8.1.6

EE24BTECH11002 - Agamjot Singh

Question:
Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.

Theoritical solution: The curves, which are given by $y = \frac{x}{\sqrt{3}}$ and $y = \sqrt{4 - x^2}$, intersect at $\left(\sqrt{3}, 1\right)$ in the first quadrant. The curve $y = \sqrt{4 - x^2}$ touches the x-axis at $(2, 0)$.
For finding the area enclosed A, we split the area into 2 parts A_1 and A_2 such that

$$A = A_1 + A_2 \tag{1}$$

$$A_1 = \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\, dx \tag{2}$$

$$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}} \tag{3}$$

$$= \frac{\sqrt{3}}{2} \approx 0.86602 \tag{4}$$

$$A_2 = \int_{\sqrt{3}}^{1} \sqrt{4 - x^2}\, dx \tag{5}$$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + 2\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2} \tag{6}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} \approx 0.18117 \tag{7}$$

$$\implies A = A_1 + A_2 \approx 1.04719 \tag{8}$$

Computational Solution: Trapezoidal rule
For finding the approximate area enclosed using numerical methods, we use the Trapezoidal method. We split the area into multiple small trapeziums (like small strips), and we sum up all the trapezium areas to find the total area.
We discretize the range of x-coordinates with uniform step-size $h \to 0$, such that the discretized points are $x_0, x_1, \ldots, x_n$ and $x_{n+1} = x_n + h$.
Let the sum of trapizoidal areas till x_n be A_n and $y = y(x)$, then we write the

difference equation,

$$A_n = \frac{h}{2}\left(y\left(x_0\right) + y\left(x_1\right)\right) + \frac{h}{2}\left(y\left(x_1\right) + y\left(x_2\right)\right) + \cdots + \frac{h}{2}\left(y\left(x_{n-1}\right) + y\left(x_n\right)\right) \tag{9}$$

$$A_n = h\left(\frac{y\left(x_0\right)}{2} + y\left(x_1\right) + y\left(x_2\right)\ldots\frac{y\left(x_n\right)}{2}\right) \tag{10}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_{n+1}\right) + y\left(x_n\right)\right),\ x_{n+1} = x_n + h \tag{11}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{12}$$

$$\tag{13}$$

By the first principle of derivative,

$$y'\left(x\right) = \lim_{h\to 0}\frac{y\left(x+h\right) - y\left(x\right)}{h} \tag{14}$$

$$y\left(x+h\right) = y\left(x\right) + h\left(y'\left(x\right)\right),\ h\to 0 \tag{15}$$

Rewriting the difference equation, we get,

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_n\right) + hy'\left(x_n\right) + y\left(x_n\right)\right) \tag{16}$$

$$A_{n+1} = A_n + h\left(y\left(x_n\right) + \frac{h}{2}y'\left(x_n\right)\right) \tag{17}$$

$$A_{n+1} = A_n + hy\left(x_n\right) + \frac{h^2}{2}y'\left(x_n\right) \tag{18}$$

Representing $y\left(x_n\right)$ as y_n, the difference equation becomes,

$$A_{n+1} = A_n + hy_n + \frac{h^2}{2}y'_n \tag{19}$$

For the given area enclosed, we take

$$y\left(x\right) = \begin{cases} \frac{x}{\sqrt{3}} & 0 < x < \sqrt{3} \\ \sqrt{4-x^2} & \sqrt{3} < x < 1 \end{cases} \tag{20}$$

Computational Area: 1.047367
Theoritical Area: 1.04719
Plotting the given equations, we get the following plot.

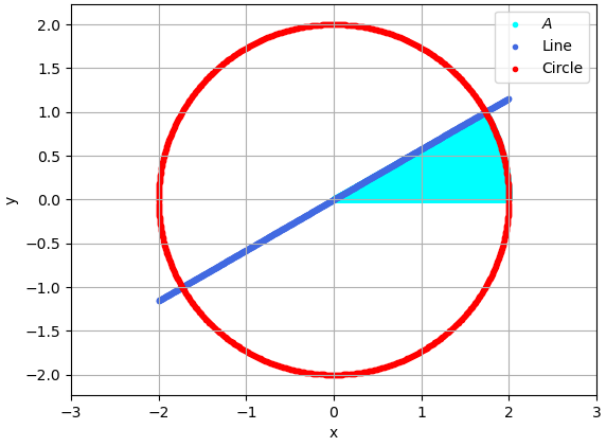

Fig. 0: Shaded area with the circle and line