Computationally solving system of linear equations (10.3.3.1.3)

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Outline

Problem

Problem Statement

Solve the following pair of linear equations,

$$3x - y = 3 \tag{2.1}$$

$$9x - 3y = 9 \tag{2.2}$$

Solution

Computation Solution - LU Decomposition Method

Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.1}$$

Expressing the system in matrix form,

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \tag{3.2}$$

$$\text{which is of the form } A\mathbf{x} = \mathbf{b} \tag{3.3}$$

Any non-singular matrix A can be expressed as a product of an upper triangular matrix U and a lower triangular matrix L, such that

$$A = LU \tag{3.4}$$

$$\implies LU\mathbf{x} = \mathbf{b} \tag{3.5}$$

U is determined by row reducing A,

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \tag{3.6}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \tag{3.7}$$

l_{21} is the multiplier used to zero out a_{21} in A.

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \tag{3.8}$$

This LU decomposition could also be computationally found using Doolittle's algorithm. The update equation is given by,

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{3.9}$$

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & j = 0, U_{jj} \neq 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} & j > 0 \end{cases} \tag{3.10}$$

$$\tag{3.11}$$

We see that there is a zero on the diagonal of the upper triangular matrix U which implies that A is singular and hence the system has either zero or infinitely many solutions.
Let $\mathbf{y} = U\mathbf{x}$,

$$L\mathbf{y} = \mathbf{b} \tag{3.12}$$

After we find $\mathbf{y}$, we find $\mathbf{x}$ using the following equation,

$$U\mathbf{x} = \mathbf{y} \tag{3.13}$$

Applying forward substitution on equation (3.12), we get,

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} \tag{3.14}$$

$$\implies \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.15}$$

Substituting **y** in equation (3.13), we get,

$$\begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.16}$$

$$\implies 0\,(x) + 0\,(y) = 0 \tag{3.17}$$

$$\text{and } 3x - y = 3 \tag{3.18}$$

This shows that the equation has infinitely many solutions.

Plot

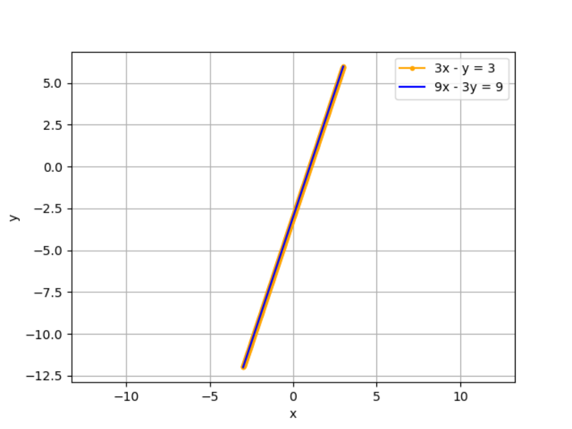

Figure 1: Plotting the two lines, which come out as parallel