Computationally solving differential equations (9.1.3)

Agamjot Singh,
EE24BTECH11002,
IIT Hyderabad.

January 8, 2025

Outline

Table of Contents

Problem

Problem Statement

Solve the differential equation:

$$\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0 \tag{2.1}$$

Solution

Theoretical Solution

The given differential equation is a second-order nonlinear ordinary differential equation and cannot be theoretcally solved using known methods.

Computation Solution - Euler's Method

By the first principle of derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \tag{3.1}$$

$$y(x+h) = y(x) + hy'(x), h \to 0 \tag{3.2}$$

For a m^{th} order differential equation, let

$$y_1 = y \ , \ y_2 = y' \ , \ y_3 = y'' \ , \ \dots \ , \ y_m = y^{m-1} \tag{3.3}$$

then we obtain the system

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \\ y_m' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(x, y_1, y_2, \dots, y_m) \end{pmatrix} \tag{3.4}$$

Here, f is described by the given differential equation. The initial conditions $y_1(x_0) = K_1$, $y_2(x_0) = K_2$, $\ldots$, $y_m(x_0) = K_m$.
Representing the system in Euler's form
(using first principle of derivative),

$$\begin{pmatrix} y_1(x+h) \\ y_2(x+h) \\ \vdots \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) + hy_2(x) \\ y_2(x) + hy_3(x) \\ \vdots \\ y_m(x) + hf(x, y_1, y_2 \ldots y_m) \end{pmatrix} \tag{3.5}$$

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \ldots, y_m) \end{pmatrix} \tag{3.6}$$

$$
\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{f(x, y_1, y_2, \dots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{3.7}
$$

$$
\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x, y_1, y_2, \dots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{3.8}
$$

Generalizing the system into an iterative format for plotting $y(x)$,

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \\ \vdots \\ y_{m,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{m,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ \vdots \\ f\left(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n}\right) \end{pmatrix} \tag{3.9}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n} \tag{3.10}$$

$$x_{n+1} = x_n + h \tag{3.11}$$

Here, the vector $\mathbf{y_n} = \begin{pmatrix} y_{1,n}(x_n) \\ y_{2,n}(x_n) \\ \vdots \\ y_{m,n}(x_n) \end{pmatrix}$ is not to be confused with y_k which is the $(k-1)^{\text{th}}$ derivative of $y(x)$.

The given differential equation can be represented as,

$$\left(y'\right)^4 + 3yy'' = 0 \tag{3.12}$$

$$y'' = -\frac{\left(y'\right)^4}{3y} \tag{3.13}$$

We see that $m = 2$, thus,

$$y_3 = y'' = -\frac{\left(y'\right)^4}{3y} = -\frac{\left({y_2}^4\right)}{3y_1} \tag{3.14}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{(y')^4}{3y} \end{pmatrix} \tag{3.15}$$

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ -\frac{(y_{2,n})^4}{3y_{1,n}} \end{pmatrix} \tag{3.16}$$

$$\tag{3.17}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h \\ 0 & 1 - \frac{(y_{2,n})^3}{3y_{1,n}} \end{pmatrix} \mathbf{y_n} \tag{3.18}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_{1,0} = 0.01 \ , \ y_{2,0} = 1 \tag{3.19}$$

we get the following plot.

Sim Plot

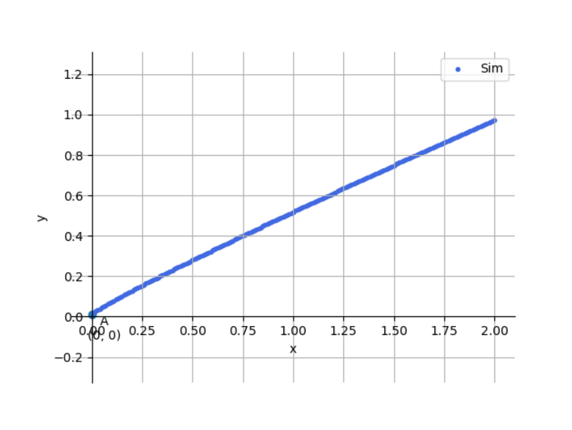

Figure 1: Computational solution for $(y')^4 + 3yy'' = 0$