Problem Set 1 - Solutions

MathWise Institute

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1 Introduction

Attempt each part of each problem yourself. Read each portion of the problem before referring to the solution set for help. If anything is unclear about the problems being asked, or if you have any question about the solution, feel free to email the author at agflores1979@gmail.com for clarification.

2 Exercises

2.1 Easy

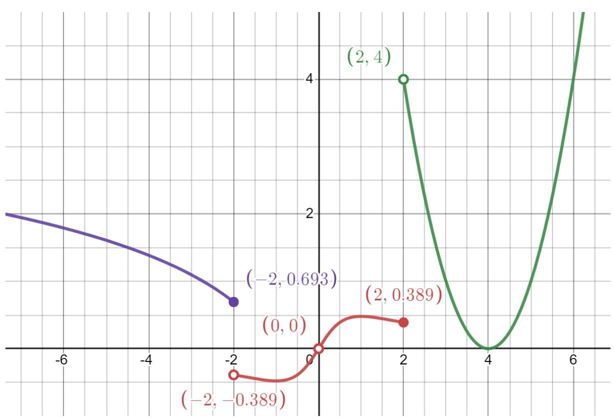


Figure 1: A graph of the function $f(x)$. An open circle means the indicated point does not belong to the function.

Refer to Fig. 1 for Exercises 1-4.

1. **Solution:** $\lim_{x\to -2^-} f(x)$ is the limit as x approaches -2 from the left. This is visually seen as the purple curve on the graph. The purple curve tends to the value 0.693 as $x \to 2$, thus, $\lim_{x\to -2^-} f(x) = 0.693$. $\lim_{x\to -2^+} f(x)$ is as $x \to 2$ from the right-hand side, the red curve. As $x \to 2$ from the the right-hand side, the graph tends to the value -0.389. Since $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$,

the limit as x approaches -2 from both sides, $\lim_{x\to -2} f(x)$, does not exist. Additionally, the value of the graph at -2, $f(-2)$, is 0.693 (not -0.389 since that circle is not filled in), and is not equal to $\lim_{x\to -2} f(x)$. The type of discontinuity seen here is a jump discontinuity.

2. **Solution:** $\lim_{x\to 2^-} f(x)$ corresponds to the red curve as it approaches 2, from the point listed on the graph we can see that the limit as x approaches 2 from the left is 0.389. $\lim_{x\to 2^-} f(x)$ corresponds to the green curve as it approaches 2 from the right side, $f(x)$ approaches the value of 4. Again, $\lim_{x\to 2} f(x)$ does not exist, because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$. From, the graph, the defined value of $f(2)$ is 0.389, not equal to $\lim_{x\to 2} f(x)$. This discontinuity is also a jump discontinuity.

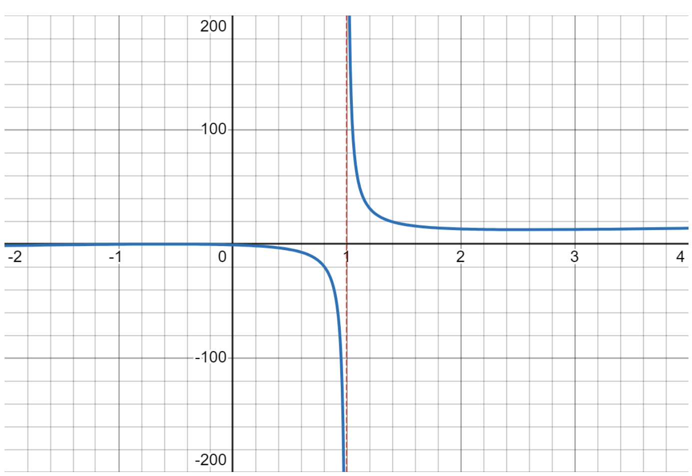
3. **Solution:** The $\lim_{x\to 0} f(x)$ is as the red curve approaches the origin from both sides, thus $\lim_{x\to 0} f(x) = 0$. $f(x)$ is undefined at $x = 0$, thus $\lim_{x\to 0} f(x) \neq f(0)$. The type of continuity present here is a removable discontinuity.

4. **Solution:** $\lim_{x\to 4} f(x)$ is as the green curve approaches $x = 4$ from both sides, we can assume that $f(x)$ approaches 0 at that point. Additionally, $f(x)$ is defined at $x = 4$, $f(4) = 0$. Thus, $\lim_{x\to 4} f(x) = f(4)$, meaning that the function is continuous at $x = 4$.

5. (a) **Solution:** As always, direct substitution should be the first attempt:

$$\begin{aligned}\lim_{x\to 3} \frac{x+1}{(x-2)^3} &= \frac{3+1}{(3-2)^3} \\ &= \frac{4}{(1)^3} \\ &= 4.\end{aligned}$$

   (b) **Solution:** Trying direct substitution for this problem yields $\frac{5}{0}$, implying that there is a vertical asymptote at $x = 1$. Looking at the graph below:



we can see that there is in fact a vertical asymptote at $x = 1$. Since we are evaluating the limit as x approaches 1 from the left, we are looking at the left of the vertical asymptote. We see that from the left of the asymptote the values of $f(x)$ are decreasing, thus $\lim_{x\to 1^-} \frac{2x^2+2x+1}{x-1} = -\infty$.

(c) **Solution:** Using the graph from part b, we see that from the right side of the vertical asymptote the values of $f(x)$ are increasing as $x \to 1$, thus $\lim_{x\to 1^+} \frac{2x^2+2x+1}{x-1} = \infty$.

(d) **Solution:** Applying direct substitution initially yields the result of $\frac{0}{0}$, we must rely on an alternative method to solve this problem. One might notice that the denominator is in the format of a difference of squares $(a^2 - b^2 = (a+b)(a-b))$, we can try expanding that out and seeing if it helps when we plug it back into the original problem:

$$\begin{aligned}
\lim_{x\to 3} \frac{x-3}{x^2-9} &= \lim_{x\to 3} \frac{x-3}{(x+3)(x-3)} \\
&\text{(replacing } x^2-9 \text{ with } (x+3)(x-3)) \\
&= \lim_{x\to 3} \frac{1}{x+3}. \\
&\text{(canceling the common factor of } x-3)
\end{aligned}$$

Using direct substitution at this point yields

$$\begin{aligned}
\lim_{x\to 3} \frac{1}{x+3} &= \frac{1}{3+3} \\
&= \frac{1}{6}.
\end{aligned}$$

(e) **Solution:** Because x is approaching $-\infty$, we must look for the "fastest growing" term. In this case, the term that is growing the fastest is the x^2 in the denominator. Thus, the limit tends to 0.

(f) **Solution:** Again, we are looking for the fastest growing term. Here we see that both the numerator and the denominator share the fastest term of x^2. We look at the coefficients of these terms for the answer:

$$\begin{aligned}
\lim_{x\to\infty} \frac{12x^2+5x+2}{6x^2-3x+8} &= \lim_{x\to\infty} \frac{12x^2}{6x^2} \\
&\text{(ignoring everything but the fastest growing terms)} \\
&= \frac{12}{6} \\
&\text{(looking only at the coefficients)} \\
&= 2.
\end{aligned}$$

2.2 Moderate

6. **Solution:** The goal of this exercise is to make $\lim_{t\to 0} p(t) = p(0)$. One should start by looking at each side separately and finding their value first:

Left-hand side: $\lim_{t\to 0} p(t)$
Applying direct substitution first gives us the result of $\frac{0}{0}$, an indeterminate form. We must alter the equation of $p(t)$ in order to find the value of the limit. We notice that every term in the equation has a t^2, we can start by factoring it out and canceling, then analyzing the remaining equation:

$$\begin{aligned}\lim_{t\to 0} p(t) &= \lim_{t\to 0} \frac{t^3 + 2t^2}{t^3 - 2t^2} \\ &= \lim_{t\to 0} \frac{t^2(t+2)}{t^2(t-2)} \\ &= \lim_{t\to 0} \frac{t+2}{t-2}.\end{aligned}$$

Evaluating the limit at this point using direct substitution yields

$$\begin{aligned}\lim_{t\to 0} \frac{t+2}{t-2} &= \frac{0+2}{0-2} \\ &= -\frac{2}{2} \\ &= -1.\end{aligned}$$

Now, let's see if we can get the right-hand side to be same value.

Right-hand side: $p(0)$
As of now, $p(0)$ is undefined (plugging 0 into p yields the result of $\frac{0}{0}$, as seen before). We can follow the same factoring steps from evaluating the left-hand side and reducing $p(t)$ to $\frac{t+2}{t-2}$. Because we were able to remove the discontinuity at $t = 0$, we can assume that the function likely had a hole at $p(0)$. The new definition of p, however, does not have a hole at $t = 0$:

$$\begin{aligned}p(0) &= \frac{0+2}{0-2} \\ &= -\frac{2}{2} \\ &= -1.\end{aligned}$$

This is equal to what we found earlier for the left-hand side as well. Thus, $p(t)$ should be defined as $p(t) = (t+2)/(t-2)$ at $t = 0$ so that $\lim_{t\to 0} p(t) = p(0)$.

7. (a) **Solution:** Assuming that the block of ice will completely melt upon contact with the flame, we can say that the volume of the ice will be 0 cm^3 when it is 0 cm from the flame, i.e. $v(0) = 0$.

   (b) **Solution:** Since the volume of the ice only changes when it is within 500 cm of the fire pit, that means it will be constant for $x \geq 500$. Thus, $v(x)$ is a constant function for $x \geq 500$. Additionally, $\lim_{x\to\infty} v(x) = 1000$ since $v(x)$ will always be 1000 cm^3 for values of $x \geq 500$.

(c) **Solution:** The limit of $v(x)$ as x approaches 0 from the right-hand side describes the volume of the block of ice as it gets closer and closer to the fire pit. The volume of the ice will decrease (until eventually the ice is completely melted) as it gets closer to the source of the fire, thus $\lim_{x \to 0^+} v(x) = 0$, which is also equal to $v(0)$ according to part a. It makes sense that $\lim_{x \to 0^+} v(x) = v(0)$ because we are comparing how the volume decreases as it gets closer to the fire pit and the value of the volume when the ice is at the fire pit.

(d) **Solution:**

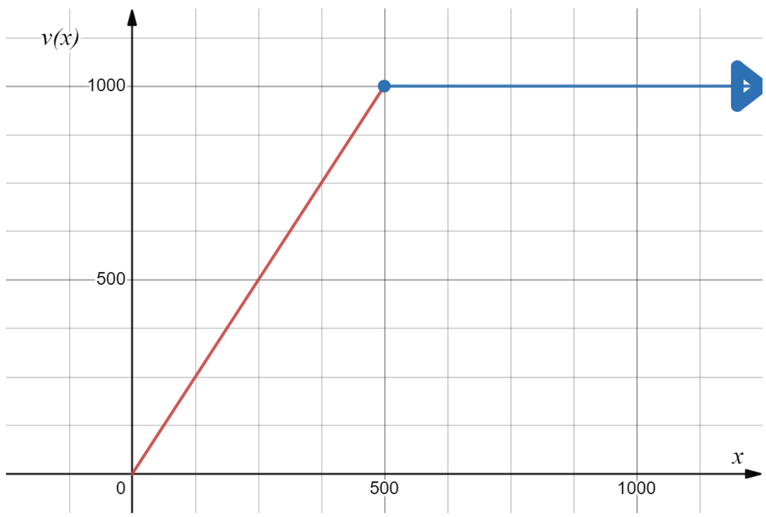


Figure 2: This is just one example of what the graph could look like. The necessary components are that from $x = 500$ and beyond the function should be constant, $v(x) = 1000$, and as $x \to 0$ from the right the function should be decreasing towards $v(0) = 0$ from $v(500) = 1000$.

2.3 Challenging

8. **Solution:** $g(x)$ currently has a discontinuity at $x = 1$ (since $g(1) = \frac{0}{0}$), thus we should simplify the definition of the function in order to remove the discontinuity. This current equation can be simplified using an understanding of rational root theorem. We apply the theorem to the numerator, $x^5 - 1$. Simply from looking at the equation one can see that $x = 1$ is a root (because plugging in 1 for x yields $1^5 - 1 = 0$). Thus, we can divide the root of $(x - 1)$ from the numerator. This can be done using long division or synthetic division. Here, we will show the synthetic division for it:

$$
\begin{array}{c|cccccc}
1 & 1 & 0 & 0 & 0 & 0 & -1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 0.
\end{array}
$$

The new polynomial in the numerator then becomes $(x-1)(x^4+x^3+x^2+x+1)$. Substituting this back into $g(x)$ and simplifying leaves us with:

$$\begin{aligned} g(x) &= \frac{x^5-1}{x-1} \\ &= \frac{(x-1)(x^4+x^3+x^2+x+1)}{x-1} \\ &= x^4+x^3+x^2+x+1. \end{aligned}$$

This is now the new definition of $g(x)$ at $x=1$ so that $\lim_{x\to 1} g(x) = g(1)$. Let us finally evaluate both sides of this equation to see if this statement holds true:

Left-hand side: $\lim_{x\to 1} g(x)$
Using direct substitution,

$$\begin{aligned} \lim_{x\to 1} g(x) &= \lim_{x\to 1} \left[x^4+x^3+x^2+x+1\right] \\ &= 1^4+1^3+1^2+1+1 \\ &= 5. \end{aligned}$$

Right-hand side: $g(1)$

$$\begin{aligned} g(1) &= (1)^4+(1)^3+(1)^2+(1)+1 \\ &= 1^4+1^3+1^2+1+1 \\ &= 5. \end{aligned}$$

Both sides of the equation equal the same value, 5, and are by extension equal to each other. This concludes the solution to the problem.

9. **Solution:** Attempting direct substitution gives us

$$\begin{aligned} \lim_{x\to 3}\left[\frac{x^2-5x+6}{x^2-7x+12} - \frac{(x^2-9)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right] &= \frac{3^2-5(3)+6}{3^2-7(3)+12} - \frac{(3^2-9)(\sqrt{3}+\sqrt{3})}{(3+3)(\sqrt{3}-\sqrt{3})} \\ &= \frac{9-15+6}{9-21+12} - \frac{(9-9)(\sqrt{3}+\sqrt{3})}{(3+3)(\sqrt{3}-\sqrt{3})} \\ &= \frac{0}{0} - \frac{0}{0}. \end{aligned}$$

Thus, we must refer to modifying the problem. First, let us start by splitting the limit into two using the sum rule:

$$\lim_{x\to 3}\left[\frac{x^2-5x+6}{x^2-7x+12}\right] - \lim_{x\to 3}\left[\frac{(x^2-9)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right]$$

We will name these limits α and β, respectively. Recall that both limits yield the indeterminate form of $\frac{0}{0}$, this means that both limits must be simplified in order to provide a solution.

Starting with limit α:

$$\lim_{x\to 3}\left[\frac{x^2-5x+6}{x^2-7x+12}\right]$$

The first step is to factor the polynomials from the numerator and denominator into their linear terms, then simplify:

$$\begin{aligned}\lim_{x\to 3}\left[\frac{x^2-5x+6}{x^2-7x+12}\right] &= \lim_{x\to 3}\left[\frac{(x-3)(x-2)}{(x-3)(x-4)}\right]\\ &= \lim_{x\to 3}\left[\frac{x-2}{x-4}\right].\end{aligned}$$

Using direct substitution here yields the result:

$$\begin{aligned}\lim_{x\to 3}\left[\frac{x-2}{x-4}\right] &= \frac{3-2}{3-4}\\ &= -\frac{1}{1}\\ &= -1.\end{aligned}$$

We can substitute this back into the original problem:

$$[-1]-\lim_{x\to 3}\left[\frac{(x^2-9)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right]$$

Now doing limit β:

$$\lim_{x\to 3}\left[\frac{(x^2-9)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right]$$

Similar to limit α, we start by factoring the quadratic in the numerator into its linear terms. Here we actually have a difference of squares, $a^2-b^2=(a+b)(a-b)$. Then, we simplify:

$$\begin{aligned}\lim_{x\to 3}\left[\frac{(x^2-9)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right] &= \lim_{x\to 3}\left[\frac{(x+3)(x-3)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right]\\ &= \lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-\sqrt{3})}\right]\end{aligned}$$

Applying direct substitution here would still yield the indeterminate result of $\frac{0}{0}$, we must do more. Let's now multiply the numerator and denominator by the conjugate of the denominator $\sqrt{x}+\sqrt{3}$ then simplify. [The conjugate of a term is simply the term itself but switching the sign between the sub-terms, since originally it was $\sqrt{x}-\sqrt{3}$, the conjugate is $\sqrt{x}+\sqrt{3}$. Additionally, remember that since we are multiplying both the numerator and the denominator by the same term, we essentially multiplying by 1, and thus not changing the value of the expression.] We do this in an attempt to rationalize the denominator:

$$\begin{aligned}
\lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-\sqrt{3})}\right] &= \lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-\sqrt{3})}\right]\cdot\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} \\
&= \lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}\right] \\
&= \lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})^2}{(\sqrt{x})^2+\sqrt{3x}-\sqrt{3x}-(\sqrt{3})^2}\right] \\
&= \lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})^2}{(\sqrt{x})^2-(\sqrt{3})^2}\right] \\
&= \lim_{x\to 3}\left[\frac{(x-3)(\sqrt{x}+\sqrt{3})^2}{x-3}\right] \\
&= \lim_{x\to 3}\left(\sqrt{x}+\sqrt{3}\right)^2
\end{aligned}$$

Here, let us try direct substitution once more:

$$\begin{aligned}
\lim_{x\to 3}\left(\sqrt{x}+\sqrt{3}\right)^2 &= \left(\sqrt{3}+\sqrt{3}\right)^2 \\
&= \left(2\sqrt{3}\right)^2 \\
&= 2^2\left(\sqrt{3}\right)^2 \\
&= 4(3) \\
&= 12.
\end{aligned}$$

Substituting this back into the original problem:

$$\begin{aligned}
\lim_{x\to 3}\left[\frac{x^2-5x+6}{x^2-7x+12}-\frac{(x^2-9)(\sqrt{x}+\sqrt{3})}{(x+3)(\sqrt{x}-\sqrt{3})}\right] &= [-1]-[12] \\
&= -13.
\end{aligned}$$

10. Refer to the following limit properties for Exercises a and b.

Sum Rule:

$$\lim_{x\to\alpha}[f(x)+g(x)] = \lim_{x\to\alpha} f(x) + \lim_{x\to\alpha} g(x)$$

Extended Sum Rule:

$$\lim_{x\to\alpha}[f_1(x)+f_2(x)+...+f_n(x)] = \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + ... + \lim_{x\to\alpha} f_n(x)$$

[Hint: Use the basic sum rule to prove both a and b. Using mathematical induction for these two exercises is ideal.]

(a) **Solution:** For all integers $n > 2$, let $P(n)$ be the statement

" $\lim_{x\to\alpha}[f_1(x)+f_2(x)+...+f_n(x)] = \lim_{x\to\alpha} f_1(x)+\lim_{x\to\alpha} f_2(x)+...+\lim_{x\to\alpha} f_n(x)$."

Basis Step: $n = 3$

$$\begin{aligned}\lim_{x\to\alpha}[f_1(x)+f_2(x)+f_3(x)] &= \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + \lim_{x\to\alpha} f_3(x)\\ &= \lim_{x\to\alpha}[f_1(x)+f_2(x)] + \lim_{x\to\alpha} f_3(x)\\ &\text{(true by the basic sum rule)}\\ &= \lim_{x\to\alpha}[f_1(x)+f_2(x)+f_3(x)]\\ &\text{(true by the basic sum rule).}\end{aligned}$$

Inductive Step: Assume that $P(12)$ is true, i.e.,

$$\begin{aligned}\lim_{x\to\alpha}[f_1(x)+f_2(x)+...+f_{12}(x)] &= \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + ... + \lim_{x\to\alpha} f_{12}(x)\\ &\text{(Inductive Hypothesis).}\end{aligned}$$

We show $P(13)$ is true, i.e.,

$$\lim_{x\to\alpha}[f_1(x)+f_2(x)+...+f_{13}(x)] = \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + ... + \lim_{x\to\alpha} f_{13}(x).$$

Right-hand side:

$$\begin{aligned}\lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + ... + \lim_{x\to\alpha} f_{13}(x) &= \left[\lim_{x\to\alpha} f_1(x) + ... + \lim_{x\to\alpha} f_{12}(x)\right] + \lim_{x\to\alpha} f_{13}(x)\\ &= \lim_{x\to\alpha}[f_1(x)+...+f_{12}(x)] + \lim_{x\to\alpha} f_{13}(x)\\ &\text{(by the inductive hypothesis)}\\ &= \lim_{x\to\alpha}[f_1(x)+...+f_{12}(x)+f_{13}(x)]\\ &\text{(by the basic sum rule).}\end{aligned}$$

By mathematical induction, $P(13)$ is true.

(b) **Solution:** For all integers $n > 2$, let $P(n)$ be the statement

" $\lim_{x\to\alpha}[f_1(x) + f_2(x) + ... + f_n(x)] = \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + ... + \lim_{x\to\alpha} f_n(x)$."

Basis Step: $n = 3$

$$\begin{aligned}\lim_{x\to\alpha}[f_1(x) + f_2(x) + f_3(x)] &= \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + \lim_{x\to\alpha} f_3(x)\\ &= \lim_{x\to\alpha}[f_1(x) + f_2(x)] + \lim_{x\to\alpha} f_3(x)\\ &\text{(true by the basic sum rule)}\\ &= \lim_{x\to\alpha}[f_1(x) + f_2(x) + f_3(x)]\\ &\text{(true by the basic sum rule).}\end{aligned}$$

Inductive Step: Assume that $P(k)$ is true for some integer $k > 2$, i.e.,

$$\begin{aligned}\lim_{x\to\alpha}[f_1(x) + f_2(x) + ... + f_k(x)] &= \lim_{x\to\alpha} f_1(x) + \lim_{x\to\alpha} f_2(x) + ... + \lim_{x\to\alpha} f_k(x)\\ &\text{(Inductive Hypothesis).}\end{aligned}$$

We show $P(k+1)$ is true, i.e.,

$$\lim_{x\to\alpha}[f_1(x) + ... + f_k(x) + f_{k+1}(x)] = \lim_{x\to\alpha} f_1(x) + ... + \lim_{x\to\alpha} f_k(x) + \lim_{x\to\alpha} f_{k+1}(x).$$

Right-hand side:

$$\begin{aligned}\lim_{x\to\alpha} f_1(x) + ... + \lim_{x\to\alpha} f_k(x) + \lim_{x\to\alpha} f_{k+1}(x) &= \left[\lim_{x\to\alpha} f_1(x) + ... + \lim_{x\to\alpha} f_k(x)\right] + \lim_{x\to\alpha} f_{k+1}(x)\\ &= \lim_{x\to\alpha}[f_1(x) + ... + f_k(x)] + \lim_{x\to\alpha} f_{k+1}(x)\\ &\text{(by the inductive hypothesis)}\\ &= \lim_{x\to\alpha}[f_1(x) + ... + f_k(x) + f_{k+1}(x)]\\ &\text{(by the basic sum rule).}\end{aligned}$$

By mathematical induction, $P(n)$ is true for all integers $n > 2$.