# Surface sampling and the intrinsic Voronoi diagram

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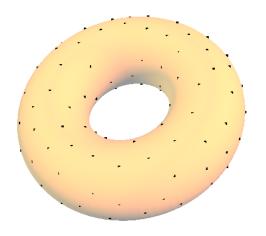
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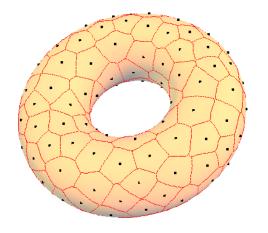




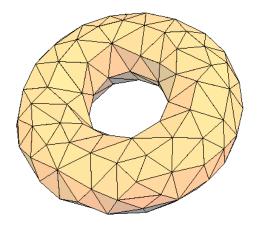








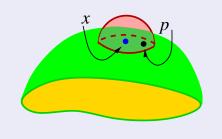


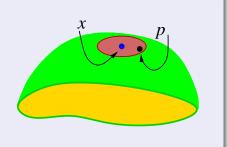




# Sampling terminology

## Extrinsic vs intrinsic





### Sampling radius

Maximum distance to the nearest sample

### Sizing function

- ullet Positive function, ho on S
- Used to specify a sampling radius:  $\epsilon \rho(x)$

### Contributions

### Our contributions

- improved intrinsic sampling criteria
- relate extrinsic and intrinsic sampling criteria



### Outline

- Introduction
- 2 Intrinsic sampling criteria
- Relating intrinsic and extrinsic criteria
- 4 Discussion



### Edelsbrunner and Shah (1994)

- V(p) is a topological disk  $\forall p \in P$
- $\mathcal{V}(p) \cap \mathcal{V}(q)$  is empty or a single Voronoi edge
- $\mathcal{V}(p) \cap \mathcal{V}(q) \cap \mathcal{V}(s)$  is empty or a single point



### Edelsbrunner and Shah (1994)

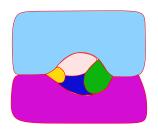
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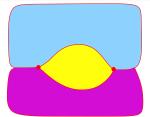
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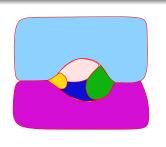
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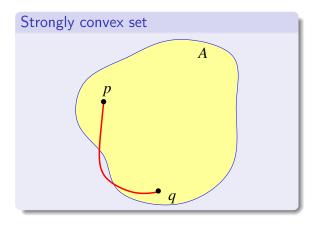
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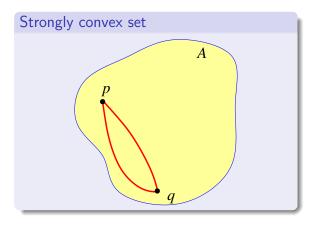




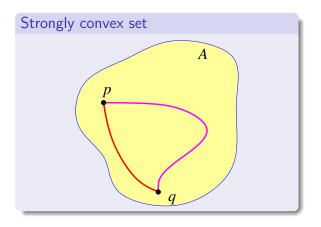
redundant



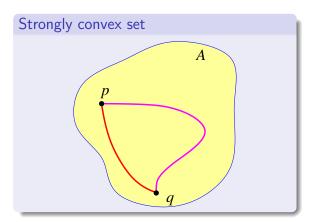












### Definition (Strong convexity radius)

An intrinsic sizing function:

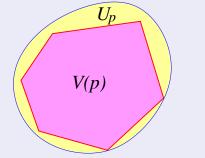
$$\rho_{\mathsf{sc}}(x) = \sup \{ \rho \mid B_{\mathcal{S}}(x; r) \text{ is strongly convex } \forall r < \rho \}$$

# Voronoi cells and convexity

# Voronoi cells are not convex • q because Voronoi boundaries are not geodesics

### Theorem

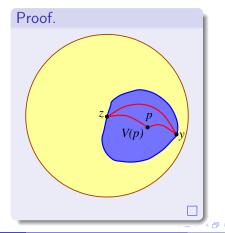
The intrinsic Voronoi diagram is well formed if each Voronoi cell can be contained in a strongly convex set.



# A sampling criterion

### A sufficient sampling radius

If P satisfies a sampling radius of  $\frac{1}{2}\rho_{sc}(x)$  then the intrinsic Voronoi diagram of P on S is well formed.





# Weakening the criterion

Is  $\rho_{sc}(x)$  a sufficient sampling radius?

### Voronoi cells are topological disks

If  $\rho_{sc}(x)$  is a sampling radius for P then the Vornonoi cells of P on S are topological disks.

unique geodesics ⇒ contractible

### The second closed-ball criterion?

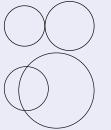
- can Voronoi cells share more than one Voronoi edge?
- not if they can share no more than 2 Voronoi vertices



### Pseudo-disks

# Pseudo-disks (Boissonnat and Oudot 2005)

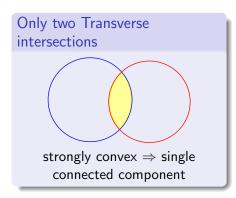
Boundaries intersect transversely twice, tangentially once or not at all.

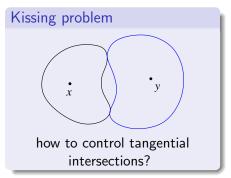


### Pseudo-disks suffice $\mathcal{V}(p)$ and Disks $\mathcal{V}(q)$ within cannot sampling $\Rightarrow$ share radius are three pseudo-Voronoi disks vertices



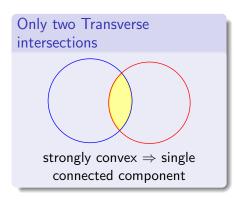
# The scr and pseudo-disks

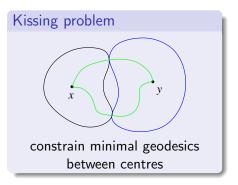






# The scr and pseudo-disks







# Main sampling result

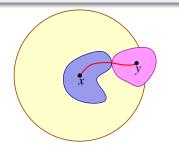
### Injectivity radius

Injectivity radius:  $\rho_i(x)$ 

 $d_S(x,y) < \rho_i(x) \Rightarrow$  unique minimal geodesic between x and y

# Intrinsic sampling radius

$$\rho_m(x) = \min\left\{\rho_{\rm sc}(x), \frac{1}{2}\rho_i(x)\right\}$$



### Theorem (Intrinsic sampling)

If  $\rho_m(x)$  is a sampling radius for P, then the intrinsic Voronoi diagram of P on S is well formed.



Introduction

Intrinsic sampling criteria

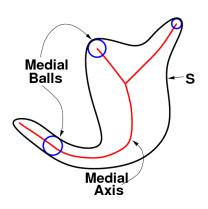
3 Relating intrinsic and extrinsic criteria

4 Discussion



### Local feature size

#### Bounded by curvature



### local feature size (Ifs)

 $\rho_f(x) = \text{distance to medial}$ axis

## Lipschitz continuity

$$|\rho_f(x)-\rho_f(z)|\leq d_{\mathbb{R}^3}(x,z)$$

### Bounded by curvature

- radius of medial ball ≤ radius of osculating ball
- $\rho_f(x) \leq \rho_\kappa(x)$



# Relating intrinsic sizing functions and the Ifs

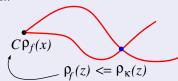
### To show

$$\rho_m(x) \ge C \rho_f(x)$$

### Proof idea

 $\rho_m(x)$  small  $\Rightarrow$  curvature large  $\Rightarrow \rho_f(x)$  small

- get curvature bound
- convert to Ifs
- bring home via Lipschitz





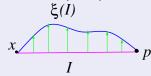
### The result

### Relating to local feature size

$$\rho_m(x) \ge \left(\frac{\pi}{4+3\pi}\right) \rho_f(x)$$

### Comparing sampling criteria

- bound geodesic distances in terms of Euclidean:  $B_S(x; \delta r) \subset B_{\mathbb{R}^3}(x; r)$
- Morvan and Thibert (2004):





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# Comparing with Leibon and Letscher (2000)

### We require

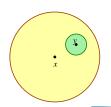
 $\rho_m(x)$  is a sampling radius for P

### (Leibon and Letscher 2000) require

For any  $y \in B_S(x; \frac{4}{5}\rho_{sc}(x))$ , there is a  $p \in P$  contained in  $B_S(y; \frac{1}{5}\rho_{sc}(x))$ .

### Comparison

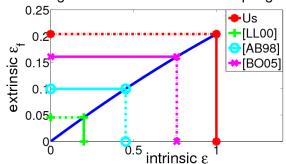
- more complicated cannot be expressed via a sampling radius
- stronger than a sampling radius of  $\frac{1}{5}\rho_{\rm sc}(x)$
- details of proof did not appear
- however, it does apply to higher dimensions





# Comparing other sampling criteria

Relating extrinsic and intrinsic sampling criteria



- Vertical axis:  $\epsilon_f$  for extrinsic sampling radius  $\epsilon_f \rho_f(x)$
- Horizontal axis:  $\epsilon$  for intrinsic sampling radius  $\epsilon \rho_m(x)$



# Open questions

### Tighter bounds

• larger C in  $\rho_m(x) \geq C\rho_f(x)$ 

### Higher dimensions

- extending the sampling criteria
- extending the sizing function comparisons



# Acknowledgments

### Thank you to:

- A. Bobenko Voronoi boundaries are not geodesics
- reviewers
- SFU Faculty of Applied Science travel funding
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# Thank You!