Analysis of the FitzHugh–Nagumo Model in Theoretical Neuroscience

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Motivation & Derivation

The Hodgkin-Huxley Model

- A thorough and precise explanation of the electrical properties of neurons.
  - Theoretically complex and computationally demanding.
- Demand for simplification.
- The goal was to make the model more accessible for theoretical analysis and simpler to utilize in simulations.
  - Enabling pattern creation and wave propagation.

Nelson, M., & Rinzel, J. (1995). The hodgkin-huxley model. The book of genesis, 29-49.

Development of the FitzHugh-Nagumo Model

- Focuses on the membrane potential and a recovery variable.
- Especially well-suited for investigating the features of excitability and oscillatory behaviors.
- Allows for bifurcation analysis and the study of various phenomena like hysteresis, excitability thresholds, and pattern formation in neural media.

Izhikevich, E.. M. (2007) Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting. MIT Press.

Derivation and Significance of Parameters

- Membrane Potential (v): electrical potential difference across the cell membrane.
- Recovery Variable (w): represents processes that are slower than the changes in membrane potential.
- External Input Current (I): external stimuli.
- a: affects the threshold for action potential initiation.
- The ratio of b to c can impacts how quickly the system recovers from an excitation.

Izhikevich, E.. M. (2007) Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting. MIT Press.

The FitzHugh-Nagumo Model

$$\frac{dv}{dt} = v(a - v)(v - 1) - w + I$$

$$\frac{dw}{dt} = bv - cw$$

- Retains the essential features of excitability and refractoriness while being more mathematically tractable.
- The chosen parameters allow the model to capture the critical dynamics of neuron behavior (like action potentials and recovery) without overcomplicating the model.

Izhikevich, E.. M. (2007) Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting. MIT Press.

Hysteresis in the FitzHugh-Nagumo Model

- Hysteresis in this context refers to the system's memory of past states.
  - Critical in understanding how neurons can exhibit different responses to the same stimulus depending on their previous states.
- Captures the essential dynamics of neuronal excitability and refractoriness.

Izhikevich, E.. M. (2007) Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting. MIT Press.

Preliminary Analysis

Fixed Point Analysis

$$
\begin{aligned}
0 &= \frac{dw}{dt}, \\
&= bv - cw, \\
\therefore w^* &= \frac{b}{c} v^*.
\end{aligned}
$$

$$
\begin{aligned}
0 &= v\,(a - v)\,(v - 1) - w + I, \\
&= \left(av - v^2\right)(v - 1) - \frac{b}{c} v + I, \\
&= av^2 - av - v^3 + v^2 - \frac{b}{c} v + I, \\
&= -v^3 + (a + 1)\,v^2 - \left(a + \frac{b}{c}\right) v + I.
\end{aligned}
$$

Fixed Point Analysis

$$v^* = \left\{q + \left[q^2 + \left(r - p^2\right)^3\right]^{1/2}\right\}^{1/3} + \left\{q - \left[q^2 + \left(r - p^2\right)^3\right]^{1/2}\right\}^{1/3} + p, \;\; w^* = \frac{b}{c} v^*,$$

where $p \equiv \dfrac{a+1}{3}$, $q \equiv \dfrac{2a^3c - 3a^2c - 3ac - 9ab - 9b + 2c + 27Ic}{54}$, and $r \equiv \dfrac{ac+b}{3c}$.

$$J = \begin{pmatrix} -3v^2 + 2(a+1)v - a & -1 \\ b & -c \end{pmatrix}$$

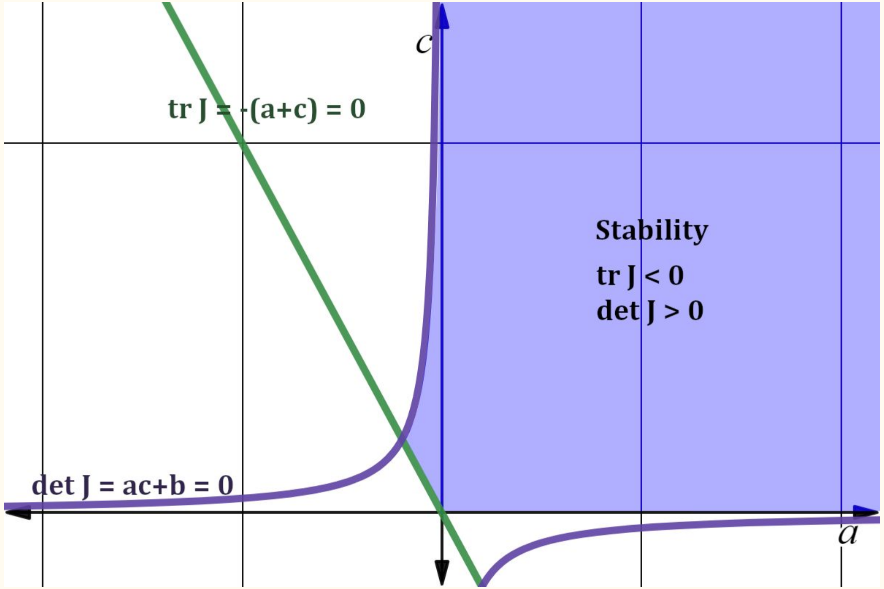
https://math.vanderbilt.edu/schectex/courses/cubic/

Further Analysis

- a = ±0.1, b = 0.01, c = 0.02, I = 0 ⇒ fixed point @ (0,0)

$$J = \begin{pmatrix} -a & -1 \\ b & -c \end{pmatrix}$$

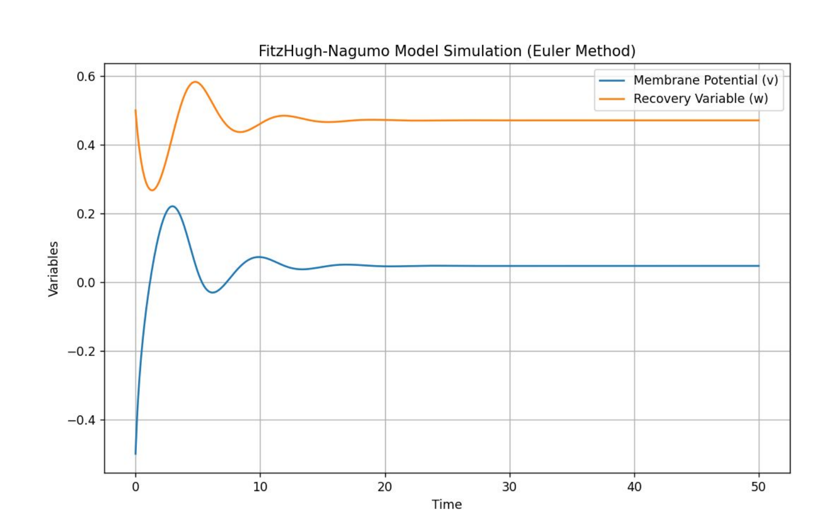
$$\lambda = \frac{-(a+c) \pm \sqrt{(a+c)^2 - 4(ac+b)}}{2}$$



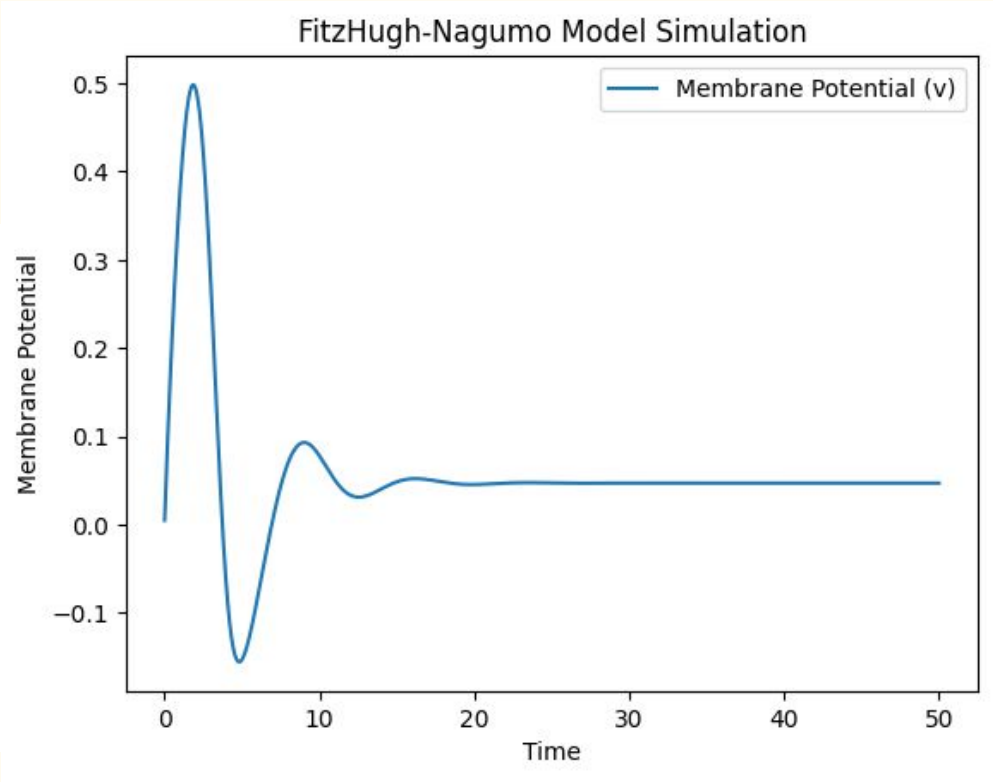
But what happens within the model? (For a Neuron)

The neuron

The activity within the single neuron is filled electrochemical activity with this activity their is placidity and then suddenly there are sinual spiking.



At v(0)=-0.5, w(0)=0.5



Euler's method for the FitzHugh-Nagumo model

Given: a=0.7, b=0.8, c=0.08, I=0.5, dt=0.01,

$v0=0.0$, $w0= 0.0$

$dv/dt=v(0.7-v)(v-1)-w+0.5$

$dw/dt=0.8v-0.08w$

$dv=v(0.7-v)(v-1)-w+0.5\cdot 0.01$

$dw=0.8v-0.08w\cdot 0.01$

$v_1=0.0+dv$

$w_1=0.0+dw$

$dv=v_1(0.7-v_1)(v_1-1)-w_1+0.5\cdot 0.01$

$dw=0.8v\text{-}0.08w_1\cdot 0.01$

$v_2=v_1+dv$

$w_2=w_1+dw$

$dv=v_2(0.7-v_2)(v_2-1)-w_2+0.5\cdot 0.01$

$dw=0.8v_2-0.08w_2\cdot 0.01$

$v3=v2+dv$

$w3=w2+dw$

What can we find?

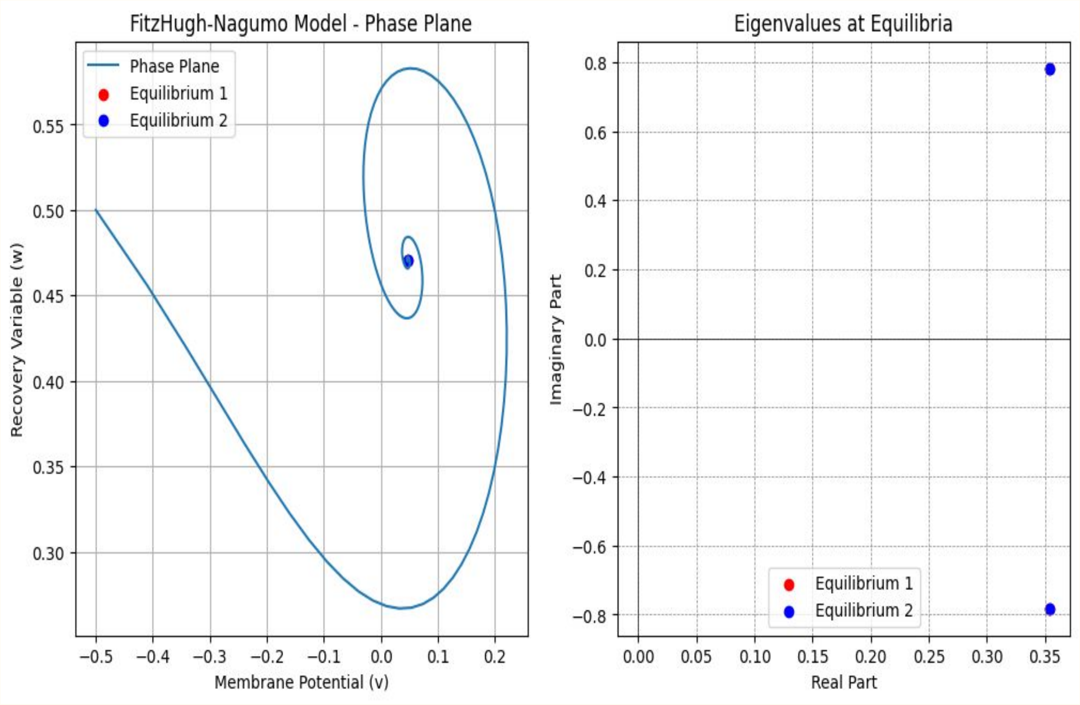
Equilibria:

Depending on the parameters: a, b, c, and I

which are within 1 and 3

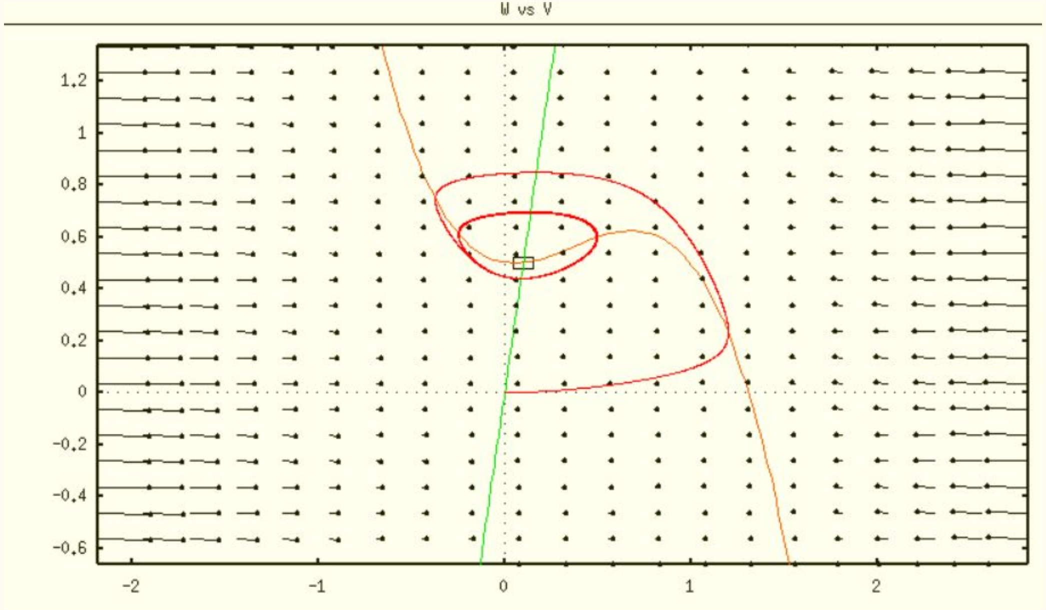
Bifurcations:

Saddle node, Hopf Bifurcations (Super critical and Sub-critical).



Bifurcation and Direction Flow

$$\frac{dv}{dt} = v(0.1 - v)(v - 1) - w + 0.5$$
$$\frac{dw}{dt} = 0.1v - 0.2w$$



Given the complex eigenvalues:

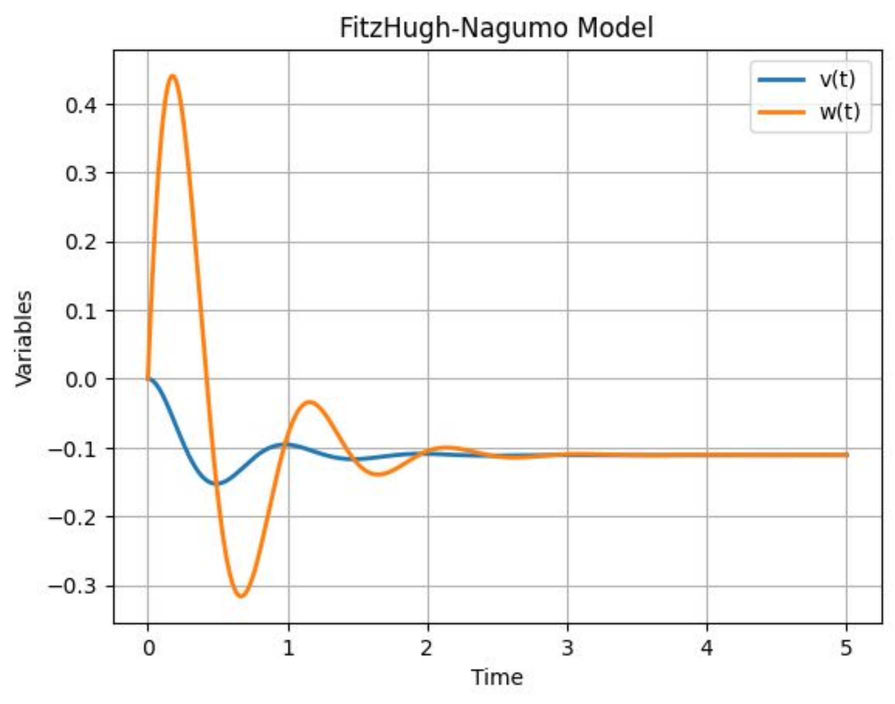
0.035040+i · 0.311401

0.035040-i · 0.311401

these eigenvalues suggest a stable oscillatory behavior in the system, potentially indicating a supercritical Hopf bifurcation. The system undergoes a transition from a stable fixed point to a stable limit cycle as a parameter is varied.

What is the Reason?

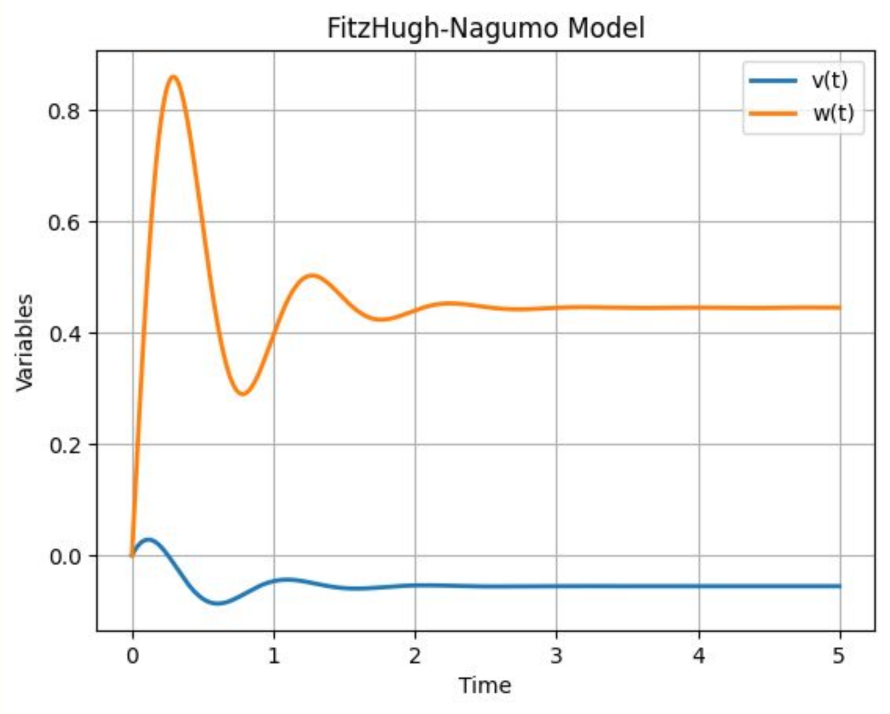
The FitzHugh-Nagumo model describes the dynamics of excitable media, often used to simulate the behavior of neurons. In the context, the focus is on the stability analysis of the equilibrium point (0, 0) in the FitzHugh-Nagumo model with the assumption $I=0$, meaning there is no external input current.



Increasing the input current(I)

The model exhibits excitability and spiking behavior. When an external input (I) is applied, it can drive the neuron from a resting state to an excited state, causing a spike in the membrane potential v. The recovery variable w influences the recovery of the neuron after spiking.

Thus, causing a hysteresis loop in the phase space illustrates how the system's trajectory depends on the past history of the input.



Further Explanation

Resting State:

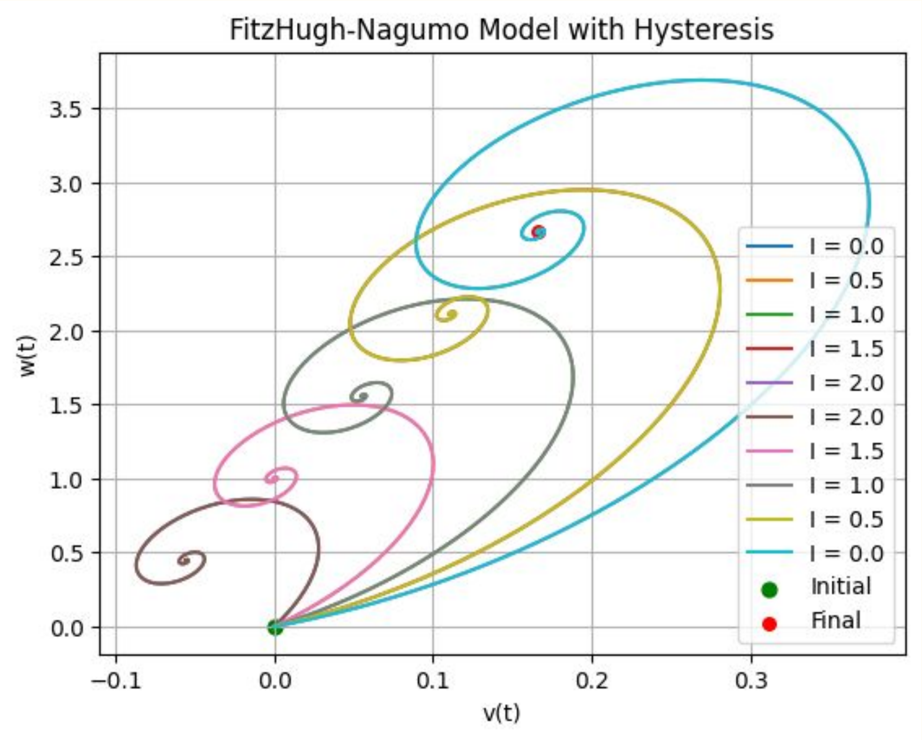
When I is initially set to zero or a low value, the system tends to a stable equilibrium point in the bottom left of the phase space.

The neuron is in a resting state with a low membrane potential (v) and a certain level of recovery (w).

Excitating/Spiking:

As I is increased, the system moves in a clockwise direction along the hysteresis loop.

The external input excites the neuron, causing an increase in the membrane potential (v).



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Excited State:

At the top right of the hysteresis loop, the neuron is in an excited state with a higher membrane potential (v).

The recovery variable (w) influences the recovery of the neuron after spiking.

Recovery:

As I is decreased, the system moves in a counterclockwise direction along the hysteresis loop. The external input is reduced, and the neuron tends to recover.

The recovery variable (w) plays a role in returning the system to a stable state.

Resting State:

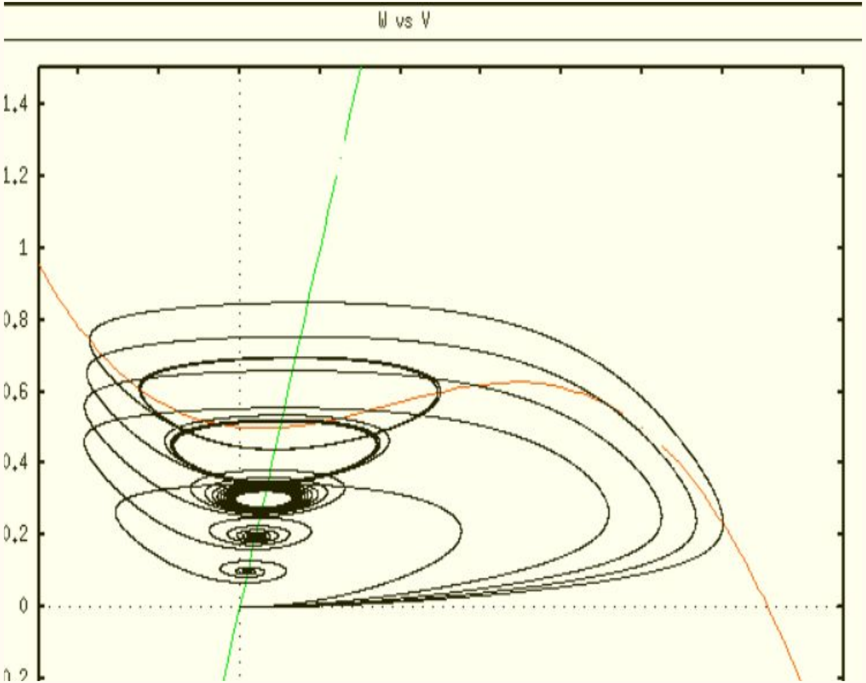
The system eventually returns to the bottom left of the hysteresis loop, representing a resting state.

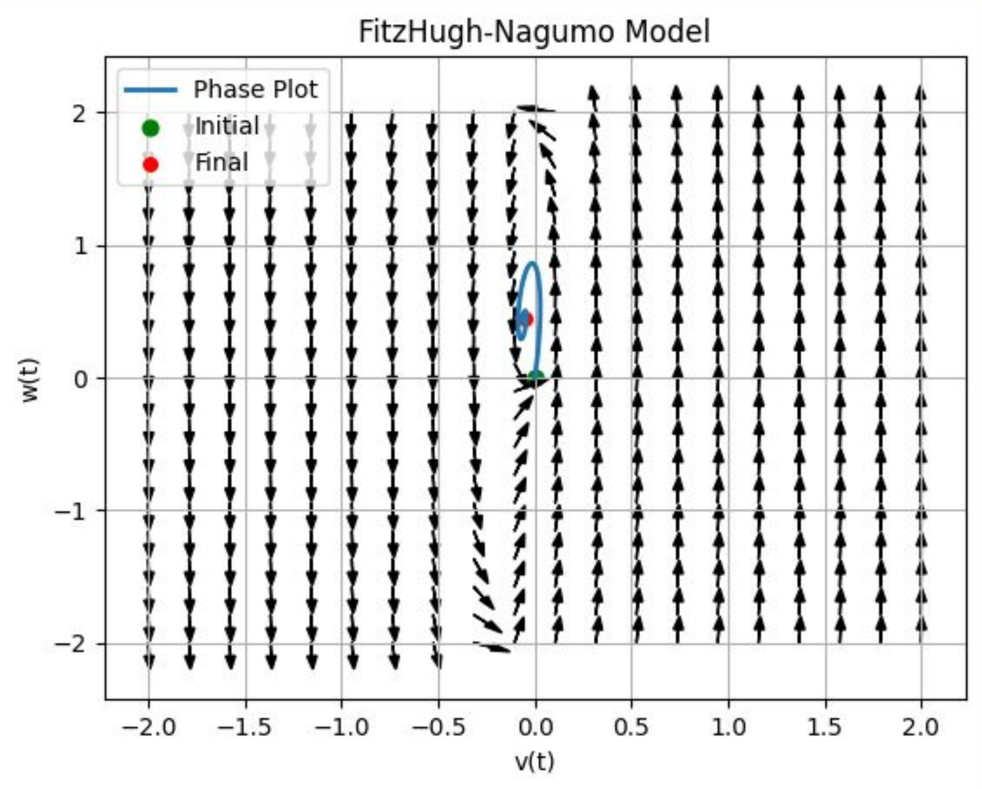
The process can be repeated as I is cycled through different values.

The loop displays the non-linear and history-dependent nature of the neuronal dynamics in response to changing inputs. The specific details of the behavior depend on the parameters of the FitzHugh-Nagumo model and the characteristics of the hysteresis loop.

Directional Flow

$$\frac{dv}{dt} = v(0.1 - v)(v - 1) - w + 0.5$$
$$\frac{dw}{dt} = 0.1v - 0.2w$$





Other Models

The Hodgkin-Huxley Model is a model that describes how action potentials are initiated and propagated in neurons. It was developed by Alan Hodgkin and Andrew Huxley in 1952 and later won them the Nobel Prize in Physiology or Medicine in 1963.

$$C_m \frac{dV_m}{dt} = I - \bar{g}_{Na} m^3 h (V_m - E_{Na}) - \bar{g}_K n^4 (V_m - E_K) - \bar{g}_L (V_m - E_L)$$

where:

$-C_m$ is the membrane capacitance,

$-I$ is the injected current,

$-\bar{g}_{Na}, \bar{g}_K, \bar{g}_L$ are the maximum conductances,

$-m, h, n$ are gating variables,

$-E_{Na}, E_K, E_L$ are reversal potentials.

The equations for the gating variables are:

Activation Variable (m):

$$\frac{dm}{dt} = \alpha_m (1 - m) - \beta_m m$$

m represents sodium (Na+) channel activation.

- α_m, β_m are rate constants.
- α_m determines activation rate, (1)
- β_m determines deactivation rate.

Inactivation Variable (h):

$$\frac{dh}{dt} = \alpha_h (1 - h) - \beta_h h$$

h represents sodium (Na+) channel inactivation.

- α_h, β_h are rate constants.
- α_h determines inactivation rate, (2)
- β_h determines reactivation rate.

Activation Variable (n):

$$\frac{dn}{dt} = \alpha_n (1 - n) - \beta_n n$$

n represents potassium (K+) channel activation.

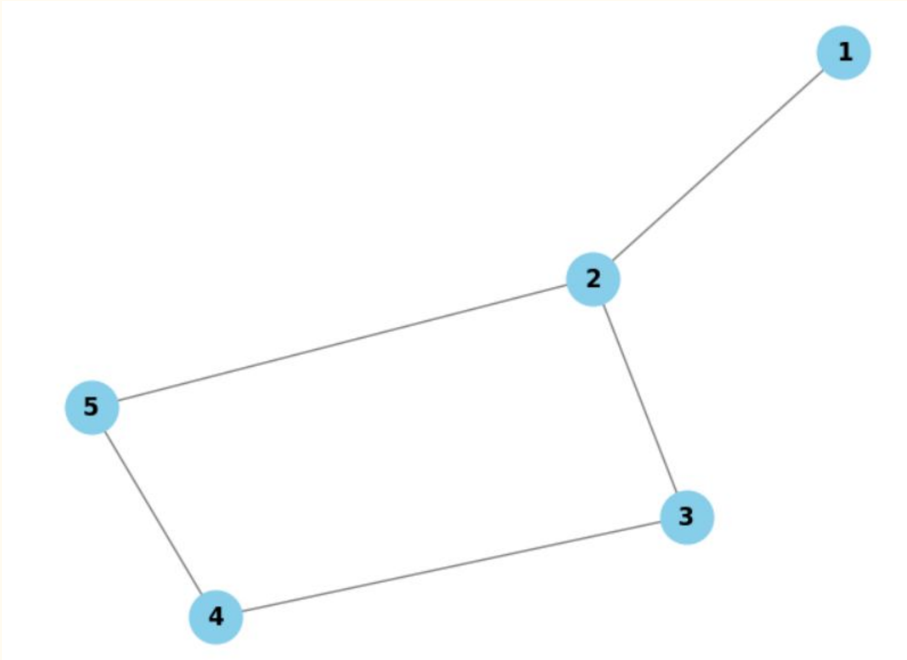
- α_n, β_n are rate constants.
- α_n determines activation rate, (3)
- β_n determines deactivation rate.

Continue

Threshold Linear Network is a type of artificial neural network where the activation function is a linear combination of the inputs, and the output is compared to a threshold to determine the final output.

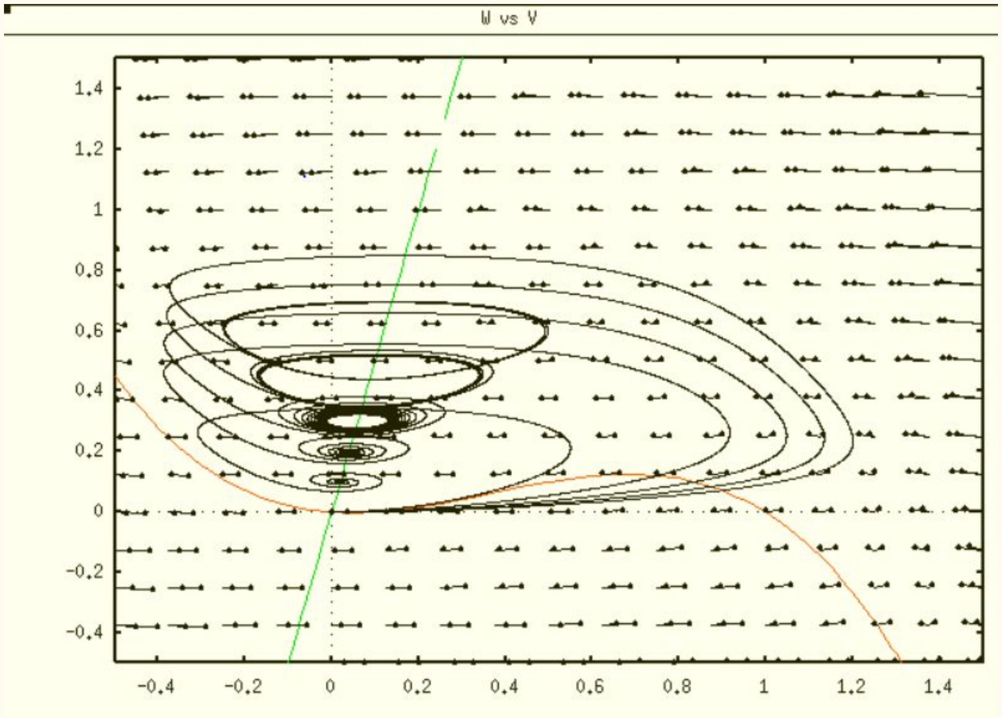
$$W_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -1+\varepsilon & \text{if } j \to i \text{ in } G, \\ -1-\delta & \text{if } j \not\to i \text{ in } G. \end{cases}$$

where 0<ε <□/□+1 and □>0.



Summary

Hysteresis in neural systems often relies on the interconnectedness of neurons and their synaptic connections. The strength and plasticity of these connections allow the network to exhibit history-dependent behavior, where the current state is influenced by past states. While individual neurons may contribute to hysteresis in their responses, the collective emergence of hysteresis on a network level is closely tied to the interactions and connectivity between neurons. The ability of the neural network to store and transmit information through synaptic plasticity is a key factor in the manifestation of hysteresis.



Thank you!