9.ex.5

EE24BTECH11002 - Agamjot Singh

Question:
Solve the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \tag{1}$$

Solution:
Theoritical solution:
The given differential equation is a second-order linear ordinary differential equation. Let $y(0) = c_1$ and $y'(0) = c_2$. By definition of Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{2}$$

Some used properties of Laplace transform include,

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - sc_1 - c_2 \tag{3}$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1} \tag{4}$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1} \tag{5}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{6}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at}f(t)\right) = F(s-a) \tag{7}$$

Applying Laplace transform on the given differential equation, we get,

$$y'' + y = 0 \tag{8}$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0 \tag{9}$$

$$s^2\mathcal{L}(y) - sc_1 - c_2 + \mathcal{L}(y) = 0 \tag{10}$$

$$\mathcal{L}(y) = \frac{sc_1 + c_2}{s^2+1} = c_1\frac{s}{s^2+1} + c_2\frac{1}{s^2+1} \tag{11}$$

$$\tag{12}$$

Taking laplace inverse on both sides, we get,

$$y = c_1\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + c_2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \tag{13}$$

$$y = c_1\cos x + c_2\sin x \tag{14}$$

$$\implies y(x) = \sqrt{(c_1)^2 + (c_2)^2}\sin\left(x + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right) \tag{15}$$

Computational Solution: Trapezoid Method
The given differential equation can be represented as

$$y'' + y = 0 \tag{16}$$

Let $y = y_1$ and $y' = y_2$, then,

$$\frac{dy_2}{dx} = -y_1 \text{ and } \frac{dy_1}{dx} = y_2 \tag{17}$$

$$\int_{y_{2,n}}^{y_{2,n+1}} dy_2 = \int_{x_n}^{x_{n+1}} -y_1\, dx \tag{18}$$

$$\int_{y_{1,n}}^{y_{1,n+1}} dy_1 = \int_{x_n}^{x_{n+1}} y_2\, dx \tag{19}$$

$$\tag{20}$$

Discretizing the steps (Trapezoid rule),

$$y_{2,n+1} - y_{2,n} = -\frac{h}{2}\left(y_{1,n} + y_{1,n+1}\right) \tag{21}$$

$$y_{1,n+1} - y_{1,n} = \frac{h}{2}\left(y_{2,n} + y_{2,n+1}\right) \tag{22}$$

Solving for $y_{1,n+1}$ and $y_{2,n+1}$, we get,

$$y_{1,n+1} = y_{1,n} + \frac{h}{2}\left(2y_{2,n} - \frac{h}{2}\left(y_{1,n} + y_{1,n+1}\right)\right) \tag{23}$$

$$\tag{24}$$

The difference equations can be written as,

$$y_{1,n+1} = \frac{\left(4 - h^2\right) y_{1,n} + 4hy_{2,n}}{(4 + h^2)} \tag{25}$$

$$y_{2,n+1} = \frac{\left(4 - h^2\right) y_{2,n} - 4hy_{1,n}}{(4 + h^2)} \tag{26}$$

$$\tag{27}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \text{ , } y_{1,0} = 0 \text{ , } y_{2,0} = 1 \tag{28}$$

we get the following plot.

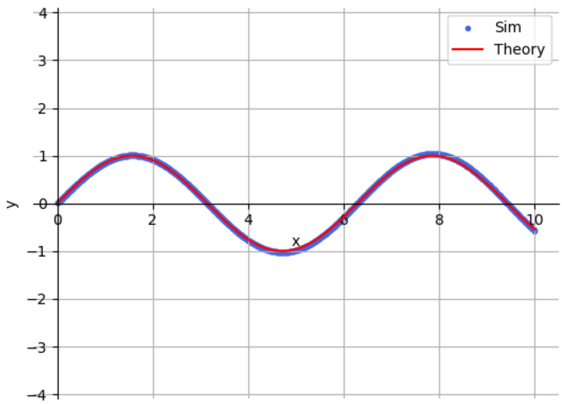

Fig. 0: Computational solution for $y'' + y = 0$