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1. Problem Statement

1.1 Convolution with a Rectangular Kernel

Compute the convolution of a given signal $f(t)$ with a rectangular kernel $h(t)$, analytically. The rectangular kernel is defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Derive the convolution expression $y(t) = (f * h)(t)$ in terms of known functions, and analyze the system's behavior for various values of the kernel duration T and the input signal $f(t)$.

Additionally, investigate the following scenarios:

(a) Modify the kernel to only consider the part for $t > 0$. How does this affect the convolution result?

(b) Shift the kernel by a time τ_0. Analyze how the shift impacts the convolution output and discuss its significance in time-delayed systems.

Choice of Input Signal:
For this analysis, we choose the **unit step function** as the input signal:

$$f(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2}$$

2. Convolution Basics

The convolution of two signals $f(t)$ and $h(t)$ is defined as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau \tag{3}$$

Interpretation:

- The output $y(t)$ represents how the shape of $f(t)$ is modified by $h(t)$.
- The rectangular kernel $h(t)$ acts as a moving average filter.

3. Solution Using Step Function

3.1 Standard Convolution $y(t) = (f * h)(t)$

Given:

- $f(t) = u(t)$ (step function)
- $h(t)$ is a rectangular pulse centered at $t = 0$ with width $2T$.

The convolution integral is evaluated in three regions:

1. **For $t < -T$:**
   - The kernel $h(t-\tau)$ and $u(\tau)$ do not overlap.
   - Thus:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau && (4) \\ &= \int_{0}^{\infty} 1 \cdot h(t-\tau) d\tau && (5) \\ &= 0 \quad (\text{No overlap when } t < -T) && (6) \end{aligned}$$

2. **For $-T \leq t \leq T$:**
   - The kernel partially overlaps $u(\tau)$ from $\tau = 0$ to $\tau = t+T$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau && (7) \\ &= \int_{0}^{\infty} 1 \cdot h(t-\tau) d\tau && (8) \\ &= \int_{0}^{t+T} 1 \cdot 1 d\tau \quad (\text{Since } h(t-\tau) = 1 \text{ when } -T \leq t-\tau \leq T) && (9) \\ &= [\tau]_0^{t+T} && (10) \\ &= t+T && (11) \end{aligned}$$

3. **For $t > T$:**
   - The kernel fully overlaps $u(\tau)$ over a width of $2T$.

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau && (12)\\
&= \int_{0}^{\infty} 1 \cdot h(t-\tau) d\tau && (13)\\
&= \int_{t-T}^{t+T} 1 d\tau \quad (\text{Since } h(t-\tau)=1 \text{ when } t-T \le \tau \le t+T) && (14)\\
&= [\tau]_{t-T}^{t+T} && (15)\\
&= (t+T)-(t-T) && (16)\\
&= 2T && (17)
\end{aligned}
$$

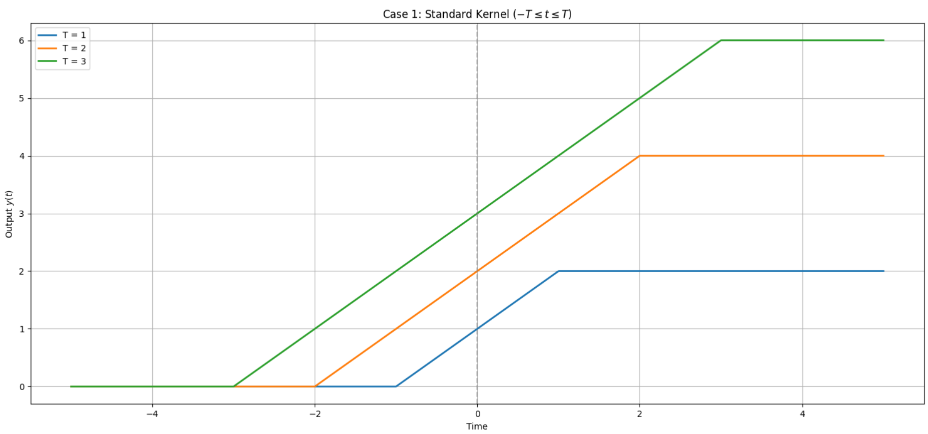
Final Result:

$$
y(t) = \begin{cases} 0, & t < -T \\ t+T, & -T \le t \le T \\ 2T, & t > T \end{cases} \tag{18}
$$

Behavior Analysis:

- The output is zero before $t = -T$.
- A linear ramp occurs from $t = -T$ to $t = T$.
- The output saturates at $2T$ for $t > T$.
- A larger T results in a wider ramp and higher saturation value.

Here is plot showing the results



3.2 Modified Kernel (Only $t>0$) (Part a)

The modified kernel is:

$$h_{mod}(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

Convolution Analysis:

1. **For $t<0$:**

$$y(t) = \int_{-\infty}^{\infty} u(\tau) h_{mod}(t-\tau) d\tau \tag{20}$$

$$= 0 \quad \text{(No overlap when } t<0\text{)} \tag{21}$$

2. **For $0 \le t \le T$:**

$$y(t) = \int_{-\infty}^{\infty} u(\tau) h_{mod}(t-\tau) d\tau \tag{22}$$

$$= \int_{0}^{t} 1 \cdot 1 d\tau \quad \text{(Overlap from } \tau=0 \text{ to } \tau=t\text{)} \tag{23}$$

$$= [\tau]_0^t \tag{24}$$

$$= t \tag{25}$$

3. **For $t>T$:**

$$y(t) = \int_{-\infty}^{\infty} u(\tau) h_{mod}(t-\tau) d\tau \tag{26}$$

$$= \int_{t-T}^{t} 1 \cdot 1 d\tau \quad \text{(Full overlap from } \tau=t-T \text{ to } \tau=t\text{)} \tag{27}$$

$$= [\tau]_{t-T}^t \tag{28}$$

$$= t-(t-T) \tag{29}$$

$$= T \tag{30}$$

Result:

$$y(t) = \begin{cases} 0, & t<0 \\ t, & 0 \le t \le T \\ T, & t>T \end{cases} \tag{31}$$

Here is a plot showing the results

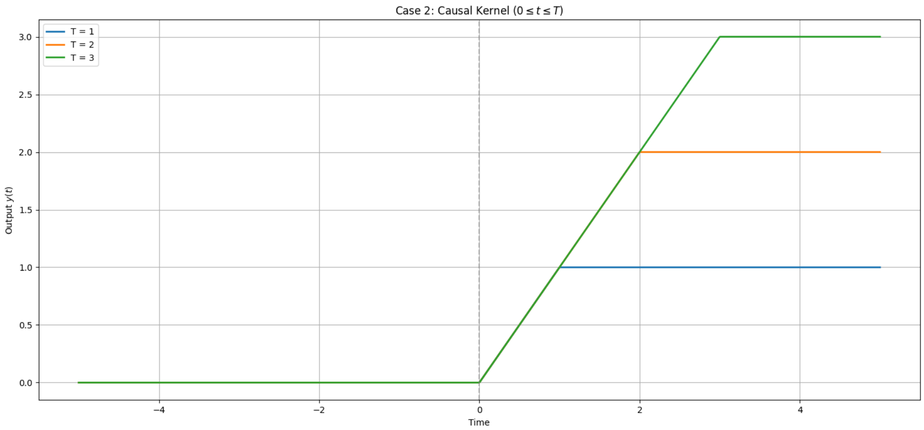


Figure 1: Modified kernel

Comparison with Original Kernel:

- The response is now **causal** (output depends only on past/current inputs).
- The ramp is shorter (from 0 to T) compared to the original ($-T$ to T).
- The steady-state value is T instead of $2T$.

3.3 Shifted Kernel by τ_0 (Part b)

The shifted kernel is:

$$h_{shift}(t) = \begin{cases} 1, & -T+\tau_0 \leq t \leq T+\tau_0 \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

Applying the time-shift property of convolution:

$$f(t)*h(t-\tau_0) = (f(t)*h(t))_{t\to t-\tau_0} \tag{33}$$

The convolution output is simply the original $y(t)$ delayed by τ_0:

$$y_{shift}(t) = y(t-\tau_0) = \begin{cases} 0, & t < -T+\tau_0 \\ (t-\tau_0)+T, & -T+\tau_0 \leq t \leq T+\tau_0 \\ 2T, & t > T+\tau_0 \end{cases} \tag{34}$$

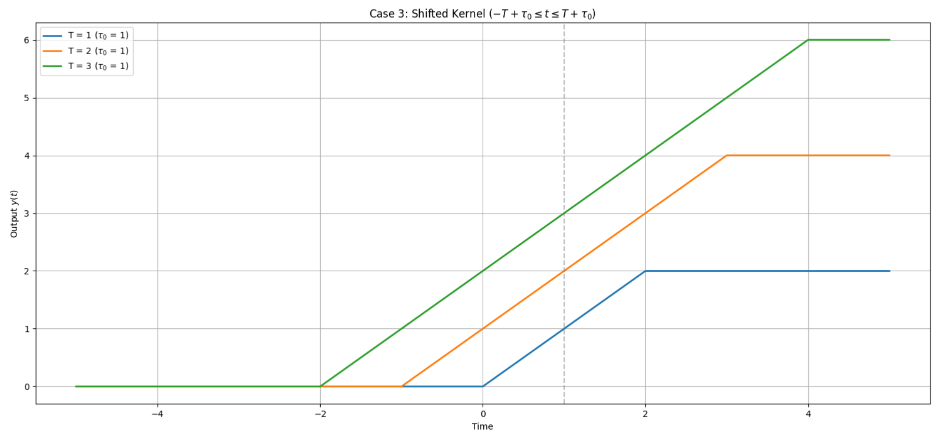


Figure 2: shifted kernel

Significance in Time-Delayed Systems:

- The shift τ_0 introduces a **time delay** in the system's response.
- For $\tau_0 > 0$, the response is delayed (system responds later).
- For $\tau_0 < 0$, the response is advanced (system responds earlier).
- Important in control systems, signal processing, and communications where delays affect stability and synchronization.

4. Conclusion

- The convolution of a step function with a rectangular kernel produces a **piecewise linear** output.
- Modifying the kernel to be **causal** changes the response to depend only on past inputs.
- Shifting the kernel introduces a **time delay**, which is crucial in real-world systems.
- The kernel width T directly affects the steady-state value and transition time of the output.

This analysis demonstrates how different kernel modifications affect signal processing outcomes and provides insight into the behavior of linear time-invariant systems.

Comparision of all cases

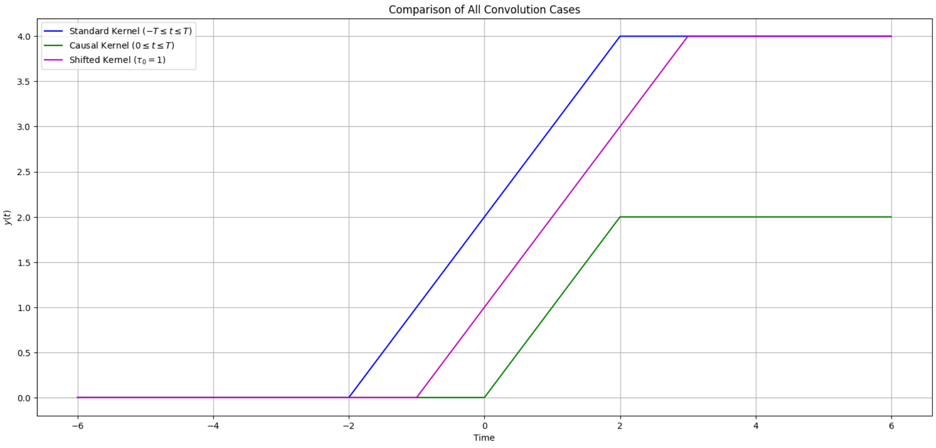


Figure 3: Comparision of all cases