THE BORSAK-ULAM THEOREM

ARAV AGARWAL

We provide a complete algebraic topological proof of the Borsak-Ulam Theorem, filling in all the details of the terse proof provided in [1]. In words, the Borsuk–Ulam theorem states that every continuous function from an n-sphere into Euclidean n-space maps some pair of antipodal points to the same point. Here, two points on a sphere are called antipodal if they are in exactly opposite directions from the sphere's center. Henceforth, whenever we say "map", we mean a continuous map.

Formally, we state the theorem as follows.

Theorem 1 (Borsak-Ulam Theorem)**.** *For any map $g : S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ such that $g(-x) = g(x)$.*

Consider two interesting illustrations of this theorem.

Example 2. Assume that the Earth's temperature varies continuously in space. Then, the case $n = 1$ can be illustrated by saying that there always exist a pair of opposite points on the Earth's equator with the same temperature. Indeed, this claim will hold for any circle.

Example 3. For the case $n = 2$, we note that Theorem 1 implies that there is always a pair of antipodal points on the Earth's surface having exactly equal temperatures and equal barometric pressures. Again, we assume both parameters vary continuously in space.

Proving Theorem 1 with all details on full display will take some time, and to make our final proof short and sweet, we begin with some lemmas.

Notation: LES stands for long exact sequence, and SES for short exact sequence.

Lemma 4. *Let $\widetilde{X}$ be a two-sheeted covering space for X with $p : \widetilde{X} \to X$. The sequence of homology groups*

$$\cdots \longrightarrow H_n(X; \mathbb{Z}_2) \xrightarrow{\tau_*} H_n(\widetilde{X}; \mathbb{Z}_2) \xrightarrow{p_*} H_n(X; \mathbb{Z}_2) \longrightarrow H_{n-1}(X; \mathbb{Z}_2) \xrightarrow{\tau_*} \cdots \tag{5}$$

is exact. Here, $\tau_\# : C_n(X;\mathbb{Z}_2) \to C_n(\widetilde{X};\mathbb{Z}_2)$ with $\tau_\# : \sigma \mapsto \widetilde{\sigma}_1 + \widetilde{\sigma}_2$, where $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are the two unique lifts of σ. Note $\sigma : \Delta^n \to X$ is a singular n-simplex of our space X.

Proof. Consider the induced map $p_\# : C_n(\widetilde{X};\mathbb{Z}_2) \to C_n(X;\mathbb{Z}_2)$. Observe that any singular simplex $\sigma : \Delta^n \to X$ always lifts to $\widetilde{X}$ by the lifting criterion, since Δ^n is simply-connected. This means $p_\#$ is surjective. Further, the lifting criterion tells us that there will be precisely two lifts $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ since $\widetilde{X}$ is a two-sheeted covering space.

Let us determine the kernel of $p_\#$. First note any singular simplex of $\widetilde{X}$ can be written as a lift of a singular simplex of X: so elements of $C_n(\widetilde{X};\mathbb{Z}_2)$ look like $\sum_i n_i\widetilde{\sigma}_i$ with $n_i \in \mathbb{Z}_2$, where $\widetilde{\sigma}_i$ is one of the two lifts of σ_i. Consider such an element and write it with $n_i = [1]_2 \neq [0]_2$ by grouping identical $\widetilde{\sigma}_i$ together. Now

$$p_\#(\sum_i \widetilde{\sigma}_i) = 0 \implies \sum_i \sigma_i = 0.$$

For $\sum_i \sigma_i$ to vanish, for any fixed σ appearing in the sum, we must have an even number of σ_is equal to σ. In particular because there are only two lifts, we know precisely two σ_is can appear in the sum, one appearing as $p_\#(\widetilde{\sigma}_{i,1})$, and the other as $p_\#(\widetilde{\sigma}_{i,2})$. So, $\ker p_\#$ is precisely generated by sums of the form $\widetilde{\sigma}_1 + \widetilde{\sigma}_2$.

Next, consider the image of $\tau_\#$. Since $\tau_\#(\sigma) = \widetilde{\sigma}_1 + \widetilde{\sigma}_2$, $\operatorname{img} \tau_\#$ is also precisely generated by sums of the form $\widetilde{\sigma}_1 + \widetilde{\sigma}_2$. So, $\operatorname{img} \tau_\# = \ker p_\#$. Finally, $\tau_\#$ clearly has trivial kernel.

Everything we have shown so far establishes that the sequences

$$0 \longrightarrow C_n(X;\mathbb{Z}_2) \xrightarrow{\tau_\#} C_n(\widetilde{X};\mathbb{Z}_2) \xrightarrow{p_\#} C_n(X;\mathbb{Z}_2) \longrightarrow 0 \tag{6}$$

are exact. So, for each n, 6 gives us an SES of chain groups.

Note that $\tau_\#$ and $p_\#$ both commute with boundary maps, i.e., they are chain maps. So, we can arrange the SESs of 6 to obtain a short exact sequence of chain complexes. At this point, a standard (purely algebraic) technique is to "stretch out" such an SES of chain complexes to obtain the associated LES of homology groups. This is described for example in [1, pp. 116]. If we do this here, we clearly see that we exactly get 5, as desired. □

Remark 7. The map on homologies τ_* induced by the chain map $\tau_\#$ is one example of what are known more generally as transfer homomorphisms (see [1, §3.G]), and because of this we henceforth refer to an LES like 5 as a **transfer sequence**.

Fact 8. S^n is a two-sheeted covering space for $\mathbb{RP}^n$. We can see this by recalling one that one way to define $\mathbb{RP}^n$ was the quotient space $S^n/(v \sim -v)$. Let $[v] = [-v]$ denote the equivalence class of the two antipodal points, so that the covering map $p : S^n \to \mathbb{RP}^n$ acts as $v \mapsto [v]$. The two unique lifts of $[v]$ are then v and $-v$.

Next, we establish the following proposition of Borsak, regarding the degree of odd endomorphisms of spheres. Note that by an odd map f, we mean $f(-x) = -f(x)$, $\forall x$.

Proposition 9. *An odd map $f : S^n \to S^n$ must have odd degree.*

Proof. Using Lemma 4 and Fact 8, we get the transfer sequence for $p : S^n \to \mathbb{RP}^n$. We suppress the coefficient group $\mathbb{Z}^2$ to simplify notation.

$$\begin{aligned}
0 \longrightarrow H_n(\mathbb{RP}^n) \xrightarrow{\tau_*} H_n(S^n) \xrightarrow{p_*} H_n(\mathbb{RP}^n) \longrightarrow H_{n-1}(\mathbb{RP}^n) \xrightarrow{\tau_*} H_{n-1}(S^n) \longrightarrow \cdots \\
\cdots \longrightarrow H_i(S^n) \xrightarrow{p_*} H_i(\mathbb{RP}^n) \longrightarrow H_{i-1}(\mathbb{RP}^n) \xrightarrow{\tau_*} H_{i-1}(S^n) \longrightarrow \cdots \\
\cdots \longrightarrow H_1(S^n) \xrightarrow{p_*} H_1(\mathbb{RP}^n) \longrightarrow H_0(\mathbb{RP}^n) \xrightarrow{\tau_*} H_0(S^n) \xrightarrow{p_*} H_0(\mathbb{RP}^n) \longrightarrow 0
\end{aligned}$$

Let us simplify. The intial 0 comes from $H_{n+1}(\mathbb{RP}^n)$ using cellular homology, which is clearly trivial because $\mathbb{RP}^n$ is an n-dimensional CW complex. Next, note that $H_i(S^n) = \mathbb{Z}_2$ for $i = 0, 1$; and $H_i(S^n) = \mathbb{Z}_2$ for $0 < i < n$. Further, we know $H_0(\mathbb{RP}^n) = \mathbb{Z}_2$ So, we have:

$$\begin{aligned}
0 \longrightarrow H_n(\mathbb{RP}^n) \xrightarrow{\tau_*} \underbrace{H_n(S^n)}_{\mathbb{Z}_2} \xrightarrow{p_*} H_n(\mathbb{RP}^n) \longrightarrow H_{n-1}(\mathbb{RP}^n) \xrightarrow{\tau_*} 0 \longrightarrow \cdots \\
\cdots \longrightarrow 0 \xrightarrow{p_*} H_i(\mathbb{RP}^n) \longrightarrow H_{i-1}(\mathbb{RP}^n) \xrightarrow{\tau_*} 0 \longrightarrow \cdots \\
\cdots \longrightarrow 0 \xrightarrow{p_*} H_1(\mathbb{RP}^n) \longrightarrow \underbrace{H_0(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow{\tau_*} \underbrace{H_0(S^n)}_{\mathbb{Z}_2} \xrightarrow{p_*} \underbrace{H_0(\mathbb{RP}^n)}_{\mathbb{Z}_2} \longrightarrow 0
\end{aligned}$$

Observing this LES, we see that we have $H_i(\mathbb{RP}^{ni} \approx H_{i-1}(\mathbb{RP}^n))$ for $2 \leq i \leq n-1$. Also, the only possible homorphisms from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ are the identity or the zero map, and using this facts and the exactness of the sequence, we can add the following isomorphism and zero symbols to the diagram:

$$0 \longrightarrow H_n(\mathbb{RP}^n) \xrightarrow{\tau_*} \underbrace{H_n(S^n)}_{\mathbb{Z}_2} \xrightarrow{p_*} H_n(\mathbb{RP}^n) \longrightarrow H_{n-1}(\mathbb{RP}^n) \xrightarrow{\tau_*} 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{p_*} H_i(\mathbb{RP}^n) \xrightarrow[\approx]{} H_{i-1}(\mathbb{RP}^n) \xrightarrow{\tau_*} 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{p_*} H_1(\mathbb{RP}^n) \xrightarrow[\approx]{} \underbrace{H_0(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow[0]{\tau_*} \underbrace{H_0(S^n)}_{\mathbb{Z}_2} \xrightarrow[\approx]{p_*} \underbrace{H_0(\mathbb{RP}^n)}_{\mathbb{Z}_2} \longrightarrow 0$$

We conclude $H_i(\mathbb{RP}^{ni} \approx H_{i-1}(\mathbb{RP}^n)) \approx \mathbb{Z}_2$ for $1 \leq i \leq n-1$. Using this fact and looking at the row in our LES, we see that $H_n(\mathbb{RP}^n)$ injects into $\mathbb{Z}_2 \approx H_n(S^n)$, and also surjects onto $\mathbb{Z}_2 \approx H_{n-1}(\mathbb{RP}^n)$. This means $H_n(\mathbb{RP}^n)$. So, we have determined the homologies of $\mathbb{RP}^n$. Our final completely described sequence looks as follows, where we notice every nonzero homology in the sequence is $\mathbb{Z}_2$.

$$0 \longrightarrow \underbrace{H_n(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow[\approx]{\tau_*} \underbrace{H_n(S^n)}_{\mathbb{Z}_2} \xrightarrow[0]{p_*} \underbrace{H_n(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow[\approx]{} \underbrace{H_{n-1}(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow{\tau_*} 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{p_*} \underbrace{H_i(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow[\approx]{} \underbrace{H_{i-1}(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow{\tau_*} 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \xrightarrow{p_*} \underbrace{H_1(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow[\approx]{} \underbrace{H_0(\mathbb{RP}^n)}_{\mathbb{Z}_2} \xrightarrow[0]{\tau_*} \underbrace{H_0(S^n)}_{\mathbb{Z}_2} \xrightarrow[\approx]{p_*} \underbrace{H_0(\mathbb{RP}^n)}_{\mathbb{Z}_2} \longrightarrow 0$$

Now for the step where we specifically use the fact that f is an odd map. Notice that $[f(v)] = [-f(-v)] = [f(-v)]$, i.e., the equivalence class (as in Fact 8) in $\mathbb{RP}^n$ of $f(v)$ is shared by $f(-v)$. This means f induces a quotient map $\bar{f} : \mathbb{RP}^n \to \mathbb{RP}^n$ with $[v] \mapsto [f(v)]$. Hence, f and $\bar{f}$ will induce maps f_* and $\bar{f}_*$ from our transfer sequence to itself.

The question is whether the squares in this diagram commute. This is when we appeal to the property of naturality [1, pp. 127], where we recall the naturality of the LES of homology associated to a SES of chain complexes. So, all we have to do is verify that the squares in

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\tau_\#} & C_n(\widetilde{S}^n;\mathbb{Z}_2) & \xrightarrow{p_\#} & C_n(\mathbb{RP}^n;\mathbb{Z}_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{f}_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle \bar{f}_\#} & & \\
0 & \longrightarrow & C_n(\mathbb{RP}^n;\mathbb{Z}_2) & \xrightarrow{\tau_\#} & C_n(\widetilde{S}^n;\mathbb{Z}_2) & \xrightarrow{p_\#} & C_n(\mathbb{RP}^n;\mathbb{Z}_2) & \longrightarrow & 0
\end{array}$$

commute, and since this was this SES (see 6 in Lemma 4) which induced the tranfer sequence we care about, that will be enough by naturality.

First, observe that

$$(pf)(v) = p(f(v)) = [f(v)] = \bar{f}([v]) = \bar{f}(p(v)) = (\bar{f}p)(v).$$

So, $pf = \bar{f}p \implies p_\# f_\# = \bar{f}_\# p_\#$.

Second, for a singular i-simplex $\sigma : \Delta_i \to \mathbb{RP}^n$ we have

$$(f_\#\tau_\#)(\sigma) = f_\#(\tau_\#(\sigma)) = f_\#(\widetilde{\sigma}_1 + \widetilde{\sigma}_2) = f\widetilde{\sigma}_1 + f\widetilde{\sigma}_2 = \tau_\#(\bar{f}(\sigma)) = \tau_\#(\bar{f}_\#(\sigma)).$$

Here, we used the fact that the two unique lifts of $\bar{f}(\sigma)$ are $f\widetilde{\sigma}_1$ and $f\widetilde{\sigma}_2$ (they are different because f maps antipodal points to antipodal points). That they are lifts can be seen by $pf\widetilde{\sigma}_1 = \bar{f}p\widetilde{\sigma}_1 = \bar{f}\sigma_1$. So, we have shown commutativity in the SES, and this allows us to conclude the squares commute in the LES diagram of our homologies.

At this point, we are able to conclude that the maps f_* and $\bar{f}_*$ in the LES of homologies are all isomorphisms, using induction on the squares. Starting from the rightmost square, the maps f_* and $\bar{f}_*$ are obviously isomorphims in 0-dimensional homology. Now if three maps are isomorphisms in a sqaure, so is the fourth (rewrite the fourth in terms of the other three to see this), and so we can propogate leftwards and conclude what that all of f_* and $\bar{f}_*$ are isomorphims. In particular, we are interested in the fact that $f_* : H_n(S^n; \mathbb{Z}_2) \to H_n(S^n; \mathbb{Z}_2)$ is an isomrphism. Now, recall if we have a map from $\phi : S^n \to S^n$, then the induced map ϕ_* on the n-th homology is multiplication by the degree of f (see [1, Lemma 2.49]). Since we are working over $\mathbb{Z}_2$, for us this means f_* is multiplication by degree of f mod 2. Since it is an isomorphism, it is multiplication by 1, and this means the degree must be odd. $\square$

The Borsak-Ulam Theorem is now an easy consequence of Proposition 9.

Proof of Theorem 1. With $g : S^n \to \mathbb{R}^n$ as in the statement, let $f = g(x) - g(-x)$, so that f is clearly an odd map. It will suffice to show f vanishes at some point, for that would imply $g(x) = g(-x)$.

Suppose f never vanishes. Then we can consider a new map $\phi : S^n \to S^{n-1}$, where $\phi(x) = \frac{f(x)}{|f(x)|}$. Note ϕ is clearly still odd. Restricting ϕ to the equator of S^n, namely S^{n-1}, we get an odd map $\phi : S^{n-1} \to S^{n-1}$. By Proposition 9, ϕ has odd degree. But this restricted ϕ is also nullhomotopic, which is seen by considering the restriction of ϕ to one of the (contractible) hemispheres bounded by the equator S^{n-1}. So the degree of ϕ is zero and odd, a contradiction. Hence, f must vanish. $\square$

References

[1] Allen Hatcher. *Algebraic Topology* (cit. on pp. 1–4).

(Arav Agarwal) Bowdoin College, Department of Mathematics, Brunswick, ME 04011 USA
Email address: aagarwal@bowdoin.edu