On topological field theories in low dimensions

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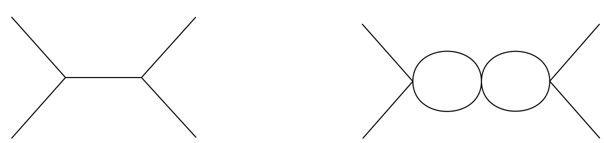
The main goal of this talk is to shed light on few basic ideas that arise in quantum field theories. The classification of topological field theories in low dimensions allows to gain some intuition behind these ideas for a sufficiently small technical price. I will begin by introducing symmetric monoidal categories and functors and then pass to the definition of a topological field theory as a symmetric monoidal functor from a cobordism category to the category of vector spaces. Then I will discuss the classification of topological field theories in dimensions one and two.

1 Introduction

1.1 Overview

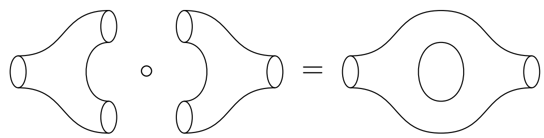
In quantum field theories dealing with point particles interactions between particles are represented using Feynman diagrams like these ones:



In topological field theories of dimension 2 one replaces point particles by circles (interpreted as closed strings), and the above diagrams are replaced by the following analogs called bordisms (or cobordisms), whose definition goes back to (Thom, 1954) and (Pontryagin, 1959):

The horizontal dimension of these pictures is interpreted as time. The above bordisms represent schematically an interaction of two strings which results in two other strings. In physics they are called *worldsheets*, in analogy with worldlines for point particles.

Mathematically, a two-dimensional bordism is a compact smooth oriented surface whose boundary is divided into two parts, called incoming and outgoing boundaries. Bordisms can be composed by gluing if they have a compatible number of outputs and inputs:



Consider a bordism X with incoming boundary Y and outgoing boundary Y'. We associate to Y and Y' complex vector spaces $\mathcal{H}_Y$ and $\mathcal{H}_{Y'}$ called the *spaces of states*. They satisfy the following axioms:

1. If Y is a union of n circles, then $\mathcal{H}_Y \simeq (\mathcal{H}_{S^1})^{\otimes n}$ (recall the same principle from quantum mechanics: the state of two particles is not simply a pair of their states, the states can be entangled, and this is mathematically formalized in the tensor product of vector spaces).

2. $\mathcal{H}_\varnothing = \mathbb{C}$

To the bordism X we associate a linear map $\Psi_X \colon \mathcal{H}_Y \to \mathcal{H}_{Y'}$. It must satisfy the following axioms:

1. Φ_X depends only on the topology of X.

2. (Functoriality) $\Psi_{X \circ X'} = \Psi_X \circ \Psi_{X'}$.

3. (Monoidality) $\Psi_{X'' \sqcup X'''} \simeq \Psi_{X''} \otimes \Psi_{X'''}$.

4. (Normalization) $\Psi_{Y \times [0,1]} = \mathrm{id}_{\mathcal{H}_Y}$.

If X is a bordism from n circles to m circles, Ψ_X is interpreted as the rule for computing the state of final m strings from the state of the initial n strings.

The pair $(\mathcal{H}_\bullet, \Psi_\bullet)$ is called a *topological field theory* in dimension 2. This definition goes back to (Atiyah, 1988). In turn, it was motivated by the definition of conformal field theory of (Segal, 1988).

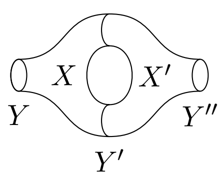
In mathematical terms the data $(\mathcal{H}_\bullet, \Psi_\bullet)$ with the above properties is a symmetric monoidal contravariant functor from the two-dimensional cobordism category to the category of vector spaces. I will clarify the meaning of these words in the following sections, where the definition will be generalized to an arbitrary dimension.

1.2 Relation to physics

Let me explain the physical nature of spaces $\mathcal{H}_\bullet$ and operators $\Psi_\bullet$. The exposition is similar to that from (Freed, 1999) and (Freed, 2006). All manifolds are supposed to be endowed with a Riemannian structure.

The first main ingredient for construction of a quantum field theory is the notion of the *field*. To each manifold X we associate a space of fields $\mathcal{F}_X$. It must satisfy some axioms:

1. (Restrictions) For each submanifold $Y \subseteq X$ there is a restriction map $\mathcal{F}_X \to \mathcal{F}_Y$, we denote the image of $\phi \in \mathcal{F}_X$ under this map by $\phi|_Y$. These maps must be compatible in the sense of the following diagram:



$$\begin{tikzcd}
 & & \mathcal{F}_{X \cup_{Y'} X'} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{F}_X \arrow[dl, "p_{\mathrm{in}}"'] \arrow[dr, "p_{\mathrm{out}}"] & & \mathcal{F}_{X'} \arrow[dl, "p_{\mathrm{in}}"'] \arrow[dr, "p_{\mathrm{out}}"] & \\
\mathcal{F}_Y & & \mathcal{F}_{Y'} & & \mathcal{F}_{Y''}
\end{tikzcd}$$

2. (Locality) We have $\mathcal{F}_{X \sqcup X'} \simeq \mathcal{F}_X \times \mathcal{F}_{X'}$.

The nature of $\mathcal{F}_X$ depends on a particular theory.

The second main ingredient for construction of a quantum field theory is the *action*. To each compact surface X it associates the functional $S_X \colon \mathcal{F}_X \to \mathbb{R}$ (the action). It must satisfy certain axioms:

1. (Gluing) If X and X' are composable bordisms with common boundary Y', then $S_{X' \circ X}(\phi) = S_X(\phi|_X) + S_X(\phi|_{X'})$.
2. (Locality) $S_{X \sqcup X'}(\phi \sqcup \phi') = S_X(\phi) + S_{X'}(\phi)$.

In many situations the action S_X admits a *Lagrangian* density L_X, i.e. it can be represented in the form $S_X(\phi) = \int_X L(\phi)$ for some differential operator L on $\mathcal{F}_X$ (typically, of order 2).

Example 1. Consider a nonlinear σ-model. Let M be a Riemannian manifold called *target manifold* and interpreted as the actual spacetime.

For a compact Riemannian manifold X, we set $\mathcal{F}_X = \mathrm{Map}(X, M)$ and call it the *configuration space*. Elements of $\mathcal{F}_X$ are called *M-valued fields* on X. For example, for a point $\bullet$ the space $\mathcal{F}_\bullet \simeq M$ represents different possibilities for the point to appear in the spacetime. In a similar way, $\mathcal{F}_{S^1}$ represents different ways for the string to appear in the spacetime. Finally, for any bordism X, $\mathcal{F}_X$ represent different ways for the abstract worldsheet (worldline, world volume) to a appear as a physical worldsheet (worldline, world volume) in the spacetime.

The action for the free theory is given by

$$S_X(\phi) = \frac{1}{2}\int_X |d\phi|^2 = \frac{1}{2}\int_X \partial_a\phi^i\partial_b\phi^j h^{ab} g_{ij}\sqrt{\det h}\,dx, \qquad (1)$$

where h is the metric on X and g is the metric on M. Physicists call $|d\phi|^2$ the *kinetic term.*

Note that $S_X(\phi)$ is the Dirichlet energy functional. The corresponding classical equation of motion is $\Delta_{\mathrm{LC}}\phi = 0$, where $\Delta_{\mathrm{LC}} = d_{\mathrm{LC}} * d$, d_{LC} stands for the Levi-Civita covariant derivative and $*$ denotes the Hodge star. If ϕ is a critical point for S_X, it is called a *harmonic map.*

If $M = \mathbb{R}^d$, the map ϕ is harmonic iff it satisfies the ordinary Laplace equation. Furthermore, if ϕ is a harmonic immersion (i.e. $d\phi$ is injective), its image is a minimal surface.

In the case of 1d σ-model (quantum mechanics) it is the action for a free particle of unit mass (I will explain it more carefully in the next section). Critical points of S_X are geodesics in M.

In the case of 2d σ-model the action S_X is *conformally invariant.* It means that if we replace the metric h_{ab} on X by a *conformally equivalent* metric $e^f h_{ab}$, where f is any real-valued function, the functional S_X will not change (it follows from formula (1) if we take into account that $h^{ab} \to e^{-f}h^{ab}$ and $\det h \to e^f \det h$). Physicists call conformal transformations *Weyl transformations.* The class of conformally equivalent metrics on X is called a *conformal structure.* On two-dimensional manifolds conformal structures are in bijective correspondence with complex structures (holomorphic altases on X or "Riemann surface structures"). Thus, the worldsheets of a 2d σ-model can be actually considered as Riemann surfaces.

We can also consider a σ-model with a fixed potential. A potential is a function $V\colon M \to \mathbb{R}$. The corresponding action is

$$S_{X,V}(\phi) = \int_X \left(\tfrac{1}{2}|d\phi|^2 - V(\phi)\right)$$

For more details, see, e.g., (Deligne, Freed, 1999).

Example 2. Consider a gauge theory. Let G be a compact semisimple Lie group on a compact Riemannian manifold X and $P \to X$ be a fixed principal G-bundle.

One defines $\mathcal{F}_X$ as the space of connections on P. Connections are smooth G-invariant distributions $H \subset TP$ complementary to the kernel of the map $TP \to TM$ induced by projection $P \to M$. Such a distribution can be described as the kernel of a G-equivariant 1-form $\omega_H \in \Omega^1(P, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

The action (for the pure Yang-Mills theory) is given by

$$S_X(\phi) = -\tfrac{1}{2} \int_X |F_\phi|^2,$$

where $F_\phi \in \Omega^2(X, \operatorname{ad} P)$ is the gauge field strength of corresponding to connection ϕ. The Euler-Lagrange equation for this action plus the Bianchi identity lead to the Yang-Mills equations for F_ϕ.

Semisimplicity of G means that the Killing form on the Lie algebra $\mathfrak{g}$ of G is non-degenerate and $|F_\phi(x)| = 0$ iff $F_\phi(x) = 0$.

We can couple our gauge fields to some fixed matter fields. A bosonic matter field φ is a section of a vector bundle $P \times_G V$ associated to P via some representation $G \to GL(V)$. The coupling for this field to the gauge field ϕ is described by the action functional

$$S_{X,\varphi}(\phi) = \tfrac{1}{2} \int_X |d_\phi \varphi|^2,$$

where $d_\phi \colon \Omega^0(M, P \times_G V) \to \Omega^1(M, P \times_G V)$ is the covariant derivative.

We can also couple our gauge fields to fermionic matter fields. Note that in order for X to admit fermionic fields, it must satisfy a specific topological *spin* condition. Recall that X is orientable iff the restriction of its tangent bundle to any embedded circle in X is trivial. In a similar way, if $\dim X > 4$ and X is simply connected, then X is spin iff the restriction of its tangent bundle to any embedded 2-sphere in X is trivial. Thus, spin condition is something like a two-dimensional analog of orientability. If X is spin, fermionic fields are sections of associated vector bundles tensored by a spinor bundle.

For more details on this example, see, e.g., (O'Farrill, 2006).

The data of fields and action allows to define a quantum field theory in the spirit of the previous subsection. For a given union of circles Y, define its space of states $\mathcal{H}_Y = L^2(\mathcal{F}_Y)$ (for some appropriate measure).

To a bordism X from Y to Y' associate the following *pull-push* operator:

$$\Psi_X \colon L^2(\mathcal{F}_Y) \to L^2(\mathcal{F}_{Y'}), \quad \Psi_X \varphi = (p_{\mathrm{out}})_*(e^{\frac{i}{\hbar} S_X} p_{\mathrm{in}}^* \varphi), \tag{2}$$

where $\hbar$ is a small parameter. The sense of operations p_{in}^*, $(p_{\text{out}})_*$ is explained in examples below. The operation $(p_{\text{out}})_*$ is the integration over the fibers of $\mathcal{F}_X \to \mathcal{F}_{Y'}$, mapping a function on $\mathcal{F}_X$ to the function on $\mathcal{F}_{Y'}$. In order to perform this operation, we need some measure in the fibers of integration. Below I will show that it is a generalization of the Feynman integral.

Example 3. Consider the case of 2d σ-model with target manifold M. Note that the space $\mathcal{F}_{S^1} = LM$ admits a so-called Wiener measure, and we can define $L^2(\mathcal{F}_{S^1})$.

Let Y, Y' be the incoming and outgoing boundaries of some bordism X. In a similar way with the wave-function of quantum mechanics, $\varphi \in L^2(\mathcal{F}_Y)$ computes the probability amplitudes for Y to appear in the space-time M in whatever way.

Then $p_{\text{in}}^*\varphi$ is a function defined on $\mathcal{F}_X$. On a given field $\gamma \in F_X$ its value is defined by $p_{\text{in}}^*\varphi(\gamma) = \varphi(p_{\text{in}}(\gamma))$. To a given worldsheet corresponding to X it associates the probability for Y to appear as the first time slice of this worldsheet. We multiply this probability by $\exp(\frac{i}{\hbar}S_X)$.

Finally, the operation $(p_{\text{out}})_*$ associates to a function Φ on $\mathcal{F}_X$ a function $\Phi' = (p_{\text{out}})_*\Phi$ on $\mathcal{F}_{Y'}$. The value $\Phi'(\gamma')$ is obtained by integrating $\Phi(\gamma)$ over all $\gamma \in \mathcal{F}_X$ such that $p_{\text{out}}(\gamma) = \gamma'$. It means that for a given way γ' for Y' to appear in the space-time it sums over different ways γ for X to appear with γ' as the last time-slice.

Example 4. Push-pull operators arise everywhere in mathematics. The Fourier transform is the simplest example:

$$\begin{tikzcd} & \mathbb{R}_x \times \mathbb{R}_\xi \arrow[dl, "p_x"'] \arrow[dr, "p_\xi"] & \\ \mathbb{R}_x & & \mathbb{R}_\xi \end{tikzcd} \qquad \mathcal{F}u = (p_\xi)_*(e^{-i\xi x}p_x^*u).$$

Another example is the classical Radon transform (X-ray transform):

$$\begin{tikzcd} & \mathbb{R}^2 \times TS^1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{R}^2 & & TS^1 \end{tikzcd} \qquad \mathcal{R}u = (p_2)_*(\delta(x_1\cos\phi + x_2\sin\phi - t)p_1^*u),$$

where $\mathbb{R}^2 = \mathbb{R}^2_{x_1,x_2}$, $TS^1 \simeq \mathbb{R}_t \times S^1_\phi$, and δ is the Dirac delta-function.

In topological field theories there are some simplifications of the general scheme. Using the definition of topological field theory, one can show (see Section 5) that spaces of states arising in topological field theories are finite-dimensional. Let me show where this finite-dimensionality comes from in the picture that we have just discussed.

In topological field theories one sets the action S_X to zero for any X.

Next, one replaces the infinite-dimensional space of fields $\mathcal{F}_X$ with a finite-dimensional moduli space $\mathcal{M}_X$. Examples include moduli space of holomorphic maps and moduli space of self-dual connections.

Finally, instead of considering infinite-dimensional spaces of states like $L^2(\mathcal{F}_X)$, one considers some finite dimensional (generalized) cohomology spaces $\mathcal{K}(\mathcal{M}_X)$. The linear map Ψ_X is defined in a similar way to (2):

$$\Psi_X \colon \mathcal{K}(\mathcal{M}_Y) \to \mathcal{K}(\mathcal{M}_{Y'}), \quad \Psi_X\varphi = (p_{\mathrm{out}})_*(p_{\mathrm{in}}^*\varphi),$$

where $p_{\mathrm{in}} \colon \mathcal{M}_X \to \mathcal{M}_Y$, $p_{\mathrm{out}} \colon \mathcal{M}_X \to \mathcal{M}_{Y'}$ are the analogs of the maps discussed above.

1.3 Quantum mechanics in the spirit of sigma model

This is an example similar to the one from (Lupercio, Uribe, 2006).

Consider the space $M = \mathbb{R}^d$ in which a free particle of unit mass lives. The particle's state is described by the wave-function $\psi_t(q)$. In particular, according to Born's interpretation, the probability to find a particle in a region $U \subset M$ at time t is $\int_U |\psi_t|^2$.

The fundamental equation of quantum mechanics, the Schrödinger equation, allows to find ψ_T from ψ_0. It states that ψ_t satisfies the following differential equation:

$$\dot{\psi}_t(q) = \tfrac{i\hbar}{2}\Delta\psi_t(q), \quad q \in M.$$

Feynman proposed a remarkable formula for solution of this equation (not only in the free case). The idea is similar to the definition of Riemann integral. We pass to the Fourier domain with variable p and back, and find that

$$\dot{\widehat{\psi}}_t(p) = -\tfrac{i\hbar}{2}p^2\widehat{\psi}_t(p),$$
$$\widehat{\psi}_T(p) = e^{-\frac{i\hbar}{2}p^2\Delta t}\widehat{\psi}_{T-\Delta t}(p),$$
$$\psi_T(q) = \left(\frac{-i}{2\pi\Delta t}\right)^{\frac{d}{2}} \int \psi_{T-\Delta t}(q_1)\exp\left(\tfrac{i}{\hbar}\left(\tfrac{q-q_1}{\Delta t}\right)^2\Delta t\right)dq_1.$$

We now substitute this expression recursively in the integral, and for $T = N\Delta t$ we get

$$\psi_T(q) = \left(\frac{-i}{2\pi\Delta t}\right)^{\frac{Nd}{2}} \int\cdots\int \psi_0(q_N)\exp\left(\tfrac{i}{\hbar}S_N\right)\,dq_1\cdots dq_N,$$
$$S_N = \left(\left(\tfrac{q-q_1}{\Delta t}\right)^2 + \left(\tfrac{q_1-q_2}{\Delta t}\right)^2 + \cdots + \left(\tfrac{q_{N-1}-q_N}{\Delta t}\right)^2\right)\tfrac{\Delta t}{2}.$$

One can interpret this integral as the integral over all piecewise linear paths $\gamma\colon [0,T] \to M$ such that $\gamma(T) = q$, broken at $t = k\Delta t$, $k = 1, \ldots, N-1$. Passing to the limit we obtain the Feynman path integral formula:

$$\psi_T(q) = \int_{\mathcal{P}_q} \mathcal{D}\gamma\ \psi_0(\gamma(0)) e^{\frac{i}{\hbar}S(\gamma)}, \tag{3}$$

where $\mathcal{P}_\gamma$ is the set of continuous paths $\gamma\colon [0,T] \to M$ with $\gamma(T) = q$, and

$$S(\gamma) = \frac{1}{2}\int_0^T |\dot{\gamma}(t)|^2\, dt$$

is the action functional for a free particle of unit mass. The path integral is interpreted as the Riemann-type limit above.

To reformulate this in the spirit of sigma model, consider an oriented Riemannian 1-manifold Y isometric to $[0,T]$. Then Y has the initial point $\bullet_i$ and the final point $\bullet_f$. Set $\mathcal{F}_{\bullet_i} = \mathrm{Map}(\bullet_i, M) \simeq M$, $\mathcal{F}_{\bullet_f} = \mathrm{Map}(\bullet_f, M) \simeq M$, $\mathcal{F}_Y = \mathrm{Map}(Y, M) \simeq \mathrm{Map}([0,T], M)$. The elements of $\mathcal{F}_{\bullet_i}$ are interpreted as different ways for our point particle to appear in the space time, whereas the elements of $\mathcal{F}_Y$ are different ways for Y to appear as the world-line for our particle.

Set $\mathcal{H}_{\bullet_i} = L^2(\mathcal{F}_i) \simeq L^2(M)$. The elements of $L^2(M)$ are ordinary wave-functions for our point particle. Define the operators $p_{\mathrm{in}}, p_{\mathrm{out}}\colon \mathrm{Map}([0,T], M) \to M$ as evaluation at 0 and T. Then for given $\psi \in L^2(M)$ and $\gamma \in \mathrm{Map}([0,T], M)$ we have $p_{\mathrm{in}}^*\psi(\gamma) = \psi(\gamma(0))$.

We make sense of operator $(p_{\mathrm{out}})_*$ using the Feynman integral. The operator $(p_{\mathrm{out}})_*$ for given function Φ on $\mathrm{Map}([0,T], M)$ and point $q \in M$ is defined as follows:

$$(p_{\mathrm{out}})_*\Phi(q) = \int_{\mathcal{P}_q} \mathcal{D}\gamma\ \Phi(\gamma).$$

Thus, formula (3) can be rewritten using the pull-push formalism:

$$\psi_T = (p_{\mathrm{out}})_*(e^{\frac{i}{\hbar}S} p_{\mathrm{in}}^*\psi_0).$$

2 Categories and functors

Probably, you are familiar with the notion of category which goes back to Aristotle. Rougly speaking, for Aristotle categories are predicatives of the most general nature, the highest genera of entities (e.g. substance, quality, quantity, action, possession). This is one of the central notions of philosophy.

The term *category* was borrowed to mathematics by S. Eilenberg and S. MacLane in 1942–1945. It emphasizes the highest level of generality and abstractness of the corresponding notion, which, however, can serve as a universal organising principle in many branches of mathematics.

Definition 1. A category $\mathcal{C}$ consists of:

- a collection of objects $\mathcal{O}_C$,
- for each pair of objects A, $B \in \mathcal{O}_C$, a collection of morphisms (arrows) $\mathbf{Mor}(A, B)$.

There is some additional data:

1. for any $A \in \mathcal{O}_C$ there is a marked *identity* morphism $A \xrightarrow{\mathrm{id}_A} A$,
2. for any morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ there is a morphism $A \xrightarrow{g \circ f} C$ called their *composition*.

This data is subject to the following axioms:

1. for any $A \xrightarrow{f} B$ we have $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$,
2. for any $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

The picture to have in mind is an oriented graph whose vertices are objects and oriented edges are morphisms. The idea to have in mind is that objects of category are sets with some additional structure and morphisms are maps of sets that preserve this structure (but in general these are not sets). Here are some examples of categories:

Example 5. Set, the category of sets. Objects of this category are sets and a morphism from a set X to a set Y is just a map from X to Y. The identity morphism on X is just an identity map on X.

Example 6. Vect(k), category of **k**-vector spaces. Objects of this category are vector spaces over **k**. A morphism from a vector space X to a vector space Y is just a linear map from X to Y.

Example 7. Top, category of topological spaces. Objects are topological spaces and morphisms are continuous maps.

Example 8. Poset, category of partially ordered sets. Objects are partially ordered sets and morphisms are order preserving maps.

Example 9. Each poset, in turn, can be considered as a category. The objects are the elements of this poset. For each pair x, y of objects there is at most one morphism $x \to y$ meaning that $x \leq y$. Other way round, any category $\mathcal{C}$ such that for any $x, y \in \mathcal{O}_C$ there is at most one morphism $x \to y$, can be considered as a poset.

The first main ingredient of the category theory is the notion of category. The second main ingredient is the notion of *functor*. A functor should be thought of as a map between categories (e.g. from **Top** to $\mathbf{Vect}(\mathbf{k})$) which respects the categorical structures.

Note that the word functor was coined by a philosopher R. Carnap in 1934 in a philosophical context. The term was borrowed to mathematics by S. Eilenberg and S. MacLane in 1942–1945.

Definition 2. A *(covariant) functor* F from category $\mathcal{C}$ to category $\mathcal{D}$ consists of:

- to each object $A \in \mathcal{O}_\mathcal{C}$ it associates an object $F(A) \in \mathcal{O}_\mathcal{D}$,
- to each morphism $A \xrightarrow{f} B$ it associates a morphism $F(A) \xrightarrow{F(f)} F(B)$.

This data is subject to the following axioms:

- for any $A \in \mathcal{O}_\mathcal{C}$ we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
- for any $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ we have $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* is defined in same way, but it sends a morphism $A \xrightarrow{f} B$ to a morphism $F(B) \xrightarrow{F(f)} F(A)$, i.e. it changes the direction of morphisms.

Example 10. We can associate to each topological space $X \in$ **Top** the algebra $C(X)$ of its continuous functions and to each map $f \colon X \to Y$ of topological spaces, a homomorphism of algebras $C(f) \colon C(Y) \to C(X)$. $C(f)$ maps a continuous function $\varphi \in C(Y)$ to a continuous function $\varphi \circ f \in C(X)$. It is an example of a contravariant functor.

Example 11. We can consider any group G as a category with a single object and with morphisms given by the elements of this group. In turn, a category with only one object and all of whose morphisms are invertible (a morphism f is called *invertible* if there exists a morhism g such that $f \circ g$ and $g \circ f$ are identities) can be considered as a group. Then a linear representation of the group G is just the functor from G to $\mathbf{Vect}(\mathbf{k})$.

Example 12. For a given topological space X we define the category $\mathbf{Ouv}(X)$ of its open sets. The objects of this category are open sets in X and a morphism $U \to V$ is just the inclusion $U \hookrightarrow V$. A *sheaf* on X is the same thing as a contravariant functor from $\mathbf{Ouv}(X)$ to $\mathbf{Set}$.

3 Symmetric monoidal categories

It turns out that many categories of interest naturally come with an additional structure similar to that of tensor product in vector spaces. These are called symmetric monoidal categories.

Definition 3. A symmetric monoidal category $\mathcal{C}$ is a category equiped with:

- a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- a marked object $I \in \mathcal{O}_{\mathcal{C}}$ called "unit",

satisfying the following axioms:

1. for any $A \in \mathcal{O}_{\mathcal{C}}$ we have $I \otimes A \simeq A$, $A \otimes I \simeq A$,
2. for any A, B, $C \in \mathcal{O}_{\mathcal{C}}$ we have $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$,
3. for any A, $B \in \mathcal{O}_{\mathcal{C}}$ we have $A \otimes B \simeq B \otimes A$.

Furthermore, the isomorphisms above must be fixed and coherent (the order of their composition must be irrelevant).

Note that we could require all the isomorphisms above to be identities but in that case even the basic category $\mathbf{Vect}(\mathbf{k})$ which motivated the definition won't satisfy these axioms. Here are some examples:

Example 13. In $\mathbf{Vect}(\mathbf{k})$ the operation $\otimes$ is the ordinary tensor product. The same for other similar "algebraic" categories.

Example 14. In $\mathbf{Top}$ the operation $\otimes$ can be defined as the disjoint union. The same for other similar "geometric" categories.

Example 15. The category of representations of a given group G. Recall that a representation of group G is a homomorphism of groups $\rho \colon G \to \mathrm{Aut}(V)$ for some vector space V. The morphism $\rho' \to \rho''$ of representations $\rho' \colon G \to \mathrm{Aut}(V')$, $\rho'' \to \mathrm{Aut}(V'')$ is a linear map $A \colon V' \to V''$ interwining representations, i.e. such that $A\rho'(g) = \rho''(g)A$. The tensor product $\rho' \otimes \rho''$ is the representation $\rho' \otimes \rho'' \colon G \to \mathrm{Aut}(V' \otimes V'')$ defined by $(\rho' \otimes \rho'')(g) \cdot (v' \otimes v'') = \rho'(g) \cdot v' \otimes \rho''(g) \cdot v''$.

Now, it is natural to require that a functor between symmetric monoidal categories respect the tensor product operation. We are led to the following definition:

Definition 4. A symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor satisfying

1. for any $A \in \mathcal{O}_{\mathcal{C}}$ we have $F(\mathrm{id}_A) \simeq \mathrm{id}_{F(A)}$,
2. for any A, $B \in \mathcal{O}_{\mathcal{C}}$ we have $F(A \otimes B) \simeq F(A) \otimes F(B)$.

Furthermore, these isomorphisms are fixed and are coherent (the order of their composition is irrelevant).

An example of such a functor is a *topological quantum field theory* (in the sense of M. Atiyah) defined below.

4 Cobordism category

In order to define the topological quantum field theory, we need to introduce the cobordism category $\mathbf{Cob}(n)$ of dimension n. The objects of this category are (compact) closed manifolds of dimension $n-1$. Morphisms are diffeomorphism classes of bordisms. The notion of bordism goes back to (Pontryagin, 1959) and (Thom, 1954).

Definition 5. A *bordism* between two closed smooth $(n-1)$-manifolds M and N is a manifold B of dimension n together with a fixed diffeomorphism $\partial B \simeq \overline{M} \sqcup N$, where $\overline{M}$ is M with opposite orientation. Bordisms B and B' are identified if there is an orientation preserving diffeomorphism $B \simeq B'$ extending the diffeomorphism $\partial B \simeq \overline{M} \sqcup N \simeq B'$.

The identity morphism id_M is defined to be the class of $M \times [0,1]$.

The composition of bordisms $M \overset{B}{\to} N$, $N \overset{B'}{\to} K$ is represented by the manifold $B \sqcup_N B'$ obtained from $B \sqcup B'$ by gluing along N.

The category $\mathbf{Cob}(n)$ is a symmetric monoidal category with respect to the tensor product $M \otimes N := M \sqcup N$ and unit given by the empty manifold.

Example 16. Let's see what is $\mathbf{Cob}(1)$. Zero-dimensional compact closed manifolds are finite unions of points with some prescribed orientations, "positive" and "negative" points, which we denote by $\bullet_+$ and $\bullet_-$. These finite unions are objects of $\mathbf{Cob}(1)$.

The only connected closed compact manifolds of dimension one are the interval and the circle. The circle does not have boundary, and it defines a bordism from $\varnothing$ to $\varnothing$. The interval can define four different bordisms: from $\bullet_+$ to itself, from $\bullet_-$ to itself, from $\varnothing$ to $\bullet_+ \sqcup \bullet_-$ and from $\bullet_+ \sqcup \bullet_-$ to $\varnothing$.

Example 17. Let's see what is $\mathbf{Cob}(2)$. The only closed compact manifolds of dimension two are finite unions of circles S^1 (note that a circle is diffeomorphic to itself with the opposite orientation). These finite unions are objects of $\mathbf{Cob}(2)$.

The morphisms in $\mathbf{Cob}(2)$ are compact surfaces with $p+q$ circular boundary components, p labeled as inputs and q as outputs. These surfaces are considered modulo orientation preserving diffeomorphisms fixing boundaries.

5 Topological Field Theories

Definition 6 (M. Atiyah)**.** A topological field theory of dimension n is a symmetric monoidal functor $Z\colon \mathbf{Cob}(n) \to \mathbf{Vect}(\mathbf{k})$.

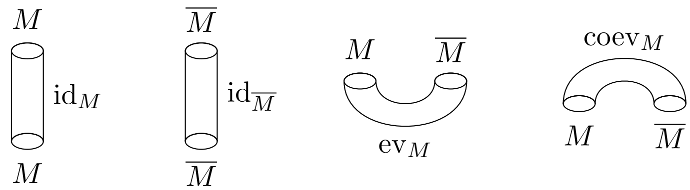
This definition allows to gather the idea of different quantum field theories which are thought as "topological". There are also some variations of this definition. Topological field theories are of some interest for mathematicians because they allow to compute topological invariants and even to discover new invariants, which were not known before (e.g. Gromov-Witten invariants in 2d, quantum Chern-Simons invariants in 3d, Donaldson and Seiberg-Witten invariants in 4d).

An example of topological field theory is the Chern–Simons theory. However, its formulation requires to introduce principal bundles and connections.

Our next goal is to draw some implications of all what was introduced before. Probably, you will be impressed by how deep is the definition of topological field theory.

Let us draw first consequences. As a functor, Z maps each closed manifold M of dimension $(n-1)$ to a vector space $Z(M)$. It must also map each bordism $M \xrightarrow{B} N$ to a linear map $Z(M) \xrightarrow{Z(B)} Z(N)$.

Consider the manifold $B = M \times [0,1]$ with boundary $\partial B = \overline{M} \sqcup M$. Note that we can consider B as four different bordisms by differently interpreting its boundary:



- considered as $M \xrightarrow{B} M$, it is the identity bordism 1_M,
- considered as $\overline{M} \xrightarrow{B} \overline{M}$, it is the identity bordism $1_{\overline{M}}$,
- considered as $M \sqcup \overline{M} \xrightarrow{B} \varnothing$, we call it the *evaluation* bordism of M and denote ev_M,
- considered as $\varnothing \xrightarrow{B} M \sqcup \overline{M}$, we call it the *coevaluation* bordism of M and denote coev_M.

Now, since Z is a functor, it must map identity maps to identity maps. So $Z(1_M)$ is just the identity map on the vector space $Z(M)$ and $Z(1_{\overline{M}})$ is the identity map on the vector space $\overline{M}$.

Next, since Z is a monoidal functor, it maps the tensor product in $\mathbf{Cob}(n)$ to the tensor product in $\mathbf{Vect}(\mathbf{k})$ and the unit in $\mathbf{Cob}(n)$ to the unit in $\mathbf{Vect}(\mathbf{k})$:

$$Z(M \sqcup \overline{M}) \simeq Z(M) \otimes Z(\overline{M}),$$
$$Z(\varnothing) \simeq \mathbf{k}.$$

Hence, $Z(\mathrm{ev}_M)$ and $Z(\mathrm{coev}_M)$ are (up to the fixed isomorphisms):

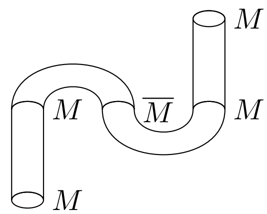
$$Z(\mathrm{ev}_M)\colon Z(M) \otimes Z(\overline{M}) \to \mathbf{k},$$
$$Z(\mathrm{coev}_M)\colon \mathbf{k} \to Z(M) \otimes Z(\overline{M}).$$

These two maps are of great importance. It turns out, that we can describe them explicitly.

Proposition 1. *The following statements are true:*

1. *$Z(M)$ is finite-dimensional,*
2. *$Z(\mathrm{ev}_M)$ defines a canonical isomorphism $Z(\overline{M}) \simeq Z(M)^\vee$,*
3. *If we further identity $Z(M) \otimes Z(M)^\vee \simeq \mathrm{End}(Z(M))$, the map $Z(\mathrm{ev}_M)$ becomes the trace and $Z(\mathrm{coev}_M)$ becomes the inclusion of the identity matrix.*

Proof. Consider an S-shaped bordism from M to M factoring through $M \sqcup \overline{M} \sqcup M$:



This bordism is clearly 1_M, and it must induce the identity map on the vector space $V = Z(M)$. This identity map factors in the following way:

$$\begin{gathered} V \to V \otimes W \otimes V \to V, \\ v \mapsto v_1 \otimes w_1 \otimes v + \cdots + v_d \otimes w_d \otimes v \mapsto v_1(w_1, v) + \cdots + v_d(w_d, v), \end{gathered} \tag{4}$$

where $W = Z(\overline{M})$, the number d is fixed and (a, b) is the pairing $Z(\mathrm{ev}_M)$. Without loss of generality one can suppose that $v_1, \ldots, v_d$ are linearly independent. Since the composite map is identity, we see that $\dim V = d$ and that $w_1, \ldots, w_d$ are linearly independent. In particular, $\dim V \leq \dim W$. Similar considerations for an S-shaped bordism from $\overline{M}$ to $\overline{M}$ show that the dimensions are equal.

Next, we see that the pairing (a, b) defines an isomorphism $W \simeq V^\vee$ by sending the basis w_j of W to the basis $(w_j, -)$ of $V^\vee$. It follows from formula (4) that $(w_j, -) = v_j^\vee$ so that under this identification the evaluation map becomes the duality pairing $v_i \otimes v_j^\vee \mapsto \langle v_i, v_j^\vee \rangle = \delta_{ij}$. If we further identify $V \otimes V^\vee \simeq \mathrm{End}(V)$, this becomes the trace.

Finally, it follows from formula (4) that under this identification the coevaluation map becomes $1_{\mathbf{k}} \mapsto v_1 \otimes v_1^\vee + \cdots + v_d \otimes v_d^\vee$. If we further identify $V \otimes V^\vee \simeq \mathrm{End}(V)$, the right hand side becomes the identity matrix. □

6 One-dimensional TFTs

Any compact closed manifold of dimension one is a finite union of points with some prescribed orientations. Denote by $\bullet_+$ and $\bullet_-$ the points with positive and negative orientations, respectively. A topological field theory associates a vector space V to $\bullet_+$ and the vector space $V^\vee$ to $\bullet_-$. To a union of p positive and q negative points it associates the vector space $V^{\otimes p} \otimes (V^\vee)^{\otimes q}$.

Further, there are only two connected compact manifolds of dimenson one, namely the interval $[0, 1]$ and the circle S^1. An arbitrary compact manifold of dimension one is a finite union of these two. We have studied all the maps associated to $[0,1]$ in the general setting above and we have noticed that they are completely determined by V.

Now, S^1 can be considered as a bordism from nothing to nothing. A TFT associates to S^1 a linear map from $\mathbf{k}$ to $\mathbf{k}$ which is uniquely determined by its value on $1_{\mathbf{k}}$. Thus, we can consider $Z(S^1)$ as a number. In order to compute this number, we cut S^1 into two parts representing it as a composition of coevaluation and evaluation of a point. Thus, $Z(S^1)$ is a composition of $1_{\mathbf{k}} \mapsto 1_{\mathrm{End}(V)}$ and of the trace map, and is equal to the dimension of V.

We see that topological field theories in dimension one are completely determined by the data of V.

Theorem 1. *There is a bijective correspondence between topological field theories in dimension* 1 *and finite-dimensional vector spaces.*

7 Two-dimensional TFTs

Any compact closed smooth manifold of dimension one is a finite union of circles (once again, note that a circle is diffeomorphic to itself with the opposite orientation). A two-dimensional topological field theory associates a vector space V to a single circle and $V^{\otimes n}$ to a disjoint union of n circles.

Bordisms are just compact smooth surfaces with $p+q$ boundary components, p of them labeled as inputs and q as outputs. Bordisms are considered modulo orientation preserving diffeomorphisms fixing boundaries.

It turns out that a two-dimensional TFT also endows the vector space V with a bilinear associative commutative product, a unit map $1_V : \mathbf{k} \to V$ and a trace map $\mathrm{tr}: V \to \mathbf{k}$, all being compatible. In other words, a TFT endows V with a structure of *commutative Frobenius algebra*.

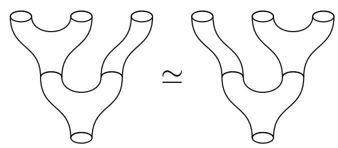
Definition 7. A commutative Frobenius algebra is a finite dimensional $\mathbf{k}$-vector space V together with an associative commutative bilinear multiplication $\nabla : V \times V \to V$ with unit $1_V : 1 \to V$ and a linear functional $\mathrm{tr}: V \to \mathbf{k}$ such that the bilinear form $(a,b) = \mathrm{tr}\,\nabla(a,b)$ is non-degenerate.

Example 18. Matrix algebras are (non-commutative) Frobenius algebras with the usual trace.

Example 19. $\mathbb{C}$ is a commutative Frobenius algebra over $\mathbb{R}$ with trace $\mathrm{tr}\, z = \mathrm{Re}\, z$.

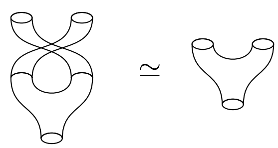
Now let's see how a TFT endows V with a structure of commutative Frobenius algebra. Indeed, a pair of pants bordism gives a bilinear map

$\nabla\colon V \times V \to V$. By composing two pairs of pants with a cylinder in two different manners, one can see that this operation is associative:



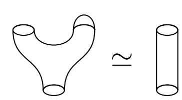
$$\nabla(\nabla(a,b),c) = \nabla(a,\nabla(b,c))$$

By composing a pair of pants with the "twisting" bordism, we see that it is also commutative.



$$\nabla(b,a) = \nabla(a,b)$$

The disk bordism from nothing to a circle gives a linear map $1_V\colon \mathbf{k} \to V$. Composing it with a pair of pants we see that it is a unit for ∇:



$$\nabla(a, 1_V) = a$$

The trace map comes from the disk bordism from a circle to nothing. Composing a pair of pants with this bordism we get the evaluation map of the circle, and we know that the corresponding bilinear map is nondegenerate.

Thus, to any TFT in dimension two one can associate a commutative Frobenius algebra. It turns out that the converse is also true.

Theorem 2. *There is a bijective correspondence between topological field theories in dimension two and commutative Frobenius algebras.*

Next, note that any closed compact surface Σ_g of genus g is a bordism from nothing to nothing. Hence, a TFT associates to it a linear map $Z(\Sigma_g)\colon \mathbf{k} \to \mathbf{k}$, which is completely determined by its value on $1_{\mathbf{k}}$. In other words, a TFT associates a number to any surface Σ_g. Let's compute this number for small values of g.

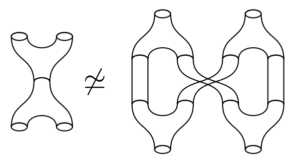
If $g = 0$, then Σ_g is a sphere. It can be considered as a composition of two discs, and hence the number $Z(\Sigma_g)$ is equal to $\mathrm{tr}(1_V)$.

If $g = 1$, then Σ_g is a torus. It can be considered as a composition of coevaluation with evaluation of the circle. Thus, the corresponding linear map $Z(\Sigma_g)$ is the composition of inclusion of the identity matrix $1_{\mathbf{k}} \mapsto 1_{\mathrm{End}(V)}$ with the matrix trace. Thus, $Z(\Sigma_g)$ is the dimension of V.

One can write down the answer for arbitrary g. Identify $\mathrm{End}(V)$ with $V \otimes V$. Thus, it makes sense to define $\omega = \nabla(1_{\mathrm{End}(V)}) \in V$. It turns out that for arbitrary g the value $Z(\Sigma_g)$ is just $\mathrm{tr}(\omega^g)$.

Remark 1. The pair of pants bordism from a circle to two circles endows V with a comultiplication $\Delta\colon V \to V \otimes V$. Rotating the above diagrams for multiplication, we obtain the diagrams which show that this comultiplication together with the trace map endow V with the structure of *cocommutative coassociative counital coalgebra*.

Hovever, in general the algebra and coalgebra structures on V are not compatible, meaning that V is *not a bialgebra*. For example, in the definition of bialgebra one requires the commutativity of the following diagram, which does not follow from our functorial definitions of Δ and ∇:



$$\begin{tikzcd} V^{\otimes 2} \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"] & V \arrow[r, "\Delta"] & V^{\otimes 2} \\ V^{\otimes 4} \arrow[rr, "1 \otimes \mathrm{swap} \otimes 1"] & & V^{\otimes 4} \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd}$$

8 Variations

8.1 Conformal Field Theory (Segal)

Our definition of the topological field theory goes back to (Atiyah, 1988). This definition, in turn, was motivated by the definition of two-dimensional conformal field theory of (Segal, 1988).

Segal introduces the category $\mathscr{C}$ whose objects are finite disjoint unions of parametrized circles. A morphism from a union of p circles to the union of q circles is a compact Riemann surface with $p+q$ boundary components, p being marked as inputs and q as outputs. Two Riemann surfaces defined the same morphism if there is a biholomorphic map between them preserving inputs and outputs. The composition is by gluing outputs to inputs and the tensor product is the disjoint union. Segal defines a conformal field theory as a symmetric monoidal functor from category $\mathscr{C}$ to $\mathbf{Vect}(\mathbf{k})$.

Thus, the linear map associated to a given surface depends not only on the topology of the surface but also on its conformal class.

8.2 Open-closed Topological Field Theory (Moore, Segal)

An open-closed topological field theory is a generalization of the topological field theory which is more relevant to applications in string theory. It was axiomatized in (Moore, 2001) and by G. Segal in his lectures in 1999.

We can interpret a circle as a closed string, and a bordism from, say, two circles to a circle as an interaction of two closed strings resulting in another closed string. An open-closed topological field theory also allows open strings which are line segments topologically and which must satisfy some boundary conditions, the latter formalized by D-branes. For a precise definition, see (Costello, 2007).

Two-dimensional open-closed TFTs were classified in (Lauda, Pfeiffer, 2008). The result is similar to classification of (closed) TFTs, but the role of commutative Frobenius algebras is played by the so-called *knowledgeable Frobenius algebras*.

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