Pricing Asian options through series expansions

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The spectral method developed by V. Linetsky and his coauthors can be applied to pricing of Asian options in the Black-Scholes-Merton framework. In the first section I give an overview of this method. The section is based on (Linetsky, 2002), (Linetsky, 2004a), (Linetsky, 2004b), (Linetsky, 2007).

In the second section I show how this approach can be directly generalized to pricing of Asian options on the short rate following the geometric Brownian motion as in (Gothan, 1978) and (Rendleman & Bartter, 1980).

In the Appendix I give a short overview of the spectral method in the general setting. The approach is illustrated by pricing a European bond option for a generic short rate process. The section is based on (Gorovoi & Linetsky, 2004).

Contents

1 Overview: Pricing Asian options using spectral expansions

1.1 Problem setting and simplification

The spectral method developed in a series of articles by V. Linetsky and his coauthors can be used to compute expectations of the form:

$$\mathbb{E}[e^{-\int_0^T r(X_u)du} f(X_u)|X_0 = x], \tag{1}$$

where r is a non-negative function of X_t and X_t is a diffusion process with time-independent drift and diffusion coefficients:

$$dX_t = b(X_t)dt + a(X_t)dW_t. \tag{2}$$

In particular, the method can be applied to pricing of Asian options in the Black-Scholes-Merton framework as explained in the present section.

In the Black-Scholes-Merton framework the risk-neutral dynamics of the underlying is given by the geometric Brownian motion:

$$S_t = S_0 e^{(r-q-\sigma^2/2)t+\sigma W_t}, \quad t \geq 0. \tag{3}$$

The goal is to compute the time zero price of an Asian put maturing at time T:

$$C(K,T) = e^{-rT}\mathbb{E}\left(K - \frac{1}{T}\int_0^T S_u\,du\right)^+. \tag{4}$$

Calls can be subsequently priced using the put-call parity for Asian options. The advantage of using puts is the boundedness of the payoff. We can standardize expression (4) as follows:

$$e^{-rT}\mathbb{E}\left(K - \frac{1}{T}\int_0^T S_u\,du\right)^+ = e^{-rT}\left(\frac{4S_0}{\sigma^2 T}\right)\mathbb{E}(k - X_\tau)^+, \tag{5}$$

$$X_\tau = \int_0^\tau e^{2(W_u+\nu u)}du, \quad k = \frac{\tau K}{S_0}, \quad \tau = \frac{\sigma^2 T}{4},\ \nu = \frac{2(r-q)}{\sigma^2} - 1. \tag{6}$$

One can show that X_t satisfies the following stochastic differential equation:

$$dX_t = \big(2(\nu+1)X_t + 1\big)dt + 2X_t\,dW_t, \quad X_0 = 0. \tag{7}$$

This reduces the problem to computation of the expectation $\mathbb{E}(k - X_\tau)^+$ for fixed k, τ, which fits into framework (1), (2).

1.2 Pricing vanilla European options on diffusions

We will compute the expectation $\mathbb{E}(k - X_\tau)^+$ for a general process X_t satisfying:

$$dX_t = b(X_t)dt + a(X_t)dW_t, \quad X_0 = 0. \tag{8}$$

For the diffusion process X_t one defines the scale density $s(x)$ and the speed density $m(x)$ as follows, see Figure 9:

$$s(x) = \exp\left(-\int \frac{2b(x)}{a^2(x)}dx\right), \quad m(x) = \frac{2}{a^2(x)s(x)}. \tag{9}$$

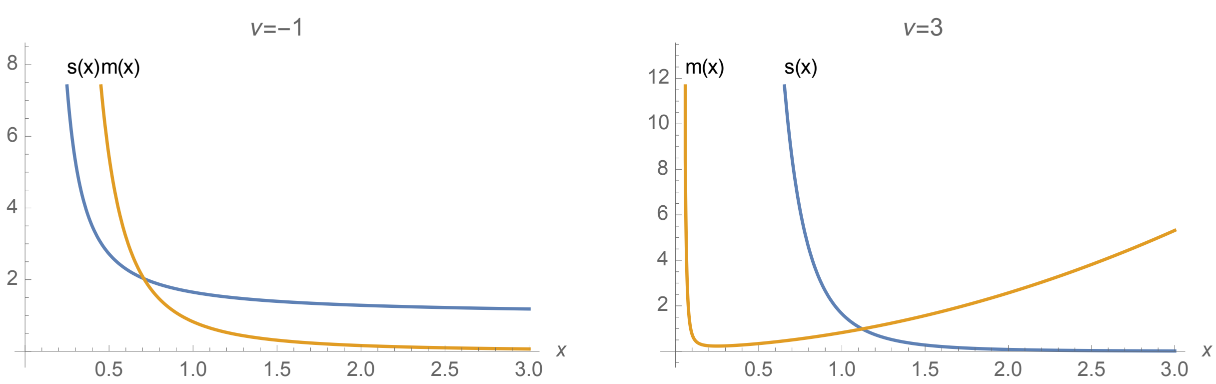

Figure 1: Functions $s(x)$ and $m(x)$ from (9) for $a(x) = 2x$, $b(x) = 2(\nu+1)x + 1$

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We denote by $p(t; x, y)$ the transition probability density for X_t with respect to the speed measure $m(x)dx$. Then $(k - x)^+ \in L^2([0, \infty), m(x)dx)$ and

$$\mathbb{E}(k - X_\tau)^+ = \int_0^k (k - y)p(\tau; 0, y)\, m(y)dy. \tag{10}$$

The spectral method is based on computing the spectral expansion of $p(\tau; 0, y)$ and plugging it into the above formula.

Remark 1 (Spectral expansions)**.** *Operators*

$$P_t f(x) = \mathbb{E}[f(X_t)|X_0 = x] = \int f(y)p(t, x, y)m(y)dy \tag{11}$$

acting on $L^2([0, \infty), m(x)dx)$ *form a strongly continuous self-adjoint contraction semigroup with non-positive infinitesimal generator:*

$$Gu(x) = \tfrac{1}{2}a^2(x)u''(x) + b(x)u'(x), \tag{12}$$

so that formally $P_t = e^{-tG}$*. The following spectral expansions are valid:*

$$-G = \int_{[0,\infty)} \lambda E(d\lambda), \quad P_t = \int_{[0,\infty)} e^{-\lambda t} E(d\lambda), \tag{13}$$

where $E(d\lambda)$ *is a projector-valued measure such that* $E(d\lambda)f$ *is the projection of* f *on the eigenspace of* $-G$ *corresponding to the eigenvalue* λ*.*

If the diffusion is confined to a bounded interval $[e_1, e_2]$*, the spectral expansion for* P_t *collapses to a sum:*

$$P_t f(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x) \int \phi_n(y)f(y)m(y)\, dy, \tag{14}$$

where λ_n *are eigenvalues and* ϕ_n *the normalized eigenfunctions of* $-G$*. Comparing with* (11) *we get the spectral expansion for the transition density with respect to the speed measure:*

$$p(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y). \tag{15}$$

To simplify the spectral expansion we will artificially confine the diffusion X_t to a bounded interval by placing an absorbing barrier at a large level B. Note that:

$$\mathbb{E}[\mathbf{1}_{\tau\le\tau_B}(k - X_\tau)^+] \to \mathbb{E}(k - X_\tau)^+, \quad B \to \infty, \tag{16}$$

where τ_B is the first hitting time of level $B > k$:

$$\tau_B = \inf\{\tau \ge 0 : X_\tau = B\}. \tag{17}$$

Let X_t^B be a diffusion obtained from X_t by putting an absorbing barrier at level B. Then:

$$\mathbb{E}[\mathbf{1}_{\tau\le\tau_B}(k - X_\tau)^+] = \mathbb{E}(k - X_\tau^B)^+. \tag{18}$$

Remark 2. *The approximation error in our example with the Asian put in the BSM framework can be estimated as follows, see (Linetsky, 2004a):*

$$\left|\mathbb{E}(k - X_\tau^B)^+ - \mathbb{E}(k - X_\tau)^+\right| \le \frac{\tau k}{B} \frac{e^{2(\nu+1)\tau} - 1}{2(\nu+1)\tau}, \quad \nu > -1. \tag{19}$$

The next goal is to compute the expectation:

$$\mathbb{E}(k - X_\tau^B)^+ = \int_0^k (k - y) p_B(\tau; 0, y) m(y)\, dy, \tag{20}$$

where $p_B(t; x, y)$ is the transition probability density of X_t^B. The diffusion X_t^B is confined to $[0, B]$. Therefore the spectral expansion of $p_B(t; 0, y)$ will only contain a sum over eigenvalues and is numerically tractable.

1.3 Sturm-Liouville problem

The spectral expansion of $p_B(t; x, y)$ is given by the series

$$p_B(t; x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y), \tag{21}$$

which converges uniformly on $[0, B] \times [0, B]$ at fixed $t > 0$. Here λ_n, ϕ_n are eigenvalues and normalized eigenfunctions of the Sturm-Liouville problem:

$$\tfrac{1}{2} a^2(x) u''(x) + b(x) u'(x) = -\lambda u(x), \quad 0 < x < B, \tag{22}$$

$$\lim_{x \to +0} \frac{u'(x)}{s(x)} = 0, \quad u(B) = 0, \tag{23}$$

where s is the scale density defined in (9). For generic functions $a(x)$ and $b(x)$ we can solve this Sturm-Liouville problem numerically.

In our particular case with $a(x) = 2x$ and $b(x) = 2(\nu + 1)x + 1$ we can solve the Sturm-Liouville problem analytically through the substitution:

$$z = \frac{1}{2x}, \quad u(x) = (2x)^{(1-\nu)/2} e^{1/(4x)} v(z). \tag{24}$$

The function u satisfies the Sturm-Liouville equation if and only if v satisfies the Schrödinger equation:

$$v_{zz} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2} \right) v = 0, \tag{25}$$

$$\kappa = \frac{1 - \nu}{2}, \quad \mu = \tfrac{1}{2}\sqrt{\nu^2 - 2\lambda} \ \text{(principal branch)}. \tag{26}$$

The basis of solutions of this Schrödinger equation is given by the Whittaker functions $W_{\kappa,\mu}(z)$ and $W_{-\kappa,\mu}(e^{i\pi}z)$. The Whittaker functions have the following asymptotic behavior:

$$W_{\kappa,\mu}(z) \sim e^{-z/2} z^{\kappa}, \quad z \to \infty, \quad |\arg z| < \frac{3\pi}{2} - 0, \tag{27}$$

so that $W_{\kappa,\mu}(z)$ decays exponentially and $W_{-\kappa,\mu}(e^{i\pi}z)$ exhibits exponential growth for $z \to \infty$, $\arg z = 0$. We set:

$$\psi(x,\lambda) = (2x)^{(1-\nu)/2} e^{1/(4x)} W_{\kappa,\mu}\left(\frac{1}{2x}\right), \tag{28}$$

so that $\psi(+0,\lambda) = 1$. It follows that $\psi(x,\lambda)$ satisfies the Sturm-Liouville equation (22) and the boundary condition at zero.

Now we will restrict our attention to the case $\nu \geq -1$. Then $\psi_(x,\lambda)$ satisfies the boundary condition $\psi(B,\lambda) = 0$ if and only if λ satisfies:

$$W_{\frac{1-\nu}{2}, ip/2}\left(\frac{1}{2B}\right) = 0, \quad \lambda = (\nu^2 + p^2)/2. \tag{29}$$

This equation has an infinite number of simple roots $0 < p_{1,B} < p_{2,B} < \cdots$ and corresponding eigenvalues $0 < \lambda_{1,B} < \lambda_{2,B} < \cdots$.

Proposition 1. *The eigenvalues $\lambda_{n,B}$ have the following asymptotic behavior, see Figure 2:*

$$\lambda_{n,B} \approx \frac{\frac{1}{2} y^2(n)}{\left(\ln\left(\frac{4By(n)}{e}\right) - \ln\ln\left(\frac{4By(n)}{e}\right) + \frac{\ln\ln(4By(n)/e)}{\ln(4By(n)/e)}\right)^2}, \quad y(n) = 2\pi(n + \frac{\nu}{4} - \frac{1}{2}). \tag{30}$$

Proof. The Whittaker function $W_{k,i\rho}(z)$ at fixed $\kappa \in \mathbb{R}$ has the following asymptotic behaviour:

$$W_{\kappa,i\rho}(z) = \sqrt{2z} e^{-\pi\rho/2} \rho^{\kappa - \frac{1}{2}} \cos\left(\rho \ln\left(\frac{z}{4\rho}\right) + \rho - (\kappa - \tfrac{1}{2})\frac{\pi}{2}\right)\left(1 + O(\rho^{-1})\right), \quad \rho \to \infty. \tag{31}$$

locally uniformly in $z > 0$. It follows that $p_{n,B} = \widetilde{p}_{n,B} + O(1)$, where $\widetilde{p}_{n,B}$ solves:

$$\widetilde{p}_{n,B}\left(\ln(4B\widetilde{p}_{n,B}) - 1\right) = y(n). \tag{32}$$

This equation can be solved in terms of the Lambert W function, defined implicitly through the equation $W(z)e^{W(z)} = z$:

$$\widetilde{p}_{n,B} = \frac{y(n)}{W\left(4By(n)/e\right)}. \tag{33}$$

This implies the statement of the Lemma taking into account the asymptotic of the Lambert W function:

$$W(z) = \ln z - \ln\ln z + \frac{\ln\ln z}{\ln z} + O\left(\left(\frac{\ln\ln z}{\ln z}\right)^2\right), \quad z \to \infty. \tag{34}$$

$\square$

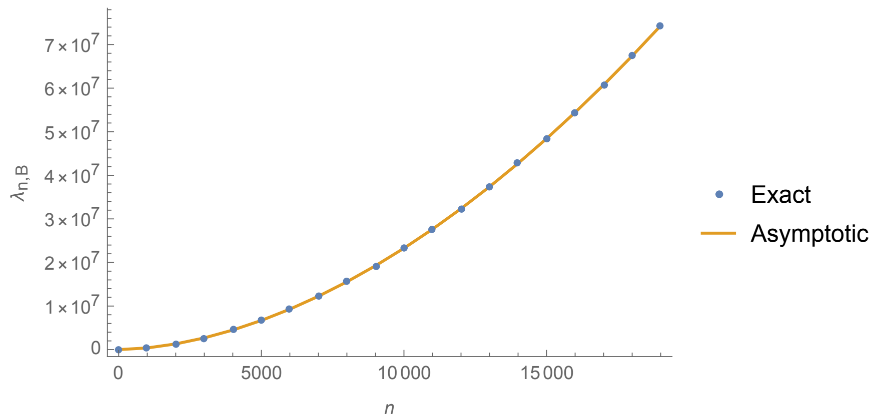

Figure 2: Eigenvalues $\lambda_{n,B}$ for $\nu = 3$, $B = 1$ and their asymptotic approximation (30) with three terms in the denominator.

Eigenvalues $\lambda_{n,B}$ and eigenfunctions $\psi(x, \lambda_{n,B})$ provide the solution to the Sturm-Liouville problem. It remains to normalize the eigenfunctions $\psi(x, \lambda_{n,B})$ by their norms $N_n^{-1} = \|\psi(\cdot, \lambda_{n,B})\|$. This gives the following explicit expression:

$$\phi_n(x) = N_n (2x)^{(1-\nu)/2} e^{1/(4x)} W_{(1-\nu)/2, ip_{n,B}/2}\left(\frac{1}{2x}\right), \tag{35}$$

$$N_n^2 = \frac{2^\nu p_{n,B} \Gamma(\frac{1}{2}(\nu + ip_{n,B})}{\Gamma(1 + ip_{n,B})\xi_{n,B}} M_{(1-\nu)/2, ip_{n,B}/2}\left(\frac{1}{2B}\right), \tag{36}$$

$$\xi_{n,B} = \frac{\partial}{\partial p} W_{(1-\nu)/2, ip/2}\left(\frac{1}{2B}\right)\Bigg|_{p=p_{n,B}}. \tag{37}$$

Some of the normalized eigenfunctions ϕ_n are illustrated in Figure 3. Eigenfunctions have a finite limit at $x = 0$ and vanish at $x = B$.

1.4 Final expansion

The desired spectral expansion of the kernel p_B reads:

$$p_B(t; x, y) = \sum_{n=1}^{\infty} e^{-\lambda_{n,B} t} \phi_n(x) \phi_n(y), \tag{38}$$

see Figure 4. The kernel is symmetric, it vanishes for $x = B$ and for $y = B$ and it tends to 0 for $t \to \infty$. For each fixed $t > 0$ the series converges uniformly on $[0, B] \times [0, B]$ and the convergence is faster for bigger t. The uniform norms of terms for different t are shown in Figure 5. We plug it into the expression for the expectation (20):

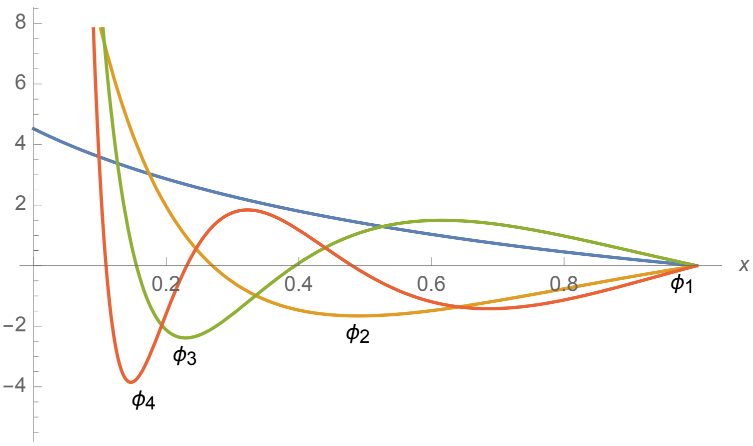

Figure 3: Normalized eigenfunctions (35) for the model (3) with $r = 0.05$, $\sigma = 0.4$, $q = 0$. The cutoff parameter is $B = 1$.

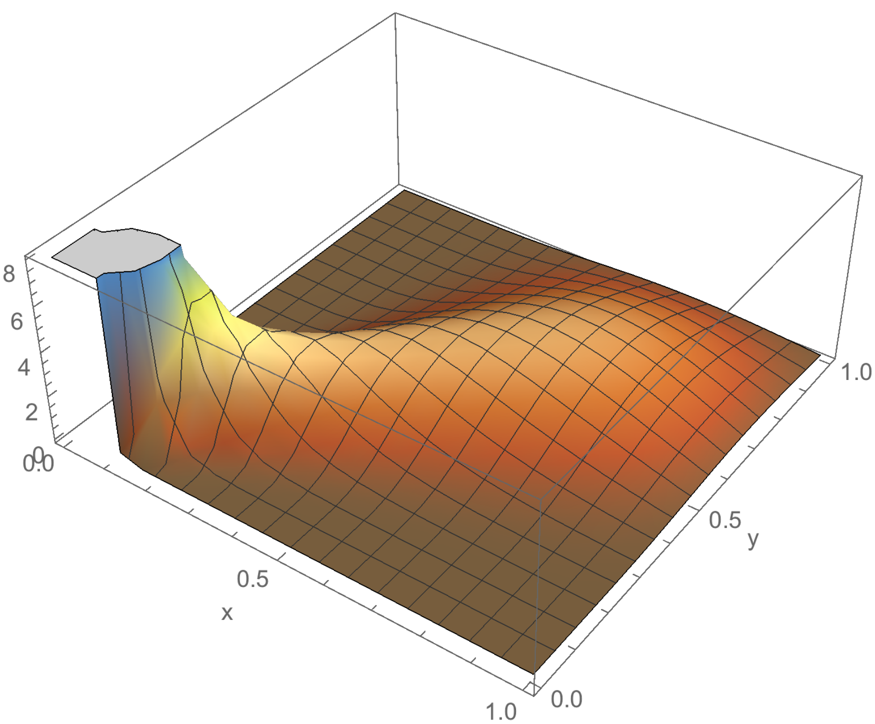

Figure 4: Kernel $p_B(t; x, y)$ of (38) for $t = \sigma^2 T/4$, $T = 1$, $r = q = 0$, $\sigma = 0.4$, $B = 1$.

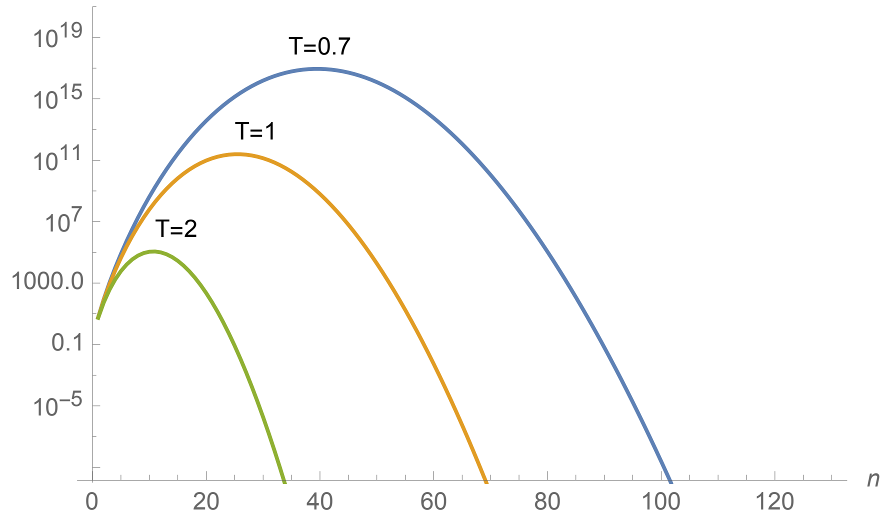

Figure 5: Uniform norms of the terms in the expansion (38) for $t = \sigma^2 T/4$ with different T. The model parameters are $r = q$, $\sigma = 0.4$, $B = 1$. All the terms after the crossing point with the n-axis have the uniform norm smaller than 10^{-10}.

$$\mathbb{E}(k - X_\tau^B)^+ = \int_0^k (k-y) p_B(\tau; 0, y) m(y) dy \tag{39}$$

$$= \sum_{n=1}^{\infty} e^{-\lambda_{n,B}\tau} \frac{1}{\|\psi(\cdot, \lambda_{n,B})\|^2} \int_0^k (k-y)\psi(y, \lambda_{n,B}) m(y) dy, \tag{40}$$

where we have used that $\psi(0, \lambda_{n,B}) = 1$. The latter integral can be found in integral tables:

$$\int_0^k (k-y)\psi(y, \lambda_{n,B}) m(y)\, dy \tag{41}$$

$$= 2^{-(\nu+1)/2} \int_0^k (k-x) x^{(\nu-1)/2} e^{-1/(4x)} W_{\frac{1-\nu}{2}, \frac{1}{2}\sqrt{\nu^2 - 2\lambda_{n,B}}} \left(\frac{1}{2x}\right) dx \tag{42}$$

$$= 2^{-(\nu+1)/2} k^{(\nu+3)/2} e^{-1/(4k)} W_{-\frac{\nu+3}{2}, \frac{1}{2}\sqrt{\nu^2 - 2\lambda_{n,B}}} \left(\frac{1}{2k}\right). \tag{43}$$

Bringing all together, we get the desired expansion for $\mathbb{E}(k - X_\tau^B)$:

$$\mathbb{E}(k - X_\tau^B)^+ = \sum_{n=1}^{\infty} e^{-\lambda_{n,B}\tau} \frac{2^{-(\nu+1)/2} k^{(\nu+3)/2} e^{-1/(4k)} W_{-\frac{\nu+3}{2}, \frac{1}{2}\sqrt{\nu^2 - 2\lambda_{n,B}}} \left(\frac{1}{2k}\right)}{\|\psi(\cdot, \lambda_{n,B})\|^2}. \tag{44}$$

This leads to the following expression for the call price of (Linetsky, 2004):

$$\begin{aligned} C(K,T) &= \frac{(1-e^{-rT})S_0}{rT} - e^{-rT}K \\ &+ \frac{4e^{-rT}S_0}{\sigma^2 T} \sum_{n=1}^{\infty} e^{-\frac{\nu^2 + p_n^2}{2}\tau} \frac{p_n \Gamma\left(\frac{\nu + ip_n}{2}\right)}{\xi_n \Gamma(1 + ip_n)} (2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2}, \frac{ip_n}{2}} \left(\frac{1}{2k}\right) {}_1F_1\left(\frac{1-\nu}{2}; \frac{ip}{2}; \frac{1}{2b}\right). \end{aligned} \tag{45}$$

2 Pricing through Lie theory and Weyl-Titchmarsch approach to spectral expansions

2.1 Reduction of the pricing PDE using Lie symmetries

We assume that the risk-neutral dynamics of the asset price is given by the geometric Brownian motion:

$$S_t = S_0 \exp\left((r - \tfrac{\sigma^2}{2})t + \sigma W_t\right), \quad t \geq 0. \tag{46}$$

Our objective is to compute the price of an Asian call:

$$C(T, K) = V(0, S_0, 0), \tag{47a}$$

$$V(t, S_t, A_t) = e^{-r(t-T)}\mathbb{E}\left[(A_T - K)^+ \,|\mathcal{F}_t\right], \quad 0 \leq t \leq T, \tag{47b}$$

$$A_t = \beta \int_0^t S_u du, \quad \beta = \frac{1}{T}. \tag{47c}$$

By the Feynman-Kac formula $V(t, x, y)$ satisfies the PDE

$$V_t + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + rxV_x + \beta x V_y - rV = 0, \quad t \in (0, T), x, y > 0, \tag{48}$$

together with the boundary conditions:

$$V(T, x, y) = (y - K)^+, \tag{49a}$$

$$V(t, 0, y) = e^{-r(T-t)}(y - K)^+, \tag{49b}$$

$$V(t, x, y) \sim \beta(T - t)e^{-r(T-t)}x, \quad x \to \infty, \tag{49c}$$

$$V(t, x, K) = \tfrac{\beta}{r}\left(1 - e^{-r(T-t)}\right)x. \tag{49d}$$

The boundary condition (49d) is specific for Asian calls and allows to reduce the domain of y to $(0, K)$. It is based on the observation that as soon as A_t reaches K it can not become smaller then K.

We will reduce the problem (48), (49) to a Sturm-Liouville problem for an ODE using the Lie theory.

Proposition 2. *Equation* (48) *and boundary conditions* (49) *are invariant with respect to the one-parameter Lie group of transformations generated by the vector field:*

$$\widehat{\Gamma} = V\partial_V + x\partial_x + (y - K)\partial_y. \tag{50}$$

Besides, a function $V(t, x, y)$ *is a solution to* (48), (49) *invariant with respect to these transformations if and only if*

$$V(t, x, y) = x\phi\left(t, \frac{y - K}{x}\right), \tag{51}$$

where $\phi(t, \xi)$ *satisfies the PDE*

$$\phi_t + \tfrac{1}{2}\sigma^2\xi^2\phi_{\xi\xi} + (b - \xi r)\phi_\xi = 0, \quad t \in (0, T), \xi < 0, \tag{52}$$

together with the boundary conditions:

$$\phi(T,\xi) = 0, \tag{53a}$$
$$\phi(t,0) = \tfrac{\beta}{r}(1 - e^{-r(T-t)}), \tag{53b}$$
$$\phi(t,\xi) \to 0, \quad \xi \to -\infty. \tag{53c}$$

Proof. Equation (48) admits six one-parameter Lie point symmetries whose infinitesimal generators are given by:

$$\begin{aligned}
\Gamma_1 &= \partial_t, \\
\Gamma_2 &= \partial_y, \\
\Gamma_3 &= V\partial_V, \\
\Gamma_4 &= x\partial_x + y\partial_y, \\
\Gamma_5 &= xy\partial_x + \tfrac{1}{2}y^2\partial_y + \sigma^{-2}(\beta x - ry)V\partial_V, \\
\Gamma_6 &= g(t,x,y)\partial_V,
\end{aligned} \tag{54}$$

where $g(t,x,y)$ is an arbitrary solution to (48). Put $\Gamma = \sum_{i=1}^{5} \alpha_i \Gamma_i$. The first boundary condition from (49) is invariant under transformations generated by Γ for $y > K$ provided:

$$\Gamma\left[V - (y-K)\right]\Big|_{V=y-K,t=T} = 0, \quad \Gamma\left[t - T\right]\Big|_{t=T} = 0, \tag{55}$$

which can be rewritten explicitly as follows:

$$-\alpha_2 + (y-K)\left(\alpha_3 - \frac{2\alpha_5 ry}{\sigma^2} + \frac{2\alpha_5\beta x}{\sigma^2}\right) - \alpha_5 y^2 - \alpha_4 y = 0, \quad \alpha_1 = 0. \tag{56}$$

This expression needs to vanish identically for all x and for all $y > K$, which leads to the following restrictions on the coefficients of Γ:

$$\alpha_1 = 0,\ \alpha_2 + \alpha_3 K = 0,\ \alpha_3 = \alpha_4,\ \alpha_5 = 0. \tag{57}$$

The remaining boundary conditions are invariant for this choice of coefficients as well. This shows that the PDE (48) and the boundary conditions (49) are invariant with repsect to the Lie group of transformations generated by the vector field

$$\widehat{\Gamma} \overset{def}{=} \Gamma|_{(57),\alpha_3=1}. \tag{58}$$

A function $V = f(t,x,y)$ is invariant with respect to the transformations generated by $\widehat{\Gamma}$ provided it satisfies the invariant surface condition:

$$\widehat{\Gamma}[V - f(t,x,y)]\Big|_{V=f(t,x,y)} = V - xV_x - (y-K)V_y = 0. \tag{59}$$

Solving this first order PDE for V we find:

$$V(t,x,y) = x\phi\left(t, \frac{y-K}{x}\right), \tag{60}$$

where $\phi(t,\xi)$ is an arbitrary function. Plugging this expression into (48) we get the PDE (52) satisfied by ϕ which guarantees that V satisfies the PDE (48). Plugging the ansatz (60) into (49) we find the boundary conditions (53) for ϕ which guarantee that V satisfies the boundary conditions (49). This finishes the proof of the present proposition. □

This reduction of Proposition 2 goes back to (Rogers & Shi, 1995), who discovered the change of variables $y \to \xi$, $\xi = (y-K)/x$ but did not deduce it from the Lie symmetry analysis. This reduction is also the starting point for the approach of (Lewis, 2002). In (Caister et al., 2010) this reduction is derived in systematic way using Lie theory.

2.2 Weyl-Titchmarsch approach to spectral expansions

It is convenient to make the following change of variables in the reduced problem (52), (53):

$$w = -\xi, \quad \tau = T - t, \quad \psi(\tau, w) = \phi(t, \xi), \tag{61}$$

so that the transformed problem takes the form:

$$\frac{1}{2}w^2\sigma^2\psi_{ww} - (\beta + \xi r)\psi_w - \psi_\tau = 0, \tag{62a}$$

$$\psi(0, w) = 0, \tag{62b}$$

$$\psi(\tau, 0) = \frac{\beta}{r}(1 - e^{-r\tau}), \tag{62c}$$

$$\psi(\tau, w) =\to 0, \quad w \to +\infty. \tag{62d}$$

We will get a spectral expansion for the solution of (62) using the Weyl-Titchmarsch approach going back to (Titchmarsch, 1962). The first step is to take the Laplace transform of $\psi(\tau, w)$ with respect to τ:

$$\Psi(s, w) = \int_0^\infty e^{-\gamma\tau}\psi(\tau, w)\, d\tau. \tag{63}$$

This transforms the problem (62) as follows:

$$\frac{1}{2}w^2\sigma^2\Psi_{ww} - (\beta + wr)\Psi_w - s\Psi = 0, \tag{64a}$$

$$\Psi(s, 0) = \frac{\beta}{s(s+r)}, \tag{64b}$$

$$\Psi(s, w) \to 0, \quad w \to +\infty. \tag{64c}$$

This Sturm-Lioville problem has the following solution:

$$\begin{gathered}\Psi(s, w) = C\left(\frac{2\beta}{\sigma^2 w}\right)^k {}_1F_1\left(k; 2\left(k + \frac{r}{\sigma^2} + 1\right); -\frac{2\beta}{w\sigma^2}\right), \\ k = \frac{1}{2}\sqrt{\frac{8s}{\sigma^2} + \left(\frac{2r}{\sigma^2} + 1\right)^2} - \frac{r}{\sigma^2} - \frac{1}{2}, \quad C = \frac{\beta}{s(s+r)}\frac{\Gamma(2 + k + 2r/\sigma^2)}{\Gamma(2\left(1 + k + r/\sigma^2\right)}.\end{gathered} \tag{65}$$

The next step of the approach consists in computing the inverse Laplace transform using the Bromwich integral:

$$\psi(\tau, w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\tau s} \Psi(s, w)\, ds, \tag{66}$$

where the integration contour lies to the right of all the singularities of $\Psi(s, w)$ in the s-plane. Computation of this integral using the residue theorem leads to the spectral expansion.

To apply the residue theorem we need to determine singularities of $\Psi(s, w)$. One can see from (65) that $\Psi(s, w)$ has first order poles at $s = 0$ and at $s = -r$ with the residues:

$$\operatorname*{Res}_{s=0}[e^{\tau\gamma}\Psi(s, w)] = \frac{\beta}{r}, \tag{67}$$

$$\operatorname*{Res}_{s=-r}[e^{\tau s}\Psi(s, w)] = -\frac{e^{-r\tau}}{r} \begin{cases} rw + \beta, & r > \frac{\sigma^2}{2}, \\ \frac{\beta\left(\frac{2\beta}{w\sigma^2}\right)^{-\frac{2r}{\sigma^2}}}{\Gamma(2-\frac{2r}{\sigma^2})} {}_1F_1(\frac{-2r}{\sigma^2}; 2 - \frac{2r}{\sigma^2}; -\frac{2\beta}{\sigma^2 w}), & r \leq \frac{\sigma^2}{2}. \end{cases} \tag{68}$$

Besides, $\Psi(s, w)$ has a branch point

$$s^* = -\frac{r^2}{2\sigma^2} - \frac{r}{2} - \frac{\sigma^2}{8}, \tag{69}$$

where the square root vanishes. To apply the residue theorem we make a branch cut along the real axis from $-\infty$ to s^* and use the integration contour shown in Fig 6.

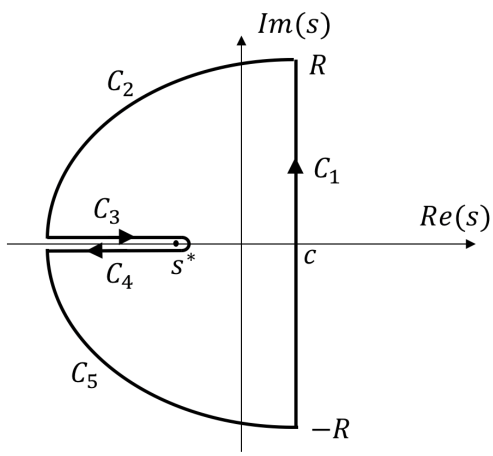

Figure 6: Integration contour for the Bromwich integral (66)

This leads to the following formula:

$$\psi(\tau, w) = -\frac{1}{2\pi i} \lim_{R\to+\infty} \int\limits_{C_2+C_3+C_4+C_5} e^{s\tau}\Psi(s, w) ds + \operatorname*{Res}_{s=0}[e^{s\tau}\Psi(s, w)] + \operatorname*{Res}_{s=-r}[e^{s\tau}\Psi(s, w)]. \tag{70}$$

The integrals over C_2 and C_5 tend to zero as $R \to +\infty$. To compute the integrals over C_3 and C_4 we make the changes of variables:

$$\begin{aligned} s &= s^* + \lambda^2 e^{i\pi} \quad \text{in } C_3, \\ s &= s^* + \lambda^2 e^{-i\pi} \quad \text{in } C_4, \end{aligned} \tag{71}$$

where $\lambda > 0$. Then:

$$\lim_{R\to+\infty} \int_{C_3+C_4} e^{s\tau}\Psi(s,w)ds \tag{72}$$

$$= e^{s^*\tau} \int_{\infty}^{0} e^{-\lambda^2\tau}\Psi(s^* + \lambda^2 e^{i\pi}, w)(-2\lambda)d\lambda + e^{s^*\tau} \int_{0}^{\infty} e^{-\lambda^2\tau}\Psi(s^* + \lambda^2 e^{-i\pi}, w)(-2\lambda)d\lambda \tag{73}$$

$$= -4ie^{s^*\tau} \int_0^\infty e^{-\lambda^2\tau} \operatorname{Im}\left[\Psi(s^* + \lambda^2 e^{-i\pi}, w)\right] \lambda d\lambda. \tag{74}$$

This leads to the following spectral expansion:

$$\psi(\tau, w) = \frac{2}{\pi} \int_0^\infty e^{(s^*-\lambda^2)\tau} \operatorname{Im}\left[\Psi(s^* + \lambda^2 e^{-i\pi}, w)\right] \lambda d\lambda + \operatorname*{Res}_{s=0}[e^{s\tau}\Psi(s,w)] + \operatorname*{Res}_{s=-r}[e^{s\tau}\Psi(s,w)]. \tag{75}$$

The integral can be transformed as follows:

$$\begin{gathered} \frac{2}{\pi} \int_0^\infty e^{(s^*-\lambda^2)\tau} \operatorname{Im}\left[\Psi(s^* + \lambda^2 e^{-i\pi}, w)\right] \lambda d\lambda \\ = \frac{\beta e^{-r\tau}}{2\pi^2\sigma^2} \int_0^\infty \frac{p\left|\Gamma\left(\frac{1}{2}(ip+\nu)\right)\right|^2 U\left(2 + \frac{ip}{2} + \frac{\nu}{2}, 1 + ip, \frac{2\beta}{w\sigma^2}\right)\sinh(p\pi)}{e^{\frac{2\beta}{w\sigma^2} + \frac{1}{8}(p^2+\nu^2)\sigma^2\tau} \left(\frac{2\beta}{w\sigma^2}\right)^{1-\frac{ip}{2}+\frac{\nu}{2}}} dp, \\ \nu = \frac{2r}{\sigma^2} - 1, \quad \lambda = \frac{p\sigma}{2\sqrt{2}}. \end{gathered} \tag{76}$$

Recalling (47a), (51) we get the desired spectral expansion for the call price of (Lewis, 2002):

$$\begin{aligned} C(K,T) = \frac{S_0}{rT} - \frac{e^{-rT}S_0}{rT} &\begin{cases} rTK/S_0 + 1, & r > \frac{\sigma^2}{2}, \\ \frac{\left(\frac{2S_0}{KT\sigma^2}\right)^{-\frac{2r}{\sigma^2}}}{\Gamma(2-\frac{2r}{\sigma^2})} {}_1F_1(\frac{-2r}{\sigma^2}; 2 - \frac{2r}{\sigma^2}; -\frac{2S_0}{KT\sigma^2}), & r \le \frac{\sigma^2}{2}. \end{cases} \\ + \frac{e^{-rT}S_0}{2\pi^2\sigma^2 T} \int_0^\infty &\frac{p\left|\Gamma\left(\frac{1}{2}(ip+\nu)\right)\right|^2 U\left(2 + \frac{ip}{2} + \frac{\nu}{2}, 1 + ip, \frac{2S_0}{KT\sigma^2}\right)\sinh(p\pi)}{e^{\frac{2S_0}{KT\sigma^2} + \frac{1}{8}(p^2+\nu^2)\sigma^2 T} \left(\frac{2S_0}{KT\sigma^2}\right)^{1-\frac{ip}{2}+\frac{\nu}{2}}} dp. \end{aligned} \tag{77}$$

Table 1 shows the call prices computed using Linetsky's series expansion (45) for different cutoffs b and using formula (77). One can see that for larger times to maturity T one needs to take larger b in the series expansion (45).

T	100.	20.	10.	2.	1.	0.5
Linetsky ($b=1$)	0.383829	0.608076	0.652146	0.350095	0.246416	0.172268
Linetsky ($b=2$)	0.383863	0.721465	0.691719	0.350095	0.246416	0.172269
Linetsky ($b=5$)	0.384927	0.781751	0.694896	0.350095	0.246416	0.172269
Linetsky ($b=10$)	0.386913	0.789367	0.694923	0.350095	0.246416	0.172269
Lewis	0.391771	0.790483	0.694923	0.350095	0.246416	0.172269

(78)

Table 1: Asian call option prices computed using Linetsky's formula (45) for different b and using Lewis' formula (77). Here $K = 2$, $r = 0.05$, $\sigma = 0.5$, $S_0 = 2$.

3 Asian options on the short rate in the Rendleman-Bartter model

3.1 Problem statement and simplification

We assume that the short rate follows the geometric Brownian motion:

$$r_t = r_0 \exp\left(\left(\mu - \tfrac{\sigma^2}{2}\right)t + \sigma W_t\right). \tag{79}$$

This model with $\mu = 0$ was used in (Gothan, 1978) and with a generic μ in (Rendleman & Bartter, 1980) and (Marsch & Rosenfeld, 1983). Our goal is to compute the expectation

$$\mathbb{E}\left[e^{-\int_0^T r_t dt}\left(K - \frac{1}{T}\int_0^T r_t\, dt\right)^+\right]. \tag{80}$$

We can rewrite this expression as follows:

$$\frac{4r_0}{\sigma^2 T}\mathbb{E}\big[e^{-\beta X_\tau}\big(k - X_\tau\big)^+\big], \quad X_t = \int_0^\tau e^{2(W_t+\nu\tau)}dt \tag{81}$$

$$\text{where} \quad k = \frac{KT\sigma^2}{4r_0},\ \tau = \frac{\sigma^2 T}{4},\ \beta = \frac{4r_0}{\sigma^2},\ \nu = \frac{2\mu}{\sigma^2} - 1. \tag{82}$$

As in (Linetsky, 2004a) one can show that X_t satisfies

$$dX_t = (2(\nu+1)X_t + 1)dt + 2X_t dW_t, \quad X_0 = 0. \tag{83}$$

This is the same diffusion as in the first section but with a different ν. The problem reduces to computation of the expectation:

$$\mathbb{E}[e^{-\beta X_\tau}(k - X_\tau)^+] \tag{84}$$

The difference with the setting of (Linetsky, 2004a) is the exponential factor.

3.2 Spectral expansion

Following the same approach as in the first section we approximate the expectation by

$$\mathbb{E}[e^{-\beta X_\tau^B}(k - X_\tau^B)^+], \tag{85}$$

where X_τ^B is obtained from X_τ by placing an absorbing barrier at some large level $B > k$. Let $p_B(t; x, y)$ be the transition probability density of the process X_t^B. Then:

$$\mathbb{E}[e^{-\beta X_\tau^B}(k - X_\tau^B)^+] = \int_0^k e^{-\beta x}(k-y)p_B(\tau; 0, y)m(y)\, dy, \tag{86}$$

$$\text{with the speed density} \quad m(y) = \tfrac{1}{2}x^{\nu-1}e^{-1/(2x)}. \tag{87}$$

Now substitute the spectral expansion for the transition density $p_B(t; x, y)$ derived in section 1:

$$p_B(t; x, y) = \sum_{n=1}^{\infty} e^{-\lambda_{n,B}t} \frac{\psi(x, \lambda_{n,B})\psi(y, \lambda_{n,B})}{\|\psi(\cdot, \lambda_{n,B})\|^2}, \tag{88}$$

$$\psi(x, \lambda) = (2x)^{(1-\nu)/2} e^{1/(4x)} W_{k,\mu}\left(\frac{1}{2x}\right), \tag{89}$$

where $\lambda_{n,B}$ are the positive solutions to following equation in λ:

$$W_{\frac{1-\nu}{2}, \frac{1}{2}\sqrt{\nu^2 - 2\lambda}}\left(\frac{1}{2B}\right) = 0. \tag{90}$$

Bringing all together we get the desired spectral expansion:

$$\mathbb{E}[e^{-\beta X_\tau^B}(k - X_\tau^B)^+] = \sum_{n=1}^{\infty} \frac{e^{-\lambda_{n,B}\tau} c_n}{\|\psi(\cdot, \lambda_{n,B})\|^2}, \tag{91}$$

$$c_n = 2^{-(\nu+1)/2} \int_0^k e^{-\beta x}(k-x)x^{(\nu-1)/2}e^{-1/(4x)} W_{\frac{1-\nu}{2}, \frac{1}{2}\sqrt{\nu^2 - 2\lambda_{n,B}}}\left(\frac{1}{2x}\right) dx. \tag{92}$$

For $\beta = 0$ this reduces to (44). For $\beta \neq 0$ this integral can be computed numerically, for example, using the `Sym` package for `Mathematica`.

3.3 Numerical examples

Figure 7 shows the Asian floor prices for the three cases listed in Table 2.

In these examples 57 terms in the spectral decomposition were used. For smaller σ and T more terms are required. The integral (92) is computed for each n and presents a computational challenge if the number of terms is greater than 100.

Figure 8 shows the prices for a different number of terms in the expansion.

Case	r_0	σ	μ
1	0.05	0.4	0
2	0.05	0.4	0.5
3	0.11	0.4	0

Table 2: Parameters of the interest rate model (79)

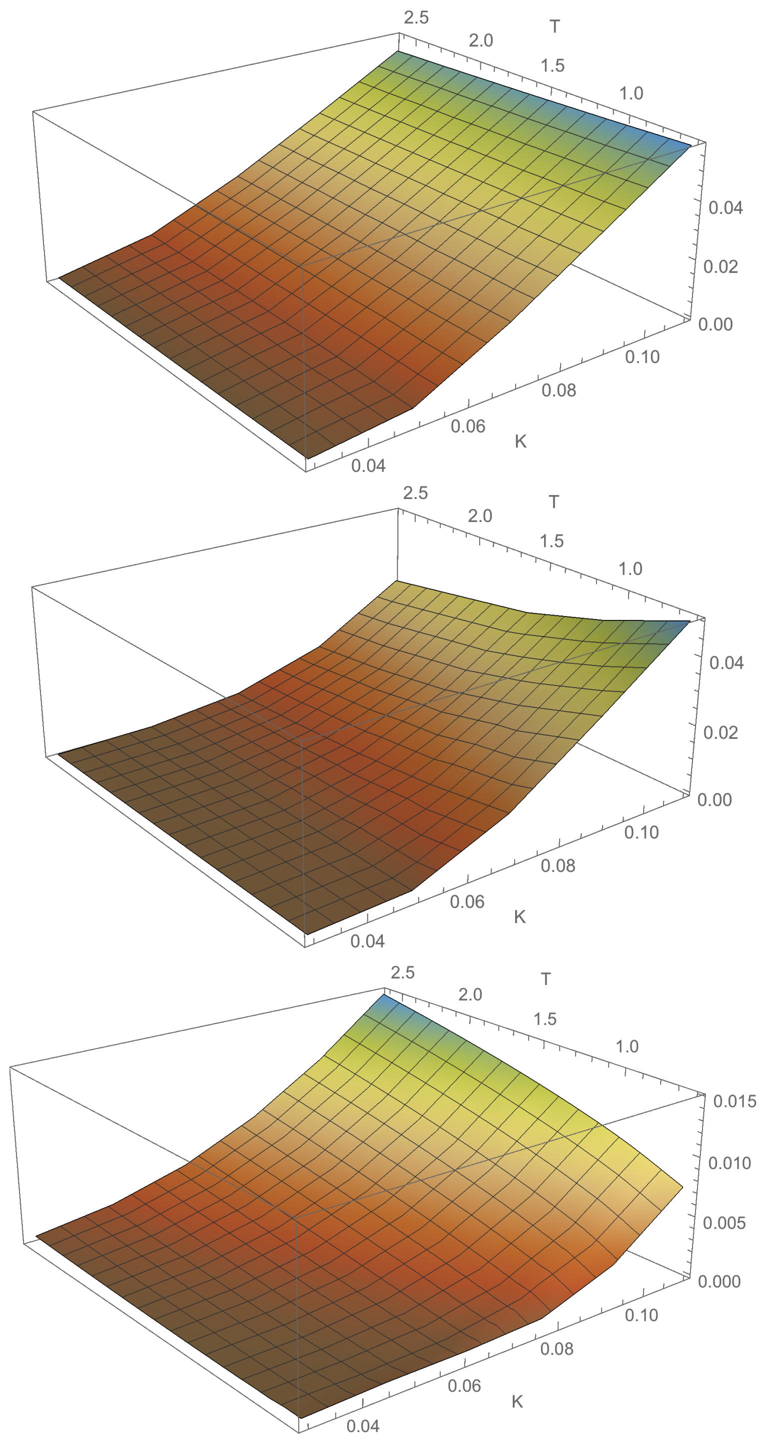

Figure 7: Asian floor prices for Cases 1, 2, 3

N	Digits	Value	Time
1	1	**0.0**000116321	0.44
3	1	**0.0**012535382	1.33
5	1	**0.0**085475689	2.24
7	1	**0.0**201919049	3.15
9	1	**0.0**280671843	4.24
11	1	**0.0**198629341	5.76
13	1	**0.0**189850489	7.53
15	2	**0.02**16720807	9.36
17	3	**0.020**6799468	11.18
19	3	**0.020**7704177	14.39
21	3	**0.020**8093347	17.98
23	5	**0.02079**36460	22.05
25	6	**0.020796**4657	25.77
27	7	**0.0207961**670	30.70
29	8	**0.02079618**46	36.48
31	9	**0.020796184**5	42.61
33	10	**0.0207961844**	48.38
35	10	**0.0207961844**	54.83
37	10	**0.0207961844**	61.96
39	10	**0.0207961844**	68.84

Figure 8: Prices of Asian floors computed using a given number N of terms, the number of stabilized digits and computation times in seconds. Stabilized digits are shown bold. Parameters correspond to Case 1 with $K = 0.07$, $T = 1$.

4 Pricing in the Vasicek model

4.1 Analytic formula for Asian floors

In the Vasicek model the short rate follows an Ornstein-Uhlenbeck process:

$$dr_t = \theta(\mu - r_t)dt + \sigma dW_t. \tag{93}$$

Our goal is to compute the price of an Asian floor:

$$\mathbb{E}\left[e^{-\int_0^T r_t dt}\big(K - \frac{1}{T}\int_0^T r_t dt\big)^+\right]. \tag{94}$$

In order to apply Linetsky's method we need to reduce the expectation to the form $\mathbb{E}f(X_\tau)$, where $\{X_t\}$ is an Itô diffusion. To this end we note that equation (93) can be solved in a closed form:

$$r_t = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma\int_0^t e^{-\theta(t-s)} dW_s. \tag{95}$$

Integrating this expression with respect to t we get:

$$\int_0^T r_t\,dt = \frac{r_0 - \mu}{\theta}(1 - e^{-\theta T}) + \mu T + \frac{\sigma}{\theta}\int_0^T (1 - e^{-\theta(T-s)})\,dW_s. \tag{96}$$

It follows that:

$$\int_0^T r_t dt \overset{(law)}{=} \alpha + \beta W_1, \tag{97}$$

$$\alpha = \frac{r_0 - \mu}{\theta}(1 - e^{-\theta T}) + \mu T, \tag{98}$$

$$\beta^2 = \frac{\sigma^2}{\theta^2}\int_0^T (1 - e^{\theta(T-s)})^2 ds = -\frac{3\sigma^2}{2\theta^3} - \frac{\sigma^2 e^{-2\theta T}}{2\theta^3} + \frac{2\sigma^2 e^{-\theta T}}{\theta^3} + \frac{\sigma^2 T}{\theta^2}. \tag{99}$$

We can now rewrite the price of the Asian floor as follows:

$$\mathbb{E}\left[e^{-\int_0^T r_t dt}\big(K - \frac{1}{T}\int_0^T r_t dt\big)^+\right] = \frac{e^{-\alpha}\beta}{T}\mathbb{E}\left[e^{-\beta W_1}(k - W_1)^+\right], \tag{100}$$

$$k = (KT - \alpha)/\beta. \tag{101}$$

Expectation $\mathbb{E}[e^{-\beta W_1}(k - W_1)^+]$ can be computed using the method of spectral expansions. But this expectation can be also computed in closed form using Girsanov's theorem:

$$\mathbb{E}[e^{-\beta W_1}(k - W_1)^+] = e^{\frac{\beta^2}{2}}\mathbb{E}[e^{-\frac{\beta^2}{2} - \beta W_1}(k - W_1)^+] \tag{102}$$

$$= e^{\frac{\beta^2}{2}}\widetilde{\mathbb{E}}(k + \beta - \widetilde{W}_1)^+ \tag{103}$$

$$= e^{\frac{\beta^2}{2}}\left(\frac{1}{2}(\beta + k)\left(\operatorname{erf}\left(\frac{\beta + k}{\sqrt{2}}\right) + 1\right) + \frac{e^{-\frac{1}{2}(\beta+k)^2}}{\sqrt{2\pi}}\right). \tag{104}$$

This leads to the following final pricing formula:

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_0^T r_t dt}\big(K - \frac{1}{T}\int_0^T r_t dt\big)^+\right] \\
&= \frac{e^{-\alpha + \frac{\beta^2}{2}}\beta}{T}\left(\frac{1}{2}(\beta + k)\left(\operatorname{erf}\left(\frac{\beta + k}{\sqrt{2}}\right) + 1\right) + \frac{e^{-\frac{1}{2}(\beta+k)^2}}{\sqrt{2\pi}}\right), \\
&k = \frac{T(K - \alpha)}{\beta}, \quad \alpha = \frac{r_0 - \mu}{\theta}(1 - e^{-\theta T}) + \mu T, \\
&\beta^2 = -\frac{3\sigma^2}{2\theta^3} - \frac{\sigma^2 e^{-2\theta T}}{2\theta^3} + \frac{2\sigma^2 e^{\theta(-T)}}{\theta^3} + \frac{\sigma^2 T}{\theta^2}.
\end{aligned} \tag{105}$$

This formula is illustrated in Figure 9.

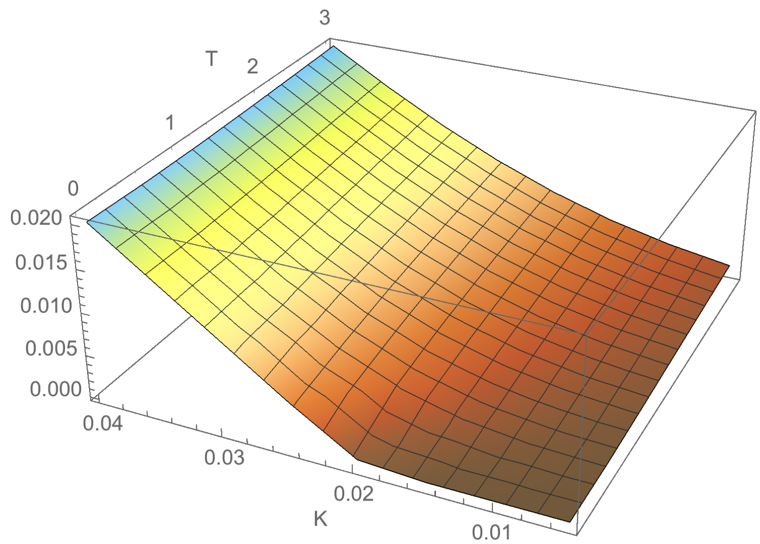

Figure 9: Asian floor prices in the Vasicek model computed using formula (105). Model parameters are: $r_0 = 0.02$, $\mu = 0.02$, $\theta = 0.1$, $\sigma = 0.02$.

4.2 Spectral expansion for Asian floors

Our next goal is to compute the expectation $\mathbb{E}[e^{-\beta W_1}(k - W_1)^+]$ using spectral expansions. The first step is to put absorbing barriers at levels $a < 0 < b$ and to consider the corresponding Sturm-Liouville problem:

$$-\frac{1}{2}u''(x) = \lambda u(x), \quad a < x < b, \tag{106}$$

$$u(a) = u(b) = 0. \tag{107}$$

The eigenvalues λ_n and the eigenfunctions $\phi_n(x)$ normalized with respect to the speed density $m(x) = 2$ are given by:

$$\lambda_n = \frac{1}{2}\left(\frac{\pi n}{b-a}\right)^2, \quad \phi_n(x) = \sqrt{\frac{1}{b-a}}\sin\left(\pi n \frac{x-a}{b-a}\right), \quad n \geq 1. \tag{108}$$

The desired spectral expansion is:

$$\mathbb{E}\left[e^{-\int_0^T r_t dt}\big(K - \frac{1}{T}\int_0^T r_t dt\big)^+\right] = \frac{e^{-\alpha}\beta}{T}\sum_{n=1}^{\infty} e^{-\lambda_n} c_n \phi_n(0), \tag{109}$$

$$\begin{aligned} c_n &= 2\int_a^k e^{-\beta x}(k-x)\phi_n(x)dx \\ &= \frac{e^{\beta(-a-k)}}{\sqrt{\frac{1}{L}}\left(\beta^2 L^2 + \pi^2 n^2\right)^2}\Big(2\pi n e^{\beta k}\left(-(a-k)\left(\beta^2 L^2 + \pi^2 n^2\right) - 2\beta L^2\right) \\ &+ 2Le^{a\beta}\left((\pi n - \beta L)(\beta L + \pi n)\sin\left(\frac{\pi n(a-k)}{L}\right) + 2\pi\beta L n\cos\left(\frac{\pi n(a-k)}{L}\right)\right)\Big), \end{aligned} \tag{110}$$

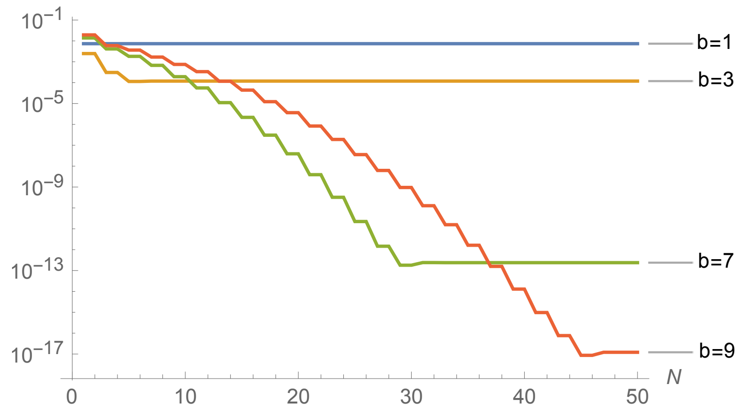

Figure 10: Pricing errors for Asian floors computed using (109) with $a = -b$ for different b and for different number of terms N. The model parameters are $\theta = 0.1$, $\mu = 0.02$, $\sigma = 0.02$, $r_0 = 0.02$, $T = 1$, $K = 0.03$.

where $L = b - a$. Figure 10 shows pricing errors in different settings.

4.3 Analytic formula for bonds and bond options

Our next goal is to price a bond call option:

$$\mathbb{E}\left[e^{-\int_0^{T_1} r_t dt}\left(P(r_{T_1}, T_2 - T_1) - K\right)^+\right], \quad 0 < T_1 < T_2, \tag{111}$$

$$P(r_0, T) = \mathbb{E}[e^{-\int_0^\tau r_t dt}]. \tag{112}$$

Using (97) and Girsanov's theorem we get the bond price:

$$\begin{aligned} P(r_0, T) &= \mathbb{E}[e^{-\alpha-\beta W_1}] = e^{-\alpha+\frac{\beta^2}{2}}\mathbb{E}[e^{-\frac{\beta^2}{2}-\beta W_1}] = e^{-\alpha+\frac{\beta^2}{2}} \\ &= \exp\left(-\frac{(r_0-\mu)\left(1-e^{-\theta T}\right)}{\theta} - \mu T + \frac{1}{2}\left(-\frac{3\sigma^2}{2\theta^3} - \frac{\sigma^2 e^{-2\theta T}}{2\theta^3} + \frac{2\sigma^2 e^{-\theta T}}{\theta^3} + \frac{\sigma^2 T}{\theta^2}\right)\right). \end{aligned} \tag{113}$$

Pricing operators $P_t f(x) = \mathbb{E}[e^{-\int_0^t r_s ds} f(r_t) | r_0 = x]$ admit the following representation through a Gaussian transition kernel, see (Borodin & Salminen, 2002), p. 252, (1.8.7):

$$\mathbb{E}[e^{-\int_0^t r_s ds} f(r_t) | r_0 = x] = \int_{-\infty}^{+\infty} f(y) q(t, x, y) dy, \tag{114}$$

$$q(t; x, y) = \frac{\exp\left(\frac{\sigma^2 t}{2\theta^2} - \mu t - \frac{\left(-\theta^2\mu+\sigma^2-e^{\theta t}\left(-\theta^2\mu+\sigma^2+\theta^2 y\right)+\theta^2 x\right)^2}{\theta^3\sigma^2\left(e^{2\theta t}-1\right)} + \frac{y-x}{\theta}\right)}{\sqrt{\pi}\sqrt{\frac{\sigma^2\left(1-e^{-2\theta t}\right)}{\theta}}}. \tag{115}$$

Besides, recall the following expression for the integral of a Gaussian:

$$I(A, B, C, R) = \int_{-\infty}^{R} e^{-A(x-B)^2+Cx} dx = \frac{\sqrt{\pi} e^{\frac{C^2}{4A}+BC} \operatorname{erfc}\left(\frac{2A(B-R)+C}{2\sqrt{A}}\right)}{2\sqrt{A}}. \tag{116}$$

Denoting by R the root of equation $P(R, T_1 - T_1) = K$ and combining formulas (113), (115), (116) we get the desired pricing formula:

$$\mathbb{E}[e^{-\int_0^{T_1} r_t dt}(P(r_{T_1}, T_2 - T_1) - K)^+] = \int_{-\infty}^{R} (P(y, T_2 - T_1) - K) q(T_1; r_0, y)\, dy \tag{117}$$

$$= F\left(I(A, B, e^{-\theta T}/\theta, R)D - I(A, B, 1/\theta, R)K\right), \tag{118}$$

$$A = \frac{\theta}{\sigma^2\left(1 - e^{-2\theta T_1}\right)}, \quad B = -\frac{\sigma^2}{\theta^2} + \mu + e^{-\theta T_1}\left(\frac{\sigma^2}{\theta^2} - \mu + r_0\right), \tag{119}$$

$$D = \exp\left(\frac{1}{2}\left(-\frac{3\sigma^2}{2\theta^3} - \frac{\sigma^2 e^{-2\theta T}}{2\theta^3} + \frac{2\sigma^2 e^{-\theta T}}{\theta^3} + \frac{\sigma^2 T}{\theta^2}\right) + \frac{\mu\left(1 - e^{-\theta T}\right)}{\theta} - \mu T\right), \tag{120}$$

$$F = \frac{\exp\left(\frac{\sigma^2 T_1}{2\theta^2} - \mu T_1 - \frac{r_0}{\theta}\right)}{\sqrt{\pi\sigma^2\left(1 - e^{-2\theta T_1}\right)/\theta}}, \quad T = T_2 - T_1. \tag{121}$$

This formula is illustrated in Figure 11 for the cases specified in Table 3.

Case	r_0	μ	θ	σ	T_2	$P(r_0, T_2)$
1	0.1	0.02	0.1	0.02	10	0.510646
2	0.1	0.02	0.1	0.025	100	0.86616

Table 3: Parameters of the model

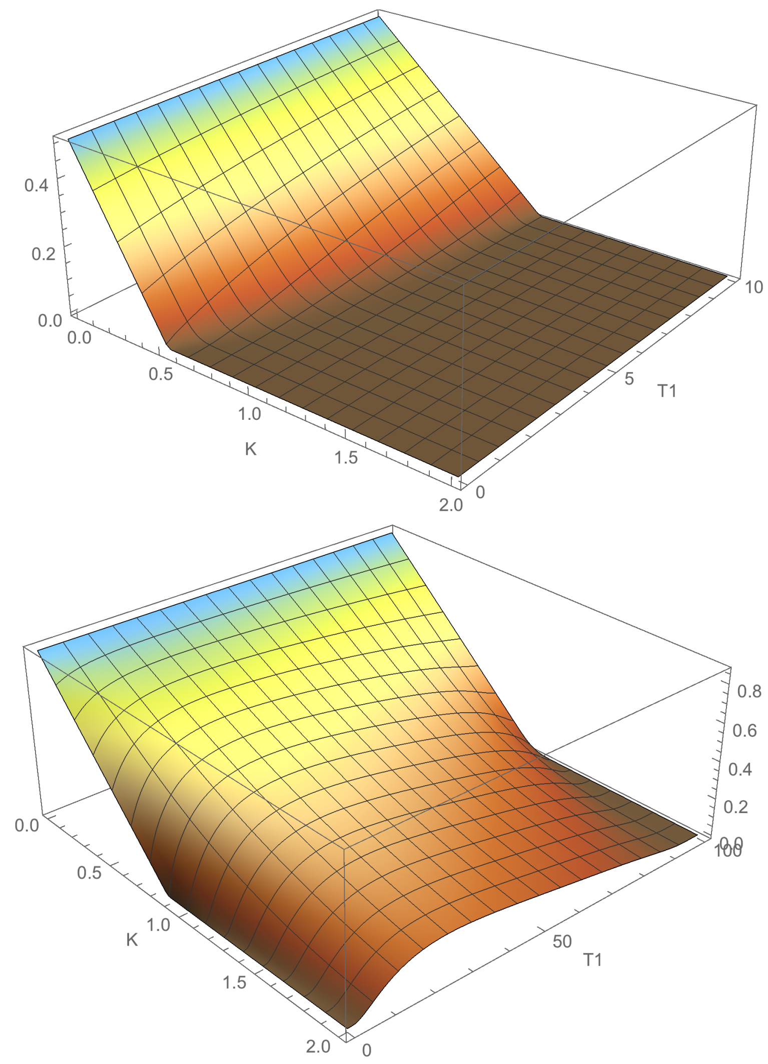

Figure 11: Bond call option prices for Cases 1 and 2 of Table 3.

4.4 Spectral expansion for bonds and bond options

The speed measure for the Ornstein-Uhlenbeck process (93) is given by:

$$m(x)dx = \frac{2}{\sigma^2} e^{-\frac{\theta(x-\mu)^2}{\sigma^2}} dx. \tag{122}$$

The following spectral expansion is valid for any $f \in L^2(\mathbb{R}, m(x)dx)$:

$$\mathbb{E}[e^{-\int_0^T r_t dt} f(r_T)] = \sum_{n=1}^{\infty} e^{-\lambda_n T} \phi_n(r_0) \int_{-\infty}^{+\infty} f(y)\phi_n(y)m(y)dy, \tag{123}$$

where the eigenvalues λ_n and the normalized eigenfunctions ϕ_n are given by:

$$\lambda_n = \mu - \theta - \frac{\sigma^2}{2\theta^2} + \theta n, \tag{124}$$

$$\phi_n(x) = N_n e^{-a\xi - \frac{a^2}{2}} H_n(\xi + a), \tag{125}$$

$$N_n^2 = \sqrt{\frac{\theta}{\pi}} \frac{\sigma}{2^n n!}, \quad \xi = \frac{\sqrt{\theta}}{\sigma}(x - \mu), \quad a = \frac{\sigma}{\theta^{3/2}}, \tag{126}$$

and H_n are Hermite polynomials, see (Gorovoi & Linetsky, 2004). The first few eigenfunctions are illustrated in Figure 12. Eigenfunctions decay exponentially for $x \to +\infty$ and grow exponentially in absolute value for $x \to -\infty$.

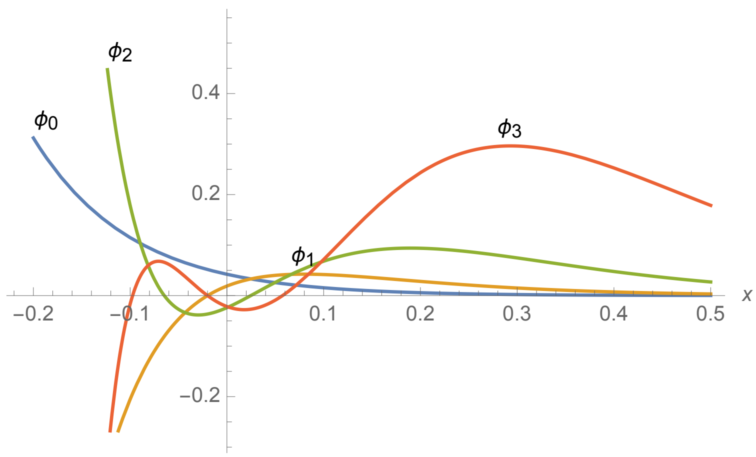

Figure 12: Eigenfunctions for the Vasicek model with $\theta = 0.1$, $\mu = 0.02$, $\sigma = 0.02$.

The spectral expansion for a zero coupon bond is given by:

$$P(r_0, T) = \mathbb{E}[e^{-\int_0^T r_t dt}] = \sum_{n=1}^{\infty} e^{-\lambda_n T} c_n \phi(r_0), \tag{127}$$

$$c_\ell = \int_{-\infty}^{+\infty} \phi_n(y) m(y) dy = \frac{2}{\sigma} \sqrt{\frac{\pi}{\theta}} N_n a^n e^{-a^2/4}. \tag{128}$$

Approximation errors for different number of terms are shown in Figure 13. Using formula (127) and denoting by R the solution to $P(R, T_2 - T_1) = K$ for $K > 0$ we get a spectral expansion for a bond call:

$$\begin{aligned}
&\mathbb{E}[e^{-\int_0^{T_1} r_t dt} \big(P(r_{T_1}, T_2 - T_1) - K\big)^+] \\
&= \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(r_0) \int_{-\infty}^{R} \big(P(y, T_2 - T_1) - K\big) \phi_n(y) m(y)\, dy \\
&= \sum_{n=1}^{\infty} e^{-\lambda_n T_1} \phi_n(r_0) \left(\sum_{\ell=1}^{\infty} e^{-\lambda_\ell (T_2 - T_1)} c_\ell \int_{-\infty}^{R} \phi_\ell(y) \phi_n(y) m(y) dy - K \int_{-\infty}^{R} \phi_n(y) m(y) dy \right).
\end{aligned} \tag{129}$$

Approximation errors for different number of terms are shown in Figure 14. For smaller times to maturity T_1 more terms of expansions are required.

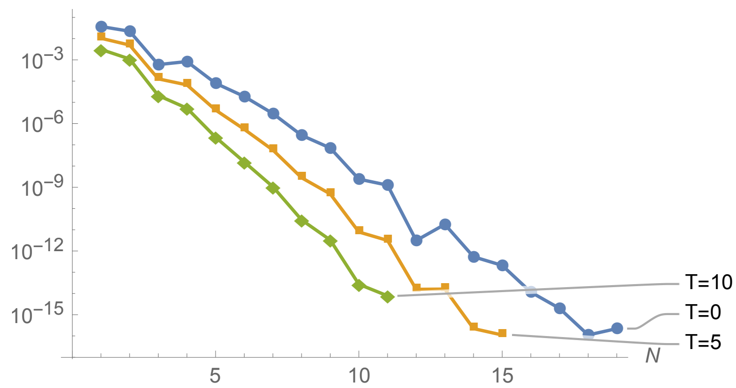

Figure 13: Absolute error in the bond price $P(r_0, T)$ for different number of terms in N the spectral expansion (127). Used model parameters are $\theta = 0.1$, $\mu = 0.02$, $\sigma = 0.02$, $r_0 = 0.02$.

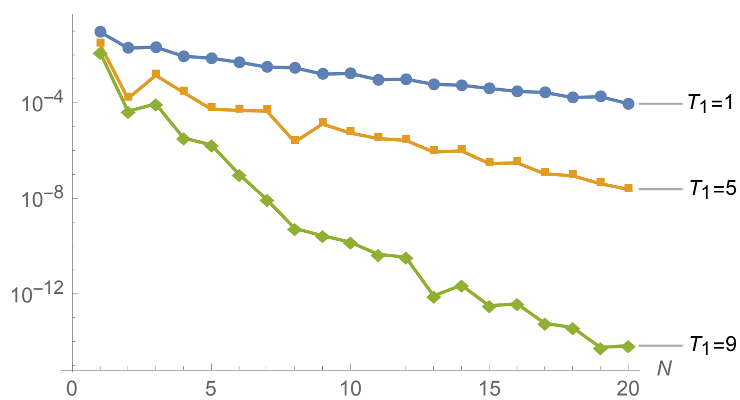

Figure 14: Absolute error in the bond call price for different number of terms N in the outer sum of expansion (129) for $L = 15$ terms in the inner sum. Model parameters are $\theta = 0.1$, $\mu = 0.02$, $\sigma = 0.02$, $r_0 = 0.02$, $K = 0.8$, $T_2 = 10$.

5 Pricing in the CIR model

5.1 Series expansions for the density and for Asian caps

In the CIR model the short rate follows a square root process:

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t. \tag{130}$$

Our objective is to compute a series expansion for an Asian cap:

$$\mathbb{E}_x\left[e^{-Y_T}(\tfrac{1}{T}Y_T - K)^+\right], \quad Y_T = \int_0^T r_s ds, \tag{131}$$

where $\mathbb{E}_x$ denotes expectation conditional on $r_0 = x$. The approach of (Nagaradjasarma, 2003) is based on the inversion of the Laplace transform. The starting point is a well known formula for the moment generating function of Y_T, see, e.g., (Lamberton & Lapeyre, 1995):

$$\begin{aligned} L(\mu) &\stackrel{def}{=} \mathbb{E}_x\left(-\mu\int_0^t r_s ds\right) = 2^{\frac{2a}{\sigma^2}}\left(\frac{\gamma e^{-\frac{\gamma t}{2}}}{(\gamma-b)e^{\gamma(-t)}+b+\gamma}\right)^{\frac{2a}{\sigma^2}} \\ &\times \exp\left(\frac{abt-x(\gamma-b)}{\sigma^2}+\frac{2\gamma x(\gamma-b)e^{\gamma(-t)}}{\sigma^2\left((\gamma-b)e^{\gamma(-t)}+b+\gamma\right)}\right), \quad \gamma=\sqrt{b^2+2\mu\sigma^2}, \end{aligned} \tag{132}$$

The first step is to derive a series expansion for the probability density of Y_t which is given by $f(t,y) = \mathcal{L}_\mu^{-1}[L(\mu)](y)$.

Proposition 3. *The following formula is valid:*

$$f(t,y) = e^{b\frac{at+x}{\sigma^2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k} u_{k,k-n,m} I_{k,k-n}\left(mt+\frac{at+x}{\sigma^2}, y\right), \tag{133}$$

$$I_{p,q}(\omega,y) \stackrel{def}{=} \mathcal{L}_\mu^{-1}\left[(\gamma-b)^p\gamma^{-q}e^{-\omega\gamma}\right], \tag{134}$$

where $I_{p,q}$ can be computed using the following recurrences:

$$\begin{aligned} I_{p,q}(y,\omega) &= I_{p-1,q-1}(y,\omega) - bI_{p-1,q}(y,\omega), \quad p,q\geq 1, \\ I_{0,q}(y,\omega) &= \frac{1}{\sigma^2(q-2)}\left(yI_{0,q-2}(y,\omega)-\sigma^2\omega I_{0,q-1}(y,\omega)\right), \quad q\geq 2, \\ I_{p,0}(y) &= \frac{e^{-\frac{b^2y^2+\sigma^4\omega^2}{2\sigma^2 y}}}{\sigma\sqrt{2\pi y}}\left(H_{p+1}(\eta)\frac{\sigma^{p+1/2}}{(2y)^{(p+1)/2}}+bH_p(\eta)\frac{\sigma^p}{(2y)^{p/2}}\right), \ \eta=\frac{\sigma}{\sqrt{2y}}\left(\omega-\frac{by}{\sigma^2}\right), \\ I_{0,0}(y) &= \frac{\sigma\omega e^{-\frac{b^2y^2+\sigma^4\omega^2}{2\sigma^2 y}}}{\sqrt{2\pi y^3}}, \quad I_{0,1}(y,\omega) = \frac{e^{-\frac{b^2y^2+\sigma^4\omega^2}{2\sigma^2 y}}}{\sqrt{2\pi y}\sigma}, \end{aligned} \tag{135}$$

where H_p are the Hermite polynomials.

Figure 15 shows the probability density $\widetilde{f}(t,y) = tf(t,ty)$ of Y_t/t for different t. The density is concentrated around $r_0 = 0.2$ for small t and becomes concentrated near the long-term mean $a/b = 0.1$ of r_t for large t.

Figure 16 shows the number of terms required for a fixed precision for the probability density of $\widetilde{f}(t,y)$ at a fixed point y. One can see that for smaller t and for larger x a bigger number of terms is required to get a fixed precision.

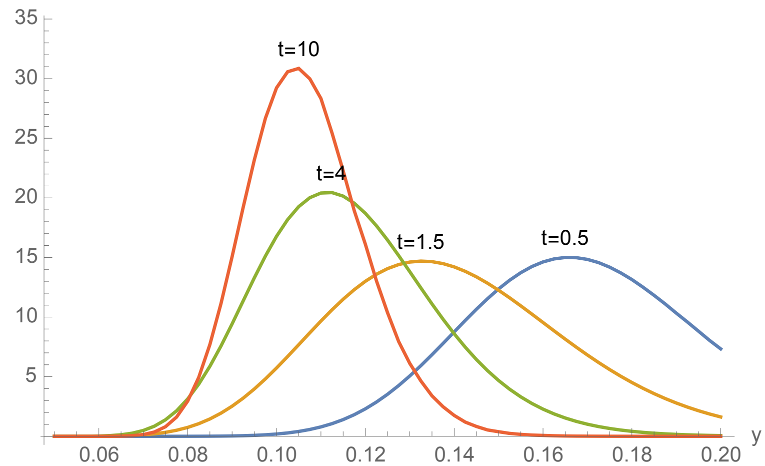

Figure 15: Probability density $\widetilde{f}(t, y)$ of Y_t/t for $a = 0.15$, $b = 1.5$, $\sigma = 0.2$, $r_0 = 0.2$.

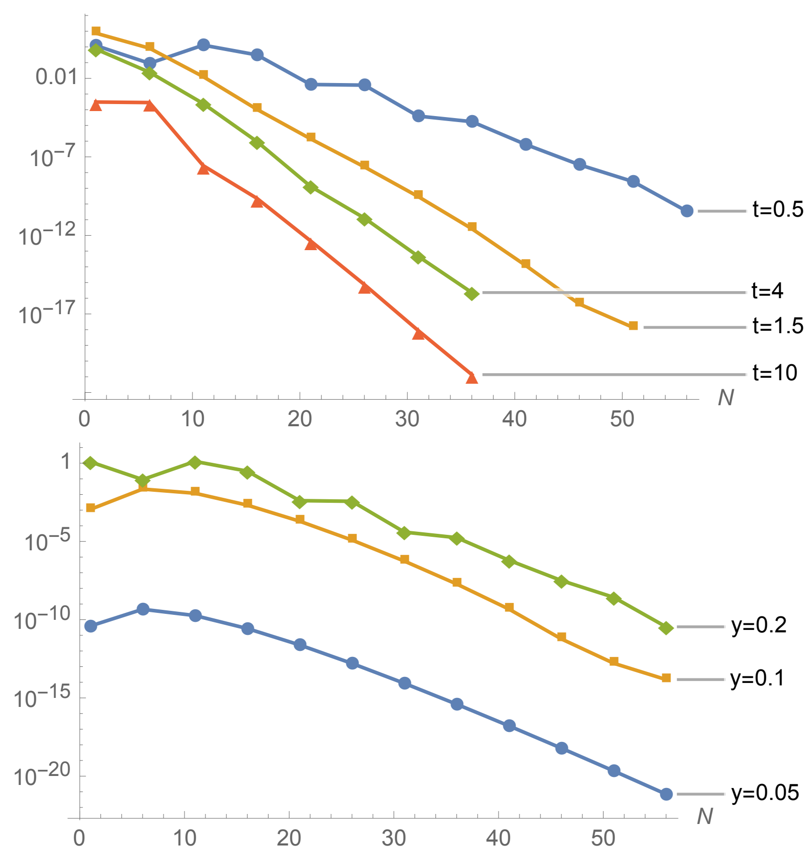

Figure 16: Number of series terms required to get a given precision for the probability density of Y_t/t at point x. Here $a = 0.15$, $b = 1.5$, $\sigma = 0.2$, $r_0 = 0.2$. The top picture corresponds to $y = 0.2$, the bottom picture corresponds to $t = 0.5$.

To compute the series expansion for Asian caps we will use the following formula:

$$\begin{aligned}\int_0^\infty e^{-\mu k}\mathbb{E}_x\left[h(Y_T)1_{\{Y_T\ge k\}}\right]dk &= \mathbb{E}_x\left[h(Y_T)\int_0^{Y_T} e^{-\mu k}dk\right]\\ &= \mu^{-1}\mathbb{E}_x\left[h(Y_T)(1-e^{-\mu Y_T})\right].\end{aligned}\tag{136}$$

Inverting the Laplace transform and recalling that $\mathcal{L}_\mu^{-1}[\mu^{-1}]=1$, we get:

$$\mathbb{E}_x\left[h(Y_T)1_{\{Y_T\ge k\}}\right]=\mathbb{E}_x[h(Y_T)]-\mathcal{L}_\mu^{-1}\left[\mathbb{E}_x\left[\mu^{-1}h(Y_T)e^{-\mu Y_T}\right]\right](k).\tag{137}$$

From this formula we get the following formula for an Asian cap:

$$\begin{aligned}&\mathbb{E}_x\left[e^{-Y_T}(Y_T/T-K)^+\right]\\ &=\mathbb{E}_x\left[e^{-Y_T}(Y_T/T-K)\right]-\mathcal{L}_\mu^{-1}\left[\mathbb{E}_x\left[\mu^{-1}(Y_T/T-K)e^{-(\mu+1)Y_T}\right]\right](KT)\\ &=-L'(1)/T-L(1)K+(\partial_\lambda G)(KT,1)/T+G(KT,1).\end{aligned}\tag{138}$$

where

$$G(y,\lambda)=\mathcal{L}_\mu^{-1}[\mu^{-1}\mathbb{E}_x\left[e^{-(\mu+\lambda)Y_T}\right]](y).\tag{139}$$

Proposition 4. *The following series expansions are valid:*

$$G(y,\lambda)=2\sigma^2 e^{b\frac{at+x}{\sigma^2}-\lambda y}\sum_{k=0}^\infty\sum_{n=0}^k\sum_{m=n}^k u_{k,k-n,m}\widetilde{I}_{k,k-n}(\omega_m,y),\tag{140}$$

$$(\partial_\lambda G)(y,\lambda)=2\sigma^2 e^{b\frac{at+x}{\sigma^2}-\lambda y}\sum_{k=0}^\infty\sum_{n=0}^k\sum_{m=n}^k u_{k,k-n,m}\left(-y\widetilde{I}_{k,k-n}(y,\omega_m)+\frac{\widehat{I}_{k,k-n}(y,\omega_m)}{\vartheta/\sigma^2}\right),\tag{141}$$

$$\omega_m=mt+(at+x)/\sigma^2,\quad \vartheta=\sqrt{b^2+2\sigma^2\lambda},\tag{142}$$

$$\widetilde{I}_{p,q}(\omega,y)=\mathcal{L}_\mu^{-1}\left[\frac{(\gamma-b)^p e^{-\omega\gamma}}{(\gamma-\vartheta)(\gamma+\vartheta)}\right],\quad \widehat{I}_{p,q}(\omega,y)=\partial_\vartheta\widetilde{I}_{p,q}(\omega,y),\tag{143}$$

where $\widetilde{I}_{p,q}$ can be computed using the following recurrences:

$$\begin{aligned}\widetilde{I}_{0,q}&=\vartheta^{-2}\left(\widetilde{I}_{0,q-2}-I_{0,q}\right),\quad q\ge 2,\\ \widetilde{I}_{p,0}&=I_{p,2}+\vartheta^2\widetilde{I}_{p,2},\quad p\ge 1,\\ \widetilde{I}_{p,q}&=\widetilde{I}_{p-1,q-1}-b\widetilde{I}_{p-1,q},\quad p,q\ge 1.\end{aligned}\tag{144}$$

$$\begin{aligned}\widetilde{I}_{0,0}(\omega,y)&=\frac{e^{-\frac{y(b-\theta)(b+\theta)}{2\sigma^2}-\theta\omega}\left(erf\left(\frac{\theta y-\sigma^2\omega}{\sqrt{2}\sigma\sqrt{y}}\right)+e^{2\theta\omega}erfc\left(\frac{\sigma^2\omega+\theta y}{\sqrt{2}\sigma\sqrt{y}}\right)+1\right)}{4\sigma^2},\\ \widetilde{I}_{0,1}(\omega,y)&=\frac{e^{-\frac{y(b-\theta)(b+\theta)}{2\sigma^2}-\theta\omega}\left(erf\left(\frac{\theta y-\sigma^2\omega}{\sqrt{2}\sigma\sqrt{y}}\right)-e^{2\theta\omega}erfc\left(\frac{\sigma^2\omega+\theta y}{\sqrt{2}\sigma\sqrt{y}}\right)+1\right)}{4\theta\sigma^2}.\end{aligned}\tag{145}$$

and $\widehat{I}_{p,q}$ *can be computed using the following recurrences:*

$$\begin{aligned}\widehat{I}_{0,q} &= -2\vartheta^{-1}\widetilde{I}_{0,q} + \vartheta^{-2}\widehat{I}_{0,q-2}, \quad q \geq 2,\\ \widehat{I}_{p,0} &= 2\vartheta\widetilde{I}_{p,2} + \vartheta^{2}\widehat{I}_{p,2}, \quad p \geq 1,\\ \widehat{I}_{p,q} &= \widehat{I}_{p-1,q-1} - b\widehat{I}_{p-1,q}, \quad p, q \geq 1,\end{aligned} \tag{146}$$

where the initial conditions are obtained by taking the derivatives of (145) *with respect to* ϑ.

Figure 17 shows Asian cap prices computed using Proposition 4. Figure 18 shows the norms of the terms in the series expansion for different times to maturity T. One can see that for smaller T more terms are required to get a desired precision.

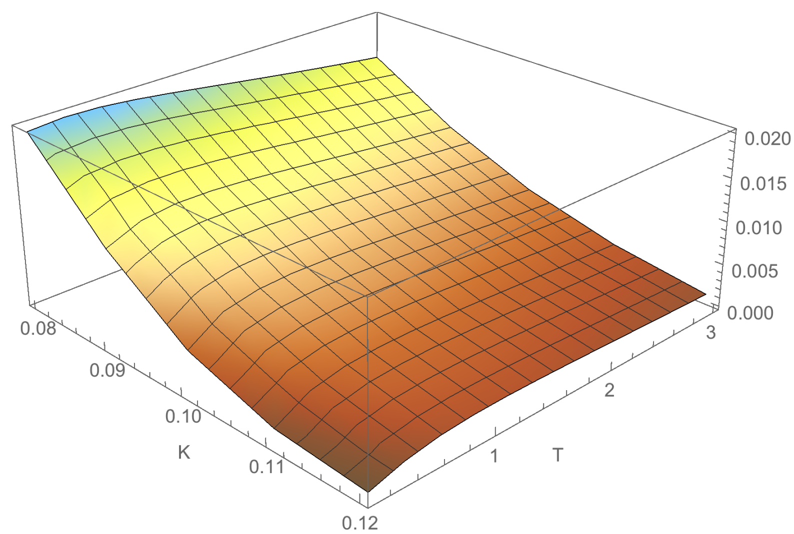

Figure 17: Asian cap prices for $a = 0.15$, $b = 1.5$, $\sigma = 0.2$, $r_0 = 0.1$.

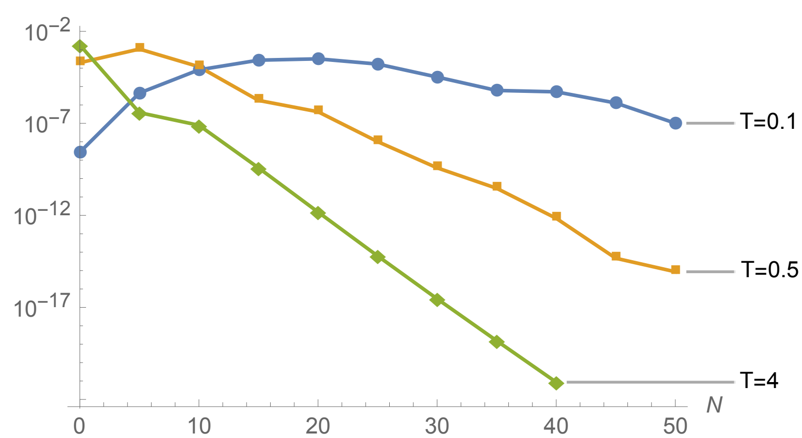

Figure 18: Norms of the terms of the series expansion for the Asian cap for $a = 0.15$, $b = 1.5$, $\sigma = 0.2$, $r_0 = 0.1$, $K = 0.1$ for different times to maturity T.

5.2 Proof of Proposition 3

5.2.1 Series expansion of the Laplace transform

The first step is to represent (132) as a series where each term has a computable inverse Laplace transform. We will use the following formulas:

$$\exp\left(\frac{x(\gamma-b)}{\sigma^2}\frac{2\gamma e^{-\gamma t}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}\right) = \sum_{n=0}^{\infty}\frac{x^n(\gamma-b)^n}{\sigma^{2n}n!}\left(\frac{2\gamma e^{-\gamma t}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}}\right)^n, \quad (147)$$

$$\frac{2\gamma e^{-\gamma t}}{(\gamma+b)+(\gamma-b)e^{-\gamma t}} = \left(1-\frac{(\gamma-b)(1-e^{-\gamma t})}{2\gamma}\right)^{-1}, \quad (148)$$

$$(1-z)^{-n-\frac{2a}{\sigma^2}} = \sum_{k=0}^{\infty}\binom{k+n+\frac{2a}{\sigma^2}-1}{k}z^k, \quad |z|<1. \quad (149)$$

Besides, we will use the following change of summation formula to reduce a double series to a single series:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}c_{n,k} = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}c_{n,k-n} = \sum_{k=0}^{\infty}\sum_{n=0}^{k}c_{n,k-n}. \quad (150)$$

Using these formulas we can rewrite (132) as follows:

$$L(\mu) = e^{-(\gamma-b)\frac{at+x}{\sigma^2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\frac{x^n(\gamma-b)^k(1-e^{-\gamma t})^{k-n}e^{-\gamma nt}}{n!\sigma^{2n}(2\gamma)^{k-n}} \quad (151)$$

$$= e^{b\frac{at+x}{\sigma^2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k}u_{k,k-n,m}e^{-(mt+\frac{at+x}{\sigma^2})\gamma}(\gamma-b)^k\gamma^{n-k}, \quad (152)$$

where

$$u_{k,n,m} = (-1)^m\frac{(-x/\sigma^2)^{k-n}}{2^n(k-n)!}\binom{k+\frac{2a}{\sigma^2}-1}{n}\binom{n}{m+n-k}, \quad (153)$$

$$\widetilde{\gamma} = \sqrt{b^2+2\sigma^2(\lambda+\mu)}, \quad \vartheta = \sqrt{b^2+2\sigma^2\lambda}. \quad (154)$$

The series converges absolutely and uniformly in $\mu\in\mathbb{C}$, $\Re\mu>1$, in view of the estimate:

$$\left|\frac{(\gamma-b)(1-e^{-\gamma t})}{2\gamma}\right| < \frac{1}{2}. \quad (155)$$

5.2.2 Inverse Laplace transforms of the terms

The next step is to compute the inverse Laplace transforms $I_{p,q}(\omega,y)$ of the functions $e^{-\omega\gamma}(\gamma-b)^p\gamma^{-q}$. Functions $I_{0,1}$ and $I_{0,2}$ can be computed analytically. To compute the terms $I_{p,0}$ we will use the following formulas:

$$(\gamma-b)^p e^{-\omega\gamma} = (-1)^p e^{-\omega b}\partial_\omega^p\left(e^{\omega b}e^{-\omega\gamma}\right), \quad (156)$$

$$I_{p,0}(y) = (-1)^p e^{-\omega b}\partial_\omega^p\left(e^{\omega b}I_{0,0}(y)\right). \quad (157)$$

Using the Rodrigues formula for the Hermite polynomials $(-1)^p e^{x^2}\partial_x^p(e^{-x^2}) = H_p(x)$, we get the following algebraic identities:

$$(-1)^p e^{\alpha\omega+\beta\omega^2}\partial_\omega^p\left(e^{-\alpha\omega-\beta\omega^2}\right) = (-1)^p e^{\beta(\omega+\frac{\alpha}{2\beta})^2}\partial_\omega^p\left(e^{-\beta(\omega+\frac{\alpha}{2\beta})^2}\right)$$
$$= H_p(\eta)\,\beta^{p/2},\quad \eta = \sqrt{\beta}(\omega+\frac{\alpha}{2\beta}), \tag{158}$$

$$(-1)^p e^{\alpha\omega+\beta\omega^2}\partial_\omega^p\left(2\beta\omega e^{-\alpha\omega-\beta\omega^2}\right) = H_{p+1}(\eta)\,\beta^{(p+1)/2} - \alpha H_p(\eta)\,\beta^{p/2}. \tag{159}$$

This leads to the explicit formula for $I_{p,0}$ in (135).

To compute the general terms $I_{p,q}$ we will use the formulas

$$\gamma^{-q}e^{-\omega\gamma} = \int_0^\infty e^{-(\omega+u)\gamma}\frac{u^{q-1}}{(q-1)!}du, \tag{160}$$

$$I_{p,q}(y,\omega) = \int_0^\infty I_{p,0}(y,\omega+u)\frac{u^{q-1}}{(q-1)!}du. \tag{161}$$

Integration by parts and explicit formulas for $I_{p,0}$ lead to the recurrent formulas for $I_{p,q}$ given in (135).

5.3 Proof of Proposition 4

The approach consists in computing a series expansion for $G(y,\lambda)$ and then in taking its derivative with respect to λ to get an expansion for $\partial_\lambda G(y,\lambda)$. In a similar way with the density expansion we get:

$$\mu^{-1}\mathbb{E}_x\left[e^{-(\mu+\lambda)Y_t}\right] = 2\sigma^2 e^{b\frac{at+x}{\sigma^2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\sum_{m=n}^{k}u_{k,k-n,m}\frac{(\widetilde{\gamma}-b)^k e^{-(mt+\frac{at+x}{\sigma^2})\widetilde{\gamma}}}{\widetilde{\gamma}^{k-n}(\widetilde{\gamma}-\vartheta)(\widetilde{\gamma}+\vartheta)}, \tag{162}$$

$$\widetilde{\gamma} = \sqrt{b^2+2\sigma^2(\mu+\lambda)}. \tag{163}$$

The next step is to compute the inverse Laplace transforms of the terms:

$$\mathcal{L}_\mu^{-1}\left[\frac{(\widetilde{\gamma}-b)^p e^{-\omega\widetilde{\gamma}}}{\widetilde{\gamma}^q(\widetilde{\gamma}-\vartheta)(\widetilde{\gamma}+\vartheta)}\right](y) = e^{-\lambda y}\widetilde{I}_{p,q}(\omega,y). \tag{164}$$

This computation of $\widetilde{I}_{p,q}$ is similar to the computation of $I_{p,q}$. We will use the following formulas:

$$\frac{e^{-\omega\gamma}}{(\gamma-\vartheta)(\gamma+\vartheta)} = \frac{1}{2\vartheta}\left(\frac{e^{-\omega\gamma}}{\gamma-\vartheta} - \frac{e^{-\omega\gamma}}{\gamma+\vartheta}\right), \tag{165}$$

$$\frac{e^{-\omega\gamma}}{\gamma\pm\vartheta} = \int_0^\infty e^{-(\omega+u)\gamma\mp\vartheta u}du, \tag{166}$$

$$\widetilde{I}_{p,q}(\omega,y) \stackrel{def}{=} \int_0^\infty I_{p,q}(\omega+u,y)\frac{e^{\vartheta u}-e^{-\vartheta u}}{2\vartheta}du. \tag{167}$$

Integration by parts leads to recurrences for $\widetilde{I}_{p,q}$ given in (144). The terms $\widetilde{I}_{0,0}$, $\widetilde{I}_{0,1}$ can be computed analytically.

To compute the series expansion for $(\partial_\lambda G)(y,\lambda)$ we take the derivative of expansion for $G(y,\lambda)$ with respect to λ. The terms of the expansion can be computed using the following formula, which follows from

$$\frac{\partial}{\partial\lambda}\mathcal{L}_\mu^{-1}\left[\frac{(\widetilde{\gamma}-b)^p e^{-\omega\widetilde{\gamma}}}{\widetilde{\gamma}^q(\widetilde{\gamma}-\vartheta)(\widetilde{\gamma}+\vartheta)}\right](y) = -ye^{-\lambda y}\widetilde{I}_{p,q}(y,\omega) + \sigma^2\vartheta^{-1}e^{-\lambda y}\widehat{I}_{p,q}(y,\omega). \tag{168}$$

By taking the derivatives of recurrences for $\widetilde{I}_{p,q}$ we get recurrences for $\widehat{I}_{p,q}$.

6 Appendix A: Spectral expansion of the Feynman-Kac semigroup

6.1 Short overview of the approach

Our goal is to compute the expectation

$$P_t f(x) = \mathbb{E}[e^{-\int_0^T r(X_t)dt} f(X_T)|X_0 = x], \tag{169}$$

where $r \geq 0$ and X_t is a diffusion process:

$$dX_t = b(X_t)dt + a(X_t)dW_t. \tag{170}$$

For simplicity and numerical tractability we will assume that the diffusion X_t has range $[e_1, e_2]$ and is absorbed at the endpoints. If $r \geq 0$, then $\{P_t\}$ is a self-adjoint contraction semigroup (called the Feynman-Kac semigroup because of the discounting factor) with nonpositive infinitesimal generator:

$$Gu(x) = \tfrac{1}{2}a^2(x)u''(x) + b(x)u'(x) - r(x)u(x). \tag{171}$$

As a corollary, the distributional kernel of P_t admits a spectral expansion:

$$p(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y), \tag{172}$$

where λ_n, ϕ_n are eigenvalues and normalized eigenfunctions of the Sturm-Liouville problem with absorbing boundary conditions:

$$-Gu(x) = \lambda u(x), \quad e_1 < x < e_2, \tag{173}$$

$$u(e_1) = u(e_2) = 0. \tag{174}$$

Solving this Sturm-Liouville problem for eigenvalues λ_n and normalized eigenfunctions ϕ_n we get the desired spectral expansion:

$$\mathbb{E}[e^{-\int_0^T r(X_t)dt} f(X_T)|X_0 = x] = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \phi_n(x), \tag{175}$$
$$c_n = \int_{e_1}^{e_2} f(y)\phi_n(y)m(y)\,dy,$$

where $m(y)$ is the speed density of diffusion X_t.

6.2 Example: Pricing a bond option

This framework can be used to price European bond options as described in (Gorovoi & Linetsky, 2004). Using (175) with $f(x) = 1$ we get the spectral expansion for the time zero price of a zero-coupon bond paying one dollar at time t:

$$P(x,t) = \mathbb{E}[e^{-\int_0^t r(X_t)dt}|X_0 = x] = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \phi_n(x), \tag{176}$$
$$c_n = \int_{e_1}^{e_2} \phi_n(y)m(y)\,dy. \tag{177}$$

Using (175) with $f(x) = (K - P(x, T-t))^+$ with $T > t$ we get the spectral expansion for a put option maturing at time t on a bond maturing at time $T > t$:

$$\mathbb{E}[e^{-\int_0^t r(X_s)ds}(K - P(X_t, T-t))^+|X_0 = x] \tag{178}$$
$$= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x) \int_{e_1}^{e_2} (K - P(y, T-t))^+ \phi_n(y)m(y)dy. \tag{179}$$

To compute the integrals we first note that $P(y, T-t)$ is a decreasing function of y. Let x^* be a solution to equation $P(x, T-t) = K$. Then:

$$\int_{e_1}^{e_2} (K - P(y, T-t))^+ \phi_n(y)m(y)dy = \int_{x^*}^{e_2} (K - P(y, T-t))\phi_n(y)m(y)dy \tag{180}$$

Now we can compute the integral by plugging the spectral expansion (176) for a zero coupon bond:

$$\int_{x^*}^{e_2} (K - P(y, T-t))\phi_n(y)m(y)dy \tag{181}$$
$$= K\int_{x^*}^{e_2} \phi_n(y)m(y)dy - \sum_{\ell=1}^{\infty} e^{-\lambda_\ell (T-t)} c_\ell \int_{x^*}^{e_2} \phi_\ell(y)\phi_n(y)m(y)dy. \tag{182}$$

This leads to the desired expression for the price of the bond option:

$$\mathbb{E}[e^{-\int_0^t r(X_s)ds}(K - P(X_t, T-t))^+ | X_0 = x] \tag{183}$$

$$= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x) \left(K \int_{x^*}^{e_2} \phi_n(y)m(y)dy - \sum_{\ell=1}^{\infty} e^{-\lambda_\ell (T-t)} c_\ell \int_{x^*}^{e_2} \phi_\ell(y)\phi_n(y)m(y)dy \right), \tag{184}$$

where c_ℓ are defined in (177).

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