9.1.3

EE24BTECH11002 - Agamjot Singh

Question:
Solve the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \tag{1}$$

Solution:
Theoritical solution:
The given differential equation is a second-order linear ordinary differential equation. Let $y(0) = c_1$ and $y'(0) = c_2$. By definition of Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{2}$$

Some used properties of Laplace transform include,

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - sc_1 - c_2 \tag{3}$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1} \tag{4}$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1} \tag{5}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{6}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at}f(t)\right) = F(s-a) \tag{7}$$

Applying Laplace transform on the given differential equation, we get,

$$y'' + y = 0 \tag{8}$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0 \tag{9}$$

$$s^2\mathcal{L}(y) - sc_1 - c_2 + \mathcal{L}(y) = 0 \tag{10}$$

$$\mathcal{L}(y) = \frac{sc_1 + c_2}{s^2+1} = c_1\frac{s}{s^2+1} + c_2\frac{1}{s^2+1} \tag{11}$$

$$\tag{12}$$

Taking laplace inverse on both sides, we get,

$$y = c_1\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + c_2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \tag{13}$$

$$y = c_1\cos x + c_2\sin x \tag{14}$$

$$\implies y(x) = \sqrt{(c_1)^2 + (c_2)^2}\sin\left(x + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right) \tag{15}$$

Computational Solution: Euler's method
By the first principle of derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \tag{16}$$

$$y(x+h) = y(x) + h\left(y'(x)\right), h \to 0 \tag{17}$$

For a m^{th} order differential equation,

Let

$$y_1 = y \ , \ y_2 = y' \ , \ y_3 = y'' \ , \ \ldots \ , \ y_m = y^{m-1} \tag{18}$$

then we obtain the system

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \\ y_m' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(x, y_1, y_2, \ldots, y_m) \end{pmatrix} \tag{19}$$

Here, f is described by the given differential equation. The initial conditions $y_1(x_0) = K_1$, $y_2(x_0) = K_2$, $\ldots$, $y_m(x_0) = K_m$.

Representing the system in Euler's form (using first principle of derivative),

$$\begin{pmatrix} y_1(x+h) \\ y_2(x+h) \\ \vdots \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) + hy_2(x) \\ y_2(x) + hy_3(x) \\ \vdots \\ y_m(x) + hf(x, y_1, y_2 \dots y_m) \end{pmatrix} \tag{20}$$

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \dots, y_m) \end{pmatrix} \tag{21}$$

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{f(x,y_1,y_2,\dots,y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{22}$$

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x,y_1,y_2,\dots,y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{23}$$

Generalizing the system into an iterative format for plotting $y(x)$,

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \\ \vdots \\ y_{m,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{m,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ \vdots \\ f(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n}) \end{pmatrix} \tag{24}$$

$$\mathbf{y_{n+1}} = \mathbf{y_n} + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n}, \text{ where } \mathbf{y_n} = \begin{pmatrix} y_{1,n}(x_n) \\ y_{2,n}(x_n) \\ \vdots \\ y_{m,n}(x_n) \end{pmatrix} \tag{25}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & h & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & h & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & h \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 + \frac{f(x_n, y_{1,n}, y_{2,n}, \ldots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n} \tag{26}$$

$$x_{n+1} = x_n + h \tag{27}$$

Here, the vector $\mathbf{y_n}$ is not to be confused with y_k which is the $(k-1)^{\text{th}}$ derivative of $y(x)$. The given differential equation can be represented as,

$$(y')^4 + 3yy'' = 0 \tag{28}$$

$$y'' = -\frac{(y')^4}{3y} \tag{29}$$

We see that $m = 2$, thus,

$$y_3 = y'' = -\frac{(y')^4}{3y} = -\frac{\left({y_2}^4\right)}{3y_1} \tag{30}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{(y')^4}{3y} \end{pmatrix} \tag{31}$$

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ -\frac{(y_{2,n})^4}{3y_{1,n}} \end{pmatrix} \tag{32}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h \\ 0 & 1 - \frac{(y_{2,n})^3}{3y_{1,n}} \end{pmatrix} \mathbf{y_n} \tag{33}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_{1,0} = 0.01 \ , \ y_{2,0} = 1 \tag{34}$$

we get the following plot.

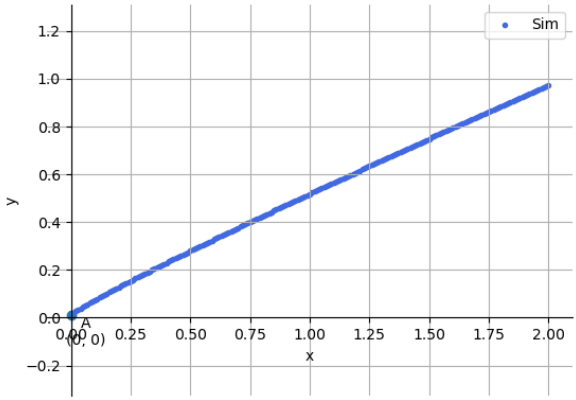

Fig. 0: Computational solution for $(y')^4 + 3yy'' = 0$