subset construction

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The subset construction is a technique of turning a non-deterministic automaton into a deterministic one, while keeping the accepting language the same. This technique shows that an NDFA is no more powerful in terms of word acceptance than a DFA.

We begin by looking at a semiautomaton (S, Σ, δ). The transition function δ is a function from $S \times \Sigma$ to $P(S)$, which maps a pair (s, a) to a subset $\delta(s, a)$ of S. Observe that δ can be extended to a function δ' from $P(S) \times \Sigma$ to $P(S)$ by setting

$$\delta'(T, a) := \bigcup \{\delta(t, a) \mid t \in T\} \tag{1}$$

for any subset T of S and $a \in \Sigma$. Note that $\delta'(\varnothing, a) = \varnothing$. What we have just done is turning a semiautomaton (S, Σ, δ) into a deterministic semiautomaton (S', Σ, δ'), where $S' = P(S)$, the powerset of S.

It is easy to see, by induction on the length of u, that $\delta'(T, u) = \bigcup \{\delta(t, u) \mid t \in T\}$.

Next, given an NDFA $A = (S, \Sigma, \delta, I, F)$, we turn it into a DFA $A' := (S', \Sigma, \delta', I', F')$ as follows:

- (S', Σ, δ') is derived from (S, Σ, δ) by the construction above,
- $I' = I$, and
- $F' = \{T \subseteq S \mid T \cap F \neq \varnothing\}$.

Since I' is an element of $S' = P(S)$, and $F' \subseteq S'$, A' is a well-defined DFA.

Proposition 1. *$L(A) = L(A')$.*

Proof. $u \in L(A)$ iff $\delta(q, u) \cap F \neq \varnothing$ (where $q \in I$) iff $\delta'(I, u) \in F'$ iff $u \in L(A')$. □

What happens if the NDFA in question contains `http://planetmath.org/EpsilonTransition`ϵ-transitions? Suppose $p \xrightarrow{\epsilon} q$ is an ϵ-transition, and $p \neq q$. Then $\delta'(\{p\}, \epsilon) = \{q\} \neq \{p\}$, which is not allowed in a DFA.

To get around this difficulty, we make a small modification on A'. First, define, for any $T \subseteq S$, the *ϵ-closure* $C_\epsilon(T)$ of T as the set

$$C_\epsilon(T) := \{t \mid t \in \delta'(T, \epsilon^k), k \geq 0\} \tag{2}$$

For any $T \subseteq S$, $\delta'(C_\epsilon(T), a) = C_\epsilon(T)$. If the automaton does not contain any ϵ-transitions, then $C_\epsilon(T) = T$.

Now, let NDFA A be an `http://planetmath.org/EpsilonAutomaton`ϵ-automaton, define $A'' := (S', \Sigma, \delta'', I'', F'')$ as follows:

- $S' = P(S)$,
- $\delta''(T, a) = \delta'(C_\epsilon(T), a)$, where δ' is defined in (1) above,
- $I'' = C_\epsilon(I)$, and
- $F'' = \{T \subseteq S \mid C_\epsilon(T) \cap F \neq \varnothing\}$.

By definition, A'' is a DFA, and it can be shown that $L(A'') = L(A)$. The proof is very similar to the one given `http://planetmath.org/EveryEpsilonAutomatonIsEquival`