FDM for Cheyette Local Volatility

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1 Implementation

1.1 Dynamics

We consider the Cheyette model:

$$dx_t = (y_t - \kappa(t)x(t))dt + \sigma(t, x_t, y_t)dW_t, \tag{1a}$$

$$dy_t = (\sigma^2(t, x_t, y_t) - 2\kappa(t)y_t)dt. \tag{1b}$$

Let $V(t, x, y)$ the the time t value of a derivative with terminal payoff $V(x_T, y_T)$:

$$V(t, x, y) = \mathbf{E}\left(e^{-\int_t^T r(s)ds} V(x_T, y_T) \mid x(t) = x, y(t) = y\right). \tag{2}$$

Then $V(t, x, y)$ satisfies the PDE:

$$\partial_t V + (L_x + L_y)V = 0, \tag{3a}$$

$$L_x = (y - \kappa(t)x)\partial_x + \frac{1}{2}\sigma^2(t, x, y)\partial_x^2 - (x + f(0, t)), \tag{3b}$$

$$L_y = \left(\sigma^2(t, x, y) - 2\kappa(t)y\right)\partial_y. \tag{3c}$$

1.2 Discretization

We will solve the pde on the rectanglar grid $[\underline{M}_1, \overline{M}_1] \times [\underline{M}_2, \overline{M}_2]$. We consider a uniform grid:

$$\begin{aligned} \Gamma_{h_1,h_2} = \{(x^{j_1}, y^{j_2})\colon x^{j_1} = \underline{M}_1 + j_1 h_1, \quad y^{j_2} = \underline{M}_2 + j_2 h_2, \\ j_1 = 0, \dots, m_1 + 1, \quad j_2 = 0, \dots, m_2 + 1\}. \end{aligned} \tag{4}$$

We put:

$$V_{j_1,j_2}(t) = V(t, x^{j_1}, y^{j_2}), \tag{5a}$$

$$\delta_x V_{j_1,j_2}(t) = \frac{V_{j_1+1,j_2}(t) - V_{j_1-1,j_2}(t)}{2h_1}, \tag{5b}$$

$$\delta_y V_{j_1,j_2}(t) = \frac{V_{j_1,j_2+1}(t) - V_{j_1,j_2-1}(t)}{2h_2}, \tag{5c}$$

$$\delta_{xx} V_{j_1,j_2}(t) = \frac{V_{j_1+1,j_2}(t) - 2V_{j_1,j_2}(t) + V_{j_1-1,j_2}(t)}{h_1^2}. \tag{5d}$$

We pass from the differential operators L_x, L_y to finite difference operators

$$\widehat{L}_x(t) = (y - k(t)x)\delta_x + \frac{1}{2}\sigma^2(t,x,y)\delta_x^2 - (x + f(0,t)), \tag{6a}$$

$$\widehat{L}_y(t) = (\sigma^2(t,x,y) - 2\kappa(t)y)\delta_y, \tag{6b}$$

acting on functions on the grid Γ_{h_1,h_2}. For a moment we omit the definitions on the boundary points. We get a second order accurate approximation of the operator $L_x + L_y$ on the grid:

$$(L_x + L_y)V(t,x,y) = (\widehat{L}_x + \widehat{L}_y)V(t,x,y) + O(h_1^2 + h_2^2). \tag{6c}$$

1.3 Time discretization and operator splitting

The pricing PDE has no mixed derivatives and can be solved using a splitting method. We use a uniform time grid $\{t_i\}$ with step h_t. On the interval $[t_i, t_{i+1}]$ we replace the time derivative with the first difference and we replace the operators L_x, L_y by their discretizations $\widehat{L}_x(\widehat{t}_i)$, $\widehat{L}_y(\widehat{t}_i)$ evaluated at the middle of the interval $\widehat{t}_i = (t_i + t_{i+1})/2$, as in the Crank-Nicolson method. After regrouping the terms, we get:

$$\begin{aligned} \left(1 - h_t \frac{1}{2}(\widehat{L}_x + \widehat{L}_y)\right) V_{j_1,j_2}(t_i) = \left(1 + h_t \frac{1}{2}(\widehat{L}_x + \widehat{L}_y)\right) V_{j_1,j_2}(t_{i+1}) \\ + O(h_t(h_1^2 + h_2^2 + h_t^2)). \end{aligned} \tag{7}$$

Note that up to an error of order $O(h_t(h_1^2 + h_2^2 + h_t^2))$:

$$\left(1 \pm h_t \frac{1}{2}(\widehat{L}_x + \widehat{L}_y)\right) \approx \left(1 \pm h_t \frac{1}{2}\widehat{L}_x\right)\left(1 \pm h_t \frac{1}{2}\widehat{L}_y\right). \tag{8a}$$

This leads to the following equality, up to $O(h_t(h_1^2 + h_2^2 + h_t^2))$ error:

$$\left(1 - h_t \frac{1}{2} \widehat{L}_x\right) \left(1 - h_t \frac{1}{2} \widehat{L}_y\right) V_{j_1,j_2}(t_i) \approx \left(1 + h_t \frac{1}{2} \widehat{L}_x\right) \left(1 + h_t \frac{1}{2} \widehat{L}_y\right) V_{j_1,j_2}(t_{i+1}). \tag{9}$$

Using basic algebra[1], we can split this second order equation into two consecutive first order equations, forgetting about the error terms:

$$\left(1 - h_t \frac{1}{2} \widehat{L}_x\right) U_{j_1,j_2} = \left(1 + h_t \frac{1}{2} \widehat{L}_y\right) V_{j_1,j_2}(t_{i+1}), \tag{10a}$$

$$\left(1 - h_t \frac{1}{2} \widehat{L}_y\right) V_{j_1,j_2}(t_i) = \left(1 + h_t \frac{1}{2} \widehat{L}_x\right) U_{j_1,j_2}, \tag{10b}$$

with intermediate variables U_{j_1,j_2}. We first determine U_{j_1,j_2} from $V_{j_1,j_2}(t_{i+1})$ using the first equation index by j_2, and then determine V_{j_1,j_2} from the second equation indexed by j_1

This approach reduces the original second order difference system with a general sparse matrix to a sequence of two first order difference equations with tridiagonal matrices with computational cost $O(m_1 m_2)$ instead of $O((m_1 m_2)^{5/4})$. The resulting method is called after Peaceman and Rachford. Besides, the first equation is perfectly parallelizable in j_2 and the second one if perfectly parallelizable in j_1 reducing the optimal parallel complexity to $O(m_1 + m_2)$.

1.4 Boundary conditions in equation (10a)

To impose boundary conditions in system (10a) we will assume that:

$$U \approx V(\widehat{t}_i), \quad \widehat{t}_i = \frac{t_i + t_{t+1}}{2}. \tag{11}$$

A better approximation is possible.

Dirichlet conditions. Dirichlet conditions in the x variable are:

$$V(t, x_0, y) = \underline{V}_1(t, y), \quad V(t, x_{m_1+1}, y) = \overline{V}_1(t, y), \tag{12}$$

for given functions $\underline{V}_1(t, y)$, $\overline{V}_1(t, y)$. To impose the lower Dirichlet boundary condition we need to do the following modifications to system (10a):

[1] Consider a formal equation of the form:

$$(1 - A)(1 - B)x = (1 + A)(1 + B)b$$

for the unknown x where A, B are abstract linear operators. Assuming that $1 - A$ is invertible and multiplying both sides by $(1 - A)^{-1}$, we can write:

$$(1 - B)x = (1 + A)(1 - A)^{-1}(1 + B)b,$$

Now, setting $u = (1 - A)^{-1}(1 + B)b$, we get two equations:

$$(1 - B)x = (1 + A)u,$$
$$(1 - A)u = (1 + B)b.$$

1. Replace the first row in the system matrix by $(1, 0, \cdots, 0)$.

2. Replace the first element in the right-hand-side by $\underline{V}(\widehat{t}_i, y)$.

In a similar way, to impose the upper Dirichlet boundary condition, the following modifications are made to system (10a):

1. Replace the last row in the system matrix by $(0, \cdots, 0, 1)$.

2. Replace the last element in the right-hand-side by $\overline{V}(\widehat{t}_i, y)$.

A possible choice for functions $\underline{V}_1(t, y)$ and $\overline{V}_1(t, y)$ is the inner value of the product, assuming zero volatility.

2 Testing

2.1 Vasicek dynamics

The Vasicek model reads:

$$dx_t = (y_t - \kappa x_t)dt + \sigma dW_t, \quad x_0 = 0, \tag{13a}$$
$$dy_y = (\sigma^2 - 2\kappa y_t)dt, \quad y_0 = 0, \tag{13b}$$

with constant κ, σ. Recall that the the time step of the finite difference method takes the form:

$$\widehat{A}_x^- U_{j_1,j_2} = \widehat{A}_y^+ V_{j_1,j_2}(t_{i+1}), \tag{14a}$$
$$\widehat{A}_y^- V_{j_1,j_2}(t_i) = \widehat{A}_x^+ U_{j_1,j_2}, \tag{14b}$$

where $\widehat{A}_x^\pm = \mathrm{Id} \pm \frac{1}{2} h_t \widehat{L}_x$, $\widehat{A}_y^\pm = \mathrm{Id} \pm \frac{1}{2} h_t \widehat{L}_y$ are the system matrices such that:

$$\widehat{L}_{x,j} = \begin{pmatrix} * & * & * & * & \cdots & * & * \\ l_{x,1,j} & d_{x,1,j} & u_{x,1,j} & 0 & \cdots & 0 & 0 \\ 0 & l_{x,2,j} & d_{x,2,j} & u_{x,2,j} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & * & * \end{pmatrix}, \tag{15a}$$

$$\widehat{L}_{y,i} = \begin{pmatrix} * & * & * & * & \cdots & * & * \\ l_{y,i,1} & d_{y,i,1} & u_{y,i,1} & 0 & \cdots & 0 & 0 \\ 0 & l_{y,i,2} & d_{y,i,2} & u_{y,i,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & * & * \end{pmatrix}. \tag{15b}$$

The coefficients are given by the following expressions:

$$l_{x,i,j} = \frac{1}{2}\frac{1}{h_x^2}(\sigma^2 - h_x(y_j - \kappa x_i)), \tag{16a}$$

$$d_{x,i,j} = -\frac{1}{h_x^2}(\sigma^2 + \frac{1}{2}h_x^2(x_i + f(0,t)) \tag{16b}$$

$$u_{x,i,j} = \frac{1}{2}\frac{1}{h_x^2}(\sigma^2 + h_x(y_j - \kappa x_i)) \tag{16c}$$

$$l_{y,i,j} = -\frac{1}{2}\frac{1}{h_y}(\sigma^2 - 2\kappa y_j), \tag{16d}$$

$$d_{y,i,j} = -\frac{1}{2}(x_i + f(0,t)), \tag{16e}$$

$$u_{y,i,j} = \frac{1}{2}\frac{1}{h_y}(\sigma^2 - 2\kappa y_j) \tag{16f}$$

2.2 Zero coupon bond (ZCB)

In the Cheyette model the price of a zero coupon bond (ZCB) maturing at time T is given by:

$$P(t,x,y;T) = \frac{P(0,T)}{P(0,t)} \exp\left(-G(t,T)x - \frac{1}{2}G^2(t,T)y\right). \tag{17}$$

This price can be computed using the finite difference method with Dirichlet boundary conditions and with the terminal condition $P(T,x,y;T) \equiv 1$. The price can be then compared to the analytical solution.