

11.16.3.8.9

EE24BTECH11002 - Agamjot Singh

Question:

Three coins are tossed at once. Find the probability of getting atmost two tails.

Solution:

Sample space (Ω) is given by,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \quad (1)$$

Event space (\mathcal{F}) is given by,

$$\mathcal{F} = 2^\Omega \quad (2)$$

Let X be the random variable,

$$X = \text{number of tails in the sequence} \quad (3)$$

We express this random variable as a sum of 3 bernoulli random variables.

$$X = X_1 + X_2 + X_3 \quad (4)$$

where,

$$X_i = \begin{cases} 0 & i^{\text{th}} \text{ toss is a Heads} \\ 1 & i^{\text{th}} \text{ toss is a Tails} \end{cases} \quad (5)$$

X models a binomial distribution.

For converting to z -domain, we use the property,

$$M_X(z) = M_{X_1}(z) M_{X_2}(z) M_{X_3}(z) \quad (6)$$

Extending this system to m tosses, we get,

$$M_X(z) = \prod_{k=1}^m M_{X_k}(z) \quad (7)$$

Let probability mass function for the bernoulli random variable X_i be given by,

$$p_{X_i}(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \\ 0 & n = \mathbb{Z} - \{0, 1\} \end{cases} \quad (8)$$

where p is the probability of getting heads.

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(n) z^{-k} = p + (1-p)z^{-1} \quad (9)$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(n) z^{-k} = p + (1-p)z^{-1} \quad (10)$$

$$\vdots \quad (11)$$

$$M_{X_m}(z) = \sum_{k=-\infty}^{\infty} p_{X_m}(n) z^{-k} = p + (1-p)z^{-1} \quad (12)$$

$$\Rightarrow M_X(z) = (p + (1-p)z^{-1})^m \quad (13)$$

$$M_X(z) = \sum_{k=-\infty}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} z^{-k} \quad (14)$$

$$(15)$$

Taking z -inverse on both sides, we get,

$$p_X(n) = {}^m C_n p^{m-n} (1-p)^n \quad (16)$$

Taking $m = 3$ and $p = \frac{1}{2}$,

$$p_X(n) = {}^3 C_n \left(\frac{1}{2}\right)^3 \quad (17)$$

Using this probability mass function, the cumulative distribution function C.D.F ($F_X(n)$) is given by,

$$F_X(n) = \begin{cases} 0 & n < 0 \\ {}^3 C_0 \left(\frac{1}{2}\right)^3 = \frac{1}{8} & 0 \leq n < 1 \\ {}^3 C_0 \left(\frac{1}{2}\right)^3 + {}^3 C_1 \left(\frac{1}{2}\right)^3 = \frac{1}{2} & 1 \leq n < 2 \\ {}^3 C_0 \left(\frac{1}{2}\right)^3 + {}^3 C_1 \left(\frac{1}{2}\right)^3 + {}^3 C_2 \left(\frac{1}{2}\right)^3 = \frac{7}{8} & 2 \leq n < 3 \\ {}^3 C_0 \left(\frac{1}{2}\right)^3 + {}^3 C_1 \left(\frac{1}{2}\right)^3 + {}^3 C_2 \left(\frac{1}{2}\right)^3 + {}^3 C_3 \left(\frac{1}{2}\right)^3 = 1 & 3 \leq n \end{cases} \quad (18)$$

Let A be an event defined as,

$$A: \text{Getting atmost two tails} \quad (19)$$

$$\Pr(A) = P(X \leq 2) = F_X(2) \quad (20)$$

$$\Pr(A) = \frac{7}{8} = 0.875 \quad (21)$$

Simulation Running a simulation requires generating random numbers with uniform probability. This is done using OpenSSL's random byte generator.

- 1) 1 byte of randomly generated uniform data is generated using OpenSSL rand.h.
- 2) This random number is scaled down from $\{0, 1, 2, \dots, 255\}$ to $[0, 1]$ by dividing by 255.

- 3) For generating the bernoulli random variable, if this normalized number is less than p then 0 is returned, else 1 is returned.

As number of trials increase, the relative frequency converges to the actual probability of the event.

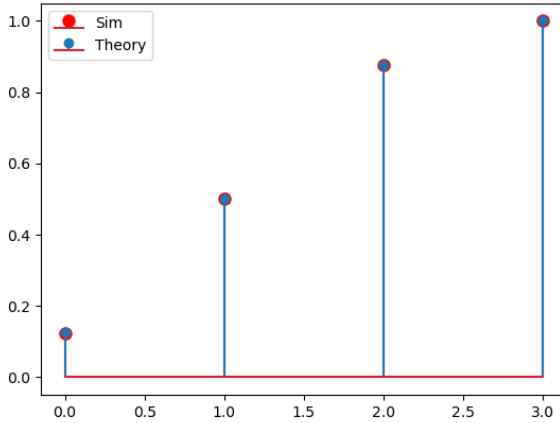


Fig. 3: CDF Plot

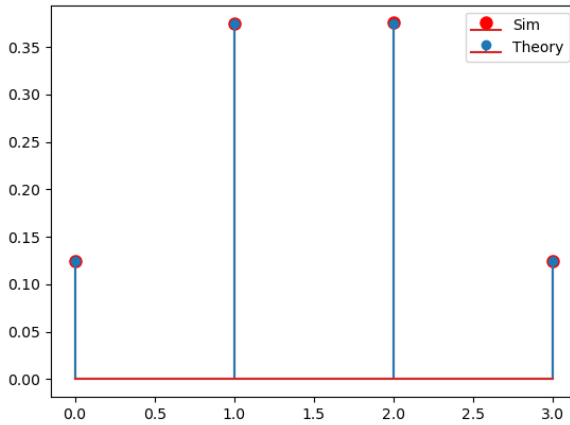


Fig. 3: PMF plot for $m = 3$

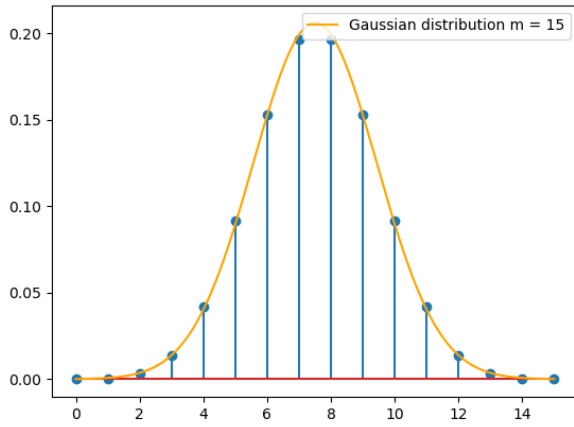


Fig. 3: PMF plot for $m = 15$ with the gaussian distribution plot