DISCRETE SUBSGROUPS OF $PSL_2(\mathbb{R})$: FUCHSIAN GROUPS

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Contents

1. The group $PSL_2(\mathbb{R})$

This paper assumes some background with the hyperbolic plane, the upper half plane model (denoted by $\mathscr{H}$), its isometries, fractional linear transformations (FLTs), and $PSL_2(\mathbb{R})$. The set of isometries of the upper half plane is written as $\mathrm{Isom}(\mathscr{H})$. Note that all isometries of $\mathscr{H}$ can be decomposed into a composition of FLTs (elements of $PSL_2(\mathbb{R})$) and the isometry $-\overline{z}$. Indeed, $PSL_2(\mathbb{R})$ contains all the orientation preserving isometries of $\mathscr{H}$, and $-\overline{z}$ supplies the non-orientation preserving component.

Definition 1.1 ($SL_2(\mathbb{R})$)**.** The special linear group $SL(2,\mathbb{R})$ or $SL_2(\mathbb{R})$ is the group of 2×2 real matrices with determinant one:

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$$

Definition 1.2 ($PSL_2(\mathbb{R})$)**.** The projective special linear group $PSL(2,\mathbb{R})$ or $PSL_2(\mathbb{R})$ is the following quotient:

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) \backslash \{\pm I\}$$

Theorem 1.3. *$PSL_2(\mathbb{R})$ acts on the upper half plane (denoted by $\mathscr{H}$), given the following choice of permutations: for $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$, we write $T(z) = \dfrac{az+b}{cz+d}$, where $z \in \mathscr{H}$. This action is what we mean by a fractional linear transformation of $\mathscr{H}$. Further, it is clear that when we interpret T as the coset $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \{\pm I\}$, it does not matter whether we make the choice of the entries being positive or negative given that $\dfrac{az+b}{cz+d} = \dfrac{-az-b}{-cz-d}$.*

Proof. We have to verify two things:

- The identity condition: $I(z) = \dfrac{1 \cdot z + 0}{0 \cdot z + 1} = \dfrac{z}{1} = z$. Here, by I we mean the identity coset in $PSL_2(\mathbb{R})$.

- The composition condition: Let $A, B \in PSL_2(\mathbb{R})$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

  Then, we have: $A \circ B(z) = A\left(\dfrac{ez+f}{gz+h}\right) = \dfrac{a\left(\dfrac{ez+f}{gz+h}\right)+b}{c\left(\dfrac{ez+f}{gz+h}\right)+d} = \dfrac{aez+af+bgz+bh}{cez+cf+dgz+dh} = \dfrac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)} = (AB)(z)$. Here, $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$.

□

Definition 1.4. Let the trace of a matrix T be given by $Tr(T) = |a+d|$. Then, we classify elements with $Tr(T) < 2, = 2$, and > 2, as *elliptic, parabolic,* and *hyperbolic* respectively.

While it is not clear at first, this classification is arrived at using the usual strategy of classifying isometries - by counting the number of fixed points. We will now describe the fixed points for *elliptic, parabolic,* and *hyperbolic* elements, and also use tools from linear algebra to give a general form for each type of element.
First, we establish some necessary background and equations:
Given an isometry $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ of the upper half plane, the fixed points are found by solving the equation $\dfrac{az+b}{cz+d} = z$.
We can rearrange this to obtain $cz^2 + (d-a)z - b = 0$. The discriminant of this quadratic is given by $(d-a)^2 + 4bc$. We can do some algebra and rewrite this as follows:

$$\begin{aligned}
(d-a)^2 + 4bc &= a^2 + d^2 - 2ad + 4bc \\
&= a^2 + d^2 + 2ad - 2ad - 2ad + 4bc \\
&= (a+d)^2 - 4(ad-bc) \\
&= (a+d)^2 - 4 \qquad\qquad (ad-bc=1)
\end{aligned}$$

Next, consider the characteristic polynomial of the matrix A:

$$\begin{aligned}
f_A(t) &= (t-a)(t-d) - bc \\
&= t^2 - (a+d)t + (ad-bc) \\
&= t^2 - (a+d)t + 1 \qquad\qquad (ad-bc=1)
\end{aligned}$$

The discriminant of the equation above is given by $(a+d)^2 - 4$.
So, we notice that the discriminant of both the characteristic polynomial, which determines eigenvalues/vectors, and that of the equation which determines the number of fixed points is the same. Namely, we have:

$$D = (a+d)^2 - 4 = (Tr(A))^2 - 4.$$

Theorem 1.5 (*Hyperbolic elements*)**.** *Hyperbolic elements have two fixed points on the boundary of $\mathscr{H}$, and are conjugate in $PSL_2(\mathbb{R})$ to matrices of the form* $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, $1 \neq \lambda \in \mathbb{R}$*. That is, they are similar to dilation matrices.*

Proof. Let $A \in PSL_2(\mathbb{R})$ be a hyperbolic element. Then we know that $Tr(A) > 2$. This means $D > 0$, which implies that there are two fixed points in $\mathbb{R} \cup \{\infty\}$. But $D > 0$ also means that the characteristic polynomial of A has two real and distinct eigenvalues, which means A is diagonalizable. That is, $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix, and $S \in GL_2(\mathbb{R})$. But notice also that $1 = det(A) = det(\Lambda)$, which means Λ must be of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, $1 \neq \lambda \in \mathbb{R}$ and an element of $SL_2(\mathbb{R})$. Now, to complete the proof that A is conjugate to matrix of the Λ in $PSL_2(\mathbb{R})$, all that is left to show is that S can be chosen to be in $PSL_2(\mathbb{R})$ instead of $GL_2(\mathbb{R})$. Suppose $S = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then write $\Delta = \sqrt{det(S)}$, and choose $S' = \dfrac{S}{\Delta} = \begin{pmatrix} p/\Delta & q/\Delta \\ r/\Delta & s/\Delta \end{pmatrix}$. It is clear that $det(S') = 1$, which means $S' \in PSL_2(\mathbb{R})$. So, we can write $A = S\Lambda S^{-1} = \left(\dfrac{S}{\Delta}\right)\Lambda(\Delta S^{-1}) = S'\Lambda(S')^{-1}$, where $S' \in PSL_2(\mathbb{R})$. □

Theorem 1.6 (*Elliptic elements*)**.** *Elliptic elements have one fixed point in $\mathscr{H}$, and are conjugate in $PSL_2(\mathbb{R})$ to clockwise rotation matrices of the form* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$*. That is, they are similar to clockwise rotation matrices.*

Proof. Let $A \in PSL(2, \mathbb{R})$ be an en elliptic element. Then we know that $Tr(A) < 2$, which is possible iff $D < 0$. This means there are two complex solutions to the equation $\dfrac{az+b}{cz+d} = z$; since complex roots always come in conjugate pairs, we can conclude that exactly one of these solutions is an element of $\mathscr{H}$. Hence, elliptic elements have one fixed point in the upper half plane.
Next, we show that A is conjugate in $PSL_2(\mathbb{R})$ to a clockwise rotation matrix. We know that A has complex eigenvalues ($\because D < 0$).
Consider the defining equation for the complex eigenvalue-eigenvector pair: $Av = \lambda v$, $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^2$. Write $v = Re(v) + iIm(v)$ and $\lambda = a + bi$. Here, $Re(v)$ and $Im(v)$ are vectors in $\mathbb{R}^2$; if $v = (a_1 + ib_1, a_2 + ib_2)$, then $Re(v) = (a_1, a_2)$ and $Im(v) = (b_1, b_2)$.
The real and imaginary parts of $Av = \lambda v$ can be written as

$$\begin{aligned} ARe(v) &= aRe(v) - bIm(v) \\ AIm(v) &= bRe(v) + aIm(v). \end{aligned}$$

Now, construct a matrix V with $Re(v), Im(v)$ as columns:

$$V = [Re(v)|Im(v)].$$

Then, we can write:

$$\begin{aligned}
AV &= A\,[Re(v)|Im(v)] \\
&= [ARe(v)|AIm(v)] \\
&= [aRe(v) - bIm(v)|bRe(v) + aIm(v)] \\
&= \left[V\begin{pmatrix} a \\ -b \end{pmatrix} |V\begin{pmatrix} b \\ a \end{pmatrix}\right] \\
&= V\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\
&= V\Lambda \qquad\qquad \left(\text{where } \Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right)
\end{aligned}$$

So, we have $AV = V\Lambda$, which implies $V^{-1}AV = \Lambda$. Notice that $det(A) = det(\Lambda) \implies 1 = det(A) = det(\Lambda) = a^2 + b^2$.
This means $|\lambda| = |a + bi| = a^2 + b^2 = 1$. Next, we write λ in polar form as $\lambda = |\lambda|(\cos\theta + \sin\theta)$ $= 1 \cdot (\cos\theta + \sin\theta) = \cos\theta + \sin\theta \implies a = \cos\theta, b = \sin\theta$.
Substituting these values for a, b in Λ, we have $\Lambda = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Finally, $V \in GL_2(\mathbb{R})$ can be 'normalized' to $V' \in PSL_2(\mathbb{R})$ in the same way S was normalized towards the end of our proof of Theorem 1.5. $\square$

Theorem 1.7 (*Parabolic elements*)**.** *Parabolic elements have a single fixed point on the boundary of $\mathscr{H}$, and are conjugate in $PSL_2(\mathbb{R})$ to matrices of the form $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. That is, they are similar to translation matrices.*

Proof. Let $A \in PSL_2(\mathbb{R})$ be a parabolic element. This means $Tr(A) = 2 \iff D = 0$, which implies A has a single repeated eigenvalue, given by $\lambda = \dfrac{a+d}{2} = 1$. Further, $D = 0$ implies that there is a single real solution to the equation $\dfrac{az+b}{cz+d} = z$. So, A has exactly one fixed point in $R \cup \{\infty\}$, the boundary of $\mathscr{H}$.
Consider the single eigenvector, say v, and its defining equation given by $Av = \lambda v \implies Av = v$, where we used $\lambda = 1$. Extend v to a basis, say v, w, and construct a matrix with these vectors as columns, say $V = [\,v \mid w\,]$.
Now, we can write:

$$\begin{aligned}
AV &= A[\,v \mid w\,] \\
&= [\,Av \mid Aw\,] \\
&= [\,v \mid Aw\,] \qquad\qquad (\because Av = v) \\
&= [\,v \mid w\,] \cdot \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix} \text{(where } sv + tw = Aw\text{, scalars } s, t \text{ exist because } v, w \text{ is a basis)} \\
&= V\Lambda \qquad\qquad \left(\Lambda = \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}\right)
\end{aligned}$$

So, we have $AV = V\Lambda$, which implies $V^{-1}AV = \Lambda$. But this means $det(A) = det(V) \implies 1 = det(A) = det(V) = t$. Hence, $\Lambda = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

Normalizing V in the same as we did in Theorem 1.5 completes our proof that A is conjugate in $PSL_2(\mathbb{R})$ to translation matrices. $\square$

Theorem 1.8 (*Subgroups of $PSL_2(\mathbb{R})$*)**.** *Hyperbolic, elliptic, and parabolic elements of $PSL_2(\mathbb{R})$ respectively form three subgroups of $PSL_2(\mathbb{R})$. That is, K, A, N are all subgroups of $PSL_2(\mathbb{R})$ of $PSL_2(R)$, where*

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\},\ A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} : \lambda > 0 \right\} \text{ and } N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$$

Proof. Left to reader. $\square$

Theorem 1.9 (*Iwasawa decomposition*)**.** *We have a decomposition $PSL_2(\mathbb{R}) = KAN$. That is, every $g \in SL_2(\mathbb{R})$ has a unique representation as $g = kan$ where $k \in K, a \in A$ and $n \in N$. In other words, every fractional linear transformation can be uniquely decomposed into a product of elliptic, hyperbolic, and parabolic transformations. (Here, K, A, N represent the factor groups, factored the same way as $PSL_2(\mathbb{R})$. Throughout this paper, from the context it is clear when the symbols are to be interpreted as coset representatives.)*

Proof. Refer to Appendix A of [2]. $\square$

Proposition 1.10 (*Exercise 2.3 from Katok*)**.** *Every transformation ins $PSL_2(\mathbb{R})$ can be written uniquely in the form TR, where R is an elliptic element fixing i, and $T(z) = az+b$ $(a, b \in \mathbb{R}, a > 0)$. Further, $PSL_2(\mathbb{R})$ as topological space is homeomorphic to $\mathbb{R}^2 \times S^1$, where S^1 is a circle.*

Proof. We know from Theorem 1.9 that $PSL_2(\mathbb{R}) = KAN$. We know $PSL_2(\mathbb{R}), K, A$ and N are all groups. So, we can take the inverse of both sides to obtain an equivalent decomposition given by $PSL_2(\mathbb{R}) = NAK$.

Now, consider $\mathcal{N} \in N$ and $\mathcal{A} \in A$, where $\mathcal{N} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $\mathcal{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$.

Next, notice that $\mathcal{N}\mathcal{A} = \begin{pmatrix} \lambda & s/\lambda \\ 0 & 1/\lambda \end{pmatrix}$. Then, the fractional linear transformation $\mathcal{N}\mathcal{A}$ corresponds to is given by $\dfrac{\lambda z + s/\lambda}{1/\lambda} = \lambda^2 z + s$. Letting $\lambda^2 = a$ and $s = b$, and using the fact that $PSL_2(\mathbb{R})$ acts on the upper half plane, we can write $\mathcal{N}\mathcal{A}(z) = T(z) = az + b$.
From $PSL_2(\mathbb{R}) = NAK$ we know that every element of $A \in PSL_2(\mathbb{R})$ can be written uniquely as a product as follows: $A = \mathcal{N}\mathcal{A}R = TR$, where we used $\mathcal{N}\mathcal{A}(z) = T(z) = az + b$, and $R \in K$. Note that since a and b depend on $\mathcal{A}$ and $\mathcal{N}$ uniquely ($a = \lambda^2$, $b = s$), uniqueness is preserved when replace $\mathcal{N}\mathcal{A}$ with T.

To complete our proof, all we have to show is that $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ fixes i:

$$\begin{aligned} R(i) &= \frac{i\cos\theta + \sin\theta}{-i\sin\theta + \cos\theta} \\ &= \frac{i(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)} \\ &= i \end{aligned}$$

$\square$

2. Discrete and Properly Discontinuous groups

Now that we have studied certain properties about $PSL_2(\mathbb{R})$ and know that it acts on the upper half plane, we have the connection between algebra and geometry we needed. The purpose of this section is to study Isom(), by studying the topological, analytic and algebraic properties of $PSL_2(\mathbb{R})$. We introduce the main object of study of Katok's book and this paper - Fuchsian groups.
First, we induce a topology on $PSL_2(\mathbb{R})$. More precisely, consider $SL_2(\mathbb{R})$ to be a topological space, where $SL_2(\mathbb{R})$ is identified as the following subset of $\mathbb{R}^2$:

$$X = \{(a, b, c, d) \in \mathbb{R}^4 \mid ad - bc = 1\}$$

Define $\delta : X \to X$, where $\delta(a, b, c, d) = (-a, -b, -c, -d)$. It is easy to verify that δ and the identity are isomorphic to $\mathbb{Z}_2$; that is, they form a cyclic group of order 2, which acts on X. Then, we can topologize $PSL_2(\mathbb{R})$ as the quotient space. Indeed, we can also introduce a norm on $PSL_2(\mathbb{R})$ from $\mathbb{R}^4$: if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $||T|| = \sqrt{a^2 + b^2 + c^2 + d^2}$. Therefore, X is now a metric space with the same normal Euclidean metric we are used to. The group of all isometries of the hyperbolic plane Isom($\mathscr{H}$) is topologized similarly

Definition 2.1. (*Discrete Set*) A set S in a larger topological space Y is said to be a *discrete set* if every point $x \in S$ has a neighborhood U such that $X \cap U = \{x\}$. The points of S are then referred to as *isolated points.*

Definition 2.2. (*Discrete Subgroup*) A subgroup Γ of Isom($\mathscr{H}$) is called a *discrete subgroup* if the induced topology (as described at the start of this section) on Γ is a discrete topology. That is, if Γ is a discrete set in the topological space Isom($\mathscr{H}$).

Proposition 2.3 (*Exercise 2.5 from Katok*)**.** *A subgroup Γ of Isom($\mathscr{H}$) is discrete if and only if $T_n \to Id_{\mathscr{H}}$, $T_n \in \Gamma$ implies $T_n = Id_{\mathscr{H}}$ for sufficiently large n.*

Proof. ($\implies$) Consider a discrete subgroup Γ of Isom($\mathscr{H}$). Let $(T_n)_1^\infty$ be a sequence of isometries such that $\lim_{n\to\infty} T_n = Id_{\mathscr{H}}$. From the formal definition of a limit at infinity, this means that for any $\epsilon > 0$, there exists $N > 0$ such that $n > N \implies ||T_n - Id_{\mathscr{H}}|| < \epsilon$. Now, since Γ is discrete, we are guaranteed to find a neighborhood, which in the case of $\mathbb{R}^4$ is a ball B, of radius say r, such that the ball contains only $Id_{\mathscr{H}}$. Then, setting $\epsilon = r$, we know we can find $N' > 0$ such that $n > N' \implies ||T_n - Id_{\mathscr{H}}|| < r$. But this means $T_n \in B$ for $n > N'$, which implies $T_n = Id_{\mathscr{H}}$ since B contained only $Id_{\mathscr{H}}$. So, $T_n = Id_{\mathscr{H}}$ for sufficiently large n.
($\impliedby$) Suppose that for any sequence $(T_n)_1^\infty$ of isometries s.t. $\lim_{n\to\infty} T_n = Id_{\mathscr{H}}$, we have $T_n = Id_{\mathscr{H}}$ for sufficiently large n. Now, suppose Γ was not discrete. Then, there exists a limit point in Γ, say Z. That is, there exists a sequence of isometries $(U_n)_1^\infty$ such that $\lim_{n\to\infty} U_n = Z$ and $U_n \neq Z \; \forall \; n$. Since Γ is a group, we can multiply everything with Z^{-1} to obtain a sequence $(Z^{-1}U_n)_1^\infty$ such that $\lim_{n\to\infty} (Z^{-1}U_n) = Z^{-1}Z = Id_{\mathscr{H}}$, and $Z^{-1}U_n \neq Z^{-1}Z = Id_{\mathscr{H}} \; \forall \; n$, which is a contradiction.

□

Definition 2.4 (*Fuchsian group*)**.** A discrete subgroup of Isom($\mathscr{H}$) is called a *Fuchsian group* if it consists only of orientation preserving transformations. That is, a Fuchsian group is a discrete subgroup of $PSL_2(\mathbb{R})$.

For any discrete subgroup Γ of Isom($\mathscr{H}$), it is not hard to see that the set of all orientation preserving transformations in Γ will be a subgroup of index equal to 1 (Γ contains only orientation preserving transformations) or 2 (to obtain any non-orientation preserving transformation R from an orientation preserving transformation T, left-multiply T by the non-orientation preserving transformation RT^{-1}). Hence, Fuchsian groups are the key entities to analyze when studying discrete subgroups of isometries of $\mathscr{H}$.

Next, we move to define properly discontinuous actions. But first, we need a few more definitions. For **2.5** through **2.8**, let X be a metric space, and G a group of isometries of X.

Definition 2.5 (*Locally finite*)**.** A family $\{M_\alpha \mid \alpha \in A\}$ of subsets of X indexed by elements of a set A is called *locally finite* if for any compact subset $K \subset X$, $M_\alpha \cap K \neq \emptyset$

Note that repetitions in the family of subsets is allowed. That is, some M_α may coincide, but they are still considered as distinct elements in the family.

Definition 2.6. For any $x \in X$, a family $Gx = \{g(x) \mid g \in G\}$ is called the G-orbit of the point x. Each point of Gx is contained with multiplicity equal to the order of $G_x = \{g \in G \mid g(x) = x\}$

Definition 2.7 (*Properly discontinuous action*)**.** We say that a group G acts *properly discontinuously* on X when the G-orbit of any point $x \in X$ is locally finite.

In Katok's errata for her book *Fuchsian groups* we also find the following remark:

Remark 2.8. Since X is locally compact, a group G acts properly discontinuously on X if and only if each orbit has no accumulation point in X, and the order of the stabilizer of each point is finite. The first condition, however, is equivalent to the fact that each orbit of G is discrete. In conclusion, G acts properly discontinuously on X if and only if each orbit is discrete and the order of the stabilizer of each point is finite.

Theorem 2.9. *G acts properly discontinuously on X if and only if each point $x \in X$ has a neighborhood V such that $T(V) \cap V \neq \emptyset$ for only finitely many $T \in G$.*

Proof. See Theorem 2.2.1 in [1]. □

Proposition 2.10 (*Exercise 2.7 from Katok*)**.**
(i) All hyperbolic and parabolic cyclic subgroups of $PSL_2(\mathbb{R})$ are Fuchsian groups.
(ii) An elliptic cyclic subgroup of $PSL_2(\mathbb{R})$ is a Fuchsian group if and only if it is finite.

Proof. (i) Consider a hyperbolic cyclic subgroup $\langle H \rangle$, where H is the generator.
Since H is hyperbolic, we know from Theorem 1.5 that we can write $H = S^{-1}TS$, where $S \in PSL_2(\mathbb{R})$ and $T = \begin{pmatrix} \lambda & 0 \\ 1 & 1/\lambda \end{pmatrix}$. Next, we will use Proposition 2.3 to show that the group is discrete, which would conclude our proof that $\langle H \rangle$ is Fuchsian.
Consider a sequence $(R_n)_1^\infty$ in $\langle H \rangle$ so that $R_n \to Id_{\mathscr{H}}$. Now, every element in R_n is of the form H^k, for some k. But, we can use $H = S^{-1}TS$ to say that R_n is the form $H^k = S^{-1}T^kS$. Rewriting the sequence by indexing k instead of R, we can say $(R_n)_1^\infty = (S^{-1}T^{k_n}S)_1^\infty$. Further, we have $S^{-1}T^{k_n}S \to Id_{\mathscr{H}}$. Left and right multiplying everything with S and S^{-1} respectively, we have a sequence $(T^{k_n})_1^\infty$ in $PSL_2(\mathbb{R})$ such that $T^{k_n} \to SS^{-1} = Id_{\mathscr{H}}$.
Suppose that for all n, $T^{k_n} \neq Id_{\mathscr{H}}$. This means for any $\epsilon > 0$, we can find sufficiently large n, say N, such that $||T^{k_N} - Id_{\mathscr{H}}|| < \epsilon$. We will attain a contradiction by showing that $||T^{k_n} - Id_{\mathscr{H}}||$ has a lower bound, which means we cannot choose ϵ sufficiently small.
Since T is a diagonal matrix, we know that $T^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & 1/\lambda^k \end{pmatrix}$. So, we have:

$$||T^k - Id_{\mathscr{H}}|| = \sqrt{(\lambda^k - 1)^2 + (\frac{1}{\lambda^k} - 1)^2}$$

Without loss of generality, we can assume $\lambda > 1$ and $k > 0$, if not then choose $\dfrac{1}{\lambda^k}$ for the same effect. Further, the case $k = 0$ trivializes the proof.
Now, as k increases, it is clear that both $(\lambda^k - 1)$ and $(\dfrac{1}{\lambda^k} - 1)$ also increase, which means $||T^k - Id_{\mathscr{H}}||$ also increases. So, $||T^k - Id_{\mathscr{H}}||$ is minimum at $k = 1$, and hence has a lower bound. This completes our proof that any hyperbolic cyclic subgroup is Fuchsian.
An identical strategy is employed in the proof that a parabolic cyclic subgroup is Fuchsian, so we omit the proof here.
(ii)
$(\Longleftarrow)$ If a subgroup is finite, it is trivial to show that the subgroup must be discrete, so we omit the proof here for brevity.
$(\Longrightarrow)$ Consider an elliptic cyclic subgroup $\langle E\rangle$ that is also Fuchsian. Since E is elliptic, we know from Theorem 1.6 that $E = S^{-1}TS$, where T is of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.
Now, from Proposition 2.3, given any sequence $(R_n)_1^\infty$ in $\langle E\rangle$ so that $R_n \to Id_{\mathscr{H}}$, we know that for sufficiently large n, $R_n = Id_{\mathscr{H}}$. Now, in an identical fashion to part (i) of this proof, write $R_n = S^{-1}T^kS$, and obtain a sequence $(T^{k_n})_1^\infty$ in $PSL_2(\mathbb{R})$ such that $T^{k_n} \to Id_{\mathscr{H}}$. So, for sufficiently large n, say N, we know that $T^k = T^{k_N} = Id_{\mathscr{H}}$, where we write $k = k_N$ for brevity. It is easy to prove $T^k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$ using induction, so we do not prove it here.
Now, from $\begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} = T^k = Id_{\mathscr{H}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we obtain $\cos k\theta = 1$ and $\sin k\theta = 0$. Since cos and sin are periodic functions, this means the matrices will start to repeat at some point, making $\langle E\rangle$ finite. □

Finally, consider the following important theorem which allows us to a key connection between the geometric study of Isom($\mathscr{H}$), and the algebraic and topological study of $PSL_2(\mathbb{R})$.

Theorem 2.11. *Let Γ be a subgroup of $PSL_2(\mathbb{R})$. Then, Γ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathscr{H}$.*

Proof. See Theorem 2.2.6 in [1]. □

References

[1] S. Katok *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
[2] K. Conrad *Decomposing* $SL_2(\mathbb{R})$.