This is an introducton to basic hypothesis testing in R. After catching up on a few items from last section, we'll work through calculating t- and F-statistics, followed by demonstrating graphically that the t-statistic $t_j = \frac{b_j - \tilde{\gamma}}{\text{se}(b_j)}$ is distributed t_{n-k}. If time remains, I'll make something up.

Last section

[TBD]

Calculate t-tests and F-tests

First, a basic overview in conducting t- and F-tests. Back to `auto.csv`! At some point I will stop using this data. But not today. We'll start with the usual preliminaries.

```
OLS <- function(y,X) {
  return(solve(t(X) %*% X) %*% t(X) %*% y)
}
data <- read.csv("auto.csv", header=TRUE)
names(data) <- c("price", "mpg", "weight")
y <- matrix(data$price)
X <- cbind(1, data$mpg, data$weight)
```

For reference, consider the regression output from `lm()`.

```
res <- lm(price ~ 1 + mpg + weight, data = data)
coef(summary(res))
summary(res)$fstatistic
```

```
               Estimate   Std. Error    t value    Pr(>|t|)
(Intercept) 1946.068668 3597.0495988  0.5410180 0.590188628
mpg          -49.512221   86.1560389 -0.5746808 0.567323727
weight         1.746559    0.6413538  2.7232382 0.008129813
   value    numdf    dendf
14.73982  2.00000 71.00000
```

Now we'll run OLS and define some useful elements for hypothesis testing using the definitions in lecture notes:

```
n <- nrow(X); k <- ncol(X)
b <- OLS(y,X)
e <- y - X %*% b
s2 <- t(e) %*% e / (n - k)
XpXinv <- solve(t(X) %*% X)
se <- sqrt(s2 * diag(XpXinv))
```

By the way, it's good practice to define intermediate variables like `XpXinv`, `s2`, and `se`. This can be useful for bug-checking and for making your code intuitive. For example, I could have defined `se` as `sqrt((t(y - X %*% b) %*% (y - X %*% b) / (n-k)) * diag(solve(t(X) %*% X)))` (or

worse!), which would have been a nightmare to debug or understand.

We can now use the vector of standard errors to calculate our `t` and `p` values for the individual t-tests:

```
t <- (b - 0) / se
(p <- apply(t, 1, function(t) {2 * pt(-abs(t), df = (n - k))}))
```

```
[1] 0.590188628 0.567323727 0.008129813
```

Great! We have replicated the `Pr(>|t|)` column of the canned output. Now let's try to replicate the full regression F-statistic. This is a joint test of coefficient significance; are the coefficients jointly different from a zero vector? Max has a great description as to why this is different from three separate tests of significance. For now, note that we are testing joint significance by setting:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

This is great. This simplifies the equation (2.81), which is fairly daunting at first:

$$F = \frac{(\mathbf{Rb} - \mathbf{r})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb} - \mathbf{r})/J}{s^2} = \frac{\mathbf{b}'(\mathbf{X}'\mathbf{X})\mathbf{b}/J}{s^2} \tag{2}$$

```
(F <- t(b) %*% (t(X) %*% X) %*% b / (s2*3))
```

```
         [,1]
[1,] 158.1714
```

Uh oh. This is much larger than the reported F-statistic of 14.74. What happened? The problem is that we also included the intercept, whereas `R` assumes that this shouldn't be included in the joint test. Simplification failed. Let's try again, redefining $\mathbf{R}$ and $\mathbf{r}$ without a restriction on the intercept:

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

Unfortunately, our formula doesn't simplify as nicely, but we still get to drop the $\mathbf{r}$ vectors.

```
R <- rbind(c(0, 1, 0), c(0, 0, 1)); J <- 2
select.var <- solve(R %*% solve(t(X) %*% X) %*% t(R))
(F <- t(R %*% b) %*% select.var %*% (R %*% b) / (s2 * J))
```

```
         [,1]
[1,] 14.73982
```

It worked! This is, of course, one of the simplest possible F-tests we could conduct, but you can see how it would be easy to construct your own F-tests using this framework.

`t` distribution proof

Max showed in class that the t-statistic $t_j = \frac{b_j - \bar{\gamma}}{\text{se}(b_j)}$ is distributed t_{n-k}. We won't go over the proof again, but we will use simulated data to visualize the distributions of z_j, q, and t_j. Part of the purpose of this exercise is to give you practice in simulating data, an immensely valuable tool for testing econometric routines and hypotheses.

$$z_j \equiv \frac{(b_j) - \bar{\gamma}}{\sqrt{\sigma^2 \cdot (\mathbf{X}'\mathbf{X})^{-1}_{jj}}} \sim \mathbf{N}(0,1) \tag{4}$$

$$q \equiv \frac{\mathbf{e}'\mathbf{e}}{\sigma^2} \sim \chi^2_{(n-k)} \tag{5}$$

$$t_j \equiv \frac{b_j - \bar{\gamma}}{\sqrt{s^2 \cdot (\mathbf{X}'\mathbf{X})^{-1}_{jj}}} = \frac{b_j - \bar{\gamma}}{\text{se}(b_j)} \sim t_{n-k} \tag{6}$$

First, we'll set `reps` (the number of times we'll randomly generate data and test statistics), `n`, and `k`. We'll also create the reps $\times$ k matrices for storing the `z`, `q`, and `t` that we'll create in each loop

```
reps <- 10000; n <- 100; k <- 2
z <- matrix(rep(0,reps*k),ncol=k)
q <- matrix(rep(0,reps),ncol=1)
t <- matrix(rep(0,reps*k),ncol=k)
```

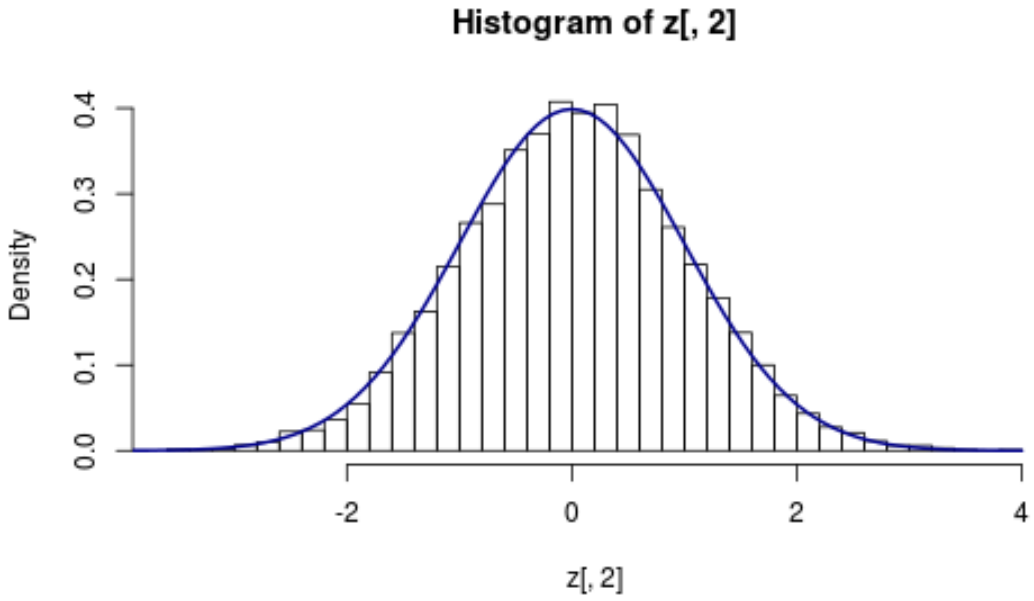
Creating `z`, `q`, and `t` in advance isn't strictly necessary but it's much more efficient to create them now than to have `R` resize them every time we run the loop. Now, the action! Once again we'll using a `for` loop:

```
for (i in 1:reps) {
  # simulate the true model
  beta <- matrix(c(42,8), nrow=2)
  X <- cbind(1, rnorm(n))
  sigma <- 1
  eps <- matrix(rnorm(n, 0, sigma), nrow=n)
  y <- X %*% beta + eps

  # run OLS and prepare everything we need to calculate z, q, and t
  b <- OLS(y,X)
  e <- y - X %*% b
  XpXinv = solve(t(X) %*% X)
  s2 <- t(e) %*% e / (n-k)
  se <- sqrt(s2 * diag(XpXinv))

  # calculate test statistics
  z[i, ] <- (b - beta) / sqrt(sigma^2 * diag(XpXinv))
  q[i] <- (t(e) %*% e) / sigma^2
  t[i, ] <- (b - beta) / se # t[i, ] <- (b - c(42,7.9)) / se # what if we have the wrong null?
}
```

There are three distinct parts to the loop above. First, we simulate a real DGP (including noise), creating `X`, `beta`, and `eps` and then constructing $\mathbf{y} = \mathbf{X}\beta + \varepsilon$. The enormous advantage of simulation is obvious here — since we know exactly what is driving our DGP we can verify that our estimating equation performs as we expect. Next, we run OLS, just as we would if we were presented with a real dataset. Finally, we calculate z_j, q, and t_j.

All that remains now is to compare the simulated distributions of z_j, q, and t_j to their expected true distributions that we demonstrated in lecture. We'll focus on z_2 and t_2, since they corresponding to our coefficient b_2, which is our randomly generated `X` variable (not the intercept). First, we'll show that $z_2 \sim N(0, 1)$:

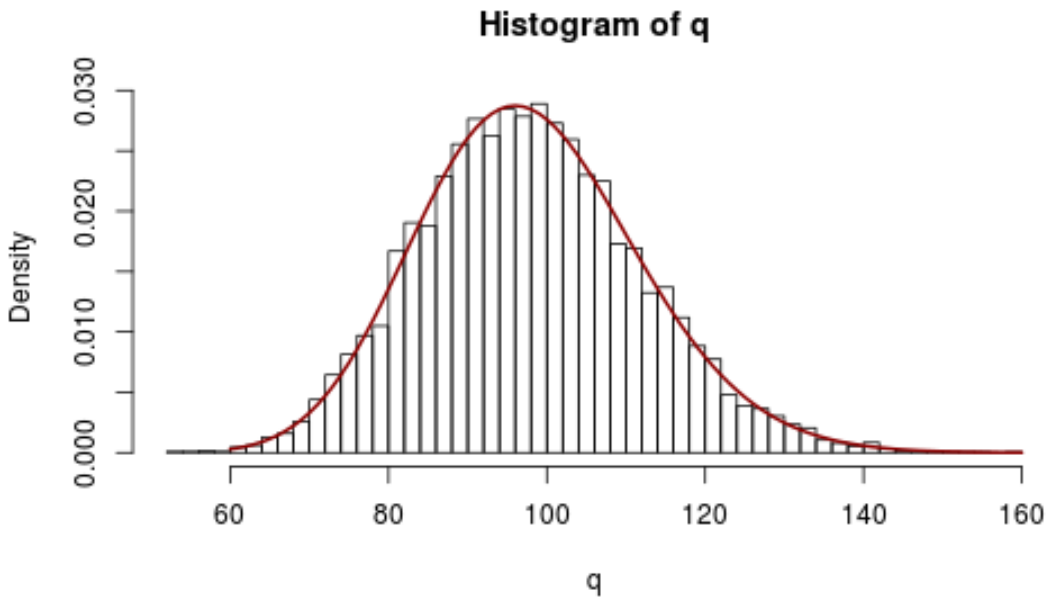
```
hist(z[ , 2], breaks = reps / 200, probability = T)
curve(dnorm(x, mean = 0, sd = 1), from = -4, to = 4, add=T, lwd=2, col="darkblue")
```



There are many prettier ways to plot a graph like this one[1], but this gets the job done. You may notice that we passed `dnorm() x`, which is a variable we haven't defined. This would normally throw an error, but since `dnorm()` is a function of `curve()`, which accepts functions of `x`, it works as we expect. Now we'll do the same for q:

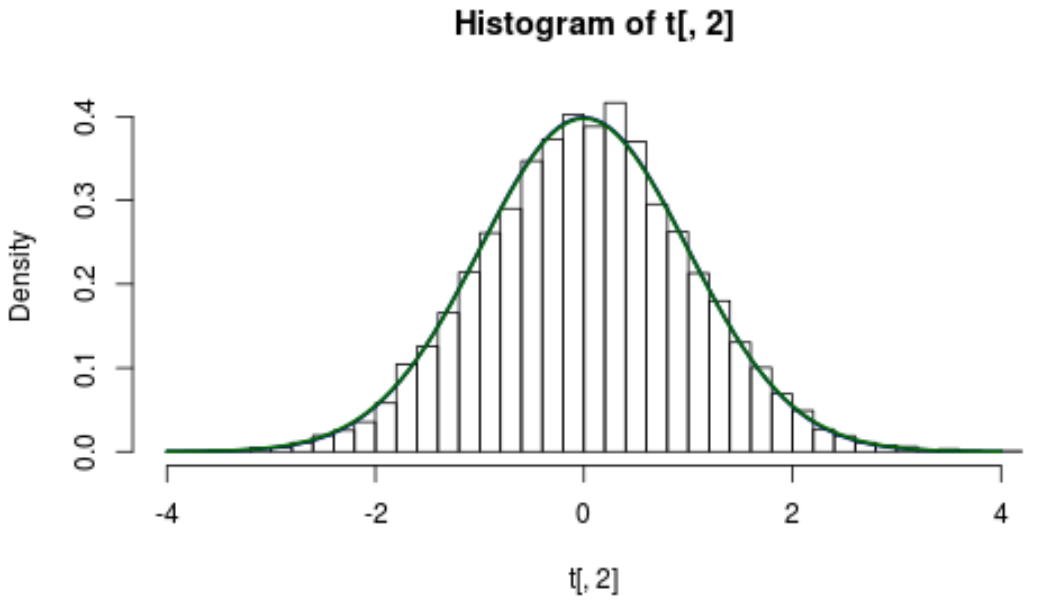
```
hist(q, breaks = 50, probability = T)
curve(dchisq(x, df = n-k), from = 60, to = 160, add=T, lwd=2, col="darkred")
```

[1] For example, using the package `ggplot2`.



We now have graphically demonstrated the truthiness of the two main conditions required to show that $t_2 \sim t_{n-k}$. But let's show that graphically as well!

```
hist(t[ ,2], breaks = reps / 200, probability = T)
curve(dnorm(x, mean = 0, sd = 1), from = -4, to = 4, add=T, lwd=2, col="darkblue")
curve(dt(x, df = n-k), from = -4, to = 4, add=T, lwd=2, col="darkgreen")
```



You'll notice that I threw another normal curve on there for good measure. You'll also notice that it lies more or less on top of the t distribution, which is what we expect with any reasonable df = n − k.

Additional puzzles

1. **Partitioned regression**: Generate a 100×4 matrix $\mathbf{X}$ *including* a column of ones for the intercept. Additionally, generate a vector $\mathbf{y}$ according to the generating process:

$$y_i = 1 + x_{1i} + 2x_{2i} + 3x_{3i} + \epsilon_i,$$

where $\epsilon_i \sim N(0, 1)$. Let $\mathbf{Q}$ be the first three columns of $\mathbf{X}$ and let $\mathbf{N}$ be the final column. In addition, let

$$\begin{aligned} \hat{\gamma}_1 &= (\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{y} \quad \text{and} \quad \mathbf{f} = \mathbf{y} - \mathbf{Q}\hat{\gamma}_1 \\ \hat{\gamma}_2 &= (\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{N} \quad \text{and} \quad \mathbf{g} = \mathbf{N} - \mathbf{Q}\hat{\gamma}_2 \\ \hat{\gamma}_3 &= \mathbf{f} \cdot \mathbf{g}/||\mathbf{g}||^2 \quad \text{and} \quad \mathbf{e} = \mathbf{f} - \mathbf{g}\hat{\gamma}_3 \end{aligned}$$

Show that $\hat{\beta} = [\hat{\gamma}_1 - \hat{\gamma}_2\hat{\gamma}_3 \quad \hat{\gamma}_3]$. Note that the total dimension of $\hat{\beta}$ is 4.