Stirling's approximation

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Stirling's formula gives an approximation for $n!$, the factorial . It is

$$n! \approx \sqrt{2n\pi}n^n e^{-n}$$

We can derive this from the gamma function. Note that for large x,

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\mu(x)} \tag{1}$$

where

$$\mu(x) = \sum_{n=0}^{\infty}\left(x+n+\frac{1}{2}\right)\ln\left(1+\frac{1}{x+n}\right)-1 = \frac{\theta}{12x}$$

with $0 < \theta < 1$. Taking $x = n$ and multiplying by n, we have

$$n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{\theta}{12n}} \tag{2}$$

Taking the approximation for large n gives us Stirling's formula.

There is also a big-O notation version of Stirling's approximation:

$$n! = \left(\sqrt{2\pi n}\right)\left(\frac{n}{e}\right)^n\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \tag{3}$$

We can prove this equality starting from (**??**). It is clear that the big-O portion of (**??**) must come from $e^{\frac{\theta}{12n}}$, so we must consider the asymptotic behavior of e.

First we observe that the Taylor series for e^x is

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

But in our case we have e to a vanishing exponent. Note that if we vary x as $\frac{1}{n}$, we have as $n \longrightarrow \infty$

$$e^x = 1 + \mathcal{O}\left(\frac{1}{n}\right)$$

We can then (almost) directly plug this in to (**??**) to get (**??**) (note that the factor of 12 gets absorbed by the big-O notation.)