Computational probability (11.16.3.8.9)

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Outline

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Problem

Problem Statement

Three coins are tossed at once. Find the probability of getting atmost two tails.

Solution

Defining the sample event space, and random variables

Sample space (Ω) is given by,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \quad (3.1)$$

Event space ($\mathcal{F}$) is given by,

$$\mathcal{F} = 2^{\Omega} \quad (3.2)$$

Let X be the random variable,

$$X = \text{number of tails in the sequence} \quad (3.3)$$

We express this random variable as a sum of 3 bernoulli random variables.

$$X = X_1 + X_2 + X_3 \tag{3.4}$$

where,

$$X_i = \begin{cases} 0 & i^{\text{th}} \text{ toss is a Heads} \\ 1 & i^{\text{th}} \text{ toss is a Tails} \end{cases} \tag{3.5}$$

X models a binomial distribution.

Z-transform to calculate PMF

For converting to z-domain, we use the property,

$$M_X(z) = M_{X_1}(z)\, M_{X_2}(z)\, M_{X_3}(z) \tag{3.6}$$

Extending this system to m tosses, we get,

$$M_X(z) = \prod_{k=1}^{m} M_{X_k}(z) \tag{3.7}$$

Let probablity mass function function for the bernoulli random variable X_i be given by,

$$p_{X_i}(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \\ 0 & n = \mathbb{Z} - \{0, 1\} \end{cases} \tag{3.8}$$

where p is the probablity of getting heads.

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(n)\, z^{-k} = p + (1-p)\, z^{-1} \tag{3.9}$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(n)\, z^{-k} = p + (1-p)\, z^{-1} \tag{3.10}$$

$$\vdots \tag{3.11}$$

$$M_{X_m}(z) = \sum_{k=-\infty}^{\infty} p_{X_m}(n)\, z^{-k} = p + (1-p)\, z^{-1} \tag{3.12}$$

$$\implies M_X(z) = \left(p + (1-p)\, z^{-1}\right)^m \tag{3.13}$$

$$M_X(z) = \sum_{k=-\infty}^{\infty} \left({}^{m}C_k p^{m-k} (1-p)^k\right) z^{-k} \tag{3.14}$$

$$\tag{3.15}$$

Taking z-inverse on both sides, we get,

$$p_X(n) = {}^{m}C_n p^{m-n} (1-p)^n \tag{3.16}$$

Taking $m = 3$ and $p = \frac{1}{2}$,

$$p_X(n) = {}^{3}C_n \left(\frac{1}{2}\right)^3 \tag{3.17}$$

Using this probability mass function, the cummulative distribution function C.D.F ($F_X(n)$) is given by,

$$F_X(n) = \sum_{k=-\infty}^{n} {}^{3}C_k \left(\frac{1}{2}\right)^3 \tag{3.18}$$

$$\tag{3.19}$$

$$F_X(n) = \begin{cases} 0 & n < 0 \\ {}^3C_0\left(\frac{1}{2}\right)^3 = \frac{1}{8} & 0 \leq n < 1 \\ {}^3C_0\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 = \frac{1}{2} & 1 \leq n < 2 \\ {}^3C_0\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_2\left(\frac{1}{2}\right)^3 = \frac{7}{8} & 2 \leq n < 3 \\ {}^3C_0\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_2\left(\frac{1}{2}\right)^3 + {}^3C_3\left(\frac{1}{2}\right)^3 = 1 & 3 \leq n \end{cases} \tag{3.20}$$

Let A be an event defined as,

$$A\text{: Getting atmost two tails} \tag{3.21}$$

$$\Pr(A) = \Pr(X <= 2) = F_X(2) \tag{3.22}$$

$$\Pr(A) = \frac{7}{8} = 0.875 \tag{3.23}$$

Simulation

Running a simulation requires generating random numbers with uniform probability. This is done using OpenSSL's random byte generator.

1. 1 byte of randomly generated uniform data is generated using OpenSSL rand.h.
2. This random number is scaled down from $\{0, 1, 2, \ldots 255\}$ to $[0, 1]$ by dividing by 255.
3. For generating the bernoulli random variable, if this normalized number is less than p then 0 is returned, else 1 is returned.

As number of trials increase, the relative frequency converges to the actual probability of the event.

Plot

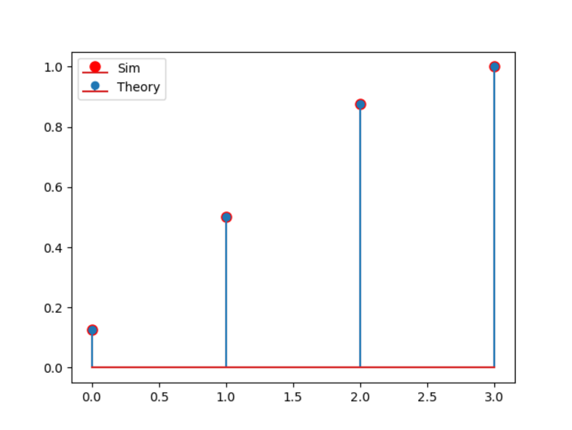

Figure 1: CDF Plot

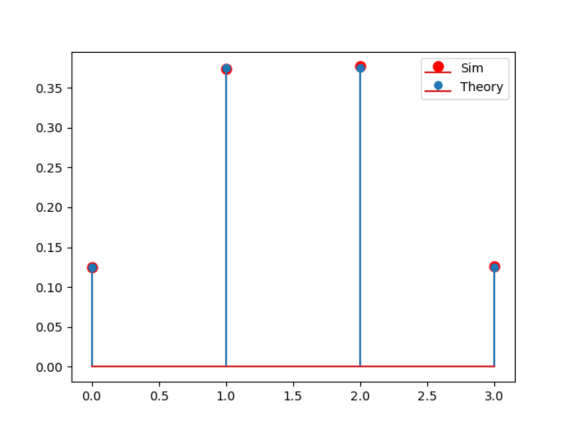

Figure 2: PMF plot for $m = 3$

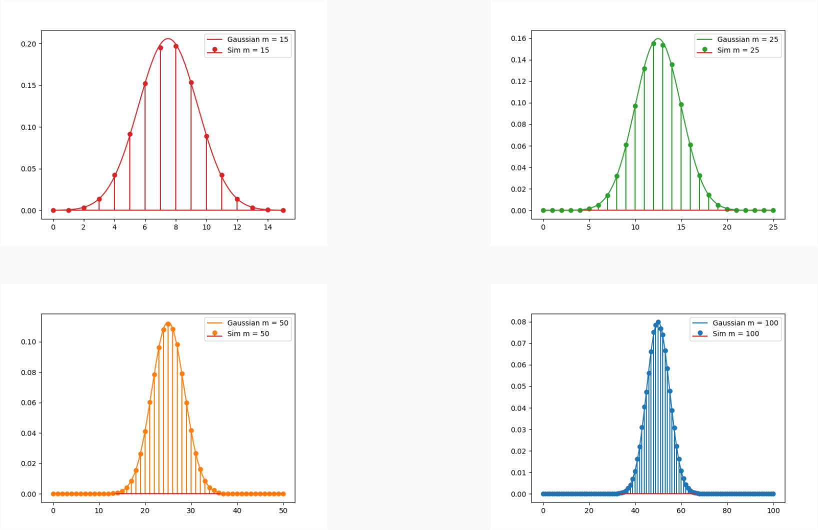

Figure 3: Simulated PMF as well as Gaussian plots for $m = 15, 20, 50, 100$. This plot shows convergence of binomial distribution to Gaussian distribution as $m \to \infty$