Post system

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Introduction

A *Post canonical system* (or *Post system* for short) $\mathfrak{P}$ is a triple (Σ, X, P), such that

1. Σ is an alphabet,
2. X is an alphabet, disjoint from Σ, whose elements we call *variables*, and
3. P is a non-empty finite binary relation on Σ^*, the Kleene star of P, such that for every $(u, v) \in P$,
   - $u \in \Sigma^* X \Sigma^*$, and
   - if S is a variable occurring in v, then it occurs in u.

   An element $(x, y) \in P$ is called a *production* of $\mathfrak{P}$, x is its *antecedent*, and y the *consequent*. $(x, y) \in P$ is often written $x \to y$.

The last condition basically says that in a production $x \to y$, x must contain at least one variable, and y can not contain any variables that are not already occurring in x. Put it more concretely, a production in a Post canonical system has the form

$$a_1 S_1 a_2 S_2 \cdots a_n S_n a_{n+1} \to b_1 S_{\phi(1)} b_2 S_{\phi(2)} \cdots b_m S_{\phi(m)} b_{m+1} \tag{1}$$

where a_i and b_j are fixed words on Σ, while S_k are variables, with $0 < n$, $0 \leq m$, $m \leq n$, and ϕ is a function (not necessarily bijective) on the set $\{1, \ldots, n\}$.

Examples. Let $\Sigma = \{a, b, c\}$ and $X = \{S, U, V, W\}$. Then (Σ, X, P) with P consisting of

$$aSb^2 \to ba, \quad cVaWaUb \to aWU, \quad a^3cUbSW \to SabU, \quad bVa \to aV^2c$$

is a Post canonical system, while (Σ, X, Y) with Y consisting of

$$ab^2 \to ba, \quad cVaWaUb \to aWU, \quad aUbSc^2W \to ScaV, \quad a \to S$$

is not, for the following reasons:

- the antecedents in the first and fourth productions do not contain a variable

- the consequents in the third and fourth productions contain variables (V in the third, and S in the fourth) which do not occur in the corresponding antecedents.

Normal systems. A Post canonical system $\mathfrak{P} = (\Sigma, X, P)$ is called a *Post normal system*, or *normal system* for short, if each production has the form $aS \to Sb$ (called a *normal production*), where a, b are words on Σ and S is a variable.

Languages generated by a Post system

Let us fix a Post system $\mathfrak{P} = (\Sigma, X, P)$. A word v is said to be *immediately derivable* from a word u if there is a production of the form (1) above, such that

$$u = a_1 u_1 a_2 u_2 \cdots a_n u_n a_{n+1} \quad \text{and} \quad v = b_1 a_{\phi(1)} b_2 a_{\phi(2)} \cdots b_m a_{\phi(m)} b_{m+1},$$

where a_i are words (not variables). This means that if we can write a word u in the form of an antecedent of a production by replacing all the variables with words, then we can "produce", or "derive" a word v in the form of the corresponding consequent, replacing the corresponding variables with the corresponding words. When v is immediately derivable from u, we write $u \Rightarrow v$. Using the example above, with the production $cVaWaUb \to aWU$, we see that

- $ca^4b = caaaab \Rightarrow a^3$ if we set $V = \lambda$ and $W = U = a$, or
- $ca^4b = caaaab \Rightarrow a^2$ if we set $V = a$ and exactly one of $W, U = a$ and the other λ.

A word v is *derivable* from a word u if there is a finite sequence of words $u_1, \ldots, u_n$ such that

$$u = u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_n = v.$$

When v is derivable from u, we write $u \overset{*}{\Rightarrow} v$. Again, following from the example above, we see that $c^2abab^2 \overset{*}{\Rightarrow} ac$, since

$$c^2abab^2 = ccababb \Rightarrow ab^2 \Rightarrow ba \Rightarrow ac.$$

Given a finite subset A of words on Σ, let T_A be the set of all words derivable from words in A. Elements of A are called *axioms* of $\mathfrak{P}$ and elements of T_A

theorems (of $\mathfrak{P}$ derived from axioms of A). Intuitively, we see that the Post system $\mathfrak{P}$ is a language generating machine that creates the language T_A via a set A of axioms. In general, we say that a language M over an alphabet Σ is *generable by a Post system* if there is a Post system $\mathfrak{P} = (\Sigma_1, X, P)$ such that $\Sigma \subseteq \Sigma_1$, a finite set A of axioms on Σ_1 such that $M = T_A \cap \Sigma^*$.

Remarks.

- If a language is generable by a Post system, it is generable by a normal system.

- A language is generable by a Post system iff it is generable by a semi-Thue system. In this sense, Post systems and semi-Thue systems are "equivalent".

- Instead of allowing for one antecedent and one consequent in any production, one can have a more generalized system where one production involves a finite number of antecedents as well as a finite number of consequents:

$$\left\{ \begin{array}{c} a_{11}S_{11}a_{12}S_{12}\cdots a_{1n_1}S_{1n_1}a_{1,n_1+1}, \\ \vdots \\ a_{p1}S_{p1}a_{p2}S_{p2}\cdots a_{pn_p}S_{pn_p}a_{p,n_p+1} \end{array} \right\} \to \left\{ \begin{array}{c} b_{11}S_{\phi_1(1)}b_{12}S_{\phi_1(2)}\cdots b_{1m}S_{\phi_1(m_1)}b_{1,m_1+1}, \\ \vdots \\ b_{q1}S_{\phi_q(1)}b_{q2}S_{\phi_q(2)}\cdots b_{qm}S_{\phi_q(m_q)}b_{q,m+1} \end{array} \right\}$$

  where each ϕ_i is a function from $\{1, \ldots, m_i\}$ to $\{(1,1), \ldots, (1,n_1), \ldots, (p,1), \ldots, (p,n_p)\}$. We may define b to be immediately derivable from a if a can be expressed using *each* of the antecedents by substituting the variables S_{ij} by words $c_{ij} \in \Sigma^*$, and b can be expressed in *at least one* of the consequents by the corresponding substitutions (of $S_{\phi_i(j)}$ into $c_{\phi_i(j)}$). It can be shown that any language generated by this more general system is in fact Post generable!

- It can be shown that a language is Post-generable iff it is recursively enumerable.

Post Computability

For any positive integer m, and an m-tuple $\overline{n} := (n_1, \ldots, n_m)$ of natural numbers, we may associate a word

$$E(\overline{n}) := ab^{n_1}ab^{n_2}a\cdots ab^{n_m}a.$$

Let $f : \mathbb{N}^m \to \mathbb{N}$ be a partial function. Define

$$L(f) := \{E(\overline{n})cE(f(\overline{n})) \mid \overline{n} \in \mathrm{dom}(f)\}.$$

We say that f is *Post-computable* if $L(f)$ is Post-generable. As expected from the last remark in the previous section, a partial function is Turing-computable iff it is Post-computable.

References

[1] M. Davis, *Computability and Unsolvability*. Dover Publications, New York (1982).

[2] N. Cutland, *Computability: An Introduction to Recursive Function Theory*. Cambridge University Press, (1980).

[3] M. Minsky, *Computation: Finite and Infinite Machines*. Prentice Hall, (1967).