syntactic congruence

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Entry type	Definition
Classification	msc 68Q70
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Defines	syntactic semigroup
Defines	syntactic monoid
Defines	syntactic morphism
Defines	maximality property of syntactic congruence

Let S be a semigroup and let $X \subseteq S$. The relation

$$s_1 \equiv_X s_2 \text{ iff } \forall l, r \in S(ls_1r \in X \text{ iff } ls_2r \in X) \tag{1}$$

is called the *syntactic congruence* of X. The quotient $S/\equiv_X$ is called the *syntactic semigroup* of X, and the natural morphism $\phi : S \to S/\equiv_X$ is called the *syntactic morphism* of X. If S is a monoid, then $S/\equiv_X$ is also a monoid, called the *syntactic monoid* of X.

As an example, if $S = (\mathbb{N}, +)$ and $X = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} \mid n = 3k\}$, then $m \equiv_X n$ if $m \mod 3 = n \mod 3$, and the syntactic monoid is isomorphic to the cyclic group of order three.

It is straightforward that $\equiv_X$ is an equivalence relation and X is union of classes of $\equiv_X$. To prove that it is a congruence, let $s_1, s_2, t_1, t_2 \in S$ satisfy $s_1 \equiv_X s_2$ and $t_1 \equiv_X t_2$. Let $l, r \in S$ be arbitrary. Then $ls_1t_1r \in X$ iff $ls_2t_1r \in X$ because $s_1 \equiv_X s_2$, and $ls_2t_1r \in X$ iff $ls_2t_2r \in X$ because $t_1 \equiv_X t_2$. Then $s_1t_1 \equiv_X s_2t_2$ since l and r are arbitrary.

The syntactic congruence is both left- and right-invariant, *i.e.*, if $s_1 \equiv_X s_2$, then $ts_1 \equiv_X ts_2$ and $s_1t \equiv_X s_2t$ for any t.

The syntactic congruence is maximal in the following sense:

- if χ is a congruence over S and X is union of classes of χ,
- then $s\chi t$ implies $s \equiv_X t$.

In fact, let $l, r \in S$: since $s\chi t$ and χ is a congruence, $lsr\chi ltr$. However, X is union of classes of χ, therefore lsr and ltr are either both in X or both outside X. This is true for all $l, r \in S$, thus $s \equiv_X t$.