Computationally solving differential equations (9.1.3)

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Problem Statement

Solve the differential equation:

$$\frac{d^2y}{dx^2} + y = 0 \tag{2.1}$$

Solution

Theoretical Solution

The given differential equation is a second-order linear ordinary differential equation.

Let $y(0) = c_1$ and $y'(0) = c_2$. By definition of Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt \tag{3.1}$$

Some used properties of Laplace transform include,

$$\mathcal{L}\left(y''\right) = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - sc_1 - c_2 \tag{3.2}$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1} \tag{3.3}$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1} \tag{3.4}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{3.5}$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}\left(e^{at} f(t)\right) = F(s-a) \tag{3.6}$$

Applying Laplace transform on the given differential equation, we get,

$$y'' + y = 0 \tag{3.7}$$

$$\mathcal{L}\left(y''\right) + \mathcal{L}\left(y\right) = 0 \tag{3.8}$$

$$s^2\mathcal{L}\left(y\right) - sc_1 - c_2 + \mathcal{L}\left(y\right) = 0 \tag{3.9}$$

$$\mathcal{L}\left(y\right) = \frac{sc_1 + c_2}{s^2 + 1} = c_1\frac{s}{s^2 + 1} + c_2\frac{1}{s^2 + 1} \tag{3.10}$$

Taking laplace inverse on both sides, we get,

$$y = c_1\mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) + c_2\mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) \tag{3.11}$$

$$y = c_1 \cos x + c_2 \sin x \tag{3.12}$$

$$\implies y\left(x\right) = \begin{cases} \sqrt{\left(c_1\right)^2 + \left(c_2\right)^2}\sin\left(x + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right) & c_2 \neq 0 \\ c_1 \cos x & c_2 = 0 \end{cases} \tag{3.13}$$

Computation Solution - Trapezoid Method

The given differential equation can be represented as

$$y'' + y = 0 \tag{3.14}$$

Let $y = y_1$ and $y' = y_2$, then,

$$\frac{dy_2}{dx} = -y_1 \text{ and } \frac{dy_1}{dx} = y_2 \tag{3.15}$$

$$\int_{y_{2,n}}^{y_{2,n+1}} dy_2 = \int_{x_n}^{x_{n+1}} -y_1 \, dx \tag{3.16}$$

$$\int_{y_{1,n}}^{y_{1,n+1}} dy_1 = \int_{x_n}^{x_{n+1}} y_2 \, dx \tag{3.17}$$

$$\tag{3.18}$$

Discretizing the steps (Trapezoid rule),

$$y_{2,n+1} - y_{2,n} = -\frac{h}{2}\left(y_{1,n} + y_{1,n+1}\right) \tag{3.19}$$

$$y_{1,n+1} - y_{1,n} = \frac{h}{2}\left(y_{2,n} + y_{2,n+1}\right) \tag{3.20}$$

Solving for $y_{1,n+1}$ and $y_{2,n+1}$, we get,

$$y_{1,n+1} = y_{1,n} + \frac{h}{2}\left(2y_{2,n} - \frac{h}{2}\left(y_{1,n} + y_{1,n+1}\right)\right) \tag{3.21}$$

The difference equations can be written as,

$$y_{1,n+1} = \frac{\left(4 - h^2\right) y_{1,n} + 4hy_{2,n}}{\left(4 + h^2\right)} \tag{3.22}$$

$$y_{2,n+1} = \frac{\left(4 - h^2\right) y_{2,n} - 4hy_{1,n}}{\left(4 + h^2\right)} \tag{3.23}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_{1,0} = 0 \ , \ y_{2,0} = 1 \tag{3.24}$$

we get the plot of the given differential equation.

Computation Solution - Bilinear Transform Method

We have to apply laplace transformation on the given differential equation. From (3.10), we get,

$$Y(s) = \frac{sc_1 + c_2}{s^2 + 1} \tag{3.25}$$

$$Y(s) = \frac{sc_1 + c_2}{s^2 + 1} \tag{3.26}$$

Applying Bilinear transform, with $T = h$, we get,

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} = \frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{3.27}$$

$$\implies Y(z) = \frac{2hc_1\left(z^2 - 1\right) + c_2 h^2 (z + 1)^2}{(h^2 + 4) z^2 + 2(h^2 - 4) z + (h^2 + 4)} \tag{3.28}$$

Note that this transformation is stable only when the poles of $Y(s)$ lie inside the unit circle centered at origin. The poles for the given Y is $s = \pm i$ which both lie on $|s| = 1$, thus this system is stable.

$$\left(z^2 + 2\frac{h^2-4}{h^2+4}z + 1\right) Y(z) \\ = \frac{2hc_1\left(z^2-1\right) + c_2h^2\left(z^2+2z+1\right)}{h^2+4} \quad (3.29)$$

$$z^2Y(z) + 2\frac{h^2-4}{h^2+4}zY(z) + Y(z) \\ = \frac{\left(2hc_1 + c_2h^2\right)z^2 + \left(2h^2c_2\right)z + \left(h^2c_2 - 2hc_1\right)}{h^2+4} \quad (3.30)$$

Some properties of one sided z transform,

$$\mathcal{Z}(y[n+2]) = z^2 Y(z) - y[1]z - y[0] \tag{3.31}$$

$$\mathcal{Z}(y[n+1]) = zY(z) - zy[0] \tag{3.32}$$

$$\mathcal{Z}(\delta[n]) = 1,\ z \neq 0 \tag{3.33}$$

$$\mathcal{Z}(y[n]) = Y(z) \implies \mathcal{Z}(y[n-n_0]) = z^{-n_0} Y(z) \tag{3.34}$$

By the time shift property (3.34),

$$\mathcal{Z}(\delta[n+2]) = z^2,\ z \neq 0 \tag{3.35}$$

$$\mathcal{Z}(\delta[n+1]) = z,\ z \neq 0 \tag{3.36}$$

Rewriting equation (3.30), we get,

$$
\begin{aligned}
& z^2 Y(z) + 2\frac{h^2-4}{h^2+4} z Y(z) + Y(z) + (-y[1]z - y[0]) \\
& + 2\left(\frac{h^2-4}{h^2+4}\right)(-zy[0]) = \frac{\left(2hc_1 + c_2 h^2\right) z^2}{h^2+4} \\
& + \frac{2h^2 c_2 - \left(h^2+4\right) y[1] - 2\left(h^2-4\right) y[0]}{h^2+4} \\
& + \frac{z\left(h^2 c_2 - 2hc_1 - \left(h^2+4\right) y[0]\right)}{h^2+4} \qquad (3.37) \\
& \text{where } z \neq 0 \qquad (3.38)
\end{aligned}
$$

Region of convergence (**ROC**) is given by $z \neq 0$.

Taking z inverse transform on both sides of equation (3.37), we get the **difference equation** which is given by,

$$
\begin{aligned}
& y[n+2]+2\left(\frac{h^2-4}{h^2+4}\right) y[n+1]+y[n] \\
& \quad=\frac{\left(2 h c_1+c_2 h^2\right) \delta[n+2]}{h^2+4} \\
& \quad+\frac{\left(2 h^2 c_2-\left(h^2+4\right) y[1]-2\left(h^2-4\right) y[0]\right) \delta[n+1]}{h^2+4} \\
& \quad+\frac{\left(h^2 c_2-2 h c_1-y[0]\right) \delta[n]}{h^2+4}
\end{aligned} \tag{3.39}
$$

Here, δ is given by,

$$
\delta\left[n-n_0\right]= \begin{cases}1 & n=n_0 \\ 0 & n \neq n_0\end{cases} \tag{3.40}
$$

As $n > 0$,

$$\delta[n+2] = \delta[n+1] = 0 \tag{3.41}$$

The equation (3.39) is now given by,

$$y[n+2] + 2\left(\frac{h^2-4}{h^2+4}\right) y[n+1] + y[n] = \frac{\left(h^2 c_2 - 2hc_1 - y[0]\right)\delta[n]}{h^2+4} \tag{3.42}$$

At this point we drop the notation $y[n]$ and replace it with y_n, and we replace $c_1 = y(0)$ and $c_2 = y'(0)$,

$$y_{n+2} + 2\left(\frac{h^2-4}{h^2+4}\right) y_{n+1} + y_n = \frac{\left(h^2 y'(0) - 2hy(0) - y_0\right)\delta[n]}{h^2+4} \tag{3.43}$$

Note that for computationally plotting the above difference equation, we need $y_0 = y(0)$ as well as y_1. To find $y_1 = y(0+h) = y(h)$ we employ first principle of derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \tag{3.44}$$

$$y(x+h) = y(x) + hy'(x), h \to 0 \tag{3.45}$$

$$y_1 = y(h) = y(0) + hy'(0) \tag{3.46}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_0 = y(0) = 0 \ , \ y'(0) = 1 \tag{3.47}$$

we get the plot of the given differential equation.

Plot

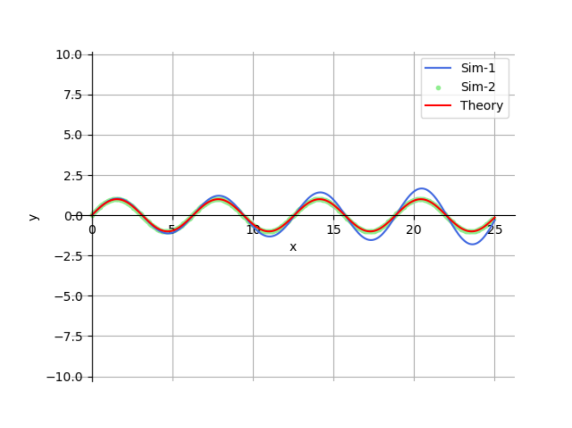

Figure 1: Here Sim-1 plot represents the plot given by Trapezoid Method, and Sim-2 which is given by Bilinear transform using the same value of *h*. This plot clearly shows the accuracy of the Bilinear transform method.