pumping lemma (context-free languages)

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Let L be a context-free language (a.k.a. type 2 language). Then there exist two integers m and n such that, if the length of a word W is greater than m, then $W = ABCDE$ where A, B, C, D, E are subwords such that

1. The length of the subword BCD is less than n.
2. BD is not be empty.
3. For all integers $k > 0$, it is the case that AB^kCD^kE belongs to L, where exponentiation denotes repetition of a subword k times.

An important use of this lemma is that it allows one to show that a language is not context-free. (Remember, just because a language happens to be described in terms of a context-sensitive grammar does not automatically preclude the possibility of describing the same language also by a context-free language.) The idea is to assume that the language is context-free, then arrive at a contradiction via this lemma.

As an illustrative example, consider the following language, which consists of but one terminal symbol, 'x' and which consists of all strings of 'x' 's whose length is a perfect square. Were this a context-free language, there would exist integers m and n as above. Choose an integer h such that $h^2 > m$. Then the word x^{h^2} belongs to our language and the lemma tells us that it can be written as $ABCDE$ so as to satisfy the conditions enumerated above. Write $A = x^a$, $B = x^b$, $C = x^c$, $D = x^d$, $E = x^e$ for suitable nonnegative integers a, b, c, d, e. Then we have $a + b + c + d + e = k^2$; by condition , $b + d > 0$ and, by condition , $a + kb + c + kd + e$ would have to be a perfect square because AB^kCD^kE would be a word of the language. This, however, leads to a contradiction: $h^2 + k(b + d)$ could not possibly be a perfect square for all k unless $b + d = 0$.

As an exercise, the reader may consider constructing a context-sensitive grammar for this language and posting it as an attachment to this entry (at which time this sentence will come down).