polynomial hierarchy is a hierarchy

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The polynomial hierarchy is a hierarchy. Specifically:

$$\Sigma_i^p \cup \Pi_i^p \subseteq \Delta_{i+1}^p \subseteq \Sigma_{i+1}^p \cap \Pi_{i+1}^p.$$

Proof

To see that $\Sigma_i^p \cup \Pi_i^p \subseteq \Delta_{i+1}^p = \mathcal{P}^{\Sigma_i^p}$, observe that the machine which checks its input against its oracle and accepts or rejects when the oracle accepts or rejects (respectively) is easily in $\mathcal{P}$, as is the machine which rejects or accepts when the oracle accepts or rejects (respectively). These easily emulate Σ_i^p and Π_i^p respectively.

Since $\mathcal{P} \subseteq \mathcal{NP}$, it is clear that $\Delta_i^p \subseteq \Sigma_i^p$. Since $\mathcal{P}^{\mathcal{C}}$ is closed under complementation for any complexity class $\mathcal{C}$ (the associated machines are deterministic and always halt, so the complementary machine just reverses which states are accepting), if $L \in \mathcal{P}^{\Sigma_i^p} \subseteq \Sigma_i^p$ then so is $\overline{L}$, and therefore $L \in \Pi_i^p$.

Unlike the arithmetical hierarchy, the polynomial hierarchy is not known to be proper. Indeed, if $\mathcal{P} = \mathcal{NP}$ then $\mathcal{P} = \mathcal{PH}$, so a proof that the hierarchy is proper would be quite significant.