} \\ 558.4x + 625.9y = 30.65(55.84x + 41.73y) \end{cases}$$

Solving numerically, we get x=-28180.1 and y=49756.9. Therefore, we need to sell 28180 units of 10-year ZCB and buy 49757 units of 15-year ZCB.

(c) Suppose that the interest rate fell to 4.8% from 6%, the hedging portfolio would change significantly. Since lower interest rate tends to produce higher bond price, especially for bond with longer durations. The 15-year ZCB is therefore more sensitive to interest rate change than the 10-year ZCB. The duration of bond portfolio will also increase as a result of interest rate fall.

So, the new hedging portfolio should include relatively more 10-year ZCB in order to compensate the price/duration change.

(d) The advantage of synthetic replication is that it produces exactly the same cash flow as the liability, which may generate exactly the same PV and duration for both assets and liabilities. Therefore, synthetic replication strategy may always provide the perfect hedge. However, there won't always be enough replicating bonds available in reality. Using approximate hedging then becomes another practical choice.