primitive recursive encoding

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Defines	sequence number

Recall that an encoding of a set L of words over some alphabet Σ is defined as an injective function $E : L \to \mathbb{N}$, the set of natural numbers (including 0 here).

Finite sequences over $\mathbb{N}$ can be thought of as words over the "infinite" alphabet $\mathbb{N}$. So the notion of word encoding directly carries over to encoding of finite sequences of natural numbers. Let $\mathbb{N}^*$ be the set of all finite sequences over $\mathbb{N}$.

Definition. Let E be an encoding for $\mathbb{N}^*$. A number is called a *sequence number* if it is in the range of E. Since E is injective, we say that $E(a)$ is *the* sequence number of the sequence a.

Once E is fixed, we also use the notation $\langle a_1, \ldots, a_k \rangle$ to mean the sequence number of the sequence $a_1, \ldots, a_k$.

Define the following operations on $\mathbb{N}$ associated with a given E:

1. $E_n := E|\mathbb{N}^*_n$, the restriction of E to the set $\mathbb{N}^*_n$ of all sequences of length n, and E_0 is defined as the number $\langle\rangle$.

2. the *length function*: $\mathrm{lh} : \mathbb{N} \to \mathbb{N}$:

$$\mathrm{lh}(x) := \begin{cases} z, & \text{if } E^{-1}(x) \text{ exists, and has length } z, \\ 0, & \text{otherwise.} \end{cases}$$

3. $(\cdot)_\cdot : \mathbb{N}^2 \to \mathbb{N}$, such that

$$(x)_y := \begin{cases} z, & \text{if } E^{-1}(x) \text{ exists, has length } \geq y, \text{ with } z \text{ its } y\text{-th number,} \\ 0, & \text{otherwise.} \end{cases}$$

4. $* : \mathbb{N}^2 \to \mathbb{N}$, such that

$$x * y := \begin{cases} E(ab), & \text{where } E(a) = x \text{ and } E(b) = y, \\ 0, & \text{otherwise.} \end{cases}$$

   The notation ab stands for the concatenation of the sequences a and b.

5. $\mathrm{ext} : \mathbb{N}^2 \to \mathbb{N}$, such that

$$\mathrm{ext}(x, y) := \begin{cases} E(ay), & \text{where } E(a) = x, \\ 0, & \text{otherwise.} \end{cases}$$

   The notation ay stands for the sequence a, extended by appending y to the right of a.

6. red : $\mathbb{N} \to \mathbb{N}$, such that

$$\operatorname{red}(x) := \begin{cases} z, & \text{where } E(a) = x \text{ and } E(b) = z \text{ such that } a = bc \text{ and } c \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

   In other words, z is the sequence number of the sequence obtained by removing the last (rightmost) number (if any) in the sequence whose sequence number is x, provided that x is a sequence number.

Definition. An encoding E for $\mathbb{N}^*$ is said to be *primitive recursive* if

- $E(\mathbb{N}^*)$ is a primitive recursive set, and
- the first three types of functions defined above are primitive recursive.

Proposition 1. *If E is primitive recursive, so are the functions $*, \operatorname{ext}$, and* red.

Proof. Let $\chi(x)$ be the characteristic function of the predicate "x is a sequence number" (via E). It is primitive recursive by assumption.

1. Let $n = \operatorname{lh}(x)+\operatorname{lh}(y)$. Then $x*y = E_n((x)_1, \ldots, (x)_{\operatorname{lh}(x)}, (y)_1, \ldots, (y)_{\operatorname{lh}(y)}) \cdot \chi(x)\chi(y)$, which is primitive recursive.

2. Let $n = \operatorname{lh}(x)+1$. Then $\operatorname{ext}(x, y) = E_n((x)_1, \ldots, (x)_{\operatorname{lh}(x)}, y)\chi(x)$, which is primitive recursive.

3. Let $n = \operatorname{lh}(x) \dot{-} 1$. Then

$$\operatorname{red}(x) = \begin{cases} E_n((x)_1, \ldots, (x)_{\operatorname{lh}(x) \dot{-} 1}), & \text{if } \operatorname{lh}(x) > 1, \text{ and } \chi(x) = 1 \\ \langle\rangle, & \text{if } \operatorname{lh}(x) = 1, \text{ and } \chi(x) = 1 \\ 0, & \text{otherwise,} \end{cases}$$

   which is primitive recursive.

□

Examples of primitive recursive encoding can be found in the parent entry (methods 2, 3, and 7).