Fixed freq. identities for the Green function and uniqueness results for passive imaging

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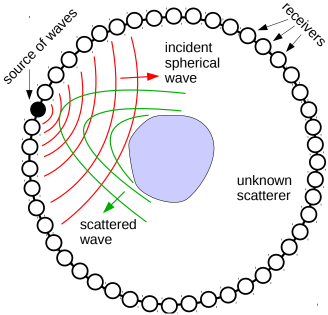
I. Motivation – Classical inverse problem



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Classical inverse problem: *Recover scatterer parameters from boundary measurements of scattered waves*

- Applications in medical diagnostics, ocean tomography, . . .

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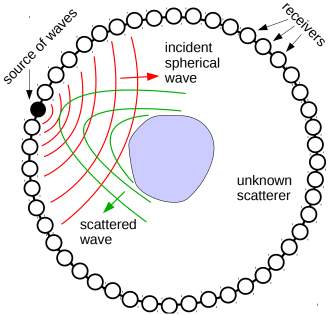

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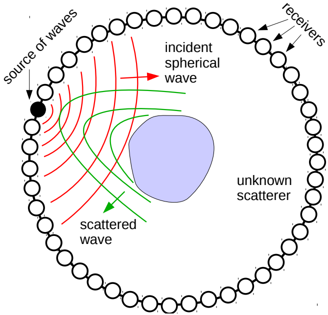

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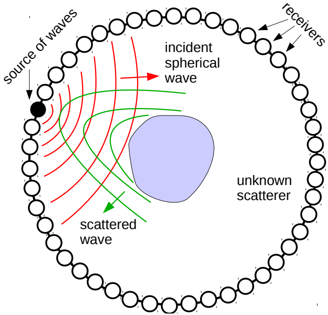

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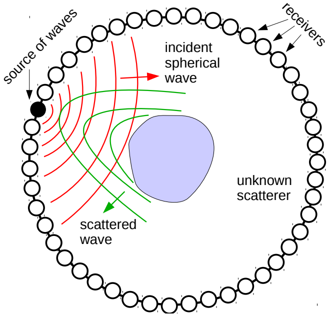

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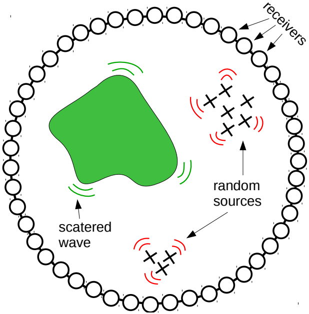

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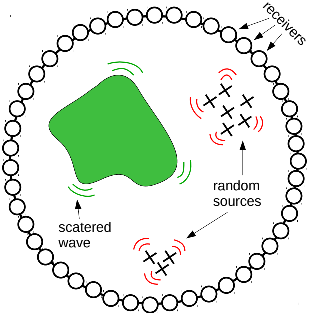

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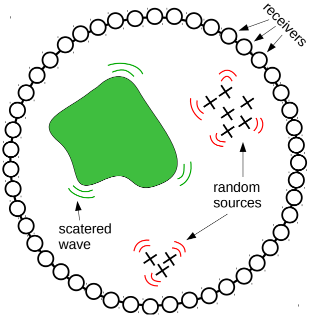

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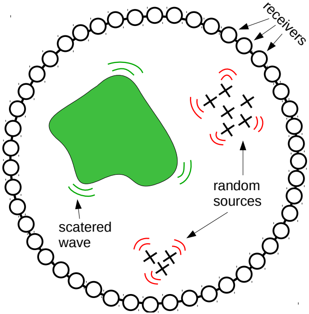

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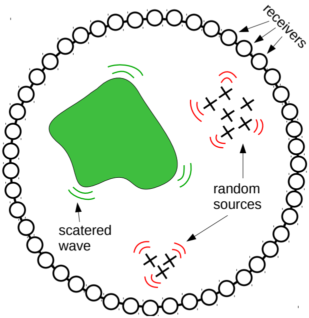

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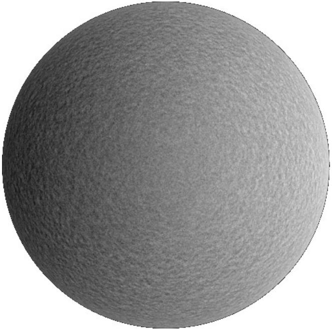

Solar Dopplergram
(https://sdo.gsfc.nasa.gov)

- Turbulent convection excites acoustic waves
- Acoustic waves travel and scatter inside of the Sun
- Acoustic field on the surface manifests as surface oscillations

Local helioseismology: *Recover the Sun parameters from the measurements of the acoustic field on its surface*

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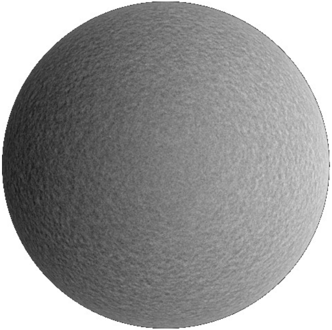

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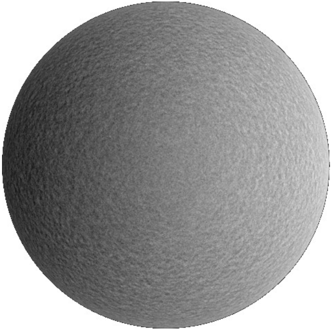

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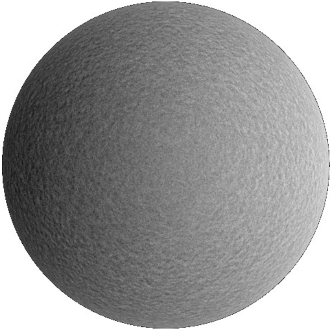

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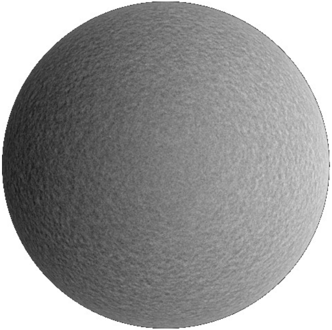

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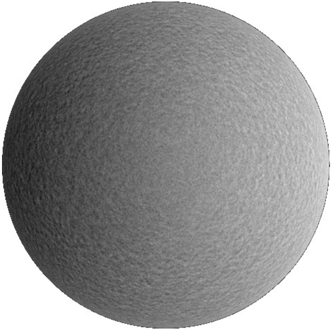

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Theorem ([9], ≈ [4, 8, 11, 5]). *The following formula holds:*

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- $G_n^+ = G_n^+(x,y)$ is the outgoing Green function

$$(-\Delta_x - k^2 n(x))G_n^+(x,y) = \delta_y(x) \quad x \in \mathbb{R}^d,$$
$$(\tfrac{\partial}{\partial|x|} - ik)G_n^+(x,y) = o(|x|^{\frac{1-d}{2}}), \quad |x| \to +\infty.$$

- $n(x)$ real, $n(x) = 1$ outside a bounded domain Ω

Inverse problem: *Given* Im $G_n^+(x_1,x_2)$ *at fixed* $k > 0$ *for all* x_1, $x_2 \in \partial\Omega$, *find* $n(x)$ *for* $x \in \Omega$

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Two important linear operators:

- Single layer potential $\mathcal{G}_n \colon H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$:

$$\mathcal{G}_n f(x) = \int_{\partial\Omega} G_n^+(x,y) f(y) dy, \quad x \in \partial\Omega$$

- Dirichlet-to-Neumann map $\Phi_n \colon H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$: $\Phi_n u_0 = \frac{\partial u}{\partial \nu}$,

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III. Algebraic relations – Derivation

- The following relations hold:

$$\mathcal{G}_1 - \mathcal{G}_n = \mathcal{G}_n(\Phi_n - \Phi_1)\mathcal{G}_1 \qquad [\mathrm{Nachman}: 88]$$

$$\Rightarrow \; \mathcal{G}_n^{-1} - \mathcal{G}_1^{-1} = \Phi_n - \Phi_1 \quad \& \quad \overline{\mathcal{G}_n^{-1}} - \overline{\mathcal{G}_1^{-1}} = \Phi_n - \Phi_1$$

$$\Rightarrow \; \mathcal{G}_n^{-1} - \mathcal{G}_1^{-1} = \overline{\mathcal{G}_n^{-1}} - \overline{\mathcal{G}_1^{-1}}$$

- Put $A = \mathrm{Re}\,\mathcal{G}_n$, $B = \mathrm{Im}\,\mathcal{G}_n$, $Q = \frac{1}{2}(\mathcal{G}_1^{-1} - \overline{\mathcal{G}_1^{-1}})$

$$\Rightarrow \boxed{\begin{array}{l} AQA + BQB = -B \\ AQB = BQA \end{array}}$$

- $Q = -T^*T$, where $T\colon L^2(\partial\Omega) \to L^2(S^{d-1})$ is a compact, injective, dense range operator (*Dirichlet-to-Farfield map*).

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It follows that with $\widetilde{A} = TAT^*$, $\widetilde{B} = TBT^*$ one has

$$\widetilde{A}^2 = \widetilde{B} - \widetilde{B}^2$$
$$\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}$$

Corollaries:

- If $\widetilde{A} > 0$, then

$$A = T^{-1}(\widetilde{B} - \widetilde{B}^2)^{1/2}(T^*)^{-1}$$
$$\widetilde{B} = TBT^*$$

- Spectrum of $\widetilde{F} = \widetilde{A} - i\widetilde{B}$ lies on $|z + \frac{i}{2}| = \frac{1}{2}$ (it is a *farfield operator*).

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$$\begin{tikzcd}
 & \mathrm{Im}\,\mathcal{G}_n \arrow[d, equal] & TBT^* \arrow[d, equal] & & TAT^* \arrow[d, equal] & \mathrm{Re}\,\mathcal{G}_n \arrow[d, equal] \\
\mathrm{Im}\,G_n^+ \arrow[r, "\text{def}"] & B \arrow[r, "\text{def}"] \arrow[drr, bend right] & \widetilde{B} \arrow[r, "\text{rel.}"] & \widetilde{A}^2 \arrow[r, red] & \widetilde{A} \arrow[r, "T\text{ inj.}"] & A \arrow[dll, bend left] \\
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 & & & n & &
\end{tikzcd}$$

Classical inverse problem. $\mathcal{G}_n$ at fixed k determines $n - 1 \in L^\infty(\Omega, \mathbb{R})$ uniquely if k^2 is not a Dirichlet eigenvalue

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IV. Uniqueness theorems

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IV. Uniqueness theorems – Helmholtz equation

Lemma. $\forall M > 0 \; \exists \varepsilon \colon \|n\|_\infty \leqslant M$ *and* $k \in (0, \varepsilon) \Longrightarrow \widetilde{A} > 0$

Theorem 1. *For any* $M > 0$ *there exists* ε *such that if* $k \in (0, \varepsilon)$*,* $\|n_{1,2}\|_\infty \leqslant M$ *and* $\operatorname{Im} \mathcal{G}_1(k) = \operatorname{Im} \mathcal{G}_2(k)$ *then* $n_1 = n_2$*.*

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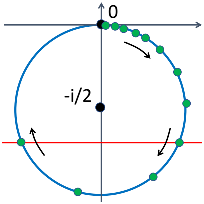
IV. Uniqueness theorems – Schrödinger equation

$$\left(-\Delta_{x_1} + v(x_1) - k^2\right) G_v^+(x_1, x_2) = \delta(x_2)$$



$(*)$ Let $\widetilde{F} = \widetilde{A} - i\widetilde{B}$. If $\widetilde{F} f_1 = \lambda_1 f_1$, $\widetilde{F} f_2 = \lambda_2 f_2$ are such that $\operatorname{Im}\lambda_1 = \operatorname{Im}\lambda_2$ then $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$

... is generic, at least, for small k, v and starlike Ω

Theorem 2. *Suppose that* $\operatorname{Re}\mathcal{G}_{v_0}(k)$ *is injective in* $H^{-\frac{1}{2}}(\partial\Omega)$, $(*)$. *Then* $\exists\delta$ *such that if* $\|v_{1,2} - v_0\|_\infty < \delta$ *and* $\operatorname{Im}\mathcal{G}_{v_1}(k) = \operatorname{Im}\mathcal{G}_{v_2}(k)$, *then* $v_1 = v_2$.

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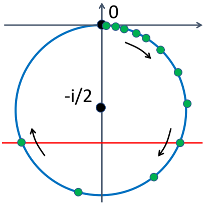

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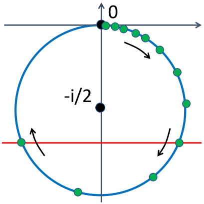

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IV. Uniqueness theorems – Schrödinger equation

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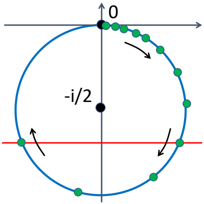

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- Acoustic equation:

$$-\nabla \cdot \left(\frac{1}{\rho_j}\nabla G_j^+(x,y)\right) - \frac{\omega^2}{\rho_j c_j^2} G_j^+(x,y) = \delta_y(x),$$

+ Sommerfeld radiation condition

Theorem 3. *For any* $\omega_{1,2}$, ρ_0, c_0 *such that* $\mathcal{G}_0(\omega_{1,2})$ *is injective in* $H^{-\frac{1}{2}}(\partial\Omega)$, $(*)$ *there exists* δ *such that if* $\|\rho_{1,2}-\rho_0\|+\|c_{1,2}-c_0\| < \delta$ *and* $\operatorname{Im}\mathcal{G}_1(\omega_{1,2}) = \operatorname{Im}\mathcal{G}_2(\omega_{1,2})$ *then* $\rho_1 = \rho_2$, $c_1 = c_2$.

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V. Conclusion

- Established results:
  - Re $\mathcal{G}_n$ and Im $\mathcal{G}_n$ are algebraically related if n is real-valued
  - Global uniqueness for PIP for Helm. equation at one small k
  - Local uniqueness for PIP for Schr. equation at one k
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- Limitations of the approach:
  - Absense of absorption
- What's next?
  - Other wave equations (more complete model of helioseismology)
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