PROBABILITY THEORY
MM 1

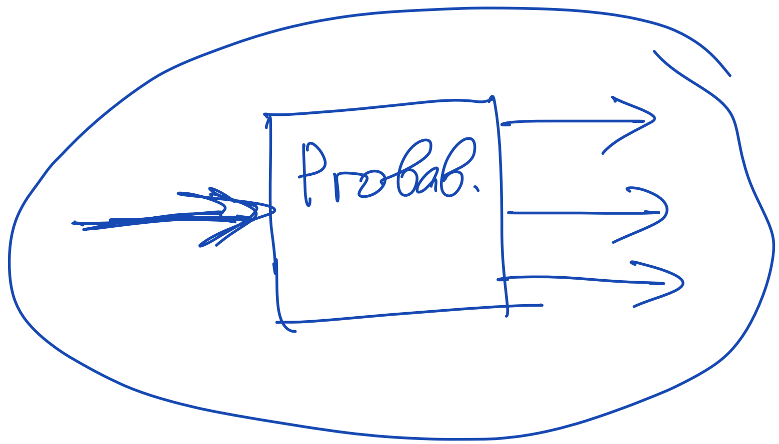
MM 1: Basic Concepts of Probability Theory

Topics:

- Introduction
- Terminology
- Axioms of probability
- How to compute probability using counting methods
- Conditional probability and Bayes' formula
- Independent events

Deterministic models vs Probabilistic models

- Deterministic model: the conditions under which an experiment is carried out determine the exact outcome of the experiment
- Probabilistic model: the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions



Frequency interpretation of probability

Pre-lecture problem: simulate a coin flipping

$$\frac{N_0(n)}{n}$$

$N_0(n) \to$ # of times we got a Head

$n \to$ # total of times

$n \to \infty \qquad \frac{N_0}{n} \to \frac{1}{2}$

$$\lim_{n \to \infty} \frac{N_0}{n}$$

Frequency interpretation of probability

$$p = \lim_{n \to \infty} \frac{N_0(n)}{n}$$

- Not possible to perform an experiment infinite number of times
- Situations when an experiment is not repeatable
- → a mathematical theory of probability

Lecture plan

- Terminology
- Axioms of probability
- How to compute probability using counting methods
- Conditional probability and Bayes' formula
- Independent events

Terminology

Experiment → Outcome → Sample space → Event

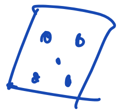
- A random experiment is an experiment in which outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.
- Outcome is a result of an experiment that can not be decomposed into other results.
- The sample space S is defined as the set of all possible outcomes.

  Discrete and continuous sample spaces
- An event is defined as a subset of S

  Certain event S = all possible outcomes

  Impossible (null) event = no outcomes

Example

Exper.	Outcomes	Sample Space	Events
	$\{5\}$	$S = \{1, 2, 3, 4, 5, 6\}$	A = outcome is an odd number $= \{1, 3, 5\}$

discrete

finite

B = outcome > 3 $= \{4, 5, 6\}$

C = outcome is 2.5 $= \{\phi\}$ impossible event

D = out. is a positive number $= S$ certain event

$S = \{ n, \ n \in [1, 2, .. \infty) \}$

driving licence

$\{3\}$

$S = \{1, 2, 3, ... \infty\}$

discrete
infinite
countable set $\sim \mathbb{N}$

E = less than 3 attemps = $\{1, 2\}$

lifetime of a computer

2 years 3 months
5 days 3 min
45 sec 5 ms...

$S = [0, \infty)$

F = still working after 3 years

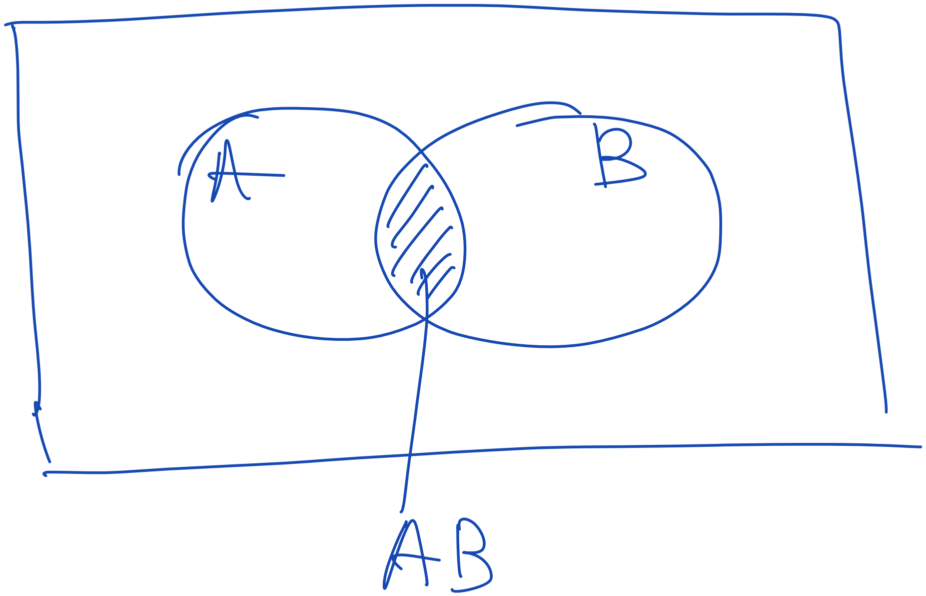
continuous

Set operations

- Union of A and B $A \bigcup B$ = {all outcomes that are either in A or B}
- Intersection of A and B = $AB = A \bigcap B$ {all outcomes that are both in A and B}
- Two events are mutually exclusive, if $AB = \emptyset$
- The complement of an event A = $A^c = \bar{A}$ {all events that are not in A}
- If all outcomes of B are in A, B is contained in A: $B \subset A$

- The definitions can be generalized for the case of n events

- Graphical representation of events can be made by Venn diagrams

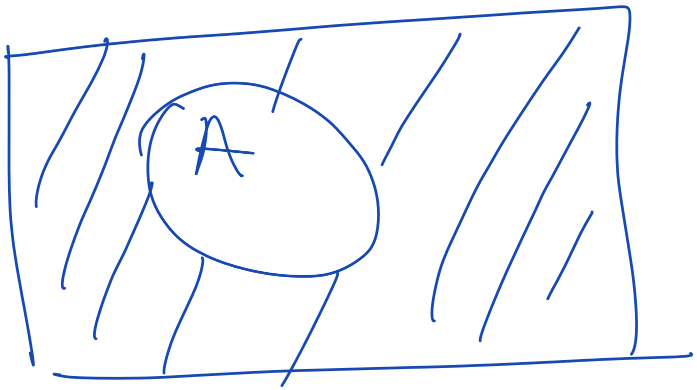


$A \cup B$

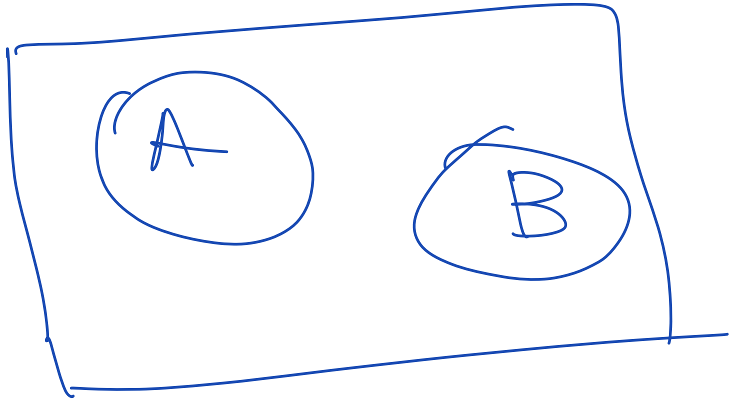
$A \cap B = AB$

$\cup \rightarrow$ "$+$"

$\cap \rightarrow$ "$\cdot$"



$\bar{A} = A^c$



$AB = \emptyset$

mutually exclusive

Set operations

Commutative law	$E \bigcup F = F \bigcup E$	$EF = FE$
Associative law	$(E \bigcup F) \bigcup G = E \bigcup (F \bigcup G)$	$(EF)G = E(FG)$
Distributive law	$(E \bigcup F)G = EG \bigcup FG$	$EF \bigcup G = (E \bigcup G)(F \bigcup G)$

$$(E \bigcup F)^c = E^c F^c$$

$$(EF)^c = E^c \bigcup F^c$$

Axioms $\longrightarrow$ Statements $\rightarrow$ Theorems

Axioms of probability

Experiment Event A $\longrightarrow$ P probability

- Let *E* be a random experiment. A probability law for the experiment *E* is a rule that assigns to each event *A* a number *p(A),* called the probability of A, that satisfies the following axioms:

- Axiom 1. $0 \leq P(A) \leq 1$

- Axiom 2. $P(S) = 1$ $S \rightarrow p = 1$

- Axiom 3. For any sequence of mutually exclusive events

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

A and B are mut. excl.

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

<u>Proposition 1.</u>

$$A \to P(A)$$

$$A^c \to \boxed{1 - P(A)}$$

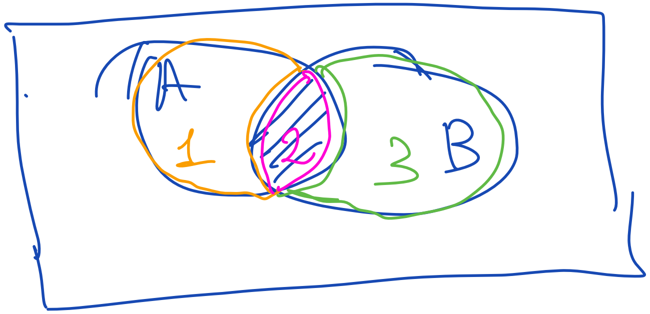
Proof: A and A^c – mut. exclusive

$$P(\underbrace{A \cup A^c}_{S}) = P(A) + P(A^c)$$

$$1 = P(S)$$

Proposition 2.

$$\boxed{P(A \cup B) = P(A) + P(B) - P(AB)}$$



$2 = AB$

$A \cup B = 1 \cup 2 \cup 3$

$$P(A \cup B) = P(1 \cup 2 \cup 3) = P(1) + P(2) + P(3)$$

$$P(A) + P(B) - P(AB) = P(1) + P(2) + \cancel{P(2)} + P(3) - \cancel{P(2)}$$

Propositions

- Proposition 1.

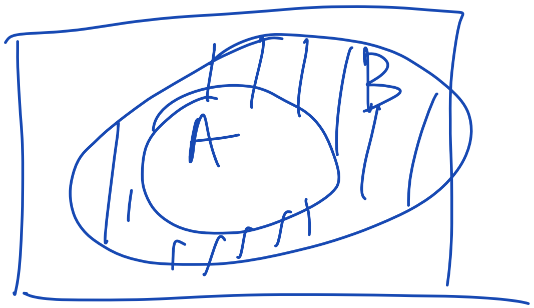
$$P(A^c) = 1 - P(A)$$

- Proposition 2.

$$P(A \bigcup B) = P(A) + P(B) - P(AB)$$

- Proposition 3.

$$\text{If } A \subset B, \text{ then } P(A) \leq P(B)$$



$B \setminus A$

$B = A \cup (B \setminus A)$

$P(B) = P(A) + P(B \setminus A) \geq 0$

$P(B) \geq P(A)$

Computing probabilities

S - discrete + finite + all outcomes are equally probable

- Sample space having equally likely outcomes
  - If S is a finite space, we ennumerate all possible outcomes $S=\{1, 2, ..., N\}$

# of outcomes in A

$$\underline{P(A)} = \frac{Number\ of\ points\ in\ A}{N}$$

total # of outcomes = size of S

- The calculation of probabilities reduces to counting the number of outcomes in the event.

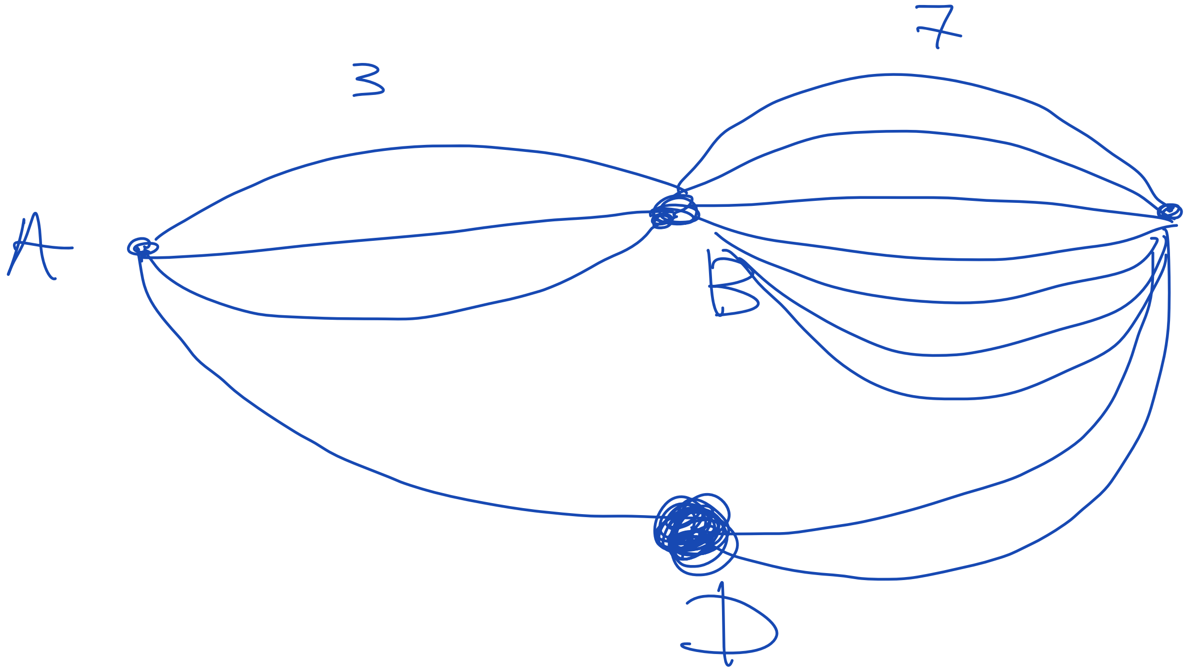
- Example: a basket with numbered balls

Principles of counting

Suppose 2 experiments are to be performed. If there are *m* possible outcomes for experiment 1, and for each possible outcome of an experiment 1, there are *k* possible outcomes for experiment 2, then there are *mk* possible outcomes of the 2 experiments.

- If third experiment is to be performed with *l* possible outcomes → sample space of 3 experiments consists of *mkl* elements.

Example: roads between villages



$3 \cdot 7 = 21$
options

$3 \cdot 7 \quad + \quad 1 \cdot 2$

Sampling with/without Replacement and with/without Ordering

N objects in the basket. We choose k objects.
Number of possible outcomes?

- Sampling with Replacement and with Ordering $\boxed{n^k}$
- Sampling without Replacement and with Ordering

permutation of k objects = k!

$$\boxed{n(n-1)\cdots(n-k+1)}$$

- Sampling without Replacement and without Ordering

$$\boxed{\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}}$$

- Sampling with Replacement and without Ordering $\boxed{\binom{n-1+k}{k}}$

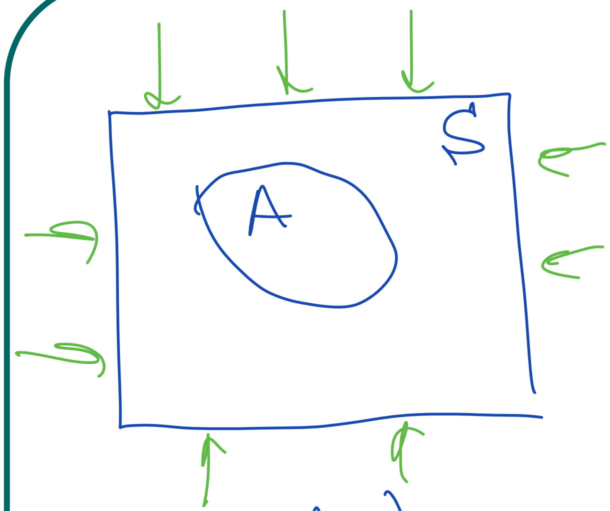
Conditional probability

- We are often interested in calculating probabilities when some partial information concerning the results of the experiment is available; or recalculating it in light of new information

$P(A|B)$

- Definition. The conditional probability is defined as

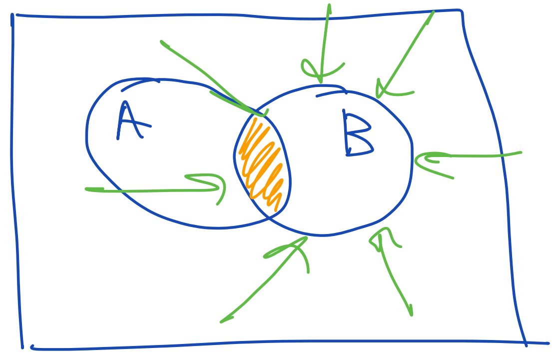
$$\boxed{P(A|B) = \frac{P(AB)}{P(B)}, \quad P(B) > 0}$$



$$\frac{P(A)}{P(S)}$$

$S \rightarrow B$

$A \rightarrow AB$

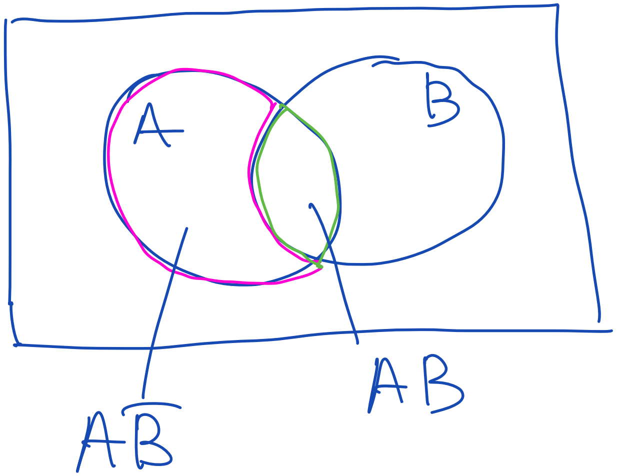


$$\frac{P(AB)}{P(B)}$$

$P(AB) = P(A|B)\cdot P(B)$ $\qquad$ $P(A|B) = \dfrac{P(AB)}{P(B)}$

Conditional probability

- It is often turns out that it is easier to compute the probability of an event if we first ”condition” on the occurence or non-occurence of a secondary event.



$$A = A\bar{B} \cup AB$$

$$P(A) = P(A\bar{B}) + P(AB) =$$

$$= P(A|\bar{B})\cdot P(\bar{B}) + P(A|B)\cdot P(B)$$

formula of total probability

Bayes' formula

- A and B are two events

$$A = AB \bigcup AB^c$$

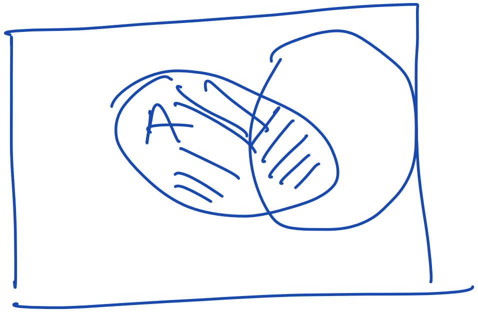
$$P(A) = P(AB) + P(AB^c) =$$
$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

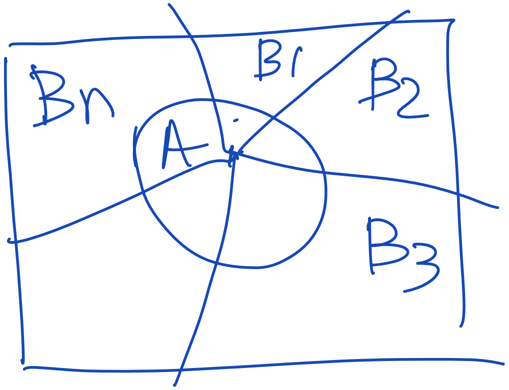
- The probability of event A is a weighted average of conditional probabilities
- Suppose that A has occured and we are interested in determined if B has also occured:

definition

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

def. formula of total probability





$B_1, B_2 \ldots B_n$ – mutually excl.

$$AB_1 \cup AB_2 \cup AB_3 \ldots \cup AB_n = A$$

Formula of total pr.

$$\boxed{P(A) = P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + \ldots + P(B_n)\cdot P(A|B_n)}$$

Example

Peter and Eric are chefs at Restaurant. Peter works 6 days a week while Erik works one day a week. In 90% of the cases Peter's food is high quality while Eric's food is high quality in 50% of the cases.

One evening Restaurant serves an awful meal.

Whom should we blame?

Is it fair to conclude that Eric prepared the food that evening?

event B = a person working at the kitchen
{ Peter; Erik }

event A = quality of food { good, bad }

$P(B = \text{Peter}) = \frac{6}{7}$

$P(B = \text{Erik}) = \frac{1}{7}$

Event B = Peter is working
$\overline{B}$ = Erik is working

$P(A = \text{good})$ – we do not know yet

$P(A = \text{good} \mid B = \text{Peter}) = 0.90$

$P(A = \text{bad} \mid B = \text{Peter}) = 0.1$

$$P(A=\text{good} \mid B=\text{Erik}) = 0.5$$
$$P(A=\text{Bad} \mid B=\text{Erik}) = 0.5$$

$$P(B=\text{Erik} \mid A=\text{Bad}) - ?$$
$$P(B=\text{Peter} \mid A=\text{Bad}) - ?$$

$$P(A=\text{bad}) = P(\text{Peter}) \cdot P(A\ \text{Bad} \mid \text{Peter}) + P(E) \cdot P(A=\text{bad} \mid E) =$$
$$= \frac{6}{7} \cdot 0.1 + \frac{1}{7} \cdot 0.5 = \frac{1.1}{7}$$

$$P(B=\text{Erik} \mid A=\text{bad}) = \frac{P(E) \cdot P(A=\text{Bad} \mid E)}{\frac{1.1}{7}} = \frac{\frac{0.5}{7}}{\frac{1.1}{7}} = \frac{5}{11}$$

$$P(B=\text{Peter} \mid A=\text{Bad}) = \frac{\frac{0.6}{7}}{\frac{1.1}{7}} = \frac{6}{11}$$

Independence of events

- Generally, knowing that B has occured, changes the chances of A's occurence. If it does not, then *P(A|B)=P(A)*

- Definition. Two events are independent, if $\boxed{P(AB)=P(A)P(B)}$

- Definition. Three events are independent, if

$$
\boxed{\begin{gathered} P(ABC)=P(A)P(B)P(C) \\ P(AB)=P(A)P(B) \\ P(BC)=P(B)P(C) \\ P(AC)=P(A)P(C) \end{gathered}}
$$

- Definition. N events are independent, if for any subset

$$
\boxed{P(A_{r_1} \dots A_{r_k}) = P(A_{r_1}) \dots P(A_{r_k})}
$$

$$\boxed{\begin{gathered}A \text{ and } B - \text{ indep.}\\ P(A\mid B) = P(A)\end{gathered}}$$

$$\Downarrow \qquad \Uparrow$$

$$P(A\mid B) \overset{\text{def.}}{=} \boxed{\frac{P(AB)}{P(B)} = P(A)}$$

$$\Downarrow \qquad \Uparrow$$

$$\boxed{P(AB) = P(A)\cdot P(B)}$$

Independent?

- You flip a coin and get a head ***and*** you flip a second coin and get a tail
- There is a sun shine ***and*** a lecture today is cancelled
- You draw one card from a deck and its black ***and*** you draw a second card and it's black.
- There is a storm ***and*** an airport is closed

How do we know that there is independence?

1) we can prove $P(AB) = P(A)P(B) \Longrightarrow$
A & B are independent

2) from physical formulation of a problem
we assume A & B are independent $\Rightarrow$
we can use formula $P(AB) = P(A)P(B)$

Example

- We roll a dice twice. Let us define A as the event that the first outcome is odd. Let B be the event that both outcomes are the same. Finally, let C be the event that the sum of outcomes is even.

  Are A and B independent?

  Are B and C independent?