Problem Set 1

MathWise Institute

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1 Introduction

Problem sets are designed to challenge the theoretical understanding of the concepts covered in the lectures. You will notice that the problem set consists of three sections: easy, moderate, and challenging. These sections describe the difficulty of the problems within them. Completing all problems should be possible based on the material covered in the corresponding and previous lectures. All questions and concerns should be directed to the author at agflores1979@gmail.com. Calculators are prohibited unless otherwise stated.

2 Exercises

2.1 Easy

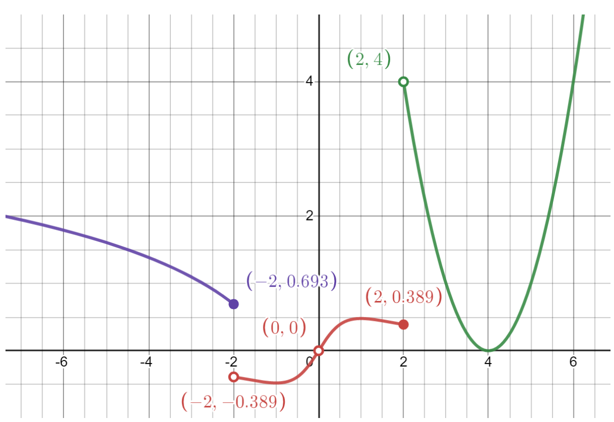


Figure 1: A graph of the function $f(x)$. An open circle means the indicated point does not belong to the function.

Refer to Fig. 1 for Exercises 1-4.

1. Find $\lim_{x\to-2^-} f(x)$, $\lim_{x\to-2^+} f(x)$, and $\lim_{x\to-2} f(x)$ if they exist. Does $\lim_{x\to-2} f(x) = f(-2)$? If not, what type of discontinuity exists at $f(-2)$?

2. Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to 2} f(x)$ if they exist. Does $\lim_{x\to 2} f(x) = f(2)$? If not, what type of discontinuity exists at $f(2)$?

3. Find $\lim_{x\to 0} f(x)$ if it exists. Does $\lim_{x\to 0} f(x) = f(0)$? If not, what type of discontinuity exists at $f(0)$?

4. Find $\lim_{x\to 4} f(x)$ if it exists. Does $\lim_{x\to 4} f(x) = f(4)$? If not, what type of discontinuity exists at $f(4)$?

5. Evaluate the following limits if they exist. If they do not exist, then state whether they are $+\infty$, $-\infty$, or undefined. [Hint: In order to solve problems b and c below, use a calculator or any online software to plot the function and then find the limit from the graph.]

   (a) $\displaystyle\lim_{x\to 3} \frac{x+1}{(x-2)^3}$

   (b) $\displaystyle\lim_{x\to 1^-} \frac{2x^2+2x+1}{x-1}$

   (c) $\displaystyle\lim_{x\to 1^+} \frac{2x^2+2x+1}{x-1}$

   (d) $\displaystyle\lim_{x\to 3} \frac{x-3}{x^2-9}$

   (e) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x}}{x^2-1}$

   (f) $\displaystyle\lim_{x\to \infty} \frac{12x^2+5x+2}{6x^2-3x+8}$

2.2 Moderate

6. How should $p(t) = (t^3+2t^2)/(t^3-2t^2)$ be defined at $t=0$ so that $\lim_{t\to 0} p(t) = p(0)$. [Hint: is $p(0)$ defined? What kind of discontinuity is this? How might one rewrite $p(t)$ so that it is defined at $t=0$?]

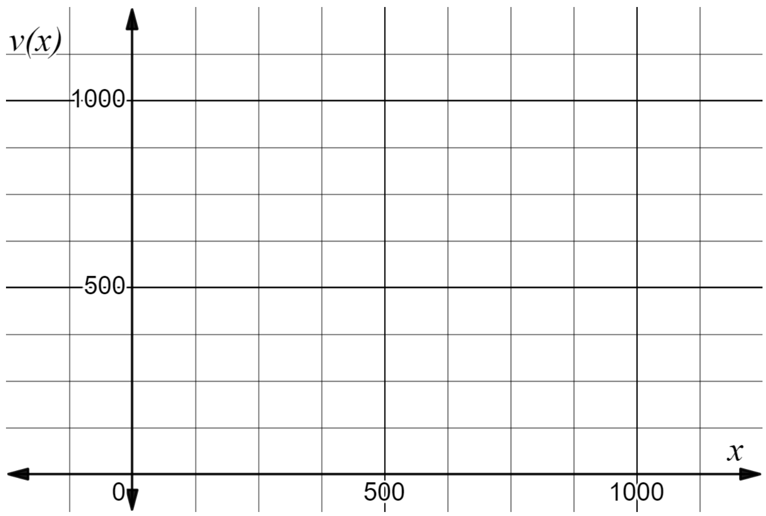
7. A block of ice is stationed at x centimeters from a fire pit. Let $v(x)$ be the volume of the block of ice in cm^3. Assume that the block of ice will completely melt upon contact with the flame.

   (a) What value would you assign to $v(0)$?

   (b) Assume the block of ice has an original volume of 1000 cm^3, and that it only melts when it is within 500 cm of the fire pit, unaffected by time and all other energy sources. What type of function is $v(x)$ for $x \geq 500$? Evaluate $\lim_{x\to\infty} v(x)$.

   (c) Describe in terms of the volume of the ice the meaning of $\lim_{x\to 0^+} v(x)$ (the limit of $v(x)$ as x approaches 0 from the right-hand side, i.e. the positive values). Should $\lim_{x\to 0^+} v(x) = v(0)$?

(d) If $v(x)$ is only defined over the interval $[0, \infty)$, draw a realistic graph of $v(x)$ in the following coordinate plane:



2.3 Challenging

8. How should $g(x) = (x^5 - 1)/(x - 1)$ be defined at $x = 1$ so that $\lim_{x \to 1} g(x) = g(1)$. [Hint: Similar to Exercise 6, simplify $g(x)$ so that the discontinuity at $x = 1$ is removed.]

9. Solve the following limit algebraically.

$$\lim_{x \to 3} \left[\frac{x^2 - 5x + 6}{x^2 - 7x + 12} - \frac{x^2 - 9}{x - 3} \right]$$

10. Refer to the following limit properties for Exercises a and b.

Sum Rule:
$$\lim_{x \to \alpha} [f(x) + g(x)] = \lim_{x \to \alpha} f(x) + \lim_{x \to \alpha} g(x)$$

Extended Sum Rule:
$$\lim_{x \to \alpha} [f_1(x) + f_2(x) + ... + f_n(x)] = \lim_{x \to \alpha} f_1(x) + \lim_{x \to \alpha} f_2(x) + ... + \lim_{x \to \alpha} f_n(x)$$

[Hint: Use the basic sum rule to prove both a and b. Using mathematical induction for these two exercises is ideal.]

(a) Assume that the extended sum rule holds for $n = 12$. Prove from your assumption that it holds for $n = 13$.

(b) Assume that the extended sum rule holds for some integer $n > 2$. Prove from your assumption that it holds for the integer $n + 1$.