Computationally solving quadratic equations (10.4.2.6)

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Outline

Table of Contents

Problem

Problem Statement

A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was 90, find the number of articles produced and the cost of each article.

Solution

Theoretical Solution

Let the number of articles produced in a day be x, then the above question can the following quadratic equation will be formed.

$$(3 + 2x)\,x = 90 \tag{3.1}$$

$$2x^2 + 3x - 90 = 0 \tag{3.2}$$

Theoretically, it can easily be solved using the quadratic formula,

$$x = \frac{-3 \pm \sqrt{729}}{4} = 6, -7.5 \tag{3.3}$$

Computation Solution - Newton Raphson's Method

Now we use the **Newton-Raphson method** to computationally find the roots.

Let

$$f(x) = 2x^2 + 3x - 90 \tag{3.4}$$

$$\implies f'(x) = 4x + 3 \tag{3.5}$$

The difference equation by the Newton-Raphson method is given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ f'(x_n) \neq 0 \tag{3.6}$$

If we, at any point, encouter a situation in which $f'(x_n) = 0$, it implies that our inital guess leads to failure of the Newton method to converge to a root and we have to abandon and restart with another intial guess.

Substituting $f(x)$ and $f'(x)$ in the difference equation, we get,

$$x_{n+1} = x_n - \left(\frac{2(x_n)^2 + 3x_n - 90}{4x_n + 3} \right) \tag{3.7}$$

$$x_{n+1} = \frac{2(x_n)^2 + 90}{4x_n + 3},\ x_n \neq -\frac{3}{4} \tag{3.8}$$

Taking initial guess $(x_0) = 8$, we get the root as $x = 6.000000476837158$. Taking initial guess $(x_0) = -8$, we get the root as $x = -7.500000476837158$.

Computation Solution - Eigenvalue Method

Frobenius **companion matrix** for a polynomial p of the form,

$$p\left(x\right) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + x^n \tag{3.9}$$

is given by

$$C\left(p\right) = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{pmatrix} \tag{3.10}$$

The eigen values of this companion matrix are the roots of the polynomial p. This is because of the characteristic polynomial for this matrix being

$$c_0 I + c_1 C + c_2 C^2 + \cdots + c_{n-1} C^{n-1} + C^n = 0 \tag{3.11}$$

$$c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots + c_{n-1} \lambda^{n-1} + \lambda^n = 0 \tag{3.12}$$

$$\tag{3.13}$$

where λ is the eigenvalue of $C(p)$.

$$p(x) = x^2 + \frac{3}{2}x - 45 = 0 \tag{3.14}$$

$$C(p) = \begin{pmatrix} 0 & 45 \\ 1 & -1.5 \end{pmatrix} \tag{3.15}$$

We find the eigenvalues using the QR algorithm. The basic principle behind this algorithm is a similarity transform,

$$A' = X^{-1}AX \tag{3.16}$$

which does not alter the eigenvalues of the matrix A.
We use this to get the Schur Decomposition,

$$A = Q^{-1}UQ = Q^{*}UQ \tag{3.17}$$

where Q is a unitary matrix $\left(Q^{-1} = Q^{*}\right)$ and U is an upper triangular matrix whose diagonal entries are the eigenvalues of A. To efficiently get the Schur Decomposition, we first householder reflections to reduce it to an upper hessenberg form.

A householder reflector matrix is of the form,

$$P = I - 2\mathbf{u}\mathbf{u}^* \tag{3.18}$$

Householder reflectors transforms any vector $\mathbf{x}$ to a multiple of $\mathbf{e_1}$,

$$P\mathbf{x} = \mathbf{x} - 2\mathbf{u}\left(\mathbf{u}^*\mathbf{x}\right) = \alpha\mathbf{e_1} \tag{3.19}$$

P is unitary, which implies that,

$$\|P\mathbf{x}\| = \|\mathbf{x}\| \tag{3.20}$$

$$\implies \alpha = \rho\|\mathbf{x}\| \tag{3.21}$$

$$\tag{3.22}$$

As $\mathbf{u}$ is unit norm,

$$\mathbf{u} = \frac{\mathbf{x} - \rho\|\mathbf{x}\|\mathbf{e_1}}{\|\mathbf{x} - \rho\|\mathbf{x}\|\mathbf{e_1}\|} = \frac{1}{\|\mathbf{x} - \rho\|\mathbf{x}\|\mathbf{e_1}\|}\begin{pmatrix} x_1 - \rho\|\mathbf{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3.23}$$

Selection of ρ is flexible as long as $|\rho| = 1$. To ease out the process, we take $\rho = \frac{x_1}{|x_1|}$, $x_1 \neq 0$. If $x_1 = 0$, we take $\rho = 1$. Householder reflector matrix (P_i) is given by,

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & I_{n-i} - 2\mathbf{u_i}\mathbf{u_i}^* \end{bmatrix} \tag{3.24}$$

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \tag{3.25}$$

Next step is to do Given's rotation to get the *QR* Decomposition.

The Givens rotation matrix $G\left(i, j, c, s\right)$ is defined by

$$G\left(i, j, c, s\right) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{3.26}$$

where $|c|^2 + |s|^2 = 1$, and G is a unitary matrix.

Say we take a vector $\mathbf{x}$, and $\mathbf{y} = G(i, j, c, s)\,\mathbf{x}$, then

$$y_k = \begin{cases} cx_i + sx_j, & k = i \\ -\bar{s}x_i + \bar{c}x_j, & k = j \\ x_k, & k \neq i, j \end{cases} \tag{3.27}$$

For y_j to be zero, we set

$$c = \frac{\overline{x_i}}{\sqrt{|x_i|^2 + |x_j|^2}} = c_{ij} \tag{3.28}$$

$$s = \frac{\overline{x_j}}{\sqrt{|x_i|^2 + |x_j|^2}} = s_{ij} \tag{3.29}$$

Using this Givens rotation matrix, we zero out elements of subdiagonal in the hessenberg matrix H.

$$
H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(1,2,c_{12},s_{12})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}
$$

$$
\xrightarrow{G(2,3,c_{23},s_{23})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,4,c_{34},s_{34})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}
$$

$$
\xrightarrow{G(4,5,c_{45},s_{45})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} = R \quad (3.30)
$$

where R is upper triangular.

For the given companion matrix,

$$c_{11} = \frac{0}{\sqrt{0^2 + 1^2}} = 0 \tag{3.31}$$

$$s_{11} = \frac{1}{\sqrt{0^2 + 1^2}} = 1 \tag{3.32}$$

Let $G_k = G\left(k, k+1, c_{k,k+1}, s_{k,k+1}\right)$, then we deduce that

$$G_4 G_3 G_2 G_1 H = R \tag{3.33}$$

$$H = G_1^* G_2^* G_3^* G_4^* R \tag{3.34}$$

$$H = QR, \text{ where } Q = G_1^* G_2^* G_3^* G_4^* \tag{3.35}$$

Using this QR algorithm, we get the following update equation,

$$A_k = Q_k R_k \tag{3.36}$$

$$A_{k+1} = R_k Q_k \tag{3.37}$$

$$= \left(G_n \dots G_2 G_1\right) A_k \left(G_1^* G_2^* \dots G_n^*\right) \tag{3.38}$$

The QR algorithm will sometimes converge to matrix which is of the form where some subdiagonal elements will not converge to 0, and there will be 2×2 blocks protruding the diagonal of the matrix. These can be solved be taking all such blocks, and then taking the eigenvalues of all these sub 2×2 matrix blocks.

Running the eigenvalue code for our companion matrix we get,

$$6 + 0.0i, -7.5 + 0.0i \tag{3.39}$$

Plot

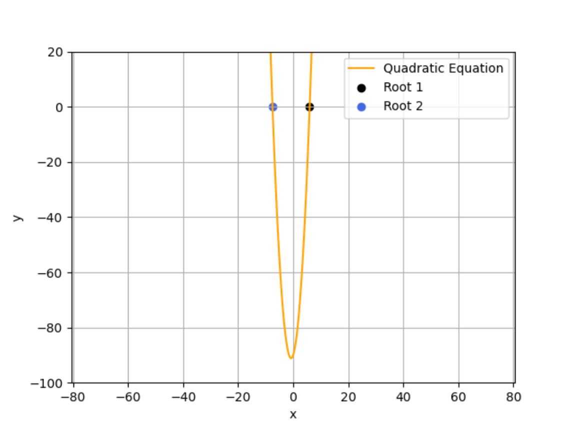

Figure 1: Objective Function with the minimum point