Generalized Radon transforms and mathematical economics

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- The first microfounded model of production goes back to (Houthakker, 1955), (Johansen, 1972)
- It was fruitfully applied to study of macroeconomic processes (Petrov, Pospelov, Shananin); e.g. impact of technological innovation on production (Johansen, 1972)
- Globalization led to qualitative changes in production processes (increase in substitutability of factors), which are out of scope of the classical model
- A natural generalization was proposed in (Sato, 1975), (Shananin, 1997), which allows to overcome this problem

The generalized Houthakker–Johansen model

- Technologies are parametrized by vectors $x \in \mathbb{R}^n_{\geq 0}$
- Each technology x has a capacity $f(x) \geq 0$ (number of production units using this technology)
- Technologies are described by the unit cost of production $q_p(x) = q(p_1 x_1, \ldots, p_n x_n)$ (here p are the prices of resources)
- The maximal possible profit for the industry is given by the *profit function* $(\Pi_q f)(p_0, p)$ (p_0 the price of the final product):

$$(\Pi_q f)(p_0, p) = \int_{\mathbb{R}^n_{\geq 0}} \max\{0, p_0 - q_p(x)\} f(x)\, dx$$

Production technologies

- Neoclassical n-input, 1-output technologies $\mathcal{Q}(n)$: smooth, 1-homogeneous $q\colon \mathbb{R}^n_{>0} \to \mathbb{R}_{>0}$ with bounded level sets
- Action of $\mathbb{R}^n_{>0}$ on $\mathcal{Q}(n)$, $(p, q) \mapsto q_p$:

$$q_p(x_1, \dots, x_n) = q(p_1 x_1, \dots, p_n x_n)$$

- Partial composition $\circ_i\colon \mathcal{Q}(m) \times \mathcal{Q}(n) \to \mathcal{Q}(m + n - 1)$:

$$\begin{aligned}&(f \circ_i g)(x_1, \dots, x_{m+n-1})\\ &= f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+m-1}), x_{i+m}, \dots)\end{aligned}$$

Main operators

Let $q \in \mathcal{Q}(n)$, $h \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

- The generalized Radon transform:

$$(R_q f)(p) = \int_{q_p^{-1}(1)} f(x) \frac{dS_x}{|\nabla q_p(x)|}$$

- Radon-type integral operators:

$$(R_q^h f)(p) = \int_{\mathbb{R}^n_{\geq 0}} h(q_p(x))\, f(x)\, dx$$

- The profit function:

$$(\Pi_q f)(p_0, p) = \int_{\mathbb{R}^n_{\geq 0}} \max\{0, p_0 - q_p(x)\} f(x)\, dx$$

Main questions

Characterization. Determine the scope of the model. When a given profit function Π of an industry can be constructed using our framework, i.e. $\Pi = \Pi_q f$ for some q, f?

Uniqueness. If $\Pi = \Pi_q f$, when such a representation is unique?

Inversion. If $\Pi = \Pi_q f$ for unique q and f, how to find q and f?

Identification. Given trade statistics, how to identify compatible q and f?

Main spaces

- Weighted spaces $L_c^r(\mathbb{R}^n_{\geq 0})$ with finite norms

$$\|f\|_{r,c} = \left(\int_{\mathbb{R}^n_{\geq 0}} |f(x)|^r x^{rc-I}\, dx \right)^{1/r}, \quad 1 \leq r < \infty,$$

$$\|f\|_{\infty,c} = \inf\{K \geq 0 \colon |f(x)x^c| \leq K \text{ for a.e. } x \in \mathbb{R}^n_{\geq 0}\},$$

  where $c \in \mathbb{R}^n_{>0}$, $I = (1, \ldots, 1)$
- R_q, R_q^h, Π_q are continious from $L_{I-c}^r(\mathbb{R}^n_{\geq 0})$ to $L_c^r(\mathbb{R}^n_{\geq 0})$
- The Mellin transform

$$(Mf)(z) = \int_{\mathbb{R}^n_{\geq 0}} x^{z-I} f(x)\, dx, \quad z \in \mathbb{C}^n$$

  is isometric from $L_c^2(\mathbb{R}^n_{\geq 0})$ to $L^2(\Re z = c)$

Question U. If $\Pi = \Pi_q f$, when such a representation is unique?

Let S, $H \subseteq \mathbb{C}^n$. We say that

- S is 1-meagre in H iff $S \cap H$ is nowhere dense in H;
- S is 2-meagre in H iff $S \cap H$ has measure zero in H;
- S is ∞-meagre in H iff $S \cap H = \varnothing$.

Theorem (A, Inverse Problems 2016)

Let $q \in \mathcal{Q}(n)$, $c \in \mathbb{R}^n_{>0}$, $r \in \{1, 2, \infty\}$. The following statements are equivalent:

- Π_q *is injective in* $L^r_{I-c}(\mathbb{R}^n_{\geq 0})$.
- R_q *is injective in* $L^r_{I-c}(\mathbb{R}^n_{\geq 0})$.
- *The nullset of* Me^{-q} *is* r*-meagre in the plane* $\Re z = c$.

Injectivity: idea

- Recall that $(R_q f)(p) = \int_{q_p(x)=1} f(x) \frac{dS_x}{|\nabla q_p(x)|}$
- If $f(x) = f_1(q_{p_1}(x)) + \cdots + f_N(q_{p_N}(x))$, then Π_q is injective:

$$\int_{\mathbb{R}^n_{\geq 0}} |f(x)|^2 dx = \sum_{k=1}^{N} \int_0^\infty \overline{f_k(t)} t^{-1} (R_q f)\left(\tfrac{p_k}{t}\right) dt.$$

- Generalizes to infinite sums. When functions of the form $\varphi(q_p(x))$ with varying φ and p span $L^1_{I-c}(\mathbb{R}^n_{\geq 0})$, $L^2_{I-c}(\mathbb{R}^n_{\geq 0})$?

Injectivity: idea

Question. When functions of the form $\varphi(q_p(\cdot))$ with varying φ, p span $L^1_{I-c}(\mathbb{R}^n_{\geq 0})$, $L^2_{I-c}(\mathbb{R}^n_{\geq 0})$?

$\Downarrow$ Wiener Tauberian theorems: translates $\phi(\cdot - a)$ of $\phi \in L^1(\mathbb{R}^n)$ span $L^1(\mathbb{R}^n)$ iff $\mathcal{F}\varphi(\xi) \neq 0\ \forall \xi$. Similar in $L^2(\mathbb{R}^n)$.

$\Downarrow$ Define $E_c \colon \mathbb{C}^{\mathbb{R}^n_{\geq 0}} \to \mathbb{C}^{\mathbb{R}^n}$ by $(E_c f)(y) = e^{cy} f(e^y)$. Then:

$$E_c \text{ is an isometry from } L^r_c(\mathbb{R}^n_{\geq 0}) \text{ to } L^r(\mathbb{R}^n),$$
$$E_c \text{ maps the Mellin transform to the Fourier transform}$$
$$E_c \text{ maps the action } (p, q) \to q_p \text{ to the additive translation}$$

Conclusion. Functions of the form $\varphi(q_p(\cdot))$ with varying φ, p span $L^1_{I-c}(\mathbb{R}^n_{\geq 0})$ iff $Me^{-q}(z) \neq 0$ for $\Re z = c$. Similar in $L^2_{I-c}(\mathbb{R}^n_{\geq 0})$.

Injectivity: example

- A CES function is a function of the form

$$q(x) = C(a_1 x_1^\alpha + \cdots + a_n x_n^\alpha)^{\frac{1}{\alpha}},$$

where $C > 0$, $a_j \geq 0$, $a_1 + \cdots + a_n = 1$ (and here $\alpha \in (0, 1]$) (Arrow, Solow et al, 1961)

- A nested CES function is obtained from CES functions using finite compositions (Sato, 1967). It allows to take into account such effects as *capital-skill complementarity* (Griliches, 1969)

Theorem (A, http://arxiv.org/abs/1508.02014)

If q is a nested CES, then Π_q injective in L^r_{I-c}, $r \in \{1, 2, \infty\}$.

Injectivity: example: idea

Question. How the injectivity behaves with respect to composition $\circ_i$ of technologies? Define $\widehat{f}(z) = Me^{-f}(z)/\Gamma(\Sigma(z))$.

- $\Downarrow$ Π_f is injective in $L^2_{I-c}(\mathbb{R}^n_{\geq 0})$ iff $\widehat{f}(z) \neq 0$ for $\Re z = c$ a.e.
- $\Downarrow$ One can show that $\widehat{f \circ_i g}(z \circ_i w) = \widehat{f}(z \circ_i \Sigma(w))\widehat{g}(w)$
- $\Downarrow$ One can show that $\widehat{q}(z) = \frac{a^{-z/\alpha}\Gamma(\Sigma(z))B(z/\alpha)}{\alpha^{n-1}C^{\Sigma(z)}}$ for CES q

Conclusion. Injectivity propagates.

Question E. Determine the scope of the model. When a given profit function Π of an industry can be constructed using our framework, i.e. $\Pi = \Pi_q f$ for some q, f?

Question U. If $\Pi = \Pi_q f$, when such a representation is unique?

$\Rightarrow$ f is uniquely determined by q iff Me^{-q} has no zeros [A, http://arxiv.org/abs/1508.02014]

Question I. If $\Pi = \Pi_q f$ for unique q and f, how to find q and f?

Question. How to decide whether Π is in image of Π_q?

$\Downarrow$ We know that $\Pi_q = R_q^{h''}$, $h''(t) = \max\{0, 1-t\}$.

Question'. How to decide whether F is in image of $R_q^{h''}$?

$\Downarrow$ We know the answer for the Laplace transform $R_{q'}^{h'}$, where $h'(t) = e^{-t}$, $q'(x) = x_1 + \cdots + x_n$

Theorem (S. Bernstein, V. Hilbert, S. Bochner)

Let $F \in \mathbb{R}^{\mathbb{R}^n_{\geq 0}}$. Then $F = R_{q'}^{h'}\mu$ for some finite Borel $\mu \geq 0$ iff F is completely monotone, i.e. F is smooth with non-negative even derivatives and non-positive odd derivatives.

Question". How to relate the image of $R_q^{h''}$ to the image of $R_{q'}^{h'}$?

Characterization

Question". How to relate the image of $R_q^{h''}$ to the image of $R_{q'}^{h'}$?

- A remarkable property

$$M(R_q^{h''}F) = \Gamma^{-1} \cdot MF \cdot Me^{-q} \cdot Mh'',$$
$$M(R_{q'}^{h'}F) = \Gamma^{-1} \cdot MF \cdot Me^{-q'} \cdot Mh'$$

- It implies

$$R_{q'}^{h'}F = M^{-1}\frac{Me^{-q'} \cdot Mh'}{Me^{-q} \cdot Mh''}M(R_q^h F)$$

- Explicit formulas:

$$Me^{-q'}(z) = \Gamma(z_1) \cdots \Gamma(z_n),$$
$$Mh'(s) = \Gamma(s),$$
$$Mh''(s) = \tfrac{1}{s(s+1)}.$$

Characterization

Set $T_q = M^{-1}\rho_q M$, where $\rho_q(z) = \frac{\Gamma(z_1+\dots+z_n+2)\Gamma(z_1)\cdots\Gamma(z_n)}{Me^{-q}(z)}$.
Denote $\mathcal{Q}_c^{\mathrm{reg}}(n) = \{q \in \mathcal{Q}(n) \colon \rho_q \in L^2 \cup L^\infty(\Re z = c)\}$

Theorem (A, Func. Ann. App. 2015)

Let $q \in \mathcal{Q}_c^{reg}(n)$, $c \in \mathbb{R}^n_{>0}$. Then $\Pi \in \mathbb{R}^{\mathbb{R}^n_{\geq 0}}$ is of the form $\Pi = \Pi_q \mu$ with Borel $\mu \geq 0$ such that $\int x^{-c} d\mu < \infty$ iff

$$\Pi \in L_c^2(\mathbb{R}^n_{\geq 0}),\ T_q \Pi \in L_c^1(\mathbb{R}^n_{\geq 0}),\ T_q \Pi \text{ is completely monotone}$$

Main questions

Question E. Determine the scope of the model. When a given profit function Π of an industry can be constructed using our framework, i.e. $\Pi = \Pi_q f$ for some q, f?

$\Rightarrow$ $\Pi = \Pi_q f$ for some f iff $T_q \Pi$ is completely monotone, where T_q is a Mellin multiplier given by an explicit formula [A, Funk. Anal. Appl. 2015]

Question U. If $\Pi = \Pi_q f$, when such a representation is unique?

$\Rightarrow$ f is uniquely determined by q iff Me^{-q} has no zeros [A, http://arxiv.org/abs/1508.02014]

Question I. If $\Pi = \Pi_q f$ for some q and f, how to find q and f?

Question. If $\Pi = \Pi_q f$ and q is known, how to find f?

- A remarkable property:

$$M(\Pi_q f) = \Gamma^{-1} \cdot Me^{-q} \cdot Mf$$

- N-smooth functions $C_c^{N,\sigma}(\mathbb{R}^n_{>0})$, N, $\sigma > n$, with finite norm

$$\|f\|_{C_c^{N,\sigma}} = \sup_{|\alpha| \le N, y \in \mathbb{R}^n} (1 + |y|)^{\frac{\sigma}{n}} \left| \frac{\partial^{|\alpha|} u}{\partial y^\alpha} \right|, \quad u(y) = e^{cy} f(e^y)$$

Theorem (A, Proc. of MIPT 2014)

Let $q \in \mathcal{Q}(n)$, $Me^{-q}(z) \neq 0$ for $\Re z = c$ a.e., $f \in C_{I-c}^{N,\sigma}(\mathbb{R}^n_{>0})$, $\Pi = \Pi_q f$. Set $s = z_1 + \cdots + z_n$. Then $f = f_R + f_R^{err}$, where

$$f_R(x) = (2\pi)^{-n} \int_{c + iB_R} \frac{x^{z-I} \Gamma(s+2)}{(Me^{-q})(z)} \cdot (M\Pi)(z)\, dz,$$

$$\|f_R^{err}\|_{C_{I-c}^{N,\sigma}} \le C(n, N, \sigma) \|f\|_{C_{I-c}^{N,\sigma}} R^{n-N}.$$

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$\Rightarrow$ explicit inversion formula [A, Proceedings of MIPT 2014]

The class $\mathcal{Q}_{CES}(n)$ of *CES* technologies

$$q(x) = C(a_1 x_1^\alpha + \cdots + a_n x_n^\alpha)^{\frac{1}{\alpha}},$$

where $C > 0$, $a_j \geq 0$, $a_1 + \cdots + a_n = 1$ (and $\alpha \in (0, 1]$).

$\Rightarrow$ If $\Pi_{q_1} f_1 = \Pi_{q_2} f_2$, $q_1 \neq q_2$ are $\mathcal{Q}_{\mathrm{CES}}(n)$, and f_1, $f_2 \geq 0$ decay fast, then $f_1 = f_2 = 0$. Otherwise, there are counter-examples [A, Proceedings of MIPT 2014]

$\Rightarrow$ There is a more simple characterization [A, Proceedings of MIPT 2013]

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