

9.ex.5

EE24BTECH11002 - Agamjot Singh

Question:

Solve the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \quad (1)$$

Solution:

Theoretical solution:

The given differential equation is a second-order linear ordinary differential equation.

Let $y(0) = c_1$ and $y'(0) = c_2$. By definition of Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad (2)$$

Some used properties of Laplace transform include,

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - sy(0) - y'(0) = s^2 \mathcal{L}(y) - sc_1 - c_2 \quad (3)$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2 + 1} \quad (4)$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2 + 1} \quad (5)$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \quad (6)$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}(e^{at} f(t)) = F(s - a) \quad (7)$$

Applying Laplace transform on the given differential equation, we get,

$$y'' + y = 0 \quad (8)$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0 \quad (9)$$

$$s^2 \mathcal{L}(y) - sc_1 - c_2 + \mathcal{L}(y) = 0 \quad (10)$$

$$\mathcal{L}(y) = \frac{sc_1 + c_2}{s^2 + 1} = c_1 \frac{s}{s^2 + 1} + c_2 \frac{1}{s^2 + 1} \quad (11)$$

$$(12)$$

Taking laplace inverse on both sides, we get,

$$y = c_1 \mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) + c_2 \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) \quad (13)$$

$$y = c_1 \cos x + c_2 \sin x \quad (14)$$

$$\implies y(x) = \sqrt{(c_1)^2 + (c_2)^2} \sin\left(x + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right) \quad (15)$$

Computational Solution: Trapezoid Method

The given differential equation can be represented as

$$y'' + y = 0 \quad (16)$$

Let $y = y_1$ and $y' = y_2$, then,

$$\frac{dy_2}{dx} = -y_1 \text{ and } \frac{dy_1}{dx} = y_2 \quad (17)$$

$$\int_{y_{2,n}}^{y_{2,n+1}} dy_2 = \int_{x_n}^{x_{n+1}} -y_1 dx \quad (18)$$

$$\int_{y_{1,n}}^{y_{1,n+1}} dy_1 = \int_{x_n}^{x_{n+1}} y_2 dx \quad (19)$$

$$(20)$$

Discretizing the steps (Trapezoid rule),

$$y_{2,n+1} - y_{2,n} = -\frac{h}{2} (y_{1,n} + y_{1,n+1}) \quad (21)$$

$$y_{1,n+1} - y_{1,n} = \frac{h}{2} (y_{2,n} + y_{2,n+1}) \quad (22)$$

Solving for $y_{1,n+1}$ and $y_{2,n+1}$, we get,

$$y_{1,n+1} = y_{1,n} + \frac{h}{2} \left(2y_{2,n} - \frac{h}{2} (y_{1,n} + y_{1,n+1}) \right) \quad (23)$$

$$(24)$$

The difference equations can be written as,

$$y_{1,n+1} = \frac{(4 - h^2)y_{1,n} + 4hy_{2,n}}{(4 + h^2)} \quad (25)$$

$$y_{2,n+1} = \frac{(4 - h^2)y_{2,n} - 4hy_{1,n}}{(4 + h^2)} \quad (26)$$

$$(27)$$

Iteratively plotting the above system taking initial conditions as

$$x_0 = 0, y_{1,0} = 0, y_{2,0} = 1 \quad (28)$$

we get the plot of the given differential equation.

Alternative Computational Solution: Bilinear transform

We have to apply laplace transformation on the given differential equation. From (11), we get,

$$Y(s) = \frac{sc_1 + c_2}{s^2 + 1} \quad (29)$$

$$Y(s) = \frac{sc_1 + c_2}{s^2 + 1} \quad (30)$$

Applying Bilinear transform, with $T = h$, we get,

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (31)$$

$$\Rightarrow Y(z) = \frac{2hc_1(z^2 - 1) + c_2h^2(z + 1)^2}{(h^2 + 4)z^2 + 2(h^2 - 4)z + (h^2 + 4)} \quad (32)$$

$$\Rightarrow \left(z^2 + 2\frac{h^2 - 4}{h^2 + 4}z + 1\right)Y(z) = \frac{2hc_1(z^2 - 1) + c_2h^2(z^2 + 2z + 1)}{h^2 + 4} \quad (33)$$

$$\Rightarrow z^2Y(z) + 2\frac{h^2 - 4}{h^2 + 4}zY(z) + Y(z) = \frac{(2hc_1 + c_2h^2)z^2 + (2h^2c_2)z + (h^2c_2 - 2hc_1)}{h^2 + 4} \quad (34)$$

Some properties of one sided z transform,

$$\mathcal{Z}(y[n + 2]) = z^2Y(z) - y[1]z - y[0] \quad (35)$$

$$\mathcal{Z}(y[n + 1]) = zY(z) - zy[0] \quad (36)$$

$$\mathcal{Z}(\delta[n]) = 1, z \neq 0 \quad (37)$$

$$\mathcal{Z}(y[n]) = Y(z) \Rightarrow \mathcal{Z}(y[n - n_0]) = z^{-n_0}Y(z) \quad (38)$$

By the time shift property (38),

$$\mathcal{Z}(\delta[n + 2]) = z^2, z \neq 0 \quad (39)$$

$$\mathcal{Z}(\delta[n + 1]) = z, z \neq 0 \quad (40)$$

Rewriting equation (34), we get,

$$\begin{aligned} z^2Y(z) + 2\frac{h^2 - 4}{h^2 + 4}zY(z) + Y(z) + (-y[1]z - y[0]) + 2\left(\frac{h^2 - 4}{h^2 + 4}\right)(-zy[0]) \\ = \frac{(2hc_1 + c_2h^2)z^2 + (2h^2c_2 - y[1] - 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[0])z + (h^2c_2 - 2hc_1 - y[0])}{h^2 + 4} \end{aligned} \quad (41)$$

$$\text{where } z \neq 0 \quad (42)$$

Region of convergence (**ROC**) is given by $z \neq 0$. Taking z inverse transform on both sides of equation (41),

$$\begin{aligned} y[n + 2] + 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[n + 1] + y[n] \\ = \frac{(2hc_1 + c_2h^2)\delta[n + 2] + (2h^2c_2 - y[1] - 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[0])\delta[n + 1] + (h^2c_2 - 2hc_1 - y[0])\delta[n]}{h^2 + 4} \end{aligned} \quad (43)$$

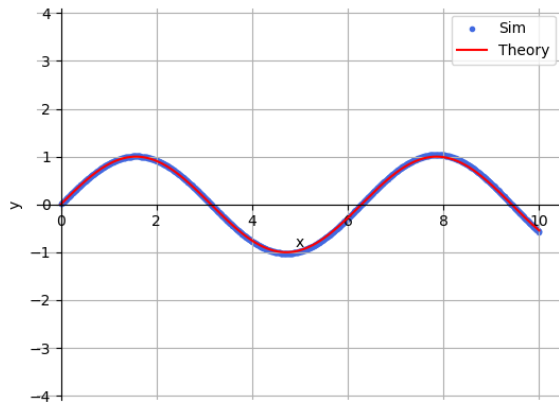


Fig. 0: Computational solution for $y'' + y = 0$