Myhill-Nerode theorem

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Let L be a language on the finite alphabet A and let $\mathcal{N}_L$ be its Nerode equivalence. The following are equivalent.

1. L is recognized by a deterministic finite automaton.

2. $A^*/\mathcal{N}_L$ is finite.

Moreover, the number of classes of $\mathcal{N}_L$ is the smallest number of states of a DFA recognizing L.

Proof. First, suppose $A^*/\mathcal{N}_L = \{q_0 = [\lambda]_{\mathcal{N}_L}, \ldots, q_{k-1}\} = Q$, where λ is the empty word on A. Construct a DFA $\mathcal{A} = \langle Q, A, q_0, \delta, F\rangle$ (called the *Nerode automaton* for L) with $\delta : Q \times A \to Q$ defined by

$$\delta(q, a) = [wa]_{\mathcal{N}_L}\ ,\ \ w \in q\ , \tag{1}$$

and

$$F = \{q \in Q \mid \exists w \in L \mid w \in q\}\ . \tag{2}$$

Then δ is well defined because $w_1\mathcal{N}_L w_2$ implies $w_1 u\mathcal{N}_L w_2 u$. It is also straightforward that $\mathcal{A}$ recognizes L.

On the other hand, let $\mathcal{A} = \langle Q, A, q_0, \delta, F\rangle$ be a DFA that recognizes L. Extend δ to $Q \times A^*$ by putting $\delta(q, \lambda) = q$ and $\delta(q, aw) = \delta(\delta(q, a), w)$ for every $q \in Q$, $a \in A$, $w \in A^*$. Define $f : Q \to A^*/\mathcal{N}_L \cup \{\emptyset\}$ as

$$f(q) = \begin{cases} [w]_{\mathcal{N}_L} & \text{if } \delta(q_0, w) = q \\ \emptyset & \text{if } \delta(q_0, w) \neq q \forall w \in A^* \end{cases} \tag{3}$$

Then f is well defined. In fact, suppose $q_1 = q_2 = q$: then either $f(q_1) = f(q_2) = \emptyset$, or there are $w_1, w_2 \in A^*$ such that $\delta(q_0, w_1) = \delta(q_0, w_2) = q$. But in the latter case, $\delta(q_0, w_1 u) = \delta(q_0, w_2 u) = \delta(q, u)$ for any $u \in A^*$, hence $w_1\mathcal{N}_L w_2$ since $\mathcal{A}$ recognizes L. Finally, for any $w \in A^*$ we have $[w]_{\mathcal{N}_L} = f\left(\delta(q_0, w)\right)$, so every class of $\mathcal{N}_L$ has a preimage according to f; consequently, $|Q| \geq |A^*/\mathcal{N}_L|$. □