commutative language

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Let Σ be an alphabet and u a word over Σ. Write u as a product of symbols in Σ:

$$u = a_1 \cdots a_n$$

where $a_i \in \Sigma$. A of u is a word of the form

$$a_{\pi(1)} \cdots a_{\pi(n)},$$

where π is a permutation on $\{1, \ldots, n\}$. The set of all permutations of u is denoted by $\mathrm{com}(u)$. If $\Sigma = \{b_1, \ldots, b_m\}$, it is easy to see that $\mathrm{com}(u)$ has

$$\frac{n!}{n_1! \cdots n_m!}$$

elements, where $n_i = |u|_{b_i}$, the number of occurrences of b_i in u.

Define a binary relation $\sim$ on Σ^* by: $u \sim v$ if v is a permutation of u. Then $\sim$ is a congruence relation on Σ^* with respect to concatenation. In fact, $\Sigma^*/\sim$ is a commutative monoid.

A language L over Σ is said to be *commutative* if for every $u \in L$, we have $\mathrm{com}(u) \subseteq L$. Two equivalent characterization of a commutative language L are:

- If $u = vxyw \in L$, then $vyxw \in L$.
- $\Psi^{-1} \circ \Psi(L) \subseteq L$, where Ψ is the Parikh mapping over Σ (under some ordering).

The first equivalence comes from the fact that if $vyxw$ is just a permutation of $vxyw$, and that every permutation on $\{1, \ldots, n\}$ may be expressed as a product of transpositions. The second equivalence is the realization of the fact that $\mathrm{com}(u)$ is just the set

$$\{v \mid |v|_a = |u|_a, a \in \Sigma\}.$$

We have just seen some examples of commutative closed langauges, such as $\mathrm{com}(u)$ for any word u, and $\Psi^{-1} \circ \Psi(L)$, for any language L.

Given a language L, the smallest commutative language containing L is called the *commutative closure* of L. It is not hard to see that $\Psi^{-1} \circ \Psi(L)$ is the *commutative closure* of L.

For example, if $L = \{(abc)^n \mid n \geq 0\}$, then $\Psi^{-1} \circ \Psi(L) = \{w \mid |w|_a = |w|_b = |w|_c\}$.

Remark. The above example illustrates the fact that the families of regular languages and context-free languages are not closed under commutative closures. However, it can be shown that the families of context-sensitive languages and type-0 languages are closed under commutative closures.

References

[1] M. Ito, *Algebraic Theory of Automata and Languages*, World Scientific, Singapore (2004).