quotient of languages

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Defines	quotient
Defines	left quotient
Defines	right quotient

Let L_1, L_2 be two languages over some alphabet Σ. The *quotient* of L_1 by L_2 is defined to be the set

$$L_1/L_2 := \{u \in \Sigma^* \mid uv \in L_1 \text{ for some } v \in L_2\}.$$

L_1/L_2 is sometimes written $L_1L_2^{-1}$. The quotient so defined is also called the *right quotient*, for one can similarly define the *left quotient* of L_1 by L_2:

$$L_1 \backslash L_2 := \{u \in \Sigma^* \mid vu \in L_1 \text{ for some } v \in L_2\}.$$

$L_1 \backslash L_2$ is sometimes written $L_2^{-1}L_1$.

Below are some examples of quotients:

- If $L_1 = \{a^nb^nc^n \mid n \geq 0\}$ and $L_2 = \{b, c\}^*$, then
  - $L_1/L_2 = \{a^mb^n \mid m \geq n \geq 0\}$
  - $L_2/L_1 = L_2$
  - $L_1 \backslash L_2 = \{\lambda\}$, the singleton consisting the empty word
  - $L_2 \backslash L_1 = L_2$
- for any language L over Σ:
  - L/Σ^* is the language of all prefixes of words of L
  - $\Sigma^*/L = \Sigma^*$
  - $L \backslash \Sigma^*$ is the language of all suffixes of words of L
  - $\Sigma^* \backslash L = \Sigma^*$
- $\lambda \in L/L \cap L \backslash L$, and if $\lambda \in L$, then $L \subseteq L/L \cap L \backslash L$.

Here are some basic properties of quotients:

1. $L_1 \subseteq (L_1/L_2)L_2 \cap L_2(L_1 \backslash L_2)$.
2. $(L_1/L_2)L_2 \subseteq (L_1L_2)/L_2$, and $L_2(L_1 \backslash L_2) \subseteq (L_2L_1) \backslash L_2$.
3. right and left quotients are related via reversal:

$$\begin{aligned}
(L_1 \backslash L_2)^R &= \{u^R \mid vu \in L_1 \text{ for some } v \in L_2\} \\
&= \{u^R \mid (vu)^R \in L_1^R \text{ for some } v^R \in L_2^R\} \\
&= \{u^R \mid u^Rv^R \in L_1^R \text{ for some } v^R \in L_2^R\} \\
&= L_1^R/L_2^R.
\end{aligned}$$

A family $\mathscr{F}$ of languages is said to be *closed under quotient by a language* L if for every language $M \in \mathscr{F}$, $M/L \in \mathscr{F}$. Furthermore, $\mathscr{F}$ is said to be *closed under quotient* if $M/L \in \mathscr{F}$ for any $M, L \in \mathscr{F}$. Closure under quotient is also termed closure under right quotient. Closure under left quotient is similarly defined.

It can be shown that the families of regular, context-free, and type-0 languages are closed under quotient (both left and right) by a regular language. The family of context-sensitive languages does not have this closure property.

Since all of the families mentioned above are closed under reversal, each of the families, except the context-sensitive family, is closed under left quotient by a regular language, according to the second property above.

References

[1] J.E. Hopcroft, J.D. Ullman, *Formal Languages and Their Relation to Automata*, Addison-Wesley, (1969).