# Part I

# INTRODUCTION

# 1 INTRODUCTION TO DIFFERENTIAL GEOMETRY

## 1.1 DIFFERENTIABLE MANIFOLDS

Roughly speaking, a manifold is a topological space that, locally, looks like the Euclidean space $\mathbb{R}^n$. This similitude is essential, and will let us control the manifold as if we were working in the Euclidean space; generally, its properties will be proved using the known properties of $\mathbb{R}^n$.

The following definition specifies the formal concept of a topological manifold:

**Definition 1 (N-dimensional topological manifold)** *Let $M^n$ be an $n$-dimensional topological space. The space $M^n$ is called a topological manifold if the following properties are satisfied:*

1. *$M^n$ is locally homeomorphic to $\mathbb{R}^n$.*
2. *$M^n$ is a Hausdorff space.*
3. *$M^n$ has a countable topological basis.*

The first property states that, for every point $p \in M^n$, there exists an open neighbourhood $U \subset M^n$ of $p$ and a homeomorphism

$$h\colon U \to V$$

with $V \subset \mathbb{R}^n$ an open set.

The Hausdorff property has to be added, as the local homeomorphism does not imply this topological characteristic. The usual counterexample is the line with two origins: let $M = \mathbb{R} \cup p$ be the union of the real line and a point $p \notin \mathbb{R}$. Define a topology in this space with $\mathbb{R} \subset M$ as an open set and the neighbourhoods of $p$ being the sets $(U \setminus \{0\}) \cup \{p\}$, where $U$ is a neighbourhood of $0 \in \mathbb{R}$. This space is locally Euclidean but not Hausdorff: the intersection of any two neighbourhoods of the points $0 \in \mathbb{R}$ and $p$ is non-empty.

The last property will prove to be key in our study, as it will let us define metrics on the manifold.

### 1.1.1 *Charts*

**Definition 2 (Coordinate chart)** *A* coordinate chart *—or* coordinate system*— in a topological manifold $M^n$ is a homeomorphism $h\colon U \to V$ from an open subset of the manifold $U \subset M$ onto an open subset of the Euclidean space $V \subset \mathbb{R}^n$.*

*We call $U$ a* coordinate neighbourhood *in $M$.*

**Definition 3 (Coordinate atlas)** *Let*

$$A = \{h_\alpha \colon U_\alpha \to V_\alpha / \alpha \in I\}$$

*be a set of coordinate charts in a topological manifold* $M^n$*, where* $I$ *is a family of indices and the open subsets* $U_\alpha \subset M$ *are the corresponding coordinate neighbourhoods.*

$A$ *is said to be an* atlas *of M if every point is covered with a coordinate neighbourhood; i.e., if* $\cup_{\alpha \in I} U_\alpha = M$*.*

### 1.1.2 *Differentiable structures*

**Definition 4 (Transition map)** *Let* $M^n$ *be a manifold and* $(U, \phi)$*,* $(V, \psi)$ *a pair of coordinate charts in* $M^n$ *with overlapping domains*

$$U \cap V \neq \emptyset$$

*The homeomorphism between the open sets of the Euclidean space* $\mathbb{R}^n$

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$$

*is called a* transition map*.*

*We say that both charts* overlap smoothly *if* $\psi \circ \phi^{-1}$ *is a homeomorphism or the This transition map is said to be* smooth *if the sets are*

**Definition 5 (Smooth overlap)** *Two charts* $(U, \phi)$*,* $(V, \psi)$ *are said to overlap smoothly if their domains are disjoint —i.e., if* $U \cap V = \emptyset$*— or if the transition map* $\psi \circ \phi^{-1}$ *is a diffeomorphism.*

**Definition 6 (Smooth coordinate atlas)**

**Definition 7 (Maximal atlas)**

**Proposition 8 (Maximal atlas uniqueness)**

**Definition 9 (Differentiable structure)**

**Definition 10 (Differentiable manifold)**