definite language

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A language is *definite* if the determination of whether a word belongs to the language completely depends on its suffix of a fixed length. Formally,

Definition. Let L be a language over an alphabet Σ. Then L is *definite* if there is a non-negative integer k such that for any word u over Σ with $|u| \geq k$, its suffix v of length k is in L iff u is in L.

Note that if $k = 0$, then L is either Σ^* or $\varnothing$.

A definite language has the following characterization:

Proposition 1. *L is definite iff there are finite languages L_1 and L_2 such that*

$$L = L_1 \cup \Sigma^* L_2.$$

Proof. Suppose first that L is definite. Let L_1 be the subset of L containing all words with length less than k and L_2 the subset of L containing all words of length k. The case when $k = 0$ is already mentioned in the note above. If $k = 1$, L_1 is either $\{\lambda\}$ or $\varnothing$, depending on whether or not $\lambda \in L$. In any case, L_1 and L_2 are both finite. We verify that $L = L_1 \cup \Sigma^* L_2$. If $u \in L$ and u has length less than k, then $u \in L_1$. Otherwise, it has length at least k. By the definiteness of L, its suffix v of length k is in L, and thus is in L_2. Therefore $u \in \Sigma^* L_2$ as a result. This shows that $L \subseteq L_1 \cup \Sigma^* L_2$. On the other hand, suppose $u \in L_1 \cup \Sigma^* L_2$. If $|u| < k$, then $u \in L_1 \subseteq L$. If $|u| \geq k$, write $u = wv$ where v is the suffix with length k. Then $v \in L_2$, which means that $u \in L$ since L is definite.

Now suppose $L = L_1 \cup \Sigma^* L_2$ with L_1, L_2 finite. Let k be the maximum of lengths of words in $L_1 \cup L_2$, plus 1. k is well-defined since $L_1 \cup L_2$ is finite. Suppose u is a word in Σ^* with $|u| \geq k$. Then $u \notin L_1$. Let v be the suffix of u with $|v| = k$. Then $v \notin L_1$ likewise. If $v \in L$, then $v \in \Sigma^* L_2$ and hence $u \in \Sigma^* L_2$ as well. If $u \in L$, then $u \in \Sigma^* L_2$, so that u has a suffix $w \in L_2$. Since $|w| < k$, w is also a suffix of v, and therefore $v \in \Sigma^* L_2 \subseteq L$ as a result. This shows that L is definite. □

From this characterization, we see that finite languages are special cases of definite languages. We also see that

Corollary 1. *Every definite language is a regular language.*

With regard to the closure properties of definite languages, we have the following: let $\mathscr{D}$ be the family of definite languages, then $\mathscr{D}$ is closed under union and complementation, and therefore any Boolean operations. However, $\mathscr{D}$ is not closed under concatenation, Kleene star, and reversal.

Remark. In fact, every definite language is locally testable. The converse, however, is not true.

References

[1] A. Salomaa, *Formal Languages*, Academic Press, New York (1973).