normalizing reduction

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Definition 1. Let X be a set and $\to$ a reduction (binary relation) on X. An element $x \in X$ is said to be in *normal form* with respect to $\to$ if $x \not\to y$ for all $y \in X$, i.e., if there is no $y \in X$ such that $x \to y$. Equivalently, x is in normal form with respect to $\to$ iff $x \notin \mathrm{dom}(\to)$. To be *irreducible* is a common synonym for 'to be in normal form'.

Denote by $\xrightarrow{*}$ the reflexive transitive closure of $\to$. An element $y \in X$ is said to be *a normal form of* $x \in X$ if y is in normal form and $x \xrightarrow{*} y$.

A reduction $\to$ on X is said to be *normalizing* if every element $x \in X$ has a normal form.

Examples.

- Let X be any set. Then no elements in X are in normal form with respect to any reduction that is either reflexive. If $\to$ is a symmetric relation, then $x \in X$ is in normal form with respect to $\to$ iff x is not in the domain (or range) of $\to$.

- Let $X = \{a, b, c, d\}$ and $\to = \{(a, a), (a, b), (a, c), (b, c), (a, d)\}$. Then c and d are both in normal form. In addition, they are both normal forms of a, while d is a normal form only for a. However, X is not normalizing because neither c nor d have normal forms.

- Let X be the set of all positive integers greater than 1. Define the reduction $\to$ on X as follows: $a \to b$ if there is an element $c \in X$ such that $a = bc$. Then it is clear that every prime number is in normal form. Furthermore, every element x in X has n normal forms, where n is the number of prime divisors of x. Clearly, $n \geq 1$ for every $x \in X$. As a result, X is normalizing.