Instituto de Física Teórica UAM-CSIC

Universidad Autónoma de Madrid

Faculty of Sciences

Master in Theoretical Physics

Master's Thesis

Higher Derivative Gravity and Holographic QCD

Author: Alejandro García Soria

Supervised by:

Juan. F. Pedraza and Hyun-Sik Jeong

September of 2025

Acknowledgments

This work was supported by a JAE Intro Scholarship from the Spanish National Research Council (CSIC), in particular within the 2024 edition through application JEINT_24_01938 and the training plan JAEINT24_EX_1112.

The author and supervisors would also like thank Matteo Baggioli and Ángel J. Murcia for their valuable discussions and kind availability for correspondence.

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Chapter 1

Introduction

The AdS/CFT correspondence has become one of the most powerful tools to study strongly coupled quantum field theories. By relating a gravity theory in an asymptotically AdS spacetime to a conformal field theory on the boundary, it provides a way to compute physical quantities that would otherwise be out of reach using standard perturbative techniques. Among the most relevant for the physics of strongly coupled field theories are the ratio of shear viscosity to entropy density, η/s, and the speed of sound, c_s. These transport and thermodynamic observables play a central role in the description of QCD-like matter.

In the simplest holographic models, η/s takes the universal value $\eta/s = 1/4\pi$, often interpreted as a lower bound for physical systems and known as the KSS bound. Likewise, in conformal plasmas such as $\mathcal{N} = 4$ SYM, the speed of sound is fixed to its conformal value $c_s^2 = 1/3$. However, real QCD deviates from this simple picture: experimental and lattice results suggest values of η/s close to, and possibly below, $1/4\pi$, as well as a temperature–dependent c_s with a characteristic dip near the deconfinement transition. Capturing these features requires moving beyond Einstein gravity to include higher–derivative terms and scalar couplings in holographic models.

The aim of this thesis is to explore how higher–derivative corrections modify the hydrodynamic and thermodynamic behavior of strongly coupled systems in the holographic framework. After reviewing the basics of the AdS/CFT dictionary and hydrodynamic response theory, we revisit the Gauss Bonnet model, where violations of the KSS bound can be computed explicitly.

The main original contribution of this work is found in Chapter 5, where we extend the analysis to a more general class of higher–curvature gravity theories with a non–minimally coupled scalar field. These dilatonic quasi–topological gravities allow for richer dynamics and lead to new formulas for η/s in terms of the dilaton profile and curvature invariants, pushing beyond the quadratic order considered in much of the literature. In parallel, we study the equation of state and compute the speed of sound c_s in these models. We show how Gauss Bonnet dilaton backgrounds can reproduce both the dip in $c_s(T)$ observed in lattice QCD and a nontrivial, temperature–dependent $\eta/s(T)$, thereby improving the phenomenological match to QCD matter compared to pure Einstein gravity.

Ultimately, this thesis provides a holographic model that illustrates the richness of the holographic landscape once higher-derivative terms are included.

The thesis is structured as follows: we begin with a review of the holographic setup (Chapter 2), then develop the dictionary and tools needed for computations (Chapter 3), apply them to hydrodynamics and transport coefficients (Chapter 4), and finally turn to higher–derivative corrections and the novel results (Chapter 5). Chapter 6 concludes with a discussion of implications and future directions.

Chapter 2

A (Nearly) String-less Introduction to AdS-CFT

This chapter aims to present a concise, yet useful, introduction to the AdS/CFT correspondence without delving too deeply into its foundations in string theory or supergravity. This approach is taken because we will be using this duality primarily as a tool; rather than focusing on the duality itself, we will apply it to problems in the field of hydrodynamics. Readers interested in exploring this fascinating topic more deeply can refer to excellent sources, but the main references underlying our understanding have been the book by Matteo Baggioli [1], and especially the lecture notes by Alfonso V. Ramallo [2]. We will start with a basic introduction to the central idea and then outline the necessary concepts needed to finally understand the duality as originally proposed by Maldacena in [3].

2.1 Introduction to the Correspondence and Holography

The AdS/CFT correspondence, proposed by Maldacena in 1997 [3], represents one of the most profound insights in theoretical physics. It provides a powerful duality between a gravity theory in a $(d+1)$-dimensional Anti-de Sitter (AdS) space and a conformal field theory (CFT) living on its d-dimensional boundary. This holographic principle has revolutionized our understanding of gauge theories, black hole physics, and strongly coupled systems.

In particular, perturbative methods in quantum field theory (QFT), which rely on small coupling expansions, often fail when interactions become strong. This limitation becomes evident in various physical systems, such as the quark-gluon plasma, where perturbative Quantum Chromodynamics (QCD) provides inaccurate results. An example of this breakdown occurs in the computation of the viscosity-to-entropy density ratio (η/s), where perturbative QCD fails to produce consistent results. This highlights the necessity of alternative methods, such as holography, to study strongly coupled systems. The AdS/CFT correspondence provides precisely that: a way to map strongly coupled QFT dynamics onto classical gravitational physics in AdS space.

The holographic principle is a profound idea in theoretical physics that asserts the information contained within a given volume of space can be entirely described by degrees of freedom residing on its boundary. This concept was first proposed by Gerard 't Hooft [4] and later refined by Leonard Susskind [5], who recognized its deep implications for black hole physics and quantum gravity. Holography thus provides a novel perspective on quantum gravity, suggesting that the fundamental degrees of freedom of spacetime itself may be encoded in a lower-dimensional field theory.

The principle finds strong motivation in black hole thermodynamics. The entropy of a black hole, given by the Bekenstein-Hawking formula [6],

$$S_{\mathrm{BH}} = \frac{\mathcal{A}}{4G_N}, \tag{2.1}$$

where $\mathcal{A}$ is the area of the event horizon and G_N is Newton's constant, scales with the area rather than the volume of the black hole. This suggests that the number of fundamental degrees of freedom in a gravitational system does not scale with its volume, as in conventional field theories, but instead with the boundary enclosing the system. The maximal entropy that can be contained within a region of space is thus given by the entropy of the largest black hole that fits within that region,

$$S_{\mathrm{max}} = S_{\mathrm{BH}}. \tag{2.2}$$

This realization led to the conjecture that a complete quantum theory of gravity should be formulated in terms of degrees of freedom living on a lower-dimensional boundary. One of the most concrete realizations of this principle is the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [3].

2.2 AdS/CFT Without Strings

As we have anticipated, the AdS/CFT correspondence states that a gravitational theory in a $(d+1)$-dimensional Anti-de Sitter (AdS) spacetime is dual to a d-dimensional conformal field theory (CFT) living on its boundary. This provides a precise and calculable realization of the holographic principle, where the bulk gravitational dynamics encode the physics of a lower-dimensional quantum field theory.

This duality is particularly powerful because it maps strongly coupled regimes of the CFT to weakly coupled gravity descriptions in AdS, allowing difficult quantum field theory problems to be studied using classical gravity. The correspondence is formalized through a dictionary that relates bulk quantities in AdS to observables in the boundary CFT. More precisely, as stated in [2], the correspondence relates the quantum physics of strongly correlated many-body systems to the classical dynamics of gravity in one higher dimension. In its original formulation [3], it related a four-dimensional CFT to the geometry of an AdS space in five dimensions.

Let us motivate the duality by considering the renormalization group (RG) flow in quantum field theories. Observables depend on the energy scale at which they are probed, since quantum fields behave differently at different energy scales. One can study the dynamics of the couplings of the theory under a change in the energy scale μ by considering the *beta function* equation,

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta_g(g(\mu)), \tag{2.3}$$

where $g(\mu)$ is the coupling constant of the theory and $\beta_g(g(\mu))$ is the beta function, which describes how g evolves with the energy scale μ. At weak coupling, the beta function β_g can be determined perturbatively. At strong coupling, the AdS/CFT correspondence suggests interpreting the energy scale μ as an extra dimension. In this new framework, the state of the QFT at a given energy scale μ corresponds to the state of a gravitational theory on a "slice" of spacetime at a particular value of the extra dimension. In this framework, the couplings g are identified with

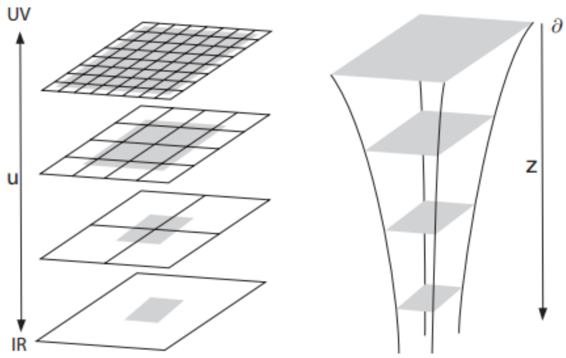

Figure 2.1: Pictorical representation of how the RG flow defines an extra dimension.The layers at different scale of the QFT (left figure) are considered as the layers of a higher dimensional space represented on the right figure. Image retrieved from [2].

the boundary values of corresponding fields ϕ in the higher-dimensional bulk. The dynamics of these fields, in the AdS/CFT paradigm, would then be determined by a gravitational theory. Moving from the boundary to the interior of AdS space then corresponds to moving from UV to IR in the field theory.

2.2.1 Degrees of Freedom qualitative

From this duality, several important questions arise about its mathematical rigor and physical relevance. One of these concerns is the fate of the degrees of freedom in both theories—the CFT and the gravity theory. For the duality to be well defined, the degrees of freedom on both sides should match.

Let us first consider the quantum field theoretic side. As is well known, the degrees of freedom are measured by the entropy. Following [2], since entropy is an extensive quantity, for a $(d-1)$-dimensional spatial region R_{d-1} at fixed time, its entropy should be proportional to its volume,

$$S_{\text{QFT}} \propto \text{Vol}(R_{d-1}). \tag{2.4}$$

On the gravitational side, according to the holographic principle, the maximum entropy of a given d-dimensional volume is determined by the entropy of the largest black hole that fits within that volume, which is proportional to its surface area, and given by the Bekenstein-Hawking formula,

$$S_{\text{BH}} = \frac{\mathcal{A}}{4G_N}, \tag{2.5}$$

This scaling with area ($d-1$-dimensional surface), rather than volume, indicates a reduction in the effective degrees of freedom, in agreement with the behavior of the QFT.

2.2.2 Degrees of Freedom quantitative

We could go further and see the explicit matching of the degrees of freedom between a CFT and the AdS geometry. Let us count the degrees of freedom in such a geometry and refine the discussion about the matching of degrees of freedom between both sides of the duality.

Let us begin with the $d-1$-dimensional QFT. For convenience, we introduce an IR and a UV cutoff, using a lattice spacing ϵ and a finite spatial box of size R (i.e., making the system finite). The number of cells in this box is $\left(\frac{R}{\epsilon}\right)^{d-1}$. If the number of degrees of freedom in each lattice cell is[1] c_{QFT}, then the total number of degrees of freedom in the QFT is

$$N_{\text{dof}}^{\text{QFT}} = \left(\frac{R}{\epsilon}\right)^{d-1} c_{\text{QFT}}. \tag{2.6}$$

For a SU(N) gauge theory (where we assume the fields are $N \times N$ matrices in the adjoint representation), the counting in the large N limit yields $c_{\text{SU}(N)} \sim N^2$.

On the other hand, for the gravitational theory in d-dimensions, we have

$$N_{\text{dof}}^{\text{AdS}} = \frac{A_\partial}{4G_N}, \tag{2.7}$$

where A_∂ is the area of the region at the boundary set at $u \to 0$. Assuming the AdS metric, that

[1] This quantity is often called the *central charge* [7].

area turns out to be

$$A_\partial = \left(\frac{L}{\epsilon}\right)^{d-1} \int_{\mathbb{R}^{d-1}} d^{d-1}x, \tag{2.8}$$

where we have integrated the volume element corresponding to the AdS metric (2.16) at a slice $u = \epsilon \to 0$. Introducing an IR cutoff R to make the integral finite, we get

$$A_\partial = \left(\frac{RL}{\epsilon}\right)^{d-1}. \tag{2.9}$$

Restoring now the Planck length, $G_N = (l_p)^{d-1}$, the number of degrees of freedom in the AdS spacetime is

$$N_{\text{dof}}^{\text{AdS}} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_p}\right)^{d-1}. \tag{2.10}$$

We then see that $N_{\text{dof}}^{\text{QFT}}$ and $N_{\text{dof}}^{\text{AdS}}$ scale in the same way with respect to the IR and UV cutoffs, allowing us to identify $\frac{1}{4}\left(\frac{L}{l_p}\right)^{d-1} \sim c_{\text{QFT}}$. The limit of classical gravity[2] implies that

$$\left(\frac{L}{l_p}\right)^{d-1} \gg 1, \tag{2.11}$$

so one concludes that a CFT has a classical gravity description whenever $c_{\text{QFT}} \gg 1$, i.e., in the large N limit.

2.2.3 The Anti-de Sitter Space

We have been using the AdS geometry as the dual for a CFT, but why is exactly this geometry the dual? Let us now further clarify why the gravity dual has AdS geometry. Generally, finding the geometry associated with a generic QFT is very challenging. However, if the theory is at a fixed point of the renormalization flow—namely, where the beta function vanishes—it necessarily exhibits conformal invariance and thus is a CFT, making the identification of the dual geometry easier. Following the discussion in [2], we start from a general $(d+1)$-dimensional metric with Poincaré invariance in d-dimensions,

$$ds^2 = \Omega^2(u)\left(-dt^2 + d\vec{x}^2 + du^2\right), \tag{2.12}$$

where u is the coordinate in the extra dimension, and $\Omega(u)$ is a function to be determined. Since we are assuming conformal invariance, the metric should also be invariant under scale transformations,

$$(t, \vec{x}) \to \lambda(t, \vec{x}), \quad u \to \lambda u. \tag{2.13}$$

[2] This limit is achieved when the coefficient multiplying the action is large, so the path integral is dominated by a saddle point.

Imposing the invariance of the metric, it is easy to see that the function $\Omega(u)$ must transform as

$$\Omega(u) \to \lambda^{-1}\Omega(u), \tag{2.14}$$

which determines

$$\Omega(u) = \frac{L}{u}, \tag{2.15}$$

where L is a constant. Putting it all together, we arrive at the general form of a $(d+1)$-dimensional conformal metric with Poincaré invariance in d dimensions,

$$ds^2 = \frac{L^2}{u^2}\left(-dt^2 + d\vec{x}^2 + du^2\right), \tag{2.16}$$

which is exactly the line element of AdS space in $(d+1)$ dimensions. The constant L is usually referred to as the *anti-de Sitter radius.*

2.3 Minimal String Ingredients for AdS/CFT

So far, we have motivated the AdS/CFT correspondence from general principles, in a more heuristic sort of manner, mainly through degrees of freedom arguments. While this reasoning provides a strong case for the duality, one may still ask why the correspondence should arise concretely in string theory, and in particular how to obtain the specific background $\mathrm{AdS}_5 \times \mathbb{S}^5$ together with $\mathcal{N} = 4$ $SU(N)$ super–Yang–Mills theory on the boundary, as it was originally proposed.

To answer this, we now turn into reviewing only a minimal set of ingredients from perturbative string theory and D–brane physics. We do not aim here to present a full string–theoretic construction, but rather to highlight those ideas that are essential for establishing the original Maldacena duality and for understanding how gauge theories emerge on branes and gravity backgrounds arise in the near–horizon limit. For further details, we refer to the reviews and lecture notes that we used for this section, [2, 8].

2.3.1 Perturbative Closed Strings and the Genus Expansion

The fundamental ingredient in string theory is the string: a one-dimensional, extended object with a characteristic length l_s. As the string propagates, it sweeps out a two-dimensional surface in spacetime called the *worldsheet* (see Figure 2.2).

The Nambu–Goto action for a classical string is

$$S_{\mathrm{NG}} = -T\int dA, \tag{2.17}$$

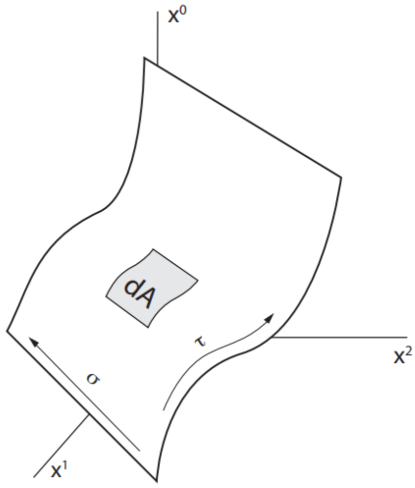

Figure 2.2: The 2-dimensional worldsheet traced out by a string evolving through spacetime. Image retrieved from [2].

where T is the *tension* of the string (with units of $length^{-2}$). It is given by

$$T = \frac{1}{2\pi\alpha'} \tag{2.18}$$

where α' is the Regge slope, a fundamental constant in the theory that controls the relationship between the mass and the spin of the string. Even more, the string length and mass are defined as

$$l_s = \sqrt{\alpha'} = \frac{1}{M_s}. \tag{2.19}$$

The action is proportional to the area of the worldsheet and regularizes the ultraviolet (UV) divergences plaguing quantum field theories of point particles. An impresive feature of string theory is that upon quantization, the massless spectrum of closed strings in flat spacetime always includes a spin-2 particle, the graviton, making string theory a candidate for a quantum theory of gravity.

Moreover, perturbative string theory systematically incorporates interactions as a sum over all possible worldsheet topologies, the so called *genus expansion* (see figure 2.3). For closed strings,

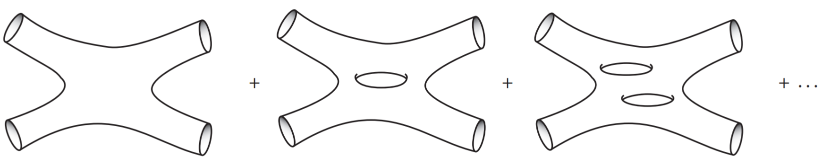

Figure 2.3: Pictorial representation of the genus expansion for the interaction of four open strings. Image retrieved from [2].

this expansion takes the form

$$\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} F_h(\alpha'), \tag{2.20}$$

where g_s is the string coupling (which weights the genus h, $i.e.$ the number of holes or "handles"). The $h = 0$ piece corresponds to the classical, tree-level amplitude, while higher h give loop (quantum) corrections.

2.3.2 The $1/N$ Expansion in Gauge Theories and Its Connection to String Loops

A foundational bridge between gauge theory and string theory is due to 't Hooft, who showed that the Feynman diagrams of an $SU(N)$ gauge theory can be organized by their topology into a $1/N$ expansion. For Yang-Mills theory with Lagrangian

$$\mathcal{L} = -\frac{N}{\lambda}\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}], \tag{2.21}$$

keeping the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed in the large-N limit, the partition function can be written as

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda), \tag{2.22}$$

where $f_h(\lambda)$ collects the sum of Feynman diagrams drawable on a surface of genus h. This precisely mirrors the genus expansion of closed string amplitudes, making the following identifications natural:

$$g_s \sim \frac{1}{N}. \tag{2.23}$$

Thus planar ($h = 0$) diagrams in the large N gauge theory map onto the classical (tree-level) graviton exchange in the dual string description.

2.3.3 Dp-branes, Gauge Fields and the DBI Action

After having introduced the connection between gauge theories and string theory, in order to move closer to the AdS/CFT correspondence itself, we also need to understand how gauge fields arise directly within string theory. This is achieved through the introduction of new extended objects: *Dp-branes.*

Dp-branes are dynamical $(p+1)$-dimensional hypersurfaces in spacetime where *open strings* can end. Their inclusion greatly enriches the theory: the closed string sector is responsible for gravity, while the dynamics of open strings ending on D-branes give rise to gauge fields. Upon quantizing open strings with Dirichlet boundary conditions along its internal excitations (parallel to its $p+1$-dimensional world-volume), one finds that the low-energy excitations are described by a

$U(1)$ gauge theory, whose dynamics are captured by the Dirac–Born–Infeld (DBI) action:

$$S_{\mathrm{DBI}} = -T_{\mathrm{Dp}} \int d^{p+1}x \sqrt{-\det\left(g_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu}\right)}, \tag{2.24}$$

where $g_{\mu\nu}$ is the induced world-volume metric, and $F_{\mu\nu}$ is the field strength of a $U(1)$ gauge field A_μ living on the brane.

If we consider a Dp-brane in flat space, the induced metric takes the following form:

$$g_{\mu\nu} = \eta_{\mu\nu} + (2\pi l_s^2)^2 \partial_\mu \phi^i \partial_\nu \phi^i \tag{2.25}$$

Expanding the action (2.24) in powers of $F_{\mu\nu}$ and $\partial_\mu \phi^i$, the quadratic terms in the expansion can be written as

$$S^{(2)}_{DBI} = -\frac{1}{g^2_{YM}} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \cdots \right). \tag{2.26}$$

The coupling g_{YM} is the Yang-Mills coupling, that can be related to l_s and g_s by the following relation:

$$g^2_{YM} = 2(2\pi)^{p-2} l_s^{p-3} g_s \tag{2.27}$$

When there are N coincident Dp-branes, the open strings stretching between them give rise to $N \times N$ matrix-valued gauge fields, and the world-volume theory is promoted from $U(1)^N$ to $U(N)$. One can think of this in terms of the freedom the open string can start and end. If there is only one Dp-brane, it can only start and end there, but if there is multiple Dp-branes, it can start in any of the branes and end in any other of them, promoting the symmetry into a $U(N)$ gauge symmetry. Actually (see [2]) one concludes that a stack of N Dp-branes realizes a $SU(N)$ gauge theory in $p+1$ dimensions.

Thus, the effective low-energy theory living on N Dp-branes is $SU(N)$ Yang-Mills theory in $p+1$ dimensions, containing both the gauge field and scalar fields describing brane fluctuations (plus their supersymmetric partners to be precise).

2.3.4 D3-branes: From Geometry to Gauge Theory and the Parameter Dictionary

Among the family of Dp-branes, the case $p = 3$ plays a particularly important role in the AdS/CFT correspondence. On the one hand, from the brane perspective, a stack of N coincident D3-branes in type IIB string theory[3] supports at low energies the dynamics of $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang–Mills theory in four spacetime dimensions — a maximally super-

[3] Type IIB is a ten-dimensional chiral $\mathcal{N} = 2$ superstring theory, which leads to maximal (32 supercharges) supersymmetry. It contains both closed strings and D-branes. Its low-energy effective description is given by type IIB supergravity.

symmetric conformal field theory.

On the other hand, from the gravity viewpoint, the supergravity (low energy description) solution for N coincident D3-branes reads

$$ds^2 = H(r)^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \tag{2.28}$$

where

$$H(r) = 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi g_s N l_s^4. \tag{2.29}$$

In the so-called *near-horizon limit* ($r \ll L$), $H(r) \simeq L^4/r^4$ and the metric becomes

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2}{r^2}\, dr^2 + L^2 d\Omega_5^2, \tag{2.30}$$

which, changing variables to $r = L^2/u$, becomes

$$ds^2 = \frac{L^2}{u^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + du^2\right) + L^2 d\Omega_5^2. \tag{2.31}$$

This is the metric of $\mathrm{AdS}_5 \times \mathbb{S}^5$: a five-dimensional anti-de Sitter (AdS) space times a five-sphere, with common radius L.

The precise parameter mapping (the "dictionary") between the gravitational theory and the gauge theory is:

$$\frac{L^4}{l_s^4} = \lambda, \qquad \left(\frac{l_p}{L}\right)^8 \sim \frac{1}{N^2}, \tag{2.32}$$

where l_p is the 10-dimensional Planck length and $\lambda = g_{YM}^2 N$ is the 't Hooft coupling of the gauge theory. In the large-N, large-λ limit, quantum and string corrections are suppressed and type IIB string theory reduces to classical supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$.

2.4 The AdS/CFT Correspondence for $\mathcal{N} = 4$ $SU(N)$ SYM

We now assemble everything above into the precise statement of the duality for the canonical example, as originally proposed by Maldacena in [3]: that is, $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ is conjectured to be equivalent to type IIB string theory in $AdS_5 \times \mathbb{S}^5$.

The first step is to relate the fundamental parameters on each side of the duality. Combining (2.27) for $p = 3$ with equation (2.29), we find

$$\left(\frac{L}{l_s}\right)^4 = N g_{YM}^2 = \lambda. \tag{2.33}$$

Moreover, the relation between the Planck length and the AdS radius is

$$\left(\frac{l_P}{L}\right)^8 = \frac{\pi^4}{2N^2}. \tag{2.34}$$

Relations (2.33) and (2.34) determine the domain of validity for the dual description in terms of classical gravity. As previously discussed, we require $l_P/L \ll 1$ to neglect quantum (loop) gravity corrections (recall Eq (2.11); this is realized when $N \gg 1$, in agreement with the genus expansion discussion earlier in the section. In addition, to neglect stringy corrections from the massive string states, we must have a curvature radius much larger than the string length: $l_s/L \ll 1$, which in turn requires a large 't Hooft coupling $\lambda \gg 1$. We thus conclude that the planar (large N), strongly coupled (large λ) SYM theory is described by classical Type IIB supergravity on $AdS_5 \times \mathbb{S}^5$, as anticipated.

Checking the symmetries on both sides, we see that $\mathcal{N} = 4$ SYM is a conformal field theory (CFT). On the gravity side, AdS_5 possesses an $SO(2,4)$ isometry group, which on the boundary coincides with the conformal group in four dimensions. Conformal symmetry is thus realized on the gravitational side by identifying the field theory as the boundary of AdS_5. Relating the energy scales, using the proper distance in the bulk d and in the field theory d_{YM}, we find (for some typical convention[4])

$$d = \frac{L}{u} d_{YM} \quad \Rightarrow \quad E = \frac{u}{L} E_{YM}. \tag{2.35}$$

Therefore, the high-energy (ultraviolet) limit of the field theory corresponds to the region near the AdS boundary $u \to 0$.

Regarding supersymmetry, $\mathcal{N} = 4$ SYM is maximally supersymmetric, with 32 fermionic supercharges and an additional six real scalars (transforming under SO(6)). The geometry $AdS_5 \times \mathbb{S}^5$ accommodates this by supporting 32 Killing spinors, corresponding to the same amount of supersymmetry, and degrees of freedom on $\mathbb{S}^5$ (rotational SO(6) symmetry) align with the scalar fields in the field theory.

With all these ingredients, we find a perfect match in the symmetries of both sides of the correspondence, justifying why this duality might hold so precisely.

Beyond the Leading Supergravity Limit: Gauss-Bonnet Corrections

It is important to stress that the discussion above describes only the strict limit in which the 't Hooft coupling λ and the number of colors N are taken to be infinitely large. In this regime, type IIB string theory reduces to classical supergravity on $AdS_5 \times \mathbb{S}^5$. At not so large values of N and λ, however, further corrections appear. Broadly speaking, there are two types:

[4] Often in literature, one uses the Poincaré patch for AdS where $u \to 0$ is the boundary.

- quantum gravity (loop) effects, suppressed by powers of $1/N^2$;
- stringy (or α') corrections, suppressed by powers of $1/\sqrt{\lambda}$.

On the gravity side, the stringy corrections take the form of higher-derivative terms added to the Einstein–Hilbert action [9]. While the true leading correction in Type IIB string theory is a very complex term of order R^4 (suppressed by $\lambda^{(-3/2)}$), it is common practice to study its qualitative effects using a simpler toy model.

The most widely used model involves adding the Gauss-Bonnet term, which is quadratic in the curvature. It is a special combination of second order curvature invariants that preserves second-order equations of motion [10]. For this reason, although it does not represent the literal correction for $\mathcal{N} = 4$ SYM, Gauss-Bonnet gravity provides a tractable framework to study the universal properties of α' corrections, capturing the first deviations from the infinite λ approximation.

It is useful to keep in mind that such higher-derivative terms provide a systematic way to go beyond the idealized large-N, large-λ limit of AdS/CFT and to explore more realistic regimes.

Finite Temperature and Black Holes in AdS

Having discussed the validity of the correspondence in the strict large-N, large-λ limit and how higher-derivative corrections offer a controlled way to go beyond this regime, let us now move to another essential generalization. So far we have considered the correspondence at zero temperature, where the dual field theory is an exact conformal theory in its vacuum state. However, many of the most interesting physical applications concern strongly coupled matter at finite temperature.

To understand the gravitational counterpart, it is helpful to recall the connection between the radial direction in AdS and the energy scale of the field theory. In a zero-temperature CFT, the existence of excitations at arbitrarily low energies (the absence of a mass gap) is holographically represented by a spacetime that extends infinitely into the interior ($u \to \infty$). Conversely, when a physical scale is introduced that cuts off the deep infrared (IR) physics, the dual geometry must reflect this by no longer being infinite. It should terminate smoothly at a finite location in the radial direction, u_{IR}.

Turning on temperature creates such IR scale in the field theory. This explicitly breaks its conformal invariance by introducing a new scale, so the system is no longer an exact CFT but only conformal in the ultraviolet, i.e., at energies much larger than T. As a consequence, physical observables such as entropy density, energy density, and transport coefficients depend explicitly on the temperature.

The holographic proposal is that this is realized by a geometry that terminates not in a singularity, but at a black hole event horizon located at a radial position u_h that is directly proportional to the temperature. Thus one replaces pure AdS space with an AdS-Schwarzschild black hole geometry [11, 12]. The temperature of the field theory is then identified with the Hawking temperature (see [13, 14]) of the black hole, which is determined by its surface gravity κ at the horizon:

$$T = \frac{\kappa}{2\pi}|_{u_h} \tag{2.36}$$

This direct correspondence between thermal field theory and black hole thermodynamics is a cornerstone of holography. It provides a geometric understanding of finite temperature quantum field theory, where the horizon acts as the IR cutoff. In particular, many of the celebrated results of AdS/CFT, such as the computation of the minimal value of the shear viscosity to entropy ratio η/s [15, 16], or the sound velocity c_s, rely crucially on this finite temperature black hole background.

2.5 Why Is It Useful?

In this chapter, we have distilled the essentials of the AdS/CFT correspondence, outlining its foundations in string theory and clarifying how the duality connects a conformal field theory to an AdS geometry by matching symmetries and fundamental parameters.

But what makes this correspondence such a valuable tool?

As we have anticipated, its true power is revealed in physical regimes where our traditional tools fail. A key example is the physics of the quark-gluon plasma (QGP). The QGP is a state of deconfined quarks and gluons formed in QCD at extremely high temperatures. Produced in heavy-ion collisions, this strongly-coupled fluid is characterized by a running coupling that remains non-perturbative at the relevant energy scales, leaving standard perturbative methods inapplicable for calculating its dynamic properties.

This is precisely where AdS/CFT provides an indispensable framework. The correspondence maps the difficult, strongly-coupled dynamics of a gauge theory onto the tractable, classical dynamics of gravity. Crucially, while the canonical example of $\mathcal{N} = 4$ SYM is not QCD (it is conformal and supersymmetric) it provides an invaluable theoretical laboratory. The working hypothesis is that certain long-wavelength hydrodynamic properties, such as shear viscosity, are universal features of a broad class of strongly-coupled gauge theory plasmas, irrespective of their specific microscopic details.

Throughout this thesis, we will exploit this correspondence to investigate such universal properties of strongly-coupled plasmas. In particular, we will focus on the ratio of shear viscosity to entropy density (η/s), which takes on a remarkably simple value in all holographic theories dual to Einstein gravity, leading to the conjecture of a universal lower bound for all relativistic fluids (see [16]). Furthermore, to assess how faithfully these holographic models capture the thermodynamics of the QGP, we will compute the speed of sound and compare the results to data from lattice QCD simulations. This provides a crucial check on the models' equation of state, ensuring they serve as physically reasonable analogues for the QGP.

Chapter 3

A Practical Understanding of the Dictionary

In the previous chapter, we introduced the AdS/CFT correspondence, motivating its origin from string theory and highlighting its remarkable applications to strongly coupled systems such as the quark-gluon plasma. We have seen how the duality matches symmetries and fundamental parameters between a conformal field theory and a higher-dimensional gravitational theory. However, to make full use of this duality as a computational tool, we must understand precisely how physical quantities on one side are related to their counterparts on the other.

This connection is made explicit through what is commonly known as the *holographic dictionary*. The dictionary provides quantitative rules for relating observables, correlation functions, and sources between the d-dimensional quantum field theory and its $d+1$-dimensional gravitational dual. In this chapter, we aim to develop a practical and working understanding of this dictionary, which serves as the foundational bridge for all applications of holography to quantum field theory.

The main references for this chapter are [1] and [2].

3.1 The Bulk and the Boundary: Bringing Them Together

The AdS/CFT dictionary is built around the map between QFT operators $\mathcal{O}_i$ and their corresponding bulk fields Φ_i.

A general bulk action for the $d+1$-dimensional gravitational theory can be written as:

$$S_{bulk}^{(d+1)}(\Phi_i : g_{\mu\nu}, A_\mu, \phi, \dots) = \int d^{d+1}x\sqrt{-g}\left[R - \frac{1}{4}F^2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \dots\right], \tag{3.1}$$

where the bulk fields Φ^I can have arbitrary masses m^I, spins J^I, and interaction terms. The fields may also be charged under various bulk symmetries[1]. In the regime we're interested in (the weakly coupled, classical gravity limit), we only consider classical fields on a curved spacetime background, neglecting all stringy and quantum corrections.

The dictionary is usually illustrated in the original case of a conformal field theory. On the CFT side, the theory is built from a collection of operators:

$$\mathcal{L}_{CFT} \equiv \sum_i c_i \mathcal{O}_i. \tag{3.2}$$

To relate bulk fields to QFT operators, consider deforming the CFT with an operator $\mathcal{O}$ and its source ϕ_0:

$$\mathcal{L} \longrightarrow \mathcal{L}_{CFT} + \phi_0(x)\mathcal{O}(x). \tag{3.3}$$

Given a source, the partition function is

$$\mathcal{Z}_{QFT}[\phi_0] = e^{W(\phi_0)} = \left\langle e^{\int \phi_0 \mathcal{O}} \right\rangle_{QFT}, \tag{3.4}$$

where $W(\phi_0)$ is the generating functional for correlation functions of $\mathcal{O}$. In any QFT, n-point functions are recovered by functional differentiation:

$$\langle \underbrace{\mathcal{O} \ldots \mathcal{O}}_{n} \rangle = \left. \frac{\delta^n W}{\delta \phi_0^n} \right|_{\phi_0 = 0}. \tag{3.5}$$

So, all observables of interest can be computed if we know W, and our main task is to relate W to a bulk gravity calculation.

This brings us to the central principle of the AdS/CFT dictionary: the GPKW (Gubser, Polyakov, Klebanov, Witten) prescription [17,18]. It states that the generating functional of the QFT with sources is equal to the partition function of the bulk theory with the fields taking prescribed boundary values:

$$\mathcal{Z}_{QFT}[\phi_0] = e^{W(\phi_0)} = \left\langle e^{\int \phi_0 \mathcal{O}} \right\rangle_{QFT} = \mathcal{Z}_{gravity}[\phi(x,u)_{boundary} \equiv \phi_0(x)], \tag{3.6}$$

where $\mathcal{Z}_{gravity}[\phi \to \phi_0]$ is the gravity path integral with boundary condition $\phi(x, u \to 0) \to \phi_0(x)$. More explicitly,

$$\mathcal{Z}_{gravity}[\phi \to \phi_0] = \sum_{\phi \to \phi_0} e^{S_{gravity}}. \tag{3.7}$$

In the classical (small g_s, large N) limit, the sum is dominated by the saddle point, thus well-approximated by evaluating the action on the classical solution, so that

$$W(\phi_0) = S_{bulk}^{on-shell}[\phi \to \phi_0]. \tag{3.8}$$

[1]These are symmetries in the bulk theory that correspond to internal symmetries in the field theory.

Therefore, to obtain n-point functions, we just take derivatives of this action with respect to ϕ_0:

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle = \left.\frac{\delta^n S^{ren}_{grav}[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\right|_{\phi_0}, \tag{3.9}$$

where $S^{ren}_{grav}[\phi]$ is the renormalized (finite) classical on-shell action, obtained after subtracting possible divergences. This statement is the backbone of holography: boundary correlators are obtained by varying the on-shell gravitational action with respect to boundary conditions.

The precise pairing of bulk fields and QFT operators is guided by symmetries: operators and fields must have matching quantum numbers under the conformal group $\mathrm{O}(2, d-1)$. For example:

$$\mathcal{L}_{CFT} + \int d^dx\sqrt{-g}\,(g_{\mu\nu}T^{\mu\nu} + A_\mu J^\mu + \phi\mathcal{O} + \cdots), \tag{3.10}$$

so that we have the correspondences:

$$\begin{array}{cc} \text{Stress tensor } T^{\mu\nu} & \text{Bulk graviton } g^{\mu\nu}, \\ \mathrm{U}(1) \text{ current } J^\mu & \text{Bulk gauge field } A^\mu, \\ \text{Scalar operator } \mathcal{O} & \text{Bulk scalar } \phi, \\ \cdots & \cdots \end{array} \tag{3.11}$$

3.2 ΛCFT, a Scalar in AdS: Renormalization and Correlators

Let us see how the dictionary works for a concrete example: a scalar field in AdS_{d+1}. This is helpful for understanding how AdS fields correspond to sources for CFT operators in practice.

We work in AdS_{d+1} with the metric

$$ds^2 = \frac{L^2}{u^2}(du^2 - dt^2 + d\mathbf{x}^2). \tag{3.12}$$

Consider the action for a massive scalar field:

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]. \tag{3.13}$$

Varying this action gives the EOM:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0. \tag{3.14}$$

In this metric, this becomes

$$u^{d+1}\partial_u(u^{1-d}\partial_u\phi) + u^2\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2L^2\phi = 0. \tag{3.15}$$

To solve near the boundary, Fourier transform along the d field-theory directions $x^\mu = (t, \mathbf{x})$:

$$\phi(u, x^\mu) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} f_k(u). \tag{3.16}$$

Close to $u = 0$, we use the power-law ansatz $f_k(u) \sim u^\beta$. Substituting (3.16) into (3.15), we get

$$\beta(\beta - d)u^\beta + \eta^{\mu\nu} k_\mu k_\nu u^{\beta+2} - m^2 L^2 u^\beta = 0.$$

Since we are close to the boundary, the terms u^β will dominate over the terms $u^{\beta+2}$, hence obtaining the indicial equation:

$$\beta(\beta - d) - m^2 L^2 = 0. \tag{3.17}$$

Solving, we find

$$\beta_\pm = \frac{d}{2} \pm \nu, \qquad \nu = \sqrt{\frac{d^2}{4} + m^2 L^2}, \tag{3.18}$$

which fixes the asymptotic powers in terms of the mass of the scalar field. Thus, the near-boundary expansion reads

$$f_k(u) \approx A(k) u^{d-\Delta} + B(k) u^\Delta, \qquad \Delta = \beta_+ = \frac{d}{2} + \nu, \qquad u \to 0. \tag{3.19}$$

Or, in position space,

$$\phi(x, u) \approx A(x) u^{d-\Delta} + B(x) u^\Delta, \qquad u \to 0. \tag{3.20}$$

Here $A(x)$ is identified as the *source* $\phi_0(x)$, while $B(x)$ encodes the expectation value of the dual operator $\langle O(x) \rangle$. For the expansion to make sense (i.e. for Δ real and the theory be stable), the mass of the scalar must satisfy the Breitenlohner-Freedman (BF) bound:

$$m^2 \geq -\left(\frac{d}{2L}\right)^2. \tag{3.21}$$

In this typical expansion for a bulk field near the boundary (3.20), the $u^{d-\Delta}$ term is usually dominant as $u \to 0$ whenever the BF bound holds. When $m^2 > 0$, however, $d - \Delta$ is negative and this term diverges at the boundary. To get a finite QFT source, we identify (up to normalization)

$$\phi_0(x) = \lim_{u \to 0} u^{\Delta - d} \phi(u, x). \tag{3.22}$$

Similarly, at leading order, $\phi(u, x) = u^{d-\Delta}\phi_0(x) + \cdots$.

What does Δ mean? It is the scaling dimension of the dual operator $\mathcal{O}$. To see this, consider the boundary action: if $\mathcal{O}$ is dual to ϕ, we have

$$S_{bdy} \sim \int d^d x \sqrt{h_\epsilon} \phi(\epsilon, x) \mathcal{O}(\epsilon, x), \tag{3.23}$$

where $u = \epsilon$ is a regulated boundary and h_ϵ as the determinant of the induced metric at such boundary. Using $\phi(\epsilon, x) = \epsilon^{d-\Delta}\phi_0(x)$, we find

$$S_{bdy} \sim L^d \int d^d x \phi_0(x) \epsilon^{-\Delta} \mathcal{O}(\epsilon, x), \tag{3.24}$$

which is finite and independent of ϵ in the limit $\epsilon \to 0$ only if

$$\mathcal{O}(\epsilon, x) = \epsilon^{\Delta} \mathcal{O}(x). \tag{3.25}$$

So, Δ is the scaling dimension: a "dilatation" in u translates to a scale transformation in the QFT.

3.3 One-point Function and Linear Response Theory

From the GKPW prescription, the one-point function of an operator in the presence of a source is (3.9)

$$\langle \mathcal{O}(x) \rangle_{\phi_0} = \frac{\delta S^{ren}_{grav}}{\delta \phi_0(x)}. \tag{3.26}$$

where S_{ren} is the renormalized on-shell action (counterterms must be included to cancel divergences near $u = 0$). Given the relationship between ϕ_0 and ϕ at the boundary, the one point function is given by:

$$\langle \mathcal{O}(x) \rangle_{\phi_0} = \lim_{u \to 0} u^{d-\Delta} \frac{\delta S^{ren}_{grav}}{\delta \phi(x)}. \tag{3.27}$$

One can show (see [2]) that the variation of the on-shell action under a change $\phi \to \phi + \delta\phi$ is a boundary term at $u = \epsilon \to 0$:

$$\delta S^{on-shell}_{grav} = \int_\epsilon^\infty du \int d^d x\, \partial_u \left(\frac{\partial \mathcal{L}}{\partial(\partial_u \phi)} \delta\phi \right) = -\int_{\partial M} d^d x\, \frac{\partial \mathcal{L}}{\partial(\partial_u \phi)} \delta\phi \Big|_{u=\epsilon}, \tag{3.28}$$

where ∂M is the AdS boundary at $u = 0$.

Defining the canonical momentum as

$$\Pi = -\frac{\partial \mathcal{L}}{\partial(\partial_u \phi)}, \tag{3.29}$$

we can write

$$\delta S^{on-shell}_{grav} = \int_{\partial M} d^d x\, \Pi(\epsilon, x)\, \delta\phi(\epsilon, x), \tag{3.30}$$

meaning

$$\frac{\delta S^{on-shell}_{grav}}{\delta \phi(\epsilon, x)} = \Pi(\epsilon, x). \tag{3.31}$$

Including necessary counterterms for UV-finiteness, the renormalized action is written as $S^{ren} = S^{on-shell}_{grav} + S^{ct}$. The renormalized canonical momentum is then

$$\Pi^{ren} = \frac{\delta S^{ren}}{\delta\phi(u,x)}, \tag{3.32}$$

so that at $u = \epsilon$,

$$\Pi^{ren}(\epsilon, x) = -\frac{\partial\mathcal{L}}{\partial(\partial_u\phi(\epsilon,x))} + \frac{\delta S^{ct}}{\delta\phi(\epsilon,x)}. \tag{3.33}$$

Combining with the one-point function formula (3.27), the expectation value of the operator $\mathcal{O}$ in presence of the source ϕ_0 can be computed from the following limit in the AdS boundary from the renormalized momentum as:

$$\langle\mathcal{O}(x)\rangle_{\phi_0} = \lim_{u\to 0} u^{d-\Delta}\Pi^{ren}(u,x). \tag{3.34}$$

This formula provides a concrete recipe: the one-point function of a boundary operator is determined by the renormalized radial momentum of its dual field, computed in the bulk and evaluated near the boundary.

Let us now connect this AdS computation with familiar QFT linear response. In field theory, in Minkowski signature, the expectation value of an operator in the presence of a source is

$$\langle\mathcal{O}(x)\rangle_{\phi_0} = \int [D\psi]\, \mathcal{O}(x)\, \exp\left(iS[\psi] + i\int d^d y\, \phi_0(y)\mathcal{O}(y)\right).$$

where ψ denotes all the fields of the QFT. Expanding to first order in ϕ_0, we can write it as:

$$\langle\mathcal{O}(x)\rangle_{\phi_0} = \langle\mathcal{O}(x)\rangle_{\phi_0=0} + i\int d^d y\, \langle\mathcal{O}(x)\mathcal{O}(y)\rangle_{\phi_0=0}\, \phi_0(y) + \cdots \tag{3.35}$$

The retarded Green's function is defined as [19]

$$iG_R(x-y) \equiv \theta(x^0 - y^0)\langle[\mathcal{O}(x)\mathcal{O}(y)]\rangle, \tag{3.36}$$

so

$$\langle\mathcal{O}(x)\rangle_{\phi_0} = \langle\mathcal{O}(x)\rangle_{\phi_0=0} - \int G_R(x-y)\phi_0(y)d^d y. \tag{3.37}$$

We then see that $\langle\mathcal{O}(x)\rangle_{\phi_0}$ measures the fluctuations of the operator away from its expectation value $i.e.$ the external response of the observable away from the expectation value. We usually assume $\langle\mathcal{O}(x)\rangle_{\phi_0=0} = 0$ (it can always be subtracted), which leads to

$$\langle\mathcal{O}(x)\rangle_{\phi_0} = -\int G_R(x-y)\phi_0(y)d^d y. \tag{3.38}$$

That is, the one-point function's linear response to an external source is governed by the retarded correlator. In momentum space,

$$\langle \mathcal{O}(k) \rangle_{\phi_0} = -G_R(k)\phi_0(k). \tag{3.39}$$

The fact that this linear response is determined by the retarded correlator is directly a consequence of causality. This is so because the source can only influence the system after it has been turned on.

In conclusion, by combining our holographic result for the one-point function (Eq. 3.34) with the definition of linear response in field theory (Eq. 3.39), we arrive at a powerful and practical recipe. The retarded Green's function of a boundary operator is directly determined by the renormalized canonical momentum of its dual field in the bulk:

$$G_R(k) = \lim_{u \to 0} \frac{u^{d-\Delta} \Pi^{ren}(u,k)}{\phi_0(k)} \,. \tag{3.40}$$

Chapter 4

Transport Coefficients, Hydrodynamics and η/s

In the previous chapter, we established the holographic dictionary: a practical framework connecting quantities in gravity and quantum field theory. We also examined how to compute correlation functions using classical bulk fields, obtaining a recipe to compute the retarded Green's function associated with linear response theory. Building on this foundation, we now turn to an important application, namely transport coefficients in the hydrodynamics of strongly coupled systems.

In this chapter, we will explore how AdS/CFT allows us to compute transport coefficients, quantities that characterize how a system responds to small disturbances, such as conductivity, diffusion, and viscosity. These coefficients appear naturally in hydrodynamics, which describes the long-wavelength, near-equilibrium behavior of many-body systems. Special attention will be given to the ratio of shear viscosity to entropy density, η/s, a key observable in the physics of the quark-gluon plasma and other strongly coupled fluids. We will discuss how this ratio emerges in holographic models and why it is of universal interest.

The chapter proceeds in several steps:

1. Introduce the hydrodynamic description of quantum field theories at finite temperature.
2. Review the framework of linear response theory and the Kubo formulas that define transport coefficients.
3. Translate these field-theoretic definitions into the holographic setup using the AdS/CFT dictionary.
4. Finish with the explicit computation of the shear viscosity to entropy ratio η/s for the Einstein-Hilbert gravity theory.

Much of the material in this chapter comes from Ramallo's lecture notes [2], the book by Baggioli [1], David Tong's lecture notes [20], and the review by Mukund Rangamani [21].

4.1 Transport Coefficients and Relativistic Hydrodynamics

Hydrodynamics provides an effective field theory description for systems close to equilibrium, valid at sufficiently long times and over large distances; instead of tracking all microscopic degrees of freedom, one keeps only a small set of conserved quantities: energy, momentum, and charges... It is a low-energy effective theory, constructed as a perturbative expansion in time and spatial gradients around equilibrium. In Fourier space, this corresponds to an expansion in powers of frequency ω and momentum k.

In this regime, the system's dynamics are dominated by so-called *hydrodynamic modes*: the longest-living, slowest decaying modes. This domain is characterized by a small set of transport coefficients; for our purposes, the shear viscosity will be especially relevant. The key equations governing hydrodynamics are the conservation of energy and any other conserved global charges. We will focus on the conservation of the energy-momentum tensor, whose equation is simply

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.1}$$

To make further progress, we need a constitutive relation: an explicit expression relating $T^{\mu\nu}$ to the local velocity of the fluid u^μ and the thermodynamic variables. Constitutive relations are mathematical prescriptions that encode how a system's internal fields, stresses, and strains respond to changes in the surrounding environment.

For an ideal (non-dissipative) fluid, the stress tensor takes the form

$$T^{\mu\nu}_{\text{ideal}} = \rho\, u^\mu u^\nu + P\left(g^{\mu\nu} + u^\mu u^\nu\right), \tag{4.2}$$

where u^μ is the normalized velocity ($g_{\mu\nu}u^\mu u^\nu = -1$), P is the pressure, and ρ is the energy density. However, ideal fluids do not capture dissipative dynamics. Real fluids exhibit dissipation, which allows the system to relax to equilibrium after being perturbed. We incorporate dissipation by adding a correction, the dissipative term,

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + P\left(g^{\mu\nu} + u^\mu u^\nu\right) + \Pi^{\mu\nu}. \tag{4.3}$$

To complete the hydrodynamical description, we express $\Pi^{\mu\nu}$ as a function of the velocity field and the thermodynamic variables. Following the effective field theory approach, we expand $\Pi^{\mu\nu}$

in gradients about equilibrium. To leading order in derivatives, one finds [1, 21]

$$T^{\mu\nu} = \underbrace{\rho\, u^\mu u^\nu + P\Delta^{\mu\nu}}_{\text{zeroth order}} \underbrace{-\eta\, \sigma^{\mu\nu} - \zeta\, \Delta^{\mu\nu}\nabla_\lambda u^\lambda}_{\text{first order}} + \cdots\,, \tag{4.4}$$

where

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu, \quad \sigma^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta}\partial_\rho u^\rho\right). \tag{4.5}$$

Notice that $\sigma^{\mu\nu}$ is first order in derivatives, meaning it measure small departures from equilibrium. The tensor $\sigma^{\mu\nu}$ is often called the *shear*, which corresponds to the symmetric, traceless, transverse part of the velocity gradient [21]. The transport coefficients η and ζ are the *shear viscosity* and *bulk viscosity*, respectively. These are the only two transport coefficients needed at this order; they quantify how momentum diffuses in the fluid. Shear viscosity η measures the fluid's resistance to transverse deformations, while bulk viscosity ζ measures resistance to uniform compression or expansion.

In simple terms, the expansion (4.4) can be viewed as follows:

$$T^{\mu\nu} = (\text{slow}) + (\text{fast}) + (\text{faster}) + \cdots\,, \tag{4.6}$$

where each term involves higher derivatives in space and time, resembling an expansion in a low-energy effective theory. In this chapter, we are mainly interested in how to compute the shear viscosity of a strongly coupled system using the AdS/CFT correspondence.

Linear Response and the Kubo Formula

Transport coefficients are most naturally defined through *linear response theory*. The idea is simple: perturb a system slightly away from its thermal equilibrium and measure its response. The proportionality constants between the perturbation and the response are the transport coefficients. When a sistem is perturbed by a small, slowly varying source $\phi_0(\omega, \vec{k})$, the response of an operator $\langle \mathcal{O} \rangle$ in momentum space has the form (Eq (3.39))

$$\langle \mathcal{O}(\omega, \vec{k}) \rangle_{\phi_0} = -G_R(\omega, \vec{k})\, \phi_0(\omega, \vec{k}), \tag{4.7}$$

where G_R is the retarded Green's function. In the long-wavelength (hydrodynamic) limit, we take the zero spatial momentum and zero frequency limit of the retarded correlator. In this limit, we are interested in the response of the system to a time-varying source $\phi_0(t)$, which can be written as

$$\langle \mathcal{O} \rangle_{\phi_0} \approx -\chi\, \partial_t \phi_0, \tag{4.8}$$

where χ is the *transport coefficient* corresponding to the operator, and it characterizes the response. In the frequency domain, with small $\omega \to 0$, this becomes

$$\langle \mathcal{O} \rangle_{\phi_0} \approx i\omega\, \chi\, \phi_0(\omega). \tag{4.9}$$

At the same time, from the general linear response result (4.7) we find

$$G_R(\omega, \vec{k} = 0) = -i\omega\chi\,, \qquad \mathrm{Im}\, G_R(\omega, \vec{k} = 0) = -\omega\chi \qquad \text{as } \omega \to 0. \tag{4.10}$$

This gives us the so-called *Kubo formula* for the transport coefficient,

$$\chi = -\lim_{\omega\to 0} \lim_{\vec{k}\to 0} \frac{1}{\omega} \mathrm{Im}\, G_R(\omega, \vec{k}). \tag{4.11}$$

We then see that it is just a matter of computing the retarded Green's function.

The beauty of the AdS/CFT correspondence is that we have seen how it gives us a new way to compute these retarded correlators: instead of evaluating them directly in the strongly coupled QFT (which is impossible with standard techniques), we compute the classical on-shell action of the corresponding bulk fields.

4.1.1 Scalar Field Perturbation

Let us now illustrate this recipe with a simple example: a scalar field in the bulk background, since we have already study it. This case will be instructive for understanding the more realistic computation of viscosity that follows in the next chapter.

Suppose we have a $(d+1)$-dimensional bulk metric of the form

$$ds^2 = g_{tt} dt^2 + g_{uu} du^2 + g_{xx}\, \delta_{ij} dx^i dx^j. \tag{4.12}$$

Assume that this time that the geometry has a horizon at $u = u_h$, and that near the horizon, the metric components behave as

$$g_{tt} \approx -c_0(u_h - u), \quad g_{uu} \approx \frac{c_u}{u_h - u}, \qquad u \to u_h, \tag{4.13}$$

where c_0 and c_u are constants. Let us consider a massless scalar field ϕ in this geometry, with the action

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{-g}\, \frac{\partial_\mu \phi \partial^\mu \phi}{q(u)}, \tag{4.14}$$

where $q(u)$ is an effective position-dependent coupling. The canonical momentum Π (cf. (3.29)) is

$$\Pi = \frac{\sqrt{-g}}{q} g^{uu} \partial_u \phi. \tag{4.15}$$

The equation of motion for ϕ is

$$\partial_u \Pi = -\frac{\sqrt{-g}}{q}\left(\frac{\partial_t^2 \phi}{g_{tt}} + \frac{\partial_i^2 \phi}{g_{xx}}\right). \tag{4.16}$$

Fourier transforming in t and $\vec{x}$, we have

$$\begin{aligned} \phi(u,t,\vec{x}) &= \int \frac{d\omega\, d^{d-1}k}{(2\pi)^d} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \phi(u,\omega,\vec{k}), \\ \Pi(u,t,\vec{x}) &= \int \frac{d\omega\, d^{d-1}k}{(2\pi)^d} e^{i(\vec{k}\cdot\vec{x}-\omega t)} \Pi(u,\omega,\vec{k}). \end{aligned} \tag{4.17}$$

For a massless scalar, the scaling dimension is $\Delta = d$. Therefore, the one-point function is just the boundary limit of the canonical momentum, and the source is the boundary value of the bulk field.

Thus, using Eqs. (3.40) and (4.11), the transport coefficient χ can be computed as

$$\chi = \lim_{u\to 0}\lim_{\omega\to 0}\lim_{\vec{k}\to 0} \mathrm{Im}\left[\frac{\Pi(u,k_\mu)}{\omega\phi(u,k_\mu)}\right] = \lim_{u\to 0}\lim_{\omega\to 0}\lim_{\vec{k}\to 0}\left[\frac{\Pi(u,k_\mu)}{i\omega\phi(u,k_\mu)}\right]. \tag{4.18}$$

It can be shown (see [2]) that in the ordered limit $\lim_{\omega\to 0}\lim_{\vec{k}\to 0}$, $\Pi/(\omega\phi)$ does not depend on u, so we can evaluate it at the horizon u_h:

$$\chi = \lim_{k_\mu \to 0} \left.\frac{\Pi(u,k_\mu)}{i\omega\phi(u,k_\mu)}\right|_{u=u_h}. \tag{4.19}$$

All that remains is to solve for the field ϕ near the horizon. When trying to get a solution for ϕ, one problem arises: in the Minkowski prescription, regularity at the horizon does not fully fix the classical solution; we need some extra boundary conditions, *i.e.* we must choose a solution that matches causal (infalling) boundary conditions. Near u_h, an ansatz $\phi = (u_h - u)^\beta$ yields, upon substitution into the equation of motion,

$$\beta = \pm i \sqrt{\frac{c_u}{c_0}}\,\omega. \tag{4.20}$$

Therefore, the two solutions are

$$\phi_\pm \sim (u_h - u)^{\pm i\sqrt{\frac{c_u}{c_0}}\omega}. \tag{4.21}$$

Fixing the field value at the boundary is then not enough, we must pick one of the above solutions. It turns out that the infalling solution, ϕ_-, is the only one compatible with causality[1], and it is the one related to the retarded Green function (see for the details [2, 22, 23]).

[1] The idea behind is that since a black hole horizon is a one-way membrane, physical perturbations on the boundary should correspond to waves that fall into the horizon, carrying energy and information inwards and not outwards

The infalling solution satisfies

$$\partial_u \phi_- = \sqrt{\frac{g_{uu}}{-g_{tt}}} i\omega\phi_-, \tag{4.22}$$

and thus

$$\left.\frac{\Pi}{i\omega\phi_-}\right|_{u_h} = \frac{1}{q(u_h)} \left.\sqrt{\frac{g}{g_{uu}g_{tt}}}\right|_{u_h}. \tag{4.23}$$

Therefore, the transport coefficient becomes

$$\chi = \frac{1}{q(u_h)} \left.\sqrt{\frac{g}{g_{uu}g_{tt}}}\right|_{u_h}. \tag{4.24}$$

It is worth noting that the square root is proportional to the horizon area $\mathcal{A}_H$ divided by the spatial volume V, so

$$\chi = \frac{1}{q(u_h)}\frac{\mathcal{A}_H}{V}. \tag{4.25}$$

This is a remarkable result. A transport coefficient χ, which describes the dissipative, long-wavelength dynamics of the boundary field theory, is determined entirely by local, geometric properties of the bulk black hole horizon $(q(u_h), g_{\mu\nu}(u_h))$.

Recalling the Bekenstein-Hawking formula for the entropy of a black hole (2.1), we can write the ratio of the transport coefficient to the entropy density s as

$$\frac{\chi}{s} = \frac{4G_N}{q(u_h)}. \tag{4.26}$$

The entire procedure we have just performed can be elegantly summarized using the formalism of Son and Starinets [23]. They show that the on-shell action in momentum space ultimately reduces to a surface term:

$$S^{\text{on-shell}} = \int \frac{d^dk}{(2\pi)^d}\phi(-k)\mathcal{F}(k,u)\phi(k)\Big|_{u_B}^{u=u_h}, \tag{4.27}$$

with $\mathcal{F}$ a flux factor and u_B the boundary $(u = 0)$. Their conjecture is that the retarded Green function is

$$G^R(k) = -2\mathcal{F}(k,u)\big|_{u_B}. \tag{4.28}$$

This provides a powerful and equivalent recipe that we will employ in our subsequent computations.

4.1.2 Shear Viscosity: Metric Perturbations

Let us now focus on the shear viscosity. In holography, the shear viscosity is obtained by perturbing the stress-energy tensor from equilibrium. The Kubo formula then reads as

$$\eta = -\lim_{\omega\to 0}\left\{\frac{1}{\omega}\mathrm{Im}[G^R_{xyxy}(\omega,\vec{k}=0)]\right\}, \tag{4.29}$$

where G^R_{xyxy} is the retarded Green function for T_{xy}.

In the holographic dictionary, the stress tensor $T_{\mu\nu}$ is dual to the bulk metric $g_{\mu\nu}$. Therefore, to compute this correlator, we must study linear perturbations of the metric, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, in the gravitational dual.

A crucial simplification occurs when we analyze these perturbations. Based on symmetry arguments, the fluctuations $h_{\mu\nu}$ decouple into three independent channels: scalar, vector, and tensor [21, 22]. Each channel is sourced by a different component of the boundary stress-energy tensor. For a perturbation assumed to be dependent only on t and one spatial component z (for the sake of simplicity), and choosing the gauge where $h_{u\mu} = 0$ for all μ, then this three classes of perturbations are:

- Tensorial: $h_{xy} \neq 0$, or $h_{xx} = -h_{yy} \neq 0$,
- Vectorial: h_{xt}, $h_{xz} \neq 0$, or h_{yt}, $h_{yz} \neq 0$,
- Scalar: h_{tz}, and diagonal elements such that $h_{xx} = h_{yy}$, etc.

We focus on the simplest case, the tensorial one: $h_{xy} \neq 0$, and define the scalar field $\phi = h^y{}_x$.

As an explicit example, let us use the action of Einstein-Hilbert with cosmological constant,

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\,[R - 2\Lambda], \tag{4.30}$$

where $16\pi G_N$ is the standard normalization. This serves as a launching point; in later chapters we will add extra terms and examine their effects.

The Einstein equations become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{4.31}$$

We consider a background metric that is asymptotically AdS, similar as in the massless scalar field discussion before, Eq. (4.12), representing the thermal state of the boundary theory. Then, we introduce a perturbation $h_{xy} = g_{xx}(u)\phi$:

$$ds^2 = g_{tt}dt^2 + g_{uu}du^2 + g_{xx}\left(\delta_{ij}dx^i dx^j + \phi dx^1 dx^2 + \phi dx^2 dx^1\right). \tag{4.32}$$

Thus, the scalar field is defined as

$$h^x_y = \phi. \tag{4.33}$$

The next step is to determine the action for the perturbation ϕ, derive the corresponding equations of motion, and obtain the on-shell action. Expanding the Einstein-Hilbert action to second order in ϕ, one finds that its dynamics are governed by the following action:

$$S = -\frac{1}{16\pi G_N}\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{4.34}$$

This is exactly the action for a massless scalar field, Eq. (4.14), with $q(u) = 16\pi G_N$. Using the previous results, the transport coefficient χ is now the shear viscosity η, which reads as:

$$\eta = \frac{1}{16\pi G_N}\sqrt{\frac{g}{g_{uu}g_{tt}}} = \frac{1}{16\pi G_N}\frac{A_H}{V}. \tag{4.35}$$

Thus the ratio between shear viscosity and entropy density is just

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.36}$$

This is a very well known result. Kovtun, Son and Starinets [16] conjectured that $1/4\pi$ is a universal lower bound for η/s. Remarkably, this value gets very close to the lowest observed values in both the quark-gluon plasma and ultracold atomic Fermi gases. Another feature of this bound is that it is universal for any theory with a gravity dual described by classical Einstein gravity and matter fields, and is valid in the infinite coupling limit $\lambda \to \infty$. Finite coupling corrections (from type IIB superstring theory, see [24] for the results) seems to only increase this ratio, supporting the conjecture even more.

As a final note, we should remark how with these computations we have demonstrated in practice how the AdS/CFT correspondence provides a powerful tool to extract hydrodynamic properties of strongly coupled quantum field theories.

Chapter 5

η/s for Gravity Theories with Higher Curvature Terms

In the previous chapter, we used holography to derive the celebrated universal result for the shear viscosity to entropy density ratio, $\eta/s = 1/4\pi$, in strongly coupled field theories at finite temperature. This bound, known as the KSS bound, arises in the simplest gravity dual described by the Einstein–Hilbert action with a negative cosmological constant. Its remarkable universality raises a natural question: what happens once Einstein gravity is extended by adding higher–derivative terms? Addressing this question is the central focus of the present chapter.

We thus turn our attention to broader classes of gravity theories that incorporate higher curvature terms in the action. These modifications, which naturally appear as finite-coupling or stringy corrections, alter the dynamics of metric fluctuations and can therefore modify the transport properties of the dual field theory, potentially leading to violations of the KSS bound. Our analysis concentrates in particular on quasi-topological gravities, which contain higher-order curvature invariants but preserve second-order field equations, as well as on extensions with a non-minimally coupled dilaton field. These dilatonic generalizations provide additional freedom for constructing realistic holographic models.

Beyond transport, a realistic model of strongly coupled QCD matter must also reproduce its thermodynamic behavior. For this reason, we will not only examine the ratio η/s, but also study the equation of state of the dual plasma, with emphasis on the speed of sound c_s. While $\mathcal{N} = 4$ SYM is conformal with a fixed $c_s^2 = 1/3$, lattice QCD simulations show a nontrivial temperature dependence characterized by a dip near the deconfinement transition. We will show that higher-derivative gravities, when coupled to a dilaton, provide the flexibility necessary to reproduce this feature, highlighting the richness of the holographic landscape when higher-derivative terms are included

5.1 Violation of the KSS Bound: Gauss-Bonnet Gravity

The KSS bound derived in the previous chapter holds for a wide variety of field theories with Einstein gravity duals. However, there exist modified gravity theories whose dual field theory may violate this bound. One of the simplest and most studied examples is Gauss-Bonnet gravity, which we anticipated in chapter 1 as a toy model for studying stringy corrections, and it introduces a specific combination of higher derivative terms. The action is

$$S = \frac{1}{16\pi G}\int d^5x\,\sqrt{-g}\left[R - 2\Lambda + \frac{\lambda_{GB}}{2}L^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\right)\right], \qquad (5.1)$$

where $\Lambda = -6/L^2$. This theory is particularly interesting because tuning the Gauss-Bonnet coupling λ_{GB} can reduce the viscosity-entropy ratio down to zero, which violates the KSS bound [25]. Following similar steps as before—by perturbing the metric, solving the equations of motion, and computing the on-shell action—one finds that the viscosity-to-entropy ratio is (see, e.g., [25, 26])

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 - 4\lambda_{GB}). \qquad (5.2)$$

This result holds for any value of the coupling λ_{GB}; the higher curvature terms do not need to be small. Thus, the KSS bound can indeed be violated.

This toy model is physically relevant because well-defined string theory constructions are known to generate corrections that can lead to $\eta/s < 1/4\pi$ [9, 27].This motivates the exploration of theories beyond simple Einstein gravity, especially in light of experimental data. Analyses of the quark-gluon plasma (QGP) created at RHIC and the LHC indicate that it is a nearly perfect fluid, with a value of η/s that is remarkably close to the $1/4\pi$ bound , and potentially below (as [28] suggested). This proximity makes the study of any correction that modifies the ratio a crucial area of research.

5.2 KSS Bound for the General Action of $d = 5$ Dimensional R^n Gravity

Motivated by this, we now move to the main original results of this thesis: studying the viscosity-to-entropy ratio for field theories dual to higher curvature gravity theories. The effects of such terms have been examined before (see, for instance, [25,27,29]). Our focus is on higher curvature gravities coupled non-minimally to a scalar field (a dilaton). While previous literature has considered R^2 terms [30–32], in this work we consider even higher order terms, providing explicit formulas for generic order n in a special class of models.

5.2.1 Obtaining a Formula for η

Let us first derive a formula for the shear viscosity in these kinds of gravity theories, using the methods developed in previous chapters. The main references for this derivation are Myers [33] and Kats and Petrov [27]. We will outline the key steps and refer to those works for full details. Here we work in five dimensions, though the procedure extends to arbitrary dimensions. The general action we consider is (see also [34]):

$$S = \frac{1}{16\pi G_N} \int d^5x \sqrt{-g} \left(R + \sum_{n=2}^{\infty} G_n(\phi_d) \beta_n \ell^{2(n-1)} \mathcal{Z}_{(n)} - V(\phi_d) - \frac{1}{2}(\nabla\phi_d)^2 \right). \tag{5.3}$$

Here, ϕ_d is a scalar field and $\mathcal{Z}_{(n)}$ denotes higher curvature invariants. The couplings β_n measure the strength of each higher curvature term of order n, and for the moment, it is important to note that they are assumed to be small, so that we work at leading order. The potential $V(\phi_d)$ allows for more general coupling of the dilaton than just a cosmological constant.

As we have mentioned, the AdS/CFT dictionary tells us that the stress-energy tensor corresponds to a metric perturbation. For a tensorial perturbation we write

$$h_x^{\ y}(t,z,u) = \int \frac{d^4k}{(2\pi)^4} \phi_k(u) e^{-i\omega t + ikz}, \tag{5.4}$$

where we take the fluctuation to depend only on t, z, and u for simplicity.

Evaluating the action (5.3) to quadratic order in fluctuations gives, aside from the background action, an effective action for the perturbation, which always takes the form [33]:

$$\begin{aligned} I_\phi = \frac{1}{16\pi G_N} \int dk du \Big(& A(u)\phi_k''\phi_{-k} + B(u)\phi_k'\phi_{-k}' + C(u)\phi_k'\phi_{-k} \\ & + D(u)\phi_k\phi_{-k} + E(u)\phi_k''\phi_{-k}'' + F(u)\phi_k''\phi_{-k}' \Big) + \mathcal{K}, \end{aligned} \tag{5.5}$$

where $\mathcal{K}$ is a generalized Gibbons-Hawking term (see [24]):

$$\mathcal{K} = \frac{1}{16\pi G_N} \int dk (K_1 + K_2 + K_3), \tag{5.6}$$

with

$$K_1 = -A\phi_k'\phi_{-k}, \;\; K_2 = -\frac{F}{2}\phi_k'\phi_{-k}', \;\; K_3 = E(p_1\phi_k' + 2p_0\phi_k)\phi_{-k}. \tag{5.7}$$

The first term, K_1, is the usual Gibbons-Hawking contribution, while K_2 and K_3 are order $\mathcal{O}(\beta)$ corrections. The coefficients p_0, p_1 in K_3 are defined from the linearized equation of motion for ϕ, which separates into a β-independent and β-dependent piece. At leading order, the equation of motion for ϕ is again second order:

$$\phi'' + p_1\phi' + p_0\phi = \mathcal{O}(\beta). \tag{5.8}$$

This boundary term ensures that the variational principle is valid up to $\mathcal{O}(\beta^2)$.

To compute the shear viscosity, we integrate the action by parts, obtaining

$$\tilde{I}_\phi = \frac{1}{16\pi G_N}\int dkdu\left[(B-A-F'/2)\phi'_k\phi'_{-k} + E\phi''_k\phi''_{-k} + \frac{D-(C-A')'}{2}\phi_k\phi_{-k}\right] + \tilde{\mathcal{K}} \tag{5.9}$$

with boundary term

$$\tilde{\mathcal{K}} = \frac{1}{16\pi G_N}\int dk\left(K_3 + \frac{1}{2}(C-A')\phi_k\phi_{-k}\right). \tag{5.10}$$

The radial canonical momentum for the effective scalar is defined by

$$\Pi_k(u) \equiv \frac{\delta\tilde{I}_\phi}{\delta\phi'_{-k}} = 2\left(B-A-F'/2\right)\phi'_k(u) - (E\phi''_k(u))'. \tag{5.11}$$

The scalar equation of motion then has the form

$$\partial_u\Pi_k(u) = M(u)\phi_k(u), \quad M(u) \equiv 2\left(D - ((C-A')'/2)\right). \tag{5.12}$$

To compute the retarded Green's function, we evaluate the effective action on-shell, which as we mentioned in Chapter 2, it reduces (after manipulation) to a boundary term:

$$I_{on-shell} = \int \frac{d^4k}{(2\pi)^4}\,\mathcal{F}_k, \tag{5.13}$$

where by the AdS/CFT recipe [23], the Green's function is given by the flux at the boundary:

$$G^R_{xy,xy} = -\left.\frac{2\mathcal{F}_k}{\phi_k(u)\phi_{-k}(u)}\right|_{\text{boundary}}. \tag{5.14}$$

The denominator ensures proper normalization, and $\phi(u)$ is subject to infalling boundary conditions. Under these conditions, and substituting (5.12) into the action, the flux factor simplifies to

$$2\mathcal{F}_k = \Pi_k\phi_{-k} + (C-A')\phi_k\phi_{-k} + Ep_0\phi'_k\phi'_{-k}. \tag{5.15}$$

As we are interested in the imaginary part, we note that the second term does not contribute, and the third term is higher order in ω (see [33] for the explicit argumentation). Therefore, the only relevant piece is the canonical momentum. Using the Kubo formula, we have

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\text{Im}G^R_{xy,xy}(\omega,\mathbf{k}=0) = \lim_{\omega\to 0}\left.\frac{\Pi(u)}{i\omega\phi(u)}\right|_{boundary}, \tag{5.16}$$

where $\Pi(u)$ is the canonical momentum at $\omega \to 0$ and $\vec{k} \to 0$. We should notice that in this regime, the equations of motion reduce to

$$\partial_u \Pi_k(u) = 0, \tag{5.17}$$

so that the canonical momentum is independent of u. This means we can evaluate it at the horizon, an approach justified because the effective mass $M(u)$ goes as $\mathcal{O}(\omega^2)$ (see [33]).

To relate $\omega\phi(u)$, we impose infalling boundary conditions at the horizon $u = u_h$. These, together with regularity, lead (see [35]) to

$$\begin{aligned}
\partial_u \phi(u_h, t) &= -i\omega \sqrt{\frac{g_{uu}}{-g_{tt}}}\Bigg|_{u_h} \phi(u_h) + \mathcal{O}(\omega^2), && (5.18)\\
\partial_u^2 \phi(u_h, t) &= -i\omega\, \partial_u \left(\sqrt{\frac{g_{uu}}{-g_{tt}}}\right)\Bigg|_{u_h} \phi(u_h) + \mathcal{O}(\omega^2). && (5.19)
\end{aligned}$$

To leading order, one can show that $\omega\phi(u) = \omega\phi(u_h)$. Thus, combining everything, we obtain the formula for the shear viscosity:

$$\eta = \frac{1}{8\pi G_N}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\left(A(u) - B(u) + \frac{F'(u)}{2}\right) + \left(E(u)\left(\sqrt{-\frac{g_{uu}}{g_{tt}}}\right)'\right)'\right]\Bigg|_{u=u_h}. \tag{5.20}$$

Equation (5.20) provides a general formula for the shear viscosity in theories with higher curvature corrections. It is important to note that this result holds when working in the low-coupling limit of β, appropriate for using these models as effective (perturbative) descriptions of more complete theories.

5.3 General Formula for η/s in Dilatonic Quasi-Topological Gravities

The general formula derived above applies in the perturbative regime, where higher–curvature couplings β are small. A natural question is whether we can go beyond this approximation. Remarkably, there exists a special class of higher–curvature theories, the *quasi-topological gravities* (QTGs) [36], where we will argue that the result can be pushed to finite couplings. We now turn to these, since they will serve as the backbone of our numerical exploration, beginning with a brief introduction.

5.3.1 Dilatonic Quasi-Topological Gravities

Quasi-topological gravities (QTGs) are a subclass of the so-called Generalized Quasi-Topological Gravities (GQTGs) [37], which are higher curvature modifications of Einstein gravity with several special properties. These properties are summarized in [38]. The most important feature for our purposes is that, when linearized around a maximally symmetric background, as with the ansatz we will use in (5.24), the equations of motion remain second order. Furthermore, these theories admit generalized black hole solutions in asymptotically flat, de Sitter, or Anti-de Sitter backgrounds, with metrics of the type

$$ds^2 = \frac{1}{u^2}\left[-e^{-\chi(u)}f(u)dt^2 + \frac{1}{f(u)}du^2 + dx_{d-1}^2\right], \tag{5.21}$$

where the functions $f(u)$, $\chi(u)$ are determined by differential equations of order at most two. This crucial property: having only second-order equations for metric perturbations, enables the derivation of a formula for shear viscosity, η, that is valid even at finite coupling.

Let us revisit the previous arguments. The coefficients p_0 and p_1 in the boundary term K_3 from (5.7) are defined through the β-linearized equation of motion for ϕ, as given in (5.8). In general, higher curvature gravities give rise to fourth or higher order fluctuation equations. In [24], this difficulty is avoided by treating the higher-derivative terms perturbatively, working only to first order in the coupling β, and using the linearized fluctuation equation.

However, for QTGs, the fluctuation equations for ϕ are genuinely second order for any β, so equation (5.8) simply becomes

$$\phi'' + p_1\phi' + p_0\phi = 0, \tag{5.22}$$

where p_1 and p_0 may in general depend on β, but the form of the equation is always second order and remains valid non-perturbatively.

This means that the expression for the flux factor (5.15) is valid at finite β and does not rely on a small-coupling approximation. Therefore, the argument for extracting η using the canonical momentum Π_k at low frequency continues to hold for all values of β in QTGs.

Additionally, we previously noted that, to leading order in β, the relation $\omega\phi(u) = \omega\phi(u_0)$ holds at the horizon. For generic higher-derivative gravities one must be perturbative, but in QTGs the differential equations are always second order and this relation remains exact even at finite coupling.

Bringing these facts together, this elevates the previous perturbative result, Eq. (5.20), to a powerful, non-perturbative formula for the shear viscosity in Quasi-Topological Gravities, making these theories especially attractive for holographic transport studies.

Obtaining a Formula for the Entropy

Let us now stop for a moment and outline how to compute the entropy density for a general higher curvature gravity theory. This step is important, as we ultimately are interested in the ratio η/s.

For higher curvature theories, the Bekenstein-Hawking entropy formula does not generally apply. Instead, we use the Wald entropy formula [39]. In a $(d+1)$-dimensional spacetime, it reads:

$$S = -2\pi \int d^{d-1}x \sqrt{-h} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}, \tag{5.23}$$

where $\epsilon_{\mu\nu}$ is the antisymmetric binormal to the horizon cross-section Σ, and h is the induced metric on the horizon.

For a plane-symmetric, static metric,

$$ds^2 = -D(u)dt^2 + B(u)du^2 + C(u)d\vec{x}^2, \tag{5.24}$$

the only nonvanishing components of the binormal are $\epsilon_{ut} = -\epsilon_{tu} = \sqrt{-g_{tt}g_{uu}}$. Plugging this into (5.23), the total black hole entropy becomes

$$S = -8\pi \int d^{d-1}x \sqrt{h} \frac{\delta \mathcal{L}}{\delta R_{utut}} g_{tt}g_{uu}. \tag{5.25}$$

The entropy density is then given by

$$s = -8\pi \left. \frac{\delta \mathcal{L}}{\delta R_{utut}} g_{tt}g_{uu} \right|_{\text{horizon}}. \tag{5.26}$$

5.3.2 KSS Bound for Dilatonic Quasi-Topological Gravities

Having now all the tools needed, let us now explicitly derive the formula for the ratio η/s in the special case of dilatonic quasi-topological gravities (QTGs). We consider the following general theory:

$$S = \frac{1}{16\pi G_N} \int d^5x\sqrt{-g} \left(\sum_{n=1}^{\infty} G_n(\phi_d)\beta_n \ell^{2(n-1)} \mathcal{Z}_{(n)} - V(\phi_d) - \frac{1}{2}(\nabla\phi_d)^2 \right). \tag{5.27}$$

In this action, the curvature terms $\mathcal{Z}_{(n)}$ are defined specifically so as to construct a QTG (order $n=1$ corresponds to $\mathcal{Z}_{(1)} = R$, i.e., Einstein gravity[1]). Explicit expressions for all orders can be found in the literature; we follow the normalization used in [40], which gives particularly simple formulas. For this theory, our ansatz for the metric—the same family as (5.24)—is

$$ds^2 = \frac{1}{u^2} \left[-e^{-\chi(u)} f(u)dt^2 + \frac{1}{f(u)}du^2 + dx_{d-1}^2 \right]. \tag{5.28}$$

The procedure for obtaining general formulas for any order n is to work out first the cases for lower n, then extrapolate a generic formula by induction. For $n=1$ this simply reproduces

[1] We also set $G_1(\phi_d)\beta_1 = 1$ to recover pure Einstein gravity.

Einstein gravity; $n = 2$ yields the Gauss-Bonnet case (previously studied), though the non-minimal coupling to ϕ_d introduces corrections.

Order $n = 2$: Gauss-Bonnet Gravity

With the general formalism in hand, let us work out explicitly the first nontrivial case, $n = 2$, corresponding to Gauss–Bonnet gravity now coupled non-minimally to a dilaton. The action reads:

$$\begin{aligned}S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big[R - \frac{1}{2}(\nabla\phi_d)^2 - V(\phi_d)\\ +\frac{1}{6}\ell^2\beta_2 G_2(\phi_d)\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\Big].\end{aligned} \tag{5.29}$$

First, let us compute the entropy density s using Wald's formula. It can easily be shown that:

$$\frac{\partial R^2}{\partial R_{abcd}} = 2Rg^{a[c}g^{d]b}, \tag{5.30}$$

$$\frac{\partial R_{\mu\nu}R^{\mu\nu}}{\partial R_{abcd}} = g^{a[c}R^{d]b} - g^{b[c}R^{d]a}, \tag{5.31}$$

$$\frac{\partial R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}}{\partial R_{abcd}} = 2R^{abcd}. \tag{5.32}$$

Combining these results and computing Wald's formula for the metric ansatz (5.28), we obtain:

$$s = \frac{1}{4\pi G_N u^3}\left(1 - 2\ell^2\beta_2 f(u)G_2(\phi_d(u))\right). \tag{5.33}$$

The crucial point here is that the entropy acquires a correction proportional to the Gauss–Bonnet coupling β_2 and to the dilaton profile at the horizon.

Now let us derive the formula for the shear viscosity η. As in earlier sections, we expand the action to second order in the perturbation $h_x^{\ y}$, identify the coefficients for (5.5), and evaluate the result in the low frequency limit using (5.20). The final result is:

$$\begin{aligned}\eta = \frac{1}{16\pi G_N u^3}\Bigg(1 + \frac{1}{3}\beta_2\ell^2\Big[uf'(u)\left(G_2(\phi_d) - u\beta_2 G_2'(\phi(u))\phi_d'(u)\right)\\ +f(u)\left(-G_2(\phi_d(u))(2 + u\chi'(u)) + uG_2'(\phi_d(u))\phi_d'(u)(4 + u\chi'(u))\right)\Big]\Bigg).\end{aligned} \tag{5.34}$$

While this expression appears complex, it has a clear structure: the leading 1 is the standard Einstein gravity result, while all terms proportional to β_2 represent the corrections from the Gauss-Bonnet term and its non-minimal coupling to the dilaton.

To compute η/s, we simply divide the formulas and evaluate them at the horizon. For that purpose, we can expand the fields near $u = u_h$:

$$\begin{aligned}
\phi_d(u) &= \phi_d(u_h) + \phi_d'(u_h)(u-u_h) + \mathcal{O}\left[(u-u_h)^2\right], && (5.35)\\
f(u) &= f'(u_h)(u-u_h) + \mathcal{O}\left[(u-u_h)^2\right], && (5.36)\\
\chi(u) &= \chi(u_h) + \chi'(u_h)(u-u_h) + \mathcal{O}\left[(u-u_h)^2\right], && (5.37)
\end{aligned}$$

where the horizon is at $u = u_h$ and $f(u_h) = 0$. This yields:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{1}{3}\ell^2\beta_2\left[u_h G_2(\phi_d(u_h)) f'(u_h) - u_h^2 f'(u_h)\phi_d'(u_h) G_2'\left(\phi_d(u_h)\right)\right]\right). \tag{5.38}$$

The interesting thing of this result is that we get a deviation from the universal value of $\eta/s = 1/4\pi$ that comes not only from the Gauss-Bonnet term (second term of the parenthesis), but also from the non-minimally coupling of the dilaton (last term of the parenthesis).

The final expression for η/s (Eq. 5.38) depends on the values of the fields and their derivatives (f', χ', ϕ_d') evaluated at the horizon. These are not free parameters; they are determined by the background solution to the full, coupled equations of motion. Therefore, the final step in our analysis is to derive and solve these equations for a given choice of potential $V(\phi_d)$ and coupling $G_2(\phi_d)$.

Varying (5.29) with respect to the metric, using the plane symmetric ansatz from (5.21), we find [34]:

$$P_{acde}{R_b}^{cde} - \frac{1}{2}g_{ab}\mathcal{L} + 2\nabla^c\nabla^d P_{acbd} = \frac{1}{2}\partial_a\phi\partial_b\phi, \tag{5.39}$$

where

$$P^{abcd} \equiv \frac{\partial\mathcal{L}}{\partial R_{abcd}}. \tag{5.40}$$

The equation of motion for the scalar field ϕ is

$$\Box\phi - V'(\phi) + \ell^2\beta_2 G_2'(\phi_d)\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) = 0, \tag{5.41}$$

where a prime denotes differentiation with respect to ϕ. For (5.29)

$$P_{abcd} = \delta R_{abcd} + \frac{1}{6}\ell^2\beta_2 G_2(\phi_d)\left(2R\,\delta R_{abcd} - 8R_{ef}\delta {Ric^{ef}}_{abcd} + 2\delta\,\mathrm{Riem}\right), \tag{5.42}$$

where

$$\delta\,\mathrm{Riem} = g_{c[a}g_{b]d}, \qquad \delta Ric_{abcdef} = g_{(a|[d}g_{c][e}g_{f]|b)}.$$

Solving (5.39) and (5.41) using the given ansatz yields four independent differential equations, which can be simplified and manipulated to get compact equations for f, χ, and ϕ_d. The result is:

$$
\begin{aligned}
f'(r) =& \frac{2V(\phi(u)) + f(r)(24 + u^2(\phi_d'(u))^2)}{2u\left[1 - 2\ell^2\beta_2 f(u)(G_2(\phi_d(u)) - 3uG_2'(\phi_d(u))\phi_d'(u))\right]} \\
&+ \frac{-\ell^2\beta_2 f(u)^2\left(8G_2(\phi_d(u)) + 4u\left((u\phi_d'(u))^2 G_2''(\phi(u)) + G_2'(\phi(u))(-2\phi_d'(u) + u\phi_d''(u))\right)\right)}{2u\left[1 - 2\ell^2\beta_2 f(u)(G_2(\phi_d(u)) - 3uG_2'(\phi_d(u))\phi_d'(u))\right]}.
\end{aligned}
\tag{5.43}
$$

$$
\chi' = \frac{u(\phi_d')^2 - 2\ell^2\beta_2 f\left(4\phi_d' G_2' + 2uG_2'\phi_d'' + 2uG_2''(\phi_d')^2\right)}{3 - 2\ell^2\beta_2 f\left[G_2(\phi_d) - 3G_2'(\phi_d)u\phi_d'\right]}
\tag{5.44}
$$

$$
\begin{aligned}
\phi_d''(u) = \frac{1}{2u^2 f(u)}\Bigg(& -4\ell^2 u^2\beta_2\left(f'(u)\right)^2 G_2'(\phi_d(u)) + 2V'(\phi_d(u)) - 2u^2 f'(u)\phi_d'(u) \\
& + uf(u)\Big[\phi_d'(u)(6 + u\chi'(u)) + 2\ell^2\beta_2 f'(u)G_2'(\phi_d(u))(16 + 5u\chi'(u)) \\
& \qquad - 4\ell^2 u\beta_2 G_2'(\phi_d(u))f''(u)\Big] \\
& - 2\ell^2\beta_2 f(u)^2 G_2'(\phi_d(u))\left(20 + 8u\chi'(u) + u^2(\chi'(u))^2 - 2u^2\chi''(u)\right)\Bigg)
\end{aligned}
$$

These coupled, non-linear ODEs encode the full dynamics of the system. Although finding analytic solutions is generally intractable, this system is well-posed for numerical integration. By specifying the input functions—the potential $V(\phi_d)$ and the coupling $G_2(\phi_d)$—we can numerically solve for the black hole geometry and extract the required near-horizon data. This data will then be used as direct input for our master formula for η/s, Eq. (5.38), allowing us to explore its behavior beyond the perturbative regime.

Equations of Motion for Quasi-Topological Gravities Non-Minimally Coupled

The analysis for $n = 2$ illustrates the method. To capture the full tower of higher-curvature corrections, however, we need the general form of the equations of motion in quasi-topological gravities, which we now present using the ansatz (5.28). The normalization we follow is from [40]. As previously explained, the equations of motion are obtained by working out the first few orders in detail and then formulating the general case via induction.

In principle, one could derive the equations using the general form (5.39); however, with higher order terms this method quickly becomes impractical. An alternative is to use the *Principle of Symmetric Criticality*, which states that for a symmetric ansatz, the field equations can be derived by restricting the variation to fields with the same symmetry (see [41, 42]). Since we are considering plane-symmetric solutions, we can evaluate each higher curvature term $\mathcal{Z}_{(n)}$ using the ansatz (5.28), then derive the equations via the Euler-Lagrange equations for each field.

Doing this for the first few orders allows us to write the equations of motion as follows.

Equation of motion for the dilaton ϕ_d

$$\begin{aligned}u^2 f\phi_d'' = &\frac{1}{2}u\phi_d'(f(u\chi'+6)-2uf')+V'(\phi_d)+\\ &\sum_{n=1}\frac{1}{2}\left[n(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^n G_n'(\phi_d)\left(\frac{40}{n}+8u\chi'+u^2(\chi')^2-2u^2\chi''\right)\right]+\\ &\sum_{n=1}\frac{1}{2}\left[n(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^{n-1}G_n'(\phi_d)u(2uf''-(16+(2n+1)u\chi')f')\right]+\\ &\sum_{n=1}\frac{1}{2}\left[2n(n-1)(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^{n-2}G_n'(\phi_d)(f')^2u^2\right]\end{aligned} \tag{5.45}$$

Equation of motion for redshift factor χ

$$\chi' = \frac{u(\phi_d')^2+\sum_{n=1}n(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^{n-1}\left(4\phi_d'G_n'+2uG_n'\phi_d''+2uG_n''(\phi_d')^2\right)}{\sum_{n=1}(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^{n-1}n\left[(5-2n)G_n(\phi_d)-(2n-1)G_n'(\phi_d)u\phi_d'\right]} \tag{5.46}$$

Equation for blackening function f

$$\begin{aligned}f' =&\frac{2V(\phi_d)+fu^2(\phi_d')^2}{2u\sum_{n=1}(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^{n-1}n\left[(5-2n)G_n(\phi_d)-(2n-1)G_n'(\phi_d)u\phi_d'\right]}\\ &+\frac{\sum_{n=1}(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^n\left[8(5-2n)G_n+4nu(-2\phi_d'G_n'+uG_n'\phi_d''+u(\phi_d')^2G_n'')\right]}{2u\sum_{n=1}(-1)^{(n+1)}\ell^{2(n-1)}\beta_n f^{n-1}n\left[(5-2n)G_n(\phi_d)-(2n-1)G_n'(\phi_d)u\phi_d'\right]}\end{aligned} \tag{5.47}$$

The green coefficients means that these coefficients actually depend on the normalization used for the curvature terms. A different normalization for the quasi-topological terms should change only the green coefficients.

As for the entropy, it is also easy to derive a general expression, that reads as:

$$s = \frac{4\pi}{u^3}\sum_{n=1}n(-1)^{n+1}\ell^{2(n-1)}\beta_n f^{n-1}G_n \tag{5.48}$$

It is worth noting that while a general, closed-form expression for the entropy density s can be derived, the same is not true for the shear viscosity. Direct computations up to order four did not reveal any clear pattern that would allow an inductive formula. Nonetheless, computing η through the master formula is not as demanding (computationally) as evaluating the entropy, so having a general expression for the entropy is of particular value for further studies of these quasi-topological gravities and already extends the results in the literature.

5.4 Equation of state and η/s in Gauss–Bonnet–Dilaton gravity

So far our focus has been on viscosity itself. In particular, we showed how Gauss–Bonnet and higher QTGs with dilatonic corrections can drive η/s below the KSS bound, showcasing a master formula for the computation of such ratio. However, a realistic holographic model must also describe the background thermodynamics: pressure, entropy, and the speed of sound. These determine the equation of state of the dual plasma and are accessible in lattice simulations or heavy-ion experiments. Only after matching both transport and thermodynamic quantities can we fairly compare to QCD data. We then turn our attention into the speed of sound c_s.

The conformal $\mathcal{N} = 4$ SYM theory has an important feature, the speed of sound c_s is always equal to $1/\sqrt{3}$. However, QCD only shows conformal behavior in the high-temperature regime. In particular, the temperature dependence of the equation of state and transport coefficients such as the bulk and shear viscosities suggest a non-conformal behavior near the de-confinement transition. Thus, we are lead to consider gravity duals that break such conformal invariance, introducing black hole solutions.

An influential approach by Gubser and collaborators [43, 44] explored black hole solutions in five-dimensional Einstein-dilaton gravity, governed by the action

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right], \tag{5.49}$$

with appropriately chosen scalar potentials $V(\phi)$. These potentials were engineered to interpolate between AdS asymptotics in the UV and exponential behaviors in the IR, such that the resulting thermodynamics mimics the lattice QCD equation of state at zero chemical potential. Remarkably, simple families of potentials such as

$$V(\phi) = -\frac{12}{L^2} \cosh(\gamma\phi) + b\phi^2, \tag{5.50}$$

with γ and b being parameters of the potential, allow for a speed of sound $c_s^2(T)$ that captures the characteristic dip near the crossover temperature T_c, a feature observed $2 + 1$-flavor QCD simulations [43].

However, in all these constructions, the shear viscosity to entropy ratio remained fixed at the universal Einstein gravity value $\eta/s = 1/4\pi$.

This limitation motivates the inclusion of higher-derivative curvature corrections in the gravitational action. In particular, we consider the same Gauss–Bonnet–dilaton black-brane backgrounds of Sec. 5.1 to compute not only the shear viscosity but also the speed of sound, trying to match lattice computations, and to track how η/s varies with temperature. We proceed in four steps:

1. **Background ansatz and couplings.**

We work in five bulk dimensions with the action

$$S = \frac{1}{16\pi G_N} \int d^5x \sqrt{-g} \left[R + \frac{12}{L^2} + G(\phi_d)\beta\, \mathcal{L}_{\rm GB} - \frac{1}{2}(\partial\phi_d)^2 - V(\phi_d) \right], \tag{5.51}$$

where we recall that $\mathcal{L}_{\rm GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. A convenient choice of scalar potential and dilaton coupling that we will use for our numerical computations is

$$G(\phi_d) = e^{\delta\phi_d}\,, \qquad V(\phi_d) = -12\cosh(\gamma\phi_d) + b\phi_d^2, \tag{5.52}$$

where δ, γ and b are free parameters of the theory. The AdS radius gets affected by the Gauss-Bonnet term, so that we get an effective radius (see [45] for example)

$$L_{GB} = \sqrt{\frac{1}{2}\left(1 + \sqrt{1 - 8\beta}\right)}, \tag{5.53}$$

where we assumed that at the boundary $G(\phi_d) \to 0$.

We then adopt the following ansatz

$$ds^2 = \frac{L_e^2}{u^2}\left[-(1-u)U(u)\,dt^2 + \frac{du^2}{(1-u)U(u)} + \mathcal{V}(u)\,dx^i dx^i \right], \qquad \phi_d(u) = u^{d-\Delta}\varphi_d(u), \tag{5.54}$$

where we set $\phi_d \to 0$ as $u^{d-\Delta}$ near the AdS asymptotic boundary $u = 0$, $\mathcal{V}(u)$controls the spatial warp factor and with $d - \Delta = 2 - \sqrt{4 + m^2 L_e^2}$ determined by the scalar mass $m^2 = 2(b - 6\gamma^2)$. Asymptotically we enforce (5.54) to be AdS:

$$U(0) = \mathcal{V}(0) = \phi(0) = 1 \tag{5.55}$$

where setting the leading coefficient $\varphi_d = 0$ comes from the desired asymptotic behavior (see [43] for a detailed construction of solutions).

2. **Horizon expansion and shooting.**

To solve this system we impose regularity at the horizon and then use a shooting method to integrate outwards to the AdS boundary. Regularity at the horizon $u_h = 1$ demands the

$$(1-u)U(u) \propto (u-1), \qquad \varphi_d(u) \text{ and } \mathcal{V}(u) \text{ finite at } u_h = 1. \tag{5.56}$$

We Taylor expand each field about $u = 1 - \varepsilon$ with unknown horizon values $\{U_h, \mathcal{V}_h, \varphi_h\}$. The derivatives in the series are fixed recursively by the full bulk equations working order by order, so that ultimately all the coefficients only depend on the three *shooting* variables $\{U_h, \mathcal{V}_h, \varphi_h\}$. We then *shoot* from $u = 1 - \varepsilon$ to $u = \varepsilon$ using a standard ODE solver, varying $\{U_h, \mathcal{V}_h, \varphi_h\}$ until the boundary normalization $\{U, \mathcal{V}, \varphi_d\}|_{u\to 0} = (1,1,1)$ is achieved to a desired precision. The various initial values give rise to solutions with different temperatures and thus different values of c_s. They are solutions of the fixed gravitational system (in the sense of the same given action).

3. **Thermodynamics and speed of sound.**

Once a family of horizon data is found, the temperature and entropy density follow by

$$T = \frac{\kappa_h}{2\pi} = \frac{1}{2\pi}\sqrt{-\frac{n^{\mu;\nu}n_{\mu;\nu}}{2}}\Bigg|_{\text{horizon}} = \frac{U_h}{4\pi}, \qquad s = 4\pi\left(L_e^2\,\mathcal{V}_h\right)^{3/2}, \tag{5.57}$$

where T is the Hawking temperature of the black hole (2.36), n^μ is the Killing field that defines the Killing horizon at the event horizon of the black hole[2], and s is the entropy density that comes from Wald's formula, as derived previously.

The repeated shoot-and-solve method over a range of $\{U_h, \mathcal{V}_h, \varphi_h\}$ produces a data set $\{T_i, s_i\}$. Sorting by T, we form an interpolating function $s(T)$ and compute the speed of sound

$$c_s^2 = \frac{d\ln T}{d\ln s} = \frac{s}{T}\,\frac{dT}{ds}. \tag{5.58}$$

In practice, the derivative dT/ds is obtained by inverting the derivative of the interpolating function $s(T)$, while the ratio s/T is extracted from the same dataset. A plot of c_s^2 versus T/T_c (with T_c defined as the temperature at which c_s^2 attains its minimum) can then be compared to lattice QCD data.

In figure 5.1 we present the resulting sound velocity curves derived for various holographic models, alongside $2+1$ flavor QCD lattice results from [46,47]. The results demonstrate the phenomenological power of including higher-derivative terms:

While the baseline Einstein–dilaton gravity (yellow curve) qualitatively reproduces the expected behavior of the equation of state, with a dip in c_s^2 near the transition temperature, the inclusion of the Gauss–Bonnet term ($\beta \neq 0$) with appropriate values of the parameters (see models B, C and D) leads to an improved quantitative agreement with lattice data; we constructed model C which resembles very closely the lattice computation done in 2014 by Hot-QCD collaboration [47]. Furthermore, we observe that introducing a non-minimal coupling of the dilaton ($\delta \neq 0$) sharpens the dip, although in these cases the numerical computation of the curves in the vicinity of T_c remains delicate and prone to instabilities from computational errors.

From figure 5.1 one also observes (see Model C) that decreasing the parameter b in the potential softens the dip in the sound velocity and delays the approach to the conformal limit $c_s^2 \to 1/3$. In addition, we can conclude from the different models that the Gauss–Bonnet coupling β serves as a knob to control the depth of the minimum: larger β raises the minimum value of c_s^2, while

[2] A Killing horizon is a surface at which a Killing vector becomes null, and if it coincides with a black hole's action event horizon, then the surface gravity κ of the killing horizon is the surface gravity of the black hole

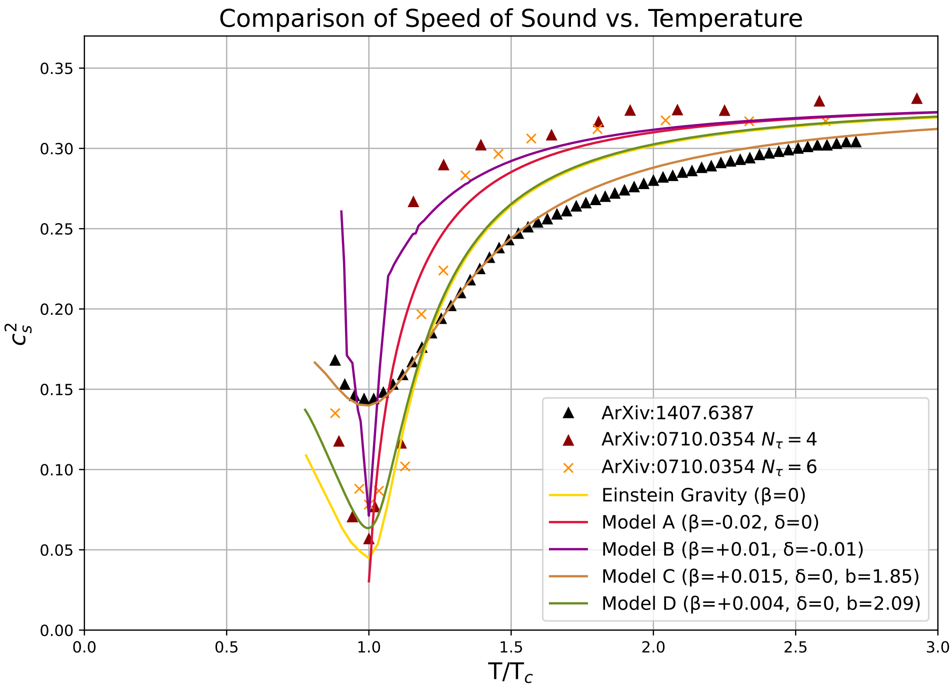

Figure 5.1: Speed of Sound as a function of Temperature, normalized to T_c. 2+1 lattice data comes from [46] (black and red triangles) and from [47] (yellow crosses). Except when noted otherwise, we are setting $\gamma = 0.606$ and $b = 2.06$. We can observe how adding the Gauss–Bonnet term ($\sim \beta \neq 0$) allows us to get a behavior that resembles more closely the equation of state than with just Einstein gravity (yellow line). Different values of the parameters allow us to modify the shape of the sound velocity and create a model that greatly resembles the lattice data, thus showcasing the versatility of higher–derivative holography.

smaller β lowers it. Another feature we have found for these models is that setting a negative Gauss–Bonnet coupling β does not replicate the dip in the sound velocity curve; it goes directly to zero.

The essential message is that higher-derivative holographic gravities provide sufficient flexibility to reproduce key features of the QCD equation of state as seen on the lattice, and allow for parametric control over the strength and position of the dip in c_s^2. Our holographic construction thus demonstrates how higher-curvature corrections enrich the phenomenological holographic landscape.

4. **Temperature-dependent η/s.**

For each numerical black hole solution, we also have all the necessary horizon data to compute the shear viscosity using our master formula (Eq. 5.20):

$$\eta = \frac{1}{8\pi G_N}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\left(A(u) - B(u) + \frac{F'(u)}{2}\right) + \left(E(u)\left(\sqrt{-\frac{g_{uu}}{g_{tt}}}\right)'\right)'\right]\Bigg|_{u=u_h}, \qquad (5.59)$$

and the ratio η/s is obtained simply by dividing the result by the entropy density. Evaluating this expression on each numerical solution yields $\eta/s(T)$.

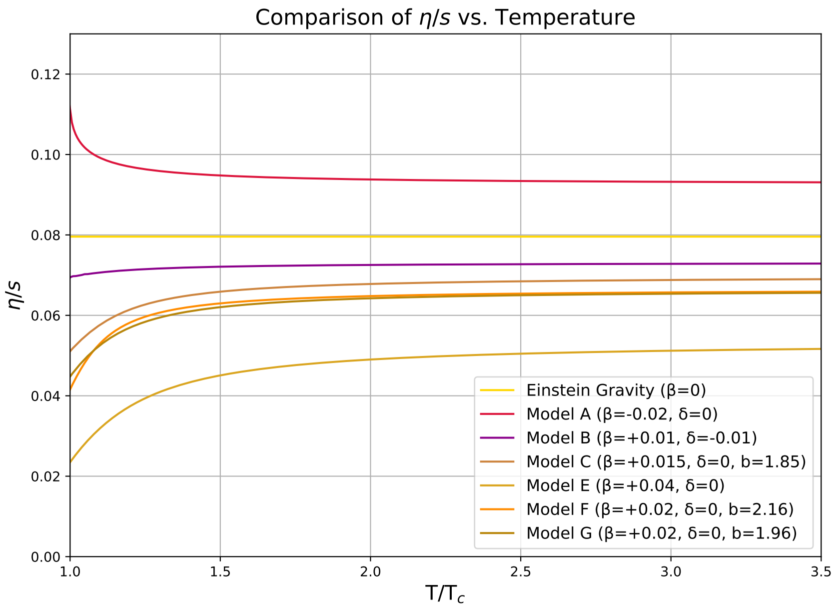

Figure 5.2: Shear viscosity to entropy ration as a function of Temperature, normalized to T_c. Except when indicated otherwise, we are setting $\gamma = 0.606$ and $b = 2.06$. We can see how adding the Gauss-Bonnet term allows us to get a T-dependent ratio, compared to the fixed value of $1/4\pi$ that one obtains with just Einstein gravity with a dilaton minimally coupled (yellow line). We observe that a negative β yields a ratio larger than the KSS bound while a positive β yields a smaller ratio than the bound.

In figure 5.2 we plot η/s against the temperature normalized to T/T_c. As expected, Einstein gravity yields the universal constant value $\eta/s = 1/4\pi$, independent of temperature. In contrast, incorporating the Gauss–Bonnet term introduces a non-trivial temperature dependence, even in the case of minimal coupling ($\delta = 0$). We find that varying β shifts the overall value of the ratio, making it larger than $1/4\pi$ if $\beta < 0$ or smaller if $\beta > 0$, while at the same time, changing the potential parameter b alters the slope of the curve. At high temperatures (going into the AdS boundary) the dilaton becomes null and the curve approach to the high-temperature limit,

$$\frac{\eta}{s} = \frac{1}{4\pi}(1 - 8\beta). \tag{5.60}$$

These findings are significant: higher-derivative terms can lead to a temperature-dependent η/s, in line with expectations from lattice QCD [48, 49] and theoretical results [50], thus extending the classical Einstein gravity result and underscoring the greater versatility of higher-curvature holography.

Ultimately, we have found that the numerical solutions show that Gauss–Bonnet–dilaton holography achieves both goals:

(i) it reproduces a QCD–like equation of state, with a dip in $c_s(T)$, and

(ii) it yields a nontrivial, temperature–dependent $\eta/s(T)$ that can even fall below $1/4\pi$.

These results highlight a central theme of this chapter: higher–derivative corrections and non-minimal scalar couplings greatly expand the holographic landscape, enabling models that capture both the thermodynamics and transport features observed in QCD. While this chapter focused specifically on Gauss–Bonnet holography, the methodology illustrates a broader principle: higher-derivative holographic theories provide a rich parameter space for phenomenological modeling of strongly coupled plasmas.

Chapter 6

Conclusions

In this thesis, we have explored the AdS/CFT correspondence as a tool to study the hydrodynamic properties of strongly coupled plasmas. Our focus has been on the ratio of shear viscosity to entropy density, η/s, and the equation of state, probed via the speed of sound c_s. The early chapters provided the motivation and technical background: first, a concise introduction to the AdS/CFT correspondence, its roots in string theory, and its motivation from black hole thermodynamics and the holographic principle; next, a detailed account of the holographic dictionary, clarifying how bulk fields map to sources and expectation values of operators in the boundary theory. Finally, an application of this dictionary to hydrodynamics, where we derived the universal Einstein gravity result $\eta/s = 1/4\pi$, the celebrated KSS bound.

The core of the thesis was the study of higher derivative gravity theories and their effects on transport. In Chapter 5 we analyzed how higher curvature corrections and non-minimal scalar couplings enrich the holographic description. We started introducing Gauss-Bonnet gravity, which shows how the universal Einstein value can be reduced, even violating the KSS bound. We then derived a general master formula for the shear viscosity valid in five-dimensional higher curvature theories coupled to a dilaton. This result holds perturbatively in general and we have argued that it extends non-perturbatively in quasi-topological gravities, thanks to the equations of motion remaining second order in these theories. Together with Wald's entropy formula, this provided a systematic framework for computing η/s across a wide range of higher-derivative models.

Our explicit analysis of Gauss-Bonnet dilaton gravity highlighted the richness of these theories. In this model the viscosity to entropy ratio acquires corrections both from the higher curvature term and from the non minimal coupling to the dilaton, leading not only to violations of the KSS bound but also to a temperature dependence of η/s. At the same time, we computed the thermodynamic background by solving the black hole equations of motion, extracting the speed of sound. We showed that while Einstein–dilaton gravity already exhibits the qualitative dip in c_s^2 near T_c, in line with lattice QCD, the inclusion of the Gauss-Bonnet term brings the holographic predictions into closer quantitative agreement with the data. Importantly, the same models predict a nontrivial temperature dependence of η/s, improving on the universal constant result of Einstein gravity and offering a more realistic description of the QGP.

Thus, the central outcome of this work is twofold: first, the development of a general framework for calculating shear viscosity and entropy in higher-derivative and dilatonic gravities; and second, the demonstration that these models can reproduce both the thermodynamics and transport features observed in strongly coupled QCD matter. These results show how higher-curvature corrections expand the holographic landscape and allow for phenomenologically relevant results beyond the universal predictions of Einstein gravity.

Future Directions.

Finally, we should notice that these results naturally invite future extensions. In particular, the inclusion of higher–derivative terms unlocks a broad parameter space for holographic model building. A natural next step is to systematically map this space, for instance by applying modern machine learning techniques to identify models that best reproduce lattice QCD thermodynamics and transport data. Extending the present analysis could bring such holographic models closer to the conditions realized in heavy–ion collisions. In this way, higher derivative gravities stand out as versatile and tunable frameworks for phenomenological applications of holography to strongly coupled plasmas.

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