BOWDOIN COLLEGE MATH DEPARTMENT

MATH 3602: ADVANCED TOPICS IN GROUP THEORY

FINAL PROJECT

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Contents

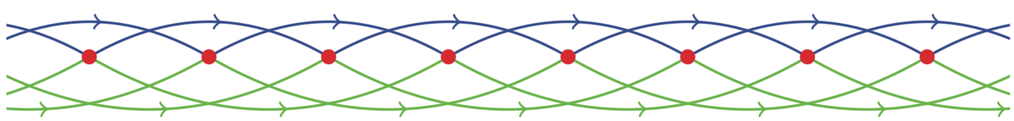
1. What is a quasi-isometry

We have considered how we can view a group as a metric space: namely, by constructing the Cayley graph of the group and taking our metric to be the shortest edge path. Equivalently, we consider the word metric on our group. There is a not so subtle problem with this metric on our group: it depends on our choice of generating set. We should now naturally wonder, why bother with thinking of a group as a metric space, if just changing the generating set changes the metric space? We are now led to try and construct a notion of equivalence between metric spaces, with a twofold hope:

(1) Our notion of equivalence is coarse enough so as to render equivalent two metric spaces for the same group using different generating sets, and
(2) is fine enough so as to distinguish between groups that are essentially dissimilar.

To motivate our definitions, we begin by considering our usual example of two different Cayley graphs for the additive integer group $\mathbb{Z}$.

We draw two Cayley graphs, $\Gamma(\mathbb{Z}, \{1\})$ which uses $\{1\}$ as the generating set, and $\Gamma(\mathbb{Z}, \{2, 3\})$ using $\{2, 3\}$ as the generating set.



Let $f : \mathbb{Z} \to \mathbb{Z}$ be the identity map. We can think of f as a map from the vertices of $\Gamma(\mathbb{Z}, \{1\})$ to the vertices of $\Gamma(\mathbb{Z}, \{2, 3\})$. We leave as an easy exercise to verify that $d_{\{1\}}(-2, 5) = 7$ and $d_{\{2,3\}}(-2, 5) = 3$. A consequence is that f will not be an isometry, since it clearly does not preserve distances. It turns out however, that this distortion of distances under f is bounded, and this is what motivates the idea of Bi-Lipschitz equivalence. Now for some definitions.

Definition 1.1 (Isometric embeddings and isometries)**.** Let (X, d_X) and (Y, d_Y) are metric spaces, a function $f : X \to Y$ is called an isometric embedding if f preserves distances, that is, for all $x_1, x_2 \in X$

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Note the term "embedding" is appropriate since preserving distances automatically implies injectivity. Now, an isometric embedding is called an isometry if it is also surjective.

We will now weaken the notion of an isometry by allowing distances to be stretched and compressed, but by bounded amounts.

Definition 1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called a bi-Lipschitz embedding if there is some constant $K \geqslant 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2).$$

Once again, we can check that these inequalities automatically imply injectivity. Of course, if $K = 1$, this will reduce to an isometric embedding. When $K > 1$, the left-hand inequality allows us to shrink distances by a factor of at most $\frac{1}{K}$, and the right-hand inequality allows us to dilate by a factor of at most K. A bi-Lipschitz embedding f is a bi-Lipschitz equivalence if it is also surjective.

Now for an exercise, which justifies our use of the term equivalence.

Problem 1.3. *Show that bi-Lipschitz equivalence is an equivalence relation on the collection of all metric spaces.*

Proof. Let $X \sim Y$ denote the existence of a surjective bi-Lipschitz embedding f from X to Y. We will show $\sim$ is an equivalence relation.
Reflexivity follows by taking f to be the identity map and $K = 1$.
Now for symmetry. Suppose $(X, d_X) \sim (Y, d_Y)$. Then, there exists a bijection f s.t.

$$\frac{1}{K} d_X(x_1, x_2) \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2).$$

We know f^{-1} exists. Consider $d_X(f^{-1}(y_1), f^{-1}(y_2))$. Then, using the fact that f is bi-Lipschitz we obtain

$$\frac{1}{K} d_X(f^{-1}(y_1), f^{-1}(y_2)) \leqslant d_Y\Big(f\big(f^{-1}(y_1)\big), f\big(f^{-1}(y_2)\big)\Big) = d_Y(y_1, y_2)$$

and similarly

$$d_Y(y_1, y_2) = d_Y\Big(f(f^{-1}(y_1), f(f^{-1}(y_2))\Big) \leqslant K d_X(f^{-1}(y_1), f^{-1}(y_2)).$$

But these can be combined to write

$$\frac{1}{K} d_Y(y_1, y_2) \leqslant d_X(f^{-1}(y_1), f^{-1}(y_2)) \leqslant K d_Y(y_1, y_2),$$

which of course by definition means $Y \sim X$.
Finally, we show transitivity. Suppose $(X, d_X) \sim (Y, d_Y)$ with $f : X \to Y$, and $(Y, d_Y) \sim (Z, d_Z)$ with $g : Y \to Z$. Then, consider $g \circ f : X \to Z$ which is a bijection because it is a composition of two bijections. We will show $g \circ f$ is also bi-Lipschitz. Consider $x_1, x_2 \in X$ and $y_1 = f(x_1), y_2 = f(x_2) \in Y$. First, because f is bi-Lipschitz we have

$$\frac{1}{K} d_X(x_1, x_2) \leqslant d_Y(y_1, y_2) \leqslant K d_X(x_1, x_2).$$

But also because g is bi-Lipschitz

$$\frac{1}{K'} d_Y(y_1, y_2) \leqslant d_Z(g(y_1), g(y_2)) \leqslant K' d_Y(y_1, y_2).$$

But we can substitute the first set of inequalities in the second to write

$$\frac{1}{KK'} d_X(x_1, x_2) \leqslant d_Z(g(y_1), g(y_2)) \leqslant KK' d_X(x_1, x_2)$$

or more explicitly

$$\frac{1}{KK'} d_X(x_1, x_2) \leqslant d_Z\Big((g \circ f)(x_1), (g \circ f)(x_2)\Big) \leqslant KK' d_X(x_1, x_2).$$

This demonstrates transitivity. □

Finally, we are in a position to state and prove the theorem relevant for to metrics on groups. In particular, this theorem tells us that two word metrics originating from two different generating sets aren't all that different - they are always bi-Lipschitz equivalent. In words, changing the generating set can stretch or compress distances, but only by a bounded amount.

Theorem 1.4. *Let G be a finitely generated group and let S and S' be two finite generating sets for G. Then the identity map $f : G \to G$ is a bi-Lipschitz equivalence from the metric space (G, d_S) to the metric space $(G, d_{S'})$.*

Proof. We want to show
$$\frac{1}{K}d_S(g,h) \leqslant d_{S'}(g,h) \leqslant K d_S(g,h)$$
for a constant $K \geqslant 1$.
But notice that $d_S(g,h) = d_S(1, g^{-1}h)$ because a group acts on its Cayley graph (through left-multiplication) by isometries, and similarly $d_{S'}(1, g^{-1}h) = d_{S'}(1, g^{-1}h)$. So it is enough to show that for all $g \in G$, $\exists K \geqslant 1$ s.t.
$$\frac{1}{K}d_S(1,g^{-1}h) \leqslant d_{S'}(1,g^{-1}h) \leqslant K d_S(1,g^{-1}h).$$
We first show the right-hand inequality holds.
Here's where it is important that S is finite, we can define
$$M = \max\{d_{S'}(1,s) \mid s \in S \cup S^{-1}\}.$$
Suppose g has word length n in S, i.e. $d_S(1,g) = n$; this means we can write $g = s_1 \cdots s_n$ for $s_i \in S \cup S^{-1}$. The triangle inequality tells us
$$\begin{aligned}
d_{S'}(1,g) &= d_{S'}(1, s_1 \cdots) \\
&\leqslant d_{S'}(1,s_1) + d_{S'}(s_1, s_1 s_2 \ldots s_k) \\
&\leqslant d_{S'}(1,s_1) + d_{S'}(s_1^{-1}s_1, s_1^{-1}s_1 s_2 \ldots s_k) \\
&\leqslant d_{S'}(1,s_1) + d_{S'}(1, s_2 \ldots s_k) \\
&\vdots \\
&\leqslant d_{S'}(1,s_1) + \cdots + d_{S'}(1,s_k) \\
&\leqslant Mk = M d_S(1,g)
\end{aligned}$$
So, $d_{S'}(1,g) \leqslant M d_S(1,g)$ shows the right-hand inequality.
A perfectly symmetric argument obtained by switching S and S' will give us the inequality $d_S(1,g) \leqslant M d_{S'}(1,g)$, which rearranged is just $\frac{1}{M}d_S(1,g) \leqslant d_{S'}(1,g)$, the left-hand inequality. □

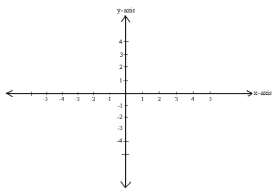
If we return to our original example of $\Gamma(\mathbb{Z}, \{1\})$ and $\Gamma(\mathbb{Z}, \{2,3\})$, then our result as applied to these two states that the identity map $f : \mathbb{Z} \to \mathbb{Z}$ is bi-Lipschitz equivalence from $\mathbb{Z}$ with the word metric with respect to 1, to $\mathbb{Z}$ with the word metric with respect to $\{2,3\}$. Equivalently, the identity map is a bi-Lipschitz equivalence from the vertex set of one Cayley graph to the other, each equipped with their respective path metrics.

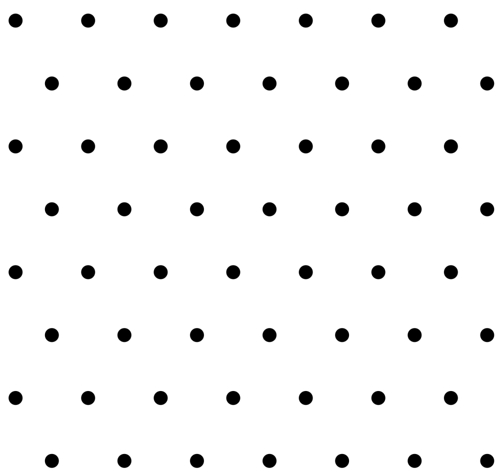
We now recall a subtle detail about our metric on a Cayley graphs. We had treated edges as part and parcel of this metric, whereby we treat edges as a copy of the unit interval and extend the metric to include points on edges. That is, we are working with the complete geometric realizations of of two Cayley graphs. So we must ask, where do we want the edges to map to, for example, when we consider equivalence between $\Gamma(\mathbb{Z}, \{1\})$ and $\Gamma(\mathbb{Z}, \{2,3\})$. It turns out to be too much to hope that this mapping will also be bi-Lipschitz, and so we now present our final (successful) attempt at equivalence.

Definition 1.5. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called a quasi-isometric embedding if there are some constants $K \geqslant 1$ and $C \geqslant 0$ so that for all $x_1, x_2 \in X$
$$\frac{1}{K}d_X(x_1,x_2) - C \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1,x_2) + C.$$
With $C = 0$ this is just a bi-Lipschitz embedding, and further if $C = 0, K = 1$ this is an isometric embedding. We think of $C > 0$ as being an error term that allows f to be bad on small scales. Note that this time the term "embedding" is slightly misleading: the $-C$ prevents the inequality from requiring f to be injective.

Definition 1.6. A quasi-isometric embedding $f : X \to Y$ is called a quasi-isometric equivalence, or just a quasi-isometry, if $\exists D > 0$ s.t. for any $y \in Y$, there is an $x \in X$ so that $d_Y(f(x), y) \leqslant D$. That is, every point in Y is within distance D of some point in the image of f. Notice that if $D = 0$, this is exactly surjectivity. For $D > 0$, we say f is coarsely surjective. We say that metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there is a quasi-isometry $f : X \to Y$.

An example of a quasi-isometry: $\varphi : \mathbb{R}^2 \to \mathbb{Z}^2$ where φ sends any point to the nearest lattice point.





We would of course like quasi-isometric equivalence to indeed be an equivalence relation. This is the purpose of solving Problem 1.9. First, a couple more definitions.

Definition 1.7. Let $f, g : X \to Y$ be maps between metric spaces. We say f and g have finite distance if there exists a real number $c \geqslant 0$ such that $d(f(x), g(x)) \leqslant c$ for all $x \in X$.

Definition 1.8. Let X and Y be metric spaces. A map $g : Y \to X$ is a quasi-inverse for $f : X \to Y$ if $gf = g \circ f$ has finite distance from Id_X and $fg = f \circ g$ has finite distance from Id_Y . That is, there exists a constant $r \geqslant 0$ so that for all $x \in X$, we have $d_X(gf(x), x) \leqslant r$, and there exists a constant $s \geqslant 0$ so that for all $y \in Y$ we have $d_Y(fg(y), y) \leqslant s$.

Problem 1.9. *Show that*

(a) A quasi-isometric embedding $f : X \to Y$ is a quasi-isometry if and only if f has a quasi-inverse $g : Y \to X$.
(b) Any quasi-inverse for f is a quasi-isometry.
(c) A composition of quasi-isometries is a quasi-isometry.

Proof. (a) ($\Longleftarrow$) Suppose $f : X \to Y$ is a quasi-isometric embedding with quasi-inverse $g : Y \to X$, then there exists $r \geqslant 0$ so that for all $y \in Y$ we have $d_Y(f(g(y)), y) \leqslant r$. So for any $y \in Y$, there is a $x \in X$ s.t. $d_Y(f(x), y) \leqslant r$, where $x = g(y)$. So, f is a quasi-isometry.
($\Longrightarrow$) Conversely, suppose $f : X \to Y$ is a quasi-isometry. So, $\exists K \geqslant 1, C \geqslant 0$ s.t.

$$\frac{1}{K} d_X(x_1, x_2) - C \leqslant d_Y(f(x_1), f(x_2)) \leqslant K d_X(x_1, x_2) + C.$$

Also, f is coarsely surjective: so for any y there is an x s.t. $d_Y(f(x), y) \leqslant D$. Set $R = \max\{C, K, D\}$, and observe that the following holds

$$\frac{1}{R}d_X(x_1, x_2) - R \leqslant d_Y(f(x_1), f(x_2)) \leqslant Rd_X(x_1, x_2) + R,$$

and for any y there is an x s.t. $d_Y(f(x), y) \leqslant R$. Let $B_R(y)$ denote the closed R-neighborhood of $y \in Y$. Then, consider the family $\{f^{-1}(B_R(y))\}_{y \in Y}$. It is clear that this family covers X, since for any $x \in X$, we see that $x \in f^{-1}(B_R(f(x)))$. We now use the axiom of choice to construct $g : Y \to X$ where we choose $g(y) \in f^{-1}\big(B_R(y)\big)$. Hence, $fg(y) \in f\Big(f^{-1}\big(B_R(y)\big)\Big) \subset B_R(y)$, which means $d_Y(fg(y), y) \leqslant R$, for any $y \in Y$. This is half of showing g is quasi-inverse. We now need to show $d_X(gf(x), x) \leqslant r$ for some r.

Since f is a quasi-isometry, we can use the left-hand inequality with $R = \max\{C, K, D\}$ as before, for $x_1 = x, x_2 = gf(x)$, to conclude that

$$\frac{1}{R}d_X(gf(x), x) - R \leqslant d_Y(f(gf(x)), f(x)) = d_Y(fg(f(x)), f(x))$$

But observing $d_Y(fg(f(x)), f(x))$, we see that by construction as shown before we must have $d_Y(fg(f(x)), f(x)) \leqslant R$. So, we can write

$$\frac{1}{R}d_X(gf(x), x) - R \leqslant d_Y(fg(f(x)), f(x)) \leqslant R$$

$$\implies d_X(gf(x), x) \leqslant 2R^2,$$

which shows g is a quasi-inverse for f.

(b) Let g be a quasi-inverse for f. Observe our definition of a quasi-inverse and notice it is symmetric in f and g. So f is quasi-inverse for g. But now by part (a), it is enough to show g is a quasi-isometric embedding to prove g is a quasi-isometry. Let $y, y' \in Y$. Set $R = \max C, K, r, s$ where C, K are the constants for f being a quasi-isometric embedding, and r, s come from the quasi-inverse relation between f and g. Note $R^2 \geqslant C$ because $R \geqslant K \geqslant 1$. Now since f is a quasi-isometry,

$$\begin{aligned}
d_X(g(y), g(y')) &\leqslant Rd_Y(fg(y), fg(y')) + R^2 \\
&\leqslant R(d_Y(fg(y), y) + d_Y(y, y') + d_Y(y', fg(y'))) + R^2 \\
&\leqslant R(R + d_Y(y, y') + R) + R^2 \\
&= Rd_Y(y, y') + 3R^2.
\end{aligned}$$

where we used the triangle inequality and also the fact that f is a quasi-inverse for g.

Rearranging the triangle inequality

$$d_Y(y, y') \leqslant d_Y(y, fg(y)) + d_Y(fg(y), fg(y')) + d_Y(fg(y'), y').,$$

we obtain

$$d_Y(fg(y), fg(y')) \geqslant d_Y(y, y') - d_Y(y, fg(y)) - d_Y(fg(y'), y').$$

We now have

$$\begin{aligned}
d_X(g(y), g(y')) &\geqslant \frac{1}{R}d_Y(fg(y), fg(y')) - 1 \\
&\geqslant \frac{1}{R}(d_Y(y, y') - d_Y(y, fg(y)) - d_Y(fg(y'), y')) - 1 \\
&\geqslant \frac{1}{R}(d_Y(y, y') - R - R) - 1 \\
&\geqslant \frac{1}{R}d_Y(y, y') - 3
\end{aligned}$$

Since $R \geqslant K \geqslant 1$, letting $K = R, C = 3R^2$ give us our desired constants for g to be a quasi-isometric embedding.

(c) Suppose $f : X \to Y$ and $g : Y \to Z$ are quasi-isometries. Then we can choose $R \geqslant 1$ (set to max of all 4 constants) so that

$$\frac{1}{R} d_X(x, x') - R \leqslant d_Y(f(x), f(x')) \leqslant R d_X(x, x') + R,$$

and

$$\frac{1}{R} d_Y(y, y') - R \leqslant d_Z(g(y), g(y')) \leqslant R d_Y(y, y') + R.$$

Now, playing with both left-hand inequalities gives us

$$\begin{aligned} \frac{1}{R^2} d_X(x, x') - 1 - R &\leqslant \frac{1}{R} d_Y(f(x), f(x')) - R \\ &\leqslant d_Z(gf(x), gf(x')) \\ &\leqslant R d_Y(f(x), f(x')) + R \\ &\leqslant R(R d_X(x, x') + R) + R \\ &= R^2 d_X(x, x') + R^2 + R. \end{aligned}$$

Note that $-R^2 - R \leqslant -1 - R$ because $R \geqslant 1$. So we can use the above to write

$$\frac{1}{R^2} d_X(x, x') - (R^2 + R) \leqslant d_Z(gf(x), gf(x')) \leqslant R^2 d_X(x, x') + (R^2 + R).$$

So, we have shown that $gf : X \to Z$ is a quasi-isometric embedding. What's left is coarse surjectivity.

Consider some $z \in Z$. Because g is coarsely surjective, $\exists y \in Y$ with $d_Z(g(y), z) \leqslant R$ and there exists $x \in X$ with $d_Y(f(x), y) \leqslant R$. So,

$$d_Z(gf(x), z) \leqslant d_Z(gf(x), g(y)) + d_Z(g(y), z) \leqslant (R d_Y(f(x), y) + R) + R \leqslant R^2 + 2R.$$

This shows gf is coarsely surjective, finishing the proof. □

Now for the point of doing all this work. After everything, this is just a corollary of Problem 1.9.

Corollary 1.10. *Quasi-isometry is an equivalence relation on the set of metric spaces.*

Proof. Reflexivity follows from taking the identity map.
Symmetry follows from parts (a) and (b) of Problem 1.9
Transitivity follows from part (c). □

We are now in a position to prove the main result, which justifies quasi-isometry as the right way of thinking about equivalence of Cayley graphs.

Theorem 1.11. *Let G be a finitely generated group and let S and S' be two finite generating sets for G. Then the geometric realization of the Cayley graph $\Gamma(G, S)$ is quasi-isometric to the geometric realization of the Cayley graph $\Gamma(G, S')$.*

Proof. First, there is a quasi-isometry from the geometric realization of any graph, to its vertex set equipped with the path metric. This is done by sending every point on an edge to a nearest vertex (choose any if there are two choices). We already showed in Theorem 1.4 that the identity map on the vertices $G \to G$ induces a bi-Lipschitz map between the vertex sets of $\Gamma(G, S)$ and $\Gamma(G, S')$. Looking at definitions, it is clear that a bi-Lipschitz equivalence is a quasi-isometry as well: inequalities are satisfied with $C = 0$, and also coarsely surjective because it is surjective. So, we have quasi-isometric equivalences as indicated in the diagram below. Corollary 1.10 showed that this is indeed an equivalence relation, and it follows that $\Gamma(G, S)$ is quasi-isometric to $\Gamma(G, S')$. □

2. Large-Scale Geometry of BS(m, n)

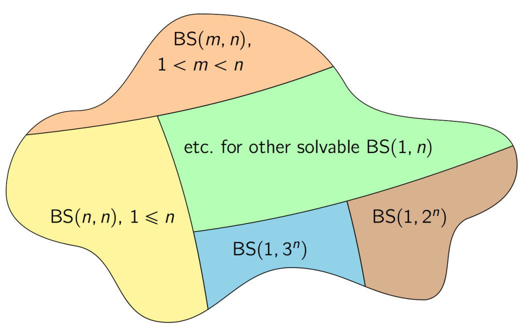
To study the Large-Scale Geometry of BS(m, n), we need to understand the quasi-isometry classes of BS(m, n). The first major result regarding this comes in the following:

Theorem 2.1. *$BS(1, p)$ is Quasi-Isometric to $BS(1, q)$ if and only if $p = n^r$ and $q = n^s$ for some $n, r, s \in \mathbb{Z}_{\geqslant 0}$.*

Note that a Baumslag-Solitar Group is solvable if and only if it is of the form BS$(1, n)$ for some $n \in \mathbb{Z}$. The result demonstrates that there are infinitely many distinct quasi-isometry classes of solvable Baumslag-Solitar groups.

In contrast, there are precisely two distinct quasi-isometry classes of solvable Baumslag-Solitar groups. This was given by Kevin Whyte.

Theorem 2.2 (2001, Whyte)**.** *$BS(2, 3)$ is Quasi-Isometric to $BS(p, q)$ for $1 < p < q$. This Quasi-Isometry class is distinct from those of the solvable Baumslag Solitar Groups.* BS(m, m) *is Quasi-Isometric to* BS(n, n) *for any $n \geqslant n$. Moreover, this Quasi-Isometry class is distinct from those discussed prior.*



Whyte's proof is technical and dense. In 2004, a more elementary proof of the same result was provided by researchers in Utah. The purpose of the remainder of this paper is to outline this proof; in the interest of brevity and readability, we do not go into complete technical depth, but instead only present the major propositions and how they all tie together to give the final result.

First, define the basic relator $ab^pa^{-1} = b^q$ of BS(p, q) as a geometric 'horobrick', infinity copies of such horobricks are glued side-by-side by matching vertical a edges to make (infinitely long) horostrips. Horostrips are glued along horocycles (b-axes) to create sheets.

Proposition 2.3. *Each sheet of* BS(p, q) *is quasi-isometric with the Euclidean ($p = q$) or hyperbolic ($p < q$) plane.*

Proposition 2.4. *The Cayley graph of* BS(p, q) *is quasi-isometric to a corresponding two-complex X, obtained by replacing each sheet with a copy of the Euclidean plane ($p = q$) or the (upper half space model of the) hyperbolic plane ($p < q$) in the obvious fashion.*

Viewed topologically, X is homeomorphic to (spine tree) $\times$ (real line). Since quasi-isometry is an equivalence relation, for results concerning quasi-isometries the previous proposition allows us to work with X, rather than the more cumbersome Cayley graph. On the other hand, the quasi-isometry between glued sheets of the graph and X ensures that any geodesic in X can be approximated (up to quasi-isometry) by a continuous 'piecewise linear' path consisting of vertical and horizontal segments just as in the Cayley graph.

We refer henceforth to the spine tree (obtained by viewing the Cayley graph from the side, as we have done in class), also known as the Basse-Serre tree, by T.

Theorem 2.5 (Whyte)**.** *For all $m, n > 1$, the groups* BS(m, m) *and* BS(n, n) *are quasi-isometric.*

Proof. We assume $m < n$. The spine trees associated to each group are regular with valences $2m$ and $2n$ (same number of sheets leaving each level as coming in), respectively, hence quasi-isometric by say the inclusion (or contraction) map f. Every sheet is a copy of the Euclidean plane. Each complex is isometric to (tree)$\times$(real line). We can then extend f to a quasi-isometry of complexes, namely $f \times$ dilation, where the second factor is a horizontal stretch by $\frac{n}{m}$. $\square$

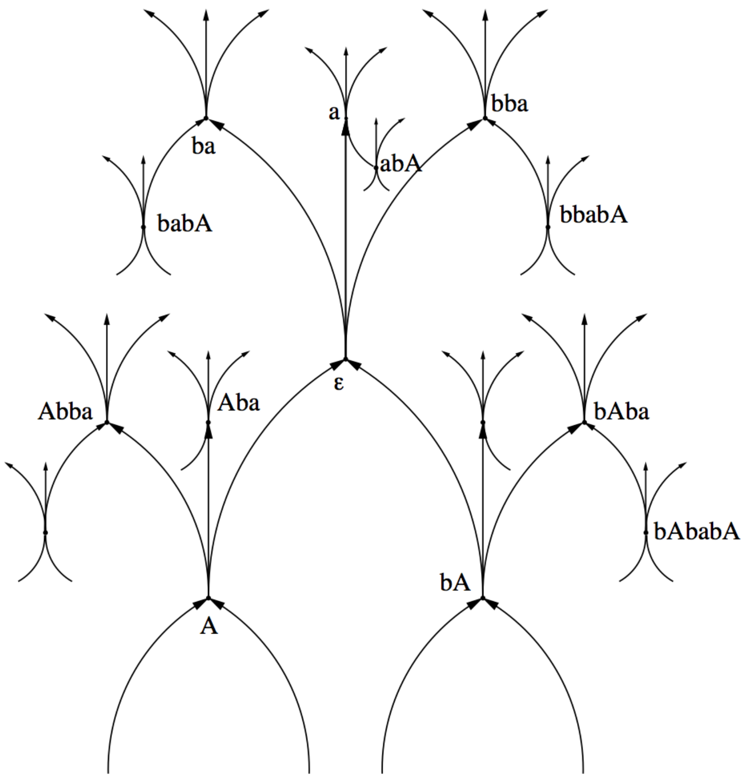
We will now focus on the quasi-isometry class for $\mathrm{BS}(m,n)$ where $m \neq n$, and provide an outline of the major constructions utilized in this proof.

The goal is to show $\mathrm{BS}(m,n)$ is quasi-isometric to $\mathrm{BS}(m+1,n)$, where $m+1 < n$. Note now that it is easily shown that $\mathrm{BS}(m,n) \cong \mathrm{BS}(n,m)$ (map a to a^{-1}); and this coupled with induction is enough to show that all $\mathrm{BS}(m,n)$ with m,n are quasi-isometric to each other.

On studying the Cayley graph of $\mathrm{BS}(p,q)$, and then its associated complex X and spine tree T, it is not hard to see that each vertex of T has p edges coming in, and q edges going out.

We now want to start with the spine tree for $\mathrm{BS}(p,q)$, and attempt to slide a horostrip over to another vertex (in a very particular fashion which we describe shortly), so as to increase the in-valence of all vertices by 1. Doing this in a metric preserving fashion will give us the tree for $\mathrm{BS}(p+1,q)$. If we can show this "algorithm" extends to the associated complexes X in a quasi-isometric fashion, then we would be done. This is the main idea, now it requires explication; what follows is an outline.

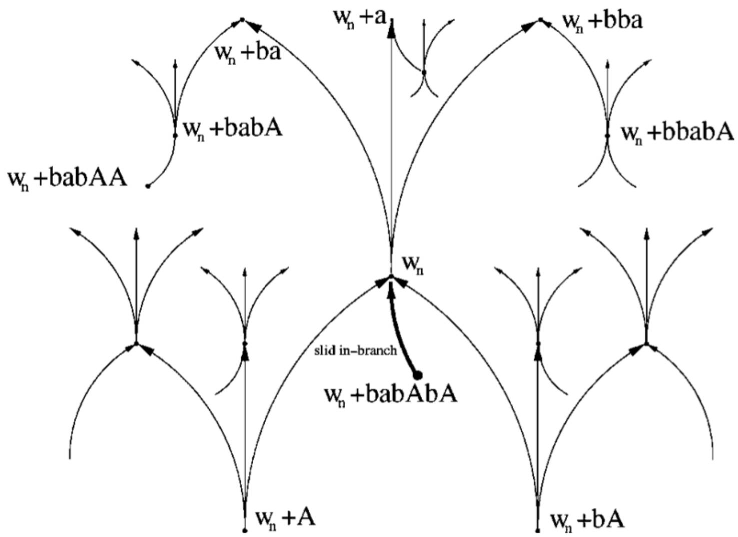
We claim (without proof) that in order to glue two horostrips of X in a metric-preserving fashion without the arbitrary scaling of one or the other, the two horostrips must come from adjacent levels of X. For a rough idea, if we tried to do this with horostrips from different levels, we would be required to move points by non-uniform distances. So, we describe a level-preserving method of sliding certain in-branches of T to increase the in-valence of each vertex from p to $p+1$. It is important that we first construct a well-ordering on the vertices of T. We describe roughly how this goes by example of working with $\mathrm{BS}(2,3)$. First, we associate a, b, a^{-1}, b^{-1} with a, b, A, B respectively. Then, observe in the following diagram how the vertices our labeled with the path taken to get to that level. So, for example, starting from the identity ϵ, we can go up (a) and stay in the same sheet; we can go over and up (ba), or over twice and up (bba), and go into different sheets; we can also down (A), or over and down (bA). In this way, we label all vertices of the tree T. We endow these labels with a short-lex well-ordering. If the length of w_1 is less than that of w_2 we define $w_1 < w_2$. If lengths are equal, use lexicographical order with the convention that $a < A < b$ (note B never appears in our labels, because it is a side view of the tree it is redundant to go left to change levels/sheets). There is more to the labels than we offer here (we don't need to for our present purposes), and they are formally known as Knuth-Bendix-Thurston (KBT) prefixes.



Here is the level-slide algorithm, which accomplishes our goal of increasing in-valence by 1.

LEVEL SLIDE ALGORITHM 3.6. Assume $1 < p < q$. Given the directed tree T with vertices labelled by KBT reduced prefixes:

For each $n > 0$ do
  Find the *nth* vertex labelled w_n
  Delete the upward edge $[w_n + babAbA, w_n + babA]$ and add the upward edge $[w_n + babAbA, w_n]$
  If the in-valence of w_n equals p
    Delete the upward edge $[w_n + bbabAbA, w_n + bbabA]$ and add the upward edge $[w_n + bbabAbA, w_n]$
Increment n.



We do not explain formally as to why this algorithm behaves as expected, but remark that it is not a hard exercise to follow it and see that it does indeed work. Next, we show this algorithm extends to complexes.

Proposition 2.6. *The level slide algorithm extends to X and preserves its quasi-isometry class.*

Proof. X is the complex for $\mathrm{BS}(p,q)$ and T is the spine tree. The result of the level-slide algorithm is a function $T \to T'$ where T' is the tree with in-valence increased by one. Now since the sliding is level-preserving, the slid half sheets are glued via the identity map on horocycles on the boundary, and since X is just tree $\times$ real line, extension to X follows. Denote this extension by $\overline{f} : X \to X'$. If we slide while leaving the terminal vertex the branches untouched, then $\overline{f}$ is a bijection.
We will show $\overline{f}^{-1}$ is a quasi-isometry. Let γ be a geodesic in X'. As discussed previously, approximate γ by a piecewise linear path. The horocyclic segments of this path are preserved isometrically by the sliding process. Vertical segments can be affected by slides: the length of slide paths has to be accommodated for. In the worst case every original vertical level-changing edge in $\gamma \in T'$ has been interrupted by slide edges. These slide edges will have length two. So the length of $\overline{f}^{-1}$ increases at most by a factor of three. Therefore $\overline{f}^{-1}$ is coarse Lipschitz. By a similar argument $\overline{f}$ is coarse Lipschitz. It is also surjective, so definitely coarsely surjective. This means it is a quasi-isometry. $\square$

The image X' of X after increasing in-valence by one is not necessarily isometric to the complex for $\mathrm{BS}(p+1,q)$, because horostrips corresponding to the same level may not be correctly scaled. But remember that this incorrect scaling is uniform, and so we can carry out a simultaneous scaling of all sheets to obtain

Proposition 2.7. *The image complex X' is quasi-isometric to $\mathrm{BS}(p+1,q)$ provided $p+1 \neq q$.*

Why do we need the $p+1 \neq q$? Because applying our algorithm to $\mathrm{BS}(p,p+1)$ can definitely not result in something quasi-isometric to $\mathrm{BS}(p+1,p+1)$, given that $\mathrm{BS}(p+1,p+1)$ has Euclidean sheets, and we cannot obtain this from hyperbolic sheets.

This is now applied inductively to obtain our main result:

Theorem 2.8 (Whyte)**.** $\mathrm{BS}(2,3)$ *is quasi-isometric to* $\mathrm{BS}(p,q)$ *for* $1 < p < q$.

References

[1] Eric M Freden, Alisha McCann *A simple proof of a theorem of Whyte.* Geom. Dedicata 108 (2004), 153–162.
[2] Matt Clay, Dan Margalit *Office Hours with a geometric group theorist.*