Uniqueness and reconstruction in a passive inverse problem of helioseismology

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Joint work with T. Hohage and R. G. Novikov

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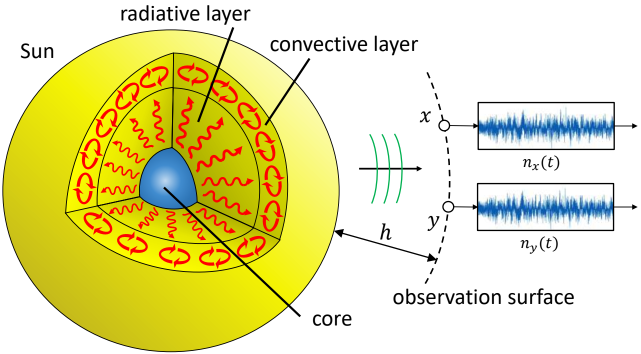
Acoustic field in the Sun



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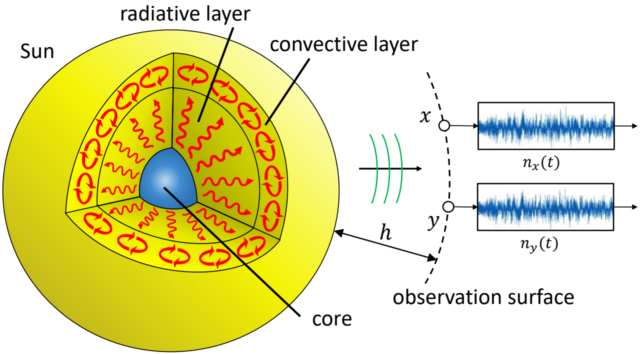

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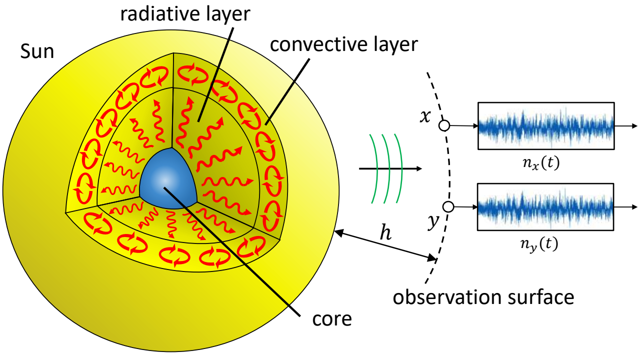

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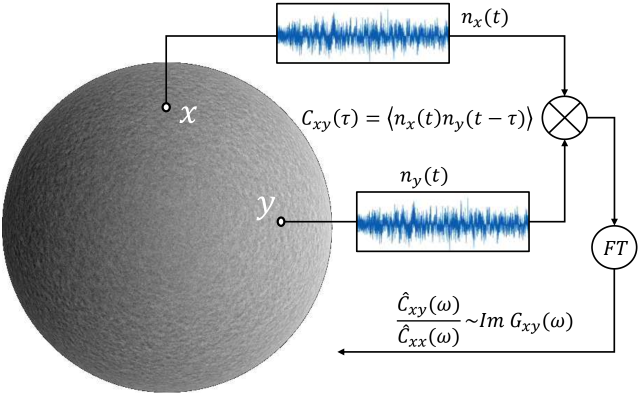
- Multi-height line-of-sight velocities can be computed from Doppler shifts in the absorption lines of the solar light *(Nagashima et al., 2014)*
- Recover solar sound speed, density and attenuation

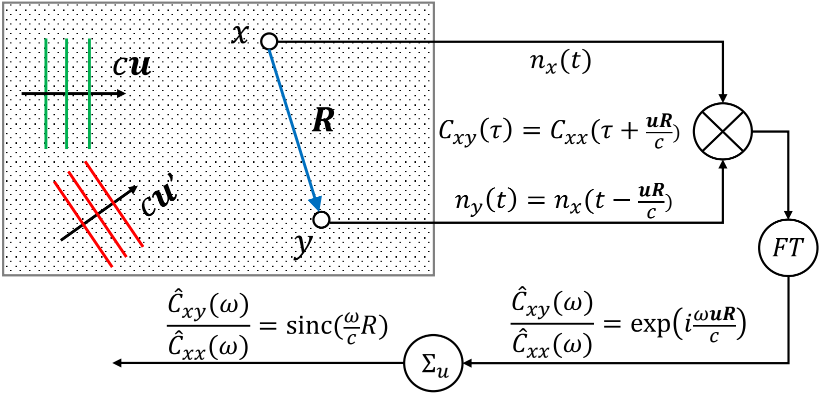
Green's function from cross-correlations



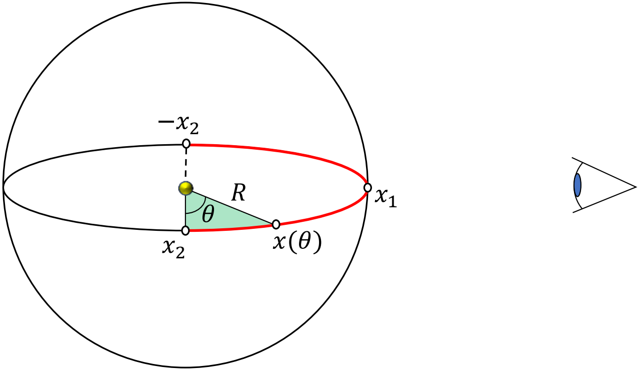
- Energy equidistribution of noise sources $\Longrightarrow$ Imaginary part of the radiation Green's function can be extracted from cross-correlations *(Eckart, 1953), (Cox, 1973), (Roux et al., 2005), (Snieder, 2007), (Gizon et al., 2017)*

Green's function from cross-correlations

Example: homogeneous medium and isotropic noise *(Eckart, 1953)*



Measurement manifold



- Great circle arc of angle π and radius R:

$$M_R^1 = \{(x(\theta), x_2) \colon \theta \in (0, \pi)\}, \quad x_j \in S_R^2,\ x_1 \cdot x_2 = 0,$$
$$x(\theta) = x_1 \sin\theta + x_2 \cos\theta$$

Equation for acoustic waves in the Sun

- Acoustic field in the Sun *(Gizon et al., 2017)*:

$$-\nabla\left(\frac{1}{\rho}\nabla\left(\sqrt{\rho}\psi_\omega\right)\right)-\frac{\sigma^2}{c^2\sqrt{\rho}}\psi_\omega=\frac{f_\omega}{\sqrt{\rho}},$$
$$\psi_\omega=\sqrt{\rho}c^2\nabla\cdot\xi_\omega,\quad \sigma^2=\omega^2+2i\omega\gamma.$$

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- Parameters: $c = c(|x|)$, $\rho = \rho(|x|)$, $\gamma = \gamma(|x|)$

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- Homogeneous higher atmosphere:

$$c(r)=c_0,\quad \rho(r)=\rho_0 e^{-(r-R_a)/H},\quad \gamma(r)=0,\quad r\geqslant R_a$$

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- **Problem.** Given "Im G_ω" at $M^1_{R_1}\cup M^1_{R_2}$, $R_1>R_2\geqslant R_a$, and for $\omega=\omega_1,\omega_2$, recover c, ρ, γ

Equivalent Schrödinger equation

- Equivalent Schrödinger equation:

$$(L_v - k^2)\psi_v = f, \quad L_v = -\Delta + v,$$
$$k^2 = \frac{\omega^2}{c_0^2} - \frac{1}{4H^2}, \quad v = k^2 - \frac{\sigma^2}{c^2} + \rho^{\frac{1}{2}}\Delta\left(\rho^{-\frac{1}{2}}\right),$$

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- Radiation radiation condition for long range potentials *(Saitō, 1974), (Agmon & Klein, 1992)*
- **Subproblem.** Given Im G_v at $M^1_{R_1} \cup M^1_{R_2}$ at fixed $k > 0$, recover v

Separation of variables

- Equation $(L_v - k^2)\psi_v = 0$ separates in spherical coordinates:

$$(L_{v,\ell} - k^2)\psi_{v,\ell}^m = 0, \quad L_{v,\ell} = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + v(r)$$

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- Incoming and outgoing solutions:

$$H^\pm_{\nu,\ell}(r) \sim \exp\big(\pm i(kr - \eta\ln(2kr) - \tfrac{1}{2}\ell\pi + \sigma_\ell(\eta))\big), \quad r \to +\infty,$$
$$H^\pm_{\nu,\ell}(r) = H^\pm_\ell(\eta, kr), \quad r \geqslant R_a,$$

where $\eta = 1/(2Hk)$ and $H^\pm_\ell$ are the Coulomb wave functions

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- Regular solution:

$$F_{v,\ell}(r) \overset{r\to 0}{\sim} r^{\ell+1}, \quad F_{v,\ell}(r) \overset{r \geqslant R_a}{=\!=} C\big(H^-(\eta, kr) - s_{v,\ell}H^+(\eta, kr)\big),$$

  where $s_{v,\ell}$ are the scattering matrix elements

Radial Green's function

- $G_{\nu,\ell}(r, r')$ is regular at zero, outgoing at ∞, and

$$(L_{\nu,\ell} - k^2) G_{\nu,\ell}(\cdot, r') = \delta_{r'}$$

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- Asymptotic expression:

$$G_{\nu,\ell}(r,r) = \frac{i}{2k}\big(H^-_\ell(\eta, kr) - s_{\nu,\ell}H^+_\ell(\eta, kr)\big)H^+_\ell(\eta, kr), \quad r \geqslant R_a$$

Recovering the potential

$$\operatorname{Im} G_v|_{M^1_{R_1}, M^1_{R_2}} \xrightarrow{\mathrm{I}} \operatorname{Im} G_{v,\ell}(r,r)|_{R_1,R_2} \xrightarrow{\mathrm{II}} s_{v,\ell} \xrightarrow{\mathrm{III}} \Lambda_{v,R} \xrightarrow{\mathrm{IV}} v$$

Step I. Extracting $\operatorname{Im} G_{v,\ell}(r,r)|_{R_1,R_2}$ from $\operatorname{Im} G_v|_{M^1_{R_1}, M^1_{R_2}}$

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$$\operatorname{Im} G_\nu|_{M^1_{R_1}, M^1_{R_2}} \xrightarrow{\mathrm{I}} \operatorname{Im} G_{\nu,\ell}(r,r)|_{R_1,R_2} \xrightarrow{\mathrm{II}} s_{\nu,\ell} \xrightarrow{\mathrm{III}} \Lambda_{\nu,R} \xrightarrow{\mathrm{IV}} \nu$$

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- Expansion in Legendre polynomials P_ℓ:

$$G_\nu(x, x_2) = \frac{1}{4\pi R^2} \sum_{\ell=0}^{\infty} (2\ell+1) G_{\nu,\ell}(R,R) P_\ell(x \cdot x_2/R^2), \quad (x, x_2) \in M^1_R,$$

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- It follows that

$$\operatorname{Im} G_{\nu,\ell}(R,R) = -2\pi R^2 \int_0^\pi \operatorname{Im} G_\nu(x(\theta), x_2) P_\ell(\cos\theta)\, d\cos\theta,$$
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- A linear system for $\operatorname{Re} s_{\nu,\ell}$, $\operatorname{Im} s_{\nu,\ell}$:

$$\operatorname{Im} G_{\nu,\ell} - \frac{1}{2k}|H^+_\ell|^2 = -\operatorname{Re} s_{\nu,\ell}\frac{\operatorname{Re}(H^+_\ell)^2}{2k} + \operatorname{Im} s_{\nu,\ell}\frac{\operatorname{Im}(H^+_\ell)^2}{2k},$$

where functions are evaluated at $r = R_1, R_2$

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- Solution to $(L_v - k^2)\psi_v = 0$ in B^3_R with $\psi_v|_{S^2_R} = Y^m_\ell$:

$$\psi_v(x) = \frac{R}{|x|}\frac{F_{v,\ell}(|x|)}{F_{v,\ell}(R)} Y^m_\ell\left(\tfrac{x}{|x|}\right)$$

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- Expression for $\Lambda_{\nu,R}$:

$$\Lambda_{\nu,R} Y^m_\ell = R\frac{\left[\frac{\partial}{\partial r}\left(\frac{1}{r}H^-_\ell(\eta,kr)\right) - s_{\nu,\ell}\frac{\partial}{\partial r}\left(\frac{1}{r}H^+_\ell(\eta,kr)\right)\right]_{r=R}}{H^-_\ell(\eta,kR) - s_{\nu,\ell}H^+_\ell(\eta,kR)} Y^m_\ell$$

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- v is uniquely determined by $\Lambda_{v,R}$ if k^2 is not a Dirichlet eigenvalue of L_v in B^3_R *(Novikov, 1988)* and *(Berezanskii, 1958), (Nachman, 1988), (Sylvester & Uhlmann, 1987)*

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- **Subproblem solved.** Given $\mathrm{Im}\, G_v$ at $M^1_{R_1} \cup M^1_{R_2}$ at fixed $k > 0$, recover v

Recovering the solar parameters

$$\nu = \omega^2\left(\frac{1}{c_0^2} - \frac{1}{c^2}\right) + \rho^{\frac{1}{2}}\Delta(\rho^{-\frac{1}{2}}) - 2i\omega\frac{\gamma}{c^2}$$

- Repeat the reconstruction procedure for $\omega = \omega_1, \omega_2$ to recover $\nu = \nu_1, \nu_2$

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- Determine $u_1 = \frac{1}{c_0^2} - \frac{1}{c^2}$, $u_2 = \rho^{\frac{1}{2}}\Delta(\rho^{-\frac{1}{2}})$, $u_3 = \frac{\gamma}{c^2}$ from ν_1, ν_2

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- Find $c = (c_0^{-2} - u_1)^{-\frac{1}{2}}$, $\gamma = c^2 u_3$ and ρ from

$$-\Delta(\rho^{-\frac{1}{2}}) + u_2\rho^{-\frac{1}{2}} = 0 \text{ in } B^2_{R_a}, \quad \rho|_{S^2_{R_a}} = \rho_0|_{S^2_{R_a}}$$

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$$v = \omega^2\Big(\frac{1}{c_0^2} - \frac{1}{c^2}\Big) + \rho^{\frac{1}{2}}\Delta(\rho^{-\frac{1}{2}}) - 2i\omega\frac{\gamma}{c^2}$$

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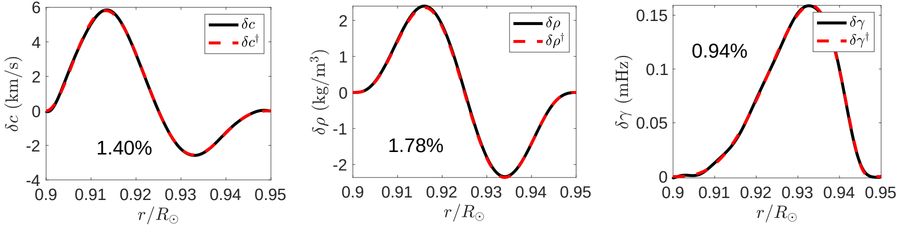
- **Theorem.** c, ρ, γ are uniquely determined from $\operatorname{Im} G_{\omega_1}$, $\operatorname{Im} G_{\omega_2}$ on $M^1_{R_1} \cup M^1_{R_2}$ and at fixed $\omega_1, \omega_2 \in (\omega_{ctf}, +\infty) \setminus \Sigma'$, $\omega_1 \neq \omega_2$, where Σ' is a discrete set without finite accumulation points

Recovery of c, ρ, γ

- Simultaneous recovery of c, ρ, γ from exact data:



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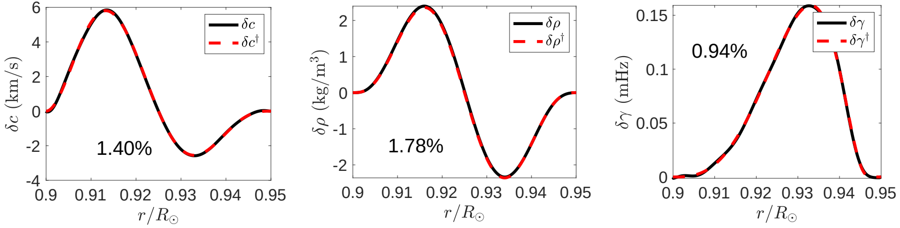
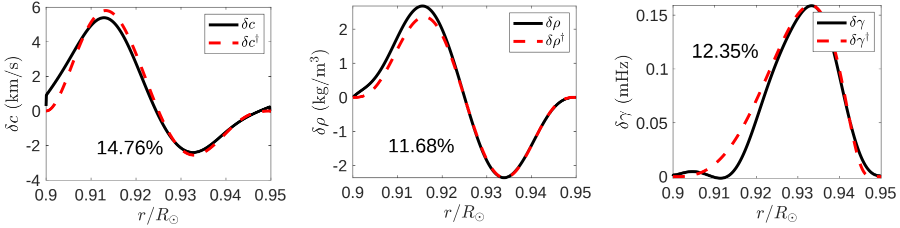


- Single parameter reconstruction for realistic noise levels :



Summary

- Spherically symmetric model for the Sun with exponentially decaying density in the atmosphere
- Acoustic field is measured at two distances above the solar surface for finite number of frequencies
- Uniqueness in the passive inverse problem for c, ρ, γ
- Confirmed by reconstructions from exact data
- Single parameter reconstructions for realistic noise levels using standard reconstruction procedures (IRGNM)

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