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Your Name

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Contents

1 Step Signal as $f(t)$

For this analysis, we choose the **unit step function** as the input signal:

$$f(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1}$$

1.0.1 Interpretation:

- The output $y(t)$ represents how the shape of $f(t)$ is modified by $h(t)$.
- The rectangular kernel $h(t)$ acts as a moving average filter.

1.1 Solution Using Step Function

1.1.1 Standard Convolution $y(t) = (f * h)(t)$

Given:

- $f(t) = u(t)$ (step function)
- $h(t)$ is a rectangular pulse centered at $t = 0$ with width $2T$.

The convolution integral is evaluated in three regions:

1. **For $t < -T$:**
   - The kernel $h(t - \tau)$ and $u(\tau)$ do not overlap.
   - Thus:

$$\begin{align} y(t) &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau \tag{2} \\ &= \int_{0}^{\infty} 1 \cdot h(t-\tau)\,d\tau \tag{3} \\ &= 0 \quad \text{(No overlap when } t < -T) \tag{4} \end{align}$$

2. **For $-T \leq t \leq T$:**
   - The kernel partially overlaps $u(\tau)$ from $\tau = 0$ to $\tau = t + T$.

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau && (5)\\
&= \int_{0}^{\infty} 1 \cdot h(t-\tau)\,d\tau && (6)\\
&= \int_{0}^{t+T} 1 \cdot 1\,d\tau \quad (\text{Since } h(t-\tau) = 1 \text{ when } -T \le t-\tau \le T) && (7)\\
&= [\tau]_0^{t+T} && (8)\\
&= t + T && (9)
\end{aligned}
$$

3. **For $t > T$:**

   - The kernel fully overlaps $u(\tau)$ over a width of $2T$.

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau && (10)\\
&= \int_{0}^{\infty} 1 \cdot h(t-\tau)\,d\tau && (11)\\
&= \int_{t-T}^{t+T} 1\,d\tau \quad (\text{Since } h(t-\tau) = 1 \text{ when } t-T \le \tau \le t+T) && (12)\\
&= [\tau]_{t-T}^{t+T} && (13)\\
&= (t+T) - (t-T) && (14)\\
&= 2T && (15)
\end{aligned}
$$

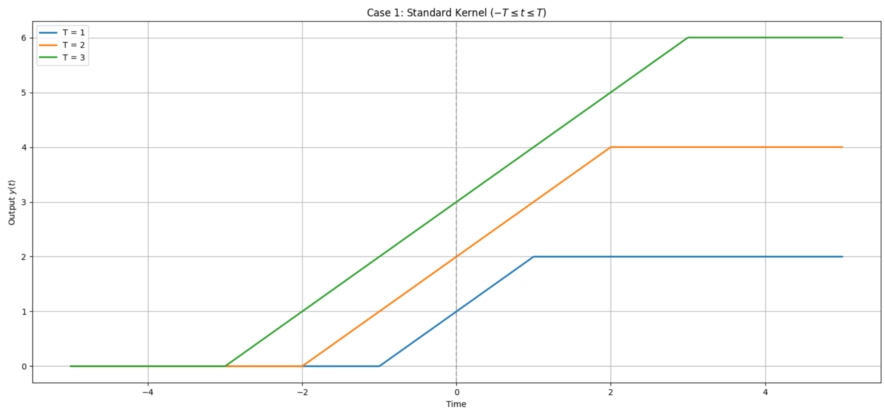
Final Result:

$$
y(t) = \begin{cases} 0, & t < -T \\ t + T, & -T \le t \le T \\ 2T, & t > T \end{cases} \qquad (16)
$$

1.1.2 Behavior Analysis:

- The output is zero before $t = -T$.
- A linear ramp occurs from $t = -T$ to $t = T$.
- The output saturates at $2T$ for $t > T$.
- A larger T results in a wider ramp and higher saturation value.

Here is plot showing the results



1.1.3 Modified Kernel (Only $t > 0$) (Part a)

The modified kernel is:

$$h_{mod}(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Convolution Analysis:

1. **For $t < 0$:**

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau) h_{mod}(t-\tau)\, d\tau & (18) \\ &= 0 \quad \text{(No overlap when } t < 0\text{)} & (19) \end{aligned}$$

2. **For $0 \le t \le T$:**

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau) h_{mod}(t-\tau)\, d\tau & (20) \\ &= \int_{0}^{t} 1 \cdot 1\, d\tau \quad \text{(Overlap from } \tau = 0 \text{ to } \tau = t\text{)} & (21) \\ &= [\tau]_0^t & (22) \\ &= t & (23) \end{aligned}$$

3. **For $t > T$:**

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} u(\tau)h_{mod}(t-\tau)\,d\tau && (24)\\
&= \int_{t-T}^{t} 1 \cdot 1\,d\tau \quad \text{(Full overlap from } \tau = t-T \text{ to } \tau = t) && (25)\\
&= [\tau]_{t-T}^{t} && (26)\\
&= t-(t-T) && (27)\\
&= T && (28)
\end{aligned}$$

Result:

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t \le T \\ T, & t > T \end{cases} \tag{29}$$

Here is a plot showing the results

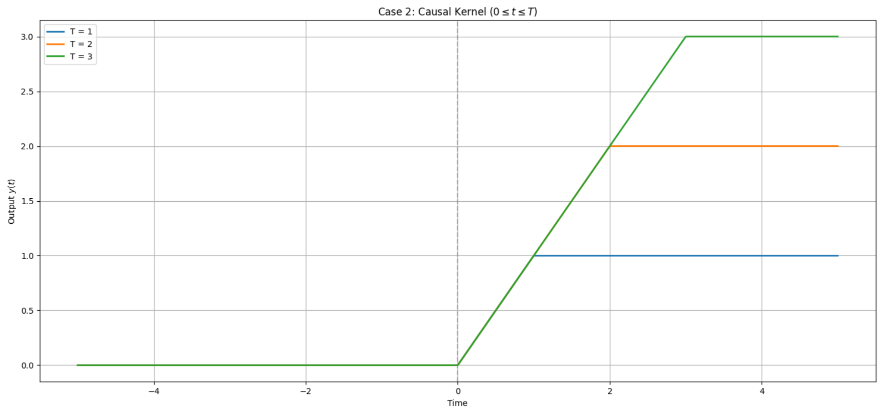


Figure 1: Modified kernel

1.1.4 Comparison with Original Kernel:

- The response is now **causal** (output depends only on past/current inputs).
- The ramp is shorter (from 0 to T) compared to the original ($-T$ to T).
- The steady-state value is T instead of $2T$.

1.1.5 Shifted Kernel by τ_0 (Part b)

The shifted kernel is:

$$h_{shift}(t) = \begin{cases} 1, & -T + \tau_0 \leq t \leq T + \tau_0 \\ 0, & \text{otherwise} \end{cases} \tag{30}$$

Applying the time-shift property of convolution:

$$f(t) * h(t - \tau_0) = (f(t) * h(t))_{t \to t - \tau_0} \tag{31}$$

The convolution output is simply the original $y(t)$ delayed by τ_0:

$$y_{shift}(t) = y(t - \tau_0) = \begin{cases} 0, & t < -T + \tau_0 \\ (t - \tau_0) + T, & -T + \tau_0 \leq t \leq T + \tau_0 \\ 2T, & t > T + \tau_0 \end{cases} \tag{32}$$

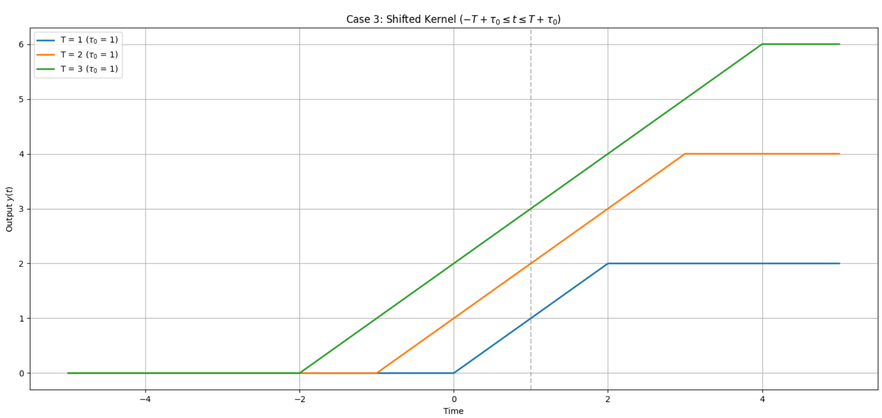


Figure 2: shifted kernel

1.1.6 Significance in Time-Delayed Systems:

- The shift τ_0 introduces a **time delay** in the system's response.
- For $\tau_0 > 0$, the response is delayed (system responds later).
- For $\tau_0 < 0$, the response is advanced (system responds earlier).
- Important in control systems, signal processing, and communications where delays affect stability and synchronization.

1.2 Conclusion

- The convolution of a step function with a rectangular kernel produces a **piece-wise linear** output.
- Modifying the kernel to be **causal** changes the response to depend only on past inputs.
- Shifting the kernel introduces a **time delay**, which is crucial in real-world systems.
- The kernel width T directly affects the steady-state value and transition time of the output.

This analysis demonstrates how different kernel modifications affect signal processing outcomes and provides insight into the behavior of linear time-invariant systems.

1.2.1 Comparision of all cases

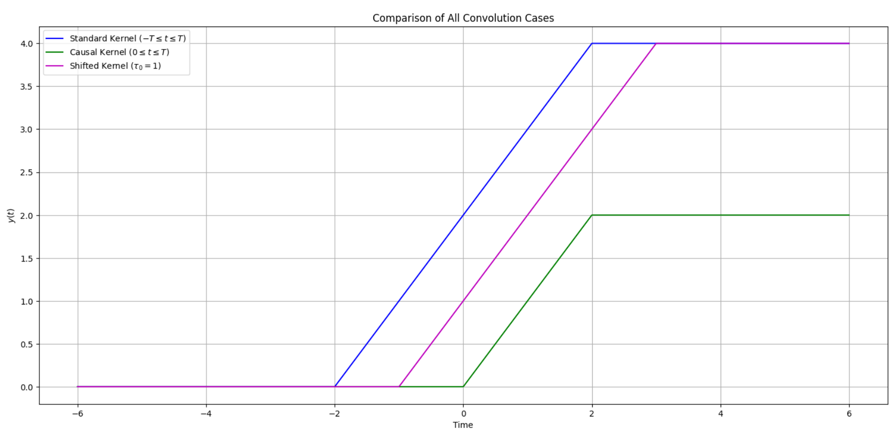


Figure 3: Comparision of all cases

2 Exponential as $f(t)$

Compute the convolution of an exponential signal $f(t) = e^{at}$ with a rectangular kernel:

$$h(t) = \begin{cases} 1 & \text{for } -T \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

We will analyze:

1. Standard convolution $y(t) = (f * h)(t)$
2. Modified kernel for $t > 0$
3. Time-shifted kernel by τ_0

2.1 Standard Convolution

2.1.1 General Solution

The convolution integral is defined as:

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \tag{34}$$

$$= \int_{-\infty}^{\infty} e^{a\tau}h(t-\tau)d\tau \tag{35}$$

Since $h(t-\tau)$ is non-zero only when $-T \le t-\tau \le T$, or equivalently, $t-T \le \tau \le t+T$, we can rewrite:

$$y(t) = \int_{t-T}^{t+T} e^{a\tau}d\tau \tag{36}$$

$$= \left[\frac{e^{a\tau}}{a}\right]_{t-T}^{t+T} \quad (\text{for } a \neq 0) \tag{37}$$

Evaluating the integral for $a \neq 0$:

$$y(t) = \frac{e^{a(t+T)}}{a} - \frac{e^{a(t-T)}}{a} \tag{38}$$

$$= \frac{e^{at}e^{aT} - e^{at}e^{-aT}}{a} \tag{39}$$

$$= \frac{e^{at}(e^{aT} - e^{-aT})}{a} \tag{40}$$

$$= \frac{2e^{at}\sinh(aT)}{a} \tag{41}$$

For the special case when $a = 0$:

$$\begin{aligned}
y(t) &= \int_{t-T}^{t+T} e^{0\cdot\tau}\, d\tau && (42)\\
&= \int_{t-T}^{t+T} 1\, d\tau && (43)\\
&= [\tau]_{t-T}^{t+T} && (44)\\
&= (t+T) - (t-T) && (45)\\
&= 2T && (46)
\end{aligned}$$

2.1.2 Time Domain Analysis

The solution is time-invariant - the same form applies for all values of t:

$$y(t) = \begin{cases} \frac{2e^{at}\sinh(aT)}{a} & a \neq 0 \\ 2T & a = 0 \end{cases} \tag{47}$$

For $a > 0$, the output grows exponentially, amplified by the factor $\frac{2\sinh(aT)}{a}$.

For $a < 0$, the output decays exponentially, with the same amplification factor. Here is a plot for different values of 'a'

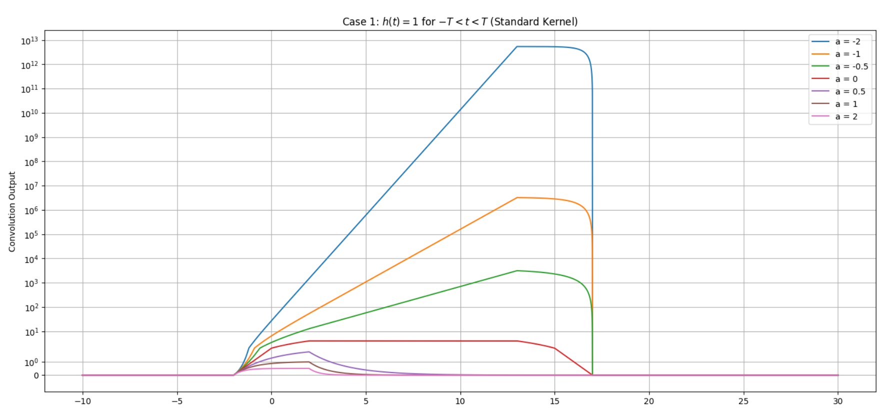


Figure 4: Varying a

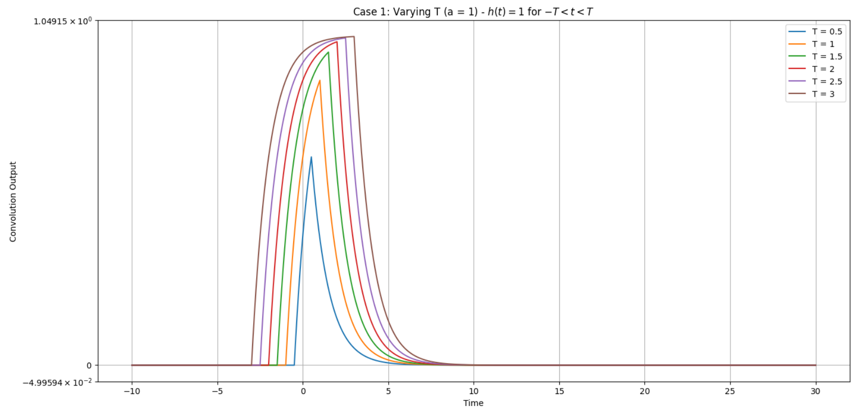


Figure 5: Varying T

2.2 Modified Kernel (t > 0)

2.2.1 Kernel Definition

The modified kernel that only considers $t > 0$ is:

$$h_{mod}(t) = \begin{cases} 1 & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

2.2.2 Convolution Result

The convolution integral becomes:

$$y(t) = \int_{-\infty}^{\infty} e^{a\tau} h_{mod}(t - \tau) d\tau \tag{49}$$

The kernel $h_{mod}(t - \tau)$ is non-zero only when $0 \le t - \tau \le T$, or equivalently, $t - T \le \tau \le t$.

We also need to consider that $e^{a\tau}$ is defined for all τ. This gives us:

$$y(t) = \int_{t-T}^{t} e^{a\tau} d\tau \tag{50}$$

However, we must also account for the domain of integration. If $t < 0$, there's no overlap between the kernel and the input. If $0 \le t < T$, we need to adjust the lower limit.

This gives us three cases:

1. **For $t < 0$:**

$$y(t) = 0 \quad \text{(No overlap)} \tag{51}$$

2. **For $0 \leq t \leq T$:**

$$\begin{aligned} y(t) &= \int_0^t e^{a\tau} d\tau && (52) \\ &= \left[\frac{e^{a\tau}}{a} \right]_0^t && (53) \\ &= \frac{e^{at} - 1}{a} \quad \text{(for } a \neq 0\text{)} && (54) \end{aligned}$$

3. **For $t > T$:**

$$\begin{aligned} y(t) &= \int_{t-T}^t e^{a\tau} d\tau && (55) \\ &= \left[\frac{e^{a\tau}}{a} \right]_{t-T}^t && (56) \\ &= \frac{e^{at} - e^{a(t-T)}}{a} && (57) \\ &= \frac{e^{at}(1 - e^{-aT})}{a} \quad \text{(for } a \neq 0\text{)} && (58) \end{aligned}$$

For $a = 0$, the results simplify to:

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq T \\ T & t > T \end{cases} \tag{59}$$

Here is a plot for different values of 'a'

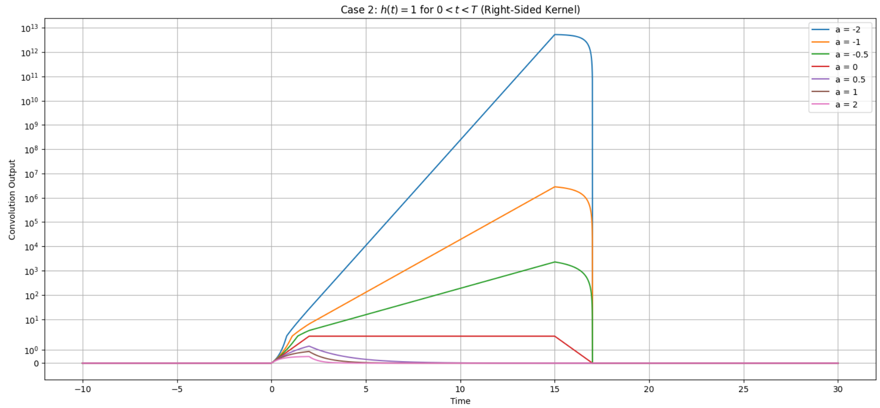


Figure 6: Varying a

Here is a plot showing the variation of T

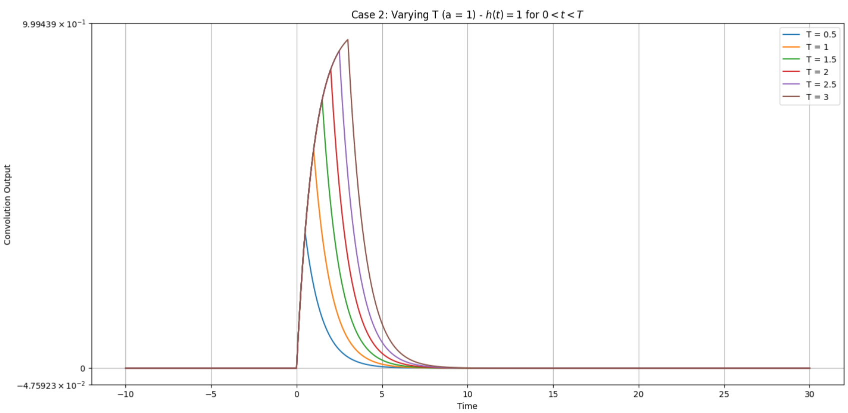


Figure 7: Varying T

2.3 Time-Shifted Kernel

2.3.1 Kernel Definition

The time-shifted rectangular kernel is:

$$h_{shift}(t) = \begin{cases} 1 & \text{for } -T + \tau_0 \leq t \leq T + \tau_0 \\ 0 & \text{otherwise} \end{cases} \tag{60}$$

2.3.2 Convolution Result

Using the time-shift property of convolution:

$$f(t) * h(t - \tau_0) = (f(t) * h(t))_{t \to t - \tau_0} \tag{61}$$

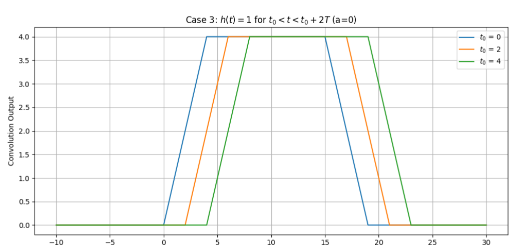
Therefore, the convolution with the shifted kernel is simply the original convolution result shifted by τ_0:

$$\begin{aligned}
y_{shift}(t) &= y(t - \tau_0) && (62)\\
&= \frac{2e^{a(t-\tau_0)}\sinh(aT)}{a} \quad (\text{for } a \neq 0) && (63)\\
&= \frac{2e^{at}e^{-a\tau_0}\sinh(aT)}{a} && (64)\\
&= e^{-a\tau_0} \cdot \frac{2e^{at}\sinh(aT)}{a} && (65)
\end{aligned}$$

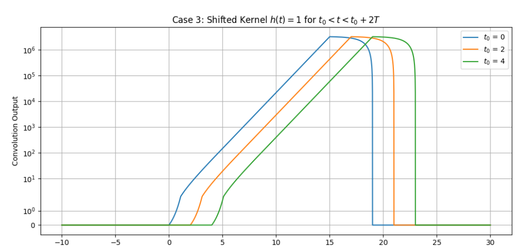
For $a = 0$:

$$y_{shift}(t) = 2T \tag{66}$$

Here are a few plots to demonstrate the result



(a) a=0



(b) a=-1

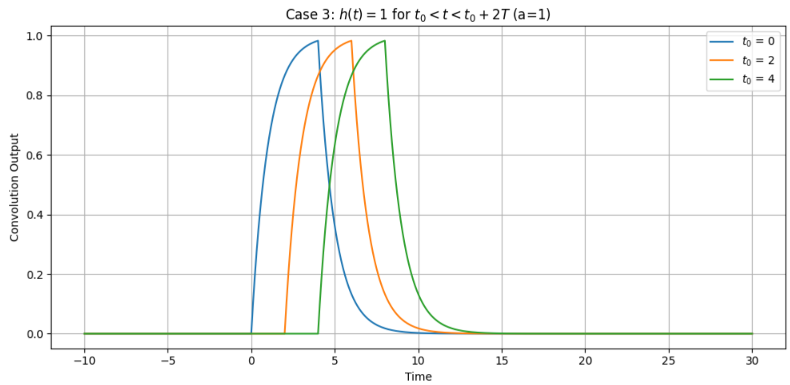


Figure 8: a=1

2.4 Conclusion

- **Standard Convolution:** The exponential input produces a scaled exponential output, with the scaling factor depending on the kernel width T and the exponential parameter a.
- **Kernel Width Effect:** The parameter T controls the amplification via the $\sinh(aT)$ term. Larger values of T result in greater amplification.
- **Time-Shift Effect:** A time-shift of τ_0 introduces a pure delay in the output, equivalent to multiplying the original output by $e^{-a\tau_0}$.
- **Causal Modification:** Making the kernel causal (only for $t > 0$) changes the initial conditions and creates a time-varying response that depends on the region of t.
- **Special Case:** When $a = 0$ (constant input), the output is simply proportional to the width of the kernel, demonstrating the averaging property of convolution.

This analysis provides insight into how convolution with a rectangular kernel affects exponential signals, which is fundamental in understanding linear time-invariant systems.

2.4.1 Time-Invariance Analysis

To prove that our system is time-invariant, we need to show that a time-shifted input results in an equally time-shifted output.

Let's consider our original input $f(t) = e^{at}$ with output $y(t) = \frac{2e^{at}\sinh(aT)}{a}$ for $a \neq 0$.

If we time-shift the input by t_0 to get $f'(t) = f(t - t_0) = e^{a(t-t_0)} = e^{at}e^{-at_0}$, the corresponding output would be:

$$y'(t) = \int_{-\infty}^{\infty} f'(\tau)h(t-\tau)d\tau \tag{67}$$
$$= \int_{-\infty}^{\infty} e^{a(\tau-t_0)}h(t-\tau)d\tau \tag{68}$$
$$= e^{-at_0}\int_{-\infty}^{\infty} e^{a\tau}h(t-\tau)d\tau \tag{69}$$
$$= e^{-at_0} \cdot y(t) \tag{70}$$

However, the time-shifted output of the original system would be:

$$y(t-t_0) = \frac{2e^{a(t-t_0)}\sinh(aT)}{a} \tag{71}$$
$$= \frac{2e^{at}e^{-at_0}\sinh(aT)}{a} \tag{72}$$
$$= e^{-at_0} \cdot \frac{2e^{at}\sinh(aT)}{a} \tag{73}$$
$$= e^{-at_0} \cdot y(t) \tag{74}$$

Since $y'(t) = y(t-t_0)$, this confirms that the system is time-invariant.

The system is time-invariant because:

- The rectangular kernel $h(t)$ depends only on the relative time difference $(t-\tau)$, not on absolute time.
- The convolution operation itself preserves time-invariance.
- The system's response at any time t depends only on the input over the interval $[t-T, t+T]$, regardless of when this interval occurs.
- The mathematical form of the output maintains the same structure regardless of time shifts in the input.

This time-invariance property is fundamental to linear systems theory and allows us to analyze the system using frequency domain techniques.

3 Sinusoidal as $f(t)$ with Half-Box Kernel

Compute the convolution of a given signal $f(t)$ with a rectangular kernel $h(t)$, analytically. The rectangular kernel is defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases}$$

...

3.1 Modify the kernel to only consider the part of the kernel for $t > 0$. How does this affect the convolution result?

3.1.1 Analytic

In this analysis, we investigate the convolution of a sinusoidal input signal

$$f(t) = A\sin(\omega t + \phi)$$

with a rectangular kernel that is modified as below. The modified rectangular kernel is defined as:

$$h(t) = \begin{cases} 1, & \text{for } 0 < t < T, \\ 0, & \text{otherwise.} \end{cases}$$

Our goal is to derive the convolution expression $y(t) = (f * h)(t)$ analytically and analyze the system's behavior.

The convolution of $f(t)$ and $h(t)$ is defined as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Substituting our functions:

$$y(t) = \int_{-\infty}^{\infty} A\sin(\omega\tau + \phi) \cdot h(t-\tau)\,d\tau$$

Since the kernel $h(t-\tau)$ is non-zero only when $0 < t - \tau < T$, we can rewrite this condition as:

$$t - T < \tau < t$$

Therefore, the limits of integration become:

$$y(t) = \int_{t-T}^{t} A\sin(\omega\tau + \phi)\,d\tau$$

Evaluating this integral: The integral of $\sin(\omega\tau + \phi)$ is:

$$\int \sin(\omega\tau + \phi)\, d\tau = -\frac{1}{\omega}\cos(\omega\tau + \phi)$$

Evaluating the integral from $t - T$ to t:

$$y(t) = A\left[-\frac{1}{\omega}\cos(\omega\tau + \phi)\right]_{t-T}^{t}$$

This simplifies to:

$$y(t) = A\left(-\frac{1}{\omega}\cos(\omega t + \phi) + \frac{1}{\omega}\cos(\omega(t - T) + \phi)\right)$$

Using the identity for the difference of cosines:

$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

$$y(t) = \frac{A}{\omega}\left(-2\sin\left(\frac{(\omega(t - T) + \phi) + (\omega t + \phi)}{2}\right)\sin\left(\frac{(\omega(t - T) + \phi) - (\omega t + \phi)}{2}\right)\right)$$

$$y(t) = \frac{A}{\omega}\left(-2\sin\left(\omega t + \phi - \frac{\omega T}{2}\right)\sin\left(\frac{\omega T}{2}\right)\right)$$

$$\boxed{y(t) = \frac{2A\sin\left(\frac{\omega T}{2}\right)}{\omega}\sin\left(\omega(t - \frac{T}{2}) + \phi\right)}$$

The convolution for the modified kernel does not change the behavior of the signal, i.e., the output still remains sinusoidal.The significant differences are as follows :

- A $\frac{T}{2}$ time shift towards right .
- The scaling of amplitude is changed from $\frac{2\sin(\omega T)}{\omega}$ to $\frac{2\sin(\frac{\omega T}{2})}{\omega}$.

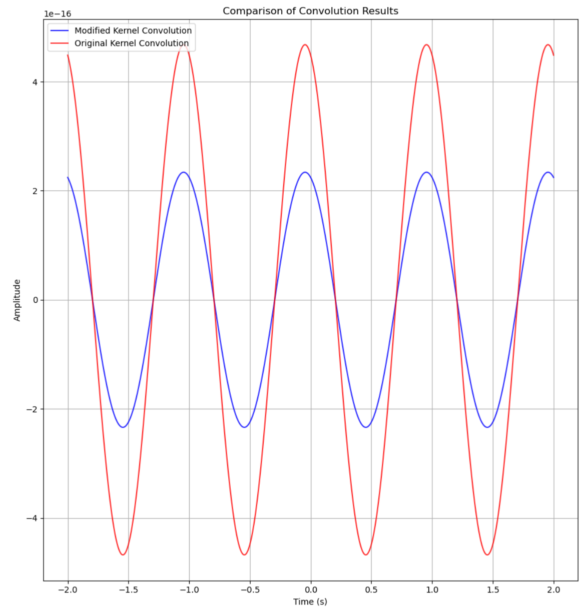


Figure 9: Plot of the convolutions

3.2 Extended Graphical and Analytical Analysis

3.2.1 Effect of Amplitude A

The first figure below demonstrates the impact of changing amplitude values on the convolution result. As seen, increasing the amplitude scales the overall magnitude of the output signal, keeping the shape unchanged.

3.2.2 Effect of Angular Frequency ω

Changing ω changes how rapidly the sinusoidal input oscillates. This figure shows how higher frequencies start diminishing in magnitude due to the sinc-type attenuation from the convolution.

3.2.3 Effect of Pulse Width T

This graph demonstrates how varying T, the width of the rectangular kernel, influences the output. Smaller T leads to lesser averaging, preserving high-frequency content. Larger T results in more smoothing.

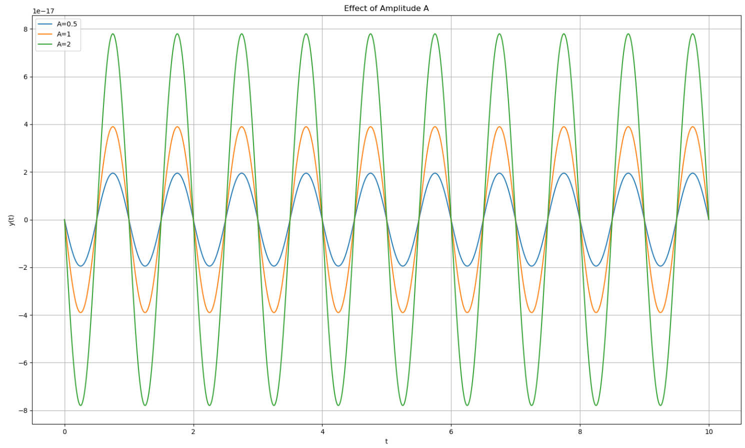


Figure 10: Effect of varying amplitude A on the convolution output.

3.2.4 Amplitude Ratio vs Angular Frequency

This plot captures the normalized amplitude ratio $\frac{A_{out}}{A} = \frac{2\sin(\omega \frac{T}{2})}{\omega}$ as a function of ω for various values of T. It reflects the sinc-function behavior, emphasizing low-pass characteristics of convolution with rectangular kernels.

3.2.5 Effect of Phase ϕ

Altering the phase ϕ of the sinusoidal signal results in horizontal shifts in the waveform. The output retains its shape but is delayed or advanced in time.

3.2.6 Special Observations

- If $\omega T = n\pi$, the output vanishes, indicating frequency nulls.
- For large T, the kernel acts as a low-pass filter.
- When $T \to 0$, the kernel approximates an impulse, and the convolution result tends to zero everywhere.

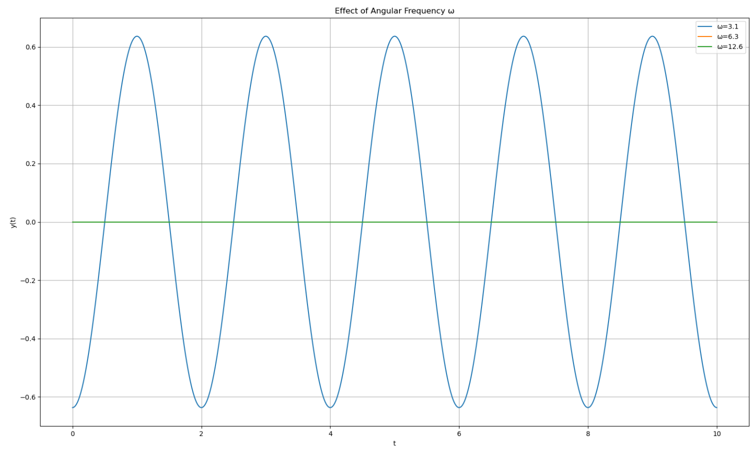


Figure 11: Effect of angular frequency ω on convolution. Higher frequencies are attenuated.

3.3 Convolution with Varying Parameters

The figure below shows the convolution output for multiple combinations of parameters including amplitude A, frequency ω, kernel width T, and phase ϕ. It highlights how these parameters influence the shape, scaling, and shift of the output signal.

Each sub-plot or curve within the figure corresponds to a unique configuration. You can observe:

- Changes in amplitude due to scaling by $\frac{2\sin(\omega T)}{\omega}$
- Phase shifts resulting from both the signal phase ϕ and kernel delay τ_0
- Smoothing effects due to increasing kernel width T
- Attenuation of high-frequency components, emphasizing the low-pass filter nature of box kernels

3.4 Conclusion

This detailed analysis of convolution using a rectangular kernel shows how parameters such as amplitude, frequency, kernel width, and phase influence the output signal.

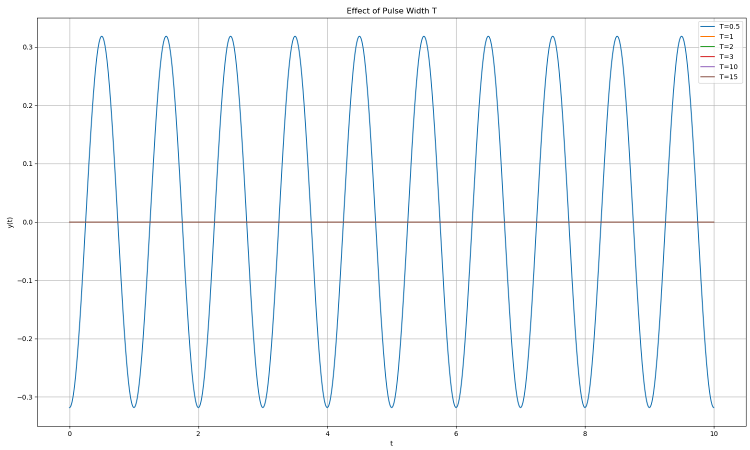


Figure 12: Convolution output for different pulse widths T.

The kernel acts as a smoothing and low-pass filter, selectively attenuating high-frequency components. Additionally, time-shifting the kernel leads to equivalent shifts in the output, demonstrating important system properties like linearity and time invariance.

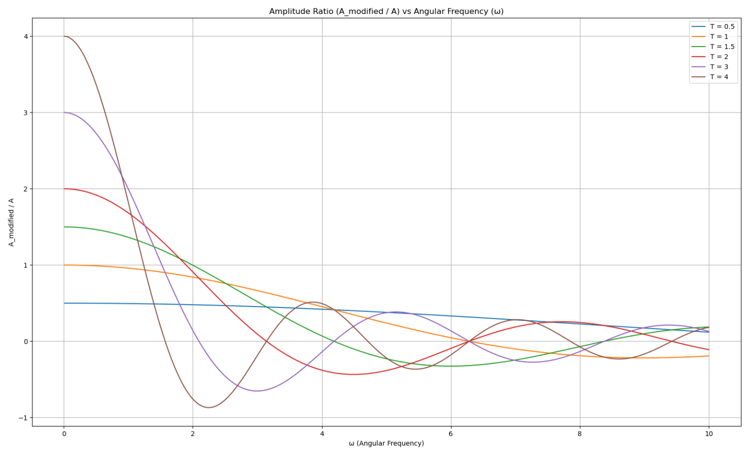


Figure 13: Amplitude ratio $\frac{A_{\text{modified}}}{A}$ versus angular frequency ω for different values of T.

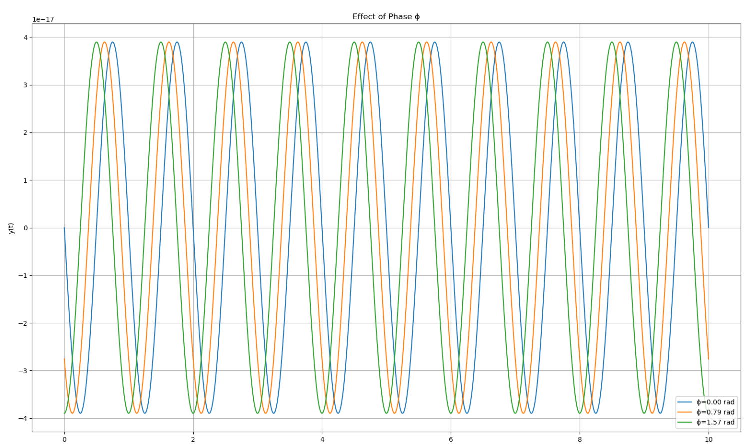


Figure 14: Phase shift ϕ changes the horizontal alignment of the output signal.

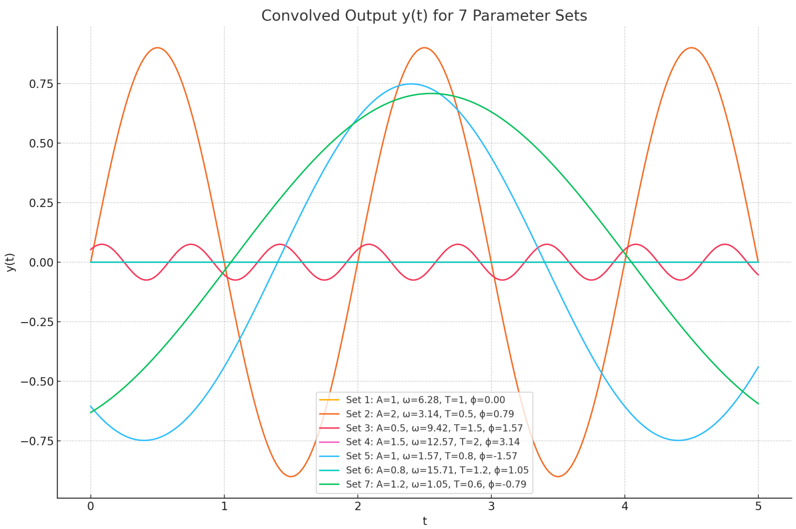


Figure 15: Convolution output for different parameter sets (A, ω, T, ϕ).