习题课

2020.06.11

课后作业01

1. 阅读绪论并给出计算如下函数的可靠数值计算方法，使其尽量达到更好的精度（10分=4+3+3）

(1). $f(x) = (a+x)^n - a^n$
(2). $f(x) = cos(a+x) - cos(a)$
(3). $f(x) = x - \sqrt{(x^2+a)}$

其中, **(1) (2)** 中的x很靠近0且$a > 0$, **(3)** 中$x >> a$

第(1)题

$$
\begin{aligned}
f(x) &= (a+x)^n - a^n \\
&= C_n^0 a^n + C_n^1 a^{n-1} x + \cdots + C_n^n x^n - a^n \\
&= C_n^1 a^{n-1} x + C_n^2 a^{n-2} x^2 + \cdots + C_n^n x^n
\end{aligned} \tag{1.1}
$$

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其中, **(1) (2)** 中的x很靠近0且$a > 0$, **(3)** 中$x >> a$

或

$$
\begin{aligned}
f(x) &= (a+x)^n - a^n \\
&= (a+x-a)((a+x)^{n-1} + a(a+x)^{n-2} + \cdots + a^{n-1}) \\
&= x((a+x)^{n-1} + a(a+x)^{n-2} + \cdots + a^{n-1})
\end{aligned}
\tag{1.2}
$$

第(2)题

$$
\begin{aligned}
f(x) &= cos(a+x) - cos(a) \\
&= cos(a+\frac{x}{2}+\frac{x}{2}) - cos(a+\frac{x}{2}-\frac{x}{2}) \\
\overset{\text{和角公式}}{=} & cos(a+\frac{x}{2})cos(\frac{x}{2}) - sin(a+\frac{x}{2})sin(\frac{x}{2}) - cos(a+\frac{x}{2})cos(\frac{x}{2}) \\
&-sin(a+\frac{x}{2})sin(\frac{x}{2}) \\
&= -2sin(a+\frac{x}{2})sin(\frac{x}{2})
\end{aligned} \tag{2}
$$

第(3)题

$$
\begin{aligned}
f(x) &= x - \sqrt{(x^2+a)} \\
&= \frac{(x-\sqrt{(x^2+a)})(x+\sqrt{(x^2+a)})}{x+\sqrt{(x^2+a)}} \\
&= \frac{x^2-(x^2+a)}{x+\sqrt{(x^2+a)}} \\
&= \frac{-a}{x+\sqrt{(x^2+a)}}
\end{aligned} \tag{3}
$$

需避免:

- **两相近数相减 –> 相对误差增大** (1, 2, 3题均属于这种情况)
- **小数作除数 –> 绝对误差增大**

2. 设有精确值$x^* = 0.0202005$, 则其近似值$x = 0.020200$有几位有效数字? 近似值x的绝对误差是多少? (5分)

解: 绝对误差 $e = x^* - x = 5 \times 10^{-7}$　　绝对误差限 $|e| = 5 \times 10^{-7} \leq \frac{1}{2} \times 10^{-6}$

有效位数从小数点后第六位开始算, $x = 0.0\underbrace{20200}_{5\text{位有效数字}}$

3. 设有插值节点$a \leq x_0 < x_1 < \cdots < x_n \leq b$. 证明与这些节点相应的$Lagrange$插值基函数 $\{l_i(x), i=0,1,\cdots,n\}$ 是线性无关的。(5分)

法一: 若$l_i(x)$线性相关, 则存在不全为0的系数a_i, 使得

$$a_0 l_0(x) + a_1 l_1(x) + \cdots + a_n l_n(x) = 0 \qquad \forall x \in \mathbb{R} \tag{1}$$

注意到

$$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} = \prod_{0\leq j\leq n, j\neq i} \frac{x-x_j}{x_i-x_j}$$

$$l_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

将x_i代入(1)中, 可得

$$a_i = 0 \qquad \forall i = 0,1,\cdots,n$$

故不存在不全为0的系数, $l_i(x)$线性无关

法二: 注意到, 若$l_i(x)$线性相关, 则线性方程组(将$x_0, x_1, \cdots, x_n$代入(1))

$$\begin{pmatrix} l_0(x_0) & l_1(x_0) & \cdots & l_n(x_0) \\ l_0(x_1) & l_1(x_1) & \cdots & l_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ l_0(x_n) & l_1(x_n) & \cdots & l_n(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = 0$$

有非零解, 故系数矩阵行列式为0

$$0 = \begin{vmatrix} l_0(x_0) & l_1(x_0) & \cdots & l_n(x_0) \\ l_0(x_1) & l_1(x_1) & \cdots & l_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ l_0(x_n) & l_1(x_n) & \cdots & l_n(x_n) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1$$

矛盾, 故$l_i(x)$线性无关

4. 利用插值数据$(-1.0, 0.0), (1.0, 1.0), (4.0, 2.0), (5.0, 4.0)$, 并构造出三次$Lagrange$插值多项式$L_3(x)$, 并计算$L_3(2.0), L_3(4.0)$. (10分)

解:

$$l_0(x) = ? \quad (\text{注意到} f(x_0) = 0, \text{可以偷懒不计算})$$

$$l_1(x) = \frac{(x+1)(x-4)(x-5)}{(1+1)(1-4)(1-5)} = \frac{1}{24}(x+1)(x-4)(x-5)$$

$$l_2(x) = \frac{(x+1)(x-1)(x-5)}{(4+1)(4-1)(4-5)} = -\frac{1}{15}(x+1)(x-1)(x-5)$$

$$l_3(x) = \frac{(x+1)(x-1)(x-4)}{(5+1)(5-1)(5-4)} = \frac{1}{24}(x+1)(x-1)(x-4)$$

$$\begin{aligned} L_3(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) + f(x_3)l_3(x) \\ &= \frac{1}{24}(x+1)(x-4)(x-5) - \frac{2}{15}(x+1)(x-1)(x-5) \\ &+ \frac{1}{6}(x+1)(x-1)(x-4) \end{aligned}$$

$$L_3(2.0) = \frac{19}{20} = 0.95 \qquad L_3(4.0) = 2.0 \ (\text{由插值性直接得出})$$

课后作业02

1. $f(x)=\sqrt{x}$ 在离散点有$f(81)=9$, $f(100)=10$, $f(121)=11$, 用插值方法计算$\sqrt{108}$的近似值, 根据误差公式给出误差界. (5分)

解: 用Lagrange插值多项式

$$l_0(x)=\frac{(x-100)(x-121)}{(81-100)(81-121)}=\frac{1}{760}(x-100)(x-121)$$
$$l_1(x)=\frac{(x-81)(x-121)}{(100-81)(100-121)}=-\frac{1}{399}(x-81)(x-121)$$
$$l_2(x)=\frac{(x-81)(x-100)}{(121-81)(121-100)}=\frac{1}{840}(x-81)(x-100)$$
$$L_2(x)=\frac{9}{760}(x-100)(x-121)-\frac{10}{399}(x-81)(x-121)+\frac{11}{840}(x-81)(x-100)$$

$$
\begin{aligned}
L_2(108) &= \frac{9}{760} \times 8 \times (-13) - \frac{10}{399} \times 27 \times (-13) + \frac{11}{840} \times 27 \times 8 \\
&= \frac{6912}{665} = 10.393984962 \\
f(108) &= \sqrt{108} = 10.392304845
\end{aligned}
$$

绝对误差: $R(x) = f(108) - L_2(108) = -0.001680117$

误差公式: $R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-81)(x-100)(x-121) \qquad \xi \in [81, 121]$

误差界: $R_2(108) = -468 f^{(3)}(\xi) = -\frac{351}{2}\xi^{-\frac{5}{2}} \in [-0.002972108, -0.001089717]$

2. 利用下面的函数值表, 作出差商表, 写出相应的牛顿插值多项式, 并计算$f(1.5)$的近似值. (5分)

x	1.0	2.0	3.0	4.0
$f(x)$	2.0	4.0	8.0	5.0

解:

i	x_i	$f(x_i)$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
0	1.0	2.0			
1	2.0	4.0	2.0		
2	3.0	8.0	4.0	1.0	
3	4.0	5.0	-3.0	-3.5	-1.5

$$\begin{aligned} N_3(x) =& f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ & + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\ =& 2 + 2(x - 1) + (x - 1)(x - 2) - 1.5(x - 1)(x - 2)(x - 3) \end{aligned}$$

$$f(1.5) \approx N_3(1.5) = \frac{35}{16} = 2.1875$$

3. 利用数据$f(0)=2.0$, $f(1)=0.5$, $f(3)=0.25$, $f'(3)=0.6$, 并构造出三次插值多项式, 写出其插值余项, 并计算$f(2)$的近似值. (5分)

解: 带导数值的为Hermite插值, 可用类似Lagrange插值或者类似Newton插值的方法求解

方法一 (Newton型):

i	x_i	$f(x_i)$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},x_{i-2},x_{i-1},x_i]$
0	0.0	2.0			
1	1.0	0.5	–1.5		
2	3.0	0.25	–0.125	$\frac{11}{24}\approx 0.4583$	
3	3.0	0.25	0.6	$\frac{29}{80}\approx 0.3625$	$-\frac{23}{720}\approx -0.031944$

得到插值多项式,

$H_3(x)=2-\frac{3}{2}x+\frac{11}{24}x(x-1)-\frac{23}{720}x(x-1)(x-3)$

插值余项为,

$$R(x)=f[x,0,1,3,3]x(x-1)(x-3)^2$$

$$f(2)\approx H_3(2)=-\frac{7}{360}=-0.019444$$

方法二 (Lagrange型):

已知$f(x_0), f(x_1), f(x_2), f'(x_2)$, 设插值多项式$H_3(x)$的基函数为$h_0(x)$, $h_1(x)$, $h_2(x)$, $g_0(x)$, 满足:

$H_3(x) = f(x_0)h_0(x) + f(x_1)h_1(x) + f(x_2)h_2(x) + f'(x_2)g_0(x)$, 并且

$$\begin{cases} h_0(0) = 1 \\ h_0(1) = 0 \\ h_0(3) = 0 \\ h_0'(3) = 0 \end{cases} \qquad \begin{cases} h_1(0) = 0 \\ h_1(1) = 1 \\ h_1(3) = 0 \\ h_1'(3) = 0 \end{cases} \qquad \begin{cases} h_2(0) = 0 \\ h_2(1) = 0 \\ h_2(3) = 1 \\ h_2'(3) = 0 \end{cases} \qquad \begin{cases} g_0(0) = 0 \\ g_0(1) = 0 \\ g_0(3) = 0 \\ g_0'(3) = 1 \end{cases}$$

接下来可推导$h_0(x)$, $h_1(x)$, $h_2(x)$, $g_0(x)$的具体形式

$$(1)\quad \begin{aligned} h_0(x) &= a_0(x-1)(x-3)^2 \qquad \text{let } x=0 \quad h_0(0) = -9a_0 = 1 \Rightarrow a_0 = -\frac{1}{9} \\ &= -\frac{1}{9}(x-1)(x-3)^2 \end{aligned}$$

$$(2)\quad \begin{aligned} h_1(x) &= a_1 x(x-3)^2 \qquad \text{let } x=1 \quad h_1(1) = 4a_1 = 1 \Rightarrow a_1 = \frac{1}{4} \\ &= \frac{1}{4}x(x-3)^2 \end{aligned}$$

$$(3)\quad \begin{cases} h_2(x) = x(x-1)(a_2+b_2x) \\ h_2'(x) = 3b_2x^2 + 2a_2x - 2b_2x - a_2 \end{cases} \begin{cases} h_2(3) = 6(a_2+3b_2) = 1 \\ h_2'(3) = 5a_2 + 21b_2 = 0 \end{cases}$$

$$\Rightarrow a_2 = \frac{7}{12} \quad b_2 = -\frac{5}{36} \quad h_2(x) = x(x-1)(\frac{7}{12} - \frac{5}{36}x)$$

$$(4)\quad \begin{aligned} g_0(x) &= a_3x(x-1)(x-3) \qquad g_0'(x) = a_3(3x^2-8x+3) \\ &= \frac{1}{6}x(x-1)(x-3) \qquad g_0'(3) = 6a_3 = 1 \Rightarrow a_3 = \frac{1}{6} \end{aligned}$$

得到插值函数,

$$H_3(x) = -\frac{2}{9}(x-1)(x-3)^2 + \frac{1}{8}x(x-3)^2 + \frac{1}{144}x(x-1)(-5x+21) \\ + \frac{1}{10}x(x-1)(x-3)$$

插值余项为,

$$R_3(x) = \frac{f^{(4)}(\xi_x)}{4!}x(x-1)(x-3)^2, \quad \xi_x \in [0,3]$$

$$f(2) \approx H_3(2) = -\frac{7}{360} = -0.019444$$

课后作业3

1. 求满足下表数据以及边界条件$S''(-2) = S''(2) = 0\ (n = 3)$的三次样条插值函数$S(x)$，并计算$S(0)$的值. 注意: 这里的$n$为小区间个数. (5分)

x	-2.0	-1.0	1.0	2.0
$f(x)$	-4.0	3.0	5.0	10.0

解: 易知,

$$h_0 = -1.0 - (-2.0) = 1.0 \quad h_1 = 1.0 - (-1.0) = 2.0 \quad h_2 = 2.0 - 1.0 = 1.0$$

由M关系式 (课本42页 1.23式), 注意到, $M_0 = S''(-2) = 0,\ M_3 = S''(2) = 0$

$$\begin{pmatrix} 2 & \lambda_1 \\ \mu_2 & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} d_1 - \mu_1 M_0 \\ d_2 - \lambda_2 M_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

其中:

$$\lambda_1 = \frac{h_1}{h_0 + h_1} = \frac{2}{3} \quad \mu_2 = \frac{h_1}{h_1 + h_2} = \frac{2}{3}$$

$$d_1 = 6f[x_0, x_1, x_2] = \frac{6}{h_0 + h_1}(\frac{f(x_2) - f(x_1)}{h_1} - \frac{f(x_1) - f(x_0)}{h_0}) = -12$$

$$d_2 = 6f[x_1, x_2, x_3] = \frac{6}{h_1 + h_2}(\frac{f(x_3) - f(x_2)}{h_2} - \frac{f(x_2) - f(x_1)}{h_1}) = 8$$

$$\Rightarrow \begin{pmatrix} 2 & \frac{2}{3} \\ \frac{2}{3} & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -12 \\ 8 \end{pmatrix} \Rightarrow \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -\frac{33}{4} \\ \frac{27}{4} \end{pmatrix} = \begin{pmatrix} -8.25 \\ 6.75 \end{pmatrix}$$

因此, 三次样条插值多项式为 (课本42页, 1.22式)

$$S(x)=\begin{cases} -\dfrac{11}{8}(x+2)^3+4(x+1)+\dfrac{35}{8}(x+2), & x\in[-2,-1] \\ \dfrac{11}{16}(x-1)^3+\dfrac{9}{16}(x+1)^3-\dfrac{17}{4}(x-1)+\dfrac{1}{4}(x+1), & x\in[-1,1] \\ -\dfrac{9}{8}(x-2)^3-\dfrac{31}{8}(x-2)+10(x-1), & x\in[1,2] \end{cases}$$

$$=\begin{cases} -1.375(x+2.0)^3+8.375x+12.75, & x\in[-2.0,-1.0] \\ 0.6875(x-1.0)^3+0.5625(x+1.0)^3-4.0x+4.5 & x\in[-1.0,1.0] \\ -1.125(x-2.0)^3+6.125x-2.25, & x\in[1.0,2.0] \end{cases}$$

得到:

$$S(0)=\frac{35}{8}=4.375$$

2. 利用下面的函数值表, 构造分段线性插值函数, 并计算$f(1.075)$和$f(1.175)$的近似值(保留4位小数). (5分)

x	1.05	1.10	1.15	1.20
$f(x)$	2.0	2.20	2.17	2.35

解: 记分段线性插值函数$p(x)=p_i(x), i=0,1,2$, 则

$$p_0(x)=\frac{x-x_1}{x_0-x_1}f(x_0)+\frac{x-x_0}{x_1-x_0}f(x_1)=4x-2.2,\quad x\in[1.05,1.10]$$

$$p_1(x)=\frac{x-x_2}{x_1-x_2}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)=-0.6x+2.86,\quad x\in[1.10,1.15]$$

$$p_2(x)=\frac{x-x_3}{x_2-x_3}f(x_2)+\frac{x-x_2}{x_3-x_2}f(x_3)=3.6x-1.97,\quad x\in[1.15,1.20]$$

$$f(1.075)=p_0(1.075)=2.1$$

$$f(1.175)=p_2(1.175)=2.26$$

3. 设$f(x)=10x^3+3x+2020$, 求$f[1,2]$和$f[1,2,3,4]$. (4分)

解: $f[x_0,x_1,\cdots,x_k]=\frac{f[x_1,x_2,\cdots,x_k]-f[x_0,x_1,\cdots,x_{k-1}]}{x_k-x_0}$ (课本26页, 1.11式)

x_i	$f(x_i)$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},x_{i-2},x_{i-1},x_i]$
1	2033			
2	2106	73		
3	2299	193	60	
4	2672	373	90	10

故 $f[1,2]=73,\quad f[1,2,3,4]=10$

另, 由于$f(x)$为三次多项式, 其三次牛顿插值多项式与$f(x)$相同, 由牛顿插值多项式的性质可知$f[1,2,3,4]$等于三次项的系数

4. 设$\{l_i(x)\}_{i=0}^6$是以$\{x_i=2i\}_{i=0}^6$为节点的6次Lagrange插值基函数. 试求$\sum_{i=0}^6(x_i^3+x_i^2+1)l_i(x)$和$\sum_{i=0}^6(x_i^3+x_i^2+1)l_i'(x)$(结果需化简). (6分)

解: 令$f(x)=x^3+x^2+1$, 则

$$L_6(x)=\sum_{i=0}^6(x_i^3+x_i^2+1)l_i(x)=\sum_{i=0}^6 f(x_i)l_i(x)=f(x)-R_6(x)$$

其中

$$R_6(x)=\frac{f^7(\xi)}{7!}(x-x_0)(x-x_1)\cdots(x-x_6)=0\quad(f^7(x)=0,\forall x)$$

因而, $\sum_{i=0}^6(x_i^3+x_i^2+1)l_i(x)=x^3+x^2+1$

同时, $\sum_{i=0}^6(x_i^3+x_i^2+1)l_i'(x)=(\sum_{i=0}^6(x_i^3+x_i^2+1)l_i(x))'=3x^2+2x$

课后作业4

1. 给出下列数据, 用最小二乘法求形如$y = ae^{bx}$的经验公式. (5分)

x_i	–0.60	–0.50	0.25	0.75
y_i	1.00	1.25	2.50	4.25

解: 对$y = ae^{bx}$两端同时作用对数函数得到, $\ln y = \ln a + bx$

令$z_i = \ln y_i$, 可得离散数据$\{(x_i, z_i)\}_{i=1}^4$

x_i	–0.60	–0.50	0.25	0.75
z_i	0.0	0.2231	0.9163	1.4469

对数组$\{(x_i, z_i)\}_{i=1}^4$进行线性拟合$z = A + Bx$, 可得**法方程**为 (课本50页, 2.1式)

$$\begin{pmatrix} 4 & \sum\limits_{i=1}^4 x_i \\ \sum\limits_{i=1}^4 x_i & \sum\limits_{i=1}^4 x_i^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^4 z_i \\ \sum\limits_{i=1}^4 x_i z_i \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & -0.1 \\ -0.1 & 1.235 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2.5863 \\ 1.2027 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0.6723 \\ 1.0283 \end{pmatrix}$$

得到 $a = e^A = 1.9587, \quad b = B = 1.0283$

即 $y = 1.9587e^{1.0283x}$

2. 在最小二乘法原理下求下列矛盾方程组:（5分）

$$\begin{cases} x_1 - 2x_2 = 4 \\ x_1 + 6x_2 = 14 \\ 3x_1 + x_2 = 7.5 \\ x_1 + x_2 = 4.5 \end{cases}$$

解: 将线性方程组写成矩阵的形式 $Ax = Y$

$$\begin{pmatrix} 1 & -2 \\ 1 & 6 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 7.5 \\ 4.5 \end{pmatrix}$$

矛盾方程组$Ax=Y$的法方程为$A^TAx=A^TY$（课本55–56页）

$$\begin{pmatrix} 1 & 1 & 3 & 1 \\ -2 & 6 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 6 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 3 & 1 \\ -2 & 6 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 14 \\ 7.5 \\ 4.5 \end{pmatrix}$$

$$\Rightarrow\begin{pmatrix} 12 & 8 \\ 8 & 42 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 45 \\ 88 \end{pmatrix}$$

解得

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} \frac{593}{220} \\ \frac{87}{55} \end{pmatrix}=\begin{pmatrix} 2.69545 \\ 1.58182 \end{pmatrix}$$

3. 利用最小二乘法构造二次多项式$y=p(x)$去拟合下列数据(这里x代表年份, y为人数), 并计算$y(2015)$, 结果精确到小数点后一位. (8分)

x	2010	2011	2012	2013	2014
y	134091	134735	135404	136072	136782

解: 设$p(x)=a(x-2012)^2+b(x-2012)+c+135404$, 得到矛盾方程组

$$\begin{pmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1313 \\ -669 \\ 0 \\ 668 \\ 1378 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 259 \\ 6719 \\ 64 \end{pmatrix}$$

解得

$$\begin{cases} a = \dfrac{131}{14} \\ b = \dfrac{6719}{10} \\ c = -\dfrac{207}{35} \end{cases} \qquad p(2015)=9a+3b+c+135404=137498.0$$

课后作业5

1. 给定$n > 1$, 给出牛顿法计算$\sqrt[n]{a}\ (a > 0)$时的迭代公式, 并用此公式来计算$\sqrt[5]{9}$, 取初值$x_0 = 2$, 迭代3次, 求x_3. (5分)

解: 令$f(x) = x^n - a$, $f(x)$的根为$\sqrt[n]{a}$, 由牛顿迭代公式 (课本65页, 式3.2)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^n - a}{n x_k^{n-1}}$$

代入$n = 5, a = 9, x_0 = 2$

$$x_1 = x_0 - \frac{x_0^5 - 9}{5x_0^4} = 1.71250$$

$$x_2 = x_1 - \frac{x_1^5 - 9}{5x_1^4} = 1.57929$$

$$x_3 = x_2 - \frac{x_2^5 - 9}{5x_2^4} = 1.55278$$

2. 写出对方程$x^3-4x^2+5x-2=0$求根时的Newton迭代公式$x_n=\varphi(x_n-1)$. 取初值$x_0=0$, 判断极限$\lim\limits_{n\to\infty}x_n$是否存在; 请给出你的理由或证明. (10分)

解: 牛顿迭代公式为,

$$x_n=x_{n-1}-\frac{x_{n-1}^3-4x_{n-1}^2+5x_{n-1}-2}{3x_{n-1}^2-8x_{n-1}+5}=x_{n-1}-\frac{(x_{n-1}-1)(x_{n-1}-2)}{3x_{n-1}-5}$$

$$\implies \varphi(x)=x-\frac{(x-1)(x-2)}{3x-5}$$

若极限$\lim\limits_{n\to\infty}x_n=x^*$存在, 则 $\varphi(x^*)=x^*$

推测$x^*=1$或$x^*=2$可能是不动点

法一：尝试证明 $\lim\limits_{n\to\infty}(1-\varphi(x_n))=0$

对 $1-\varphi(x)$ 进行化简, 得到

$$1-\varphi(x)=1-x-\frac{(1-x)(x-2)}{3x-5}=\frac{(1-x)(2x-3)}{3x-5}$$

当$0\leq x\leq 1$时, (注意$x_0=0$)

$$\frac{1}{2}\leq\frac{2x-3}{3x-5}\leq\frac{3}{5}\implies 0\leq 1-\varphi(x)\leq 1\implies 0\leq\varphi(x)\leq 1$$

于是当$n\geq 1, 0\leq x_{n-1}\leq 1$时, 迭代得到的下一项$0\leq x_n\leq 1$

由$x_n=\varphi(x_{n-1})$知,

$$\begin{aligned}0\leq 1-x_n=1-\varphi(x_{n-1})&=\frac{(1-x_{n-1})(2x_{n-1}-3)}{3x_{n-1}-5}\\&\leq(1-x_{n-1})\times\frac{3}{5}\\&\leq(1-x_{n-2})\times(\frac{3}{5})^2\\&\leq(1-x_0)\times(\frac{3}{5})^n=(\frac{3}{5})^n\end{aligned}$$

当$n\to+\infty$时, $\lim\limits_{n\to+\infty}x_n=1$

法二：利用压缩映射定理 (课本62页, 定理3.1), 需要满足两个条件

1. 当$x \in [a,b]$时, 有$a \le \varphi(x) \le b$
2. $\varphi(x)$在$[a,b]$上可导, 并且存在正数$L<1$, 使对任意的$x \in [a,b]$, 有$|\varphi'(x)| \le L$

注意到,

$$\varphi'(x) = \frac{f(x)f''(x)}{(f'(x))^2} = \frac{(x^3-4x^2+5x-2)(6x-8)}{(3x^2-8x+5)^2} = \frac{(x-2)(6x-8)}{(3x-5)^2}$$

则$\varphi(x)$的极值点位于$x=\frac{4}{3}$和$x=2$, 于是$\varphi(x)$在$x \in [0,1]$时单调递增且连续, 又因为$\varphi(0)=\frac{2}{5}, \varphi(1)=1$, 于是 $x \in [0,1]$时, $0 \le \varphi(x) \le 1$

为了估计$\varphi'(x)$的范围, 接下来求$\varphi(x)$的二阶导数

$$\varphi''(x) = \frac{4}{(3x-5)^3}$$

知$x \in [0,1]$时, $\varphi''(x)<0$, 于是$\varphi'(x)$在$[0,1]$上单调递减

而$\varphi'(0)=\frac{16}{25}, \varphi'(1)=\frac{1}{2}$, 所以, $x \in [0,1]$时 $|\varphi'(x)| \le \frac{16}{25} < 1$

由压缩映射定理知, 在$[0,1]$区间上, $\varphi(x)$必有唯一的不动点, 即$x=1$

课后作业6

1. 分别计算下列矩阵的$\|\cdot\|_1$, $\|\cdot\|_\infty$范数: (4分)

$$(a)\ A=\begin{pmatrix} 5 & -2 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -6 \end{pmatrix},(b)\ A=\begin{pmatrix} 4 & -1 & 1 \\ -2 & 3 & 1 \\ -1 & 1 & 6 \end{pmatrix}$$

解: 记忆方式,

- 1是竖着的所以是`列`(绝对值)和的最大值
- ∞是横着的所以是`行`(绝对值)和的最大值

(a) $\|A\|_1=\max\{7,7,8\}=8$

$\|A\|_\infty=\max\{8,5,9\}=9$

(b) $\|A\|_1=\max\{7,5,8\}=8$

$\|A\|_\infty=\max\{6,6,8\}=8$

2. 分别计算矩阵$A = \begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$的谱半径及其$\|\cdot\|_2$范数。(4分)

解:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 2 \\ 1 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda - 1) - 2 = \lambda^2 - 5\lambda + 2$$

$$\Rightarrow \lambda_1 = \frac{5 + \sqrt{17}}{2} = 4.56155 \qquad \lambda_2 = \frac{5 - \sqrt{17}}{2} = 0.43845$$

所以A的谱半径, $\rho(A) = \frac{5+\sqrt{17}}{2} = 4.56155$

根据2范数的定义, $\|A\|_2 = \sqrt{\rho(A^T A)}$, 所以

$$A^T A = \begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 17 & -9 \\ -9 & 5 \end{pmatrix}$$

$$\det(\lambda I - A^T A) = \begin{vmatrix} \lambda - 17 & 9 \\ 9 & \lambda - 5 \end{vmatrix} = \lambda^2 - 22\lambda + 4 = 0$$

$$\Rightarrow \lambda_1 = 11 + 3\sqrt{13} \qquad \lambda_2 = 11 - 3\sqrt{13}$$

从而

$$\|A\|_2 = \sqrt{11 + 3\sqrt{13}} = \frac{\sqrt{26} + 3\sqrt{2}}{2} = 4.67083$$

3. 用Doolittle分解法解下列线性方程组（请给出详细的解题过程，包括矩阵分解）（LU分解6分，解方程4分，共10分）

$$\begin{cases} 5x_1 + x_2 + 2x_3 = 10 \\ x_1 + 3x_2 - x_3 = 5 \\ 2x_1 + x_2 + 5x_3 = 20 \end{cases}$$

解: Doolittle分解, $A = LU$, 其中L为单位下三角阵, U为上三角阵 (课本85页, 4.7式)

$$\begin{aligned}
\begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 1 & 5 \end{pmatrix} &= \begin{pmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ & u_{22} & u_{23} \\ & & u_{33} \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{2}{5} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ & \frac{14}{5} & -\frac{7}{5} \\ & & u_{33} \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{2}{5} & \frac{3}{14} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 5 & 1 & 2 \\ & \frac{14}{5} & -\frac{7}{5} \\ & & \frac{9}{2} \end{pmatrix}}_{U}
\end{aligned}$$

下面利用分解结果解线性方程组, 首先求y使得$Ly = b$

$$\begin{pmatrix} 1 & & \\ \frac{1}{5} & 1 & \\ \frac{2}{5} & \frac{3}{14} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ \frac{215}{14} \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 15.3571 \end{pmatrix}$$

再求x使得$Ux = y$

$$\begin{pmatrix} 5 & 1 & 2 \\ & \frac{14}{5} & -\frac{7}{5} \\ & & \frac{9}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ \frac{215}{14} \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{63} \\ \frac{25}{9} \\ \frac{215}{63} \end{pmatrix} = \begin{pmatrix} 0.0794 \\ 2.7778 \\ 3.4127 \end{pmatrix}$$

课后作业7

1. 设有线性代数方程组$Ax = b$, 其中,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

(a) 求Jacobi迭代的迭代矩阵及相应的迭代格式; (4分)

(b) 讨论此时Jacobi迭代（方法）的收敛性. (6分)

解: (a) Jacobi迭代矩阵 (课本99页)

$$R = I - D^{-1}A$$

$$R = I - D^{-1}A$$

$$= I - \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$= I - \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$= I - \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

迭代常数项向量 $g = D^{-1}b = (1,1,1,1)^T$

迭代格式的矩阵形式 $X^{(k+1)} = RX^{(k)} + g$

分量形式:

$$\begin{cases} x_1^{(k+1)} = \frac{1}{2}x_2^{(k)} + 1 \\ x_2^{(k+1)} = \frac{1}{2}x_1^{(k)} + \frac{1}{2}x_3^{(k)} + 1 \\ x_3^{(k+1)} = \frac{1}{2}x_2^{(k)} + \frac{1}{2}x_4^{(k)} + 1 \\ x_4^{(k+1)} = \frac{1}{2}x_3^{(k)} + 1 \end{cases}$$

(b) 注意, A不是严格对角优, 不能以此作为收敛性的判断条件. 迭代收敛的 **充分必要** 条件是迭代矩阵的谱半径$\rho(R)=\max\limits_{1\le i\le n}|\lambda_i|<1$

$$\begin{aligned}\det(\lambda I-R)&=\begin{vmatrix}\lambda & -\frac{1}{2} & & \\ -\frac{1}{2} & \lambda & -\frac{1}{2} & \\ & -\frac{1}{2} & \lambda & -\frac{1}{2}\\ & & -\frac{1}{2} & \lambda\end{vmatrix}\\ &=\lambda\begin{vmatrix}\lambda & -\frac{1}{2} & \\ -\frac{1}{2} & \lambda & -\frac{1}{2}\\ & -\frac{1}{2} & \lambda\end{vmatrix}+\frac{1}{2}\begin{vmatrix}-\frac{1}{2} & -\frac{1}{2} & \\ & \lambda & -\frac{1}{2}\\ & -\frac{1}{2} & \lambda\end{vmatrix}\\ &=\lambda(\lambda^3-\frac{1}{2}\lambda)+\frac{1}{2}(-\frac{1}{2}(\lambda^2-\frac{1}{4}))\\ &=\lambda^4-\frac{3}{4}\lambda^2+\frac{1}{16}\end{aligned}$$

$$\Rightarrow \lambda^2=\frac{3\pm\sqrt{5}}{8}$$

$$\Rightarrow |\lambda_1|=|\lambda_2|=\sqrt{\frac{3+\sqrt{5}}{8}}=0.809017,\ |\lambda_3|=|\lambda_4|=\sqrt{\frac{3-\sqrt{5}}{8}}=0.309017$$

故而, $\rho(R)=\max\limits_{1\le i\le 4}|\lambda_i|=0.809017<1$, Jacobi迭代收敛.

2. 设有线性代数方程组

$$\begin{cases} 5x_1 - 3x_2 + 2x_3 = 5 \\ -3x_1 + 5x_2 + 2x_3 = 5 \\ 2x_1 + 2x_2 + 7x_3 = 7 \end{cases}$$

(a) 写出Gauss–Seidel迭代的分量形式; (5分)

(b) 求Gauss–Seidel迭代的分裂矩阵(splitting matrix)及迭代矩阵(iteration matrix); (4分)

(c) 讨论Gauss–Seidel迭代的收敛性(请给出理由或证明). (6分)

解: (a) 原方程组可写为

$$\begin{cases} x_1 = \frac{1}{5}(3x_2 - 2x_3 + 5) \\ x_2 = \frac{1}{5}(3x_1 - 2x_3 + 5) \\ x_3 = \frac{1}{7}(-2x_1 - 2x_2 + 7) \end{cases}$$

$\Longrightarrow$ Gauss–Seidel迭代的分量形式 (课本102页, 5.6式)

$$\begin{cases} x_1^{(k+1)} = \frac{1}{5}(3x_2^{(k)} - 2x_3^{(k)} + 5) \\ x_2^{(k+1)} = \frac{1}{5}(3x_1^{(k+1)} - 2x_3^{(k)} + 5) \\ x_3^{(k+1)} = \frac{1}{7}(-2x_1^{(k+1)} - 2x_2^{(k+1)} + 7) \end{cases}$$

(b) Gauss–Seidel的分裂矩阵为 (PPT ch5, 23页)

$$Q = D + L = \begin{pmatrix} 5 & 0 & 0 \\ -3 & 5 & 0 \\ 2 & 2 & 7 \end{pmatrix}$$

Gauss–Seidel的迭代矩阵为

$$\begin{aligned} G &= I - Q^{-1}A = -(D+L)^{-1}U = -Q^{-1}U \\ &= -\begin{pmatrix} 5 & 0 & 0 \\ -3 & 5 & 0 \\ 2 & 2 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & \frac{9}{25} & -\frac{16}{25} \\ 0 & -\frac{48}{175} & \frac{52}{175} \end{pmatrix} \end{aligned}$$

(c) 法一: 根据书上定理5.4, 当A对称正定时, Gauss–Seidel迭代收敛. 而实对称正定的等价条件之一便是所有顺序主子式大于0, 下面即验证A的所有顺序主子式大于零.

$$D_1 = \begin{vmatrix} 5 \end{vmatrix} = 5 > 0$$

$$D_2 = \begin{vmatrix} 5 & -3 \\ -3 & 5 \end{vmatrix} = 16 > 0$$

$$D_3 = \begin{vmatrix} 5 & -3 & 2 \\ -3 & 5 & 2 \\ 2 & 2 & 7 \end{vmatrix} = 16 \times 7 - 2 \times \begin{vmatrix} 5 & 2 \\ -3 & 2 \end{vmatrix} + 2 \times \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = 48 > 0$$

故A对称正定, Gauss–Seidel迭代收敛

法二: 计算迭代矩阵G的谱半径

$$\det(\lambda I - G) = \begin{vmatrix} \lambda & -\frac{3}{5} & \frac{2}{5} \\ 0 & \lambda - \frac{9}{25} & \frac{16}{25} \\ 0 & \frac{48}{175} & \lambda - \frac{52}{175} \end{vmatrix} = \lambda \begin{vmatrix} \lambda - \frac{9}{25} & \frac{16}{25} \\ \frac{48}{175} & \lambda - \frac{52}{175} \end{vmatrix}$$

$$= \lambda(\lambda^2 - \frac{23}{35}\lambda - \frac{12}{175}) = 0$$

$$\Rightarrow \ \lambda_1 = 0, \lambda_2 = \frac{23 + \sqrt{865}}{70}, \lambda_3 = \frac{23 - \sqrt{865}}{70}$$

$$\rho(G) = \frac{23 + \sqrt{865}}{70} = 0.748727 < 1$$

故Gauss-Seidel迭代收敛

另可以, 将$\lambda = -1, \lambda = 0, \lambda = 1$代入$f(\lambda) = \lambda^2 - \frac{23}{35}\lambda - \frac{12}{175}$中验证

得到$f(-1) > 0, f(0) < 0, f(1) > 0$得到$f(\lambda)$在$(-1, 1)$间必有两个零点, 所以$|\lambda_2| < 1, |\lambda_3| < 1$

课后作业8

1. 给定函数$f(x)$离散值如下, 分别用复化梯形和复化Simpson公式计算$\int_{1.0}^{1.8} f(x)dx$. (6分)

x	1.0	1.2	1.4	1.6	1.8
$f(x)$	3.2	3.5	5.0	5.2	4.8

解: 可以看到这里对积分区间做了4等分, 其对应的复化梯形公式为 (课本118页, 6.6式)

$$
\begin{aligned}
T_4(f) &= h[\frac{1}{2}f(a) + \sum_{i=1}^{3} f(a+ih) + \frac{1}{2}f(b)] \\
&= 0.2[\frac{1}{2}f(1.0) + \sum_{i=1}^{3} f(1.0+0.2i) + \frac{1}{2}f(1.8)] \\
&= 0.2[\frac{1}{2} \times 3.2 + 3.5 + 5.0 + 5.2 + \frac{1}{2} \times 4.8] \\
&= 3.54
\end{aligned}
$$

4等分区间对应的复化Simpson公式为（课本119页, 6.8式）

$$
\begin{aligned}
S_4(f) &= \frac{h}{3}[f(a) + 4\sum_{i=0}^{2-1} f(x_{2i+1}) + 2\sum_{i=1}^{2-1} f(x_{2i}) + f(b)] \\
&= \frac{0.2}{3}[f(1.0) + 4f(1.2) + 2f(1.4) + 4f(1.6) + f(1.8)] \\
&= \frac{0.2}{3}[3.2 + 4 \times 3.5 + 2 \times 5.0 + 4 \times 5.2 + 4.8] \\
&= 3.52
\end{aligned}
$$

2. 构造积分$\bar{I}(f)=\int_{-h}^{2h} f(x)dx$的数值积分公式, $I(f)=a_{-1}f(-h)+a_0f(0)+a_1f(2h)\ (h>0)$使其具有尽可能高的代数精度, 该公式的代数精度是多少? (6分)

解: 由于数值积分公式中有三个未知数a_{-1},a_0,a_1, 可假设其满足两阶代数精度, 令$f(x)$分别为$1,x,x^2$代入积分公式中

$$
\begin{cases}
a_{-1}+a_0+a_1=3h=\int_{-h}^{2h} 1\cdot dx \\
-ha_{-1}+2ha_1=\frac{3}{2}h^2=\int_{-h}^{2h} xdx \\
h^2a_{-1}+4h^2a_1=3h^3=\int_{-h}^{2h} x^2dx
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
a_{-1}=0 \\
a_0=\frac{9}{4}h \\
a_1=\frac{3}{4}h
\end{cases}
$$

此时我们知道所求格式至少满足两阶代数精度, 但是最高的代数精度为多少?

不要忘了再将$f(x) = x^3, x^4 \cdots$代入验证, 看所得格式的代数精度为多少

$$I(x^3) = \frac{3}{4}h \times 8h^3 = 6h^4 \neq \frac{17}{4}h^4 = \int_{-h}^{2h} x^3 dx$$

故该公式的代数精度为2

3. 记$I(f)=\int_{-2}^{2}f(x)dx$, 设$S(f(x))$为其数值积分公式, 其中, $I(f)\approx S(f(x))=Af(-\alpha)+Bf(0)+Cf(\alpha)$,

a) 试确定参数A,B,C,α使得该数值积分公式具有尽可能高的代数精度, 并求该公式的代数精度(需给出求解过程); (5分)

b) 设$f(x)$足够光滑(可微). 求该数值积分公式的误差. (5分)

解: a) 法一: 将$f(x)=1,x,x^2,x^3$分别代入原积分和数值积分公式, 得

$$\begin{cases} A+B+C=4=\int_{-2}^{2}1\cdot dx \\ -\alpha A+\alpha C=0=\int_{-2}^{2}xdx \\ \alpha^2A+\alpha^2C=\frac{16}{3}=\int_{-2}^{2}x^2dx \\ -\alpha^3A+\alpha^3C=0=\int_{-2}^{2}x^3dx \end{cases}$$

通过观察可发现，方程$-\alpha A+\alpha C=0$两端同时乘以α^2，可得方程$-\alpha^3 A+\alpha^3 C=0$故上述线性方程组线性相关，条件不足以解出4个未知数，需添加$f(x)=x^4$时的方程

$$\begin{cases} A+B+C=4=\int_{-2}^{2} 1\cdot dx \\ -\alpha A+\alpha C=0=\int_{-2}^{2} x dx \\ \alpha^2 A+\alpha^2 C=\frac{16}{3}=\int_{-2}^{2} x^2 dx \\ \alpha^4 A+\alpha^4 C=\frac{64}{5}=\int_{-2}^{2} x^4 dx \end{cases} \Rightarrow \begin{cases} A=\frac{10}{9} \\ B=\frac{16}{9} \\ C=\frac{10}{9} \\ \alpha=\pm\sqrt{\frac{12}{5}} \end{cases}$$

为进一步确定代数精度，将$f(x)=x^5,x^6,\cdots$代入积分公式

$$-\alpha^5 A+\alpha^5 C=0=\int_{-2}^{2} x^5 dx$$

$$\alpha^6 A+\alpha^6 C=\frac{768}{25}\neq\frac{256}{7}=\int_{-2}^{2} x^6 dx$$

可知该积分格式的代数精度为5阶

法二: 注意观察可知, 该积分公式的积分区间对称, 且三点Gauss–Legendre积分公式包含零点, 故可直接查询书上131页的表6.4得到

$$\alpha = \frac{(a+b)+(b-a)x_1^{(3)}}{2} = 2x_1^{(3)} = 1.549193$$

$$A = \frac{(b-a)}{2}\alpha_1^{(3)} = 2\alpha_1^{(3)} = 1.111111$$

$$B = \frac{(b-a)}{2}\alpha_2^{(3)} = 2\alpha_2^{(3)} = 1.777778$$

$$C = \frac{(b-a)}{2}\alpha_3^{(3)} = 2\alpha_3^{(3)} = 1.111111$$

代数精度应为$2n-1=5$阶

b) Gauss积分误差为 (PPT ch62_4, 50页)

$$
\begin{aligned}
E_n(f) = I(f) - S(f) &= \frac{f^{(2n)}(\xi)}{(2n)!}\int_a^b W(x)\omega_n^2(x)dx \\
&= \frac{f^{(6)}(\xi)}{6!}\int_{-2}^{2}(x-\alpha)^2x^2(x+\alpha)^2dx \\
&= \frac{f^{(6)}(\xi)}{6!}\int_{-2}^{2}(x^2-\alpha^2)^2x^2dx \\
&= \frac{f^{(6)}(\xi)}{6!}\int_{-2}^{2}(x^2-\frac{12}{5})^2x^2dx \\
&= \frac{f^{(6)}(\xi)}{6!}\int_{-2}^{2}x^6-\frac{24}{5}x^4+\frac{144}{25}x^2dx \\
&= \frac{f^{(6)}(\xi)}{6!}\frac{1024}{175} = \frac{64}{7875}f^{(6)}(\xi),\ \xi\in[-2,2]
\end{aligned}
$$

课后作业9

1. 给定函数$f(x)$的离散值表, 分别用向前、向后及中心差商公式计算$f'(0.02), f'(0.04)$. (6分)

x	0.0	0.02	0.04	0.06
$f(x)$	6.0	4.0	2.0	8.0

解: 向前、向后及中心差商的公式分别定义如下,

$$
\begin{cases}
f'(x_0) = \dfrac{f(x_0+h)-f(x_0)}{h} & \text{向前差商} \\
f'(x_0) = \dfrac{f(x_0)-f(x_0-h)}{h} & \text{向后差商} \\
f'(x_0) = \dfrac{f(x_0+h)-f(x_0-h)}{2h} & \text{中心差商}
\end{cases}
$$

故, 由以上定义可知

	向前差商	向后差商	中心差商
$f'(0.02)$	$\frac{f(0.04)-f(0.02)}{0.04-0.02}=-100.0$	$\frac{f(0.02)-f(0.0)}{0.02-0.0}=-100.0$	$\frac{f(0.04)-f(0.0)}{0.04-0.0}=-100.0$
$f'(0.04)$	$\frac{f(0.06)-f(0.04)}{0.06-0.04}=300.0$	$\frac{f(0.04)-f(0.02)}{0.04-0.02}=-100$	$\frac{f(0.06)-f(0.02)}{0.06-0.02}=100.0$

2. 求积分$\int_0^1 x^2 f(x)dx$的2点Gauss积分公式, 这里x^2为权重函数. (6分)

解: 注意这一题是带**权重函数**的Gauss积分, **不能**直接套书上关于Gauss–Legendre积分的公式

求解Gauss积分的三个步骤,

1. 求出区间$[a,b]$上权函数为$W(x)$的正交多项式$p_n(x) \perp \mathbb{P}_{n-1}$.
2. 求出$p_n(x)$的n个零点$x_1, x_2, \cdots, x_n$即为Gauss积分节点.
3. 计算积分系数$\alpha_i = \int_a^b W(x) l_i(x) dx$.

Gauss积分公式即为,

$$G_n(f) = \sum_{i=1}^{n} \alpha_i f(x_i)$$

注意这里的正交性由关于权函数的内积定义,

$$(f,g) \triangleq \int_a^b W(x)f(x)g(x)dx, \qquad f(x) \perp g(x) \Leftrightarrow (f,g) = 0$$

下面按照上述三步骤求解Gauss积分:

步骤一

$$p_0(x) = 1$$

$$p_1(x) = x - \frac{(x, p_0(x))}{(p_0(x), p_0(x))} p_0(x) = x - \frac{3}{4}$$

其中, $(x, p_0(x)) = \int_0^1 x^3 dx = \frac{1}{4}$, $(p_0(x), p_0(x)) = \int_0^1 x^2 dx = \frac{1}{3}$

$$p_2(x) = x^2 - \frac{(x^2, p_0(x))}{(p_0(x), p_0(x))} p_0(x) - \frac{(x^2, p_1(x))}{(p_1(x), p_1(x))} p_1(x) = x^2 - \frac{4}{3}x + \frac{2}{5}$$

其中, $(x^2, p_0(x)) = \int_0^1 x^4 dx = \frac{1}{5}$, $(x^2, p_1(x)) = \int_0^1 x^5 - \frac{3}{4}x^4 dx = \frac{1}{60}$
$(p_1(x), p_1(x)) = \int_0^1 x^2(x - \frac{3}{4})^2 dx = \frac{1}{80}$

步骤二

求解$p_2(x)$的两个零点 $\Longrightarrow \quad x_0=\frac{2}{3}+\frac{\sqrt{10}}{15}=0.87749,\ x_1=\frac{2}{3}-\frac{\sqrt{10}}{15}=0.45585$

步骤三

Gauss积分系数为,

$$
\alpha_0=\int_0^1 x^2\frac{x-x_1}{x_0-x_1}dx=\frac{8+\sqrt{10}}{48}=0.23255
$$
$$
\alpha_1=\int_0^1 x^2\frac{x-x_0}{x_1-x_0}dx=\frac{8-\sqrt{10}}{48}=0.10079
$$

最后得到Gauss积分公式为,

$$
G_2(f)=0.23255f(0.87749)+0.10079f(0.45585)
$$

3. 设有常微分初值问题

$$\begin{cases} y'(x) = -y(x), (0 \le x \le 1) \\ y(0) = 1 \end{cases}$$

假设求解区间$[0,1]$被n等分(n充分大), 令$h = \frac{1}{n}, x_k = \frac{k}{n}(k = 0, 1, \cdots, n)$,

(a) 分别写出用**向前Euler公式**, **向后Euler公式**, **梯形格式**以及**改进的Euler公式**求上述微分方程数值解时的差分公式 (即分别写出此四种方法/公式下, y_{k+1}与y_k之间的递推关系式). (4分, 每种方法给1分)

(b) 设$y_0 = y(0)$, 分别求此四种公式(方法)下的近似值y_n的表达式; (注: 这里的y_n即是$y(x_n) \equiv y(1)$的近似值). (4分, 每种方法给1分)

(c) 当n充分大(即区间长度$h \to 0$)时, 分别判断四种方法下的近似值y_n是否收敛到原问题的真解$y(x)$在$x = 1$处的值(i.e., $y(1)$). (8分, 每种方法给2分)

解: (a) 注意这里的$f(x,y)=-y \implies f(x_k,y_k)=f(x_{k+1},y_k)=-y_k\neq -y(x_k)$

向前Euler公式为 (课本140页, 7.2式)

$$y_{k+1}=y_k+hf(x_k,y_k)=y_k-\frac{1}{n}y_k=(1-\frac{1}{n})y_k$$

向后Euler公式为 (课本142页, 7.3式)

$$y_{k+1}=y_k+hf(x_{k+1},y_{k+1})=y_k-\frac{1}{n}y_{k+1} \implies y_{k+1}=(1-\frac{1}{n+1})y_k$$

梯形格式为 (课本146页)

$$y_{k+1}=y_k+\frac{h}{2}(f(x_k,y_k)+f(x_{k+1},y_{k+1}))=y_k-\frac{1}{2n}(y_k+y_{k+1})$$
$$\implies y_{k+1}=(1-\frac{2}{2n+1})y_k$$

改进的Euler公式为 (课本146页, 7.8式)

$$\begin{aligned} y_{k+1}&=y_k+\frac{h}{2}(f(x_k,y_k)+f(x_{k+1},y_k+hf(x_k,y_k)))\\ &=y_k+\frac{1}{2n}(-y_k+(-y_k+\frac{1}{n}(-y_k)))\end{aligned}$$
$$\implies y_{k+1}=(1-\frac{1}{n}+\frac{1}{2n^2})y_k$$

(b) 由(a)中的y_{k+1}与y_k之间的递推关系式, 以及$y_0 = 1$

向前Euler公式

$$y_n = (1-\frac{1}{n})^n y_0 = (1-\frac{1}{n})^n$$

向后Euler公式

$$y_n = (1-\frac{1}{n+1})^n y_0 = (1-\frac{1}{n+1})^n$$

梯形公式

$$y_n = (1-\frac{2}{2n+1})^n y_0 = (1-\frac{2}{2n+1})^n$$

改进Euler公式

$$y_n = (1-\frac{1}{n}+\frac{1}{2n^2})^n y_0 = (1-\frac{1}{n}+\frac{1}{2n^2})^n$$

(c) 易知原方程的解析解为$y(x)=e^{-x}$，则真解$y(1)=\frac{1}{e}$，下面验证$n\to\infty$时，四种方法下近似值y_n的收敛性，这里用到了数学分析中的一个极限

$$\lim_{n\to\infty}(1+\frac{1}{n})^n=e \implies \lim_{n\to\infty}(1-\frac{1}{n})^n=\frac{1}{e}$$

向前Euler公式

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}(1-\frac{1}{n})^n=\frac{1}{e}$$

向后Euler公式

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}(1-\frac{1}{n+1})^n=\lim_{n\to\infty}(1-\frac{1}{n+1})^{(n+1)\times\frac{n}{n+1}}=\frac{1}{e}$$

梯形公式

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}(1-\frac{2}{2n+1})^n=\lim_{n\to\infty}(1-\frac{2}{2n+1})^{\frac{2n+1}{2}\times\frac{2n}{2n+1}}=\frac{1}{e}$$

改进Euler公式

$$\begin{aligned}\lim_{n\to\infty}y_n&=\lim_{n\to\infty}(1-\frac{1}{n}+\frac{1}{2n^2})^n=\lim_{n\to\infty}(1-\frac{2n-1}{2n^2})^n\\&=\lim_{n\to\infty}(1-\frac{2n-1}{2n^2})^{\frac{2n^2}{2n-1}\times\frac{2n^2-n}{2n^2}}=\frac{1}{e}\end{aligned}$$

课后作业10

1. 试推导例题7.4 (第3版教材151–152页)中的差分格式的局部截断误差, (10分)

$$y_{n+1} = y_{n-1} + \frac{h}{3}[7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})]$$

即验证

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

(提示: 将差分格式右端的某些项在某点处同时作Taylor展开)

解: 计算局部截断误差时,

- 假设之前的计算都是精确的, 即$y_{n-1} = y(x_{n-1})$
- 观察差分格式两端, 选择公共项最多的节点展开 (这里选择x_{n-1})

各项Taylor展开的结果为，

$$f(x_n, y_n) = y'(x_n)$$
$$= y'(x_{n-1}) + hy''(x_{n-1}) + \frac{1}{2}h^2y^{(3)}(x_{n-1}) + \frac{1}{6}h^3y^{(4)}(x_{n-1}) + \frac{1}{24}h^4y^{(5)}(x_{n-1})$$
$$+ O(h^5)$$

$$f(x_{n-1}, y_{n-1}) = y'(x_{n-1})$$

$$f(x_{n-2}, y_{n-2}) = y'(x_{n-2})$$
$$= y'(x_{n-1}) - hy''(x_{n-1}) + \frac{1}{2}h^2y^{(3)}(x_{n-1}) - \frac{1}{6}h^3y^{(4)}(x_{n-1}) + \frac{1}{24}h^4y^{(5)}(x_{n-1})$$
$$+ O(h^5)$$

$$y(x_{n+1})$$
$$= y(x_{n-1}) + 2hy'(x_{n-1}) + 2h^2y''(x_{n-1}) + \frac{4}{3}h^3y^{(3)}(x_{n-1}) + \frac{2}{3}h^4y^{(4)}(x_{n-1})$$
$$+ \frac{4}{15}h^5y^{(5)} + O(h^6)$$

根据Taylor展开结果制成如下表格

	$y(x_{n-1})$	$y'(x_{n-1})$	$y''(x_{n-1})$	$y^{(3)}(x_{n-1})$	$y^{(4)}(x_{n-1})$	$y^{(5)}(x_{n-1})$
y_{n-1}	1					
$f(x_n, y_n)$		1	h	$\frac{1}{2}h^2$	$\frac{1}{6}h^3$	$\frac{1}{24}h^4$
$f(x_{n-1}, y_{n-1})$		1				
$f(x_{n-2}, y_{n-2})$		1	$-h$	$\frac{1}{2}h^2$	$-\frac{1}{6}h^3$	$\frac{1}{24}h^4$
$y_{n+1} = y_{n-1} + \frac{h}{3}[7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})]$	1	$2h$	$2h^2$	$\frac{4}{3}h^3$	$\frac{1}{3}h^4$	$\frac{1}{9}h^5$
$y(x_{n+1})$	1	$2h$	$2h^2$	$\frac{4}{3}h^3$	$\frac{2}{3}h^4$	$\frac{4}{15}h^5$
$y(x_{n+1}) - y_{n+1}$					$\frac{1}{3}h^4$	$O(h^5)$

通过表格最后一项可看出

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

2. 试用线性多步法构造$p=1,q=2$时的隐式差分格式, 求该格式局部截断误差的误差主项并判断它的阶(即精度), 最后为该隐式格式设计一种合适的预估–校正格式. (10+1+4=15分)

解: $p=1 \implies$ 积分区间为$[x_{n-1},x_{n+1}]$

$q=2$ 的隐格式 $\implies$ 积分节点为$\{x_{n+1},x_n,x_{n-1}\}$

故有差分格式:

$$y_{n+1}=y_{n-1}+[\alpha_0 f(x_{n+1},y_{n+1})+\alpha_1 f(x_n,y_n)+\alpha_2 f(x_{n-1},y_{n-1})]$$

下一步, 根据$\alpha_j=\int_{x_{n-1}}^{x_{n+1}} l_j(x)dx$确定积分系数, 这里可作变换$y=x-x_n$来简化计算

$$\alpha_0=\int_{x_{n-1}}^{x_{n+1}}\frac{(x-x_n)(x-x_{n-1})}{(x_{n+1}-x_n)(x_{n+1}-x_{n-1})}dx=\frac{1}{2h^2}\int_{-h}^{h}y(y+h)dy=\frac{1}{3}h$$

$$\alpha_1=\int_{x_{n-1}}^{x_{n+1}}\frac{(x-x_{n+1})(x-x_{n-1})}{(x_n-x_{n+1})(x_n-x_{n-1})}dx=-\frac{1}{h^2}\int_{-h}^{h}(y-h)(y+h)dy=\frac{4}{3}h$$

$$\alpha_2=\int_{x_{n-1}}^{x_{n+1}}\frac{(x-x_{n+1})(x-x_n)}{(x_{n-1}-x_{n+1})(x_{n-1}-x_n)}dx=\frac{1}{2h^2}\int_{-h}^{h}(y-h)ydy=\frac{1}{3}h$$

从而得到差分格式,

$$y_{n+1}=y_{n-1}+h[\frac{1}{3}f(x_{n+1},y_{n+1})+\frac{4}{3}f(x_n,y_n)+\frac{1}{3}f(x_{n-1},y_{n-1})]$$

下一步求所得格式的局部截断误差，将右端各项均在x_{n-1}处Taylor展开

$$y_{n-1}=y(x_{n-1})$$

$$f(x_{n-1},y_{n-1})=y'(x_{n-1})$$

$$f(x_n,y_n)=y'(x_n)$$

$$=y'(x_{n-1})+hy''(x_{n-1})+\frac{1}{2}h^2y^{(3)}(x_{n-1})+\frac{1}{6}h^3y^{(4)}(x_{n-1})+\frac{1}{24}h^4y^{(5)}(x_{n-1})$$

$$+O(h^5)$$

注意对于节点x_{n+1}我们无法假设计算是精确的，故$y_{n+1}\neq y(x_{n+1})$，

因而$f(x_{n+1},y_{n+1})\neq f(x_{n+1},y(x_{n+1}))=y'(x_{n+1})$

为了计算$f(x_{n+1},y_{n+1})$在x_{n-1}处的Taylor展开，我们先计算$f(x_{n+1},y(x_{n+1}))$在x_{n-1}处的Taylor展开，再计算$f(x_{n+1},y_{n+1})$与$f(x_{n+1},y(x_{n+1}))$的关系

$$f(x_{n+1},y(x_{n+1}))=y'(x_{n+1})$$

$$=y'(x_{n-1})+2hy''(x_{n-1})+2h^2y^{(3)}(x_{n-1})+\frac{4}{3}h^3y^{(4)}(x_{n-1})+\frac{2}{3}h^4y^{(5)}(x_{n-1})$$

$$+O(h^5)$$

随后, 记 $T_{n+1}=y(x_{n+1})-y_{n+1}$

$$f(x_{n+1},y_{n+1})=f(x_{n+1},y(x_{n+1})-T_{n+1})=f(x_{n+1},y(x_{n+1}))-T_{n+1}f_y(x_{n+1},\xi)$$

对$y(x_{n+1})$进行Taylor展开

$$\begin{aligned} y(x_{n+1})=&\ y(x_{n-1})+2hy'(x_{n-1})+2h^2y''(x_{n-1})+\frac{4}{3}h^3y^{(3)}(x_{n-1})+\frac{2}{3}h^4y^{(4)}(x_{n-1})\\ &+\frac{4}{15}h^5y^{(5)}(x_{n-1})+O(h^6)\end{aligned}$$

通过合并Taylor展开的各项, 局部截断误差为

$$T_{n+1}=y(x_{n+1})-y_{n+1}=-\frac{1}{90}h^5y^{(5)}(x_{n-1})+\frac{h}{3}T_{n+1}f_y(x_{n+1},\xi)+O(h^6)$$

$$\begin{aligned}\implies T_{n+1}&=\frac{-\frac{1}{90}h^5y^{(5)}(x_{n-1})+O(h^6)}{1-\frac{h}{3}f_y(x_{n+1},\xi)}\\ &=(-\frac{1}{90}h^5y^{(5)}(x_{n-1})+O(h^6))(1+\frac{h}{3}f_y(x_{n+1},\xi)+\cdots)\\ &=(\underbrace{-\frac{1}{90}h^5y^{(5)}(x_{n-1})}_{\text{误差主项}}+O(h^6))\end{aligned}$$

该格式局部截断误差为$O(h^5) \to$ 具有4阶精度, 起步计算时使用不少于 **三阶** 的格式

可使用三阶Runge–Kutta公式

$$\begin{cases} \bar{y}_{n+1} = y_n + \dfrac{h}{6}(k_1 + 4k_2 + k_3) \\ \qquad k_1 = f(x_n, y_n) \\ \qquad k_2 = f(x_n + \dfrac{1}{2}h, y_n + \dfrac{1}{2}hk_1) \\ \qquad k_3 = f(x_n + h, y_n - hk_1 + 2hk_2) \\ y_{n+1} = y_{n-1} + h[\dfrac{1}{3}f(x_{n+1}, \bar{y}_{n+1}) + \dfrac{4}{3}f(x_n, y_n) + \dfrac{1}{3}f(x_{n-1}, y_{n-1})] \end{cases}$$