PROBABILITY THEORY
MM 2

MM 2: RANDOM VARIABLES

Topics:

Discrete and continuous random variables.

Cumulative distribution function.

Probability mass function and probability density function.

Jointly distributed random variables.

Independent random variables.

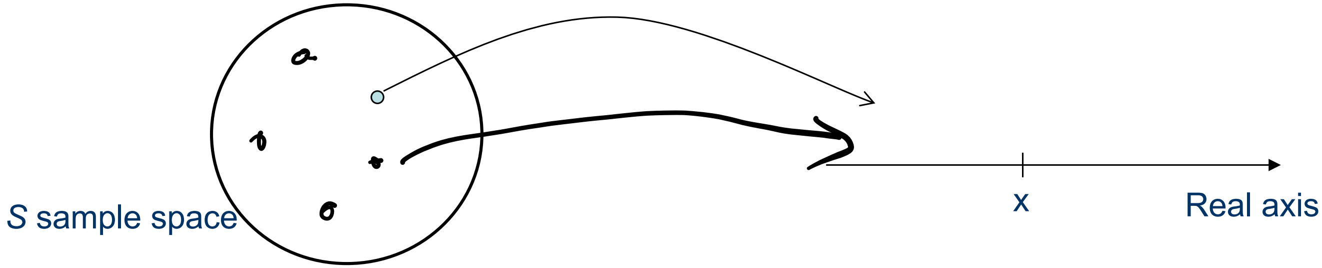
What should we learn today?

- What is a random variable?
- What types of random variables exists – how are they different and similar?
- How to work with distributions:
  - Cumulative density function
  - Probability mass function
  - Probability density function
- How to handle jointly distributed multiple random variables?

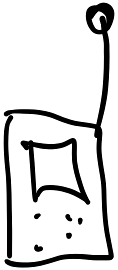
pdf

Random variable

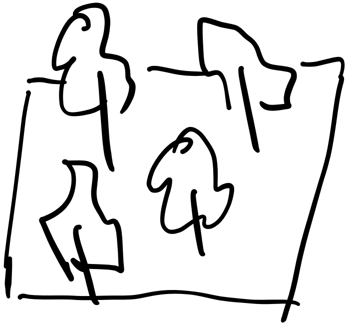
- Definition. A r.v. X is a function that assigns a real number to each outcome of a random experiment.



- Example. A coin is flipped 3 times. A r.v. X is a number of heads in 3 trials.
- Example. A r.v. X is a lifetime of a system component.



time to receive a call τ



# of trees

H $\longrightarrow$ "0"

T $\longrightarrow$ "1"

Altern.

H $\to$ 21

T $\to$ -3

Mat. notation

X - r.v. = a number of heads

Exp.: flip a coin 3 times

Possible values for $X = \{0, 1, 2, 3\}$

Example: coin flipping

Possible outcomes:

HHH	3
HHT	2
HTH	2
THH	2
HTT	1
THT	1
TTH	1
TTT	0

$P(X=3) = \frac{1}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=0) = \frac{1}{8}$

# of outcomes in total $= 2 \cdot 2 \cdot 2 = 2^3 = 8$

# of outcomes $X = k$

k out of 3

$$\binom{3}{k} = \frac{3!}{k!(3-k)!}$$

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

Example: coin flipping

Possible outcomes:

HHH	3
HHT	2
HTH	2
THH	2
HTT	1
THT	1
TTH	1
TTT	0

$P(X=3) = \frac{1}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=0) = \frac{1}{8}$

Total number of outcomes =

$= 2^3 = 8$

# of outcomes when # of H is i

$$\binom{3}{i} = \frac{3!}{(3-i)!\,i!} = \begin{cases} 1, & i=3 \\ 3, & i=2 \\ 3, & i=1 \\ 1, & i=0 \end{cases}$$

$$\boxed{P(X<2)} = P(X=0) + P(X=1) + P(X=2) = \frac{7}{8}$$

$X=0$

$X=1$

$X = 2$

Random variable

How to describe a behavior of a r.v.?

distribution functions

cdf

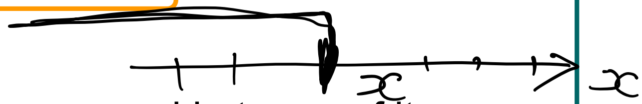
pdf → cont. r.v.

pmf → discrete r.v.

Cumulative distribution function

- The cdf of a r.v. *X* is defined for any real number *x* as the probability of the event *{ X≤x }*

$$\boxed{F(x) = P(X \leq x)} = P(X \in (-\infty, x])$$



- *F* is a function of *x*.
- All probability questions about *X* can be answered in terms of its distribution function
- Example: how to compute *P(a<X ≤b)* ?

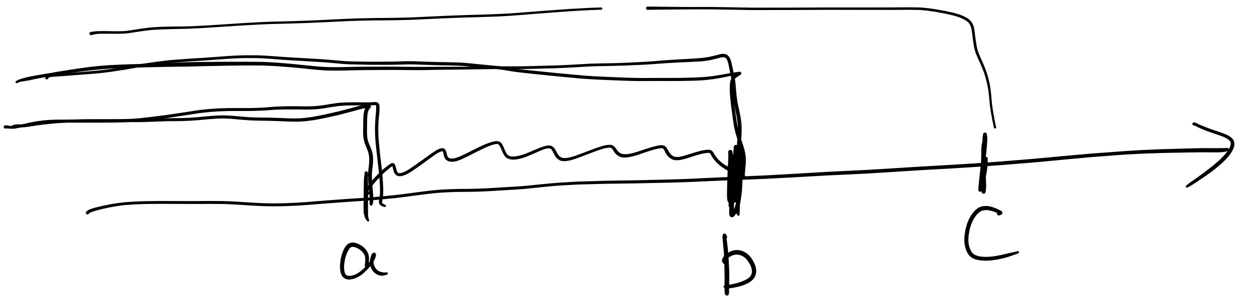
$$\{X \leq b\} = \{X \leq a\} \bigcup \{a < X \leq b\}$$

$$P\{X \leq b\} = P\{X \leq a\} + P\{a < X \leq b\}$$

$$\boxed{P\{a < X \leq b\} = F(b) - F(a)}$$

$$P(X \le a) = F(a) = P(X \in (-\infty, a])$$

$$P(X \in (a, b])$$



$$(-\infty, b] = (-\infty, a] \cup (a, b]$$

$$\underbrace{P(X \in (-\infty, b])}_{F(b)} = \underbrace{P(X \in (-\infty, a])}_{F(a)} + P(X \in (a, b])$$

$$P(X \in (a, b]) = F(b) - F(a)$$

Properties of cdf

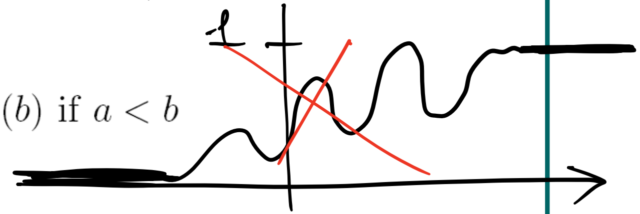
1. $0 \le F(x) \le 1$

$F(x) = P(X \in \underbrace{(-\infty, x]}) \qquad P(X \in (-\infty, \infty)) = 1$

2. $\lim_{x \to \infty} F(x) = 1$

3. $\lim_{x \to -\infty} F(x) = 0$

$x \to -\infty$



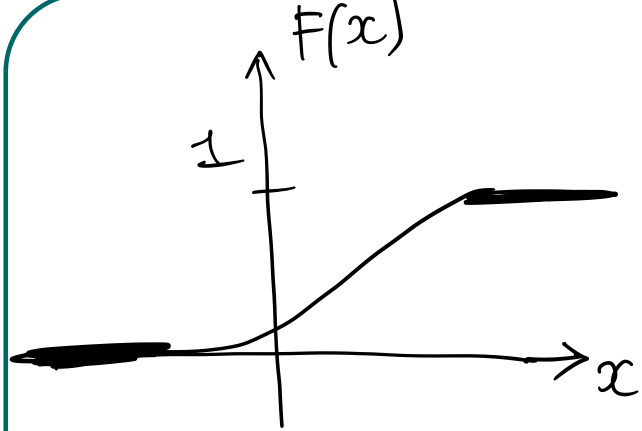
4. $F(x)$ is a nondecreasing function: $F(a) \le F(b)$ if $a < b$

5. $F(x)$ is a continuous from the right:

$$F(b) = \lim_{h \to 0} F(b + h) = F(b+)$$

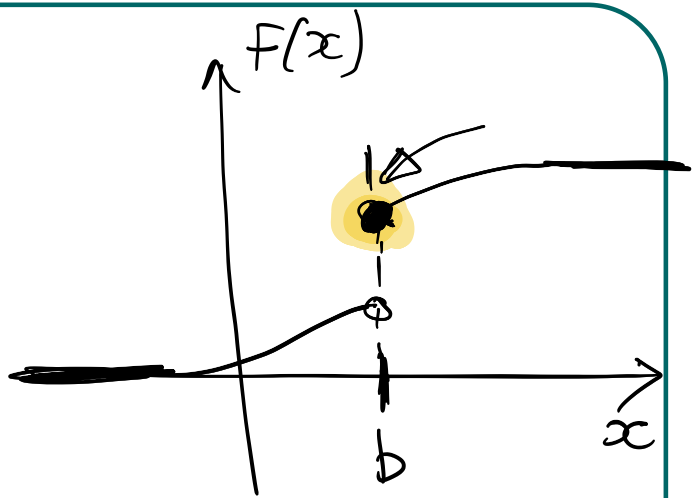
6. Probability that a r.v. X takes on a specific value b is equal to the jump (step) of cdf at the point b:

$$P(X = b) = F(b+) - F(b-)$$



continuous

X - continuous



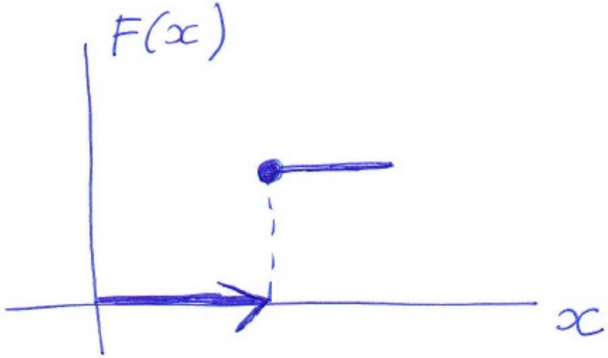
$$F(b) = F(b+)$$

$$F(x) = P(X \in (-\infty, x])$$

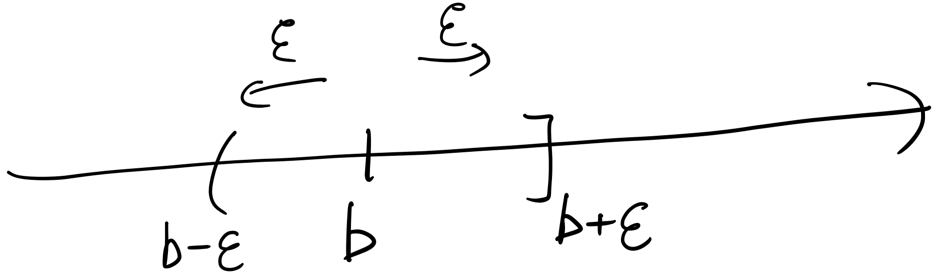
X - "continuous from the right"

Property no 5

Continuous from the right:



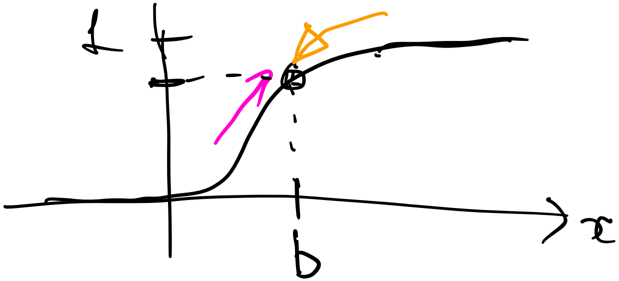
$P(X=b)$ $\qquad \varepsilon \to 0$



$$P(X \in (b-\varepsilon; b+\varepsilon]) = F(b+\varepsilon) - F(b-\varepsilon)$$

$\varepsilon \to 0$

$$P(X=b) = F(b+) - F(b-)$$



continuous r.v.

$$F(b) - F(b) = 0$$

$$\boxed{P(X=b)=0}$$

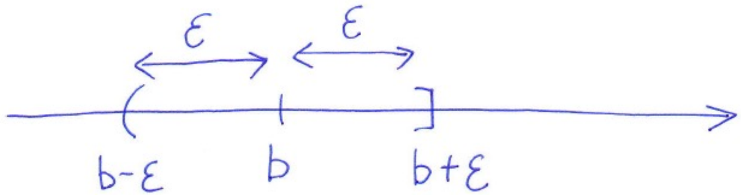
for all values b

Property no 6

We know $P\{a < X \leq b\} = F(b) - F(a)$

$$P\{X = b\} = \lim_{\varepsilon \to 0} \{b - \varepsilon < X \leq b + \varepsilon\} =$$

$$= \lim_{\varepsilon \to 0} [F(b+\varepsilon) - F(b-\varepsilon)] = F(b+) - F(b-)$$

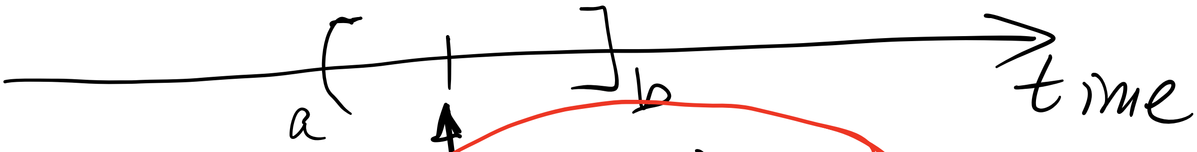


Example X – lifetime of a computer

$X \in [0, \infty)$

$P(X = 1 \text{ year } 5 \text{ days } 30 \text{ min } 35 \text{ sec}) =$

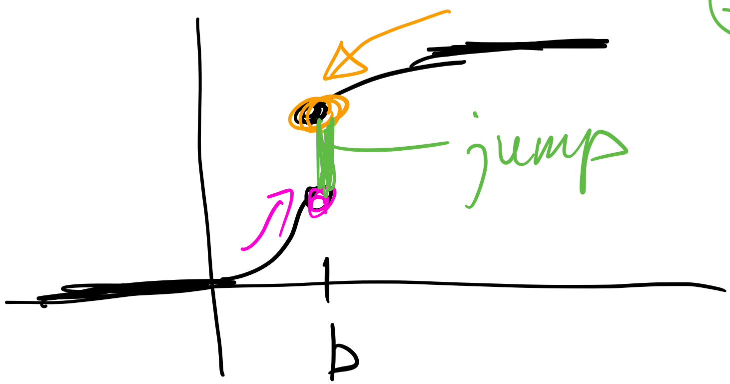
$= P(X \in (35\text{sec}, 36\text{sec}]) \neq 0$



$P(X = b) = 0$

$P(X \in (a, b]) \neq 0$

"jump"

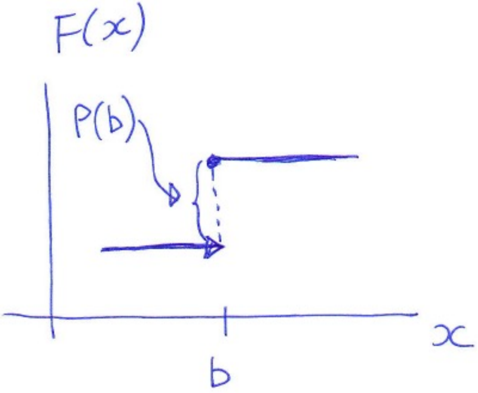


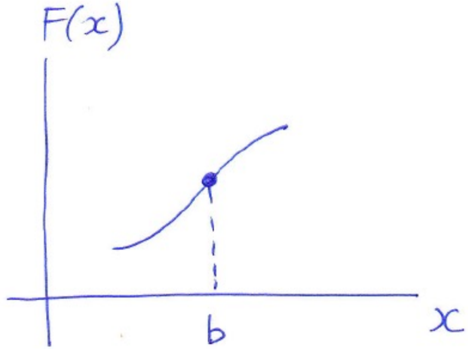
$\boxed{P(X=b)}$ = "jump"

$P(X=b) = \underline{F(b+)} - \underline{F(b-)}$

discrete r.v.

Property no 6





discrete r.v.

$P(b) =$ "jump"

continuous r.v.

$P(b) = 0$

Discrete r.v.

Ex. $X \in \{1,2,3,4,5,6\}$

$X \in \{1,2,\ldots\}$

- Definition. A r.v. whose set of possible values is a sequence is said to be discrete.
  countable
- For a discrete r.v. we define probability mass function

$$p(a) = P(X = a)$$

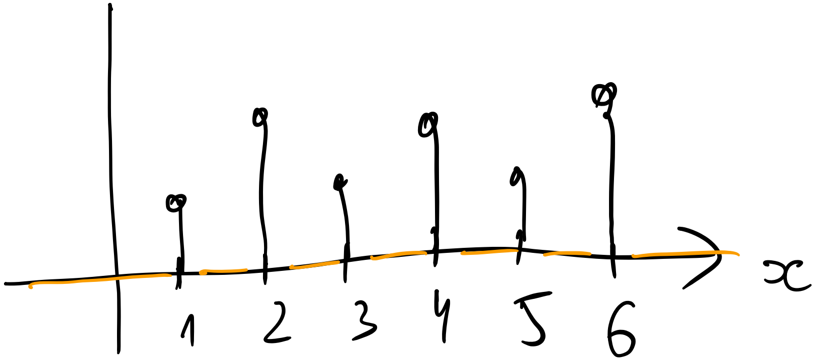
$$p(x_i) > 0, \quad i = 1, 2, \ldots$$
$$p(x) = 0, \quad otherwise$$

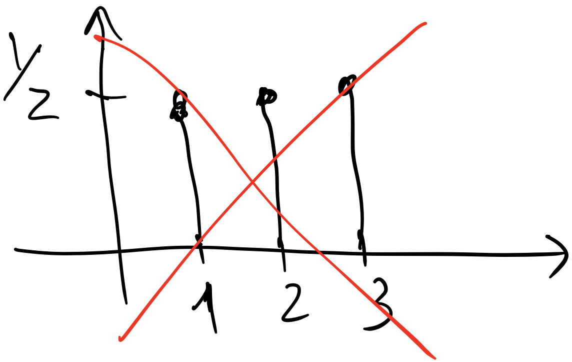
$$\sum_{i=1}^{\infty} p(x_i) = 1$$

- The cdf is a step-function and can be expressed as

$$F(a) = \sum_{all\ x \leq a} p(x)$$

$p(x)$ pmf

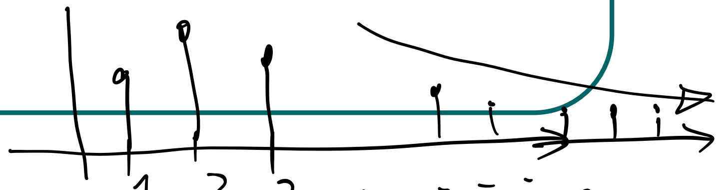




- pmf — yes / no

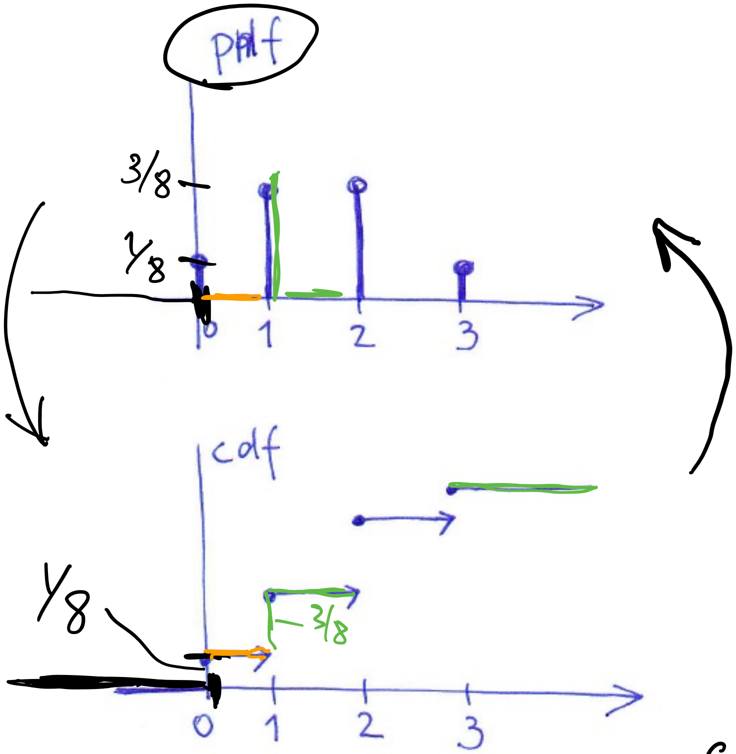
$$\sum_{i=1}^{3} p(i) = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2} > 1$$

$$\sum_{\text{all } a} p(a) = 1$$



$\sum_i = 1$

Example coin flipping



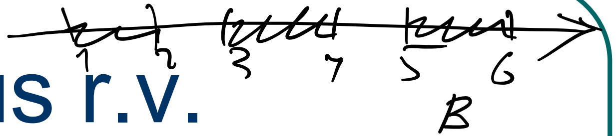
$$\frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

$$\sum_i p(i) = 1$$

$$F(a) = \sum_{\text{all } x \leq a} p(a)$$

discrete r.v. $\rightarrow$

cdf is a step function



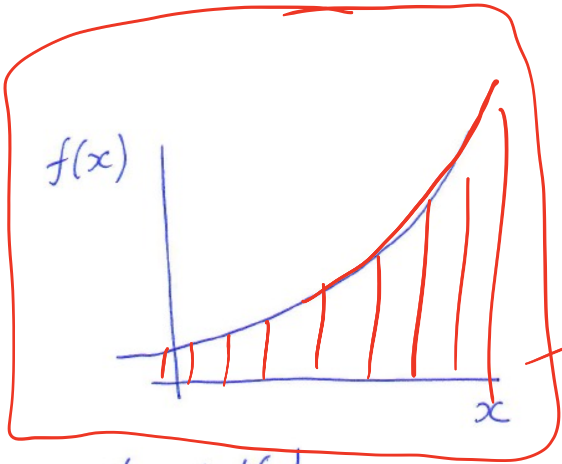
Continuous r.v.

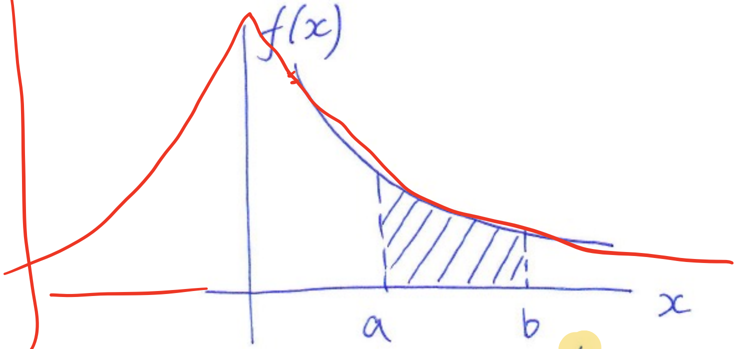
- Definition. X is a continuous r.v. if there exists a nonnegative function f(x), defined for all x having the property that for any set B of real numbers

$$P(X \in B) = \int_B f(x)dx$$

- Function f(x) is called the probability density function of X
- Additionally, f(x) should satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$
- Different probability statements can be expressed using pdf
- If we let a=b, then $P(a \le X \le b) = \int_a^b f(x)dx$ $\quad P(X = a) = \int_a^a f(x)dx = 0$
- Probability that a continuous r.v. will assume any particular value is zero

Continuous r.v.





not pdf!

$$\underbrace{\int_{-\infty}^{+\infty} f(x)\,dx}_{= P(X \in (-\infty; +\infty))} = 1$$

$$P\{a < X \le b\} = \int_a^b f(x)\,dx$$

$$P(X = a) = \int_a^a f(x)\,dx = 0$$

$$P(X = b) = \int_b^b f(x)\,dx = 0$$

cdf $\leftrightarrow$ pdf

Continuous r.v

$$\int_{-\infty}^{x} f(x)dx$$

- The relationship between cdf and pdf is expressed by

$$\boxed{F(a) = P(X \in (-\infty, a]) = \int_{-\infty}^{a} f(x)dx}$$

$$\boxed{\frac{d}{da}F(a) = f(a)}$$

- Example: The pdf of the samples of the amplitide of speech waveforms is found to decay exponentially at rate alpha

$$f(x) = ce^{-\alpha|x|}$$

Find constant c and find probability $P(|X| < v)$

Example speech amplitude

- Use normalization condition:

$$1 = \int_{-\infty}^{+\infty} c e^{-\alpha|x|} dx = 2c \int_{0}^{\infty} e^{-\alpha x} dx =$$

$$= 2c \left. \frac{e^{-\alpha x}}{-\alpha} \right|_{0}^{\infty} = 2c\left(0 + \frac{1}{\alpha}\right) = \frac{2c}{\alpha}$$

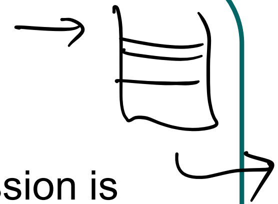
$$\Rightarrow \quad c = \frac{\alpha}{2}$$

- Find probability

$$P\{|X| < v\} = \frac{\alpha}{2} \int_{-v}^{v} e^{-\alpha|x|} dx = 2 \cdot \frac{\alpha}{2} \int_{0}^{v} e^{-\alpha x} dx =$$

$$= \alpha \left. \frac{e^{-\alpha x}}{-\alpha} \right|_{0}^{v} = \left[1 - e^{-\alpha v}\right]$$

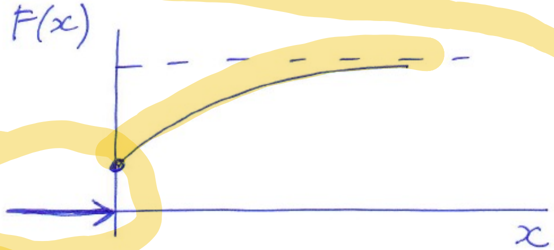
Example of a mixed r.v.

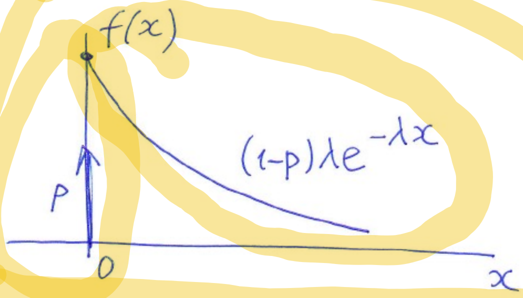


- The delay (= waiting time in a queue) for a packet transmission is zero if the queue is empty, and if the queue is not empty, the delay is an exponentially distributed r.v. with cdf $F(x) = 1 - e^{-\lambda x}$
- The probability that the queue is empty is ***p*** and busy ***1-p***.
- Cdf of the delay X:

$$F(x) = P(X \le x) = P(X \le x \mid \text{idle}) \cdot p + P(X \le x \mid \text{busy}) \cdot (1-p)$$

$$= p + (1-p)(1 - e^{-\lambda x})$$



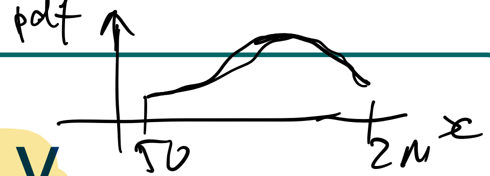


Types of r.v.

Discrete r.v.	Continuous r.v.	Mixed type
Pmf p(x) $P\{a < X < b\} = \sum_{x_i \in (a,b)} p(x_i)$	Pdf f(x) $P(a \leq X \leq b) = \int_a^b f(x)dx$	$F(x) = p_1 F_1(x) + \ldots p_n F_n(x)$

$X \in (50\text{cm}, 2\text{M})$

$Z \in (0, 100\text{ y.o})$



Multiple r.v.

$P(X = 20\text{ y.o}, Z = 50\text{cm}) = 0$

- So far, we were speaking about calculation of probabilities of events involving a single r.v. in isolation. Now we will look at the techniques for probability calculations of events that involve the joint behavior of two or more r.v.
- Example: height, weight and age of a person from a group
  X Y Z
- To spesify the relationship between two r.v., we define the joint cumulative probability distribution function of X and Y

$F(x,y)$

$$\boxed{F(\cancel{X,Y}) = P(X \le x, Y \le y)}$$

joint cdf

- A knowledge of the joint cdf enables us to calculate the distribution function of r.v. X:

$$F_X(x) = P(X \le x) = P(X \le x, Y < \infty) = F(x, \infty)$$

joint cdf $\longrightarrow$ individual cdf

$F_{X,Y}(x, y) \longrightarrow$ $F_X(x)$

$F_Y(y)$

Jointly distributed discrete r.vs.

joint pmf → individual pmf

- If X and Y are discrete r.vs., we define joint probability mass function:

$$\boxed{p(x_i, y_i) = P(X = x_i, Y = y_i)}$$

- The individual mass functions are easily obtained from the joint pmf:

$$\{X = x_i\} = \bigcup_j \{X = x_i, Y = y_i\}$$

$$P\{X = x_i\} = P\bigcup_j \{X = x_i, Y = y_i\} = \sum_j P\{X = x_i, Y = y_i\} = \sum_j p(x_i, y_j)$$

- Example from the last lecture
- The joint probabilities can be presented in tabular form. Because the individual probabilities appear in the margin of the table, they are often called marginal probabilities.

Example X, Y joint pmf

$Y \backslash X$	1	2	3	
0	$1/8$	$1/3$	$1/8$	$7/12$
1	$1/8$	$1/6$	$1/8$	$5/12$
	$1/4$	$1/2$	$1/4$	

pmf for Y

marginal pmf

$7/12 + 5/12 = 1$

marginal pmf for X

“Restaurant” example

X \ Y	Peter	Eric	P{X = bad, good}
bad	0.086	0.07	0.156
good	0.774	0.07	0.844
P{Y = Peter, Eric}	0.86	0.14	

marginal probabilities

pmf $\sum_i$ $\longrightarrow$ pdf $\int$

Jointly distributed continous r.vs.

- We say that X and Y are jointly continous, if there exist a function f(x,y) defined for all real x and y, having the property that for every set C in the 2-dimentional plane

$$P\{(X,Y) \in C\} = \iint_{(x,y)\in C} f(x,y)\,dx\,dy$$

- f(x,y) is called joint probability density function

$$P\{X \in [a,b], Y \in [c,d]\} = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- The marginal pdfs are obtained by integrating out the variables that are not of interest.

ind pdf for r.v. X →

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_Y = \int_{-\infty}^{\infty} f(x,y)\,dx = f_Y(y)$$

- The joint cumulative distribution function is defined as $\boxed{F(x,y) = P(X \le x, Y \le y)}$

joint distribution → individual distribution

cdf → cdf

pmf → pmf

pdf → pdf

more information

?

←

~~always yes~~ ←

no ←

not always

Example

Y \ X	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

Y \ X	0	1	
0	1/2	0	1/2
1	0	1/2	1/2
	1/2	1/2	

indiv. distr. $\not\longrightarrow$ joint distr.

Question

- As we have seen, specifying the joint probability mass function or probability density function determines the individual distribution functions.
- Is reverse true? If I know individual mass functions, can I determine the joint mass function?

indiv. dist. $\rightarrow$ joint dist.

+ extra info

independency

Example: transmission system

System 1 – independent

useless com. syst.

$Y_R \backslash X_{Tr}$	0	1	
0	1/4	1/4	1/2
1	1/4	1/4	1/2
	1/2	1/2	

System 2 – ideal com. system

$Y_R \backslash X_{Tr}$	0	1	
0	1/2	0	1/2
1	0	1/2	1/2
	1/2	1/2	

$$p(x,y) = p_X(x)\,p_Y(y)$$

indep. + ind. $\rightarrow$ joint distr.

Independent r.v.

- Definition. X and Y are independent, if for any two sets of real numbers A and B

$$\boxed{P(X \in A, Y \in B) = P(X \in A)P(Y \in B)}$$

- In terms of joint cumulative distribution function:

$$\boxed{F(a,b) = F_X(a)F_Y(b)}$$

- In terms of pmf (discrete r.v.) and pdf (continuous r.v.)

$$\boxed{p(x,y) = p_X(x)p_Y(y)} \qquad \boxed{f(x,y) = f_X(x)f_Y(y)}$$

- Basically, X and Y are independent, if knowing the value of one does not change the distribution of another

Multiple r.v.

- Let $X_1, \ldots, X_n$ be the jointly distributed random variables.
- The joint cumulative distribution function is defined as

$$\boxed{F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)}$$

- Example. A computer system receives messages over three communication lines. Let X_i be the number of messages received on line i in one hour. The joint pmf is given by

$$p(x_1, x_2, x_3) = (1 - a_1)(1 - a_2)(1 - a_3) a_1^{x_1} a_2^{x_2} a_3^{x_3}$$

Find individual pmfs

Example 3 communication lines

$$p_{X_3}(x_3) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (1-a_1)(1-a_2)(1-a_3)\, a_1^{i}\, a_2^{k}\, a_3^{x_3} =$$

$$= (1-a_1)(1-a_2)(1-a_3)\, a_3^{x_3} \underbrace{\sum_{i=1}^{\infty} a_1^{i}}_{\frac{1}{1-a_1}} \underbrace{\sum_{k=1}^{\infty} a_2^{k}}_{\frac{1}{1-a_2}} =$$

$$= (1-a_3)\, a_3^{x_3}$$

- Q: are lines independent?

Conditional distributions

- The relationship between two random variables can often be clarified by consideration of the conditional distribution of one given the value of the other.
- The conditional pmf of X given that Y=y is defined by

$$\boxed{p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}}$$

- If X and Y have a joint pdf, then the conditional pdf of X given that Y=y is defined as

$$\boxed{f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}}$$

Example conditional distributions

- Joint pmf of X and Y is given:

Y \ X	0	1	
0	0.4	0.1	0.5
1	0.2	0.3	0.5
	0.6	0.4	

- Calculate the conditional pmf of X given that Y=1:

$$P\{X=0 \mid Y=1\} = \frac{p(0,1)}{P\{Y=1\}} = \frac{0.2}{0.5} = 0.4$$

$$P\{X=1 \mid Y=1\} = \frac{p(1,1)}{P\{Y=1\}} = \frac{0.3}{0.5} = 0.6$$