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1 Fundamental theorem of linear programming

1.1 Polyhedra

A *convex combination* of points $x_1, \dots, x_N \in \mathbb{R}^n$ is any point of the form

$$x = \alpha_1 x_1 + \cdots + \alpha_N x_N$$

where $\alpha_1 + \cdots + \alpha_N = 1$, $\alpha_1 \geq 0, \dots, \alpha_N \geq 0$. The set of all the convex combinations of points from a set $S \subseteq \mathbb{R}^n$ is called the *convex hull* of S and is denoted $\operatorname{conv} S$. A set is called *convex* if it coincides with its convex hull.

Exercise 1.1. Let $S \subseteq \mathbb{R}^n$. Show that $\operatorname{conv} S$ is the intersection of all the convex sets containing S.

Exercise 1.2. Show that a set $S \subseteq \mathbb{R}^n$ is convex if and only if for any points x_1, $x_2 \in S$ and for any $\alpha \in (0, 1)$ the point $x = \alpha x_1 + (1 - \alpha) x_2$ also belongs to S.

Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $x \in S$ is called a *corner* point (or extreme point) if it can not be represented as a convex combination of two other points from S.

Theorem. *(Krein-Milman). Let $S \subseteq \mathbb{R}^n$ be a compact convex set. Then S is the convex hull of its corner points.*

For a given matrix $A \in \mathbb{R}^{m \times n}$ we denote by A_i its i-th row. For a vector $b \in \mathbb{R}^m$ the notation b_i is used to denote the i-th element. The notation $Ax \leq b$ is used for the system of inequalities $A_i x \leq b_i$ for $i = 1, \dots, m$.

A *polyhedron* is defined as the set

$$\mathcal{P} = \mathcal{P}_{A,b} = \{x \in \mathbb{R}^n \colon Ax \leq b\}$$

for some matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$. Equivalently, a polyhedron is the intersection of a finite number of closed half-spaces. A bounded polyhedron is called *polytope*.

Problem 1.3. (Smale's 9th problem, open). Does there exist a polynomial $Q \in \mathbb{R}[m, n]$ and an algorithm which takes a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^m$ as input and determines if the polyhedron $\mathcal{P}_{A,b}$ is empty using at most $Q(m, n)$ basic arithmetic operations?

Inequalities $A_{i_1} x \leq b_{i_1}, \dots, A_{i_k} x \leq b_{i_k}$ are called *linearly independent* if row-vectors $A_{i_1}, \dots, A_{i_k}$ are linearly independent. If for some point $x \in \mathcal{P}$ the equality $A_i x = b_i$ holds, the constraint $A_i x \leq b_i$ is called *binding* or *active* at x.

Proposition 1.4. *Let $x \in \mathcal{P} = \mathcal{P}_{A,b}$. Then x is a corner point of $\mathcal{P}$ if and only if there are n linearly independent constraints which are active at x.*

Proof. ($\Longleftarrow$) Let $\widetilde{A}$ be a submatrix of A whose rows are n linearly independent constraints and let $\widetilde{b}$ be the corresponding subvector of b, so that $\widetilde{A}x = \widetilde{b}$. Suppose that there are some y, $z \in \mathcal{P}$ and $\alpha \in (0,1)$ such that $x = \alpha y + (1-\alpha)z$. Note that:

$$\widetilde{A}x = \alpha \widetilde{A}y + (1-\alpha)\widetilde{A}z \leq \alpha\widetilde{b} + (1-\alpha)\widetilde{b} = \widetilde{b},$$

with equality if and only if $\widetilde{A}y = \widetilde{A}z = \widetilde{b}$. But invertibility of $\widetilde{A}$ implies that $x = y = z = \widetilde{A}^{-1}\widetilde{b}$, which shows that x is a corner point.

($\Longrightarrow$) Let $x \in \mathcal{P}$ be a corner point and let $\widetilde{A}$ be the submatrix of A corresponding to all the constraints which are active at x. Assume that $\operatorname{rank} \widetilde{A} \leq n-1$.

Since $\widetilde{A}$ is not of full row rank, there is a vector $d \in \ker \widetilde{A}$, $d \neq 0$. Put $x_\pm = x \pm \varepsilon d$ and choose $\varepsilon > 0$ so small that all the non-active constraints are still valid for $x_\pm$. Then $x = (x_+ + x_-)/2$, showing that x is not a corner point. The contradiction proves the claim.

□

Proposition 1.5. *A bounded polyhedron $\mathcal{P} = \mathcal{P}_{A,b}$ is a convex hull of its corner points.*

Proof. Let V be the set of corner points of $\mathcal{P}$. One can see that $\mathcal{P}$ is convex showing that $\operatorname{conv} V \subseteq P$. We will show that for any $x \in \mathcal{P}$ we have $x \in \operatorname{conv} V$ using induction on the number of constraints active at x. Let $\widetilde{A}$ be the matrix of all the constraints which are active at x.

Induction basis. If $\operatorname{rank} \widetilde{A} = n$, then x is a corner point according to Proposition 1.4.

Induction hypothesis. Suppose that if $\operatorname{rank} \widetilde{A} \geq k+1$, then x is a convex combination of some points from V.

Inductive step. Suppose that rank $= k < n$, let $d \in \ker \widetilde{A}$, and put $x(\varepsilon) = x + \varepsilon d$. Since $\mathcal{P}$ is bounded, $x(\varepsilon) \in \mathcal{P}$ if and only if $\varepsilon_- \leq \varepsilon \leq \varepsilon_+$ for some $\varepsilon_- < 0 < \varepsilon_+$. But then $x(\varepsilon)$ is a convex combination of $x(\varepsilon_-)$ and $x(\varepsilon_+)$, and there are at least $k+1$ constraints which are active at $x(\varepsilon_\pm)$. We use the inductive hypothesis to represent $x(\varepsilon_\pm)$ as a convex combination of some points from V, which completes the proof.

□

Proposition 1.6. *A non-empty polyhedron $\mathcal{P} = \mathcal{P}_{A,b}$ has a corner point if and only if it does not contain lines.*

Proof. ($\Longrightarrow$) Suppose that $\mathcal{P}$ contains a line $x(t) = x + td$ for some x, $d \in \mathbb{R}^d$, $t \in \mathbb{R}$. It means that: $A(x+td) \leq b$ for all d, or:

$$tAd \leq b - Ax \quad \forall t.$$

It follows that $Ad = 0$ and $\operatorname{rank} A \leq n-1$, so that $\mathcal{P}$ can not have corner points according to Proposition 1.4.

($\Longleftarrow$) Suppose that $\mathcal{P}$ does not contain lines. Choose a point $x \in \mathcal{P}$ with the maximal number of active constraints, and denote the corresponding submatrix of A by $\widetilde{A}$.

Suppose that A does not have corner points. According to Proposition 1.4 it means that $\operatorname{rank} \widetilde{A} < n$. Let $d \in \ker \widetilde{A}$ and put:

$$x(t) = x + td,$$
$$t_- = \inf_{x(t) \in \mathcal{P}} t, \quad t_+ = \sup_{x(t) \in \mathcal{P}} t.$$

Since $\mathcal{P}$ does not contain lines, either t_-, or t_+ is finite. Assume without loss of generality that $t_+ < \infty$. Then there are more active constraints at $x(t_+)$ than at x which contradicts the choice of x and proves the claim. $\square$

1.2 Fundamental theorem of linear programming

A linear program in general form is an optimization problem of the form:

$$\begin{array}{rl} \text{maximize} & c_1^T x_1 + c_2^T x_2 \\ \text{subject to} & A_{11} x_1 + A_{12} x_2 \leq b_1 \\ & A_{21} x_1 + A_{22} x_2 = b_2 \\ & x_1 \geq 0, \end{array}$$

where $c_j \in \mathbb{R}^{n_j}$, $b_i \in \mathbb{R}^{m_i}$, $A_{ij} \in \mathbb{R}^{m_i \times n_j}$. A point $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ is *feasible* if it satisfies all the constraints. The set of all the feasible points is the *feasible region*.

Exercise 1.7. Show that a linear program in general form is equivalent to a linear program in symmetric form:

$$\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b, \\ x \geq 0. & \end{array} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Deduce from this equivalence that a feasible region of a linear program (in symmetric form and in general form as well) is a polyhedron.

Note that due to the inequalities $x \geq 0$ the feasible region of a linear program in symmetric form does not contain straight lines and contains a corner point according to Proposition 1.6.

Theorem 1.8. *(Fundamental theorem of LP). Consider the linear program in symmetric form* (1)*. One and only one of the following statements holds true:*

1. *The feasible region is empty.*

2. *The objective function is unbounded in the feasible region.*
3. *An optimal solution exists and is attained at some corner point of the feasible region.*

The theorem follows from the following lemma.

Lemma 1.9. *Consider the linear program in symmetric form (1). If the feasible region $\mathcal{P}$ is non-empty and the objective function is bounded from above in the feasible region, then an optimal solution exists and it attained in some corner point of the feasible region.*

Proof. Let $x \in \mathcal{P}$. We will show that there exists a corner point $x^* \in P$ for which $c^T x^* \geq c^T x$. Since a polyhedron can have only finite number of corner points, it will prove the theorem.

Let $\widetilde{A}$ be the submatrix of A corresponding to all the constraints which are active at x. We will prove the lemma using induction on $\operatorname{rank} \widetilde{A}$.

Basis of induction. If $\operatorname{rank} \widetilde{A} = n$, then x is a corner point and the theorem is proved.

Inductive hypothesis. Assume that the statement is valid if $\operatorname{rank} \widetilde{A} \geq k + 1$.

Inductive step. Suppose that $\operatorname{rank} \widetilde{A} = k < n$. Let $d \in \ker C$, $d \neq 0$ be such that $c^T d \geq 0$. Put

$$x(t) = x + td,$$
$$t_- = \inf_{x(t) \in \mathcal{P}} t, \quad t_+ = \sup_{x(t) \in \mathcal{P}} t$$

and consider three possible cases.

1. Suppose that $c^T d = 0$. Then either t_-, or t_+ is finite, since P does not contain lines because of the inequalities $x \geq 0$. Without loss of generality assume that $t_+ < \infty$. Then $x(t_+)$ has the same objective value as x but has a bigger number of active inequalities. The inductive hypothesis implies the claim.
2. Suppose that $c^T d > 0$ and $t_+ = \infty$. Then the objective is unbounded in the feasible region, contradicting the assumptions.
3. Suppose now that $c^T d > 0$ and $t_+ < \infty$. Then $x(t_+)$ has a greater objective value than x and satisfies the assumptions of the inductive hypothesis, which proves the claim.

□

2 Simplex method

2.1 Linear programs in standard form

Consider a linear program in standard form:

$$\begin{aligned} \text{maximize} \quad & z = c^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0, \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{m\times(m+n)}$ is a matrix of rank m, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^{m+n}$. Note that the feasible region of a linear program in standard form is a polyhedron:

$$\mathcal{P} = \{x \in \mathbb{R}^{m+n} \colon \widetilde{A}x \leq \widetilde{b}\},$$
$$\widetilde{A} = \begin{bmatrix} A \\ -A \\ -I(n+m) \end{bmatrix}, \quad \widetilde{b} = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix},$$

where $I(n+m)$ is the identity matrix of size $n+m$.

For a given subset of indices $S = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n+m\}$ we put:

$$x_S = (x_{j_1}, \dots, x_{j_k}),$$
$$A_S = (A_{j_1}, \dots, A_{j_k}),$$

where x_j denotes the j-th element of x, and A_j denotes the j-th row of A.

Proposition 2.1. *Let $A \in \mathbb{R}^{m\times(n+m)}$ be a matrix of rank m and let $b \in \mathbb{R}^m$. Then $x \in \mathbb{R}^{n+m}$ is a corner point of the feasible polyhedron $\mathcal{P}$ of the linear program (2) if and only if one can find m indices $B = \{j_1, \dots, j_m\}$ such that the matrix A_B has rank m and $x_B = A_B^{-1} b \geq 0$, $x_N = 0$, where $N = \{1, \dots, n+m\} \setminus B$.*

Proof. ($\Longrightarrow$) Suppose that x is a corner point of $\mathcal{P}$. Let J be the subset of indices such that $x_J > 0$ and $x_S = 0$ for $S = \{1, \dots, n+m\} \setminus J$. Assume without loss of generality that $J = \{1, \dots, k\}$. We will show that $\operatorname{rank} A_J = k$.

Note that $x = (x_J, x_S)$, $A = (A_J, A_S)$. The matrix of constraints, which are active at x, has the form:

$$\widetilde{A}^{\text{active}} = \begin{bmatrix} A_J & A_S \\ -A_J & -A_S \\ 0 & I(n+m-k) \end{bmatrix}.$$

Since x is a corner point of $\mathcal{P}$, using Proposition 1.4 we have that

$$n + m = \operatorname{rank} \widetilde{A}^{\text{active}} = \operatorname{rank} A_J + \operatorname{rank} I(n+m-k) = \operatorname{rank} A_J + n + m - k,$$

which shows that $\operatorname{rank} A_J = k$ so that A_J is of full column rank. Since $\operatorname{rank} A = m$ we can always extend J to some set of indices B of size m such that $\operatorname{rank} A_B = m$. Put $N = \{1, \ldots, n+m\} \setminus B$. Since $x \in \mathcal{P}$, we have that:

$$b = Ax = A_B x_B + A_N x_N = A_B x_B,$$

where we have used that $x_N = 0$. Since $\operatorname{rank} A_B = m$, it follows that $x_B = A_B^{-1} b$, which proves the claim.

($\Longleftarrow$) Let B be a subset of indices of size m such that $\operatorname{rank} A_B = m$ and let $x_B = A_B^{-1} b \geq 0$, $x_N = 0$. Clearly, x is a feasible point. The matrix of constraints which are active at x has the form:

$$\widetilde{A}^{\text{active}} = \begin{bmatrix} A_B & A_N \\ -A_B & -A_N \\ * & 0 \\ 0 & I(n) \end{bmatrix},$$

so that

$$\operatorname{rank} \widetilde{A}^{\text{active}} \geq \operatorname{rank} A_B + \operatorname{rank} I(n) = m + n.$$

This shows that x is a corner point by Proposition 1.4. $\square$

We return to the linear program (2). Let B be a subset of indices of size m such that the matrix A_B has full rank and $x_B = A_B^{-1} b \geq 0$. Without loss of generality we suppose that $B = (n+1, \ldots, n+m)$, and let $x = (x_N, x_B)$, $A = (A_N, A_B)$, $c = (c_N, c_B)$. The problem (2) can be rewritten in the form:

$$\begin{array}{rl} \text{maximize} & z = c_N^T x_N + c_B^T x_B, \\ \text{subject to} & A_N x_N + A_B x_B = b, \\ & x_N, x_B \geq 0. \end{array}$$

Since $\operatorname{rank} A_B = m$, we can express x_B in terms of variables x_N reducing the problem to the *canonical form* for the simplex method:

$$\begin{array}{rl} \text{maximize} & z = \bar{c}_N^T x_N + \bar{z}, \\ \text{subject to} & \overline{A}_N x_N + x_B = \bar{b}, \quad \bar{b} \geq 0, \\ & x_N, x_B \geq 0, \end{array} \tag{3}$$

where

$$\overline{A}_N = A_B^{-1} A_N, \quad \bar{b} = A_B^{-1} b, \quad \bar{c}_N = c_N - (A_B^{-1} A_N)^T c_B, \quad \bar{z} = c_B^T \bar{b}.$$

It is also convenient to rewrite (3) in coordinate form:

$$\begin{array}{r|ccccccccc} \text{maximize } z & \bar{c}_1 x_1 & + & \ldots & + & \bar{c}_n x_n & & & & +\bar{z} \\ \hline \text{subject to} & \bar{a}_{11} x_1 & + & \ldots & + & \bar{a}_{1n} x_n & +x_{n+1} & & & = \bar{b}_1 \\ & \bar{a}_{21} x_1 & + & \ldots & + & \bar{a}_{2n} x_n & & +x_{n+2} & & = \bar{b}_2 \\ & \bar{a}_{m1} x_1 & + & \ldots & + & \bar{a}_{mn} x_n & & & +x_{n+m} & = \bar{b}_m \end{array} \tag{4}$$

2.2 Simplex method

The simplex method is the most known algorithm for solving general linear programs. The simplex method requires a linear program in canonical form (3). The variables x_B are called *basic* and the variables x_N are called *non-basic*.

The *basic solution* corresponding to the basic variables x_B is obtained by setting $x_N = 0$, so that $x_B = \overline{b}$ and the objective value is $z = \overline{z}$. This solution is feasible if and only if $\overline{b} \geq 0$. Using Proposition 2.1 we immediately get the following.

Proposition 2.2. *x is a basic feasible solution of the problem (2) if and only if x is a corner point of the feasible polyhedron.*

The optimality condition follows from the formulation (3).

Proposition 2.3. *(Optimality certificate). Consider the linear program (4). If $\overline{c}_j \leq 0$ for all j, then the basic feasible solution $x_B = \overline{b}$, $x_N = \overline{0}$ is optimal.*

Remark 2.4. *If a basic feasible solution x is optimal, it does not necessarily imply that $c_j \leq 0$ for all j.*

If the optimality condition does not hold on a given corner point x of the feasible polyhedron, simplex method selects another corner point with the objective value which is not worse.

Assume that $\overline{c}_j > 0$ for some j and put $x_j = \Delta$ while keeping the other non-basic variables from $x_1, \ldots, x_n$ equal to zero. It follows from (4) that this will uniquely determine the values of the basic variables $x_{n+1}, \ldots, x_{n+m}$ and the objective value z as:

$$z = z(\Delta) = \overline{c}_j \Delta + \overline{z},$$
$$x_{n+i} = x_{n+i}(\Delta) = \overline{b}_i - \overline{a}_{ij}\Delta,$$

where $i \in \{1, \ldots, n\} \setminus \{j\}$, and let $x = x(\Delta)$ be the corresponding solution. From this formula we get the following proposition.

Proposition 2.5. *Consider the linear program (4). If $\overline{c}_j > 0$ and $\overline{a}_{ij} \leq 0$ for all i, then the objective value is unbounded from above in the feasible region.*

Now assume that

$$I_j = \{i \colon \overline{a}_{ij} > 0\} \neq \varnothing.$$

Feasibility of $x(\Delta)$ for $\Delta \geq 0$ is equivalent to the system:

$$\overline{b}_{ij} - \overline{a}_{ij}\Delta \geq 0, \quad \forall i \in I_j.$$

Let $i = \operatorname{argmin}\left(\overline{b}_{ij}/\overline{a}_{ij},\ i \in I_j\right)$, which is called the *min-ratio rule* for choosing i, and put

$$\Delta^\sharp = \frac{\overline{b}_{ij}}{\overline{a}_{ij}}, \quad x^\sharp = x(\Delta^\sharp), \quad z^\sharp = \overline{z} + c_j \Delta^\sharp.$$

Proposition 2.6. *Consider the linear program* (4). *Suppose that $\bar{c}_j > 0$ for some j, $I_j \neq \varnothing$, and i be choosen by the min-ratio rule. Then $x^\sharp$ is a corner point of the feasible region and $z^\sharp \geq \bar{z}$. Besides, if $\bar{b}_i > 0$, then $z^\sharp > \bar{z}$.*

Proof. Let $\widetilde{\mathcal{B}} = (j, n+1, \ldots, n+i-1, n+i+1, \ldots, n+m)$. Then the submatrix of $\overline{A} = [\overline{A}_N, I(m)]$ corresponding to these indices is given by:

$$\overline{A}_{\widetilde{\mathcal{B}}} = \begin{bmatrix} \bar{a}_{1j} & 1 & & & & & \\ \vdots & & \ddots & & & & \\ \bar{a}_{i-1,j} & & & 1 & 0 & & \\ \bar{a}_{i,j} & & & 0 & 0 & & \\ \bar{a}_{i+1,j} & & & 0 & 1 & & \\ \vdots & & & & & \ddots & \\ \bar{a}_{m,j} & & & & & & 1 \end{bmatrix}$$

On can see that rank $\overline{A}_{\widetilde{B}} = m$. Besides, since $x^\sharp \in P$, we also have

$$\bar{b} = \overline{A}x^\sharp = \overline{A}_{\widetilde{B}} x^\sharp_{\widetilde{B}}$$

which shows that

$$x^\sharp = A_{\widetilde{B}}^{-1}\bar{b} = \geq 0.$$

It follows from Proposition 2.1 that $x^\sharp$ is a corner point of the feasible region. The remaining statements are straightforward. □

This procedure constitutes a step of the simplex method. Note that it starts from some corner point of the feasible polyhedron and finds another corner point with the objective value which is not worse. Besides, if $\bar{b}_i > 0$, the objective value is improved. If $\bar{b} > 0$, the corresponding basic feasible solution is called non-degenerate. If all basic feasible solutions are *non-degenerate*, the linear program is called *non-degenerate*. Using Proposition 2.1 we get the following convergence result.

Proposition 2.7. *Suppose that the linear program* (3) *is non-degenerate. Then the simplex method converges to the optimal solution after at most $\binom{n}{m}$ steps.*

Remark 2.8. *(Bland's rule). Consider the linear program* (3) *without the assumption that it is non-degenerate. At each step of the simplex method choose the entering and the exiting variables with the smallest allowed indices. Then, if the problem has an optimal solution, simplex method will converge to an optimal solution such that $c_j \leq 0$ for all j.*

3 Duality

3.1 Bijection between corner points

Consider a linear problem in symmetric form:

$$\begin{array}{rl} \text{maximize} & c_D^T x_D \\ \text{subject to} & A_D x_D \leq b, \\ & x_D \geq 0, \end{array} \tag{5}$$

where $x_D = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Introducing slack variables $x_S = (x_{n+1}, \ldots, x_{n+m})$ we get an equivalent linear program in standard form:

$$\begin{array}{rl} \text{maximize} & c^T x_D \\ \text{subject to} & A_D x_D + x_S = b, \\ & x_D, x_S \geq 0. \end{array} \tag{6}$$

Note that x_D is feasible for program (5) if and only if $x = (x_D, b - Ax_D)$ is feasible for program (6).

Proposition 3.1. *A point $x_D \in \mathbb{R}^n$ is a corner point of program (5) if and only if the point $x = (x_D, b - A_D x_D) \in \mathbb{R}^{n+m}$ is a corner point of program (6) .*

Proof. Let x_D be a feasible point of program (5). Note that x_D is a corner point of program (5) if and only if there exist subsets of indices $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$ such that $|I| + |J| = n$, and x is the unique solution of the system:

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i, \quad i \in \mathcal{I}, \\ x_j &= 0, \quad j \in \mathcal{J}, \end{aligned}$$

But this is true if and only if $x = (x_D, b - A_D x_D)$ is the unique solution of the system:

$$\begin{aligned} A_D x_D + x_S &= b, \\ x_{n+i} &= 0, \quad i \in \mathcal{I}, \\ x_j &= 0, \quad j \in \mathcal{J}. \end{aligned}$$

We can rewrite this system in the form:

$$\begin{aligned} A_B x_B &= b, \\ x_N &= 0, \end{aligned}$$

where $A = [A_D, I(m)]$,

$$N = \mathcal{J} \cup \{n + i \colon i \in \mathcal{I}\}, \quad B = \{1, \ldots, n+m\} \setminus N.$$

Note that x is the unique solution to this system if and only if $\operatorname{rank} A_B = m$, $x_B = A_B^{-1} b$, $x_N = 0$. By Proposition 2.1 x is a corner point in program (6). $\square$

Remark 3.2. *It follows that x_D is an optimal solution to (5) if and only if $(x_D, b - A_D x_D)$ is an optimal solution to (6).*

3.2 Weak and strong duality

Consider a linear program in symmetric form:

$$\begin{array}{ll} \text{maximize} & c_D^T x \\ \text{subject to} & A_D x_D \le b, \\ & x_D \ge 0, \end{array} \tag{7}$$

where $A_D \in \mathbb{R}^{m\times n}$, $c_D \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. The *dual* linear program is defined by:

$$\begin{array}{ll} \text{minimize} & y^T b, \\ \text{subject to} & y^T A_D \ge c_D^T. \\ & y \ge 0. \end{array} \tag{8}$$

The original linear program is also called *primal* with respect to its dual.

Proposition 3.3. *(Weak duality). Let $x_D \in \mathbb{R}^n$ be a feasible solution of the primal problem and $y \in \mathbb{R}^m$ be a feasible solution of the dual problem. Then $c_D^T x_D \le y^T b$.*

Proof.

$$c_D^T x_D \le (y^T A_D) x_D = y^T (A_D x_D) \le y^T b.$$

□

As a corollary, if the primal objective is not bounded from above, the dual problem is not feasible. In a similar way, if the the objective in the dual problem is not bounded from below, the primal problem is not feasible.

Theorem 3.4. *(Strong duality) Assume that one of the following statements is valid:*

1. *The primal or the dual program has an optimal solution.*
2. *The primal and the dual programs are feasible.*

Then both the primal and the dual problems have optimal solutions, and optimal objective values coincide.

Proof. We will prove that if the primal problem has an optimal solution, then the dual also has an optimal solution, and optimal values coincide. We will use the simplex method to construct the optimal solution of the dual. The case of a dual problem admitting an optimal solution is left as an exercise.

The second statement of the present theorem follows from the fundamental theorem of linear programming, from weak duality, and from the first statement of the present theorem.

Assume that problem (5) has an optimal solution. It follows from Proposition 3.1 that the problem

$$\begin{array}{rl} \text{maximize} & c_D^T x_D \\ \text{subject to} & A_D x_D + x_S = b \\ & x_D = (x_1, \dots, x_n) \geq 0 \\ & x_S = (x_{n+1}, \dots, x_{n+m}) \geq 0, \end{array} \tag{9}$$

also has an optimal solution and the optimal objective values coincide. It is convenient to introduce notations $x = (x_D, x_S)$, $A = [A_D, I(m)]$, $c = (c_D, 0) \in \mathbb{R}^n$ and to rewrite (9) in the form:

$$\begin{array}{rl} \text{maximize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0. \end{array}$$

Apply the simplex method to this problem and let $B \subset \{1, \dots, n+m\}$, $B = |m|$ be the found indices of optimal basic variables, and $N = \{1, \dots, n+m\} \setminus B$. The problem can be rewritten as:

$$\begin{array}{rl} \text{maximize} & c_N^T x_N + c_B^T x_B \\ \text{subject to} & A_N x_N + A_B x_B = b \\ & x_N, x_B \geq 0, \end{array}$$

and, after pivoting,

$$\begin{array}{rl} \text{maximize} & (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1} b \\ \text{subject to} & A_B^{-1} A_N x_N + x_B = A_B^{-1} b, \\ & x_N, x_B \geq 0. \end{array}$$

From the optimality condition it follows that:

$$c_N^T - c_B^T A_B^{-1} A_N \leq 0.$$

Besides, we also have trivial inequalities

$$c_B^T - c_B^T A_B^{-1} A_B \leq 0.$$

Putting $y^T = c_B^T A_B^{-1} \in \mathbb{R}^m$ we can rewrite these two groups of inequalities together as:

$$c^T - y^T A \leq 0.$$

Restricting this to the sets of indices $D = \{1, \dots, n\}$ and $S = \{n+1, \dots, n+m\}$, we find that:

$$c_B^T - y^T A_B \leq 0, \quad -y^T \leq 0,$$

which shows that y is feasible in (8). Besides,

$$y^T b = c_B^T A_B^{-1} b = c_B^T x_B = c^T x = c_D^T x_D.$$

It follows by weak duality that y is optimal in the dual problem, and the optimal objective values coincide in the primal and dual problems coincide. □

Remark 3.5. *It follows from the proof of Theorem 3.4 that the constructed vector y is the vector of shadow prices of constraints in* (5)*.*

Proposition 3.6. *(Complementary slackness). Let x_D be feasible in* (7) *and y be feasible in* (8)*. Then x_D and y are optimal if and only if they satisfy the complementary slackness conditions:*

$$\begin{aligned} y^T(b - A_D x_D) &= 0, \\ (y^T A_D - c_D^T)x_D &= 0. \end{aligned}$$

Proof. ($\Longrightarrow$) Assume that x_D and y are optimal. Using Theorem 3.4 we get:

$$(y^T A_D - c_D^T)x_D + y^T(b - A_D x) = y^T b - c_D^T x_D = 0.$$

Besides, since x_D and y are feasible, we also have that:

$$\begin{aligned} (y^T A_D - c_D^T)x_D &\geq 0, \\ y^T(b - A_D x_D) &\geq 0. \end{aligned}$$

This implies the claim.

($\Longleftarrow$) Suppose that x_D is primal feasible, y is dual feasible, and that the complementary slackness conditions are valid. Then:

$$y^T b - c_D^T x_D = (y^T A_D - c_D^T)x_D + y^T(b - A_D x_D) = 0.$$

It follows from Proposition 3.3 that x and y are optimal. □

Now consider a general linear program:

$$\begin{array}{lll} \text{maximize} & c_1 x_1 + \cdots + c_n x_n & \\ \text{subject to} & a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, & i \in M_1, \\ & a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i, & i \in M_2, \\ & a_{i1}x_1 + \cdots + a_{in}x_n = b_i, & i \in M_3, \\ & x_j \geq 0, & j \in N_1, \\ & x_j \leq 0, & j \in N_2, \\ & x_j \in \mathbb{R}, & j \in N_3, \end{array}$$

where

$$M_1 \coprod M_2 \coprod M_3 = \{1, \dots, m\},$$
$$N_1 \coprod N_2 \coprod N_3 = \{1, \dots, n\}.$$

The *dual* linear program is defined as follows:

$$\begin{array}{lll} \text{minimize} & b_1 y_1 + \cdots + b_m y_m & \\ \text{subject to} & a_{1j} y_1 + \cdots + a_{mj} y_m \geq c_j, & j \in N_1, \\ & a_{1j} y_1 + \cdots + a_{mj} y_m \leq c_j, & j \in N_2, \\ & a_{1j} y_1 + \cdots + a_{mj} y_m = c_j, & i \in N_3, \\ & y_i \geq 0, & i \in M_1, \\ & y_i \leq 0, & j \in M_2, \\ & y_i \in \mathbb{R}, & i \in M_3. \end{array}$$

Remark 3.7. *By reducing a general linear program to an equivalent linear program in symmetric form one can show that weak and strong duality hold true in general, see [2].*

Remark 3.8. *Complementary slackness conditions hold true in general. For the pair of dual problems above the complementary slackness conditions take the form:*

$$y_i \big(b_i - a_{i1} x_1 - \cdots - a_{in} x_n \big) = 0, \quad i \in M_1 \cup M_2,$$
$$x_i \big(a_{1j} y_1 + \cdots + a_{mj} y_m - c_j \big) = 0, \quad j \in N_1 \cup N_2.$$

4 Bimatrix games

4.1 Games in normal form

A game in normal form is a triple $\Gamma = (P, S, u)$, where $P = \{1, \dots, M\}$ is a finite set of players, $S = S_1 \times \cdots \times S_n$ is the set of *game outcomes*, S_i is the set of *pure strategies* of player i, and $u_i\colon S \to \mathbb{R}$, $u_i = u_i(s_1, \dots, s_n)$ is the *payoff function* of player i. In what follows we will assume that the sets of strategies S_1, ..., S_n are finite.

It is convenient to separate the strategies of player i and other players by using the notation:

$$S \cong S_i \times S_{-i}, \quad S_{-i} = \prod\nolimits_{j \neq i} S_j,$$

so that the payoff of player i can be written as:

$$u_i(s) = u_i(s_i, s_{-i}), \quad s_i \in S_i,\ s_{-i} \in S_{-i}.$$

Note that the payoff of player i depends upon her decision as well as upon decisions of the other players.

Assume now that the game is played repeatedly. Then the players can choose their pure strategies according to some probability distributions. For a given finite set X let $\Delta(X)$ be the set of probability distributions on X. A *mixed strategy* of player i is a distribution $\sigma_i \in \Delta(S_i)$.

Note that the choice of mixed strategies σ_1, ..., σ_n determines a probability distribution σ on the set of game outcomes $\Delta(S)$ by the product rule:

$$\sigma(i_1, \dots, i_N) = \sigma_1(i_1) \cdots \sigma_n(i_n),$$

This allows to compute the expected payoff of player i by the formula:

$$u_i(\sigma) = \mathbb{E}^\sigma u_i = \sum_{s \in S} \sigma(s) u_i(s).$$

It is also convenient to separate the strategies of player i and other players by using the notation:

$$\sigma = (\sigma_i, \sigma_{-i}), \quad \sigma_i \in \Delta(S_i),\ \sigma_{-i} \in \prod\nolimits_{j \neq i} \Delta(S_j).$$

If the players use their mixed strategies, player i can estimate the probability distributions of the other players and adapt his strategy. Strategy $\sigma_i^* = \sigma_i^*(\sigma_{-i}) \in \Delta(S_i)$ is *the best response* to strategies σ_{-i} of the other players if:

$$u_i(\sigma_i^*, \sigma_i) \geq u_i(\sigma_i, \sigma_{-i}), \quad \forall \sigma_i \in \Delta(S_i).$$

The choice of strategies σ_1^*, ..., σ_n^* is a *Nash equilibrium* if strategy σ_i^* is the best response to strategies σ_{-i}^* for all $i = 1$, ..., n.

4.2 Bimatrix games

Assume that $P = 2$ and that $S_1 = \{1, \ldots, m\}$, $S_2 = \{1, \ldots, n\}$. Then the game is called *bimatrix game*. Note that the payoffs of the players are given by the matrices:

$$u_1(i,j) = a_{ij}, \quad u_2(i,j) = b_{ij},$$

where $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times n}$. It is also useful to specify the payoffs using a bimatrix notation:

	1	...	n
1	a_{11}, b_{11}	...	a_{1n}, b_{1n}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
m	a_{m1}, b_{m1}	...	a_{mn}, b_{mn}

.

Given mixed strategies $\sigma_1 = (\sigma_1(1), \ldots, \sigma_1(m)) \in \Delta(S_1)$, $\sigma_2 = (\sigma_2(1), \ldots, \sigma_2(n)) \in \Delta(S_2)$, the expected payoffs of the players are:

$$u_1(\sigma_1, \sigma_2) = \sigma_1^T A \sigma_2, \quad u_2(\sigma_1, \sigma_2) = \sigma_1^T B \sigma_2.$$

Theorem 4.1. *Consider a bimatrix game with payoff matrices A, $B \in \mathbb{R}^{m \times n}$. Then a pair of strategies $\sigma_1 \in \Delta(S_1)$, $\sigma_2 \in \Delta(S_2)$ is a Nash equilibrium if and only if there exist u, $v \in \mathbb{R}$ such that:*

$$\begin{aligned} \sigma_1^T \left(1(m)u - A\sigma_2\right) = 0, &\quad 1(m)u \geq A\sigma_2 \\ \left(1(n)^T v - \sigma_1^T B\right)^T \sigma_2 = 0, &\quad 1(n)^T v \geq \sigma_1^T B, \end{aligned}$$

where $1(k) = (1, \ldots, 1) \in \mathbb{R}^k$.

Proof. Note that $\sigma_1 \in \Sigma(S_1)$ is the best response to $\sigma_2 \in \Sigma(S_2)$ if and only if σ_1 solves the linear program:

$$\begin{array}{rl} \text{maximize} & \sigma_1^T A \sigma_2 \\ \text{subject to} & \sigma_1^T 1(m) = 1, \\ & \sigma_1 \geq 0. \end{array}$$

The dual linear program has the form:

$$\begin{array}{rl} \text{minimize} & u \\ \text{subject to} & 1(m)u \geq A\sigma_2. \end{array}$$

It follows from the complementary slackness conditions (Proposition 3.6) that $\sigma_1(\Sigma_1)$ is the optimal solution to the primal problem if and only if the first group of conditions in the present theorem is valid.

In a similar way, one can show that σ_2 is the best response to σ_1 if and only if the second group of conditions is valid. □

Example 4.2. Consider the bimatrix game

	1	2
1	2,1	0,2
2	1,2	3,0

To check if it has Nash equilibrium in pure strategies, we compute the best responses $s_1(s_2)$ and $s_2(s_1)$ of the first and second players, respectively.

$$\begin{aligned} s_1(1) = 1, \quad s_1(2) = 2, \\ s_2(1) = 2, \quad s_2(2) = 1. \end{aligned}$$

We see that the game does not have Nash equilibria in pure strategies.

To find Nash equilibrium in mixed strategies, assume that the mixed strategy of the first player is $(p, 1-p)$ with $p > 0$, and the mixed strategy of the second player is $(q, 1-q)$ with $q > 0$. To find the Nash equilibrium, we use Theorem 4.1. We have to find q and p from the following equations:

$$\begin{aligned} 2q + 0(1-q) = q + 3(1-q) \quad (= u) \\ p + 2(1-p) = 2p + 0(1-p) \quad (= v), \end{aligned}$$

which gives $q = 3/4$, $p = 2/3$.

4.3 The minimax theorem

A *matrix game* is a bimatrix game with payoff matrices A and $B = -A$. This game is also-called called *zero-sum* because the profit of the first player is the loss of the second player and vice versa.

Theorem 4.3. *Consider a matrix game with the first player payoff matrix $A \in \mathbb{R}^{m \times n}$. This game always admits a Nash equilibrium and each Nash equilibrium (σ_1, σ_2) can be found by solving the following linear programs dual to each other:*

$$\begin{array}{ll|ll} \textit{maximize} & t & \textit{minimize} & s \\ \textit{subject to} & 1(n)^T t \le \sigma_1^T A & \textit{subject to} & 1(m)s \ge A\sigma_2 \\ & \sigma_1^T 1(m) = 1 & & 1(n)^T \sigma_2 = 1 \\ & \sigma_1 \ge 0 & & \sigma_2 \ge 0. \end{array} \tag{10}$$

Proof. Note that t, σ_1 is feasible for (10) (left) if and only if $\sigma_1 \in \Delta(\{1, \dots, m\})$ and

$$t \le \min_{\sigma_2} \sigma_1^T A \sigma_2.$$

It follows that t^*, σ_1^* is an optimal solution of (10) (left) if and only if

$$t^* = \max_{\sigma_1} \min_{\sigma_2} \sigma_1^T A \sigma_2 = \min_{\sigma_2} (\sigma_1^*)^T A \sigma_2.$$

In a similar way, s^*, σ_2^* is an optimal solution of (10) (right) if and only if

$$s^* = \min_{\sigma_2} \max_{\sigma_1} \sigma_1^T A \sigma_2 = \max_{\sigma_1} \sigma_1 A \sigma_2^*.$$

Note that:

$$t^* = \min_{\sigma_2} (\sigma_1^*)^T A \sigma_2 \leq (\sigma_1^*)^T A \sigma_2^* \leq \max_{\sigma_1} \sigma_1^T A \sigma_2^* = s^*.$$

Since the problems (10) are dual to each other, we get $t^* = s^*$ and:

$$\min_{\sigma_2} (\sigma_1^*)^T A \sigma_2 = (\sigma_1^*)^T A \sigma_2^* = \max_{\sigma_1} \sigma_1^T A \sigma_2^*.$$

It means that σ_1^* is the best response to σ_2^*, and σ_2^* is the best response to σ_1^*. □

Remark 4.4. *If the first player plays strategy σ_1^* he will get at least t^* if he does not know the strategy of the second player. The number t^* is called the maximin value of the game and σ_1^* is called the maximin strategy of the first player.*

5 Finite Markov chains

5.1 Definition and main properties

This lecture is based on the book [7].

Let $I = \{1, \dots, N\}$ be a finite collection of states and let $\Delta(I)$ denote the set of probability distributions on I:

$$\Delta(I) = \{x \in \mathbb{R}^N : x \geq 0,\ x_1 + \cdots + x_N = 1\}.$$

A finite discrete-time *Markov chain* is a collection of random variables (i.e. a random process) $\{X_n\}_{n=1}^{\infty}$ with values in the set I satisfying the following properties:

1. Markov property:

$$\mathbb{P}(X_{n+1} = j \mid X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = j | X_n = i_n),$$
$$\forall n \geq 1, \forall i_1, \dots, i_n, j \in I$$

2. Time-homogeneity:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_2 = j \mid X_1 = i) = p_{ij}, \quad \forall n \geq 1,\ \forall i, j \in I$$

The matrix $P = (p_{ij}) \in \mathbb{R}^{N \times N}$ is called the *transition matrix*. Note that the transition matrix is a *stochastic matrix*: its elements are non-negative and the sum of the elements in each row is equal to one.

The Markov chain is completely characterized by its transition matrix and and by the distribution of X_1.

Example 5.1. Let $\{X_n\}_{n=1}^{\infty}$ be a Markov chain with transition matrix P and with $X_1 \overset{d}{\sim} \pi$, $\pi = (\pi_1, \dots, \pi_N) \in \Delta(I)$. We can compute the distribution of X_2 as follows:

$$\mathbb{P}(X_2 = j) = \textstyle\sum_i \mathbb{P}(X_2 = j | X_1 = i)\mathbb{P}(X_1 = i) = \sum_i \pi_i p_i j = (\pi P)_j,$$

which shows that $X_2 \overset{d}{\sim} \pi P$. It follows by induction that $X_{n+1} \overset{d}{\sim} \pi P^n$.

Exercise 5.2. Let P be the transition matrix of a Markov chain $\{X_n\}_{n=1}^{\infty}$. Using the law of total probability show that

$$\mathbb{P}(X_{n+k} = j | X_n = i) = p_{ij}^{(k)},$$

where $P^k = (p_{ij}^{(k)})$ is the k-th power of matrix P.

A state $j \in I$ is *accessible* from state $i \in I$ if $p_{ij}^{(k)} > 0$ for some $k > 0$. States $i \in I$ and $j \in I$ communicate if each of them is accessible from the other one. Note that communication is an equivalence relation on states. The corresponding equivalence classes are called *communicating classes*. A Markov chain is called *irreducible* if it has only one communicating class, and it is called reducible otherwise.

Exercise 5.3. Show that a Markov chain is reducible if and only if one can rename its states in such a way that the transition matrix P takes the upper block-triangular form:

$$P = \begin{bmatrix} *_{k\times k} & * \\ 0 & * \end{bmatrix}.$$

Exercise 5.4. Show that renaming the states of a Markov chain corresponds to conjugation of the transition matrix by a permutation matrix.

5.2 Invariant distribution

For a given Markov chain $\{X_n\}_{n=1}^{\infty}$ we call a probability distribution $\pi \in \Delta(I)$ an *invariant distribution* (or steady distribution, or equilibrium distribution) if $\pi = \pi P$. Note that if π is taken as the distribution of X_n, then X_{n+1} will also have this distribution:

$$\begin{aligned} \mathbb{P}(X_{n+1} = j) &= \sum_{i=1}^{N} \mathbb{P}(X_{n+1} = j | X_n = i)\mathbb{P}(X_n = i) \\ &= \sum_{i=1}^{N} p_{ij}\pi_i = \pi_j. \end{aligned}$$

Next we will show the existence of an invariant distribution constructively.

We need to introduce some notations. Put

$$\begin{aligned} \mathbb{E}^k Y &= \mathbb{E}(Y | X_1 = k), \\ \mathbb{P}^k(A) &= \mathbb{P}(A | X_1 = k), \end{aligned}$$

where $k \in I$. Also let T_k be the time of the first visit to state k counting from time step two:

$$T_k = \min\{n \geq 2 : X_n = k\}, \quad k \in I.$$

Lemma 5.5. *Let $\{X_n\}_{n=1}^{\infty}$ be an irreducible Markov chain with transition matrix P. Let γ_j^k be the expected time spent in state j before returning to state k:*

$$\gamma_j^k = \mathbb{E}^k \left(\sum_{n=1}^{T_k - 1} 1_{X_n = j} \right).$$

Then $\gamma_j^k < \infty$ for all k,j; $\gamma_k^k = 1$ for all k; and $\gamma^k = (\gamma_1^k, \ldots, \gamma_N^k)$ satisfies $\gamma^k = \gamma^k P$.

Proof. Formula $\gamma_k^k = 1$ follows from the definition of γ_k^k. Assuming that $\gamma_i^k < \infty$ for all k, i the equation $\gamma^k = \gamma^k P$ follows from the following chain of transformations:

$$\begin{aligned}
\gamma_j^k &= \mathbb{E}^k\big(\textstyle\sum_{n=2}^{T_k} 1_{X_n=j}\big) \\
&= \mathbb{E}^k\big(\textstyle\sum_{n=2}^{\infty} 1_{X_n=j} 1_{T_k \geq n}\big) \\
&= \mathbb{E}^k\big(\textstyle\sum_{n=2}^{\infty} \sum_{i=1}^{N} 1_{X_n=j} 1_{X_{n-1}=i} 1_{T_k \geq n}\big) \\
&= \textstyle\sum_{n=2}^{\infty} \sum_{i=1}^{N} \mathbb{P}^k\big(X_n = j, X_{n-1} = i, T_k \geq n\big) \\
&= \textstyle\sum_{n=2}^{\infty} \sum_{j=1}^{N} p_{ij} \mathbb{P}^k\big(X_{n-1} = i, T_k \geq n\big) \\
&= \textstyle\sum_{i=1}^{N} p_{ij} \mathbb{E}^k\big(\sum_{n=2}^{T_k} 1_{X_{n-1}=i}\big) \\
&= \textstyle\sum_{i=1}^{N} p_{ij} \mathbb{E}^k\big(\sum_{n=1}^{T_k-1} 1_{X_n=i}\big) = \sum_{i=1}^{N} p_{ij} \gamma_i^k.
\end{aligned}$$

Since all the terms are non-negative, this chain of equations is also valid without the assumption that $\gamma_i^k < \infty$ for all k, i. Iterating the formula s times we get:

$$\gamma_j^k = \textstyle\sum_{i=1}^{N} p_{ij}^{(s)} \gamma_i^k.$$

For a given pair of indices k, i let $s \geq 1$ be such that $p_{ik}^{(s)} > 0$. Such $s \geq 1$ always exists by the definition of irreducibility of $\{X_n\}_{n=1}^{\infty}$. Now put $k = j$ in the above formula and recall that $\gamma_k^k = 1$ to obtain that $\gamma_i^k < \infty$. Since the pair i, k is arbitrary, this proves the lemma. $\square$

Theorem 5.6. *Let $\{X_n\}_{n=1}^{\infty}$ be an irreducible discrete-time Markov chain with a finite state space. Then the following statements are valid:*

1. *X_n admits a unique steady distribution $\pi = (\pi_1, \ldots, \pi_N) \in \Delta(I)$.*
2. *$\pi_i > 0$ for $i = 1, \ldots, N$.*
3. *$\pi_i = 1/m_i$, where $m_i = \mathbb{E}^i T_i$ is the mean return time to state i.*

Proof. (*Positivity*). Assume that $\{X_n\}_{n=1}^{\infty}$ is an irreducible Markov chain and let π be some steady distribution. Assume that π contains some zero entries. Renaming the states if necessary, we can suppose that $\pi = (\pi_1, \ldots, \pi_k, 0, \ldots, 0)$, where $\pi_1, \ldots, \pi_k > 0$. Since π is steady, it satisfies the equation $\pi = \pi\mathcal{P}$. Writing this equation element-wise, we get:

$$0 = \sum_{i=1}^{N} \pi_i p_{ij}, \quad j = k+1, \ldots, N.$$

Since $\pi_1, \ldots, \pi_k > 0$ this also implies that $p_{ij} = 0$ for $i = 1, \ldots, k$, and $j = k+1, \ldots, N$. It follows from exercise 5.3 that $\{X_n\}_{n=1}^{\infty}$ is reducible. Contradiction!

(*Uniqueness*). Let $\{X_n\}$ be an irreducible Markov chain and suppose that $\{X_n\}_{n=1}^{\infty}$ admits two steady distributions $\pi \neq \nu$. We proved that $\pi > 0$, $\nu > 0$. Let $i^* = \operatorname{argmin}_i(\pi_i/\nu_i)$ and put

$$\widetilde{\mu} = \pi - \nu\frac{\pi_{i^*}}{\nu_{i^*}},$$
$$\mu = \frac{\widetilde{\mu}}{\widetilde{\mu}_1 + \cdots + \widetilde{\mu}_N}.$$

Note that μ is a also a steady distribution and $\mu_{i^*} = 0$. Contradiction!
(*Construction*). Fix k and put

$$\pi_i^{(k)} = \gamma_i^k/(\gamma_1^k + \cdots + \gamma_N^k)$$

It follows from Lemma 5.5 that $\pi^{(k)}$ is a steady distribution for $\{X_n\}$. Uniqueness of the steady distribution implies that that $\pi^{(1)} = \cdots = \pi^{(n)} \stackrel{def}{=\!=} \pi$. It follows that for each fixed k:

$$\pi_k = \pi_k^{(k)} = 1/(\gamma_k^1 + \cdots + \gamma_k^N) = 1/m_k,$$

which concludes the proof of the present theorem. □

Sometimes it is very easy to find a steady distribution. A discrete-time Markov chain $\{X_n\}_{n=1}^{\infty}$ with transition matrix $P = (p_{ij})$ and a distribution $\pi = (\pi_i)$ are *in detailed balance* if the following equations hold true:

$$\pi_i p_{ij} = \pi_j p_{ji}, \quad i, j = 1, \ldots, N.$$

Exercise 5.7. Suppose that a Markov chain $\{X_n\}_{n=1}^{\infty}$ and a distribution π are in detailed balance. Show that π is a steady distribution for $\{X_n\}_{n=1}^{\infty}$.

A graph is a pair $G = (V, E)$, where $V = \{1, \ldots, N\}$ is a finite set of nodes (or vertices), and $E \subset V \times V$ is the set of edges such that $(i, j) \in E$ if and only if $(j, i) \in E$. If $e = (i, j) \in E$ the edge e is called *incident* to nodes i and j. The total number of edges e which are incident to a given node i is *the valency* v_i of node i. The graph is called *connected* if each pair of nodes i, j can be connected by a sequence of edges.

A symmetric random walk on a graph G is a Markov chain X_n, $n \geq 0$, with transition matrix $\mathcal{P} = (p_{ij})$ such that $p_{ij} = 1/v_i$. It means that it jumps from state i along each of the incident edges with equal probability.

Proposition 5.8. *Let X_n be a symmetric random walk on a connected graph $G = (V, E)$. Then X_n is an irreducible Markov chain, and its steady distribution is given by:*

$$\pi_i = \frac{v_i}{v_1 + \cdots + v_N}, \quad i = 1, \ldots, N.$$

Proof. It follows from the definition of a connected graph and from the definition of a symmetric random walk that X_n is irreducible. The detailed balance equations for the Markov chain X_n are as follows:

$$\frac{\pi_i}{v_i} = \frac{\pi_j}{v_j}, \quad i, j = 1, \ldots, N.$$

These equations together with exercise 5.7 imply the statement of the present proposition. □

5.3 Ergodic theorem

We will show that for irreducible chains the proportion of time spent in state i in the long run converges to the equilibrium probability of this state. We will need the strong law of large numbers:

Theorem. *(Strong Law of Large Numbers). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of non-negative independent identically distributed random variables with mean m. Then:*

$$\mathbb{P}\left(\lim_{n\to+\infty} \frac{X_1 + \cdots + X_n}{n} = m\right) = 1.$$

We need to introduce some notations. Let $V_i(n)$ be the number of visits of a Markov chain $\{X_n\}_{n=1}^{\infty}$ to state i by time n:

$$V_i(n) = \sum_{k=1}^{n} 1_{X_k = i}.$$

Note that $V_i(n)/n$ is the proportion of time spent in state i by time n. We also need the following notations:

$$T_i^{(0)} = 0, \quad T_i^{(r+1)} = \min\{n > T_i^{(r)} : X_n = k\},\ r \geq 0,$$
$$S_i^{(r)} = T_i^{(r+1)} - T_i^{(r)},\ r \geq 0.$$

Theorem 5.9. *Let $\{X_n\}_{n=1}^{\infty}$ be an irreducible Markov chain and let $\pi = (\pi_1, \ldots, \pi_N) \in \Delta(I)$ be its invariant distribution. Then:*

$$\mathbb{P}\left(\lim_{n\to+\infty} \frac{V_i(n)}{n} = \pi_i\right) = 1, \quad \forall i \in I.$$

Proof. Note that $S_i^{(r)}$, $r \geq 2$, are independent and identically distributed non-negative random variables with mean m_i. It follows from the strong law of large numbers that:

$$\mathbb{P}\left(\lim_{n\to+\infty} \frac{S_i^{(1)} + \cdots + S_i^{(n)}}{n} = m_i\right) = 1, \quad \forall k \in I.$$

Also note that:

$$S_i^{(1)} + \cdots + S_i^{(V_i(n))} = T_i^{(V_i(n)+1)} \geq n+1,$$
$$S_i^{(1)} + \cdots + S_i^{(V_i(n)-1)} = T_i^{(V_i(n))} \leq n,$$

which leads to the inequalities

$$\frac{S_i^{(1)} + \cdots + S_k^{(V_i(n)-1)}}{V_i(n)} \leq \frac{n}{V_i(n)} \leq \frac{S_i^{(1)} + \cdots + S_k^{(V_i(n))}}{V_i(n)}.$$

Note that irreducibility of $\{X_n\}_{n=1}^{\infty}$ implies that $V_i(n) \to +\infty$ as $n \to +\infty$ for each $i \in I$. Tending $n \to +\infty$ in the above inequalities and using Theorem 5.6 we conclude the proof of the present theorem. □

Corollary 5.10. *Let $\{X_n\}_{n=1}^{\infty}$ be an irreducible Markov chain with invariant distribution $\pi = (\pi_1, \ldots, \pi_N)$ and let $f \colon I \to \mathbb{R}$. Then:*

$$\mathbb{P}\left(\lim_{n\to+\infty} \frac{f(X_1) + \cdots + f(X_n)}{n} = \overline{f}\right) = 1,$$
$$\overline{f} = \sum_{i\in I} f(i)\pi_i.$$

Proof. The claim follows from Theorem 5.9 and from the inequality

$$\left|\frac{1}{n}\sum\nolimits_{k=1}^{n} f(X_k) - \overline{f}\right| = \left|\sum\nolimits_{i\in I}\left(\frac{V_i(n)}{n} - \pi_i\right) f(i)\right|$$
$$\leq \sum\nolimits_{i\in I}\left|\frac{V_i(n)}{n} - \pi_i\right| \max_{i\in I} |f(i)|.$$

□

6 Absorbing Markov chains

6.1 Definition and main properties

This lecture is based on the book [4].

Let X_n be a discrete-time Markov chain with a finite number of states $I = \{1, \dots, N\}$. State i is called *absorbing* if, once entered, it can not be left:

$$\mathbb{P}(X_2 = i | X_1 = i) = 1.$$

Let A be the set of absorbing states. The Markov chain $\{X_n\}_{n=1}^{\infty}$ is *absorbing* if:

1. $A \neq \varnothing$
2. For each $i \in I \setminus A$ there exists $j \in A$ and $k \geq 1$ such that:

$$\mathbb{P}(X_{k+1} = j | X_1 = i) > 0.$$

States $T = I \setminus A$ are called *transient*.

Exercise 6.1. Let $\{X_n\}_{n=1}^{\infty}$ be an absorbing Markov chain. Show that $X_n \to X_\infty$ almost surely, where X_∞ is a random variable with values in A.

Without loss of generality we can assume that $A = \{1, \dots, m\}$, $T = \{m + 1, \dots, N\}$, so that the absorbing states are listed first. Then the transition matrix has the form:

$$P = \begin{bmatrix} I(m) & 0 \\ R & Q \end{bmatrix}, \tag{11}$$

where $I(m) \in \mathbb{R}^{m \times m}$ is the identity matrix, $Q = (q_{ij}) \in \mathbb{R}^{(N-m)\times(N-m)}$, $R = (r_{ij}) \in \mathbb{R}^{(N-m)\times m}$.

Lemma 6.2. *The following formulas are valid:*

$$P^k = \begin{bmatrix} I(m) & 0 \\ R_k & Q^k \end{bmatrix}, \quad R_k = R + QR_{k-1}.$$

Proof. We will prove using the induction on k. For $k = 1$ the statement is trivial. Suppose that the statement is true for some fixed k. Then we have:

$$P^{k+1} = P^k P = \begin{bmatrix} I(m) & 0 \\ R & Q \end{bmatrix} \begin{bmatrix} I(m) & 0 \\ R_k & Q^k \end{bmatrix} = \begin{bmatrix} Id_m & 0 \\ R + QR_k & Q^{k+1} \end{bmatrix}$$

which proves the claim.

□

To prove the next lemma we need some facts from linear algebra.

Exercise 6.3. For a given matrix $A \in \mathbb{R}^{N\times N}$ we put

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|.$$

Show that for each pair of matrices A, $B \in \mathbb{R}^{N\times N}$ the following inequality is valid:

$$\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty.$$

Exercise 6.4. Let $A \in \mathbb{R}^{N\times N}$ be such that $\|A^k\|_\infty < 1$ for some $k \ge 1$. Show that $A^n \to 0$ as $n \to +\infty$. *Hint: Represent n as $n = qk + r$, where $r \in \{0, \dots, k-1\}$, and use Exercise 6.3.*

Exercise 6.5. Let $A \in \mathbb{R}^{N\times N}$ be such that $A^k \to 0$ as $k \to +\infty$. Show that the matrix $I - A$ is invertible and $(I - A)^{-1}$ is given by the convergent Neumann series:

$$(I - A)^{-1} = I + A + A^2 + \dots.$$

Lemma 6.6. *Let X_n be an absorbing Markov chain. Then the matrix $Id - Q$ is invertible.*

Proof. It follows from the definition of transient states that there exists $k \ge 1$ such that

$$\max_{i\in T} \mathbb{P}(X_{n+k} \in T | X_n = i) < 1.$$

From this inequality together with Lemma 6.2 we get

$$1 > \max_{i\in T} \mathbb{P}(X_{n+k} \in T | X_n = i) = \max_{i\in T} \sum_{j\in T} p_{ij}^{(k)} = \max_{i\in T} \sum_{j\in T} q_{ij}^{(k)} = \|Q^k\|_\infty.$$

Using Exercise 6.4 we obtain that $Q^n \to 0$ as $n \to +\infty$. The claim now follows from Exercise 6.5. □

The matrix $N = (I - Q)^{-1}$ is called the *fundamental matrix* of the Markov chain $\{X_n\}_{n=1}^\infty$. Let τ_j be the time spent by the Markov chain $\{X_n\}_{n=1}^\infty$ in state $j \in T$ before being absorbed and τ be the total time spent in transient states before being absorbed:

$$\tau_j = \sum_{k=1}^\infty 1_{X_k = j}, \quad \tau = \sum_{i\in T} \tau_i.$$

Proposition 6.7. *Let $\{X_n\}_{n=1}^\infty$ be an absorbing Markov chain, $N = (n_{ij})$ its fundamental matrix, and T the set of transient states. Then*

$$\mathbb{E}^i \tau_j = n_{ij},$$
$$\mathbb{E}^i \tau = \sum_{j\in T} n_{ij},$$

where $i \in T$.

Proof. The first formula follows from the following transformations:

$$\begin{aligned}
\mathbb{E}^i \tau_j &= \textstyle\sum_{k=0}^{\infty} \mathbb{P}\big(X_k = j | X_0 = i\big) \\
&= \textstyle\sum_{k=0}^{\infty} p_{ij}^{(k)} && \text{(Exercise 5.2)} \\
&= \textstyle\sum_{k=0}^{\infty} q_{ij}^{(k)} && \text{(Lemma 6.2)} \\
&= n_{ij}. && \text{(Exercise 6.5)}
\end{aligned}$$

The second formula is a direct corollary of the first one. □

Proposition 6.8. *Let X_n be an absorbing Markov chain with transition matrix P in the canonical form (11) and let $N = (Id - Q)^{-1}$ be its fundamental matrix. Then $B = NR$, where $B = (b_{ij})$, and b_{ij} is the probability that the Markov chain will be absorbed by state $j \notin T$ if it starts in state $i \in T$:*

$$b_{ij} = \mathbb{P}\big(X_\infty = j \mid X_0 = i\big).$$

Proof. Fix $i \in T$, $j \in A$. By the law of total probability:

$$\begin{aligned}
b_{ij} &= \mathbb{P}\big(X_2 = j \mid X_1 = i\big) \\
&+ \sum_{\ell \in T} \mathbb{P}\big(X_2 = \ell \mid X_1 = i\big)\mathbb{P}\big(X_\infty = j \mid X_2 = \ell\big) \\
&= p_{ij} + \textstyle\sum_{\ell \in T} p_{i\ell} b_{\ell k} \\
&= r_{ij} + \textstyle\sum_{\ell \in T} q_{i\ell} b_{\ell k},
\end{aligned}$$

In matrix form it means that $B = R + QB$. Solving for B, we get that $B = NR$. □

6.2 Linear constant coefficient recurrences

A linear homogeneous recurrence relation of degree d with constant coefficients is given by:

$$x_{n+d} + a_1 x_{n+d-1} + a_2 x_{n+d-2} + \cdots + a_d x_n = 0, \quad n \geq 1. \tag{12}$$

where $a_1, \ldots, a_d \in \mathbb{C}$, $a_d \neq 0$, are known coefficients, and $\{x_n\}_{n=1}^{\infty}$ is the unknown sequence. The theory of linear recurrences is parallel to the theory of linear ordinary differential equations.

Example 6.9. Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$ are defined as the solution to the recurrence:

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_1 = F_2 = 1.$$

Exercise 6.10. Show that the space of solutions $\{x_n\}_{n\geq 1}$ of a linear homogeneous recurrence relation is a linear vector space of dimension d. *Hint: one can arbitrarily specify $x_1, \ldots, x_d$.*

It is convenient to introduce the *forward shift* operator E by the rule $Ex_n = x_{n+1}$ for $n \geq 1$. Then the recurrence (12) can be rewritten as:

$$(E^d + a_1 E^{d-1} + \cdots + a_d)x_n = 0, \quad n \geq 1,$$
$$\text{or} \quad \chi(E)x_n = 0, \quad n > d,$$

where χ is the *characteristic polynomial* of the recurrence defined by

$$\chi(\lambda) = \lambda^d + a_1\lambda^{d-1} + \cdots + a_d.$$

Example 6.11. Let $x_n = r^n$. Then $\chi(E)x_n = \chi(r)r^n$. It follows that $x_n = r^n$ solves (12) if and only if r is a root of χ. It follows that if all the roots of the characteristic polynomial are different, the general solution to the recurrence is given by the linear combinations of power functions.

Theorem 6.12. *Let χ be the characteristic polynomial of recurrence* (12) *and let $r_1, \ldots, r_\ell$ be its different roots so that*

$$\chi(\lambda) = (\lambda - r_1)^{m_1} \ldots (\lambda - r_\ell)^{m_\ell}.$$

Then the basis of solutions of the homogeneous recurrence (12) *is given by the functions:*

$$r_k^n, nr_k^n, \ldots, n^{m_k-1}r_k^n, \quad k = 1, \ldots, \ell.$$

Proof. Let r be a root of multiplicity m of χ. Then r is a root of multiplicity m of the polynomial

$$\lambda^n\chi(\lambda) = \lambda^{n+d} + a_1\lambda^{n+d-1} + \cdots + a_d\lambda^n.$$

Then r is also a root of multiplicity at least $m-1$ of the polynomial $\frac{d}{d\lambda}(\lambda^n\chi(\lambda))$ as well as of the polynomial

$$\lambda\frac{d}{d\lambda}(\lambda^n\chi(\lambda)) = (n+d)\lambda^{n+d} + a_1(n+d-1)\lambda^{n+d-1} + \cdots + a_d n\lambda^n.$$

But this implies that $x_n = nr^n$ solves the original recurrence. Repeating the reasoning, we get that r is a root of polynomials

$$\left(\lambda\frac{d}{d\lambda}\right)^k(\lambda^n\chi(\lambda)) = (n+d)^k\lambda^{n+d} + a_1(n+d-1)^k\lambda^{n+d-1} + \cdots + a_d n^k\lambda^n$$

for $k = 0, 1, \ldots, m-1$. This proves the theorem □

A linear constant coefficient non-homogeneous recurrence relation of degree d is a system of equations:

$$x_{n+d} + a_1x_{n+d-1} + a_2x_{n+d-2} + \cdots + a_dx_n = f_n, \quad n \geq 1. \tag{13}$$

where $a_1, \ldots, a_d \in \mathbb{C}$, $a_d \neq 0$, are known coefficients, $\{f_n\}_{n=1}^\infty$ is a known sequence, and $\{x_n\}_{n=1}^\infty$ is the unknown sequence.

Exercise 6.13. Show that the space of solutions to the nonhomogeneous recurrence (13) is a linear manifold of dimension d. Besides, a general solution to this recurrence is a sum of a fixed particular solution and of the general solution to the homogeneous recurrence (12).

Example 6.14. We want to find a particular solution to the recurrence:

$$x_{n+2} - 2x_{n+1} - 3x_n = 2^n, \quad n \geq 1.$$

We can rewrite the recurrence as:

$$(E+1)(E-3)x_n = 2^n, \quad n \geq 1.$$

Note that $(E-2)2^n = 0$. Applying $(E-2)$ to both sides of the recurrence we find that x_n also satisfies the recurrence:

$$(E-2)(E+1)(E-3)x_n = 0.$$

A general solution to this recurrence is given by

$$x_n = 2^n C_1 + (-1)^n C_2 + 3^n C_3.$$

Since we are interested in some particular solution to the original recurrence we can drop the term $(-1)^n C_2 + 3^n C_3$ as it satisfies the homogeneous recurrence associated to the original recurrence. It follows that we can seek x_n in the form $x_n = 2^n C_1$. Substituting it into the original recurrence we find $C_1 = -\frac{1}{3}$ so that a particular solution is $x_n = -\frac{1}{3}2^n$.

Example 6.15. We want to find a particular solution to the recurrence

$$x_{n+2} - 8x_{n+1} + 12x_n = n^2 6^n$$

which can be rewritten as:

$$(E-6)(E-2)x_n = n^2 6^n.$$

Note that by Theorem 6.12 the right-hand side is annihilated by $(E-6)^3$ but not by $(E-6)^2$. we apply $(E-6)^3$ to both sides and find that

$$(E-6)^4(E-2)x_n = 0.$$

The general solution to this homogeneous recurrence is

$$x_n = 2^n C_1 + 6^n(C_2 + nC_3 + n^2 C_4 + n^3 C_5),$$

but the terms with C_1 and C_2 can be dropped since they are annihilated by $(E-6)(E-2)$. We seek a particular solution in the form $x_n = n6^n(C_3 + nC_4 + n^2 C_5)$.

Proposition 6.16. *Consider the non-homogeneous recurrence* (13) *and let*

$$f_n = P_k(n) r^n.$$

where $P_k(n)$ is a polynomial of degree k. Assume that r is a root of the characteristic polynomial of multiplicity $m \geq 0$. Then the recurrence (13) *admits a particular solution of the form:*

$$x_n = n^m Q_k(n) r^n,$$

where $Q_k(n)$ is a polynomial of degree at most k.

Proof. Assume that x_n is a solution to (13) so that $\chi(E)x_n = f_n$. We can rewrite it as:

$$(E-r)^m \chi'(E) x_n = f_n,$$

where $\chi'(\lambda) = \chi(\lambda)/(\lambda - r)^m$.

Note that $(E-r)^{k+1} f_n = 0$ by Theorem 6.12. It follows that x_n satisfies:

$$(E-r)^{m+k+1} \chi'(E) x_n = 0.$$

By Theorem 6.12 x_n has the form:

$$x_n = n^m Q_k(n) r^n + y_n,$$

where Q_k is a polynomial of degree at most k and y_n is some solution to the homogeneous recurrence

$$\chi(E) y_n = 0.$$

Since we are interested in some particular solution, we can subtract this solution to the homogeneous recurrence y_n from x_n and still get some particular solution to the non-homogeneous recurrence. □

Example 6.17. Find the solution to the recurrence:

$$x_{n+2} + 4x_n = \cos(n) 2^n, \quad n \geq 1,$$

with initial conditions $x_1 = 0$, $x_2 = 0$.

We can rewrite the recurrence as:

$$(E - 2i)(E + 2i) x_n = \cos(n) 2^n, \quad n \geq 1.$$

It follows that the general solution to the homogeneous recurrence is given by

$$y_n = C_1 (2i)^n + C_2 (-2i)^n = \widetilde{C}_1 2^n \cos(\tfrac{\pi}{2} n) + \widetilde{C}_2 2^n \sin(\tfrac{\pi}{2} n).$$

A general procedure for solving a linear non-homogeneous recurrence with boundary or initial conditions is as follows:

1. Find a general solution y_n of the homogeneous recurrence.
2. Find a particular solution x_n of the non-homogeneous recurrence. The general solution of the non-homogeneous recurrence is given by $z_n = x_n + y_n$.
3. Identify the coefficients in $z_n = x_n + y_n$ by requiring that it satisfies the boundary or initial conditions.

6.3 Gambler's ruin

Two players, A and B, flip a biased coin with probability of heads of p. Each time the coin shows heads the second player gives \$1 to the first player. When the coin shows tails, the first player gives \$1 to the second player. The players start with K_1 and K_2 dollars, respectively, and the game ends when one of the players goes bankrupt. Questions of interest are:

1. What is the probability that the first player will run out of money?
2. What is the expected game duration?
3. What is the variance of the game duration?

Let X_n be the fortune of the first player at time step n. Then $\{X_n\}_{n=1}^{\infty}$ is the absorbing Markov chain with the set of states $I = \{0, \ldots, K_1 + K_2\}$, where the states 0 and $K_1 + K_2$ are absorbing.

1. (Probability of ruin for $p = 0.5$). Let p_i be the probability that the first player will run out of money if his current fortune is i dollars:

$$p_i = \mathbb{P}\big(\lim_{n\to\infty} X_n = 0 | X_1 = i\big).$$

By the law of total probability for each $i \in \{1, \ldots, K_1 + K_2 - 1\}$:

$$\begin{aligned} p_i = \mathbb{P}\big(\lim_{n\to\infty} X_n = 0 | X_2 = i+1\big)\mathbb{P}\big(X_2 = i+1 | X_1 = i\big) \\ +\mathbb{P}\big(\lim_{n\to\infty} X_n = 0 | X_2 = i-1\big)\mathbb{P}\big(X_2 = i-1 | X_1 = i\big) \\ = p_{i+1}p + p_{i-1}(1-p). \end{aligned}$$

It follows that we have the following recurrence:

$$\begin{aligned} &pp_{i+1} - p_i + (1-p)p_{i-1} = 0, \quad 0 < i < K_1 + K_2, \\ &p_0 = 1, \quad p_{K_1+K_2} = 0. \end{aligned}$$

This is a homogeneous linear constant coefficient recurrence of degree two. The characteristic polynomial is $\chi(\lambda) = p\lambda^2 - \lambda + (1-p)$ and its roots are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1-p}{p} \stackrel{def}{=} q.$$

We consider the case $p = 0.5$ so that $\lambda_1 = \lambda_2 = 1$. The general solution of the recurrence is given by $p_i = C_1 + C_2 i$. Plugging this into the initial conditions we find $C_1 = 1$, $C_2 = -1/(K_1 + K_2)$ so that:

$$p_i = \frac{K_1 + K_2 - i}{K_1 + K_2}$$

and $p_{K_1} = \frac{K_2}{K_1+K_2}$.

Exercise 6.18. Let $p \in (0,1)$, $p \neq 0.5$. Show that

$$p_i = \frac{q^i - q^{K_1+K_2}}{1 - q^{K_1+K_2}}, \quad 0 \leq i \leq K_1 + K_2.$$

2. (Expected game duration for $p = 0.5$). Let T be the game duration:

$$T = \min\{k \geq 1 \colon X_{k+1} = 0 \text{ or } X_{k+1} = K_1 + K_2\}$$

and put $T_i = \mathbb{E}(T \mid X_1 = i)$. From the law of total expectation and from the initial conditions we have:

$$\begin{aligned}
&pT_{i+1} - T_i + (1-p)T_{i-1} = -1, \quad 0 < i < K_1 + K_2,\\
&T_0 = 0, \quad T_{K_1+K_2} = 0.
\end{aligned}$$

Since $\lambda = 1$ is a double root of the characteristic polynomial when $p = 0.5$, we seek a particular solution in the form $T_i^{\text{part}} = Ci^2$.

Exercise 6.19. Let $\chi(\lambda) = 0.5\lambda^2 - \lambda + 0.5$. Show that:

$$\chi(E)i^2 = 1, \quad \chi(E)i^3 = 3(i+1), \quad \chi(E)i^4 = 6(i+1)^3 + 1.$$

Using this exercise we find that

$$\chi(E)T_i^{\text{part}} = C\chi(E)i^2 = C = -1,$$

so that $T_i^{\text{part}} = -i^2$. The general solution to the non-homogeneous recurrence is given by

$$T_i = A + iB - i^2.$$

Plugging it into the boundary conditions we find:

$$T_i = i(K_1 + K_2 - i).$$

In particular, $T_{K_1} = K_1K_2$.

3. (Variance of the game duration for $p = 0.5$). To compute the variance of the game duration we first compute the quantities

$$(T^2)_i = \mathbb{E}(T^2 \mid X_1 = i).$$

Proceeding as before we get the recurrence:

$$\begin{aligned}
&p(T^2)_{i+2} - (T^2)_{i+1} + (1-p)(T^2)_i = 1 - 2T_{i+1}, \quad 0 < i < K_1 + K_2,\\
&(T^2)_0 = 0, \quad (T^2)_{K_1+K_2} = 0,
\end{aligned}$$

where $T_{i+1} = (i+1)(K_1+K_2) - (i+1)^2$ when $p = 0.5$.

Exercise 6.20. Solve the recurrence of $(T^2)_i$ and find the variance of the game duration. *Hint:* $\operatorname{Var} T_{K_1} = (T^2)_{K_1} - (T_{K_1})^2$.

7 Poisson process

7.1 Exponential distribution

A nonnegative random variable S has *exponential distribution* with rate parameter $\lambda > 0$, denoted $S \sim E(\lambda)$, if

$$\mathbb{P}(S > t) = e^{-\lambda t},\ t \geq 0.$$

Exercise 7.1. Let $S \sim E(\lambda)$, $\lambda > 0$. Show that $\mathbb{E}T = 1/\lambda$, $\operatorname{Var} T = 1/\lambda^2$.

We say that a non-negative random variable S has a *memoryless property* if:

$$\mathbb{P}(S > t + s \mid S > s) = \mathbb{P}(S > t), \quad t, s > 0.$$

Example 7.2.

1. Inter-arrival times for bank customers are memoryless.
2. Time to failure of a car is not memoryless. Older cars break more often.

Proposition 7.3. *Let $S \neq 0$ be a non-negative random variable. Then S has the memoryless property if and only if $S \sim E(\lambda)$ for some $\lambda > 0$.*

Proof. ($\Longleftarrow$) Let $S \sim E(\lambda)$ for some $\lambda > 0$. Then:

$$\begin{aligned}
\mathbb{P}(S > t + s \mid S > s) &= \frac{\mathbb{P}(S > t + s, S > s)}{\mathbb{P}(S > s)} \\
&= \frac{\mathbb{P}(S > t + s)}{\mathbb{P}(S > s)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\
&= e^{-\lambda t} = \mathbb{P}(S > t).
\end{aligned}$$

($\Longrightarrow$) Let S be a non-zero random variable satisfying the memoryless condition. Then $\mathbb{P}(S > t) > 0$ for all $t \geq 0$ and

$$\mathbb{P}(S > t + s) = \mathbb{P}(S > t)\mathbb{P}(S > s), \quad t, s \geq 0.$$

Put $Q(t) = \log \mathbb{P}(S > t)$. Then:

1. $Q(t)$ is right-continuous.
2. $Q(t)$ is non-decreasing.
3. $Q(+\infty) = -\infty$.
4. $Q(t)$ satisfies the *Cauchy functional equation*

$$Q(t + s) = Q(t) + Q(s), \quad t, s > 0.$$

It follows from this that:

1. $Q(nt) = nQ(t)$ for any $t > 0$ and any integer $n \geq 1$.
2. Replacing t by m/n and dividing the equation by n we also get $Q(\frac{m}{n}) = \frac{m}{n}Q(1)$.
3. Using the right-continuity of $Q(t)$ we get that $Q(t) = tQ(1)$ for all $t \geq 0$. Since $Q(t)$ is non-increasing and $Q(+\infty) = -\infty$ we get $Q(1) = -\lambda$, $\lambda > 0$.

It follows that $\mathbb{P}(S > t) = e^{Q(t)} = e^{-\lambda t}$. $\square$

Example 7.4. (Feller's waiting time paradox). Assume that buses arrive at a fixed bus stop every 15 min on average. You leave your house at random. What is the expected waiting time for a bus?

1. Assume that inter-arrival times are independent and exponentially distributed. Then the expected waiting time is 15 min.
2. Assume that inter-arrivals are deterministic. Then you will have only 7.5 minutes on average.

Next we will show that the minimum of exponential random variables is again an exponential random variable.

Proposition 7.5. *Let $S_1, \dots, S_N$ be independent random variables such that $S_i \sim E(\lambda_i)$ for some $\lambda_i > 0$. Put*

$$S = \min\{S_1, \dots, S_N\}.$$

Then the following statements are valid:

1. *$S \sim E(\lambda)$ with $\lambda = \lambda_1 + \cdots + \lambda_N$.*
2. *$\mathbb{P}(S = S_i) = \lambda_i/\lambda$.*

Proof. The first statement can proved as follows:

$$\begin{aligned}
\mathbb{P}(S > t) &= \mathbb{P}(S_1 > t, \dots, S_N > t) \\
&= \mathbb{P}(S_1 > t) \cdots \mathbb{P}(S_N > t) \\
&= e^{-\lambda_1 t} \cdots e^{-\lambda_N t} \\
&= e^{-(\lambda_1 + \cdots + \lambda_N)t}.
\end{aligned}$$

To prove the second statement fix $i \in \{1, \dots, N\}$ and put

$$F_i(t) = \mathbb{P}(S_i \leq t) = 1 - e^{-\lambda_i t}.$$

Then:

$$
\begin{aligned}
\mathbb{P}(S = S_i) &= \mathbb{P}(S_j \geq S_i \ \forall j \neq i) \\
&= \int_0^\infty \mathbb{P}(S_j \geq t \ \forall j \neq i) dF_i(t) && \text{(law of total probability)} \\
&= \int_0^\infty \prod_{j \neq i} \mathbb{P}(S_j \geq t) dF_i(t) && \text{(independence)} \\
&= \int_0^\infty \prod_{j \neq i} e^{-\lambda_1 t} \cdots e^{-\lambda_N t} \lambda_i e^{-\lambda_i t} dt \\
&= \int_0^\infty e^{-(\lambda_1 + \cdots + \lambda_N)t} \lambda_i dt = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_N}.
\end{aligned}
$$

□

7.2 Continuous limit of Bernoulli trials

Recall that an integer-valued random-variable N has *Poisson distribution* with rate parameter $\lambda > 0$, denoted $N \sim Pois(\lambda)$, if

$$\mathbb{P}(N = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \ldots$$

Exercise 7.6. Let $N \sim Pois(\lambda)$. Show that $\mathbb{E}N = \lambda$, $\operatorname{Var} N = \lambda$.

The Poisson process is a model for a stream of uncoordinated events in continuous time. The Poisson process can be considered as a continuous limit of a sequence of independent Bernoulli trials. We consider the following example.

Customers arrive to a store. Let $h > 0$ and assume that:

1. Only one customer can arrive during the interval $[kh, (k+1)h)$, $k \geq 0$
2. Let be $X_k^{(h)}$ be the indicator that some customer arrives in the time interval $[kh, (k+1)h)$. Then $\{X_k^{(h)}\}_{k=0}^\infty$ are independent and

$$X_k^{(h)} = \begin{cases} 1 \text{ with probability } \lambda h, \\ 0 \text{ otherwise,} \end{cases}$$

for some fixed $\lambda > 0$.

Fix $t = nh$ and let $N_t^{(h)}$ be the total number of customers who arrives during $[0, t)$:

$$N_t^{(h)} = \textstyle\sum_{k=0}^{n-1} X_k^{(h)}.$$

Then $N_t^{(h)}$ has the Binomial distribution $Bin(n, \lambda h)$, that is,

$$\mathbb{P}(N_t^{(h)} = k) = \binom{n}{k} (\lambda h)^k (1 - \lambda h)^{n-k}.$$

Passing $h \to +0$ at fixed t we find that:

$$\begin{aligned}\lim_{h \to +0} \mathbb{P}(N_t^{(h)} = k) &= \lim_{n \to +\infty} \frac{n^k}{k!} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \\ &= \frac{(\lambda t)^k e^{-\lambda t}}{k!},\end{aligned}$$

that is, $N_t^{(h)} \to N_t$ in distribution, where $N_t \sim Pois(\lambda t)$.

Now let $J_k^{(h)} = \min\{t \geq 0 \colon N_t^{(h)} = k\}$, $k \geq 0$, be the customer arrival times and $S_k^{(h)} = J_k^{(h)} - J_{k-1}^{(h)}$, $k \geq 0$, be the customer inter-arrival times. The random variables $\{S_k^{(h)}\}_{k=1}^\infty$ are independent and

$$\mathbb{P}(S_k^{(h)} > t) = (1 - \lambda h)^n, \quad t = nh,$$

which shows that

$$\lim_{h \to +0} \mathbb{P}(S_k^{(h)} > t) = e^{-\lambda t},$$

so that $S_k^{(h)} \to S_k$ in distribution, where $S_k \sim E(\lambda)$.

7.3 Poisson process

Motivated by these observations, we introduce the Poisson process $\{N_t\}_{t \geq 0}$. Let $\{S_n\}_{n=1}^\infty$ be a sequence of independent $E(\lambda)$-distributed random variables (*holding times*). We define the *jump times* by the formula:

$$J_0 = 0, \quad J_n = S_1 + \cdots + S_n,\ n \geq 1.$$

A *Poisson process* of rate λ is a random process $\{N_t\}_{t \geq 0}$ defined by:

$$N_t = n, \quad J_n \leq t < J_{n+1}.$$

Thus, N_t is the number of jumps by time t. The next exercise shows how to compute the distribution of the time of the n-th jump.

Exercise 7.7. (Erlang distribution). Let $\{S_n\}_{n=1}^\infty$ be a sequence of independent $E(\lambda)$-distributed random variables and let $J_n = S_1 + \cdots + S_n$. Then $J_n \sim Erlang(k, \lambda)$, that is, the probability density function of J_n is given by:

$$f_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0.$$

Hint: The probability density function of the sum of independent random variables is the convolution of probability densities of the terms.

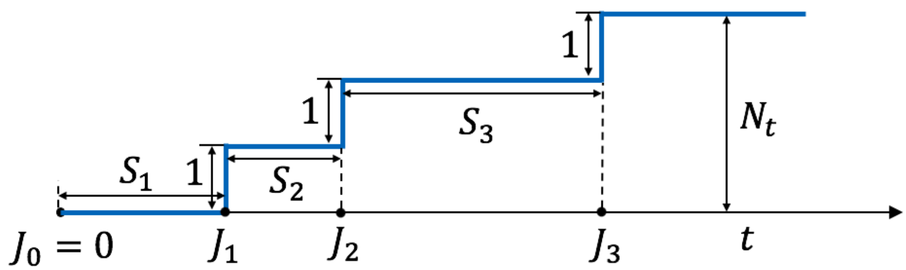


Figure 1: Poisson process

Exercise 7.8. Let $\{N_t\}_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$. Then $N_t \sim Pois(\lambda t)$, that is,

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, \ldots.$$

Hint: $\mathbb{P}(N_t = k) = \mathbb{P}(J_k \leq t < J_k + S_{k+1})$. *Use the law of total probability, conditioning on the value of* J_k.

Proposition 7.9. *(Markov property and time-homogeneity). Let* $\{N_t\}_{t\geq 0}$ *be a Poisson process of rate* λ. *Then* $\{N_{t+s} - N_s\}_{t\geq 0}$ *is a Poisson process of rate* λ *independent of* N_t, $t \in [0, s]$.

Proof. Put $\widetilde{N}_t = N_{t+s} - N_s$. It is enough to prove the statement conditioned on $N_s = i$ for each fixed i. Note that the event $\{N_s = i\}$ can be represented as:

$$\{N_s = i\} = \{J_i \leq s\} \cap \{S_{i+1} \geq s - J_i\}.$$

Conditioned on this event, the holding times $\{\widetilde{S}_n\}_{n=1}^{\infty}$ for process $\{\widetilde{N}_t\}_{t\geq 0}$ are given by:

$$\widetilde{S}_1 = S_{i+1} - (s - J_i), \quad \widetilde{S}_n = S_{n+i}, \; n \geq 2.$$

Fix $u \leq s$. Then using the law of total probabilty we have:

$$\begin{aligned}
\mathbb{P}(\widetilde{S}_1 > t, J_i \leq u | N_s = i) &= \mathbb{P}(S_{i+1} > t + s - J_i | N_s = i) \\
&= \int_0^s \mathbb{P}(S_{i+1} > t + s - v | S_{i+1} > s - v) 1_{v \leq u} d\mathbb{P}(J_i \leq v | N_s = i) \\
&= \int_0^s \mathbb{P}(S_{i+1} > t) 1_{v \leq u} \mathbb{P}(J_i \leq v | N_i = s) \\
&= \mathbb{P}(S_{i+1} > t) \mathbb{P}(J_i \leq u | N_s = i).
\end{aligned}$$

It follows that conditioned on $\{N_s = i\}$ we have $\widetilde{S}_1 \sim E(\lambda)$ and $\widetilde{S}_{i+1}$ is independent of J_i. Together with independence of $\{S_n\}_{n=1}^{\infty}$ this shows that $\{\widetilde{S}_n\}_{n=1}^{\infty}$ are independent from each other and from $S_1, \ldots, S_n$. This shows that $\{\widetilde{N}_t\}_{t\geq 0}$ is a Poisson process of rate λ. Note also that conditioned on $\{N_s = i\}$ one has

$$N_t = \textstyle\sum_{k=1}^{i} 1_{J_k \leq t}, \quad 0 \leq t \leq s.$$

This shows that $\{\widetilde{N}_t\}_{t\geq 0}$ is independent of N_t, $0 \leq t \leq s$.

$\square$

8 Continuous-time Markov chains

8.1 Definition and first properties

Let I be a finite or countable collection of states. A *continuous-time Markov chain* with values in I is a right-continuous random process $\{X_t\}_{t\geq 0}$ with values in I satisfying the following properties:

1. Markov property:

$$\mathbb{P}(X_{t_n} = j | X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}) = \mathbb{P}(X_{t_n} = j | X_{t_{n-1}} = i_{n-1}),$$
$$\forall t_n > \cdots > t_1 \geq 0, \quad \forall i_1, \ldots, i_n \in I.$$

2. Time-homogeneity:

$$\mathbb{P}(X_{t+s} = j | X_s = i) = \mathbb{P}(X_t = j | X_0 = i),$$
$$\forall t, s \geq 0, \quad \forall i, j \in I.$$

3. Smoothness: Put $p_{ij}(t) = \mathbb{P}(X_t = i | X_0 = j)$. Then:

$$p_{ij} \in C^1[0, +\infty) \quad i, j \in I.$$

The matrix $P(t) = (p_{ij}(t))$ is called the *transition matrix*. Put:

$$Q = \lim_{t \to +0} \frac{dP(t)}{dt}.$$

The matrix $Q = (q_{ij})$ is called the *generator* of the Markov chain $\{X_t\}_{t\geq 0}$ and q_{ij}, $i \neq j$, is called the *transition rate* from state i to state j. We draw a transition diagram for $\{X_t\}_{t\geq 0}$ as a directed graph $\Gamma = (I, E)$ with the set of arcs $E = \{(i,j) \in I \times I \colon q_{ij} > 0\}$ and we indicate the transition rate q_{ij} above the arc (i, j).

From now on we will consider the case $I = \{1, \ldots, N\}$. Then we will briefly discuss the case of countable I.

Exercise 8.1. Let $\{X_t\}_{t\geq 0}$ be a Markov chain with generator $Q = (q_{ij})$. Show that Q has the following properties:

1. $q_{ii} \leq 0$ for all $i \in I$.
2. $q_{ij} \geq 0$ for all $i, j \in I$, $i \neq j$.
3. $\sum_{j \in I} q_{ij} = 0$ for all $i \in I$.

Proposition 8.2. *Let $\{X_t\}_{t\geq 0}$ be a Markov chain with transition matrix $P(t)$. Then:*

1. *$\{P(t)\}_{t\geq 0}$ is a semigroup:*

$$P(0) = I, \quad P(t+s) = P(s)P(t), \ t, s \geq 0.$$

2. *Transition matrix satisfies the equations:*

$$\frac{dP}{dt} = QP(t) \qquad \text{(backward Kolmogorov equation)}$$
$$\frac{dP}{dt} = P(t)Q \qquad \text{(forward Kolmogorov equation).}$$

Proof. By definition $P(0) = I$. The semigroup property follows from by the law of total probability:

$$\begin{aligned} p_{ij}(t+s) &= \mathbb{P}(X_{t+s} = j | X_0 = i) \\ &= \textstyle\sum_{\ell \in I} \mathbb{P}(X_{t+s} = j | X_s = \ell) \mathbb{P}(X_s = \ell | X_0 = i) \\ &= \textstyle\sum_{\ell \in I} p_{\ell j}(t) p_{i\ell}(s). \end{aligned}$$

Taking the derivative in the semigroup property with respect to s and setting $s = 0$ we get the backward Kolmogorov equation. Differentiating with respect to t and setting $t = 0$ we get the forward Kolmogorov equation. □

Corollary 8.3. *Let $\{X_t\}_{t \geq 0}$ be a Markov chain with a finite set of states I, transition matrix $P(t)$ and generator matrix Q. Then $P(t) = \exp(Qt)$ for all $t \geq 0$.*

Corollary 8.4. *Let $\{X_t\}_{t \geq 0}$ be a Markov chain with generator matrix Q and let $X_0 \sim \pi$, $\pi \in \Delta(I)$. Then $X_t \sim \pi_t$, where $\pi_t = \pi P(t)$. Besides, π_t can be found by solving the Kolmogorov forward equation:*

$$\frac{d\pi_t}{dt} = \pi_t Q, \ t > 0, \quad \pi_0 = \pi.$$

Proof. By the law of total probability

$$\begin{aligned} [\pi_t]_j &= \mathbb{P}(X_t = j | X_0 \sim \pi) \\ &= \sum_{i \in I} \mathbb{P}(X_t = j | X_0 = i) \pi_i \\ &= \sum_{i \in I} p_{ij}(t) \pi_i = [\pi P(t)]_j. \end{aligned}$$

Taking the derivative with respect to t and using the Kolmogorov forward equation we get:

$$\frac{d\pi_t}{dt} = \pi \frac{dP(t)}{dt} = \pi P(t) Q = \pi_t Q.$$

□

Exercise 8.5. Let $\{X_t\}_{t \geq 0}$ be a Markov chain with generator Q and let $f = (f_i) \in \mathbb{R}^N$. Show that

$$\mathbb{E}[f_{X_t} | X_0 = i] = f_i + t[Qf]_i + o(t), \quad t \to +0.$$

8.2 Embedded jump process

Let $\{X_t\}_{t\geq 0}$ be a continuous-time Markov chain without absorbing states. As a right-continuous process, $\{X_t\}_{t\geq 0}$ jumps from one state to another and the time spent in each state is random. The *jump times* are defined as follows:

$$J_1 = 0, \quad J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}.$$

The *holding times* (also called *sojourn times*) are defined by the formula:

$$S_n = J_n - J_{n-1}, \quad n \geq 1.$$

The random process $\{Y_n\}_{n=1}^{\infty}$, $Y_n = X_{J_n}$, is called the (embedded) *jump process* (or the *jump chain*) of process $\{X_t\}_{t\geq 0}$. The following theorem shows how that a continuous-time Markov chain is uniquely determined by its holding times and jump chain. We will prove this result for a simpler case where absorbing states are not allowed.

In what follows, for a given generator $Q = (q_{ij})$ we denote $q_i = -q_{ii}$, $i \in I$.

Lemma 8.6. *Let $\{Y_n\}_{n=1}^{\infty}$ be a discrete-time Markov chain with transition matrix $\Pi = (\pi_{ij})$ such that $\pi_{ii} = 0$ for all $i \in I$. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of random variables such that conditioned on $Y_1 = i_1, \ldots, Y_n = i_n$ the random variables $S_1, \ldots, S_n$ are independent and $S_k \sim E(q_{i_k})$, $k = 1, \ldots, n$. Put $J_0 = 0$, $J_n = S_1 + \cdots + S_n$,*

$$X_t = Y_n, \quad J_n \leq t < J_{n+1},\ n \geq 0.$$

Then $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain with generator matrix $Q = (q_{ij})$ such that:

$$q_{ii} = -q_i, \quad q_{ij} = q_i \pi_{ij},\ i \neq j.$$

Proof. (Markov property and time-homogeneity). Fix $s > 0$ and let $\widetilde{X}_t = X_{t+s}$, $t \geq 0$. In a similar way with Proposition 7.9 one can show that $\{\widetilde{X}_t\}_{t\geq 0}$ is a continuous-time Markov process such that conditional on $\{X_s = i\}$ it is independent of $\{X_t\}_{0\leq t<s}$ and its joint distributions coincide with the joint distributions of $\{X_t\}_{t\geq 0}$ conditional on $X_0 = i$.

(Smoothness and generator). We will show that the following formula is valid for each $i, j \in I$:

$$\mathbb{P}(X_t = j | X_0 = i) = \delta_{ij} + q_{ij}t + o(t), \quad t \to +0,$$

where δ_{ij} is the Kronecker delta. Together with Markov property and time-homogeneity, this will imply that $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain with generator matrix Q.

Fix $i \in I$. Then:

$$\begin{aligned}\mathbb{P}(X_t = i | X_0 = i) &\geq \mathbb{P}(S_1 > t | Y_1 = i) \\ &= e^{-q_i t} = 1 + q_{ii}t + o(t), \quad h \to +0.\end{aligned}$$

Let $j \in I$, $j \neq i$. Then:

$$\begin{aligned}
\mathbb{P}(X_t = j | X_0 = i) &\geq \mathbb{P}(S_1 \leq t, Y_2 = j, S_2 > t | Y_1 = i) \\
&\geq \mathbb{P}(S_1 \leq t, S_2 > t | Y_1 = i, Y_2 = j)\mathbb{P}(Y_2 = j | Y_1 = i) \\
&= (1 - e^{-q_i t})e^{-q_j t}\pi_{ij} = q_{ij}t + o(t), \quad t \to +0.
\end{aligned}$$

It follows that:

$$\mathbb{P}(X_t = j | X_0 = i) \geq \delta_{ij} + q_{ij}t + o(t), \quad t \to +0, \quad \forall i, j.$$

Taking the sum over all $j \in I$ we get $1 \geq 1 + o(t)$ which shows that all the inequalities hold as equalities. This proves the theorem. □

A state $i \in I$ of Markov chain $\{X_t\}_{t \geq 0}$ is called absorbing if $\mathbb{P}(X_t = i | X_0 = i) = 1$ for all $t \geq 0$.

Exercise 8.7. Let $\{X_t\}_{t \geq 0}$ be a Markov chain with generator $Q = (q_{ij})$. Show that state $i \in I$ is absorbing if and only if $q_{ii} = 0$.

Theorem 8.8. *Let $\{X_t\}_{t \geq 0}$ be a Markov chain with generator $Q = (q_{ij})$ and without absorbing states and put $q_i = -q_{ii}$. Let $\{Y_n\}_{n=1}^{\infty}$ be the associated jump chain and $\{S_n\}_{n=1}^{\infty}$ the holding times. Then:*

1. *$\{Y_n\}_{n=1}^{\infty}$ is a discrete-time Markov chain with jump matrix $\Pi = (\pi_{ij})$:*

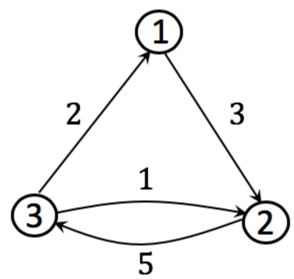
$$\pi_{ii} = 0 \ \forall i \in I, \quad \pi_{ij} = q_{ij}/q_i, \ \forall i, j \in I, i \neq j.$$

2. *Conditional on $Y_1 = i_1, \ldots, Y_n = i_n$, the holding times $\{S_k\}_{k=1}^{n}$ are independent and $S_k \sim E(q_k)$.*

Proof. Let $\{\widetilde{Y}_n\}_{n=1}^{\infty}$ be a discrete-time Markov chain with jump matrix Π and $\{\widetilde{S}_n\}_{n=1}^{\infty}$ be such that conditional on $Y_1 = i_1, \ldots, Y_n = i_n$ $\{\widetilde{S}_k\}_{k=1}^{n}$ are independent and $\widetilde{S}_k \sim E(q_k)$. Construct a continuous-time Markov chain $\{\widetilde{X}_t\}_{t \geq 0}$ using Lemma 8.6. Then $\{X_t\}_{t \geq 0}$ and $\{\widetilde{X}_t\}_{t \geq 0}$ have the same generators and, as a corollary, the same joint distributions. It follows that the distributions of their holding times and the joint distribution of their jump chains also coincide. □

Corollary 8.9. *Let $\{X_t\}_{t \geq 0}$ be a Markov chain with generator $Q = (q_{ij})$, holding times $\{S_n\}_{n=1}^{\infty}$ and jump chain $\{Y_n\}_{n=1}^{\infty}$. Then $S_n = \frac{1}{q_{Y_n}} T_n$, where $q_i = -q_{ii}$ is the holding rate at state i and $\{T_n\}_{n=1}^{\infty}$ are independent random variables from $E(1)$.*

Example 8.10. Consider a continuous-time Markov chain $\{X_t\}_{t \geq 0}$ with diagram:



Assume that the chain starts from state three at time zero. Then the chain jumps from state three at time $S \sim E(3 = 2 + 1)$ which is $1/3$ on average. At time S the chain jumps to state one with probability $2/3 = 2/(2+1)$ and to state two with probability $1/3 = 1/(2+3)$.

8.3 Invariant distribution

Let $\{X_t\}_{t\geq 0}$ be a continuous-time Markov chain with state space $I = \{1, \dots, N\}$ and generator matrix Q. Distribution $\pi \in \Delta(I)$ such that $\pi Q = 0$ is called *invariant*.

Example 8.11. Let $\{X_t\}_{t\geq 0}$ be a Markov chain with generator matrix Q. Assume that $X_0 \sim \pi$, $\pi \in \Delta(I)$, and $\pi Q = 0$. Then for each $t \geq 0$ we have $X_t \sim \pi_t$, where

$$\pi_t = \pi P(t) = \pi \exp(Qt) = \textstyle\sum_{k=1}^{\infty} \pi Q^k / k! = \pi.$$

Lemma 8.12. *Let $\{X_t\}_{t\geq 0}$ be a Markov chain with generator matrix Q. Let $\{Y_n\}_{n=1}^{\infty}$ be the associated jump chain with transition matrix Π. Let $x \in \mathbb{R}^N$ and put $y = (q_1 x_1, \dots, q_N x_N) \in \mathbb{R}^N$. Then $xQ = 0$ if and only if $y = y\Pi$.*

Proof. It follows from Lemma 8.6 that $q_i(\pi_{ij} - \delta_{ij}) = q_{ij}$. The claim follows from the transformations:

$$[y(\Pi - I)]_j = \textstyle\sum_{i\in I} y_i(\pi_{ij} - \delta_{ij}) = \sum_{i \in I} x_i q_i (\pi_{ij} - \delta_{ij}) = [\nu Q]_j.$$

$\square$

The Markov chain $\{X_t\}_{t\geq 0}$ is called *irreducible* if the associated jump chain is irreducible. Let T_i denote the *first passage time* to state i:

$$T_i = \inf\{t \geq S_1 \colon X_t = i\}.$$

We will also use the notation $\mathbb{E}^i \xi = \mathbb{E}(\xi | X_0 = i)$.

Theorem 8.13. *Let $\{X_t\}_{t\geq 0}$ be an irreducible Markov chain with generator Q. Then $\{X_t\}_{t\geq 0}$ admits a unique invariant distribution $\pi = (\pi_1, \dots, \pi_N) \in \Delta(I)$ and $\pi_i = 1/(q_i m_i)$, where $m_i = \mathbb{E}^i T_i$.*

Proof. (Uniqueness). Let $\{X_t\}_{t\geq 0}$ be irreducible and let $\{Y_n\}_{n=1}^{\infty}$ be the associated irreducible jump chain. Assume that π^1, π^2 are two different invariant distributions for $\{X_t\}_{t\geq 0}$. Put

$$\widetilde{\pi}_i^k = q_i \pi_i^k / \textstyle\sum_{j=1}^N q_j \pi_j^k.$$

By Lemma 8.12 $\widetilde{\pi}^1$ and $\widetilde{\pi}^2$ are two different invariant distributions for $\{Y_n\}_{n=0}^{\infty}$ which contradicts its irreducibility.

(Construction). Let $\{Y_n\}_{n=1}^{\infty}$ be the jump chain of $\{X_t\}_{t\geq 0}$ and Π its transition matrix. Let $\{S_n\}_{n=1}^{\infty}$ be the holding times of $\{X_t\}_{t\geq 0}$. Put:

$$\begin{aligned} N_i &= \min\{n > 1 \colon Y_n = i\}, \quad i \in I, \\ \gamma_j^i &= \mathbb{E}^i \big(\textstyle\sum_{n=1}^{N_i - 1} 1_{Y_n = j}\big), \quad i, j \in I, \end{aligned}$$

so that N_i is the first passage time to state i for $\{Y_n\}_{n=1}^{\infty}$ and γ_j^i is the expected time spent in state j by $\{Y_n\}_{n=1}^{\infty}$ before returning to state i. Also put

$$\mu_j^i = \mathbb{E}^i \int_0^{T_i} 1_{X_s = j} ds, \quad i, j \in I.$$

Note that μ_j^i is the expected time spent by $\{X_t\}_{t\geq 0}$ in state j before returning to state i. We will show that $\mu_j^i = \gamma_j^i / q_j$. This relation follows from the transformations:

$$\begin{aligned} \mu_j^i &= \mathbb{E}^i \big(\textstyle\sum_{n=1}^{N_i - 1} 1_{Y_n = j} S_n\big) \\ &= \mathbb{E}^i \big(\textstyle\sum_{n=1}^{\infty} 1_{N_i > n} 1_{Y_n = j} S_n\big) \\ &= \textstyle\sum_{n=1}^{\infty} \mathbb{E}^i (S_n | N_i > n, Y_n = j) \mathbb{P}(N_i > n, Y_n = j | Y_1 = i) \\ &= \tfrac{1}{q_i} \mathbb{E}^i \big(\textstyle\sum_{n=1}^{N_i - 1} 1_{Y_n = j}\big) = \gamma_j^i / q_i. \end{aligned}$$

Now let $\mu^i = (\mu_1^i, \ldots, \mu_N^i)$ and $\gamma^i = (\gamma_1^i, \ldots, \gamma_N^i)$. It follows from Theorem 5.6 that $\gamma^i = \gamma^i \Pi$, where Π is the transition matrix of $\{Y_n\}_{n=1}^{\infty}$. Together with Lemma 8.12 and relation $\mu_j^i = \gamma_j^i / q_j$ it shows that $\mu^i Q = 0$. Put

$$\pi^i = (\pi_1, \ldots, \pi_N), \quad \pi_k^i = \mu_k^i / (\mu_1^i + \cdots + \mu_N^i).$$

Then π^i is the invariant distribution for $\{X_t\}_{t\geq 0}$ and by uniqueness we get $\pi^1 = \cdots = \pi^N \stackrel{def}{=} \pi$. It follows that for each $i \in I$:

$$\pi_i = \pi_i^i = \mu_i^i / (\mu_1^i + \cdots + \mu_N^i) = \mathbb{E}^i S_1 / \mathbb{E}^i T_i = 1/(q_i m_i),$$

which proves the theorem. □

8.4 Ergodic theorem

In a similar way to irreducible discrete-time Markov chains, for irreducible continuous-time Markov chains the share of time in state i in the long run is equal to the invariant probability of this state.

Theorem 8.14. *Let $\{X_t\}_{t\geq 0}$ be an irreducible Markov chain with invariant distribution $\pi = (\pi_1, \dots, \pi_N) \in \Delta(I)$. Then for each $i \in I$:*

$$\frac{1}{t}\int_0^t 1_{X_s=i}\, ds \to \pi_i \text{ almost surely as } t \to +\infty.$$

Proof. Fix $i \in I$. We need to introduce some notations:

$$T_i^{(0)} = 0, \quad T_i^{(n)} = \min\{t \geq T_i^{(n-1)} + M_i^{(n)} \colon X_t = i\},\ n \geq 1,$$
$$M_i^{(n)} = \min\{t > T_i^{(n-1)} \colon X_t \neq i\} - T_i^{(n-1)},\ n \geq 1,$$
$$L_i^{(n)} = T_i^{(n)} - T_i^{(n-1)},\ n \geq 1.$$

Note that $M_i^{(n)}$ is the n-th holding time at state i, $T_i^{(n)}$ is the n-th passage time to state i, and $L_i^{(n)}$ is the time between visits $n-1$ and n to state i. Then $\{M_i^{(n)}\}_{n=1}^{\infty}$ are independent identically distributed from $E(q_i)$ and $\{L_i^{(n)}\}_{n=1}^{\infty}$ are independent identically distributed with mean $m_i = 1/(q_i\pi_i)$. By the strong law of large numbers:

$$(L_i^{(1)} + \dots + L_i^{(n)})/n = T_i^{(n)}/n \to m_i \quad \text{almost surely as } n \to +\infty,$$
$$(M_i^{(1)} + \dots + M_i^{(n)})/n \to 1/q_i \quad \text{almost surely as } n \to +\infty.$$

Let t be such that $T_i^{(n)} \leq t < T_i^{(n+1)}$:

$$\frac{T_i^{(n)}}{T_i^{(n+1)}} \frac{M^{(1)} + \dots + M^{(n)}}{L_i^{(1)} + \dots + L_i^{(n)}} \leq \frac{1}{t}\int_0^t 1_{X_s=i}\, ds \leq \frac{T_i^{(n+1)}}{T_i^{(n)}} \frac{M_i^{(1)} + \dots + M_i^{(n+1)}}{L_i^{(1)} + \dots + L_i^{(n+1)}}.$$

The claim follows by taking the limit as $n \to +\infty$. □

Corollary 8.15. *Let $\{X_t\}_{t\geq 0}$ be an irreducible Markov chain with invariant probability $\pi = (\pi_1, \dots, \pi_N) \in \Delta(I)$. Then for each function $f\colon I \to \mathbb{R}$:*

$$\frac{1}{t}\int_0^t f(X_s) ds \to \overline{f} \quad \text{almost surely as } t \to +\infty,$$
$$\overline{f} = \pi_1 f(1) + \dots + \pi_N f(N).$$

Proof. Note that

$$f(X_s) = \textstyle\sum_{i\in I} f(i) 1_{X_s=i}.$$

The claim follows from theorem 8.14 and from the representation:

$$\frac{1}{t}\int_0^t f(X_s) ds - \overline{f} = \textstyle\sum_{i\in I} f(i)\left(\frac{1}{t}\int_0^t 1_{X_s=i} ds - \pi_i\right).$$

□

8.5 Countable state space

Let $I = \mathbb{N} = \{1, 2, \dots\}$ and put

$$\Delta(I) = \{\pi = (\pi_1, \pi_2, \dots) \in \mathbb{R}^{\mathbb{N}} \colon \textstyle\sum_{i=1}^{\infty} \pi_i = 1,\ \pi \geq 0\}.$$

Let $\{X_t\}_{t\geq 0}$ be a Markov chain with state space I. Let $\{S_n\}_{n\geq 1}$ be the holding times of $\{X_t\}_{t\geq 0}$. $\{X_t\}_{t\geq 0}$ is called *non-explosive* if $\sum_{n=1}^{\infty} S_n = \infty$ almost surely.

Lemma 8.16. *Let $\{T_n\}_{n=1}^{\infty}$ be independent random variables with $T_n \sim E(q_n)$, $q_n > 0$. Then:*

1. *If $\sum_{n=1}^{\infty} 1/q_n < \infty$, then $\sum_{n=1}^{\infty} T_n < \infty$ almost surely.*
2. *If $\sum_{n=1}^{\infty} 1/q_n = \infty$, then $\sum_{n=1}^{\infty} T_n = \infty$ almost surely.*

Proof. (*Part 1*). Assume that $\sum_{n=1}^{\infty} 1/q_n < \infty$. Then by monotone convergence:

$$\mathbb{E}\big(\textstyle\sum_{n=1}^{\infty} T_n\big) = \sum_{n=1}^{\infty} \frac{1}{q_n} < \infty.$$

It follows that $\mathbb{P}(\sum_{n=1}^{\infty} T_n = \infty) = 0$.
(*Part 2*). Assume that $\sum_{n=1}^{\infty} 1/q_n = \infty$. Then:

$$\prod_{n=1}^{\infty}\big(1 + \tfrac{1}{q_n}\big) \geq 1 + \sum_{n=1}^{\infty} \frac{1}{q_n} = \infty.$$

By monotone convergence and independence:

$$\mathbb{E}\exp\big(-\textstyle\sum_{n=1}^{\infty} T_n\big) = \prod_{n=1}^{\infty} \mathbb{E}e^{-T_n} = \prod_{n=1}^{\infty}\big(1 + \tfrac{1}{q_n}\big)^{-1} = 0.$$

It follows that $\mathbb{P}(\sum_{n=1}^{\infty} T_n = \infty) = 1$.

□

Proposition 8.17. *Let $\{X_t\}_{t\geq 0}$ be a Markov chain with generator $Q = (q_{ij})$. Assume that $\sup_i q_i \leq q < \infty$, where $q_i = -q_{ii}$. Then $\{X_t\}_{t\geq 0}$ is non-explosive.*

Proof. Let $\{Y_n\}_{n=1}^{\infty}$ be the and $\{S_n\}_{n=1}^{\infty}$ be the holding times of $\{X_t\}_{t\geq 0}$. Then $S_n = T_n / q_{Y_n}$, where $\{T_n\}_{n=1}^{\infty}$ are independent and $E(1)$ distributed. Using Lemma 8.16 we get:

$$\sum_{k=1}^{n} S_k \geq \frac{1}{q} \sum_{k=1}^{n-1} T_k \to +\infty \quad \text{as } n \to +\infty \text{ almost surely.}$$

□

Let $\{X_t\}_{t\geq 0}$ be a Markov chain with state space I and generator $Q = (q_{ij})$, and let $\{Y_n\}_{n=1}^{\infty}$ be its . State $i \in I$ is called:

1. *recurrent* if $\mathbb{P}(Y_n = i \text{ for infinitely many } n) = 1$.

2. *transient* if $\mathbb{P}(Y_n = i \text{ for infinitely many } n) = 0$.

A recurrent state i is called *positive recurrent* if $\mathbb{E}^i T_i < \infty$ or $q_i = 0$, where $q_i = -q_{ii}$ and $T_i = \min\{t \geq S_1 \colon X_t = i\}$ be the first passage time to state i.

Remark 8.18. *One can show that in an irreducible Markov chain either all the states are recurrent (resp. transient, resp. positive recurrent) or none of them. In the former case the Markov chain itself is called recurrent (resp. transient, resp. positive recurrent).*

Theorem 8.19. *Let $\{X_t\}_{t\geq 0}$ be an irreducible Markov chain with generator $Q = (q_{ij})$ and transition matrix $P(t) = (p_{ij}(t))$. Then $\{X_t\}_{t\geq 0}$ is positive recurrent if and only if it is non-explosive and has an invariant distribution $\pi = (\pi_1, \pi_2, \dots)$. In this case the following formulas are valid:*

1. *(Time to return) $\pi_i = 1/(q_i m_i)$, where $q_i = -q_{ii}$, $m_i = \mathbb{E}^i T_i$, for all $i \in I$.*
2. *(Convergence) $p_{ij}(t) \to \pi_j$ as $t \to +\infty$ for all i, $j \in I$.*
3. *(Ergodicity) $\frac{1}{t}\int_0^t 1_{\{X_t = i\}} \to \pi_i$ as $t \to +\infty$ almost surely for all $i \in I$.*

8.6 Time-reversal

Theorem 8.20. *Let $\{X_t\}_{t\geq 0}$ be an irreducible, non-explosive Markov chain with generator Q and such that $X_0 \sim \pi$, where $\pi = (\pi_1, \pi_2, \dots)$ is the invariant distribution. Fix $T > 0$ and put $\widehat{X}_t = X_{T-t}$, $t \in [0, T]$. Then $\{\widehat{X}_t\}_{0\leq t\leq T}$ is an irreducible, non-explosive Markov chain with generator $\widehat{Q} = (\widehat{q}_{ij})$ and invariant distribution π, such that:*

$$\lambda_i q_{ij} = \lambda_j \widehat{q}_{ji}.$$

Proposition 8.21. *(Detailed balance). Let $\{X_t\}_{t\geq 0}$ be a Markov chain with generator Q and let $\pi \in \Delta(I)$ be a distribution such that:*

$$\pi_i q_{ij} = \pi_j q_{ij} \quad \forall i, j \in I.$$

Then π is an invariant distribution for $\{X_t\}_{t\geq 0}$.

Proof. Fix $j \in I$. Then:

$$[\pi Q]_j = \sum_{i\in I} \pi_i q_{ij} = \sum_{i\in I} \pi_j q_{ji} = \pi_j \sum_{i\in I} q_{ji} = 0.$$

□

Corollary 8.22. *Let $\{X_t\}_{t\geq 0}$ be an irreducible, non-explosive Markov chain with generator $Q = (q_{ij})$ and invariant measure π satisfying the detailed balance equations, and such that $X_0 \sim \pi$. Fix $T > 0$ and put $\widehat{X}_t = X_{T-t}$. Then $\{\widehat{X}_t\}_{0\leq t\leq T}$ is a Markov chain with generator Q and $\widehat{X}_0 \sim \pi$.*

9 Memoryless queues

9.1 Queueing theory

Queueing theory is a mathematical theory describing waiting in queues. In a queueing system customers arrive, wait for a service, are being served, and leave as shown in Figure 2.

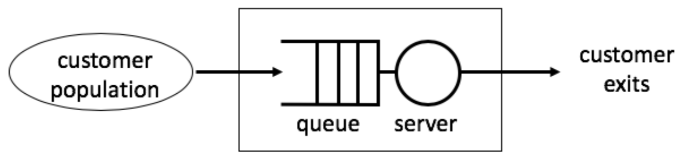


Figure 2: Queueing system

Typical questions of interest are:

1. How many servers should be employed?
2. What is the optimal service discipline?
3. Is the waiting area adequate?

In 1953 D. G. Kendall proposed to classify queues by notation A/S/c, where A is the code for the distribution of *inter-arrival times of customers*, S is the code for the distribution of *service times*, and c is the *number of servers*. The most common codes used for distributions are M (memoryless), G (general distribution) and D (deterministic).

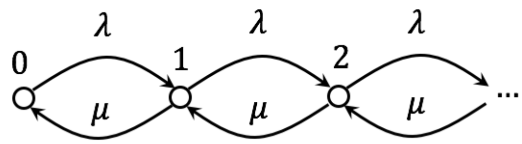
9.2 M/M/1 queues

An M/M/1 queueing system is characterized by the properties:

1. Inter-arrival times of customers are independent and $E(\lambda)$ distributed with the *arrival rate* $\lambda > 0$.
2. Service times are independent and $E(\mu)$ distributed with the *service rate* $\mu > 0$.
3. At most one customer can be served at a time.

Example 9.1. Assume that there are currently two customers in a store. Then a new customer arrives after time $S \sim E(\lambda)$ and the currently served customer leaves after time $Y \sim E(\mu)$. The number of customers changes after time $\min(S, Y) \sim E(\lambda + \mu)$ and at that time there will be one customer with probability $\mu/(\lambda + \mu)$ or three customers with probability $\lambda/(\lambda + \mu)$.

Let X_t be the number of customers in the system at time t. Then $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain with transition diagram:



To study the long-run behaviour of the queue we compute the invariant distribution of the Markov chain. The detailed balance equation reads:

$$\pi_i \lambda = \pi_{i+1}\mu \quad i \geq 0,$$

so that the invariant distribution is exists provided $\lambda < \mu$ and is given by:

$$\pi_i = (1-\rho)\rho^i, \quad \rho = \lambda/\mu,$$

where ρ is called the *utilization factor*.

Let $X \sim \pi$, where π is the invariant distribution. Let $L_s = \mathbb{E}X$ be the average number of customers in the system in the long run and $L_q = \mathbb{E}(X-1)_+$ be the average queue lenght in the long run.

Proposition 9.2. *Consider an M/M/1 queue with the arrival rate λ and service rate $\mu > \lambda$. Let $\rho = \lambda/\mu$ be the utilization factor. Then:*

$$L_s = \frac{\rho}{1-\rho}, \quad L_q = \frac{\rho^2}{1-\rho}.$$

Proof. A simple computation shows that:

$$L_s = \sum_{i=0}^{\infty} i\pi_i = (1-\rho)\sum_{i=0}^{\infty} i\rho^i = \rho/(1-\rho).$$

Another computations verifies the second claim:

$$\begin{aligned} L_q &= \mathbb{E}^{\pi}(X_t - 1 + 1_{X_t=0}) \\ &= (\mathbb{E}^{\pi} X_t) - 1 + \pi_0 \\ &= L_s - \rho = \rho^2/(1-\rho). \end{aligned}$$

□

Assume that a customer arrives when there are already i customers in the system. Then the waiting time of the arriving customer is $W_q^i \sim Erlang(i, \mu)$ and his time spent in the system is $W_s^i \sim Erlang(i+1, \mu)$. Let $X \sim \pi$, where π is the equilibrium distribution. Then in long run the average waiting time is $W_q = \mathbb{E}W_q^X$ and the average time spent by a customer in the system is $W_s = \mathbb{E}W_s^X$.

Proposition 9.3. *(Little's law). Consider an M/M/1 queue with arrival rate λ and service rate $\mu > \lambda$. Then:*

$$L_s = \lambda W_s, \quad L_q = \lambda W_q.$$

Proof. The first formula follows from computations:

$$\begin{aligned} W_s &= \textstyle\sum_{i=0}^{\infty} \mathbb{E}(W|X=i)\pi_i && \text{(law of total probability)} \\ &= \textstyle\sum_{i=0}^{\infty} (i+1)\pi_i/\mu \\ &= (L_s+1)/\mu = L_s/\lambda. \end{aligned}$$

The second formula follows from this formula and from $W_q = W_s - 1/\mu$. □

Example 9.4. *(Loading trucks).* A brick distributor employs one worker whose job to load bricks on outgoing trucks. On average $\lambda = 3$ trucks per hour arrive to the station following the Poisson process. The worker loads $\mu = 4$ trucks per hour on average following the exponential distribution.

The exploitation factor is $\rho = 0.75$ and the invariant distribution is:

i	0	1	2	3
π_i	.25	.19	.14	.11
$\mathbb{P}(X \geq i) = \rho^i$	1	.75	.56	.42

One can see that 56% of the time there is at least one truck waiting in the queue and 42% of the time there are at least two trucks waiting in the queue. Also, we have the following average quantities:

L_s	L_q	W_s	W_q
3 trucks	2.25 trucks	1 h	45 min

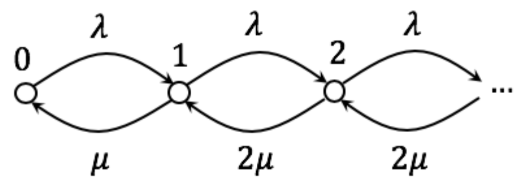
Assume that brick loaders earn \$6 per hour and truck drivers earn \$10 per hour. Then the idle time of truck drivers costs 3 x \$10 = \$30 per hour. *Would employing a second brick loader be economically attractive?*

9.3 M/M/2 queues

An M/M/2 queue is characterized by the following properties:

1. Customer inter-arrival times are independent and $E(\lambda)$ distributed
2. Service times are independent and $E(\mu)$ distributed
3. At most two customers can be served in parallel

Let X_t be the number of customers in the queueing system at time t. Then $\{X_t\}_{t\geq 0}$ is a Markov chain with transition diagram:



To study the long-run behaviour of the system we compute the invariant distribution using the detailed balance equation:

$$\pi_0\lambda = \pi_1\mu,$$
$$\pi_{i+1}\lambda = \pi_i 2\mu, \quad i \geq 1.$$

Let $\rho = \lambda/(2\mu)$ be the *traffic intensity*. It follows that:

$$\pi_1 = 2\rho\pi_0, \quad \pi_{i+1} = 2\rho^{i+1}\pi_0,\ i \geq 1.$$

Exercise 9.5. Let X be a random variable with values in $\mathbb{Z}_{\geq 0}$. The *probability generating function* (pgf) of X is defined by:

$$G_X(z) \stackrel{def}{=} \pi_0 + \pi_1 z + \pi_2 z^2 + \cdots$$

where $\pi_i = \mathbb{P}(X = i)$. Show that $G_X(1) = 1$ and $G'_X(1) = \mathbb{E}X$.

Let $X \sim \pi$ be the number of customers in the queueing system in the equilibrium and let $G_X(z)$ be the probability generating function for X:

$$\begin{aligned} G_X(z) &= \pi_0 + \pi_1 z + \pi_2 z^2 + \cdots \\ &= \pi_0\big(1 + 2\rho z + 2\rho^2 z^2 + 2\rho^3 z^3 + \cdots\big) \\ &= \pi_0\big(1 + 2\rho z/(1-\rho z)\big) = \pi_0(1+\rho z)/(1-\rho z). \end{aligned}$$

Using $G_X(1)$ we get $\pi_0 = (1-\rho)/(1+\rho)$. This is the fraction of time in the long run when the servers are idle.

Proposition 9.6. *Let $X \sim \pi$, where π is the equilibrium distribution for an M/M/2 queue with arrival rate λ and service rate μ, and let $\rho = \lambda/(2\mu)$ be the traffic intensity. Let $L_s = \mathbb{E}X$ be the number of customers in the system and $L_q = \mathbb{E}(X-2)_+$ be the queue length in the long run . Then:*

$$L_s = 2\rho/(1-\rho^2), \quad L_q = 2\rho^3/(1-\rho^2).$$

Proof. By Exercise 9.5 we have $L_s = G'_X(1)$ which proves the first formula. The second formula can be proved as follows:

$$\begin{aligned} L_q &= \mathbb{E}(X-2)_+ \\ &= 0\pi_0 + 0\pi_1 + 0\pi_2 + \pi_3 + 2\pi_4 + \cdots \\ &= \mathbb{E}X - 2 + 2\pi_0 + \pi_1 \\ &= L_s - 2\rho = 2\rho^3/(1-\rho^2). \end{aligned}$$

□

Next we compute the distribution of the waiting time of a customer in the long run.

Proposition 9.7. *Let W_q^i be the waiting time of an arriving customer assuming that there are i customers in the system at the arrival. Then $W_q^0 = W_q^1 = 0$, $W_q^i \sim Erlang(i-1, 2\mu)$ for $i \geq 2$.*

Proof. Formulas $W_q^0 = W_q^1 = 0$ are straightforward. To prove the formulas for $i \geq 2$ we use induction on i.

(*Basis*). Assume that $i = 2$ and let $Y_1 \sim E(\mu)$, $Y_2 \sim E(\mu)$ be the remaining service times of the first and second customers in the system. Then by Proposition 7.5 $W_q^2 = \min(Y_1, Y_2) \sim E(2\mu) = Erlang(1, 2\mu)$.

(*Hypothesis*). Let $i \geq 3$ and assume that $W_q^k \sim Erlang(k, 2\mu)$, $2 \leq k < i$.

(*Step*). $W_q^{i+1} = W_q^{i-1} + W_q^2$, where W_q^{i-1} is the waiting time of the previous customer and W_2 is the remaining waiting time. It follows that $W^{i+1} \sim Erlang(i, 2\mu)$. □

Denote by W_s^i the time spent in the system by a customer who arrives when there are already i customers in the system. Let $X \sim \pi$, where π is the equilibrium distribution. In the long run the average waiting time is $W_q = \mathbb{E}W_q^X$ and the average time spent in the system is $W_s = \mathbb{E}W_s^X$.

Exercise 9.8. (Little's law). Consider an M/M/2 queue with arrival rate λ and service rate μ such that $\rho = \lambda/(2\mu) < 1$. The following formulas are valid:

$$L_q = \lambda W_q, \quad L_s = \lambda W_s.$$

10 M/G/1 queues

10.1 Definition and main properties

An M/G/1 queueing system is characterized by the following properties:

1. Customers arrive according to a Poisson process with the *arrival rate* $\lambda > 0$.
2. Service times are independent copies of a given random variable $S \not\equiv 0$ with cumulative distribution function $F_S(t) = \mathbb{P}(S \leq t)$.
3. There is a single server.

Example 10.1. *(Hyperexponential service time).* Assume that the share π_i of customers has service time $S_i \sim E(\lambda_i)$. Let S be the service time of a random customer. Then:

$$\begin{aligned}\mathbb{P}(S > t) &= \sum\nolimits_i \mathbb{P}(S_i > t)\mathbb{P}(\textit{type is } i) \\ &= \sum\nolimits_i e^{-\lambda_i t}\pi_i.\end{aligned}$$

Example 10.2. *(Deterministic service times).* Traffic at hub airports can be modeled as a queueing system with Poisson arrivals and deterministic service times, see [6]. Poisson arrivals and deterministic service times are also used to model traffic at network switches.

Example 10.3. *(Erlang service times).* Assume that the service routine consists of k phases and the durations of phases are independent and $E(\lambda)$ distributed. Then the service time is $Erlang(k, \lambda)$ distributed.

Let N_t be the number of customers in the system at time $t \geq 0$.

Exercise 10.4. Show that $\{N_t\}_{t\geq 0}$ is not a Markov chain.

Let X_n^A (resp. X_n^D) be the number of customers in the system observed by the n-th customer upon his arrival (resp. departure).

Theorem 10.5. *(Poisson Arrivals See Time Averages – PASTA, see [9]). Let $B \subseteq \{0, 1, \dots\}$. Put:*

$$V_B(t) = \frac{1}{t}\int_0^t 1_{\{N_s \in B\}} ds, \quad U_B(n) = \frac{1}{n}\sum_{k=1}^n 1_{\{X_k^A \in B\}}, \quad i \geq 0.$$

Then $V_B \to V_B^\infty$ a.s. for all B if and only if $U_B \to V_B^\infty$ a.s. for all B.

Proposition 10.6. *Consider an $M/G/1$ queue with arrival observations $\{X_n^A\}_{n=1}^\infty$ and departure observations $\{X_n^D\}_{n=1}^\infty$. Then*

$$\lim_{n\to+\infty} \mathbb{P}(X_n^A \leq k) = \lim_{n\to+\infty} \mathbb{P}(X_n^D \leq k), \quad k \geq 0.$$

Proof. Note that $X_n^D = k$ implies $X_{n+k+1}^A \leq k$, see Figure 3 (left). Also note that $X_{n+k+1}^A = k$ implies $X_n^D \leq k$, see Figure 3 (right). The claim follows. □

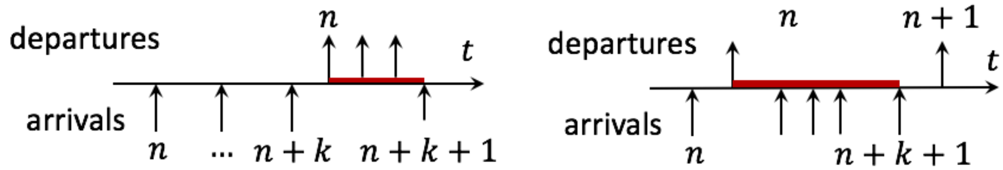


Figure 3: Arrivals and departures.

10.2 M/G/1 embedded chain

$\{X_n^D\}_{n=1}^{\infty}$ is called the *M/G/1 embedded chain*.

Lemma 10.7. *Let Y be the number of customers arriving during the service time S of some customer. Then $\mathbb{E}Y = \lambda \mathbb{E}S$,*

$$\mathbb{P}(Y = k) = p_k = \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} dF_S(t) \quad k \geq 0,$$
$$\mathbb{P}(Y > k) = a_k = \int_0^{\infty} \left(1 - F_S\left(\frac{t}{\lambda}\right)\right) \frac{t^k}{k!} e^{-t} dt.$$

Proof. Conditional on $S = t$ we have $Y \sim Pois(\lambda t)$. By the law of total probability:

$$\begin{aligned}\mathbb{P}(Y = k) &= \int_0^{\infty} \mathbb{P}(Y = k | S = t) dF_S(t) \\ &= \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} dF_S(t).\end{aligned}$$

By the law of total expectation:

$$\mathbb{E}Y = \int_0^{\infty} \mathbb{E}(Y | S = t) dF_S(t) = \int_0^{\infty} \lambda t \, dF_S(t) = \lambda \mathbb{E}S.$$

Let $T_{k+1} \sim Erlang(k+1, \lambda)$ be the arrival time of the $(k+1)$-st customer arriving during the service time. Then $Y > k$ if and only if $S > T_{k+1}$. By the law of total probability:

$$\begin{aligned}\mathbb{P}(Y > k) = \mathbb{P}(S > T_{k+1}) &= \int_0^{\infty} \mathbb{P}(S > t) \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} dt \\ &= \int_0^{\infty} \left(1 - F_S\left(\frac{t}{\lambda}\right)\right) \frac{t^k}{k!} e^{-t} dt.\end{aligned}$$

□

Proposition 10.8. *Let $\{X_n^D\}_{n=1}^{\infty}$ be the M/G/1 embedded chain. Then $\{X_n^D\}_{n=1}^{\infty}$ is a Markov chain with transition matrix*

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \\ 0 & 0 & p_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Let S_n be the service time of customer n. Let Y_n be the number of customers arriving during the service time of customer n. The claim follows from the formulas illustrated in Figure 4:

$$\mathbb{P}(X_{n+1}^D = j | X_n^D = 0) = \mathbb{P}(Y_{n+1} = j) = p_j, \quad j \geq 0,$$
$$\mathbb{P}(X_{n+1}^D = j | X_n = i) = \mathbb{P}(i - 1 + Y_{n+1} = j) = p_{j-i+1}, \quad i \geq 1,\ j \geq i - 1.$$

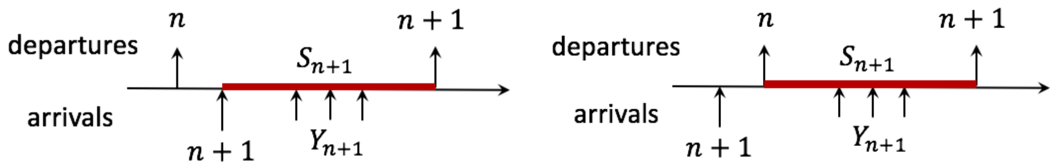


Figure 4: Customers arriving during the service time of customer $n + 1$.

□

Exercise 10.9. Show that $\{X_n^D\}_{n=1}^\infty$ is irreducible. *Hint: Show that* $p_i > 0$, $i \geq 0$.

Exercise 10.10. Compute the transition matrix for $\{X_n^D\}_{n=1}^\infty$ if $S = D$, $D = const$.

10.3 Phase-type distributions

Let $\{Z_t\}_{t\geq 0}$ be a continuous-time Markov chain with states $\{0, 1, \ldots, N\}$, where state 0 is absorbing, states $\{1, \ldots, N\}$ are transient and are called *phases*. Let $Z_0 \sim (\pi_0, \pi)$, $\pi = (\pi_1, \ldots, \pi_N)$, and let Q be the generator of $\{Z_t\}_{t\geq 0}$ so that:

$$Q = \begin{pmatrix} 0 & 0 \\ g & G \end{pmatrix}$$

where $g \in \mathbb{R}^N$ and $G \in \mathbb{R}^{N\times N}$. Put $S = \min\{t \geq 0 \colon Z_t = 0\}$. Then S has a *phase-type distribution* with parameters π and G, denoted $S \sim PH_N(\pi, G)$.

Example 10.11. *(Phase diagram).* Hyperexponential and Erlang distributions are examples of phase-type distributions. It is convenient to represent a phase-type distribution by its *phase diagram* as shown in Figure 5.

Phase-type distributions are widely used to model and fit service times in queueing theory and claim sizes (severities) in insurance models. One reason is that their Laplace transforms are rational functions.

Proposition 10.12. *Let $S \sim PH_N(\pi, G)$ and put $\pi_0 = 1 - \pi\mathbb{1}$. Then:*

$$\mathbb{P}(S > t) = \pi e^{Gt}\mathbb{1},$$

$$L_S(w) \stackrel{def}{=} \mathbb{E}e^{-wS} = \pi_0 - \pi(wI - G)^{-1}G\mathbb{1} \quad \textit{(Laplace transform).}$$

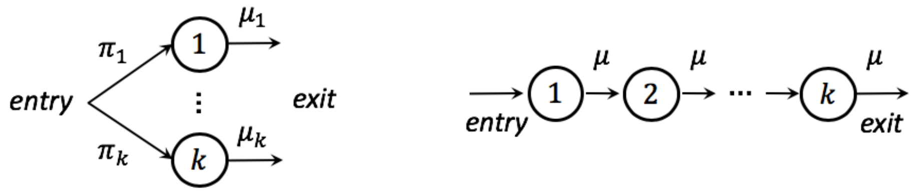


Figure 5: Phase diagram for the hyperexponential distribution (left) and for the Erlang distribution (right)

Proof. (1). Let $\{Z_t\}_{t\geq 0}$ be the associated Markov chain and put $p_i(t) = \mathbb{P}(Z_t = i)$, $p(t) = (p_1(t), \ldots, p_N(t))$ By the Kolmogorov forward equation:

$$\tfrac{d}{dt}\big(p_0(t), p(t)\big) = \big(p_0(t), p(t)\big)\begin{pmatrix} 0 & 0 \\ g & G \end{pmatrix} = \big(p(t)g, p(t)G\big).$$

Together with $p(0) = \pi$ this shows that $p(t) = \pi e^{Gt}$. It follows that:

$$\mathbb{P}(S > t) = \mathbb{P}(Z_t \neq 0) = p(t)\mathbb{1} = \pi e^{Gt}\mathbb{1}.$$

(2). The following transformations are valid:

$$\begin{aligned} \tfrac{d}{dt}p(t) = p(t)G \implies & \int_0^\infty e^{-wt}\tfrac{d}{dt}p(t)\,dt = \int_0^\infty e^{-wt}p(t)Gdt \\ \implies & -\pi + w\int_0^\infty e^{-wt}p(t)dt = \int_0^\infty e^{-wt}p(t)\,dtG \\ \implies & \int_0^\infty e^{-wt}p(t)\,dt = \pi(wI - G)^{-1}. \end{aligned}$$

To compute $L_S(w)$ note that:

$$\begin{aligned} L_S(w) = \int_0^\infty e^{-wt}d\mathbb{P}(S \leq t) &= \pi_0 - \int_0^\infty e^{-wt}\pi e^{Gt}G\mathbb{1}\,dt \\ &= \pi_0 - \int_0^\infty e^{-wt}p(t)dt\,G\mathbb{1} \\ &= \pi_0 - \pi(wI - G)^{-1}G\mathbb{1}. \end{aligned}$$

□

Remark 10.13. *Fitting of phase-type distributions to sample data is often done using the EM (expectation maximization algorithm), see [3]. It is also common to approximate a theoretical density, such as log-normal or Pareto density, by a phase type distribution to facilitate analysis.*

11 Equilibrium in M/G/1 queues

11.1 Maximum of a random walk

We will derive a condition for existence of equilibrium in stochastic dynamical systems of special kind given by Lindley's equation. We will use these results to find the equilibrium queue length and waiting time in an M/G/1 queueing system. This subsection is based on the article [5].

Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of independent copies of a random variable Z. A *random walk* with respect to $\{Z_n\}_{n=1}^{\infty}$ is a process $\{X_n\}_{n=0}^{\infty}$ defined by:

$$X_0 = 0, \quad X_n = Z_1 + \cdots + Z_n. \tag{14}$$

Put

$$M_n = \max\{X_0, \dots, X_n\}, \quad M = \max_{k \geq 0} X_k. \tag{15}$$

Lemma 11.1. *(Maximum of a random walk). Assume that $\mathbb{E}Z \in [-\infty, 0)$. Then $M < \infty$ almost surely and $M_n \to M$ in distribution.*

Proof. (*Finiteness of M*). By the strong law of large numbers:

$$\frac{X_n}{n} = \frac{Z_1 + \cdots + Z_n}{n} \to \mathbb{E}Z < 0 \quad \text{a.s..}$$

It follows that $X_n \to -\infty$ almost surely.

Exercise 11.2. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of random variables. Show that $\xi_n \to -\infty$ a.s. if and only if:

$$\lim_{N \to +\infty} \mathbb{P}(\xi_n < M \text{ for } n \geq N) = 1 \quad \forall M \in \mathbb{R}.$$

By the exercise for each $\varepsilon > 0$ there exist $N \geq 0$, $C \geq 0$ such that:

$$\mathbb{P}(X_n \leq 0 \text{ for } n > N) \geq 1 - \varepsilon,$$
$$\mathbb{P}(X_n \leq C \text{ for } 0 \leq n \leq N) \geq 1 - \varepsilon.$$

By the inclusion-exclusion principle:

$$\begin{aligned}
\mathbb{P}(M \leq C) &\geq \mathbb{P}(\{X_n \leq C \text{ for } n \leq 1\} \cap \{X_n \leq 0 \text{ for } n > N\}) \\
&\geq \mathbb{P}(X_n \leq C \text{ for } n \leq N) + \mathbb{P}(X_n \leq 0 \text{ for } n > N) - 1 \\
&\geq 1 - 2\varepsilon.
\end{aligned}$$

It follows that $\mathbb{P}(M < \infty) = 1$.

(*Convergence*). Fix $t \in \mathbb{R}$. By continuity of probability on non-decreasing sequences of events:

$$\begin{aligned}
\mathbb{P}(M_n \leq t) &= \mathbb{P}(X_k \leq t \text{ for } 0 \leq k \leq n) \\
&\to \mathbb{P}(X_k \leq t \text{ for } k \geq 0) = \mathbb{P}(M \leq t), \quad n \to \infty.
\end{aligned}$$

□

A process $\{W_n\}_{n=0}^{\infty}$ satisfies the *Lindley equation* with respect to $\{Z_n\}_{n=1}^{\infty}$ if:

$$W_{n+1} = \max\{W_n + Z_{n+1}, 0\}, \quad n \geq 0. \tag{16}$$

We will show that the solution to the Lindley equation is distributed as the maximum of a random walk.

Lemma 11.3. *Let $\{W_n\}_{n=0}^{\infty}$ satisfy (16) with $W_0 = 0$. Then W_n has the same distribution as M_n for all $n \geq 0$.*

Proof. Note that $W_1 = \max\{0, Z_1\}$,

$$\begin{aligned} W_2 &= \max\{0, \max\{0, Z_1\} + Z_2\} \\ &= \max\{0, \max\{Z_2, Z_1 + Z_2\}\} = \max\{0, Z_2, Z_1 + Z_2\}. \end{aligned}$$

By induction:

$$\begin{aligned} W_n &= \max\{0, Z_n, Z_{n-1} + Z_n, \ldots, Z_1 + \cdots + Z_n\} \\ &\overset{d}{=} \max\{0, Z_1, Z_1 + Z_2, \ldots, Z_1 + \cdots + Z_n\} = M_n. \end{aligned}$$

□

Proposition 11.4. *Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of iid copies of a random variable Z with $\mathbb{E}Z < 0$. Let W_n satisfy (16) with $W_0 = 0$. Then:*

$$W_n \xrightarrow{d} W, \quad W < \infty \ \text{a.s.}, \quad W \overset{d}{=} (W + Z)_+.$$

Proof. By Lemma 11.3 W_n has the same distribution as M_n, where M_n is defined by (15). By Lemma 11.1 $W_n \to W = M$ in distribution, $W < \infty$ a.s. Fix $t \geq 0$. The last claim follows from the transformations:

$$\begin{aligned} \mathbb{P}(W_{n+1} \leq t) &= \mathbb{P}(W_n + Z_n \leq t) \\ &= \int_{\mathbb{R}} P(M_n \leq t - z) d\mathbb{P}(Z \leq z) && \text{(LTP)} \\ &\to \int_{\mathbb{R}} \mathbb{P}(M \leq t - z) d\mathbb{P}(Z \leq z) && \text{(monotone conv.)} \\ &= \mathbb{P}(M + Z \leq t) = \mathbb{P}(\max\{W + Z, 0\} \leq t). \end{aligned}$$

□

11.2 Invariant distribution

Lemma 11.5. *(Cut property). Let $\{X_n\}_{n=1}^{\infty}$ be a Markov chain with state space I and transition matrix $P = (p_{ij})$. Then $\pi = (\pi_i) \in \Delta(I)$ is an invariant distribution if and only if for each I_1, $I_1 \subseteq I$ such that $I = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$, I_1, $I_2 \neq \varnothing$:*

$$F(I_1, I_2) = F(I_2, I_1)$$

where $F(A,B)$ denotes the probability flow from A to B in one time step:

$$F(I_1, I_2) = \sum_{i \in I_1, j \in I_2} \pi_i p_{ij}.$$

Proof. Note that:

$$\begin{aligned} F(\{j\}, I) &= \sum_{i \in I} \pi_j p_{ji} = \pi_j, \\ F(I, \{j\}) &= \sum_{i \in I} \pi_i p_{ij}. \end{aligned}$$

It follows that :

$$\pi \text{ is an invariant distribution if and only if } F(I, \{j\}) = \pi_j \text{ for all } j \in I. \tag{17}$$

Let I_1, I_2 be as in the formulation. Then:

$$\begin{aligned} F(I_1, I_2) - F(I_2, I_1) &= F(I, I_2) - F(I, I_2) \\ &= \sum_{j \in I_2} \big(F(I, \{j\}) - F(\{j\}, I)\big) \\ &= \sum_{j \in I_2} \big(F(I, \{j\}) - \pi_j\big). \end{aligned}$$

Together with (17) this proves the claim. □

Consider an M/G/1 queueing system with the arrival rate λ and service time S with cdf $F_S(t) = \mathbb{P}(S \leq t)$. Recall the notations:

$$p_k = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dF_S(t), \quad a_k = \int_0^\infty (1 - F_S(\tfrac{t}{\lambda})) \frac{t^k}{k!} e^{-t} dt.$$

Theorem 11.6. *Let $\{X_n^D\}_{n=1}^\infty$ be an M/G/1 embedded chain corresponding to the arrival rate λ and service time S. Assume that $\rho = \lambda \mathbb{E} S < 1$. Then $\{X_n^D\}_{n=1}^\infty$ admits a unique invariant distribution $\pi = (\pi_0, \pi_1, \dots)$ and $X_n^D \to X$ in distribution, where $X \sim \pi$. Besides, π is given by:*

$$\pi_i = \frac{1}{p_0} \left(a_{i-1} \pi_0 + \sum_{j=1}^{i-1} a_{i-j} \pi_j \right), \ i \geq 1.$$

Proof. (*Convergence*). Let Y_n be the number of customers arriving during the service time of customer n. Note that:

$$\begin{aligned} X_n^D = 0 &\implies X_{n+1}^D = Y_{n+1} \\ X_n^D \geq 1 &\implies X_{n+1}^D = X_n^D + Y_{n+1}^D - 1. \end{aligned}$$

This can be rewritten as:

$$X_{n+1}^D = (X_n^D - 1)_+ + Y_{n+1} = Q_n^D + Y_{n+1},$$

where $Q_n^D = (X_n^D - 1)_+$ be the queue size observed by customer n upon his departure. Then $\{Q_n^D\}_{n=1}^{\infty}$ satisfies the Lindley equation

$$Q_{n+1}^D = (Q_n^D + Y_{n+1} - 1)_+.$$

Let Y be the number of customers arriving during the service time of some customer. By assumption and Lemma 10.7 $\mathbb{E}(Y-1) = \lambda ES - 1 < 0$. By Proposition 11.4:

$$\begin{aligned} Q_n^D &\xrightarrow{d} Q, \quad Q < \infty \ a.s., \quad Q \stackrel{d}{=} (Q + Y - 1)_+, \\ \implies X_n^D &\xrightarrow{d} X, \quad X < \infty \ a.s., \quad X \stackrel{d}{=} Q + Y. \end{aligned}$$

Let $X_n^D \stackrel{d}{=} X$. Then:

$$X_{n+1}^D \stackrel{d}{=} (Q + Y - 1)_+ + Y \stackrel{d}{=} Q + Y \stackrel{d}{=} X.$$

It follows that $X \sim \pi$, where π is an invariant distribution for $\{X_n^D\}_{n=1}^{\infty}$.
(*Invariant distribution*). We use Lemma 11.5 for $I_1 = \{0\}, \{0,1\}, \dots$, see Figure 6.

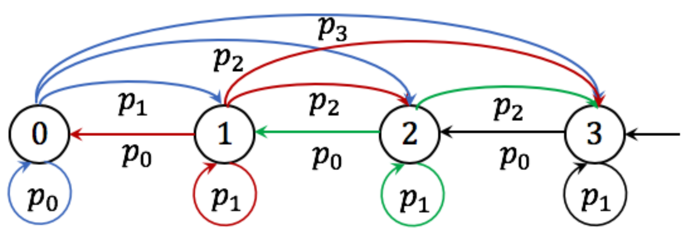


Figure 6: Transition diagram for an M/G/1 queueing system.

This gives the equations:

$$\begin{aligned} \pi_1 p_0 &= \pi_0(p_1 + p_2 + \cdots) \\ \pi_2 p_0 &= \pi_0(p_2 + p_3 + \cdots) + \pi_1(p_2 + p_3 + \cdots) \\ \pi_3 p_0 &= \pi_0(p_3 + p_4 + \cdots) + \pi_1(p_3 + p_4 + \cdots) + \pi_2(p_2 + p_3 + \cdots) \\ \cdots &= \dots \\ \pi_i p_0 &= \pi_0(p_i + p_{i+1} + \cdots) + \pi_1(p_i + p_{i+1} + \cdots) + \cdots + \pi_{i-1}(p_2 + p_3 + \cdots). \end{aligned}$$

Recalling that $a_i = p_{i+1} + p_{i+2} + \cdots$ we get the desired formulas for π_i, $i \geq 1$. $\square$

12 Pollaczek–Khinchine formulas

12.1 Equilibrium queue length

We consider an M/G/1 queueing system with the arrival rate λ and service time S with cumulative distribution function $F_S(t) = \mathbb{P}(S \le t)$. The quantity $\rho = \lambda \mathbb{E} S$ is called the *utilization rate*.

Let Y be the number of customers arriving during the service time S.

Lemma 12.1. *The following formula is valid:*

$$G_Y(z) = L_S(\xi), \quad \xi = \lambda(1-z),$$

where $G_Y(z) = \mathbb{E} z^Y$ is the probability generating function of Y and $L_S(\xi) = \mathbb{E} e^{-\xi S}$ is the Laplace transform of S.

Proof. Conditional on $S = t$ we have $Y \sim Pois(\lambda t)$. Then:

$$\mathbb{E}[z^Y | S = t] = \sum_{k=0}^{\infty} z^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = e^{\lambda(z-1)t}.$$

By the law of total expectation:

$$\mathbb{E} z^Y = \int_0^\infty \mathbb{E}(z^Y | S = t) dF_S(t) = \int_0^\infty e^{\lambda(z-1)t} dF_S(t) = \mathbb{E} e^{-\xi S}.$$

$\square$

Proposition 12.2. *Assume that $\rho < 1$. Let X be the equilibrium number of customers in the system and $\pi_0 = \mathbb{P}(X = 0)$. Then*

$$G_X(z) = \frac{\pi_0 \xi L_S(\xi)}{\lambda L_S(\xi) - \lambda + \xi}, \quad \xi = \lambda(1-z), \tag{18}$$

where $G_X(z) = \mathbb{E} z^X$. Besides, $\pi_0 = 1 - \rho$.

Proof. (*Probability generating function*). Recall that:

$$\begin{aligned} X &\overset{d}{=} (X-1)_+ + Y = X + Y - 1 + 1_{\{X=0\}} \\ z^X &\overset{d}{=} z^{-1} z^Y z^{X + 1_{\{X=0\}}}. \end{aligned}$$

It follows that

$$\begin{aligned} G_X(z) &= z^{-1} G_Y(z) \mathbb{E} z^{X + 1_{\{X=0\}}} \\ &= z^{-1} G_Y(z) \big(\pi_0 z + \sum_{k \ge 1} \pi_k z^k\big) \\ &= z^{-1} G_Y(z) \big(\pi_0 (z-1) + G_X(z)\big) \end{aligned}$$

Expressing $G_X(z)$ we get:

$$G_X(z) = \frac{\pi_0 G_Y(z)(z-1)}{z - G_Y(z)}.$$

Substituting $G_Y(z) = L_S(\xi)$, $z = 1 - \xi/\lambda$, we get the required formula.
(*Expression for* π_0). Note that:

$$\pi_0 = \frac{G_X(z)}{G_Y(z)} \frac{z - G_Y(z)}{z-1}.$$

The claim follows by the l'Hôpital rule:

$$\pi_0 = \frac{G_X(1)}{G_Y(1)} \lim_{z \to 1} \frac{z - G_Y(z)}{z-1} = 1 - G_Y'(1) = 1 - \rho.$$

□

Corollary 12.3. *(Expected queue length). Assume that* $\rho < 1$, $\mathbb{E}[S^2] < \infty$. *Let* X *be the equilibrium number of customers in the system. Then:*

$$\mathbb{E}X = \rho + \frac{\lambda^2 \mathbb{E}[S^2]}{2(1-\rho)}, \quad \mathbb{E}(X-1)_+ = \frac{\lambda^2 \mathbb{E}[S^2]}{2(1-\rho)}.$$

Proof. By Lemma 12.1:

$$G_Y(z) = \mathbb{E}e^{\lambda(z-1)S} = 1 + \lambda \mathbb{E}[S](z-1) + \tfrac{1}{2}\lambda^2 \mathbb{E}[S^2](z-1)^2 + \cdots$$

By Proposition 12.2:

$$\begin{aligned} G_X(z) &= \frac{(1-\rho)G_Y(z)(z-1)}{z - G_Y(z)} \\ &= \frac{(1-\rho)\big(1 + \rho(z-1) + \cdots\big)(z-1)}{z - 1 - \rho(z-1) - \frac{1}{2}\lambda^2 \mathbb{E}[S^2](z-1)^2 + \cdots} \\ &= (1-\rho)\frac{1 + \rho(z-1) + \cdots}{1 - \rho - \frac{1}{2}\lambda^2 \mathbb{E}[S^2](z-1) + \cdots} \\ &= 1 + \left[\rho + \frac{\lambda^2 \mathbb{E}[S^2]}{2(1-\rho)}\right](z-1) + \cdots \end{aligned}$$

This proves the first formula if we recall that $G_X'(1) = \mathbb{E}X$. For the second formula note that

$$\mathbb{E}(X-1)_+ = \mathbb{E}(X - 1 + 1_{X=0}) = \mathbb{E}X - 1 + \pi_0 = \mathbb{E}X - \rho.$$

□

12.2 Equilibrium waiting time

Proposition 12.4. *Let W_n be the waiting time of customer n. Assume that $\rho < 1$. Then:*

$$W_n \xrightarrow{d} W, \quad W < \infty \text{ a.s.}, \quad W \stackrel{d}{=} (W + S - T)_+ \text{ (Wiener-Hopf equation)},$$

where $T \sim E(\lambda)$. Besides:

$$L_W(\xi) = \frac{(1-\rho)\xi}{\lambda L_S(\xi) - \lambda + \xi}, \quad \xi = \lambda(1-z). \tag{19}$$

Proof. (*Convergence*). Let S_n be the service time of customer n. Let T_n be the time between the arrivals of customers $n-1$ and n. Note that:

$$W_{n+1} = (W_n + S_n - T_{n+1})_+, \quad n \geq 1,$$

see Figure 7.

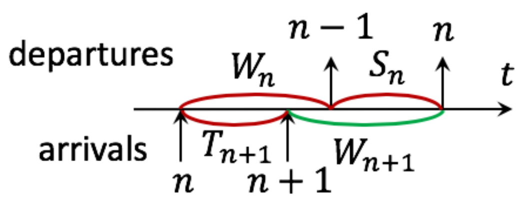


Figure 7: Waiting time of customer n

By assumption

$$\mathbb{E}S_n - \mathbb{E}T_{n+1} = \mathbb{E}S - \lambda < 0.$$

By Proposition 11.4 $W_n \to W$ in distribution, where $W < \infty$ a.s. and W has the same distribution as $(W + S - T)_+$.

(*Probability generating function*). Let $\{X_n^D\}_{n=0}^{\infty}$ be the M/G/1 embedded chain. Note that X_n^D is the number of customers who arrive during the time spent by customer n in the system $W_n + S_n$, see Figure 7. Conditional on $W_n + S_n = t$, $X_n^D \sim Pois(\lambda t)$. Thus:

$$\begin{aligned}
G_{X_n^D}(z) &= \int_0^\infty \mathbb{E}(z^{X_n^D} | W_n + S_n = t) dF_{W_n+S_n}(t) \\
&= \int_0^\infty e^{-\lambda t(1-z)} dF_{W_n+S_n}(t) \\
&= \mathbb{E}\left[e^{-\lambda(W_n+S_n)(1-z)}\right] = L_{W_n}(\xi) L_{S_n}(\xi),
\end{aligned}$$

Let $n \to +\infty$ and let $X \sim \pi$, where π is the equilibrium distribution for $\{X_n^D\}_{n=1}^{\infty}$. Then $G_X(z) = L_W(\xi) L_S(\xi)$. Substituting the expression for $G_X(z)$ from Proposition 12.2 we get the required expression for $L_W(\xi)$.

$\square$

Corollary 12.5. *(Expected waiting time). Assume that $\rho = \lambda ES < 1$, $\mathbb{E}[S^2] < \infty$. Let W be the equilibrium waiting time. Then*

$$\mathbb{E}W = \frac{\lambda \mathbb{E}S^2}{2(1-\rho)}.$$

Proof. Recall that:

$$L_S(\xi) = 1 - \mathbb{E}[S]\xi + \tfrac{1}{2}\mathbb{E}[S^2]\xi^2 + \cdots$$

The required formula follows from the transformations:

$$\begin{aligned}
L_W(\xi) &= \frac{(1-\rho)\xi}{\lambda L_S(\xi) - \lambda + \xi} \\
&= \frac{(1-\rho)\xi}{\lambda\left(1 - \mathbb{E}[S]\xi + \frac{1}{2}\mathbb{E}[S^2]\xi^2 + \cdots\right) - \lambda + \xi} \\
&= \frac{1-\rho}{1 - \rho + \frac{1}{2}\mathbb{E}[S^2]\lambda\xi + \cdots} = 1 - \frac{\mathbb{E}[S^2]\xi}{2(1-\rho)} + \cdots
\end{aligned}$$

□

Computation of the equilibrium waiting time distribution by formula (19) requires to invert the Laplace transform. This can be done analytically for rational functions of w by the partial fraction expansion. This holds in particular if the service time has a phase-type distribution.

Example 12.6. *(M/M/1).* Let $S \sim Exp(\mu)$. Then:

$$L_S(\xi) = \int_0^\infty e^{-\xi t}\lambda e^{-\lambda t} dt = \frac{\mu}{\xi + \mu}.$$

Using $\rho = \lambda/\mu$ and formula (19):

$$L_W(\xi) = \frac{(1-\rho)\xi}{\lambda\mu/(\xi+\mu) - \lambda + \xi} = 1 - \rho + \rho\frac{\mu(1-\rho)}{\xi + \mu(1-\rho)}.$$

It follows that:

$$\mathbb{P}(W \leq t) = 1 - \rho + \rho(1 - e^{-\mu(1-\rho)t}), \quad t \geq 0.$$

In particular, the waiting time is zero with probability $1-\rho$ and the waiting time is distributed according to $Exp(\mu(1-\rho))$ conditional on $W > 0$.

12.3 Ruin of an insurance company

Consider the following model of an insurance company:

1. Claims arrive according to a Poisson process $\{N_t\}_{t\geq 0}$ with arrival rate λ and arrival times $\{T_n\}_{n=1}^{\infty}$.
2. Claim *severities* $\{S_n\}_{n=1}^{\infty}$ are independent copies of a random variable S with cumulative distribution function $F_S(t) = \mathbb{P}(S \leq t)$.
3. Premiums are paid continuously with the rate of r dollars per year.

The cumulative net loss right after claim n is:

$$L_n = \sum_{k=1}^{n} S_k - rT_n = \sum_{k=1}^{n} (S_k - r\Delta T_k),$$

where $\Delta T_1 = T_1$, $\Delta T_n = T_n - T_{n-1}$ for $n \geq 2$. The maximum loss experienced by the company is:

$$M = \max\{0, L_1, L_2, \dots\}.$$

Proposition 12.7. *Assume that $r > \lambda \mathbb{E} S$. Then $M < \infty$ a.s. and $M \stackrel{d}{=} W$, where W is the equilibrium waiting time in an M/G/1 queue with arrival rate λ/r and service time S.*

Proof. By Lemma 11.3 M_n has the same distribution as W_n such that $W_0 = 0$ and

$$W_n = \left(W_{n-1} + S_k - rT_k\right)_+, \quad n \geq 1.$$

This equation coincides with Lindley's the equation for the waiting time an M/G/1 queue with arrival rate λ/r and with service time S. By Proposition 11.4 $W_n \to W$, where $W < \infty$ almost surely and W is the equilibrium waiting time in an M/G/1 queue with arrival rate λ/r and service time S. □

13 Basics of nonlinear optimization

13.1 Convex functions

Set $X \subseteq \mathbb{R}^n$ is called *convex* if for any $x, y \in X$ and $\alpha \in [0,1]$ it follows that $\alpha x + (1-\alpha)y \in C$. Let $X \subseteq \mathbb{R}^n$ be a convex set and let $f\colon X \to \mathbb{R}$. Function f is called *convex* (resp. *strictly convex*) if for all $x, y \in X$ and for all $\alpha \in (0,1)$:

$$f(\alpha x + (1-\alpha)y) \quad \leq \text{(resp. } <\text{)} \;\; \alpha f(x) + (1-\alpha) f(y).$$

f is called *strictly convex* if the inequality is strict for $\alpha \in (0,1)$.

Proposition 13.1. *Let $X \subseteq \mathbb{R}^n$ be a convex set, $f\colon X \to \mathbb{R}$ be a convex function and let x^* be a local (non-strict) minimum of f. Then x^* is a global minumum. In addition, if f is strictly convex, then x^* is the unique minimum*

Proof. (*Global minimum*). There exists an open neighborhood $U(x^*)$ of x^* such that $f(x^*) \leq f(x)$ for all $x \in U(x^*) \cap X$. Fix $y \in X$. Then there exists $t > 0$ such that $ty + (1-t)x^* \in X \cap U(x^*)$. By convexity:

$$f(x^*) \leq t f(y) + (1-t) f(x^*)$$

which is equibalent to $f(x^*) \leq f(y)$.
(*Uniqueness*). Assume that f is strictly convex and suppose that $x, y \in X$ are two global minima and $f(x) = f(y) = w$. By convexity of X we have $z = (x+y)/2 \in X$. By strict convexity of f:

$$f(z) < \tfrac{1}{2} f(x) + \tfrac{1}{2} f(y) = w.$$

This contradicts the optimality of x and y. $\square$

Let $f\colon X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is an open set. Assume that $f \in C^2(X)$, that is, f has continuous derivatives up to order two. The *Hessian* of f at $x \in X$ is a matrix $Hf(x) \in \mathbb{R}^{n \times n}$ defined by:

$$[Hf(x)]_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

Exercise 13.2. Let $g \in C^2(a,b)$. Show that g is convex if and only if $g''(t) \geq 0$ for all $t \in (a,b)$. Show that if $g''(t) > 0$ for all $t \in \mathbb{R}$, then g is strictly convex.

Proposition 13.3. *Let $X \subseteq \mathbb{R}^n$ be an open convex set and $f \in C^2(X)$. Then f is convex if and only if $Hf(x)$ is positive semi-definite for all $x \in X$. Besides, if $Hf(x)$ is positive definite for all $x \in X$, then f is strictly convex.*

Proof. Assume that f is convex. Let $x \in X$, $y \in \mathbb{R}^n \setminus \{0\}$ and put $g(t) = f(x + ty)$, where $t \in T$,

$$T = \{t \in \mathbb{R} \colon x + ty \in X\}.$$

Note that $g \in C^2(T)$ and g is convex. By Exercise 13.2 it satisfies $g''(t) \geq 0$ for all $t \in T$. Then:

$$g''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} y_i y_j = y^T (Hf(x)) y \geq 0.$$

It follows that $Hf(x)$ is positive semi-definite.

The proof of the remaining statements can be obtained by reversing the argumentation. □

Example 13.4. Let $\Sigma \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Put $f(x) = \frac{1}{2} x^T \Sigma x$, where $x \in \mathbb{R}^n$. Then $\nabla f(x) = \Sigma x$, $Hf(x) = \Sigma$ for all x. It follows that f is strictly convex.

13.2 Subgradient

Let $X \subseteq \mathbb{R}^n$ be a convex set and $f \colon X \to \mathbb{R}$ be a convex function. Vector $g \in \mathbb{R}^n$ is called *subgradient* of f at $x \in X$ if for all $y \in X$:

$$f(y) \geq f(x) + g^T (y - x).$$

The set of subgradients at point $x \in X$ is called the *subdifferential* of f at x and its denoted $\partial f(x)$.

Example 13.5. Assume that $\nabla f(x)$ exists at $x \in X$. Then $\partial f(x) = \{\nabla f(x)\}$.

Example 13.6. Assume that $f(x) = |x|$, $x \in \mathbb{R}$. Then $\partial f(0) = [-1, 1]$.

Proposition 13.7. *Let $X \subseteq \mathbb{R}^n$ be a convex set, $f \colon X \to \mathbb{R}$ be a function and $x^* \in X$. Then x^* is the global minimum of f if and only if $0 \in \partial f(x^*)$.*

Proof. Follows from the definition of subgradient. □

Let $f \colon (a, b) \to \mathbb{R}$ and $t \in (a, b)$. The *left derivative* (resp. *right derivative*) of f at t is defined by:

$$f'_-(t) = \lim_{\Delta t \to +0} \frac{f(t - \Delta t) - f(t)}{\Delta t} \qquad \left(\text{resp. } f'_+(t) = \lim_{\Delta \to +0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \right)$$

Exercise 13.8. Let $f \colon (a, b) \to \mathbb{R}$. Fix $t \in (a, b)$ and put

$$g(s, t) = \frac{f(s) - f(t)}{s - t}, \quad s \in (a, t) \cup (t, b).$$

Show that f is convex if and only if $g(s, t)$ is non-decreasing in s for each fixed $t \in (a, b)$.

Lemma 13.9. *Let $f\colon (a,b) \to \mathbb{R}$ be a convex function and $t \in (a,b)$. Then $f'_-(t)$ and $f'_+(t)$ exist and $f'_-(t) \le f'_+(t)$.*

Proof. By Exercise 13.8:

$$\frac{f(u)-f(t)}{u-t} \le \frac{f(v)-f(t)}{v-t} \quad u < t < v.$$

The left-hand side is a non-decreasing function of $u \in (a,t)$ and the right-hand-side is a non-decreasing function of $v \in (t,b)$. The claim follows by passing $u \to t$ and $v \to t$ and recalling that a bounded monotone function admits a limit. □

Proposition 13.10. *Let $f\colon (a,b) \to \mathbb{R}$ be a convex function and $t \in (a,b)$. Then $\partial f(t) = [f'_-(t), f'_+(t)]$.*

Proof. Note that $p \in \partial f(t)$ if and only if:

$$\begin{aligned} & f(s) \ge f(t) + p(s-t) \quad \forall s \in (a,b) \\ \iff \quad & \begin{cases} p \le \frac{f(s)-f(t)}{s-t} & s \in (t,b) \\ p \ge \frac{f(s)-f(t)}{s-t} & s \in (a,t) \end{cases} \end{aligned}$$

By Lemma 13.9 and Exercise 13.8 this is equivalent to $p \in [f'_-(t), f'_+(t)]$. □

Corollary 13.11. *Let $u\colon (a,b) \to \mathbb{R}$ be a non-decreasing function and $c \in (a,b)$. Put $f(t) = \int_c^t u(s)ds$. Then $f(t)$ is convex and $\partial f(t) = [u(t-0), u(t+0)]$.*

Proof. (*Convexity*). Let $a < t_1 < t_2 < b$ and put $t_\alpha = \alpha t_1 + (1-\alpha)t_2$ for some $\alpha \in (0,1)$. Then:

$$\begin{aligned} \alpha f(t_1) + (1-\alpha) f(t_2) - f(t_\alpha) &= \int_a^b \big(\alpha 1_{t \le t_1} + (1-\alpha) 1_{t \le t_2} - 1_{t \le t_\alpha}\big) u(t)dt \\ &= \int_a^b \big(-\alpha 1_{t_1 < t \le t_\alpha} + (1-\alpha) 1_{t_\alpha < t \le t_2}\big) u(t)dt \\ &\ge -\alpha u(t_\alpha)(t_\alpha - t_1) + (1-\alpha) u(t_\alpha)(t_2 - t_\alpha) = 0. \end{aligned}$$

(*Subdifferential*). Let $\Delta t > 0$. Note that:

$$u(t+0) \le \frac{1}{\Delta t}(f(t+\Delta t) - f(t)) = \frac{1}{\Delta t}\int_t^{t+\Delta t} u(s)ds \le u(t+\Delta t).$$

Passing $\Delta t \to +0$ we get $f'_+(t) = u(t+0)$. In a similar way one can show that $f'_-(t) = u(t-0)$. □

14 Portfolio theory

14.1 Constrained optimization

Let $X \subseteq \mathbb{R}^n$ be an open set, $f \in C^2(X, \mathbb{R}^n)$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Consider the problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b. \end{array} \tag{20}$$

Proposition 14.1. *(Lagrange multipliers). Let $x^* \in X$ be a local minimum in (20). Then $\nabla f(x^*) = A^T\lambda$ for some $\lambda \in \mathbb{R}^m$.*

Conversely, let $x^* \in X$ be a feasible point in (20) such that $\nabla f(x^*) = A^T\lambda$ for some $\lambda \in \mathbb{R}^m$. Assume that $Hf(x^*)$ exists and is positive definite. Then x^* is a local minimum in (20).

Proof. (*Necessary condition*). Let $y \in \ker A \setminus \{0\}$ and put $g(t) = f(x^* + ty)$. Then $g(t)$ is well defined for in a neighborhood of zero and $g(t)$ has a maximum at $t = 0$ and $g'(0) = y \cdot \nabla f(x^*) = 0$, that is:

$$\nabla f(x^*) \in (\ker A)^\perp \in \operatorname{im} A^T.$$

(*Sufficient condition*). Assume that $\nabla f(x^*) = 0$ and $Hf(x^*)$ is positive definite. Let $y \in \ker A \setminus \{0\}$ and put $g(t) = f(x^* + ty)$. Then $g(t)$ is well-defined in a neighborhood of zero and:

$$g'(0) = y \cdot \nabla f(x^*) = y \cdot A^T\lambda = Ay \cdot \lambda = 0.$$

Besides, $g''(0) = y^T[Hf(x^*)]y > 0$. It follows that $g(t)$ has a local minimum at $t = 0$. Since y is arbitrary, it follows that f has a local minimum at x^*. □

Example 14.2. Let $a \in \mathbb{R}^N \setminus \{0\}$ and let $\mathbf{V} \in \mathbb{R}^{N\times N}$, $\mathbf{V} \succ 0$. Consider the problem:

$$\begin{array}{ll} \text{minimize} & f(x) = x^T\mathbf{V}x \\ \text{subject to} & a^Tx = 1. \end{array}$$

Note that $\nabla f(x) = 2\mathbf{V}x$ and $Hf(x) = 2\mathbf{V} \succ 0$, $x \in \mathbb{R}^N$. We seek x^* such that:

$$2\mathbf{V}x^* = \lambda a \implies x^* = \tfrac{\lambda}{2}\mathbf{V}^{-1}a.$$

Substituting into the constraint we find:

$$\tfrac{\lambda}{2}a^T\mathbf{V}^{-1}a = 1 \implies \lambda = \frac{2}{a^T\mathbf{V}^{-1}a} \implies x^* = \frac{\mathbf{V}^{-1}a}{a^T\mathbf{V}^{-1}a}.$$

By Proposition 14.1 and strict convexity of f it is the global minimum.

14.2 Mean-variance optimization (proofs)

Proposition 14.3. *Assume that* $\mathbf{V} \succ 0$ *and* $\mu_{MR} > 0$*. Then:*

$$x_{Sh} = \frac{\mathbf{V}^{-1}\mu}{\mathbf{1}^T\mathbf{V}^{-1}\mu}.$$

Proof. x_{Sh} solves:

$$\begin{array}{ll} \text{maximize} & \frac{\mu^T x}{\sqrt{x^T\mathbf{V}x}} \\ \text{subject to} & \mathbf{1}^T x = 1. \end{array}$$

Since $\mu_{MR} = \mu^T\mathbf{V}^{-1}\mathbf{1} > 0$, the problem is equivalent to:

$$\begin{array}{ll} \text{maximize} & \frac{\mu^T x}{\sqrt{x^T\mathbf{V}x}} \\ \text{subject to} & \mathbf{1}^T x = 1, \\ & \mu^T x > 0. \end{array}$$

Put $z = x/(\mu^T x)$ so that $x = z/(\mathbf{1}^T z)$. Then the problem is equivalent to:

$$\begin{array}{ll} \text{maximize} & \frac{\mu^T z}{\sqrt{z^T\mathbf{V}z}} \\ \text{subject to} & \mathbf{1}^T z > 0, \\ & \mu^T z = 1. \end{array} \iff \begin{array}{ll} \text{minimize} & z^T\mathbf{V}z \\ \text{subject to} & \mathbf{1}^T z > 0, \\ & \mu^T z = 1. \end{array}$$

Drop the constraint $\mathbf{1}^T z > 0$. Then by Example 14.2 the optimal solution is:

$$z^* = \frac{\mathbf{V}^{-1}\mu}{\mu^T\mathbf{V}^{-1}\mu} \iff x^* = \frac{\mathbf{V}^{-1}\mu}{\mathbf{1}^T\mathbf{V}^{-1}\mu}.$$

Besides, z^* satisfies the dropped constraint since $\mathbf{1}^T z = \mu\mathbf{V}^{-1}\mathbf{1} > 0$. □

Proposition 14.4. *(Two mutual fund theorem). Assume that* $\mathbf{V} \succ 0$*,* $\mu_{MR} > 0$*, and* $x_{MR} \neq x_{Sh}$*. Let* x^* *be an efficient portfolio with expected return* E*. Then:*

$$x^* = (1-\lambda)x_{MR} + \lambda x_{Sh},$$
$$\lambda = \frac{E - \mu_{MR}}{\mu_{Sh} - \mu_{MR}}.$$

Proof. Portfolio x^* solves:

$$\begin{array}{ll} \text{minimize} & x^T\mathbf{V}x \\ \text{subject to} & Ax = b, \end{array}$$

where $A = [\mu, \mathbb{1}]^T \in \mathbb{R}^{2\times N}$, $b = [E, 1]^T \in \mathbb{R}^2$. By Proposition 14.1 we seek a solution to such that

$$2\mathbf{V}x^* = \lambda A^T\lambda \implies x^* = \tfrac{1}{2}\mathbf{V}^{-1}A^T\lambda.$$

Substituting into the constraints we find:

$$\tfrac{1}{2}A\mathbf{V}^{-1}A^T\lambda = b \implies \lambda = 2(A\mathbf{V}^{-1}A^T)^{-1}b.$$

It follows that:

$$x^* = \mathbf{V}^{-1}A^T(A\mathbf{V}^{-1}A^T)^{-1}b.$$

Note that x^* is an affine function of E. This proves the claim. $\square$

14.3 Capital asset pricing model (proofs)

Proposition 14.5. *The following formula is valid:*

$$\mu_i - r_f = \beta_i(\mu_M - r_f),$$
$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\sigma_M^2}.$$

Proof. Note that:

$$\begin{aligned} x_M &= \frac{\mathbf{V}^{-1}(\mu - r_f\mathbf{1})}{\mathbf{1}^T\mathbf{V}^{-1}(\mu - r_f\mathbf{1})} \\ \implies \mathbf{V}x_M &= \frac{\mu - r_f\mathbf{1}}{\mathbf{1}^T\mathbf{V}^{-1}(\mu - r_f\mathbf{1})} \\ \implies x_M^T\mathbf{V}x_M &= \frac{x_M^T(\mu - r_f\mathbf{1})}{\mathbf{1}^T\mathbf{V}^{-1}(\mu - r_f\mathbf{1})} = \frac{\mu_M - r_f}{\mathbf{1}^T\mathbf{V}^{-1}(\mu - r_f\mathbf{1})} \end{aligned}$$

It follows that:

$$\begin{aligned} \mu - r_f\mathbf{1} &= [\mathbf{1}^T\mathbf{V}^{-1}(\mu - r_f\mathbf{1})]\mathbf{V}x_M \\ &= \frac{\mathbf{V}x_M}{x_M^T\mathbf{V}x_M}(\mu_M - r_f) = \boldsymbol{\beta}(\mu_M - r_f), \end{aligned}$$

where $\boldsymbol{\beta} = \mathbf{V}x_M/(x_M^T\mathbf{V}x_M)$. The claim follows by observing that $[\mathbf{V}x_M]_i = \mathrm{Cov}(R_i, R_M)$ and $x_M^2\mathbf{V}x_M = \sigma_M^2$. $\square$

15 Dynamic portfolio management

15.1 Bet sizing problem

You repeatedly toss a biased coin making bets on heads. Each time the coin shows tails you lose your wager. Each time the coin shows heads you get your wager back and win d dollars for each dollar bet[1]. How much should you bet?

Let $\{\xi(t)\}_{t=1}^{\infty}$ be iid random variables such that

$$\xi(t) = \begin{cases} d & \text{with probability } p > 0.5, \\ -1 & \text{with probability } q = 1 - p. \end{cases}$$

Let $x(t) \in [0,1]$ be the fraction of your current wealth that you wager at round $t \geq 1$. Let your initial wealth be \$1 and let $W_x(t)$ be your wealth after round $t \geq 1$ when using strategy $\{x(t)\}_{t=1}^{\infty}$.

Then:

$$\begin{aligned} W_x(T) &= W_x(t-1)(1 - x(t)) + W_x(t-1)x(t)(1 + \xi(t)) \\ &= \prod_{t=1}^{T} \big[1 + \xi(t)x(t)\big]. \end{aligned}$$

Strategy $\{x(t)\}_{t=1}^{\infty}$ is called *causal* if for each $T \geq 1$, $x(T)$ is independent of $\{\xi(t)\}_{t=T}^{\infty}$. We want to find a causal strategy outperforming other causal strategies. Which objective should we optimize?

Example 15.1. *(Expected wealth maximization)*. Fix $T \geq 1$. Consider the problem:

$$\begin{array}{rl} \text{maximize} & \mathbb{E}W_x(T) \\ \text{subject to} & \{x(t)\}_{k=1}^{\infty} \text{ is causal,} \\ & x(t) \in [0,1],\ 1 \leq t \leq T, \end{array}$$

Let $A = \big\{\xi(1) = a_1, \ldots, \xi(T-1) = a_{T-1}\big\}$. Then:

$$\begin{aligned} \mathbb{E}[W_x(T)|A] &= W_x(T-1)\mathbb{E}[1 + \xi(T)x(T)|A] \\ &= W_x(T-1)\big(1 + \mathbb{E}\xi(T)\mathbb{E}[x(T)|A]\big) \\ &\leq W_x(T-1)\big(1 + \mathbb{E}\xi(T)\big). \end{aligned}$$

It follows that:

$$\mathbb{E}W_x(T) \leq \mathbb{E}W_x(T-1)\big(1 + \mathbb{E}\xi(T)\big) \leq \prod_{t=1}^{T}\big(1 + \mathbb{E}\xi(t)\big) = [p(d+1)]^T.$$

Besides, the equality is attained if $x(t) = 1$, $t \geq 1$. However, if we bet all our wealth at each round we will be ruined almost surely as $T \to +\infty$:

$$\mathbb{P}(W_x(T) > 0) = p^T \to 0, \quad n \to +\infty.$$

1. This is referred to as "d:1 odds"

15.2 Fixed fraction betting

Consider a strategy $\{x(t)\}_{t=1}^{\infty}$ such that $x(t) = f \in [0,1]$ for each $t \geq 1$. Put

$$S(T) = \sum_{t=1}^{T} 1_{\{\xi(t)>0\}} \quad \text{(number of successes in } T \text{ trials).}$$

Then:

$$\begin{aligned} W_x(T) &= (1+df)^{S(T)}(1-f)^{1-S(T)} = \exp(Tg_f(T)), \\ g(f,T) &= \frac{S(T)}{T}\log(1+df) + \frac{1-S(T)}{T}\log(1-f) \end{aligned}$$

By the strong law of large numbers:

$$\frac{S(T)}{T} \to p \ a.s., \quad \frac{T-S(T)}{T} \to q \ a.s. \quad as\ T \to +\infty.$$

It follows that $g(f,T) \to g(f)$ a.s., where:

$$g(f) = p\log(1+df) + q\log(1-f).$$

Example 15.2. $g(f) = 0.05$ corresponds to $W_x(T) \approx (1+0.05)W_x(T-1)$ for large T. That is, the fortune grows at the rate of 5% per round.

Consider the following problem:

$$\begin{array}{rl} \text{maximize} & g(f) \\ \text{subject to} & f \in [0,1]. \end{array} \tag{21}$$

Proposition 15.3. *(Kelly criterion). Problem (21) admits a unique solution f^* given by*

$$f^* = \frac{pd-q}{d} = \frac{edge}{odds} \quad \textit{(Kelly fraction).}$$

Besides, $g(f^*) > 0$.

Proof. Note that:

$$g'(f) = \frac{pd(1-f) - q(1+fd)}{(1+fd)(1-f)} = \frac{pd-q-fd}{(1+fd)(1-f)}.$$

One can see that:

$$\begin{aligned} g'(f) = 0 &\iff f = f^* \\ g'(f) > 0 &\iff f < f^* \\ g'(f) < 0 &\iff f > f^*. \end{aligned}$$

It follows that $f = f^*$ is a global maximum of $g(f)$ and $g'(f^*) > g(0) = 0$. $\square$

Corollary 15.4. *Let $\{x(t)\}_{t=1}^{\infty}$ is given by $x(t) = f^*$, $t \geq 1$, where f^* is the Kelly fraction. Then $W_x(T) \to \infty$ a.s. as $T \to +\infty$.*

Proof.

$$\begin{aligned}\mathbb{P}\big(\lim_T W_x(T) = +\infty\big) &= \mathbb{P}(\lim_T e^{Tg(f^*,T)} = \infty)\\ &\geq \mathbb{P}\big(\lim_T g(f^*,T) = g(f^*)\big) = 1.\end{aligned}$$

□

Example 15.5. You are at a blackjack table and your current wealth is \$1000. You use a basic strategy for which the winning probability is 50.05% (see [8] for the strategy). How much should you wager? In this game $d = 1$ (you win one dollar on the top of each dollar wagered) and

$$f^* = p - q = 0.5005 - 0.4995 = 0.001.$$

That is, \$1 should be wagered.

15.3 Optimal growth portfolio

We now consider a more general situation. Consider a universe with N assets. Let $R_k(t) > -1$ be the return on asset k in period $t \geq 1$. Put $\mathbf{R}(t) = (R_1(t), \ldots, R_N(t))$.

Let $x_k(t) \geq 0$ be the amount invested at asset k in period t per \$1 of own capital. The vector $x(t) = (x_1(t), \ldots, x_N(t))$ is called *portfolio* for period t. Portfolio $x(t)$ is called *fully invested* if $\mathbf{1}^T x(t) = 1$. Investment strategy $\{x(t)\}_{t=1}^{\infty}$ is called *causal* if $x(T)$ is independent of $\mathbf{R}(T)$, $\mathbf{R}(T+1)$, ... for each fixed $T \geq 1$.

Let the initial wealth be \$1 and let $W_x(t)$ be the wealth at the end of period k corresponding to investment strategy $\{x(t)\}_{t=1}^{\infty}$. Then:

$$\begin{aligned}W_x(t) &= W(t-1)\big(1 + \mathbf{R}(t)^T x(t)\big),\\ W_x(T) &= \textstyle\prod_{t=1}^{T}\big(1 + \mathbf{R}(t)^T x(t)\big), \quad T \geq 1.\end{aligned}$$

The *average growth rate* $g_x(T)$ for the periods $\{1, \ldots, T\}$ is defined as

$$W_x(T) = e^{Tg_x(T)} \iff g_x(T) = \tfrac{1}{T}\log W_x(T).$$

Consider the growth rate maximization problem:

$$\begin{array}{ll}\text{maximize} & \mathbb{E}\log W(T)\\ \text{subject to} & \{x(t)\}_{t=1}^{T} \text{ is causal,}\\ & \mathbf{1}^T x(t) = 1,\ x(t) \geq 0,\ 1 \leq t \leq T.\end{array} \tag{22}$$

Note that $\{x(t)\}_{t=1}^{\infty}$ is a feasible strategy in (22) if and only if it is causal and $x(t)$ is fully invested for each $t \geq 1$.

Proposition 15.6. *(Myopic policy).* $\{x(t)\}_{t=1}^{T}$ *is an optimal solution to* (22) *if and only for each* $t \geq 1$*,* $x(t)$ *is an optimal solution to:*

$$\begin{array}{ll} \textit{maximize} & \mathbb{E}\log\big(1+\mathbf{R}(t)^T v\big) \\ \textit{subject to} & \mathbf{1}^T v = 1,\ v \geq 0. \end{array}$$

Proof. The claim follows from the formula:

$$\mathbb{E}\log W_x(t) = \textstyle\sum_{t=1}^{T} \mathbb{E}\log\big(1+\mathbf{R}(t)^T x(t)\big).$$

$\square$

Remark 15.7. *Assume that* $\{\mathbf{R}(t)\}_{t=1}^{\infty}$ *are identically distributed. Then maximizing the growth rate* $g_x(T)$ *corresponds to using a fixed fraction strategy independent of* T*.*

We will show that on average the wealth provided by the optimal growth rate strategy exceeds the wealth corresponding to any other causal strategy.

Proposition 15.8. *Let* $\{x^*(t)\}_{t=1}^{\infty}$ *be a feasible strategy in* (22)*. Then* $\{x^*(t)\}_{t=1}^{\infty}$ *is an optimal solution to* (22) *if and only if*

$$\mathbb{E}\left[\frac{W_x(t)}{W_{x^*}(t)}\right] \leq 1, \quad t = 1,\ldots,T, \tag{23}$$

for each strategy $\{x(t)\}_{t=1}^{\infty}$ *which is feasible in* (22)*.*

Proof. ($\Longleftarrow$) Let $\{x^*(t)\}_{t=1}^{\infty}$ satisfy (23) for each feasible $\{x(t)\}_{k=1}^{\infty}$. By Jensen's inequality:

$$\mathbb{E}\log W_x(t) - \mathbb{E}\log W_{x^*}(t) = \mathbb{E}\log\left(\frac{W_x(t)}{W_{x^*}(t)}\right) \leq \log\mathbb{E}\left(\frac{W_x(t)}{W_{x^*}(t)}\right) \leq 0.$$

It follows that $\{x^*(t)\}_{k=1}^{\infty}$ is an optimal solution to (22).
($\Longrightarrow$) Assume that $\{x^*(t)\}_{t=1}^{\infty}$ is an optimal solution to (22). Let $\varepsilon > 0$, $v \in \mathbb{R}^n$, $\mathbb{1}^T v = 1$, and put $v^{\varepsilon} = x^*(t) + \varepsilon(v - x^*(t))$. By Proposition 15.6:

$$\begin{aligned} 0 &\geq \mathbb{E}\log\big(1+\mathbf{R}(t)^T v^{\varepsilon}\big) - \mathbb{E}\log\big(1+\mathbf{R}(t)^T x^*(t)\big) \\ &= \mathbb{E}\log\left[\frac{1+\mathbf{R}(t)^T v^{\varepsilon}}{1+\mathbf{R}(t)^T x^*(t)}\right] = \mathbb{E}\log\left[1+\varepsilon\left(\frac{1+\mathbf{R}(t)^T v}{1+\mathbf{R}(t)^T x^*(t)} - 1\right)\right]. \end{aligned}$$

Divide by ε and let $\varepsilon \to 0$. By dominated convergence:

$$0 \geq \lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\mathbb{E}\log\left[1+\varepsilon\left(\frac{1+\mathbf{R}(t)^T v}{1+\mathbf{R}(t)^T x^*(t)} - 1\right)\right] = \mathbb{E}\left[\frac{1+\mathbf{R}(t)^T v}{1+\mathbf{R}(t)^T x^*(t)} - 1\right]$$
$$\Longrightarrow\ \mathbb{E}\left[\frac{1+\mathbf{R}(t)^T v}{1+\mathbf{R}(t)^T x^*(t)}\right] \leq 1.$$

Let $A = \{\mathbf{R}_1 = a_1, \dots, \mathbf{R}_{t-1} = a_{t-1}\}$. It follows that:

$$\mathbb{E}\left[\frac{\mathbf{R}(t)^T x(t)}{\mathbf{R}(t)^T x^*(t)} \,\middle|\, A \right] \leq 1,$$

where $\{x(t)\}_{t=T}^{\infty}$ is any feasible strategy in(22). Using this inequality we get:

$$\mathbb{E}\left[\frac{W_x(t)}{W_{x^*}(t)} \,\middle|\, A \right] = \frac{W_x(t-1)}{W_{x^*}(t-1)} \mathbb{E}\left[\frac{1 + \mathbf{R}(t)^T x(t)}{1 + \mathbf{R}(t)^T x^*(t)} \,\middle|\, A \right] \leq \frac{W_x(t-1)}{W_{x^*}(t-1)}.$$

By the law of total expectation and induction:

$$\mathbb{E}\left[\frac{W_x(t)}{W_{x^*}(t)} \right] \leq \mathbb{E}\left[\frac{W_x(t-1)}{W_{x^*}(t-1)} \right] \leq \cdots \leq 1.$$

$\square$

16 Deterministic inventory management

16.1 Economic Order Quantity

Consider the following model:

1. A resource is consumed continuously at rate $\lambda > 0$.
2. The resource is ordered in quantity q when the warehouse is empty and is immediately supplied. Ordering q units of good costs $K + cq$ dollars. *Example: c can include cost of labour and materials, K can include paperwork and transportation costs.*
3. Holding cost is h dollars per unit time per unit of good. *Examples: warehouse rent, insurance, heat/electricity/security costs.*

We are interested in finding an order quantity q minimizing the operational costs. Inventory level at time t is given by $x(t) = q - \lambda t$, $t \in [0, T)$, where $T = q/\lambda$ is the cycle length, see Figure 8 (left). The total cost per unit time in each cycle is:

$$\begin{aligned} C(q) &= \frac{1}{T}\big(K + cq + h\int_0^T (q - \lambda t)dt\big) \\ &= \tfrac{1}{T}(K + cq) + hq - \tfrac{1}{2}h\lambda T = \tfrac{\lambda K}{q} + \lambda c + \tfrac{1}{2}hq. \end{aligned}$$

Proposition 16.1. *(Economic order quantity). The following formula is valid:*

$$\min_{q>0} C(q) = C(q^*), \quad q^* = \sqrt{2K\lambda/h}.$$

Proof. Note that

$$C'(q) = -\tfrac{\lambda K}{q^2} + \tfrac{1}{2}h, \quad c''(q) = \tfrac{2\lambda K}{q^3} > 0.$$

First order condition $C'(q) = 0$ gives $q = \sqrt{2K\lambda/h}$. Since $C(q)$ is convex, this is the global minimum. □

Remark 16.2. *Assume that the lead time (order delivery time) is $L \in (0, T)$. Then the order must be placed when the inventory level is $R = \lambda L$, see Figure 8 (left).*

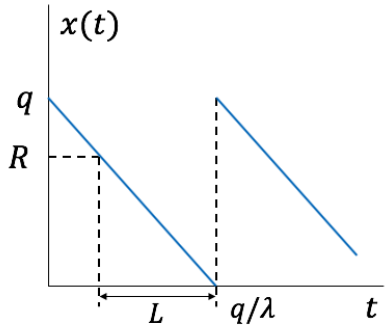
Now we assume that the lead time is $L \in (0, T)$ and that the demands are random. The *lead time demand* is the demand during the replenishment cycle, see Figure 8 (right). The *service level* is the probability of not stocking out:

$$SL = \mathbb{P}(LTD \le R),$$

where $R = \lambda L$ is the amount of stock at the beginning of the replenishment cycle.

Proposition 16.3. *Assume that $LTD \sim N(\lambda L, \sigma^2)$ and let $\alpha \in (0,1)$. Then $SL = 1 - \alpha$ if and only if $R = \lambda L + \sigma z_\alpha$, where*

$$z_\alpha = \Phi^{-1}(1 - \alpha), \quad \Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{s^2}{2}} ds.$$



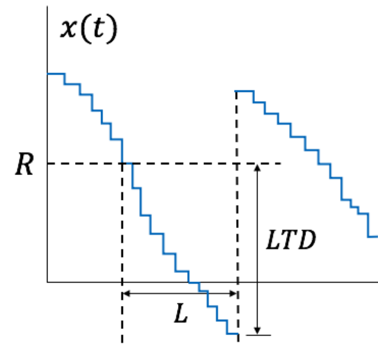


Figure 8: Stock level for positive lead time: deterministic demands (left) and random demands (right).

Proof. Note that $LTD = \lambda L + \sigma\xi$, where $\xi \sim N(0,1)$. Then:

$$SL = \mathbb{P}(LTD \leq R) = \mathbb{P}(\lambda L + \sigma\xi \leq R) = \mathbb{P}\big(\xi \leq (R - \lambda L)/\sigma\big).$$

It follows that:

$$SL = 1 - \alpha \iff \frac{R - \lambda L}{\sigma} = z_\alpha \iff R = \lambda L + \sigma z_\alpha.$$

$\square$

The extra stock $z_\alpha \sigma$ required to compensate for randomness of demands is called the *safety stock*.

16.2 Uncapacitated lot sizing

Consider the *uncapacitated lot sizing* model:

1. The resource is consumed in amount d_t at time $t \in \{1, \ldots, T\}$.
2. The resource is ordered in quantity q_t at time $t \in \{1, \ldots, T\}$ and is immediately supplied. Ordering q_t units at time t costs $K_t + c_t q_t$.
3. Holding cost is h_t dollars per unit of good for the period $[t, t+1]$

Let s_t be the amount of good left in the warehouse right after time t. The flow of goods in represented in Figure 9. The objective is to minimize the costs:

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T}(K_t y_t + c_t q_t + h_t s_t) \\ \text{subject to} & 0 \leq q_t \leq M y_t, \quad 1 \leq t \leq T, \\ & q_1 = d_1 + s_1, \\ & s_t + q_{t+1} = d_{t+1} + s_{t+1}, \quad t = 1, \ldots, T-1 \\ & s_{T-1} + q_T = d_T, \\ & y_t \in \{0,1\}, \quad 1 \leq t \leq T, \\ & s_t \geq 0, \quad 1 \leq t \leq T. \end{array} \tag{24}$$

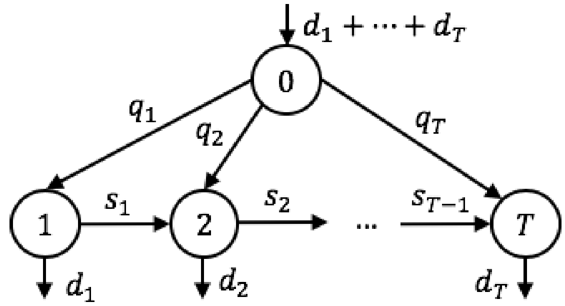
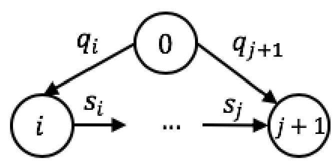


Figure 9: Product flow in the uncapacitated lot sizing model

where $M > d_1 + \cdots + d_T$ is an arbitrary constant. Variables $\{s_t\}_{t=0}^T$ and $\{y_t\}_{t=1}^T$ are auxiliary and are uniquely determined by $\{q_t\}_{t=1}^T$. Note that the feasible set is compact and the objective function is continuous which shows existence of an optimal solution.

Lemma 16.4. *Assume that $K_t > 0$ for all $t \in \{1, \ldots, T\}$. Let $\{q_t\}_{t=1}^T$, $\{y_t\}_{t=1}^T$, $\{s_t\}_{t=1}^{T-t}$ be an optimal policy. Then $s_t q_{t+1} = 0$ for all $t \in \{1, \ldots, T-1\}$.*

Proof. Assume that $s_j q_{j+1} > 0$ for some $j \in \{1, \ldots, T-1\}$. Then there exists $i \leq j$ such that $q_i > 0$, $q_{j+1} > 0$ and $s_t > 0$ for $t \in \{i, \ldots, j\}$:



Fix $u \in \mathbb{R}$ and consider a policy $\{q_t(u)\}_{t=1}^T$, $\{y_t\}_{t=1}^T$, $\{s_t(u)\}_{t=1}^{T-1}$ where:

$$\begin{aligned} q_t(u) &= q_t, \quad t \notin \{i, j+1\} \\ q_i(u) &= q_i + u, \quad q_{j+1}(u) = q_{j+1} - u, \end{aligned}$$

and:

$$\begin{aligned} s_t(u) &= s_t, \quad t \notin \{i, \ldots, j\}, \\ s_t(u) &= s_t + u, \quad t \in \{1, \ldots, j\}. \end{aligned}$$

Let $C(u)$ be the corresponding objective value. Note that the policy is feasible for

$$\begin{gathered} u_{\min} \leq u \leq u_{\max}, \\ u_{\min} = \max\{-q_i, -s_i, \ldots, -s_j\} < 0, \\ u_{\max} = q_{j+1} > 0. \end{gathered}$$

Besides, the $C(u)$ depends on u linearly. Since $C(0)$ is a global minimum by assumption, $C(u) = C(0)$ for $u \in [u_{\min}, u_{\max}]$. Put:

$$\widetilde{y}_t = \begin{cases} y_t, & t \neq j+1, \\ 0, & t = j+1. \end{cases}$$

Then the policy $\{q_t(u_{\max})\}_{t=1}^T$, $\{s_t(u_{\max})\}_{t=1}^T$, $\{\widetilde{y}_t\}_{t=1}^T$ is feasible and the corresponding objective value is $C(0) - K_{j+1} < C(0)$. Contradiction. $\square$

Using Lemma 16.4 we get:

Theorem 16.5. *Assume that $K_t > 0$ for $t \in \{1, \dots, T\}$. Let $\{q_t\}_{t=1}^T$ be an optimal ordering policy. Then there exist $1 = t_1 < \cdots < t_n = T+1$ such that:*

$$q_t = \begin{cases} d_t + \cdots + d_{s-1}, & t = t_k,\ s = t_{k+1}, \\ 0, & t \neq t_k,\ k = 1, \dots, n. \end{cases}$$

16.3 Example

Consider a model with $T = 4$ and with the following parameters:

t	1	2	3	4
d_t	1	3	5	2
$K_t \equiv K$	8	8	8	8
$c_t \equiv c$	1	1	1	1
$h_t \equiv h$	2	2	2	$-$

It is convenient to represent different ordering policies by paths from node 1 to node 5 in the graph in Figure 10. Path $(t_1 = 1, \dots, t_n = 5)$ corresponds to

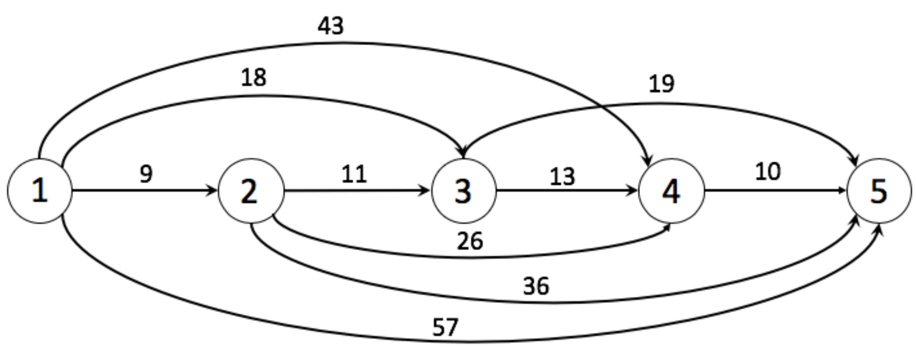


Figure 10: Costs for different policies $C(t, s)$.

ordering points $t_1, \dots, t_{n-1}$.

Let $C(i,j)$ be the cost of satisfying the demands d_t for $i \le t < j$ by ordering only at time $t = i$:

$$C(i,j) = K + c(d_i + d_{i+1} + \cdots + d_{i+k}) + h(d_{i+1} + 2d_{i+2} + \cdots + kd_{i+k}),$$

where $i + k = j - 1$. For example:

$$\begin{aligned}
C(1,2) &= 8 + 1 \times 1 = 9 \\
C(2,3) &= 8 + 1 \times 3 = 11 \\
C(1,3) &= 8 + 1 \times (1 + 3) + 2 \times 3 = 18.
\end{aligned}$$

We assign weight $C(i,j)$ to arc (i,j) in the graph Figure 10. By Theorem 16.5 the optimal ordering policy corresponds to the shortest path from node 1 to node 5 in this graph. Let $V(t)$ be the minimal path length from node t to node 5. Note that:

$$\begin{aligned}
&V(t) = \min\{C(t,s) + V(s) \colon s = t+1, \ldots, T+1\}, \qquad \text{(Bellman equation)} \\
&V(T+1) = 0.
\end{aligned}$$

Define $s(t)$ by:

$$V(t) = C(t, s(t)) + V(s(t)), \quad t = 1, \ldots, T.$$

Note that $s(t)$ gives the next optimal ordering point for the problem starting at time t. We will compute $V(t)$ and $s(t)$ backwards in time:

$$\begin{aligned}
V(4) &= C(4,5) = 10, \quad s(4) = 5, \\
V(3) &= \min\{C(3,5), C(3,4) + V(4)\} \\
&= \min\{19, 23\} = 19, \quad s(3) = 5, \\
V(2) &= \min\{C(2,5), C(2,4) + V(4), C(2,3) + V(3)\} \\
&= \min\{36, 36, 30\} = 30, \quad s(2) = 3, \\
V(1) &= \min\{C(1,5), C(1,4) + V(4), C(1,3) + V(3), C(1,2) + V(2)\} \\
&= \min\{57, 53, 37, 39\} = 37, \quad s(1) = 3.
\end{aligned}$$

Optimal ordering points are 1 and $s(1) = 3$, the optimal objective is $V(1) = 37$. The optimal ordering policy is:

t	1	2	3	4
q_t	4	0	7	0

17 Newsvendor problem

We recall three facts from the lecture on basics of non-linear optimization.:

(Fact 1) Let $f\colon (a,b) \to \mathbb{R}$ be a convex function. The subdifferential of f at x is given by $\partial f(x) = [f'_-(x), f'_+(x)]$.

(Fact 2) Point x^* is a global minimum of f if and only if $0 \in \partial f(x^*)$.

(Fact 3) Let $F\colon (a,b) \to \mathbb{R}$ be a non-decreasing function and let $c \in (a,b)$. Put:

$$h(x) = \int_c^x F(t)dt.$$

Then h is convex and $h'_\pm(x) = F(x \pm 0)$ for all $x \in (a,b)$.

Also recall that function f is *concave* if and only if $-f$ is convex.

17.1 Classical newsvendor problem

Consider the following model:

1. Demand D for newspapers is random with cdf $F_D(t) = \mathbb{P}(D \le t)$.
2. A newsboy orders x newspapers in the morning paying c dollars per paper. He sells $\min\{D, x\}$ newspapers during the day earning p dollars per paper.
3. Papers left over at the end of the day are thrown away.

The newsboy wants to maximize his expected profits:

$$\begin{array}{rl} \text{maximize} & \pi(x) = p\mathbb{E}\min\{D,x\} - cx \\ \text{subject to} & x > 0. \end{array} \tag{25}$$

Proposition 17.1. *$x^* > 0$ is an optimal solution to (25) if and only if:*

$$F_D(x^* - 0) \le \frac{p-c}{p} \le F_D(x^*) \quad \textit{(see Figure 11).}$$

In particular, if F_D is continuous and invertible, this is equivalent to:

$$x^* = F_D^{-1}\left(\frac{p-c}{p}\right).$$

Proof. We will need the following exercise:

Exercise 17.2. Let $X \ge 0$ be a continuous random variable. Then

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt.$$

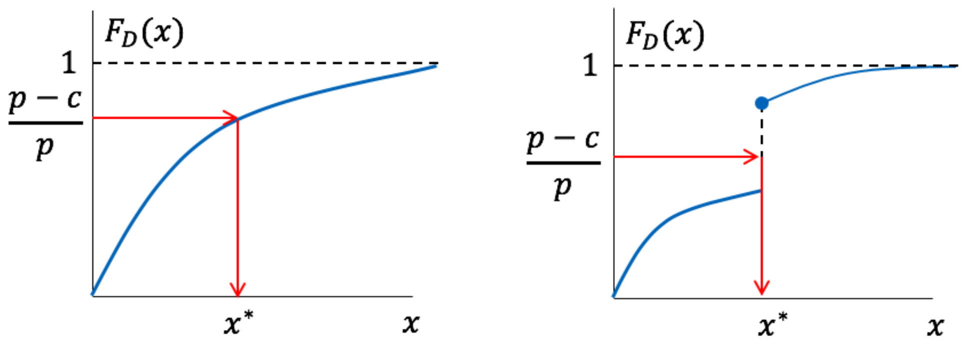


Figure 11: Optimal quantity for a continuous demand cdf (left) and for a discontinuous demand cdf (right).

Using Exercise 17.2 we get:

$$\begin{aligned}\pi(x) &= p\int_0^\infty \mathbb{P}(\min\{D,x\} > t)\,dt - cx\\ &= p\int_0^x \mathbb{P}(D>t)dt - cx = (p-c)x - p\int_0^x F_D(t)\,dt.\end{aligned}$$

$F_D(t)$ is non-decreasing as a cdf. By Facts 3 $\pi(x)$ is concave. By Fact 1 its subdifferenatial is $\partial\pi(x) = [\pi'_+(x), \pi'_-(x)]$, where by Fact 3:

$$\pi'_+(x) = p - c - pF_D(x), \quad \pi'_-(x) = p - c - pF_D(x-0).$$

By Fact 2, x^* is a global maximum of π if and only if $0 \in \partial\pi(x^*)$, which is equivalent to desired the claim. $\square$

Example 17.3. (*Pareto demand*) Assume that:

$$\mathbb{P}(D > x) = \begin{cases} \frac{1}{\sqrt{x}} & x \geq 1,\\ 1 & \text{otherwise.}\end{cases}$$

Note that the optimal order quantity x^* satisfies $x^* \geq 1$. Fix $x \geq 1$. Then:

$$\begin{aligned}\pi(x) &= p\mathbb{E}\min\{D,x\} - cx\\ &= p\int_0^x \mathbb{P}(D>t)dt - cx\\ &= p + p\int_1^x \tfrac{1}{\sqrt{y}}dy - cx = p(2\sqrt{x}-1) - cx,\\ \pi'(x) &= \frac{p}{\sqrt{x}} - c, \quad f'(x) = 0 \iff x = \left(\tfrac{p}{c}\right)^2,\\ \pi'(x) &= -\frac{p}{2x^{3/2}} < 0.\end{aligned}$$

It follows that f is concave and the global maximum is $x^* = (p/c)^2$.

We perform 10 000 simulations of newsvendor's capital as a function of time if the newsvendor starts with $c = \$10$, $p = \$100$, the newsvendor starts with $K = \$5000$, and the time horizon is $T = 100$ trading periods. For the optimal order quantity 15.5% of simulations ended by bankruptcy, see Figure 12 (left). However, ordering only 50% or optimal quantity leads to bankruptcy only in 0.02% of cases, see Figure 12 (right). This illustrates the fact that the optimal

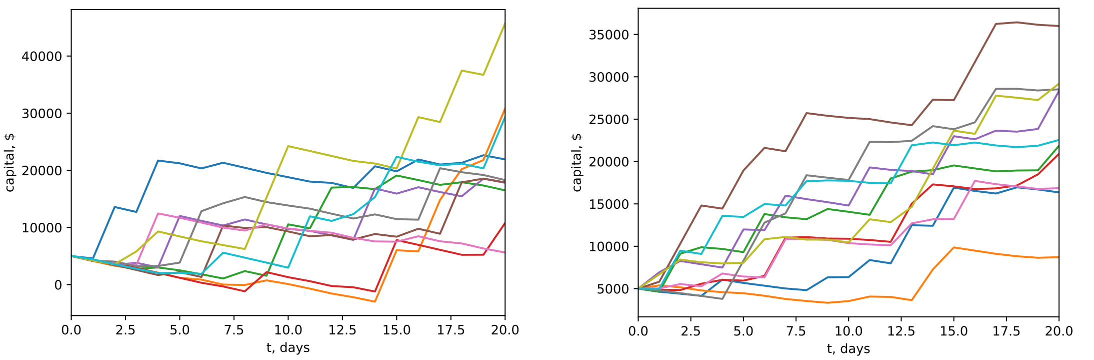


Figure 12: Newsvendor's capital as a function of time for the optimal order quantity (left) and 50% of the optimal quantity (right).

solution to (25) can expose the newsboy to high risk of bankrupcy if the demand distribution has heavy tails (in our case $\mathbb{E}X = +\infty$, $\operatorname{Var} X = +\infty$).

17.2 General newsvendor model

Consider the following model:

1. The demand D is random with cdf $F_D(t) = \mathbb{P}(D \le t)$.
2. x units of product are ordered.
3. For each unit of unsatisfied demand the *underage cost* c_u is incurred. For each unit of products left over the *overage cost* c_o is incurred.

The agent minimizes the expected costs:

$$\begin{aligned} &\text{minimize} \quad C(x) = c_u\mathbb{E}(D-x)_+ + c_o\mathbb{E}(x-D)_+ \\ &\text{subject to} \quad x \ge 0. \end{aligned} \tag{26}$$

Proposition 17.4. *$x^* > 0$ is an optimal solution to (26) if and only if:*

$$F_D(x^* - 0) \le \frac{c_u}{c_u + c_o} \le F_D(x^*).$$

In particular, if F_D is continuous and invertible this is equivalent to

$$x^* = F_D^{-1}\left(\frac{c_u}{c_u + c_o}\right).$$

Proof. Note that:

$$\mathbb{E}(D-x)_+ = \int_0^\infty \mathbb{P}(D-x>t)dt = \int_x^\infty (1-F_D(t))dt,$$
$$\mathbb{E}(x-D)_+ = \int_0^\infty \mathbb{P}(x-D>t)dt = \int_0^x F_D(t)dt.$$

$F_D(t)$ is non-decreasing as a cdf. By Facts 3 (x) is convex. By Fact 1 $\partial C(x) = [C'_-(x^*), C'_+(x^*)]$, where by Fact 3

$$C'_-(x^*) = (c_u+c_o)F(x^*-0) - c_u, \quad C'_+(x) = (c_u+c_o)F(x^*) - c_u.$$

By Fact 2 x^* is a global minimum of C if and only if $0 \in \partial C(x^*)$, which is equivalent to the claimed inequalities. □

Example 17.5. *(Fashion store, see [1]).* A store can order items in advance of the fashion season for \$100 per unit or during the season for \$190 per unit. Items are sold to customers for \$250 per unit during the season. Items left over after the season are sold to outlet stores for \$80 per unit. Demand data is shown in Figure 13.

Demand	73	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	97	Σ
frequency	1	1	2	1	1	3	5	7	7	8	10	10	9	8	5	6	5	3	3	2	2	1	100
probability	.01	.01	.02	.01	.01	.03	.05	.07	.07	.08	.10	.10	.09	.08	.05	.06	.05	.03	.03	.02	.02	.01	1
cdf	.01	.02	.04	.05	.06	.09	.14	.21	.28	.36	.46	.56	.65	.73	.78	.84	.89	.92	.95	.97	.99	1	1

Figure 13: Historical demand data for items

Let $\pi(x,D)$ be the profit when x units are ordered. Then $\pi(x,D) = \pi(D,D) - C(x,D)$, where $C(x,D) = c_u(D-x)_+ + c_o(D-x)_-$. Each unit of non-satisfied demand leads to a profit loss of $c_u = \$190 - \$100 = \$90$. Each non-sold unit leads to a loss of $c_o = \$100 - \$80 = \$20$.

Let $c_u/(c_u+c_o) = 90/110 \approx 0.81$. Inspecting Figure 13 we find the optimal order quantity $x^* = 89$.

Remark 17.6. *(Censored demands). In practice only sales* $\min\{D,x\}$ *are observable.*

17.3 Demand management

Consider the following toy model:

1. There are N places in the airplane, x_i of which are sold at fare p_i, $i \in \{1,\dots,K\}$.
2. Demand D_i for fare class i is random with cdf $F_i(t) = \mathbb{P}(D_i \le t)$.

3. Travelers preferring class i will not buy tickets at other fares. *Example: economy class travelers would rather book another flight than buy a business class ticket.*

The airline wants to determine the number of places sold at each fare maximizing the expected profits:

$$\begin{array}{rl} \text{maximize} & \pi(x) = \sum_{i=1}^{K} p_i \mathbb{E}\min\{D_i, x_i\}, \\ \text{subject to} & x_1 + \cdots + x_K = N, \\ & x = (x_1, \ldots, x_N) \geq 0. \end{array} \tag{27}$$

Exercise 17.7. Assume that $F_1, \ldots, F_N$ are invertible. Let λ be such that:

$$\sum_{i=1}^{K} p_i F_i^{-1}\left(\tfrac{p_i - \lambda}{p_i}\right) = N.$$

Put $x_i^* = F_i^{-1}(\frac{p_i - \lambda}{p_i})$. Then $x^* = (x_1^*, \ldots, x_K^*)$ is a global maximum in (27).

Now consider a model with nested booking limit:

1. There are two fares $p_1 > p_2$ and at most x out of N tickets are of class two.
2. Demand D_i for fare class i is random with cdf $F_i(t) = \mathbb{P}(D_i \leq t)$.
3. Travelers of the second class will not buy tickets of the first class if there are no tickets of the second class left.

We are interested in finding a booking policy $x \in [0, N]$ maximizing the expected profit. Let Q_i be the number of booked tickets of class i. We also consider the worst-case scenario that demand for tickets of the second class is realized earlier than demand for tickets of the first class. Then:

$$Q_2 = \min\{D_2, x\}, \quad Q_1 = \min\{D_1 + Q_2, u\} - Q_2.$$

The optimal policy is determined by solving:

$$\begin{array}{rl} \text{maximize} & p_1 \mathbb{E} Q_1 + p_2 \mathbb{E} Q_2 \\ \text{subject to} & x \in [0, N]. \end{array}$$

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