shuffle of languages

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Defines	shuffle closure

Let Σ be an alphabet and u, v words over Σ. A *shuffle* w of u and v can be loosely defined as a word that is obtained by first decomposing u and v into individual pieces, and then combining (by concatenation) the pieces to form w, in a way that the order of the pieces in each of u and v is preserved.

More precisely, a *shuffle* of u and v is a word of the form

$$u_1 v_1 \cdots u_k v_k$$

where $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$. In other words, a shuffle of u and v is either a k-insertion of either u into v or v into u, for some positive integer k.

The set of all shuffles of u and v is called *the* shuffle of u and v, and is denoted by

$$u \diamond v.$$

The shuffle operation can be more generally applied to languages. If L_1, L_2 are languages, the shuffle of L_1 and L_2, denoted by $L_1 \diamond L_2$, is the set of all shuffles of words in L_1 and L_2. In short,

$$L_1 \diamond L_2 = \bigcup \{u \diamond v \mid u \in L_1, v \in L_2\}.$$

Clearly, $u \diamond v = \{u\} \diamond \{v\}$. We shall also identify $L \diamond u$ and $u \diamond L$ with $L \diamond \{u\}$ and $\{u\} \diamond L$ respectively.

A language L is said to be *shuffle closed* if $L \diamond L \subseteq L$. Clearly Σ^* is shuffle closed, and arbitrary intersections shuffle closed languages are shuffle closed. Given any language L, the smallest shuffle closed containing L is called the *shuffle closure* of L, and is denoted by $L^{\diamond}$.

It is easy to see that $\diamond$ is a commutative operation: $u \diamond v = v \diamond u$. It is also not hard to see that $\diamond$ is associative: $(u \diamond v) \diamond w = u \diamond (v \diamond w)$.

In addition, the shuffle operation has the following recursive property: for any u, v over Σ, and any $a, b \in \Sigma$:

1. $u \diamond \lambda = \{u\}$,
2. $\lambda \diamond v = \{v\}$,
3. $ua \diamond vb = (ua \diamond v)\{b\} \cup (u \diamond vb)\{a\}$.

For example, suppose $u = aba$, $v = bab$. Then

$$
\begin{aligned}
u \diamond v &= [aba \diamond ba]\{b\} \cup [ab \diamond bab]\{a\} \\
&= [(aba \diamond b) \cup (ab \diamond ba)]\{ab\} \cup [(ab \diamond ba) \cup (a \diamond bab)]\{ba\} \\
&= (ab \diamond ba)\{ab, ba\} \cup (aba \diamond b)\{ab\} \cup (a \diamond bab)\{ba\} \\
&= \{abba, baab, abab, baba\}\{ab, ba\} \cup \{baba, abba, abab\}\{ab\} \cup \{abab, baab, baba\}\{ba\} \\
&= \{abba, baab, abab, baba\}\{ab, ba\} \\
&= \{abbaab, baabab, ababab, babaab, abbaba, baabba, ababba, bababa\}
\end{aligned}
$$

Remark. Some closure properties with respect to the shuffle operation: let $\mathscr{R}$ be the family of regular languages, and $\mathscr{F}$ the family of context-free languages. Then $\mathscr{R}$ is closed under $\diamond$. $\mathscr{F}$ is not closed under $\diamond$. However, if $L_1 \in \mathscr{R}$ and $L_2 \in \mathscr{F}$, then $L_1 \diamond L_2 \in \mathscr{F}$. The proofs of these closure properties can be found in the reference.

References

[1] M. Ito, *Algebraic Theory of Automata and Languages*, World Scientific, Singapore (2004).