Nerode equivalence

Canonical name	NerodeEquivalence
Date of creation	2013-03-22 18:52:11
Last modified on	2013-03-22 18:52:11
Owner	Ziosilvio (18733)
Last modified by	Ziosilvio (18733)
Numerical id	4
Author	Ziosilvio (18733)
Entry type	Definition
Classification	msc 68Q70
Classification	msc 20M35
Defines	maximality property of Nerode equivalence

Let S be a semigroup and let $X \subseteq S$. The relation

$$s_1 \mathcal{N}_X s_2 \iff \forall t \in S (s_1 t \in X \iff s_2 t \in X) \tag{1}$$

is an equivalence relation over S, called the *Nerode equivalence* of X.

As an example, if $S = (\mathbb{N}, +)$ and $X = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} \mid n = 3k\}$, then $m \mathcal{N}_X n$ iff $m \mod 3 = n \mod 3$.

The Nerode equivalence is right-invariant, *i.e.*, if $s_1 \mathcal{N}_X s_2$ then $s_1 t \mathcal{N}_X s_2 t$ for any t. However, it is usually not a congruence.

The Nerode equivalence is maximal in the following sense:

- if η is a right-invariant equivalence over S and X is union of classes of η,
- then $s \eta t$ implies $s \mathcal{N}_X t$.

In fact, let $r \in S$: since $s \eta t$ and η is right-invariant, $sr \eta tr$. However, X is union of classes of η, therefore sr and tr are either both in X or both outside X. This is true for all $r \in S$, thus $s \mathcal{N}_X t$.

The Nerode equivalence is linked to the syntactic congruence by the following fact, whose proof is straightforward:

$$s_1 \equiv_X s_2 \text{ iff } l s_1 \mathcal{N}_X l s_2 \; \forall l \in S \; .$$