Statistics MM2: Parameter estimation

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AALBORG UNIVERSITY
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Connectivity

Schedule

1. Introduction to statistics
2. **Parameter estimation**
3. Confidence intervals
4. Hypothesis testing 1
5. Hypothesis testing 2
6. Regression
7. Workshop: wrap-up and exam problems

Outline

Recap on sampling

Types of estimation

Estimating the mean and variance

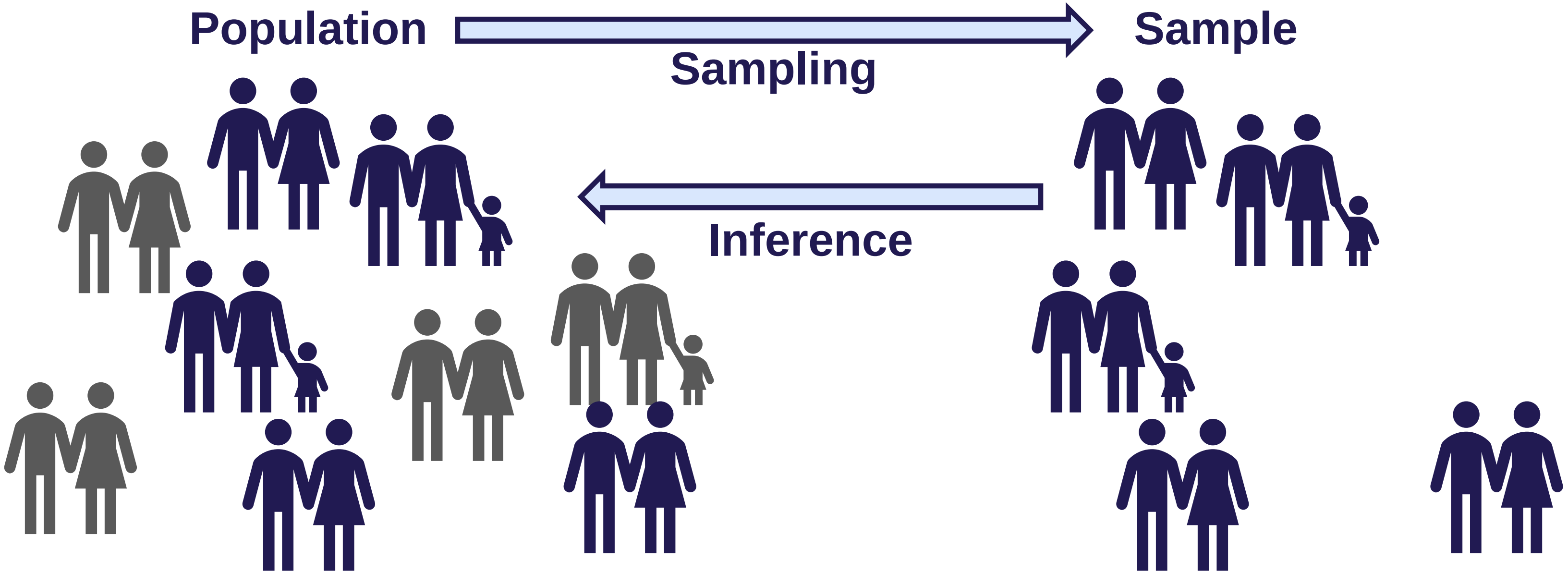
Evaluating estimators

Maximum likelihood Estimation (MLE)

Recap on sampling

Sampling

If we cannot measure the whole population, we use a smaller sample

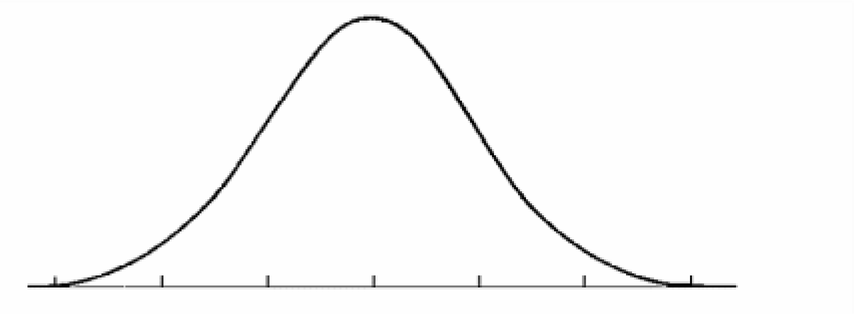


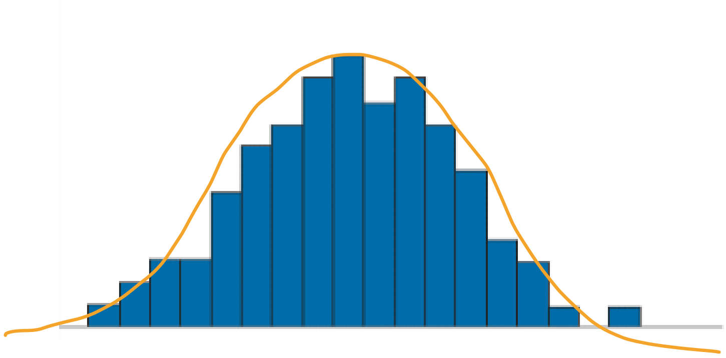
How do we create a sample?

We randomly draw n values from the population

Each value X_i is a random variable with distribution F

We use the sample to estimate some parameter of F (inference)





Implications and assumptions during sampling

If we randomly draw n values from the population

And each value X_i is a random variable with distribution F

Then a sufficiently large sample will ***look like*** the whole population

What does it mean *looking like*?

The parameters of the population are similar to the statistics

The parameters describe the population and the statistics describe the sample

- Mean
- Variance
- Quantiles

Types of estimation

Parametric estimation

We observe a sample $X_1, X_2, \ldots, X_n$

Each value X_i is a random variable with **known distribution** F with parameter θ

The parameter θ is a fixed value and not a RV

From the sample with n points, we create an estimate of θ, denoted as

$$\hat{\theta}_n = h(X_1, X_2, \ldots, X_n)$$

The estimate $\hat{\theta}_n$ is our statistic derived from the sample data

Since $\hat{\theta}_n$ depends on the sampled data, it is a RV

What are the properties of $\hat{\theta}_n$?

We hope that the estimator $\hat{\theta}_n$ is close to the real value of θ

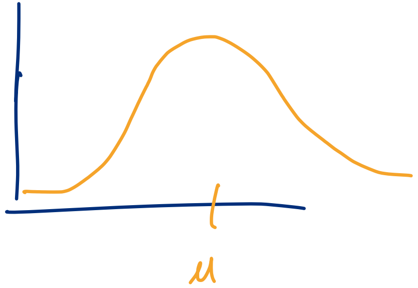
Non-parametric estimation

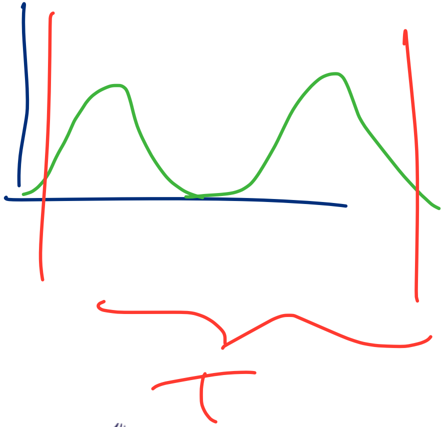
There are no assumptions on the parameters for distribution F

We don’t need to know F

When compared to parametric estimation

- Much more general
- Much more difficult to do





Example:

Histograms and order statistics (quantiles) are non-parametric

Can be used to show the shape of the distribution

But don’t give us any mathematical description of it

Estimating the mean and variance

Properties of the sample mean pt. 1

If $X_1, X_2, \ldots, X_n$ are random variables, the **sample mean** is the random variable

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Assuming the samples are drawn independently:

- The expectation of the sample mean is

$$\mathbb{E}(\bar{X}_n) = \mu$$

- The variance of the sample mean is $\mathrm{var}(\bar{X}_n) = \frac{\sigma^2}{n}$

Comes from the **central limit theorem**
and the fact that the variance of each sample is $\mathrm{var}(X_i) = \sigma^2$

Properties of the sample mean pt. 2

The law of large numbers

$\bar{X}_n$ **converges in probability** to $\mu = \mathbb{E}(X_i)$ as $n \to \infty$

$$\bar{X}_n \xrightarrow{P} \mu = \mathbb{E}(X_i)$$

This means that, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$$

So, $\bar{X}_n$ is close to μ with high probability if the sample size is large

Properties of the sample mean pt. 3

The central limit theorem

If n is large:

$$\sum_{i=1}^{n} X_i = X_1 + X_2 + \cdots + X_n \approx N(n\mu, n\sigma^2)$$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \approx N\left(\mu, \frac{\sigma^2}{n}\right)$$

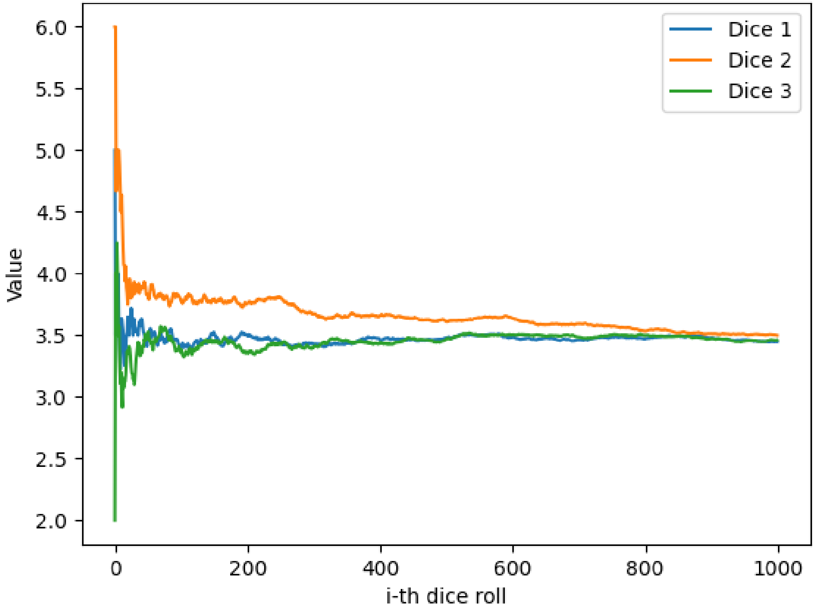
$$\frac{1}{n} N(n\mu, n\sigma^2)$$

$$= N\left(\frac{n}{n}\mu, \frac{n\sigma^2}{n^2}\right)$$

Recall that $F(x;\mu,\sigma^2) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

Dice example

We roll a dice $i = 1, 2, \dots, n$ times and compute the sample mean X_i



Estimating the variance

Sample variance

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

The term $n-1$ is needed to have an unbiased estimate

$$\mathbb{E}[S_n^2] = \sigma^2$$

Evaluating estimators

Metrics to evaluate estimators

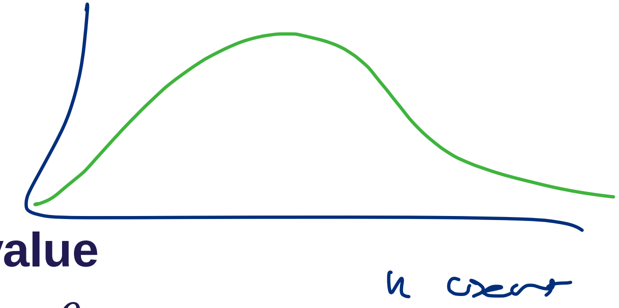
An estimator should be as close as possible to the true value
This should happen as frequently as possible

What does this mean?

Metrics for evaluation:

- **Bias:** measures accuracy
- **Variance:** measures precision
- **Mean square error:** measures both accuracy and precision

Bias

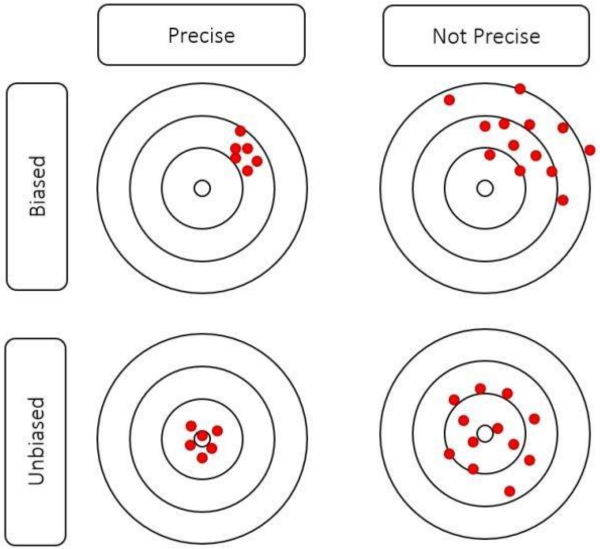


Measure of how close the estimate is to the true value

Let $\hat{\theta}_n = h(X_1, X_2, \dots, X_n)$ be the estimator for parameter θ

The bias is defined as

$$\text{bias}\left(\hat{\theta}_n\right) = \mathbb{E}\left(\hat{\theta}_n\right) - \theta$$



Unbiased estimator:

An estimator with $\text{bias}\left(\hat{\theta}_n\right) = 0$

This is what we hope for

There are tricks to correct the bias

Variance and standard error

Measure of how precise the estimator is for different samples

Let $\hat{\theta}_n = h(X_1, X_2, \dots, X_n)$ be the estimator for parameter θ

The variance is

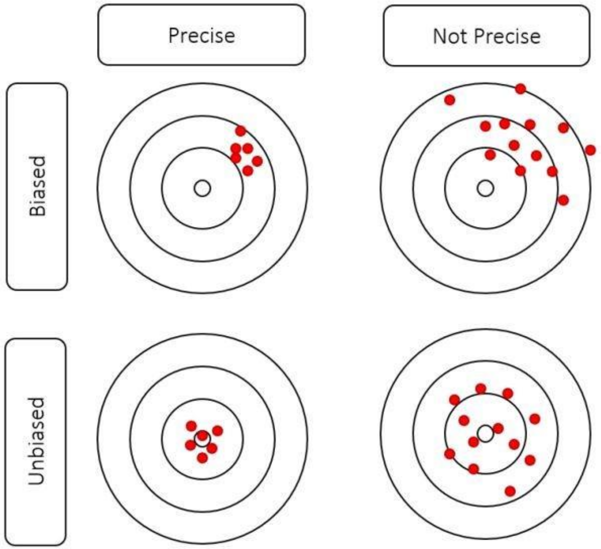
$$\mathrm{var}(\hat{\theta}_n) = \mathbb{E}\left[\left(\hat{\theta}_n - \mathbb{E}(\hat{\theta}_n)\right)^2\right]$$

The standard error is

$$\mathrm{se}(\hat{\theta}_n) = \sqrt{\mathrm{var}(\hat{\theta}_n)}$$

but $\mathrm{se}(\hat{\theta}_n)$ depends on F and might be unknown

The estimated standard error is $\widehat{\mathrm{se}}(\hat{\theta}_n)$



Example: Point estimator for Bernoulli RVs pt. 1

Let $X_1, X_2, \ldots, X_n \sim \mathrm{Bernoulli}(p)$. We don't know the real value of p
We have to create an estimate for p, defined as

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n$$

Then

$$\mathbb{E}(\hat{p}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}(X_i)}_{=p} = p$$

Is this estimator biased or not? $\mathrm{bias}(\hat{p}_n) = \mathbb{E}(\hat{p}_n) - p = p - p = 0$

Example: Point estimator for Bernoulli RVs pt. 2

The **variance of a RV** is $\mathrm{var}(X_i) = \mathbb{E}[(X_i - \mu_i)^2] = \mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2$

$$\mathrm{var}\left(\sum_{i=1}^{n} a_i X_i + b\right) = \sum_{i=1}^{n} a_i^2 \mathrm{var}(X_i)$$

The **variance** for our estimator is

$$\mathrm{var}(\hat{p}_n) = \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} p(1-p) = \frac{p(1-p)}{n}$$

Then, the **estimated standard error** is $\widehat{\mathrm{se}} = \sqrt{\mathrm{var}(\hat{p}_n)} = \sqrt{p(1-p)/n}$

The Mean Squared Error (MSE)

We define the **Mean Squared Error (MSE)** of an estimator as

$$\mathrm{MSE} = \mathbb{E}\left[\hat{\theta}_n - \theta\right]^2 = \mathrm{bias}^2\left(\hat{\theta}_n\right) + \mathrm{var}(\hat{\theta}_n)$$

Combines the bias and variance

An estimator is consistent if $\mathrm{bias}^{2}\left(\hat{\theta}_n\right) \to 0$ and $\mathrm{var}\left(\hat{\theta}_n\right) \to 0$ as $\mathrm{n} \to \infty$

This means that $\hat{\theta}_n \xrightarrow{P} \theta$

Is our estimator for parameter p of a Bernoulli RV, namely $\hat{p}_n$, consistent?

Recall that $\mathbb{E}(\hat{p}_n) = p$, $\mathrm{bias}(\hat{p}_n) = \underbrace{\cancel{\mathbb{E}(\hat{p}_n)} - \cancel{p}}_{=0}$, and $\mathrm{var}(\hat{p}_n) = p(1-p)/n \to 0$

Example

Let $X_1, X_2, \dots, X_n$ be i.i.d. RVs with mean $\mathbb{E}[X_i] = \theta$ and variance $\mathrm{var}(X_i) = \sigma^2$

Consider the following two estimators for θ

1. $\hat{\theta}_1 = X_1$
2. $\hat{\theta}_n = \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Which one is a better estimator?

For $\hat{\theta}_1 = X_1$: $\mathrm{bias}(\hat{\theta}_1) = \mathbb{E}[X_1] - \theta = 0$ and $\mathrm{var}(X_1) = \sigma^2$

For $\hat{\theta}_n = \bar{X}_n$: $\mathrm{bias}(\hat{\theta}_n) = \mathbb{E}[\bar{X}_n] - \theta = 0$ and $\mathrm{var}(\bar{X}_n) = \sigma^2/n$

Since $\mathrm{var}(\bar{X}_n) = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$, **the estimator $\hat{\theta}_n = \bar{X}_n$ is consistent**

The estimator $\hat{\theta}_1 = X_1$ **is not consistent**

The bias-variance trade-off

Present in many cases in statistics and machine learning

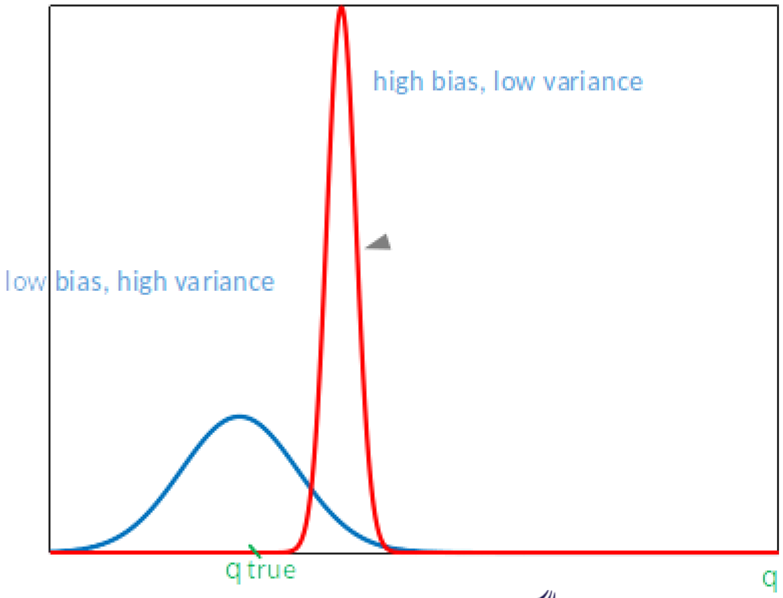
The variance of a parameter estimation can be decreased by increasing the bias

Increased complexity

High accuracy but sensitive to variations

Reduced complexity

Low accuracy but resilient to variations



Maximum Likelihood Estimation (MLE)

Maximum Likelihood Estimation (MLE)

The most common method for parametric estimation

The **likelihood function for** $X_1, X_2, \ldots, X_n$ i.i.d RVs with PDF/pmf $f(x;\theta)$ is

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta)$$

Usually, we use the **log-likelihood function** is $\ell_n(\theta) = \log \mathcal{L}_n(\theta)$

We treat the likelihood function as a function of parameter θ

Maximum Likelihood Estimator (MLE) $\hat{\theta}_n$: the value of θ that maximizes $\mathcal{L}_n(\theta)$

The MLE is consistent

MLE for discrete and continuous RVs

Suppose we observe the outcomes $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$

If $X_1, X_2, \dots, X_n$ are **discrete RVs**, the likelihood function is the joint pmf

$$\mathcal{L}_n(\theta) = \mathcal{L}(x_1, x_2, \dots, x_n; \theta) = P_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n; \theta)$$

If $X_1, X_2, \dots, X_n$ are **continuous RVs**, the likelihood function is the joint PDF

$$\mathcal{L}_n(\theta) = \mathcal{L}(x_1, x_2, \dots, x_n; \theta) = f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n; \theta)$$

Example: MLE for Bernoulli RV pt. 1

Let $X_1, X_2, \ldots, X_n \sim \text{Bernoulli}(p)$ with pmf

$$f(x;p) = p^x (1-p)^{1-x},$$

$p^{X_1} \times p^{X_2} \times p^{X_3} \cdots p^{X_n} =$

The likelihood function is

$$\mathcal{L}_n(p) = \prod_{i=1}^{n} f(X_i;p) = \boxed{\prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}}$$

$p^{X_1+X_2+\ldots,X_n}$ ☆

$(1-p)^{1-X_1} \times (1-p)^{1-X_2} \ldots$

If we define $\boxed{X = \sum_{i=1}^{n} X_i}$ we get the likelihood and log-likelihood functions:

linear →

$$\mathcal{L}_n(p) = p^X(1-p)^{n-X}$$

log →

$$\ell_n(p) = \log(p^X(1-p)^{n-X}) = X\log p + (n-X)\log(1-p)$$

Which one should we work with?

Example: MLE for Bernoulli RV pt. 2

How do we find the value of p that maximizes the likelihood $\mathcal{L}_n(p)$ or $\ell_n(p)$?

1. Take the derivative with respect to p and make it equal to 0

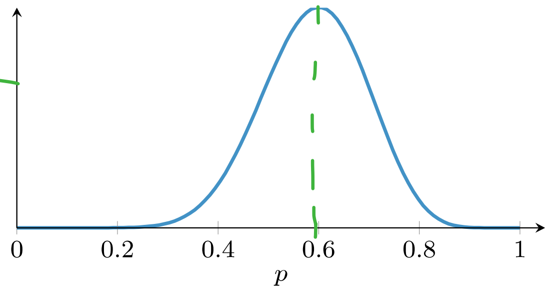
$$\frac{d\mathcal{L}_n(p)}{dp} = Xp^{X-1}(1-p)^{n-X} - (n-X)(1-p)^{n-X-1}p^X = 0$$

$$\frac{d\ell_n(p)}{dp} = \frac{X}{p} - \frac{n-X}{(1-p)} = 0$$

2. Solve for p

$$\hat{p}_n = \frac{X}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$$

$n = 20$ and $X = 12$



Finally, an MLE example with numbers pt. 1

There's a bag with 3 balls. Each ball is either red or blue.

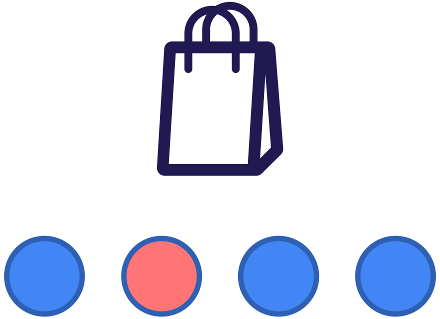
We denote the number of blue balls as θ, whose value can be 0, 1, 2, or 3.

We estimate θ by grabbing and putting back a ball 4 times

This is a process of selection with replacement

Let X_1, X_2, X_3, X_4 be the RVs of the color of the i-th ball

$$X_i = \begin{cases} 1, & \text{if the ball is blue} \\ 0, & \text{if the ball is red} \end{cases}$$



Then, $X_i \sim \text{Bernoulli}\left(\frac{\theta}{3}\right)$

The outcomes are $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1$

That is, 3 out of 4 of the balls are blue

Finally, an MLE example with numbers pt. 2

1. For each possible value of θ, find $P_{X_1,X_2,X_3,X_4}(1,0,1,1;\theta)$

$$P_{X_1,X_2,X_3,X_4}(1,0,1,1;\theta=0)=\prod_{i=1}^{4}P(X_i=x_i;\ \theta=0)=0\times1\times0\times0=0$$

$$P_{X_1,X_2,X_3,X_4}(1,0,1,1;\theta=1)=\prod_{i=1}^{4}P(X_i=x_i;\ \theta=1)=\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{2}{81}=0.0247$$

$$P_{X_1,X_2,X_3,X_4}(1,0,1,1;\theta=2)=\prod_{i=1}^{4}P(X_i=x_i;\ \theta=1)=\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{8}{81}=0.0987$$

$$P_{X_1,X_2,X_3,X_4}(1,0,1,1;\theta=3)=1\times0\times1\times1=0$$

Finally, an MLE example with numbers pt. 3

2. Use the definition of MLE to estimate θ with n experiments given $p = \theta/3$

Previously, we defined $\ell_n(p) = \log(p^X(1-p)^{n-X}) = X\log p + (n-X)\log(1-p)$

$$\ell_n(\theta) = \log\left(\left(\frac{\theta}{3}\right)^X \left(1-\frac{\theta}{3}\right)^{n-X}\right) = X(\log\theta - \log 3) + (n-X)\log\left(1-\frac{\theta}{3}\right)$$

$$\frac{d\ell_n(\theta)}{d\theta} = \frac{X}{\theta} - \frac{n-X}{3\left(1-\frac{\theta}{3}\right)} = 0$$

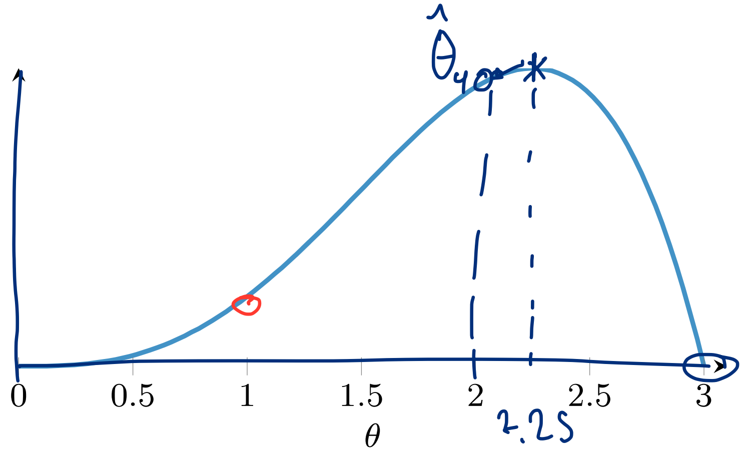
$$X(3-\theta) = \theta(n-X) \Longrightarrow 3X - \theta X = n\theta - \theta X \Longrightarrow \theta = \frac{3X}{n} = 3\bar{X}_n$$

Finally, an MLE example with numbers pt. 4

For $n = 4$ and $X = 3$ we have $\bar{X}_4 = 3/4$ and the MLE becomes

$$\hat{\theta}_4 = 3\bar{X}_4 = 3 \times 3/4 = 9/4 = 2.25$$

Since $\hat{\theta}_n$ must be an integer, we take the closest integer value and $\hat{\theta}_4 = 2$



Summary

Summary

Sampling is essential for estimation

In **parametric estimation**, we know the distribution but not the parameter

- The estimator is a RV that takes a value based on the sample

In **non-parametric estimation**, we don't need to know the distribution

There can be an infinite number of estimators

We find the best ones using **metrics: bias, variance, and MSE**

If the MSE goes to 0, the estimator is **consistent**

The Maximum Likelihood Estimator (MLE) is consistent

Appendix

Useful derivative formulas and rules

$$\frac{da^x}{dx} = \log(a)a^x$$

$$\frac{d\log_a(x)}{dx} = \frac{1}{x\log(a)}$$

$$\frac{d\log(x)}{dx} = \frac{1}{x}$$

Product rule: $h(x) = f(x)g(x)$

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient rule: $h(x) = f(x)/g(x)$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Chain rule: $h(x) = f\big(g(x)\big)$

$$h'(x) = f'\big(g(x)\big)g'(x)$$

Derivations for Bernoulli MLE (log-likelihood)

$$\frac{d\ell_n(p)}{dp} = \frac{X}{p} + \frac{n-X}{(1-p)}(-1) = \frac{X}{p} - \frac{n-X}{(1-p)} = 0$$

$$\frac{X}{p} = \frac{n-X}{(1-p)}$$

$$X(1-p) = p(n-X) \Rightarrow X - Xp = np - Xp$$

$$X = np \Rightarrow p = \hat{p}_n = \frac{X}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$