6.5.16

EE24BTECH11002 - Agamjot Singh

prim **Question:**
Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
Solution:

Theoritical solution:
Let the two numbers be x and y, $x, y \geq 0$. It is given that,

$$x + y = 16 \tag{1}$$

and we have to minimize

$$f(x, y) = x^3 + y^3, , \; x, y > 0 \tag{2}$$

Writing y in terms of x, we get,

$$f(x) = x^3 + (16 - x)^3 = 3x^2 - 48x + 256, \; 0 < x < 16 \tag{3}$$

$$f'(x) = 3x^2 - 3(16 - x)^2 \tag{4}$$

$$f''(x) = 6x + 6(16 - x) \tag{5}$$

For a minimum to occur, $f'(x) = 0$ and $f''(x) > 0$,

$$f'(x) = 0 \tag{6}$$

$$\implies 3x^2 - 3(16 - x)^2 = 0 \tag{7}$$

$$\implies x = 8 \tag{8}$$

To verify if it is a minimum,

$$f''(8) = 96 > 0 \tag{9}$$

$$\implies x = 8, \; y = 16 - x = 8 \text{ is where the minimum occurs} \tag{10}$$

$$\implies f_{\text{min}} = 8^3 + 8^3 = 1024 \tag{11}$$

Computational Solution: Gradient Descent algorithm
By the gradient descent algorithm, the difference equation is given by,

$$x_{n+1} = x_n - \mu f'(x) \tag{12}$$

$$\implies x_{n+1} = x_n - \mu(6x_n - 48) \tag{13}$$

$$\implies x_{n+1} = (1 - 6\mu) x_n + 48\mu \tag{14}$$

where f is the objective function given by equation (3) and $\mu > 0$ is the step size. Taking one sided Z-transform on both sides of (14),

$$zX(z) - zx_0 = (1-6\mu)X(z) + 48\mu \tag{15}$$

$$(z+6\mu-1)X(z) = 48\mu + zx_0 \tag{16}$$

$$X(z) = \frac{48\mu + zx_0}{z+6\mu-1} \tag{17}$$

$$X(z) = \frac{48\mu z^{-1}}{1+(6\mu-1)z^{-1}} + \frac{x_0}{1+(6\mu-1)z^{-1}} \tag{18}$$

$$X(z) = 48\mu \sum_{n=0}^{\infty} (1-6\mu)^n z^{-(n+1)} + x_0 \sum_{n=0}^{\infty} (1-6\mu)^n z^{-n} \tag{19}$$

$$X(z) = \left(48\mu z^{-1} + x_0\right) \sum_{n=0}^{\infty} (1-6\mu)^n z^{-n} \tag{20}$$

By (20), ROC is given by,

$$\left|\frac{1-6\mu}{z}\right| < 1 \tag{21}$$

$$\implies |z| > |1-6\mu| \tag{22}$$

$$\implies |1-6\mu| > 0 \tag{23}$$

$$\implies \mu \in \mathbb{R} \setminus \left\{\frac{1}{6}\right\} \tag{24}$$

If the sequence x_n has to converge,

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0 \tag{25}$$

$$\implies \lim_{n\to\infty} |-6\mu x_n + 48\mu| = 0 \tag{26}$$

$$\implies \mu \lim_{n\to\infty} |-6x_n + 48| = 0,\ \mu > 0 \tag{27}$$

$$\implies \lim_{n\to\infty} x_n = 8 \tag{28}$$

We take the initial guess = 7, step size = 0.01, tolerance = 0.0001.
Scipy Solution: 7.999999930756325
Using the gradient descent algorithm, we get x_min = 7.999989986419678

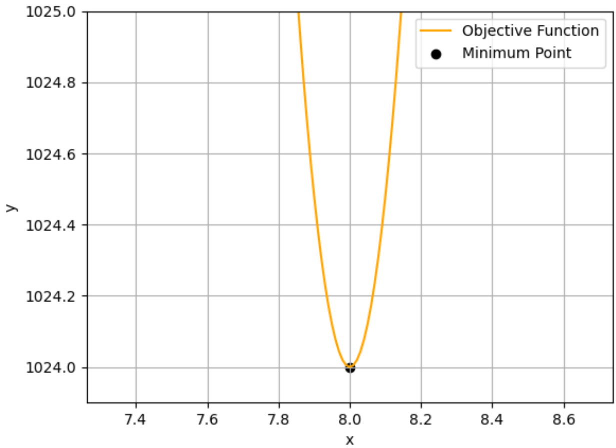

Fig. 0: Objective Function with the minimum point