terminating reduction

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Let X be a set and $\to$ a reduction (binary relation) on X. A *chain* with respect to $\to$ is a sequence of elements $x_1, x_2, x_3, \ldots$ in X such that $x_1 \to x_2$, $x_2 \to x_3$, etc... A chain with respect to $\to$ is usually written

$$x_1 \to x_2 \to x_3 \to \cdots \to x_n \to \cdots .$$

The length of a chain is the cardinality of its underlying sequence. A chain is finite if its length is finite. Otherwise, it is infinite.

Definition. A reduction $\to$ on a set X is said to be *terminating* if it has no infinite chains. In other words, every chain *terminates*.

Examples.

- If $\to$ is reflexive, or non-trivial symmetric, then it is never terminating.
- Let X be the set of all positive integers greater than 1. Define $\to$ on X so that $a \to b$ means that $a = bc$ for some $c \in X$. Then $\to$ is a terminating reduction. By the way, $\to$ is also a normalizing reduction.
- In fact, it is easy to see that a terminating reduction is normalizing: if a has no normal form, then we may form an infinite chain starting from a.
- On the other hand, not all normalizing reduction is terminating. A canonical example is the set of all non-negative integers with the reduction $\to$ defined by $a \to b$ if and only if
  - either $a, b \neq 0$, $a \neq b$, and $a < b$,
  - or $a \neq 0$ and $b = 0$.

  The infinite chain is given by $1 \to 2 \to 3 \to \cdots$, so that $\to$ is not terminating. However, $n \to 0$ for every positive integer n. Thus every integer has 0 as its normal form, so that $\to$ is normalizing.

Remarks.

- A reduction is said to be *convergent* if it is both terminating and confluent.
- A relation is terminating iff the transitive closure of its inverse is well-founded.

To see this, first let R be terminating on the set X. And let S be the transitive closure of R^{-1}. Suppose A is a non-empty subset of X that contains no S-minimal elements. Pick $x_0 \in A$. Then we can find $x_1 \in A$ with $x_1 \neq x_0$, such that $x_1 S x_0$. By the assumption on A, this process can be iterated indefinitely. So we have a sequence $x_0, x_1, x_2, \ldots$ such that $x_{i+1} S x_i$ with $x_i \neq x_{i+1}$. Since each pair (x_i, x_{i+1}) can be expanded into a finite chain with respect to R, we have produced an infinite chain containing elements $x_0, x_1, x_2, \ldots$, contradicting the assumption that R is terminating.

On the other hand, suppose the transitive closure S of R^{-1} is well-founded. If the chain $x_0 R x_1 R x_2 R \cdots$ is infinite, then the set $\{x_0, x_1, x_2, \ldots\}$ has no S-minimal elements, as $x_i S x_j$ whenever $i > j$, and j arbitrary.

- The reflexive transitive closure of a terminating relation is a partial order.

A closely related concept is the descending chain condition (DCC). A reduction $\to$ on X is said to satisfy the *descending chain condition (DCC)* if the only infinite chains on X are those that are eventually constant. A chain $x_1 \to x_2 \to x_3 \to \cdots$ is eventually constant if there is a positive integer N such that for all $n \geq N$, $x_n = x_N$. Every terminating relation satisfies DCC. The converse is obviously not true, as a reflexive reduction illustrates.

Another related concept is acyclicity. Let $\to$ be a reduction on X. A chain $x_0 \to x_1 \to \cdots x_n$ is said to be cyclic if $x_i = x_j$ for some $0 \leq i < j \leq n$. This means that there is a "closed loop" in the chain. The reduction $\to$ is said to be *acyclic* if there are no cyclic chains with respect to $\to$. Every terminating relation is acyclic, but not conversely. The usual strict inequality relation on the set of positive integers is an example of an acyclic but non-terminating relation.