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# Part I

# INTRODUCTION

# 1 LORENTZIAN VECTOR SPACES

Let $V$ be a real vector space with $\dim V = n (\geqslant 2)$ and let $g$ be a non-degenerate symmetric bilinear form on $V$ with index 1, i.e., $g(u, v) = 0$ for all $v \in V$ implies $u = 0$, and the maximum dimension of a subspace $L$ of $V$ such that $g(u, u) \leqslant 0$ for all $u \in L$ with equality if and only if $u = 0$ is 1. We will say that $g$ is a *Lorentzian product* and $(V, g)$ a *Lorentzian vector space*. A vector $v$ is said to be *spacelike* (resp. *timelike*, *null*) if $g(v, v) > 0$ or $v = 0$ (resp. $g(v, v) < 0$, $g(v, v) = 0$ and $v \neq 0$). The *light cone* is the subset of $V$ consisting of all null vectors of $(V, g)$.

The basic tool we will use to study Lorentzian products is the following result [7, Lemma 5.26]

**Lemma 1** *If $v$ is a timelike vector of $(V, g)$ then we have the orthogonal decomposition*

$$V = \mathrm{Span}\{v\} \oplus v^{\perp},$$

*where $v^{\perp} = \{u \in V : g(u, v) = 0\} (= \mathrm{Span}\{v\}^{\perp})$. Moreover, the restriction of $g$ on $v^{\perp}$ is positive definite (i.e. $g_{|_{v^{\perp}}}$ is a Euclidean product) and the restriction of $g$ to $\mathrm{Span}\{v\}$ is negative definite (i.e. $(-g)_{|_{\mathrm{Span}\{v\}}}$ is a Euclidean product).*

**Remark 2** This result is not true if $v$ is assumed to be null. In fact, in this case $v \in v^{\perp}$. On the other hand, for $v \neq 0$ spacelike, the analogous decomposition holds but if $\dim V \geqslant 3$ then $g_{|_{v^{\perp}}}$ is also Lorentzian.

**Remark 3** The classical Euclidean geometry holds in positive definite subspaces of a Lorentzian vector space; in particular, the Schwarz inequality holds true in $v^{\perp}$, for any timelike vector $v$,

$$| g(u, w) | \leqslant \sqrt{g(u, u)} \sqrt{g(w, w)}$$

for all $u, w \in v^{\perp}$, with equality if and only if $u$ and $w$ are linearly dependent.

One sees that any two timelike vectors of $(V, g)$ are never orthogonal. However, for null vectors we have [10, Cor. 1.1.5], [7, p. 155].

**Proposition 4** *Given two null vectors $x, y$ of a Lorentzian vector space $(V, g)$ we have $g(x, y) = 0$ if and only if $x$ and $y$ are linearly dependent.*

*Proof.* Using Lemma 1 we write $V = \mathrm{Span}\{v\} \oplus v^{\perp}$, where $g(v,v) = -1$. Therefore we have

$$x = av + u, \quad u \in v^{\perp}, \quad u \neq 0, \quad g(u,u) = a^2, \quad (a \neq 0),$$

$$y = bv + w, \quad w \in v^{\perp}, \quad w \neq 0, \quad g(v,v) = b^2, \quad (b \neq 0).$$

If $g(x,y) = 0$ then $\mid g(u,w) \mid = \sqrt{g(u,u)}\sqrt{g(w,w)}$ and therefore $u = kw$ for some $k \in \mathbb{R}$. In fact, $k = a/b$. The converse is trivial. $\square$

Let $\mathcal{T}(V,g)$ be the subset of $V$ consisting of all timelike vectors of $(V,g)$. For each $v \in \mathcal{T}(V,g)$ we put

$$C(v) = \{u \in \mathcal{T}(V,g) \,:\, g(u,v) < 0\}.$$

Observe that $v \in C(v)$ for any $v \in \mathcal{T}(V,g)$, hence $C(v) \neq \emptyset$. Moreover, given another $w \in \mathcal{T}(V,g)$ we know $g(v,w) \neq 0$. Therefore, either $w \in C(v)$ or $w \in C(-v)$ and so

$$\mathcal{T}(V,g) = C(v) \cup C(-v)$$

for any $v \in \mathcal{T}(V,g)$. We call $C(v)$ the *time cone* defined by $v$. The following result characterizes when two timelike vectors lie to the same time cone [7, Lemma 5.29]

**Lemma 5** *Given* $u,v \in \mathcal{T}(V,g)$, *they belong to the same time cone if and only if* $g(u,v) < 0$.

*Proof.* Clearly, if $g(u,v) < 0$ then $u \in C(v)$, and so $u$ and $v$ belong to the time cone defined by $v$. Conversely, using again Lemma 1 we write

$$u = aw + y, \quad g(w,w) = -1, \quad y \in w^{\perp}, \quad a = -g(u,w) > 0,$$

$$v = bw + z, \quad g(w,w) = -1, \quad z \in w^{\perp}, \quad b = -g(v,w) > 0.$$

Taking into account

$$\sqrt{g(y,y)} < a \quad \text{and} \quad \sqrt{g(z,z)} < b,$$

we have

$$g(u,v) \leqslant -ab + \mid g(y,z) \mid \leqslant -ab + \sqrt{g(y,y)}\sqrt{g(z,z)} < -ab + ab = 0.$$

$\square$

**Corollary 6** *Given* $w \in \mathcal{T}(V,g)$, $u,v \in C(w)$ *and* $a,b \in \mathbb{R}$, $a,b \geqslant 0$, $a^2 + b^2 \neq 0$, *we have* $au + bv \in C(w)$.

*Proof.* Compute $g(au + bv, w) = ag(u,w) + bg(v,w) < 0$ and use previous result. $\square$

**Remark 7** Time cones are then convex subsets of $V$. On the other hand, note that if $u \in C(v)$ then $C(u) = C(v)$.

We have seen in Remark 3 that the classical Schwarz inequality holds true in positive definite subspaces of a Lorentzian vector space. In general, this inequality does not hold for any pair of vectors. However, if we pay attention only on timelike vectors we have the so-called *wrong-way Schwarz inequality* [7, Prop. 5.30]

**Proposition 8** *For any $u, v \in \mathcal{T}(V, g)$ we have*

$$| g(u,v) | \geqslant \sqrt{-g(u,u)}\sqrt{-g(v,v)}$$

*and equality holds if and only if $u$ and $v$ are linearly dependent.*

*Proof.* We write using Lemma 1

$$u = av + x, \quad a \in \mathbb{R}, \quad x \in v^{\perp},$$

and therefore

$$g(u,v)^2 = a^2 g(v,v)^2, \qquad g(u,u) = a^2 g(v,v) + g(x,x) < 0,$$

which gives

$$g(u,v)^2 = \{g(u,u) - g(x,x)\}g(v,v) \geqslant g(u,u)g(v,v),$$

and equality holds if and only if $g(x,x) = 0$, i.e. $x = 0$, which means $u = av$.

□

Taking into account Lemma 5, if two timelike vectors $u$ and $v$ belong to the same time cone we have, from Proposition 8,

$$-g(u,v) \geqslant \sqrt{-g(u,u)}\sqrt{-g(v,v)},$$

and therefore, there exists a unique $\theta \in \mathbb{R}$, $\theta \geqslant 0$, such that

$$\cosh\theta = \frac{-g(u,v)}{\sqrt{-g(u,u)}\sqrt{-g(v,v)}},$$

which is called the *hyperbolic angle* between $u$ and $v$.

As a consequence of Proposition 8, taking in mind Lemma 5 we get the so-called *wrong-way Minkowski inequality* [7, Cor. 5.31]

**Corollary 9** *For any $u, v \in \mathcal{T}(V, g)$ in the same time cone we have*

$$\sqrt{-g(u+v,u+v)} \geqslant \sqrt{-g(u,u)} + \sqrt{-g(v,v)}$$

*and equality holds if and only if $u$ and $v$ are linearly dependent.*

Now, we will explain a genuine notion in Lorentzian geometry. Let $(V, g)$ be a Lorentzian vector space and let $B = (v_1, .., v_{n-1}, v_n)$ be a basis of $V$ such that

$$M_B(g) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

i.e. any two vectors of $B$ are $g$-orthogonal, each $v_j$, $1 \leqslant j \leqslant n-1$, is unitary spacelike and $v_n$ is unitary timelike. The basis $B$ is called an *orthonormal basis* of $(V, g)$. We will say that two orthonormal basis $B$ and $B'$ of $(V, g)$ define the same *time orientation* of $(V, g)$ when $g(v_n, v'_n) < 0$, i.e. if and only if $v_n$ and $v'_n$ lie in the same time cone of $(V, g)$.

This clearly defines an equivalence relation on the set of all orthonormal basis of $(V, g)$, which possesses only two equivalence classes (independently of the dimension of V): the class defined by $B = (v_1, .., v_{n-1}, v_n)$ and the class defined by $\tilde{B} = (v_1, .., v_{n-1}, -v_n)$. A *time orientation* on $(V, g)$ is each of these two classes. Equivalently, a time orientation on $(V, g)$ is given by each time cone of $(V, g)$.

It should be noticed that the notion of an orientation on V does not depend on $g$; i.e. it is not a metric concept, whereas the notion of time orientation on $(V, g)$ is defined making use of the Lorentzian product $g$, although a time orientation does not change if we change $g$ to a conformally related Lorentzian product $ag$, $a \in \mathbb{R}$, $a > 0$.

We end this section explaining some topological remarks. If V is an $n$-dimensional vector space and $B$ is a basis of V, then we have a linear isomorphism

$$b_B : V \longrightarrow \mathbb{R}^n,$$

defined by $b_B(v) = (a_1, .., a_n)$, where $(a_1, .., a_n)$ are the coordinates of $v$ in $B$. By using $b_B$ a topology $\mathfrak{T}_B$ can be defined in V in such a way that the $\mathfrak{T}_B$-open subsets of V are $b_B^{-1}(O)$, where O is an open subset of $\mathbb{R}^n$. It is not difficult to see that $\mathfrak{T}_B = \mathfrak{T}_{B'}$ for any basis $B, B'$ of V. Endowed with this topology, the function $V \longrightarrow \mathbb{R}$, given by $v \mapsto g(v, v)$ is continuous. Hence $\mathcal{T}(V, g)$ is an open subset of V and each time cone is also an open subset of V. Moreover, the set of nonzero spacelike vectors $\{v \in V : g(v, v) > 0\}$ is an open subset of V and the light cone with the zero vector $\{v \in V : g(v, v) = 0\}$ is a closed subset of V. Finally, every null vector can be obtained as the limit of a sequence of timelike vectors as well of spacelike vectors (or both types of vectors). Thus, $\{v \in V : g(v, v) < 0\} \cup \{v \in V : g(v, v) > 0\}$ is a dense subset of V.

# 2 INTRODUCTION TO DIFFERENTIAL GEOMETRY

## 2.1 DIFFERENTIABLE MANIFOLDS

Roughly speaking, a manifold is a topological space that, locally, looks like the Euclidean space $\mathbb{R}^n$. This similitude is essential, and will let us control the manifold as if we were working in the Euclidean space; generally, the definitions concerning manifolds and the properties proved from them will be based on the known properties of $\mathbb{R}^n$.

The following definition specifies the formal concept of a topological manifold:

**Definition 1 (N-dimensional topological manifold)** *Let $M^n$ be an $n$-dimensional topological space. The space $M^n$ is called a topological manifold if the following properties are satisfied:*

1. *$M^n$ is locally homeomorphic to $\mathbb{R}^n$.*
2. *$M^n$ is a Hausdorff space.*
3. *$M^n$ has a countable topological basis.*

The first property states that, for every point $p \in M^n$, there exists an open neighbourhood $U \subset M^n$ of $p$ and a homeomorphism

$$h \colon U \to V$$

with $V \subset \mathbb{R}^n$ an open set.

One could think that the Hausdorff property is redundant, as the local homeomorphism may imply this topological characteristic. This is not true, and the usual counterexample is the line with two origins.

Let $M = \mathbb{R} \cup p$ be the union of the real line and a point $p \notin \mathbb{R}$. Define a topology in this space with $\mathbb{R} \subset M$ as an open set and the neighbourhoods of $p$ being the sets $(U \setminus \{0\}) \cup \{p\}$, where $U$ is a neighbourhood of $0 \in \mathbb{R}$. This space is locally Euclidean but not Hausdorff: the intersection of any two neighbourhoods of the points $0 \in \mathbb{R}$ and $p$ is non-empty.

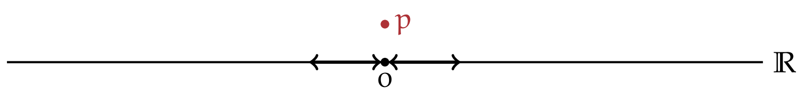



Figure 1: Line with two origins.

The last property of the definition will prove to be key in our study, as it will let us define metrics on the manifold.

### 2.1.1 *Charts*

The main characteristic of the manifolds, its ressemblance to the Euclidean space, have to be exploited in order to understand the nature of the mathematical object.

The conceptual space where the manifolds live can be thought as the Plato's world of Ideas, where everything is pure but cannot be understood without studying particular examples.

The idea of the manifold will be understood, then, taking pieces of the manifold and lowering them to the real word; i. e., the Euclidean space, where we will be able to *physically* touch the manifold.

The essential tool to make this happen will be the coordinate charts. These tools are like prisms to see the manifold from the Euclidean perspective, and they will let us grasp the nature of the ideal concept of a manifold.

**Definition 2 (Coordinate chart)** *A* coordinate chart *—or* coordinate system*— in a topological manifold* $M^n$ *is a homeomorphism* $h\colon U \to V$ *from an open subset of the manifold* $U \subset M$ *onto an open subset of the Euclidean space* $V \subset \mathbb{R}^n$*.*

*We call* $U$ *a* coordinate neighbourhood *in* $M$*.*

One single chart may not cover the whole manifold. In order to completely understand it, we need a set of charts that describe it completely.

**Definition 3 (Coordinate atlas)** *Let*

$$A = \{h_\alpha\colon U_\alpha \to V_\alpha / \alpha \in I\}$$

*be a set of coordinate charts in a topological manifold* $M^n$*, where* $I$ *is a family of indices and the open subsets* $U_\alpha \subset M$ *are the corresponding coordinate neighbourhoods.*

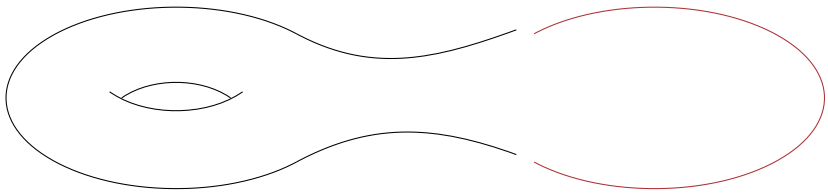
$A$ *is said to be an* atlas *of M if every point is covered with a coordinate neighbourhood; i. e., if* $\cup_{\alpha \in I} U_\alpha = M$*.*

### 2.1.2 *Differentiable structures*

The concept of manifold is quite general and includes a vast set of examples. We can impose, however, some properties on the smoothness of the manifold to restrict the objects we will work with.

This section introduces the concept of differentiable structure, whose definition is key in the later description of differentiable manifolds, the core concept of this chapter.

The first question in this study is the following: a chart describe perfectly a single piece of the manifold, but what happens when the domains of a pair of charts overlap? The following two definitions precise the concepts involved in this question.



**Definition 4 (Transition map)** *Let* $M^n$ *be a manifold and* $(U, \phi)$, $(V, \psi)$ *a pair of coordinate charts in* $M^n$ *with overlapping domains, that is:*

$$U \cap V \neq \emptyset$$

*The homeomorphism between the open sets of the Euclidean space* $\mathbb{R}^n$,

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V),$$

*is called a* transition map.

**Definition 5 (Smooth overlap)** *Two charts* $(U, \phi)$, $(V, \psi)$ *are said to overlap smoothly if their domains are disjoint —i.e., if* $U \cap V = \emptyset$*— or if the transition map* $\psi \circ \phi^{-1}$ *is a diffeomorphism.*

The description of two charts that overlap smoothly can be naturally extended to the concept of smooth atlas, that will make possible to do calculus on the manifold.

**Definition 6 (Smooth coordinate atlas)** *An atlas* $A$ *is said to be smooth if every pair of charts in* $A$ *overlap smoothly.*

But what happens if we define two different atlases in the manifold? Will the calculus depend on this choice? Fortunately we can find, for each manifold, one particular atlas that contain every other atlas defined there. It is formally described in the following definition and its uniqueness is proved in Proposition 8.

**Definition 7 (Complete atlas)** *A* complete atlas *—or* maximal atlas*— on* $M^n$ *is a smooth atlas that contains each coordinate chart in* $M^n$ *that overlaps smoothly with every coordinate chart in* $M^n$.

**Proposition 8 (Complete atlas uniqueness)** *Let* $M$ *be a topological manifold.*

- *Every smooth atlas on* $M$ *is contained in a complete atlas.*
- *Two smooth atlas on* $M$ *determine the same complete atlas if and only if its union is a smooth atlas.*

**Proof.**

■

**Definition 9 (Differentiable structure)**

**Definition 10 (Differentiable manifold)**

# 3 THE NOTION OF SPACETIME

In this section we will follow [9]

A *Lorentzian metric* on an $n(\geqslant 2)$-dimensional manifold $M$[1] is a symmetric 2-covariant tensor field $g$ such that

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

is a Lorentzian product for all $p \in M$. A *Lorentzian manifold* is a pair $(M, g)$ consisting of an $n(\geqslant 2)$-dimensional manifold $M$ and a Lorentzian metric $g$ on $M$.

It should be noticed that if a manifold $M$ admits a symmetric 2-covariant tensor field $g$ such that $g_p$ is non-degenerate for all $p \in M$, then $g$ has a Levi-Civita connection $\nabla$. This assertion follows from the classical Koszul formula (see Section 5) which defines $\nabla$ just from the non-degeneracy property (equivalently, note also that the Christofell symbols $\Gamma^k_{ij}$ may be defined using only the non-degeneration property). Therefore, from the connectedness of $M$, for each two points $p_0, p_1 \in M$ there exists a piece-wise smooth curve

$$\gamma : [a, b] \longrightarrow M,$$

$a, b \in \mathbb{R}$, $a < b$ such that $\gamma(a) = p_0$ and $\gamma(b) = p_1$. Therefore, we have the corresponding parallel transport

which is a linear isometry between $(T_{p_0}M, g_{p_0})$ and $(T_{p_1}M, g_{p_1})$.

Consequently, index$(g_{p_0})$ =index$(g_{p_1})$, and we may speak of the *index of* $g$.

A non-degenerate symmetric 2-covariant tensor field $g$ is called a *semi-Riemannian metric*, thus, $g$ is Riemannian if its index is zero and Lorentzian if its index is 1 and $\dim M \geqslant 2$.

A semi-Riemannian metric of index $s$ such that $0 < s < \dim M$ is said to be *indefinite*.

Thus, a *semi-Riemannian* (resp. *Riemannian*, *indefinite Riemannian*) *manifold* is a pair $(M, g)$, where $g$ is a semi-Riemannian (resp. Riemannian, indefinite Riemannian) metric.

Now we will explain the concept of time orientation of a Lorentzian manifold.

Let $(M, g)$ be a Lorentzian manifold and denote by $C_p(M, g)$ the set consisting of the two time cones of $(T_pM, g_p)$, $p \in M$. Put

1 Unless otherwise is specified, a manifold will be assumed to be of class $C^\infty$, connected and with a countable basis in its topology.

A *time orientation* on $(M, g)$ is a map

such that $\tau(p) \in C_p(M, g)$, i.e. $\tau(p)$ is a time cone of $(T_pM, g_p)$, and such that for each $p_0 \in M$ there exist an open neighborhood $U$ of $p_0$ and $X \in \mathfrak{X}(U)$ which satisfies

If a Lorentzian manifold $(M, g)$ admits a time orientation, it is called *time orientable*. A time orientable Lorentzian manifold $(M, g)$ produces two *time oriented Lorentzian manifolds* $(M, g, \tau)$ and $(M, g, \tau')$, where $\tau'(p)$ is the opposite cone of $\tau(p)$ in $(T_pM, g_p)$. A 4-dimensional time oriented Lorentzian manifold is called a *spacetime*.

The following result characterizes the existence of a time orientation [7, Lemma 5.32]

**Proposition 10** *A Lorentzian manifold* $(M, g)$ *is time orientable if and only if there exists* $Y \in \mathfrak{X}(M)$ *such that* $g(Y, Y) < 0$.

*Proof.* If such a vector field exists, then we can choose $\tau(p)$ as the time cone of $(T_pM, g_p)$ such that $Y_p \in \tau(p)$ for all $p \in M$. Conversely, let $\tau$ be

a time orientation on $(M, g)$. For each $p_0 \in M$ there exist a neighborhood $U^{p_0}$

and $X_U \in \mathfrak{X}(U^{p_0})$ such that $(X_U)_p \in \tau(p)$ for all $p \in U^{p_0}$.

Let $\{f_\alpha\}$ be a smooth partition of unity subordinate to the open covering $\{U^p : p \in M\}$,

i.e. $\{supp(f_{\alpha)}\}$ is locally finite, $f_\alpha \geqslant 0$, $\sum_\alpha f_\alpha = 1$

and $supp(f_\alpha) \subset U_\alpha$ for some $U_\alpha$ of the covering of $M$.

The vector field

$$Y := \sum f_\alpha X_{U_\alpha}$$

is then well-defined and for each $p \in M$

there exists an open neighborhood $V(p)$ such that $V \cap supp(f_\alpha) = \emptyset$ for all

$\alpha \neq i_1, .., i_k$; therefore

$$Y_{|V} = f_{i_1} X_{U_{i_1}} + .. + f_{i_k} X_{U_{i_k}},$$

with $\sum_j f_{i_j} = 1$.

Then, using the convexity of time cones, Corollary 6, the vector field $Y$ is timelike everywhere.

□

**Example 11** (1) Let $Ł^n$ be the $n$-dimensional Lorentz-Minkowski space, i.e. $Ł^n$ is $\mathbb{R}^n$

endowed with the Lorentzian metric $g = dx_1^2 + ... + dx_{n-1}^2 - dx_n^2$, where $(x_1, .., x_n)$ is the

usual coordinate system of $\mathbb{R}^n$. The coordinate vector field $\partial/\partial x_n$ is unitary timelike

and hence, Proposition 10, $Ł^n$ is time orientable.

(2) Let $\S_1^n$ be the $n$-dimensional De Sitter space; i.e. $\S_1^n = \{p \in Ł^{n+1} : g(p,p) = 1\}$,

where $g$ denotes the Lorentzian metric of $Ł^{n+1}$. For each $p \in \S_1^n$, we have $T_p\S_1^n = \{v \in Ł^{n+1} : g(p,v) = 0\}$

and denote by $g_p$ the restriction of $g$ to $T_p\S_1^n$, which is Lorentzian because $Ł^{n+1} = T_p\S_1^n \oplus \mathrm{Span}\{p\}$,

the direct sum is also $g$-orthogonal and $p$ is spacelike. Observe that a vector field on $\S_1^n$ can be contemplated as a smooth

map

$$X : \S_1^n \longrightarrow Ł^{n+1}$$

such that at each point $p \in \S_1^n$ we have $X_p$ is $g$-orthogonal to $p$. Thus, if we put

$p = (y,t) \in \S_1^n$, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$, then $X_p = (\frac{t}{1+t^2} y, 1)$ is a well-defined timelike vector field on $\S_1^n$. Therefore, Proposition 10, the Lorentzian manifold $\S_1^n$ is time orientable.

(3) Let

The following result gives a geometric characterization of time orientability **[S-W]**

**Corollary 12** *A Lorentzian manifold $(M,g)$ is time orientable if and only if for any piece-wise smooth curve $\gamma : [a,b] \longrightarrow M$ such that $\gamma(a) = \gamma(b) = p$, we have*

$$g(P^{\gamma}_{a,b}(v), v) < 0$$

*for all $v \in \mathcal{T}(T_pM, g_p)$, for all $p \in M$.*

*Proof.* Assume $(M,g)$ is time orientable and consider $X \in \mathfrak{X}(M)$ such that $g(X,X) < 0$. Changing $X$ to $-X$, if necessary, we may assume $g(X_p, v) < 0$. Let $Y$ be a vector field along

$\gamma$ such that $\frac{DY}{dt} = 0$ and $Y(a) = v$.

Note that we have $Y(b) = P^{\gamma}_{a,b}(v)$. Consider the function $f : [a,b] \longrightarrow \mathbb{R}$ given by

which is continuous and never vanishes because $X_{\gamma(t)}$ and $Y(t)(= P^{\gamma}_{a,t}(v))$ are timelike. Therefore $f(t) < 0$ for all $t \in [a,b]$ and, in particular, $f(b) < 0$. This means, taking into account Lemma 5, that $Y(b)$ and $X_{\gamma(b)}$ lie in the same time cone.

Conversely, let us consider, for two arbitrary points $p$ and $q$ of $M$, two piece-wise smooth curves $\alpha$ and $\beta$ from $p$ to $q$. We want to show that for any $v \in \mathcal{T}(T_pM, g_p)$ the parallel transported vectors $P^{\alpha}(v)$, $P^{\beta}(v)$ lie in the same time cone of $(T_qM, g_q)$.

In order to achieve this conclusion we construct a piece-wise smooth curve $\gamma : [a, b] \longrightarrow M$ from $\alpha$ and $\beta$ in a standard way such that $\gamma(a) = \gamma(b) = p$. Note that

$$g(P^{\alpha}(v), P^{\beta}(v)) = g((P^{\beta})^{-1}P^{\alpha}(v), v) = g(P^{\gamma}_{a,b}(v), v) < 0,$$

which means that $P^{\alpha}(v)$ and $P^{\beta}(v)$ lie in the same time cone of $(T_qM, g_q)$.

Therefore, we have a well-defined way to chose a time cone $\tau(p)$ at any $p \in M$. Finally, we will to show the smoothness. Given $p_0 \in M$ and the time cone $\tau(p_0)$ consider $v \in \tau(p_0)$. Let $X$ be a vector field which extends $v$; i.e. such that $X_{p_0} = v$. Note that $X$ remains timelike in some (connected) open neighborhood $U$ of $p_0$. For each $q \in U$ we construct a piece-wise smooth curve $\alpha : [a, b] \longrightarrow U$ satisfying $\alpha(a) = p_0$, $\alpha(b) = q$ and consider the function $h : [a, b] \longrightarrow \mathbb{R}$ given by

which is continuous and never vanishes. Therefore $h(t) < 0$ for all $t \in [a, b]$ and, in particular, $h(b) < 0$. This means, taking into account Lemma 5, that $X_q$ and $P^{\alpha}_{a,b}(v)$ lie in the same time cone of $(T_qM, g_q)$ and thus $X_q \in \tau(q)$ for all $q \in U$.

$\square$

Now assume each closed piece-wise smooth curve is null homotopic by means of a piecewise smooth homotopy.

In this case, Corollary 12 says that $(M, g)$ must be time orientable. But it is known that this fact holds true whenever $M$ is assumed to be simply connected. Therefore, we have

**Corollary 13** *If $M$ is simply connected and $g$ is a Lorentzian metric on $M$, then the Lorentzian manifold $(M, g)$ must be time orientable.*

A well-known non time orientable Lorentzian manifold is the following Lorentzian cylinder **[S-W]**

**Example 14** Let $g$ be the Lorentzian metric on $\mathbb{R}^2$ given by

$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)_{(x,y)} = -g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)_{(x,y)} = \cos 2y, \quad g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)_{(x,y)} = \sin 2y.$$

Observe that

$$\det\begin{pmatrix} \cos 2y & \sin 2y \\ \sin 2y & -\cos 2y \end{pmatrix} = -1 < 0$$

everywhere, which implies that $g$ is Lorentzian. The map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, defined by $f(x,y) = (x, y+\pi)$,

is clearly an isometry of $(\mathbb{R}^2, g)$. Put $M := \mathbb{R}^2/$, where the action of on $\mathbb{R}^2$ is defined via $f$ as follows

$$\big(m, (x,y)\big) \mapsto f^m(x,y) = (x, y + m\pi).$$

Then $M$ is a cylinder and the metric $g$ may be induced to a Lorentzian metric $\tilde{g}$ in $M$. We want to show that $(M, \tilde{g})$ is not time orientable. If we choose a time cone at $(0,0)$ then along the axis $x = 0$ it changes its position in the counterclockwise rotation sense. Note that $(0,0)$ and $(0,\pi)$ represent the same point of $M$ but the time cones at these points are not compatible with the equivalence relation in $\mathbb{R}^2$ induced by $f$. Note that $Y = -\sin y \frac{\partial}{\partial x} + \cos y \frac{\partial}{\partial y}$ is a timelike vector field on $(\mathbb{R}^2, g)$ (of course, $(\mathbb{R}^2, g)$ is time orientable from Corollary 13) which satisfies $Y_{(0,0)} = \frac{\partial}{\partial y}\mid_{(0,0)}$ and $Y_{(0,\pi)} = -\frac{\partial}{\partial y}\mid_{(0,\pi)}$.

Taking into account that $df_{(0,0)}Y_{(0,0)} = -Y_{(0,\pi)}$, $Y$ cannot be induced on $M$. On the other hand, assume there exists $\tilde{X} \in \mathfrak{X}(M)$ such that $\tilde{g}(\tilde{X}, \tilde{X}) < 0$ and let $X \in \mathfrak{X}(\mathbb{R}^2)$, $g(X,X) < 0$, which projects onto $\tilde{X}$. Necessarily $df_{(x,y)}X_{(x,y)} = X_{(x,y+\pi)}$ and $g(Y_{(x,y)}, X_{(x,y)}) \neq 0$ for all $(x,y) \in \mathbb{R}^2$.

Therefore, either $g(Y,X) > 0$ or $g(Y,X) < 0$ everywhere. But this is incompatible with

$$g\left(Y_{(0,\pi)}, X_{(0,\pi)}\right) = -g\left(df_{(0,0)}Y_{(0,0)}, df_{(0,0)}X_{(0,0)}\right) =$$

$$= -g\left(Y_{(0,0)}, X_{(0,0)}\right).$$

Previous example shows a (connected) orientable manifold $M$ which admits a Lorentzian metric $\tilde{g}$

such that $(M, \tilde{g})$ is not time orientable. It is possible to have a time orientable Lorentzian

manifold $(N, g)$ where $N$ is not (topologically) orientable. Even more, it is also easy to construct

a non time orientable Lorentzian manifold $(P, g')$ such that $P$ is not (topologically) orientable.

As in the non orientable case, a Lorentzian manifold $(M, g)$ which is not time orientable admits a double

Lorentzian covering manifold $(\hat{M}, \hat{g})$ which is time orientable. Note that

$(\hat{M}, \hat{g})$ and $(M, g)$ have the same local geometry, but the first one possesses a

globally defined timelike vector field and the second one does not.

It is classical that, by using a partition of the unity on a (paracompact) manifold $M$, we can always construct a Riemannian metric on $M$. But, the same procedure does not work in the Lorentzian case. In fact, although we can consider a Lorentzian metric on each coordinate open subset of $M$, it may be not possible to glue the locally defined Lorentzian metrics, as in the Riemannian case, to produce a Lorentzian metric defined on the whole manifold $M$. Therefore, it is natural to ask when a manifold admits a Lorentzian metric.

The answer is the well-known result [14]

**Proposition 15** *An* $n(\geqslant 2)$*-dimensional manifold* $M$ *admits a Lorentzian metric if and only if it admits a* $1$*-dimensional distribution.*

*Proof*. First consider a Lorentzian metric $g$ on $M$, and let $g_R$ be an arbitrarily chosen Riemannian metric on $M$.

A $(1,1)$-tensor field $P$ on $M$ can be defined by setting, for each $u \in T_P M$, $P(u)$ the unique vector of $T_p M$ such that

for all $v \in T_p M$, $p \in M$. Clearly, $P$ is $g_R$-selfadjoint and, therefore, at any point $p \in M$, there exists a $g_R$-orthonormal basis of $T_p M$ consisting of eigenvectors of $P$. Observe that none of the eigenvalues is zero, $n-1$ are positive and one is negative. Put $\mathcal{D}_p$ the eigenspace associated to the negative eigenvalue of $P$ at $p$, then $\mathfrak{D}$ defines a 1-dimensional distribution (or line field) on $M$. It should be noted that $\mathfrak{D}$ clearly depends on the arbitrary Riemannian metric $g_R$.

Conversely, if a 1-dimensional distribution $\mathfrak{D}$ on $M$ is given, fix an arbitrary Riemannian metric $g_R$ on $M$. We know that there exist an open covering $\{U_\alpha\}$ of $M$ and vector fields $X_\alpha \in \mathfrak{X}(U_\alpha)$ such that, locally,

By putting

$$g_L(u,v) := g_R(u,v) - 2\, g_R\big(u, X_\alpha(p)\big)\, g_R\big(v, X_\alpha(p)\big),$$

for any tangent vectors $u, v \in T_p M$ with $p \in U_\alpha$, it is easily seen that $g_L$ does not depend on $\alpha$ and therefore, it is a Lorentzian metric on all $M$.

$\square$

**Remark 16** Instead of a 1-dimensional distribution if we have $X \in \mathfrak{X}(M)$ such that $X_p \neq 0$ for all $p \in M$, we can construct a Lorentzian metric $g_L$, starting from a Riemannian metric $g_R$ on $M$, as follows

$$g_L(u,v) := g_R(u,v) - 2\,\frac{g_R(u,X_p)g_R(v,X_p)}{g_R(X_p,X_p)}$$

where $u, v \in T_p M$, $p \in M$.

Proposition 15 can be generalized to obtain [14]

**Proposition 17** *An $n$-dimensional manifold $M$*
*admits an indefinite Riemannian metric of index $s$, $0 < s < n$, if and only if it admits a $s$-dimensional distribution.*

As an application, any parallelizable manifold (in particular any Lie group) admits an indefinite Riemannian metric of any index.

**Remark 18** In Propositions 15, 17 the usual notion of manifold which assume the existence of a countable basis in its topology has been considered.

In the more general terminology of [5] i.e. without the assumption of having a countable basis in its topology, it can be shown [6] that if a manifold admits a Lorentzian metric, then it must be paracompact.

More generally, the same conclusion holds [13, Cor. 25] if one assumes the existence of an affine connection (therefore, paracompactness is also derived from the assumption of the existence of an indefinite Riemannian metric.

Any non-compact manifold admits a non-vanishing vector field. In fact, it can be taken as the gradient, with respect to any Riemannian metric, of a smooth function with no critical points.

Thus any $n(\geqslant 2)$-dimensional non-compact manifold admits a Lorentzian metric.

On the other hand, an $n(\geqslant 2)$-dimensional compact manifold $M$ admits a 1-dimensional distribution if and only if its Euler-Poincaré characteristic $\chi(M)$ is zero. Therefore, any $(2n+1)$-dimensional compact orientable manifold admits a Lorentzian metric.

The existence of a 1-dimensional distribution on a manifold is closely related to the existence of a non-vanishing vector field. In fact, it is a standard topological result that

**Proposition 19** *An $n(\geqslant 2)$-dimensional compact manifold $M$ admits a non-vanishing vector field if and only if $\chi(M) = 0$.*

On a simply connected manifold (compact or not), every 1-dimensional distribution on $M$ arises from a global non-vanishing vector field $X \in \mathfrak{X}(M)$. However, a 1-dimensional distribution cannot be lifted in general to a global non-vanishing vector field as the following example shows [14].

Consider the special orthogonal group of order 3, $SO(3)$, and put $M = \mathbb{S}^1 \times SO(3)$.

$M$ is a 4-dimensional compact manifold. Moreover it is parallelizable, and therefore every vector field $X \in \mathfrak{X}(M)$ can be contemplated as a smooth map

$$X : M \to \mathbb{R}^4$$

and, by fixing a diffeomorphism $\psi : \mathbb{R}P^3 \to SO(3)$, a 1-dimensional distribution $\mathfrak{D}$ can be seen as a smooth map

$$\mathfrak{D} : M \to SO(3).$$

In particular, the canonical projection on the second factor $\mathfrak{D}_2$ defines a natural 1-dimensional distribution on $M = \S^1 \times SO(3)$. If we assume that $\mathfrak{D}_2$ lifts to a vector field $X$ without any zero, then, taking into accountthat $\mathbb{R}^4 - \{0\}$ is simply connected, one easily shows that $SO(3)$ would be also simply connected, which is not true. Hence $\mathfrak{D}_2$ cannot be lifted to a global vector field on $\S^1 \times SO(3)$.

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