University of Massachusetts, Amherst

Mathematics Journal Club

Feroz & Agustin

These are lecture used during the meetings of the Mathematics Journal Club at UMass Amherst. The presentations were made by Feroz and Agustin in alternating weeks. The notes were prepared by the presenter for that week.

Contents

2025-11-07: Orbits, Stabilizers, and Fixed Points

by *FAM*

Background & Motivation

Group actions are a useful way to study orbits, stabilizers, and fixed points. Consider a group G and an arbitrary set S. Group actions are functions $\phi : G \to \text{Perm}(S)$ that describe how elements of the group G permute the elements of the set S; how G "acts" on the set S.[1] Recall that group actions can be left or right actions:

- left action: $s.\phi(g) = g \cdot s$,
- right action: $\phi(g).s = s \cdot g$.

Suppose we continue with this "action" of G on S. For $s \in S$, if $\phi(g)$ "acts" on s, two questions can be asked:

1. What other "states" in S are reachable from s via the actions of G?
2. Which elements of G leave s unchanged?

We think of the states that are reachable from s as the *orbit* of s, denoted $\text{Orb}(s)$. The elements of G that leave s unchanged are called the *stabilizer* of s, denoted $\text{Stab}(s)$. Finally, we can also ask which states in S are left unchanged by all elements of G; these states are called the *fixed points* of the action of G on S, denoted $\text{Fix}(G)$.

More formally, suppose that G acts on a set S (on the right) via $\phi : G \to S$. Then:

Definition 1 (Orbit)**.** The **orbit** of an element $s \in S$ under the action of a group G is the set

$$\text{Orb}(s) = \{s.\phi(g) \mid g \in G\}.$$

Similarly, for a left action, the orbit is defined as

$$\text{Orb}(s) = \{\phi(g).s \mid g \in G\}.$$

Definition 2 (Stabilizer)**.** The **stabilizer** of an element $s \in S$ under the action of a group G is the set

$$\text{Stab}(s) = \{g \in G \mid s.\phi(g) = s\}.$$

Similarly, for a left action, the stabilizer is defined as

$$\text{Stab}(s) = \{g \in G \mid \phi(g).s = s\}.$$

The definition of the stabilizer is simply the set of all group elements s.t. when you press the g button, $\phi(g)$, associated with them, the stabilizers are all of the group elements that fix the state s.[2] Lastly, we have:

Definition 3. The **fixed points** of a group G acting on a set S are the elements of S that remain unchanged under the action of all elements of G:

$$\text{Fix}(G) = \{s \in S \mid s.\phi(g) = s, \forall g \in G\}.$$

[1] Reminder: ϕ is a "homomorphism." ϕ sends the group identity to the permutation identity, and preserves group operation: $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. Thus, since $\phi(g)$ has to be the identity permutation, it must send s to s.

[2] Think of a group action as a group "switch board" and every element has a "button."

Similarly, for a left action, the fixed points are defined as

$$\mathrm{Fix}(G) = \{s \in S \mid \phi(g).s = s, \forall g \in G\}.$$

In more complicated terms, the fixed points of the action are the orbits of size 1; that is, the states that do not change under any group action.

Note: orbits of ϕ are the connected components in the action diagram. Stabilizers are the loops at each state in the action diagram.

Examples

Consider the group $D_4 = \langle r, f \rangle$; *i.e.*, the dihedral group of order 8 generated by a rotation r of 90° and a reflection f across a diagonal.

Now suppose G acts on the set of 2×2 matrices with all zero entries:

$$S_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

In this case, the only matrix in S_1 is the zero matrix. Applying any group action to this matrix leaves it unchanged. Thus, the orbit of this matrix is simply itself, $\left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \right\}$, and the stabilizer is the entire group D_4. The fixed points of this action are also $\left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \right\}$.

Still considering the group D_4, now let it act on the set of 2×2 matrices with entries of 1 either along the diagonal or the anti-diagonal:

$$S_2 = \left\{ s_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, s_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

We see that applying the group actions to either matrix permutes them:

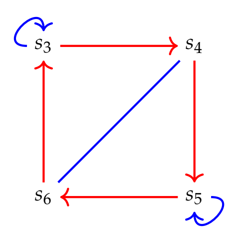
$$s_1.\phi(r) = s_2, \quad s_1.\phi(f) = s_2, \quad \text{and} \quad s_2.\phi(r) = s_1, \quad s_2.\phi(f) = s_1.$$

Thus, the orbit of either matrix is the entire set S_2, $\{s_1, s_2\}$. Additionally, since r and f behave similarly here, the stabilizers of both matrices are the same. Namely, the stabilizers are $\{e, r^2, rf, r^3 f\}$. Finally, there are no fixed points in this action since no matrix remains unchanged under all group actions.

Finally, consider the set of 2×2 matrices with entries of 1 along either a row or a column:

$$S_3 = \left\{ s_3 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, s_4 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, s_5 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, s_6 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Since this is a larger set, the action diagram is more complicated.



In the figure to the right, the red arrows represent the action of r and the blue arrows represent the action of f. Directional arrows indicate how the states are permuted under each group action, while undirected edges indicate bi-directional actions. The orbit of any state in this set is the entire set S_3, since any state can be reached from any other state via some combination of group actions. Both s_3 and s_5 have stabilizers $\{e, f\}$, since applying f to either state leaves it unchanged. On the other hand, s_4 and s_6 have stabilizers $\{e, r^2 f\}$. Finally, there are no fixed points in this action since no matrix remains unchanged under all group actions.

With all of these examples, we make a few observations:

1. the stabilizers are subgroups of G, and
2. as the orbits increase in size, the stabilizers decrease in size (and vice versa?).

Particularly, notice that the product of the sizes of the orbit and stabilizer for each state is equal to the size of the group G: That is to say, a bigger orbit means more

	S_1	S_2	S_3
$\lvert\mathrm{Orb}(\cdot)\rvert$	1	2	4
$\lvert\mathrm{Stab}(\cdot)\rvert$	8	4	2
Product	8	8	8

possibilities for the element to get mapped to, which takes more group elements to do so, so fewer elements are left to fix/stabilize these elements.

Orbit-Stabilizer Theorem

The observations made in the previous section can be formalized in the Orbit-Stabilizer Theorem, which states that for a group G acting on a set S, the size of the orbit of an element $s \in S$ multiplied by the size of its stabilizer is equal to the size of the group G:

Theorem 4 (Orbit-Stabilizer Theorem)**.** *Let G be a finite group acting on a set S. For any element $s \in S$, the following holds:*

$$|\mathrm{Orb}(s)| \cdot |\mathrm{Stab}(s)| = |G|.$$

Proposition for the Proof

Proposition 5. *$\exists s \in S$, s.t. the set* $\mathrm{Stab}(s)$ *is a "subgroup" of S.*

As an outline for the proof of this proposition, we need to show that $\mathrm{Stab}(s)$ satisfies the subgroup criteria:

1. it contains the identity element, $s.\phi(e) = s$,
2. inverses are contained, $\forall g \in \mathrm{Stab}(s), s.\phi(g^{-1}) = s$, and
3. it is closed under the group operation, if $\forall g_1, g_2 \in \mathrm{Stab}(s)$, $s.\phi(g_1) = s$ and $s.\phi(g_2) = s$, then $s.\phi(g_1 g_2) = s$.

The act of showing these three properties will prove the proposition and is left as an exercise to the reader.

Remark 6. The kernel of the group action ϕ is the set of all group elements that "fix" everything in S: informally, $\phi : G \to \mathrm{Perm}(S)$, and formally, $\ker(\phi) = \{g \in G \mid \phi(g) = e\}$, or equivalently, $\ker(\phi) = \{g \in G \mid s.\phi(g) = s, \forall s \in S\}$. Notice that $\ker(\phi) = \bigcap_{s \in S} \mathrm{Stab}(s)$.

2025-11-14: Lie Groups, Algebras, and Brackets

by *AGF*

A bit of history

In his short life, Évariste Galois (1811–1832) laid the foundations of group theory and its applications to the theory of equations. Galois introduced the concept of a group as a set of permutations that describe the symmetries of the roots of polynomial equations. In particular, these were discrete symmetries, as the roots of polynomials are finite, isolated points. For example, the roots of the polynomial $x^2 - 2 = 0$ are $\sqrt{2}$ and $-\sqrt{2}$, and the group of symmetries consists of two elements: the identity permutation and the permutation that swaps the two roots.

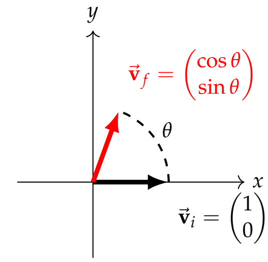


Figure 1: Showing the 2D rotation of $\hat{\mathbf{x}}$, the unit vector along the x-axis, by an angle θ.

Ten years after Galois's death, Sophus Lie (1842–1899) extended the concept of groups to continuous symmetries. His motivation was to use symmetries to solve differential equations. He noticed that, without providing initial conditions, differential equations will have infinitely many solutions, which can be viewed as continuous families of solutions. For example, the differential equation $y'(x) = 2x$ has the general solution $y(x) = x^2 + C$, where C is an arbitrary constant which creates a continuous family of parabolas. That is, *there is a vertical translational symmetry in the set of solutions*. Unfortunately, Lie's work was as recognized as Galois's during their lifetimes. However, though Lie theory was initially developed to study differential equations, it has since found applications in various fields, including geometry, physics, and representation theory.

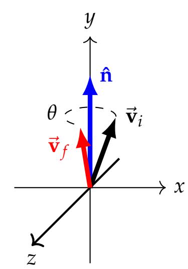


Figure 2: Showing the 3D rotation of a vector $\vec{\mathbf{v}}_i$ around an axis $\hat{\mathbf{n}}$ by an angle θ counterclockwise when looking down the axis from the tip of $\hat{\mathbf{n}}$.

Rotations

In order to rotate a vector in 2D space, all we must specify is an angle θ which tells us how much to rotate the vector counterclockwise, starting at the positive x-axis. In 3D, we must now also specify the axis of rotation, which can be represented by a unit vector $\hat{\mathbf{n}}$, in addition to the angle θ which now tells us how much to rotate around that axis. If we wish to extend this to higher dimensions, how can we generalize it? Furthermore, how can we go beyond vectors in $\mathbb{R}^n$ and into $\mathbb{C}^n$?

To begin this generalization, we list the properties of rotations in $\mathbb{R}^n$ and see how we can realise them in the complex case:

1. **Rotation is linear**: Consider two vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ in $\mathbb{R}^n$. We represent the rotation operation by Rotate (very creative, I know). Then, $\text{Rotate}(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2) = \text{Rotate}(\vec{\mathbf{v}}_1) + \text{Rotate}(\vec{\mathbf{v}}_2)$. Additionally, for any scalar λ, $\text{Rotate}(\lambda\vec{\mathbf{v}}) = \lambda\,\text{Rotate}(\vec{\mathbf{v}})$. From these two properties, we see that rotation is a linear transformation, and thus can be represented by a matrix multiplication: $\text{Rotate}(\vec{\mathbf{v}}) = \mathbf{R}\vec{\mathbf{v}}$, where $\mathbf{R}$ is the rotation matrix.[3]

[3] We write matrices (and operators in general) in **boldface**.

Prior to proceeding with property 2 in the list to the left, recall that a property of the transpose operation is that $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$.

2. **Rotations preserve length and angles**: Rotations do not change the length (or norm) of vectors, nor do they change the angles between them. The last part of this previous sentence tells us that the scalar product of two vectors, say $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, is preserved under rotation: $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = (\mathbf{R}\vec{\mathbf{v}}) \cdot (\mathbf{R}\vec{\mathbf{w}})$. Notice that, in the case of

$\vec{\mathbf{v}}$, $\vec{\mathbf{w}} \in \mathbb{R}^n$, the scalar product is equivalently written as $\vec{\mathbf{v}}^\top \vec{\mathbf{w}} = (\mathbf{R}\vec{\mathbf{v}})^\top (\mathbf{R}\vec{\mathbf{w}}) = \vec{\mathbf{v}}^\top \mathbf{R}^\top \mathbf{R} \vec{\mathbf{w}} \implies \mathbf{R}^\top \mathbf{R} = \mathbf{I}_n$, with $\mathbf{I}_n$ representing the $n \times n$ identity matrix.

The collection of all matrices $\mathbf{R}$ satisfying $\mathbf{R}^\top \mathbf{R} = \mathbf{I}_n$ is denoted by the *orthogonal group* $O(n)$. That is,

$$O(n) = \left\{ \mathbf{R} \in \mathbb{R}^{n\times n} \mid \mathbf{R}^\top \mathbf{R} = \mathbf{I}_n \right\}.$$

3. **Rotations preserve orientation**: From property 2, rotations are orthogonal transformations. However, being orthogonal is not sufficient to be a rotation. The previous two properties are also satisfied by reflections, which flip the orientation of the space. Thus, the matrix corresponding to a reflection will also be in $O(n)$. We enforce this third property so that we only capture rotations and not reflections. Linear algebra tells us that the determinant of a rotation matrix is positive. Additionally, since rotation matrices preserve lengths and angles (by property 2), they also preserve areas and volumes. Thus, the determinant of a rotation matrix must be $+1$: $\det \mathbf{R} = +1$.

In particular, if a matrix $\mathbf{R}$ satisfies all three properties above, then it is a rotation matrix. The collection of all such matrices is represented by the *special orthogonal group* $SO(n)$:

$$SO(n) = \left\{ \mathbf{R} \in \mathbb{R}^{n\times n} \mid \mathbf{R}^\top \mathbf{R} = \mathbf{I}_n, \det \mathbf{R} = +1 \right\}.$$

Using the above collection of properties, we can start to build our understanding of rotations in complex spaces. We denote a rotation in $\mathbb{C}^n$ by a matrix $\mathbf{U}$. Property 1 still holds, as linearity is independent of the field over which we are working. Property 2 must be modified slightly, as the scalar product in complex spaces is defined differently. Specifically, for $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{C}^n$, the scalar product is defined as $\vec{\mathbf{v}}^\dagger \vec{\mathbf{w}}$, where $\vec{\mathbf{v}}^\dagger$ is the conjugate transpose of $\vec{\mathbf{v}}$: $\vec{\mathbf{v}}^\dagger \equiv (\vec{\mathbf{v}}^\top)^*$. Thus, property 2 becomes $\vec{\mathbf{v}}^\dagger \vec{\mathbf{w}} = (\mathbf{U}\vec{\mathbf{v}})^\dagger (\mathbf{U}\vec{\mathbf{w}}) \implies \mathbf{U}^\dagger \mathbf{U} = \mathbf{I}_n$.[4] Finally, even though there is no notion of orientation in complex spaces, we would still want the complex rotation matrices to preserve the structure of the space. Thus, Property 3 remains unchanged. Therefore, the collection of all rotation matrices in $\mathbb{C}^n$ is represented by the *special unitary group* $SU(n)$:

$$SU(n) = \left\{ \mathbf{U} \in \mathbb{C}^{n\times n} \mid \mathbf{U}^\dagger \mathbf{U} = \mathbf{I}_n, \det \mathbf{U} = +1 \right\}.$$

[4] Similar to the orthogonal group, the collection of all matrices $\mathbf{U}$ satisfying $\mathbf{U}^\dagger \mathbf{U} = \mathbf{I}_n$ is denoted by the *unitary group* $U(n)$:

$$U(n) = \left\{ \mathbf{U} \in \mathbb{C}^{n\times n} \mid \mathbf{U}^\dagger \mathbf{U} = \mathbf{I}_n \right\}.$$

The three properties of rotations in real and complex spaces can then be summarized in the following table:

Property	Real Rotations	Complex Rotations
Linearity	Rotate $(\mathbf{v}) = \mathbf{R}\mathbf{v}$	Rotate $(\mathbf{v}) = \mathbf{U}\mathbf{v}$
Norm & Angle Preservation	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}_n$, $\mathbf{R} \in O(n)$	$\mathbf{U}^\dagger \mathbf{U} = \mathbf{I}_n$, $\mathbf{U} \in U(n)$
Orientation Preservation	$\det \mathbf{R} = +1$	$\det \mathbf{U} = +1$

Table 1: Summary of rotation groups in real spaces and their complex analogues.

One can then think of the equations $\mathbf{R}^\top \mathbf{R} = \mathbf{I}_n$ and $\mathbf{U}^\dagger \mathbf{U} = \mathbf{I}_n$ as *quadratic* constraints on the matrix elements of $\mathbf{R}$ and $\mathbf{U}$, respectively. However, in n dimensions, there are n^2 matrix elements, making it difficult to visualize these constraints.[5] Luckily, the set of these matrices, whether it is $O(n)$, $SO(n)$, $U(n)$, or $SU(n)$, form what is called a *Lie group*, dealt by *Lie theory*. This theory will help us get closer to a general form of these matrices.

[5] For the $n = 2$ case, the constraints can be reduced to a set of simple linear equations thanks to the Property 3 on the determinant.

Lie Theory

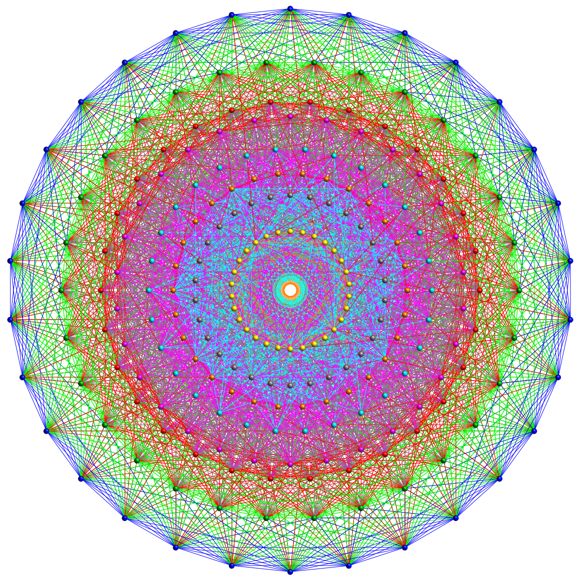


Figure 3: From TOE: "E_8 4_{21} Petrie projection with 8 concentric rings of 30 vertices which are in the same color palette used to color the edges, which uses an algorithm that assigns colors based on the norm of the projected edge. The vertex ring colors from smallest to largest are: 1=Yellow, 2=Gray, 3=Orange, 4=Cyan, 5=Magenta, 6=Red, 7=Green, 8=Blue."

The spirit of Lie theory is to impose a *coordinate system* on complicated systems so that they are easier to handle. For example, consider the set of complex numbers with unit norm, *i.e.*, the set $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. This set can be visualized as the unit circle in the complex plane. However, working directly with this geometric representation can be cumbersome. Instead, we can impose a coordinate system using the angle θ that a complex number z makes with the positive real axis. Specifically, any $z \in \mathbb{S}^1$ can be represented as $z = e^{i\theta}$, where $\theta \in [0, 2\pi)$. This parametrization simplifies operations such as multiplication and inversion within $\mathbb{S}^1$. In the following figure, we illustrate this polar representation of unit length complex numbers. Thinking about it deeply, this is the essence of Lie theory: finding a way to parametrize complex structures in a manner that makes them more tractable.

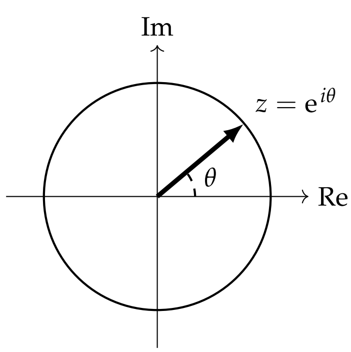


Figure 4: Showing the polar representation of unit length complex numbers.

The circle shown above is an example of a *Lie group*, specifically denoted by $U(1)$. This group represents the set of all complex numbers with unit norm under the operation of multiplication. The parametrization $z = e^{i\theta}$ provides a smooth way to navigate through this group using the angle θ.

Lie Group

A *Lie group* is a mathematical structure that combines the properties of a group with those of a smooth manifold. This means that a Lie group is not only a set equipped with a group operation (*e.g.*, multiplication, addition) but also has a geometric structure that allows for smooth transitions between its elements. The key features of Lie groups include:

- **Group Structure**: A Lie group has a binary operation that satisfies the group axioms: closure, associativity, identity element, and existence of inverses.
- **Smooth Manifold**: The elements of a Lie group can be represented as points in a smooth manifold, allowing for the application of calculus. This means that one can define concepts like tangent spaces and differentiable functions on the group.

To get a better understanding of Lie groups, we look closer at manifolds. A *manifold* is a topological space that locally resembles Euclidean space near each point. More formally, an n-dimensional manifold is a space where each point has a neighborhood that is homeomorphic (topologically equivalent) to an open subset of $\mathbb{R}^n$. Consider the circle in Figure 4. Suppose we want to study the properties of this circle as a manifold. We see that if we consider some arbitrary point on the circle, we can look at a small neighborhood around that point and "zoom in" enough such that this neighborhood looks like a straight line segment, $\mathbb{R}^1$. This process of zooming in is akin to taking a tangent line at that point. The tangent line is a linear approximation of the manifold at that point, and it allows us to use tools from linear algebra and calculus to study the manifold's properties. In the figure below, we illustrate this concept by showing how a small neighborhood around a point on the unit circle can be smoothly deformed to a tangent line and back to the original curve.

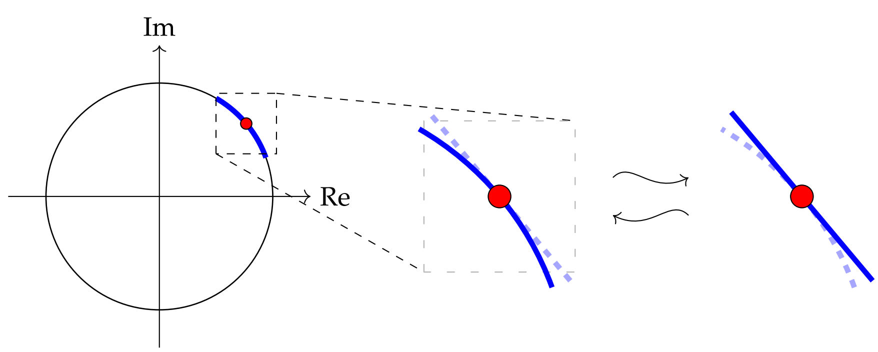


Figure 5: Showing how the neighborhood of a point on the complex numbers of unit length can be smoothly deformed to a tangent line and back to the original curve.

References