characteristic monoid

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Given a semiautomaton $M = (S, \Sigma, \delta)$, the transition function is typically defined as a function from $S \times \Sigma$ to S. One may instead think of δ as a set $C(M)$ of functions

$$C(M) := \{\delta_a : S \to S \mid a \in \Sigma\}, \qquad \text{where} \qquad \delta_a(s) := \delta(s, a).$$

Since the transition function δ in M can be extended to the domain $S \times \Sigma^*$, so can the set $C(M)$:

$$C(M) := \{\delta_u : S \to S \mid u \in \Sigma^*\}, \qquad \text{where} \qquad \delta_u(s) := \delta(s, u).$$

The advantage of this interpreation is the following: for any input words u, v over Σ:

$$\delta_u \circ \delta_v = \delta_{vu},$$

which can be easily verified:

$$\delta_{vu}(s) = \delta(s, vu) = \delta(\delta(s, v), u) = \delta(\delta_v(s), u) = \delta_u(\delta_v(s)) = (\delta_u \circ \delta_v)(s).$$

In particular, δ_λ is the identity function on S, so that the set $C(M)$ becomes a monoid, called the *characteristic monoid* of M.

The characteristic monoid $C(M)$ of a semiautomaton M is related to the free monoid Σ^* generated by Σ in the following manner: define a binary relation $\sim$ on Σ^* by $u \sim v$ iff $\delta_u = \delta_v$. Then $\sim$ is clearly an equivalence relation on Σ^*. It is also a congruence relation with respect to concatenation: if $u \sim v$, then for any w over Σ:

$$\delta_{uw}(s) = \delta_u(\delta_w(s)) = \delta_v(\delta_w(s)) = \delta_{vw}(s)$$

and

$$\delta_{wu}(s) = \delta_w(\delta_u(s)) = \delta_w(\delta_v(s)) = \delta_{wv}(s).$$

Putting the two together, we see that if $x \sim y$, then $ux \sim vx \sim vy$. We denote $[u]$ the congruence class in $\Sigma^*/\sim$ containing the word u.

Now, define a map $\phi : C(M) \to \Sigma^*/\sim$ by setting $\phi(\delta_u) = [u]$. Then ϕ is well-defined. Furthermore, under ϕ, it is easy to see that $C(M)$ is anti-isomorphic to $\Sigma^*/\sim$.

Remark. In order to avoid using anti-isomorphisms, the usual practice is to introduce a multiplication $\cdot$ on $C(M)$ so that $\delta_u \cdot \delta_v := \delta_v \circ \delta_u$. Then $C(M)$ under $\cdot$ is isomorphic to $\Sigma^*/\sim$.

Some properties:

- If M and N are isomorphic semiautomata with identical input alphabet, then $C(M) = C(N)$.
- If N is a subsemiautomaton of M, then $C(N)$ is a homomorphic image of a submonoid of $C(M)$.
- If N is a homomorphic image of M, so is $C(N)$ a homomorphic image of $C(M)$.

References

[1] A. Ginzburg, *Algebraic Theory of Automata*, Academic Press (1968).

[2] M. Ito, *Algebraic Theory of Automata and Languages*, World Scientific, Singapore (2004).