

6.5.16

EE24BTECH11002 - Agamjot Singh

prim Question:

Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Solution:

Theoretical solution:

Let the two numbers be x and y , $x, y \geq 0$. It is given that,

$$x + y = 16 \quad (1)$$

and we have to minimize

$$f(x, y) = x^3 + y^3, \quad x, y > 0 \quad (2)$$

Writing y in terms of x , we get,

$$f(x) = x^3 + (16 - x)^3 = 3x^2 - 48x + 256, \quad 0 < x < 16 \quad (3)$$

$$f'(x) = 3x^2 - 3(16 - x)^2 \quad (4)$$

$$f''(x) = 6x + 6(16 - x) \quad (5)$$

For a minimum to occur, $f'(x) = 0$ and $f''(x) > 0$,

$$f'(x) = 0 \quad (6)$$

$$\implies 3x^2 - 3(16 - x)^2 = 0 \quad (7)$$

$$\implies x = 8 \quad (8)$$

To verify if it is a minimum,

$$f''(8) = 96 > 0 \quad (9)$$

$$\implies x = 8, y = 16 - x = 8 \text{ is where the minimum occurs} \quad (10)$$

$$\implies f_{\min} = 8^3 + 8^3 = 1024 \quad (11)$$

Computational Solution: Gradient Descent algorithm

By the gradient descent algorithm, the difference equation is given by,

$$x_{n+1} = x_n - \mu f'(x) \quad (12)$$

$$\implies x_{n+1} = x_n - \mu(6x_n - 48) \quad (13)$$

$$\implies x_{n+1} = (1 - 6\mu)x_n + 48\mu \quad (14)$$

where f is the objective function given by equation (??) and $\mu > 0$ is the step size. Taking

one sided Z-transform on both sides of (??),

$$zX(z) - zx_0 = (1 - 6\mu)X(z) + 48\mu \quad (15)$$

$$(z + 6\mu - 1)X(z) = 48\mu + zx_0 \quad (16)$$

$$X(z) = \frac{48\mu + zx_0}{z + 6\mu - 1} \quad (17)$$

$$X(z) = \frac{48\mu z^{-1}}{1 + (6\mu - 1)z^{-1}} + \frac{x_0}{1 + (6\mu - 1)z^{-1}} \quad (18)$$

$$X(z) = 48\mu \sum_{n=0}^{\infty} (1 - 6\mu)^n z^{-(n+1)} + x_0 \sum_{n=0}^{\infty} (1 - 6\mu)^n z^{-n} \quad (19)$$

$$X(z) = (48\mu z^{-1} + x_0) \sum_{n=0}^{\infty} (1 - 6\mu)^n z^{-n} \quad (20)$$

By (??), ROC is given by,

$$\left| \frac{1 - 6\mu}{z} \right| < 1 \quad (21)$$

$$\implies |z| > |1 - 6\mu| \quad (22)$$

$$\implies |1 - 6\mu| > 0 \quad (23)$$

$$\implies \mu \in \mathbb{R} \setminus \left\{ \frac{1}{6} \right\} \quad (24)$$

If the sequence x_n has to converge,

$$\lim_{n \rightarrow \infty} |x_{n+1} - x_n| = 0 \quad (25)$$

$$\implies \lim_{n \rightarrow \infty} |-6\mu x_n + 48\mu| = 0 \quad (26)$$

$$\implies \mu \lim_{n \rightarrow \infty} |-6x_n + 48| = 0, \mu > 0 \quad (27)$$

$$\implies \lim_{n \rightarrow \infty} x_n = 8 \quad (28)$$

We take the initial guess = 7, step size = 0.01, tolerance = 0.0001.

Scipy Solution:

$$f_{\min} = 1024.0000000000005, x_{\min} = 7.999999930756325 \quad (29)$$

Using the gradient descent algorithm, we get,

$$f_{\min} = 1024.000000004813, x_{\min} = 7.999989986419678 \quad (30)$$

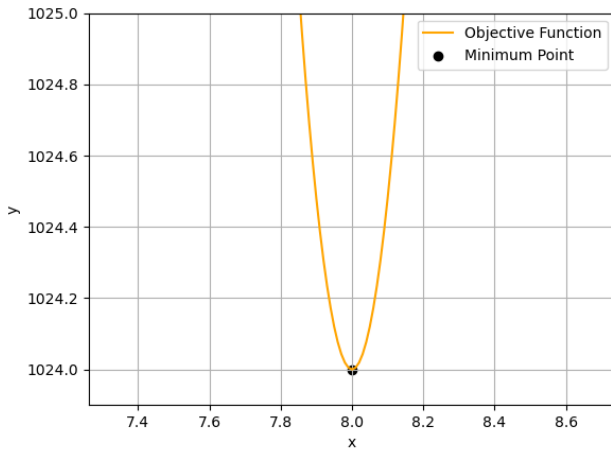


Fig. 0: Objective Function with the minimum point