BOWDOIN COLLEGE MATH DEPARTMENT

MATH 3702: ALGEBRAIC GEOMETRY

THE COORDINATE RING

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This project assumes knowledge about basic ideas in Algebraic Geometry including affine varieties, polynomial ideals, multivariate polynomial Division Algorithm, Gröbner bases, Elimination Theory, Ideal-Variety Correspondence, Hilbert Basis Theorem, and the Nullstellensatz. More precisely, we assume familiarity with most of the theory (not the algorithms) covered in the first 4 chapters of Cox, Little and O'Shea's *Ideals, Varieties, and Algorithms.* In addition to this, it will be useful to know ideas from basic Abstract Algebra such as rings, fields, homomorphisms and quotient rings.

In some texts, affine varieties are required to be defined by prime ideals, and varieties defined by ideals in general are referred to as affine algebraic sets. Here, we follow the convention adopted in Cox et al., where we refer to the former as *irreducible* varieties and the latter general collection as affine varieties.

1. Polynomial Mappings

Definition 1.1 (Polynomial Mapping)**.** Let $V \subseteq k^m, W \subseteq k^n$ be affine varieties.
Then, $\phi : V \to W$ is a **polynomial mapping** (or a **regular mapping**) if $\exists$ polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_m]$ such that $\phi(\overrightarrow{a}) = (f_1(\overrightarrow{a}), \dots, f_n(\overrightarrow{a}))$, $\forall \overrightarrow{a} \in V \subseteq k^m$.
We say that the tuple $(f_1, \dots, f_n)$ **represents** ϕ, where the f_i are **components** of this representation.

Using the word "represents" here may remind readers familiar with abstract algebra of coset representatives - which arise when dealing with quotients of algebraic objects. Indeed, the following proposition acknowledges that representatives of ϕ are rarely unique, and points us to defining the correct type of equivalency relation, and finally working with quotients.

Proposition 1.2. *Let $V \subseteq k^m$ be an affine variety. Then:*

(i) f and $g \in k[x_1, \dots, x_m]$ represent the same polynomial function on V if and only if $f - g \in \mathbf{I}(V)$.
(ii) $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ represent the same polynomial mapping from V to k^n if and only if $f_i - g_i \in \mathbf{I}(V)$ for each $i, 1 \leqslant i \leqslant n$.

Definition 1.3 (Coordinate ring)**.** The **coordinate ring** of an affine variety $V \subseteq k^n$ is denoted by $k[V]$, and is given by the collection of polynomial functions $\phi : V \to k$.

Remark 1.4. It is easy to check that $k[V]$ is commutative ring with identity by defining the sum and product to be point-wise addition and multiplication on its image, which we can do since its image is in a field. Further, for any $\phi, \psi \in k[V]$, choosing representatives f, g respectively, give us representatives $f + g$ and $f \cdot g$ for $\phi + \psi$ and $\phi \cdot \psi$, respectively. This guarantees that $\phi + \psi \in k[V]$ and $\phi \cdot \psi \in k[V]$.
It is useful to restrict our attention to $k[V]$, since general polynomial mappings $\phi : V \to k^n$ can be built up using n elements of $k[V]$; in other words, elements of $k[V]$ are the components for any general polynomial mapping.

An easy to verify consequence of Proposition 1.2 is the following theorem, which will be key to the story this project is trying to tell. At this point, familiarity with basic ring theory is important.

Theorem 1.5. *Let $V \subseteq k^n$ be an affine variety. The ring $k[V]$ is isomorphic (as a ring) to the quotient ring $k[x_1, \dots, x_n]/\mathbf{I}(V)$. We denote this as $k[V] \cong k[x_1, \dots, x_n]/\mathbf{I}(V)$.*

Proof. Consider the natural map $\Theta : k[V] \to k[x_1, \dots, x_n]/\mathbf{I}(V)$ as $\Theta(\phi) = [f]$, where f is polynomial representative of ϕ, and $[f]$ is the equivalency/congruence class of f in the quotient ring $k[x_1, \dots, x_n]/\mathbf{I}(V)$.
First, we verify that Θ is well-defined. Suppose f, g are both polynomial representatives of ϕ. From Proposition 1.2, this means $f - g \in \mathbf{I}(V) \iff f \equiv g \bmod I \iff [f] = [g]$. So, Θ is well-defined.
Next, we show that Θ is a bijection. For any $[f] \in k[x_1, \dots, x_n]/\mathbf{I}(V)$, choose $\phi \in k[V]$ represented by f. So, Θ is surjective. Further, suppose $\Theta(\phi) = \Theta(\psi)$. Let ϕ and ψ have representatives g and

h respectively. Then, we have $[g] = [h] \iff g \equiv h \bmod I \iff g - h \in \mathbf{I}(V)$. We again use Proposition 1.2 (this time in the other direction) to conclude that $\phi = \psi$. So, Θ is injective.
Finally, we need to check that Θ preserves the sum and product operations on $k[V]$, which we defined and described in Remark 1.4. We can then write $\Theta(\phi + \psi) = [f + g]$ and $\Theta(\phi \cdot \psi) = [f \cdot g]$. From the definitions of the sum and product operations in $k[x_1, \ldots, x_n]/\mathbf{I}(V)$, we can write $[f + g] = [f] + [g]$ and $[f \cdot g] = [f] \cdot [g]$. It follows now that $\Theta(\phi + \psi) = [f] + [g] = \Theta(\phi) + \Theta(\psi)$ and $\Theta(\phi \cdot \psi) = [f] \cdot [g] = \Theta(\phi) \cdot \Theta(\psi)$. □

Given the above result, we will tend to work with $k[V]$ as the quotient ring $k[x_1, \ldots, x_n]/\mathbf{I}(V)$, for the latter is an algebraic object we understand fairly well. In Section 2 we will also use Theorem 1.5 to understand how to carry out the usual computations in $k[V]$. We will also often denote $[f]$ to be the polynomial function represented by f in $k[V]$. At this point, we can also explain the reasoning behind the terminology "Coordinate" Ring. Each variable x_i gives a polynomial mapping $[x_i] : V \to k$; this is a projection map, which we refer to as the i-th coordinate map on V. Theorem 1.5 then tells us that the individual coordinate maps (from 1 to n) "generate" $k[V]$ (more precisely, they generate $k[V]$ as a k-algebra), since any map is a k-linear combination of products of $[x_i]$.

Remark 1.6. The canonical homomorphism from $k[x_1, \ldots, x_n]$ to $k[V]$ is given by $\Phi(f) = \phi$, where $\phi \in k[V]$ is a polynomial mapping represented by f. It is also obviously surjective. Using Proposition 1.2, we show in Proposition 1.7 that Φ is injective iff $\mathbf{I}(V)$ is trivial.
Indeed, using Theorem 1.5, which tells us $k[V] \cong k[x_1, \ldots, x_n]/\mathbf{I}(V)$, it becomes clear that Proposition 1.7 is just an acknowledgement of the basic ring theory fact that a homomorphism (here, the canonical map from $k[x_1, \ldots, x_n]$ to $k[x_1, \ldots, x_n]/\mathbf{I}(V)$) is injective iff the kernel is trivial (here, kernel is $\mathbf{I}(V)$), and the proof supplied is a disguised proof of that fact (this is because we don't use anything specific about the rings we are working with, and so the reasoning will work over any ring). Problem 1.8 will then tell us exactly when $\mathbf{I}(V)$ is trivial.

Proposition 1.7. *Let Φ be as in Remark 1.6, then Φ is injective if and only if $\mathbf{I}(V)$ is trivial.*

Proof. ($\implies$) Suppose Φ is injective. First, notice that the zero polynomial, i.e., $0 \in k[x_1, \ldots, x_n]$ maps under Φ to $0 \in k[V]$, where 0 is the polynomial mapping represented by the polynomial 0. Now, Proposition 1.2 tells us that for $g \in k[x_1, \ldots, x_n]$, $\Phi(g) = 0 \iff g - 0 \in \mathbf{I}(V) \iff g \in \mathbf{I}(V)$. So, for any $g \in \mathbf{I}(V), \Phi(g) = 0$. But Φ is injective, which implies $g = 0$. So, $\mathbf{I}(V) = \{0\}$.
($\impliedby$) Suppose $\mathbf{I}(V)$ is trivial. Let $\Phi(g) = \Phi(h)$. Recall that for any homomorphism, $-\Phi(h) = \Phi(-h)$, and so we can write $\Phi(g) = \Phi(h) \iff \Phi(g) - \Phi(h) = 0 \iff \Phi(g) + \Phi(-h) = 0 \iff \Phi(g - h) = 0$. Using Proposition 1.2 in a similar way as before, we can say that $g - h \in \mathbf{I}(V)$. Since $\mathbf{I}(V) = \{0\}$, we have $g - h = 0 \iff g = h$. So, Φ is injective. □

Problem 1.8 (Exercise §5.1 7)**.**

(a) If k is an infinite field and $V \subseteq k^n$ is a variety, then $\mathbf{I}(V) = \{0\}$ if and only if $V = k^n$.
(b) On the other hand, if k is finite, then $\mathbf{I}(V)$ is never equal to $\{0\}$.

Proof. (a) ($\implies$) Suppose $\mathbf{I}(V) = \{0\}$. Then, $\mathbf{V}(\mathbf{I}(V)) = \mathbf{V}(\{0\})$. From the Ideal-Variety Correspondence, we know that $\mathbf{V}(\mathbf{I}(V)) = V$. Further, $\mathbf{V}(\{0\}) = k^n$, because the zero polynomial trivially vanishes everywhere. So, $V = k^n$.
($\impliedby$) Suppose $V = k^n$. Consider any $f \in \mathbf{I}(V)$. Then, f vanishes over k^n. When k is infinite, we know that this means f is identically zero, and so $f = 0$. Since f was arbitrary, $\mathbf{I}(V) = \{0\}$.

(b) $k \backslash \{0\}$ forms a multiplicative group or order $q - 1$, where k has order q. We can use Fermat's Little Theorem for finite groups (easily proved using Lagrange's Theorem) which says $a^q = a$ for all $a \in k$. But this implies the polynomial $x^q - x$ vanishes for all $x \in k$. Hence, $x^q - x \in \mathbf{I}(V)$ for any variety V, which means $\mathbf{I}(V)$ will always contain at least one non-zero element, and can thus never equal $\{0\}$.

□

The following proposition tells us that studying the algebraic properties of the coordinate ring of a variety can reveal geometric properties of the variety, and is a hint towards the extended Algebra-Geometry dictionary we develop in this project.

Proposition 1.9. *Let $V \subseteq k^n$ be an affine variety. The following are equivalent:*

(i) V is irreducible.
(ii) $\mathbf{I}(V)$ is a prime ideal.
(iii) $k[V]$ is an integral domain.

Remark 1.10. Notice that given (i) $\iff$ (iii) in the above Proposition, we can supply an alternate proof to (i) $\iff$ (ii) than the one given in Cox et al. (Prop. 3, §5.4). For this, we use Theorem 1.5, (i) $\iff$ (iii), tells us that V is irreducible iff $k[x_1, \ldots, x_n]/\mathbf{I}(V)$ is an integral domain. Recall the ring theory fact that a quotient ring is an integral domain exactly when the ideal is prime, which implies that V is irreducible iff $\mathbf{I}(V)$ is prime.

The next example illustrates how polynomial mappings can be used to introduce a sense of "sameness" of varieties.

Example 1.11. Suppose we have variety $V \subseteq k^n$ parameterized by

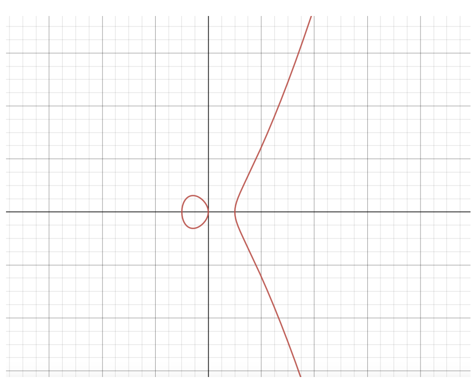
$$x_1 = x_1, x_2 = f_2(x_1), \ldots, x_n = f_n(x_1)$$

That is, it is parameterized solely in terms of the first coordinate. This is visualized geometrically as some sort of a one-dimensional curve in the space k^n which spans across the x_1 axis. Then, we have polynomial maps:

$$\pi : V \to k,\ (x_1, \ldots, x_n) \mapsto x_1$$
$$\phi : k \to V,\ x \mapsto (x, f_2(x), \ldots, f_n(x))$$

We can check easily that π and ϕ are polynomial inverses of each other. So, we have established a bijective correspondence using polynomials between the varieties V and k. In such a case, we will say that the varieties $V \subseteq k^n$ and $k \subseteq k$ are isomorphic. This is geometrically not surprising, for even as topological 1-manifolds V and k are clearly homeomorphic.
Since we are doing algebraic geometry, it is not surprising that the corresponding algebraic structures, namely, the coordinate rings will also be the "same". Indeed, substituting $x_i = f_i(x_1)$ for $i \geqslant 2$ in any polynomial mapping $\phi \in k[V]$ gives us a (unique) polynomial in terms of x_1 alone, i.e. in $k[x_1]$. Further, clearly any polynomial in $k[x_1]$ can be obtained using this substitution (just choose that polynomial itself as the pre-image). This tells us $k[V] \cong k[x_1]$. Similarly, using x_1 as the coordinate on, we have the obvious relation $k[k] \cong k[x_1]$. So, by transitivity, we see that the coordinate rings $k[V]$ and $k[k]$ are also isomorphic.

Problem 1.12 shows that the converse intuition is also true, i.e., if two varieties do not "appear" the same geometrically, then the varieties are not isomorphic in the sense we introduced. We expect the variety below not to be the "same" as the real number line.



Problem 1.12 (Exercise §5.1 10)**.** *In this problem, we will see that there are no non-constant polynomial mappings from $V = \mathbb{R}$ to $W = \mathbf{V}(y^2 - x^3 + x) \subseteq \mathbb{R}^2$. Thus, these varieties are not isomorphic (i.e., they are not "the same" in the sense introduced in this section).*

(a) Suppose $\phi : R \to W$ is a polynomial mapping represented by $\phi(t) = (a(t), b(t))$ where $a(t), b(t) \in R[t]$. Explain why it must be true that $b(t)^2 = a(t)(a(t)^2 - 1)$.
(b) Explain why the two factors on the right of the equation in part (a) must be relatively prime in $\mathbb{R}[t]$.
(c) Using the unique factorizations of a and b into products of powers of irreducible polynomials, show that $b^2 = ac^2$ for some polynomial $c \in \mathbb{R}[t]$ relatively prime to a.
(d) From part (c) it follows that $c^2 = a^2 - 1$. Deduce from this equation that c, a, and, hence, b must be constant polynomials.

Proof.

(a) Since the image is to be contained in W, we must have $(a(t), b(t)) \in W = \mathbf{V}(y^2 - x^3 + x)$, which is true iff $b(t)^2 - a(t)^3 + a(t) = 0 \iff b(t)^2 = a(t)(a(t)^2 - 1)$.
(b) Suppose f is a polynomial which is a common factor of a and a^2-1. Then, $f|a, f|a^2-1 \implies f|a^2, f|a^2-1 \implies f|a^2-(a^2-1) \implies f|1$. But $f|1$ means that f is a constant polynomial. Since f was arbitrary, a and $a^2 - 1$ must be relatively prime.
(c) Let $d = a^2 - 1$. Then, from part (a), we have $b^2 = ad$, where a, d are relatively prime. Suppose the unique factorization of b into irreducibles is $b = lf_1^{a_1} \cdots f_s^{a_s}$. Then, the unique factorization of $b^2 = ad$ is $l^2 f_1^{2a_1} \cdots f_s^{2a_s}$, where $l \in \mathbb{R}, f_i \in \mathbb{R}[x], s \in \mathbb{N}$.
Now, the product of the unique factorizations of a and d must also evaluate to $l^2 f_1^{2a_1} \cdots f_s^{2a_s}$. This means the irreducible factors of a and d are the f_i; but, since they are relatively prime, they cannot share irreducible factors, and it follows that (after appropriately renumbering the f_i) we can write $a = l_1 f_1^{2a_1} \cdots f_m^{2a_m}$ and $d = l_2 f_{m+1}^{2a_{m+1}} \cdots f_s^{2a_s}$. Notice that if a is a constant polynomial, we are already done with part (d), which is where we want to get. So, suppose a is non-constant. Then, $\mathrm{LC}(d) = \mathrm{LC}(a^2 - 1) = \mathrm{LC}(a)^2 \implies l_2 > 0$. Now, choosing $c = \sqrt{l_2} f_{m+1}^{a_{m+1}} \cdots f_s^{a_s}$, we see that $d = c^2$, and c is clearly relatively prime to a. So, $b = ac^2$ as required.
(d) We have $c^2 = a^2 - 1 \implies a^2 - c^2 = 1 \implies (a-c)(a+c) = 1 \implies (a-c)|1, (a+c)|1$, which means $a - c, a + c$ are both constant polynomials. The sum of two constant polynomials is also constant, and so $(a-c)+(a+c) = 2a$ is constant which implies a is constant. Similarly, $c = ((a + c) - (a - c))/2$ is also constant, but this a contradiction to what we assumed in part (c).

□

2. Understanding and Using the Coordinate Ring

The Coordinate Ring, $k[V]$, of an affine variety V is an algebraic construction. In this section, we will spend a short while understanding this object using algebraic methods, state a few results, and take advantage of our knowledge of Gröbner bases to allow us to do computations in $k[V]$.

The following Proposition, and its proof, will serve as a refresher for Gröbner bases and their key properties.

Proposition 2.1 (Exercise §2.6 1, Unique remainders)**.** *Fix a monomial ordering and let $I \subseteq k[x_1 \ldots, x_n]$ be an ideal. Suppose that $f \in k[x_1, \ldots, x_n]$.*

(i) f can be written in the form $f = g + r$, where $g \in I$ and no term of r is divisible by any element of $\mathrm{LT}(I)$.
(ii) r and g are uniquely determined.

Proof. Dickson's Lemma guarantees the existence of a Gröbner basis for every ideal I. So, consider a G.B. for I. Recall that a G.B. of I is defined as a finite subset $G = \{g_1, \dots, g_t\}$ of I so that $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(G) \rangle$. This means that for any $p \in I$, $\mathrm{LT}(g_i) \mid \mathrm{LT}(p)$ for some $g_i \in G$.

(i) Divide f by $g_1, \dots, g_t$ using the Division Algorithm, and write $f = g + r$, where r is the remainder. Suppose r was divisible by $\mathrm{LT}(h)$, for some $h \in I$. Now, $\mathrm{LT}(g_i) \mid \mathrm{LT}(h)$, for some $g_i \in G$; this implies (by transitivity) that $\mathrm{LT}(g_i) \mid r$. But the Division Algorithm ensures that r is not divisible by any element of $\mathrm{LT}(G)$, so we have a contradiction, and hence r is not divisible by any element of $\mathrm{LT}(I)$.

(ii) Suppose $f = g + r = g' + r'$ as in (i). This implies $g - g' = r' - r$. Also, $g, g' \in I, r' - r = g - g' \in I$. Since no term of r or r' is divisible by any element of $\mathrm{LT}(I)$, no term of $r - r'$ is divisible by any element of $\mathrm{LT}(I)$ either. But $r - r' \in I \implies \mathrm{LT}(r - r') \in \mathrm{LT}(I)$, which means $r - r'$ must be zero, implying $r = r'$.

□

Proposition 2.1 shows once a monomial order is fixed, we can define a unique "remainder of f on division by I," denoted by either $\overline{f}^I$, or $\overline{f}^G$ where G is some G.B. for I. Proposition 2.2 relates the Proposition 2.1 to quotient rings.

Proposition 2.2 (Modified Exercise §5.3 1, Unique representatives)**.** *Fix a monomial ordering on $k[x_1, \dots, x_n]$ and let $I \subseteq k[x_1, \dots, x_n]$ be an ideal.*
(i) Every $[f] \in k[x_1, \dots, x_n]/I$ can be given by a unique representative $[r] = [f]$, for a unique polynomial r which is a k-linear combination of the monomials in the complement of $\langle \mathrm{LT}(I) \rangle$, denoted as $\langle \mathrm{LT}(I) \rangle^c$.
(ii) The elements of $\langle \mathrm{LT}(I) \rangle^c$ are "linearly independent modulo I," i.e., if we have $\sum_\alpha c_\alpha x^\alpha \equiv 0 \bmod I$, where $x^\alpha \in \langle \mathrm{LT}(I) \rangle^c$, then $c_\alpha = 0$ for all α.

Proof.

(i) We claim $r = \overline{f}^I$. Since $f = g + r$ for some $g \in I$, we have $f - r \in I \iff [f] = [r]$. Clearly, r is a k-linear combination of the monomials in $\langle \mathrm{LT}(I) \rangle^c$. Next, we need to show this is well-defined, i.e., $\overline{h}^I = \overline{f}^I$ for $h \in [f]$. We know $h \in [f] \iff f - h \in I$. Say, $f - h = g' \in I$. We know $f = g + r$, then $h = (g - g') + r$, and $(g - g'), r$ satisfy the conditions given in (i) of Proposition 2.1, and (ii) of the same tells us these polynomials are uniquely determined. So, $\overline{h}^I = r$, for any $h \in [f]$.

(ii) Suppose $\sum_\alpha c_\alpha x^\alpha \equiv 0 \bmod I$. Let $f = \sum_\alpha c_\alpha x^\alpha$. Then, $f \in I$. Let $\mathrm{LT}(f) = c_\lambda x^\lambda$, for some $c_\lambda \neq 0$. But, $f \in I \implies \mathrm{LT}(f) \in \langle \mathrm{LT}(I) \rangle \implies x^\lambda \in \langle \mathrm{LT}(I) \rangle$, which is a contradiction, implying that f cannot have non-zero leading term, which means $c_\alpha = 0, \forall\, \alpha$.

□

Note that $\langle \mathrm{LT}(I) \rangle^c$ is not the complement of $\langle \mathrm{LT}(I) \rangle$ as a subset of $k[x_1, \dots, x_n]$, but rather just the monomials in the complement.

The next two results are easy to verify Corollaries of part (i) of Proposition 2.2.

Corollary 2.3. *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. Then $k[x_1, \dots, x_n]/I$ is isomorphic as a k-vector space to $S = \mathrm{Span}(x^\alpha \mid x^\alpha \in \langle \mathrm{LT}(I) \rangle^c)$.*

Corollary 2.4. *Let I be an ideal in $k[x_1, \dots, x_n]$ and let G be a Gröbner basis of I with respect to any monomial order. For each $[f] \in k[x_1, \dots, x_n]/I$, we get the standard representative $\overline{f} = \overline{f}^G$ in $S = \mathrm{Span}(x^\alpha \mid x^\alpha \in \langle \mathrm{LT}(I) \rangle^c)$. Then:*

(i) $[f] + [g]$ is represented by $\overline{f} + \overline{g}$.
(ii) $[f] \cdot [g]$ is represented by $\overline{\overline{f} \cdot \overline{g}} \in S$.

Remark 2.5. Proposition 1.9 is the first instance of how the new algebraic structure we are considering, namely $k[V]$ (or equivalently $k[x_1, \dots, x_n]/\mathrm{I}(V)$), can tell us something about the geometry. Theorems 2.6 and 2.7 are another instance of how the algebra can tell us something about the geometry. To see this clearly, we can restate (part of) Theorems 2.6 and 2.7 as "Over an algebraically closed field k, a variety V is finite if and only if its coordinate ring, $k[V]$, is finite-dimensional as a k-vector space; and the number of points in the variety is exactly the dimension of $k[V]$".
The proof (which we omit) uses Corollary 2.3 extensively, and also relies on thinking about $k[x_1, \dots, x_n]/I$ coordinate-wise, further emphasizing the terminology "Coordinate" Ring. The next two theorems also allow us to completely understand zero-dimensional varieties (finite points). Indeed, Theorems 2.6 and 2.7 hint at how we might define in general the dimension of an affine variety.

Theorem 2.6 (Finiteness Theorem). *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and fix a monomial ordering on $k[x_1, \dots, x_n]$. Consider the following five statements:*

(i) For each i, $1 \leqslant i \leqslant n$,there is some $m_i \geqslant 0$ such that $x_i^{m_i} \in \langle \mathrm{LT}(I) \rangle$.
(ii) Let G be a Gröbner basis for I. Then for each i, $1 \leqslant i \leqslant n$, there is some $m_i \geqslant 0$ such that $x_i^{m_i} = \mathrm{LM}(g)$ for some $g \in G$.
(iii) The set $\langle \mathrm{LT}(I) \rangle^c$ is finite.
(iv) The k-vector space $k[x_1, \dots, x_n]/I$ is finite-dimensional.
(v) $\mathbf{V}(I) \subseteq k^n$ is a finite set.

Then (i)–(iv) are equivalent and they all imply (v).
Furthermore, if k is algebraically closed, then (i)–(v) are all equivalent.

Theorem 2.7 (Bound for 0-dimensional varieties). *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal such that for each i, some power $x_i^{m_i} \in \langle LT(I) \rangle$, and set $V = \mathbf{V}(I)$. Then:*

(i) The number of points of V is at most $\dim k[x_1, \dots, x_n]/I$ *(where "dim" means dimension as a vector space over k).*
(ii) The number of points of V is at most $m_1 \cdot m_2 \cdots m_n$.
(iii) If I is radical and k is algebraically closed, then equality holds in part (i), i.e., the number of points in V is exactly $\dim k[x_1, \dots, x_n]/I$.

3. Algebra-Geometry Dictionary version2.0

Remark 2.6 summarizes how we can understand a variety (the geometry) using its coordinate ring (the algebra), and vice-versa. This section will build upon this, and supply and Algebra-Geometry dictionary between subvarieties of a variety V, and ideals in $k[V]$. Indeed, the Algebra-Geometry dictionary we studied in Chapter 4 of Cox. et al is the special case of the previous statement when $V = k^n$ and $k[V] = k[k^n] = k[x_1, \dots, x_n]$.

Definition 3.1. Let $V \subseteq k^n$ be an affine variety.

(i) For any ideal $J = \langle \phi_1, \dots, \phi_s \rangle \subseteq k[V]$, we define
$$\mathbf{V}_V(J) = \{(a_1, \dots, a_n) \in V \mid \phi(a_1, \dots, a_n) = 0, \forall\, \phi \in J\}$$
We call $\mathbf{V}_V(J)$ a subvariety of V.
(ii) For each subset $W \subseteq V$, we define
$$\mathbf{I}_V(W) = \{\phi \in k[V] \mid \phi(a_1, \dots, a_n) = 0, \forall\, (a_1, \dots, a_n) \in W\}$$

The maps we considered for our original Algebra-Geometry Dictionary were $\mathbf{V}$ and $\mathbf{I}$; as expected, here we work with $\mathbf{V}_V$ and $\mathbf{I}_V$. Then, our first result is as follows.

Proposition 3.2 (Exercise §5.4 3). *Let $V \subseteq k^n$ be an affine variety.*

(i) For each ideal $J \subseteq k[V]$, $W = \mathbf{V}_V(J)$ is an affine variety in k^n contained in V.

(ii) For each subset $W \subseteq V$, $\mathbf{I}_V(W)$ is an ideal of $k[V]$.
(iii) If $J \subseteq k[V]$ is an ideal, then $J \subseteq \sqrt{J} \subseteq \mathbf{I}_V(\mathbf{V}_V(J))$.
(iv) If $W \subseteq V$ is a subvariety, then $W = \mathbf{V}_V(\mathbf{I}_V(W))$.

Proof.

(i) Recall from Ring Theory the general bijective correspondence between ideals of a R/I and ideals of R containing I, where R is any ring. Applying this here, we consider the ideal $\widetilde{J}$ in $k[x_1, \ldots, x_n]$ corresponding to J, given by $\widetilde{J} = \{j \in k[x_1, \ldots, x_n] \mid [j] \in J\}$. We claim $\mathbf{V}(\widetilde{J}) = W$. This is clear because $\vec{a} \in \mathbf{V}(\widetilde{J}) \iff j(\vec{a}) = 0, \forall j \in \widetilde{J} \iff [j](\vec{a}) = 0, \forall [j] \in J \iff \vec{a} \in W$. Finally, W is obviously contained in V by the definition of $\mathbf{V}_V(J)$.

(ii) Clearly $[0] \in \mathbf{I}_V(W)$, because $[0](\vec{a}) = 0, \forall \vec{a}$. Suppose $[f], [g] \in \mathbf{I}_V(W)$. Then, $([f] + [g])(\vec{a}) = [f+g](\vec{a}) = (f+g)(\vec{a}) = f(\vec{a}) + g(\vec{a}) = 0 \implies [f] + [g] \in \mathbf{I}_V(W)$. Similarly, $[h] \cdot [f] \in \mathbf{I}_V(W)$.

(iii) $J \subseteq \sqrt{J}$ is well-known and true in general. So, we only need to show $\sqrt{J} \subseteq \mathbf{I}_V(\mathbf{V}_V(J))$. Let $[f] \in \sqrt{J}$. Then, $[f]^m \in J \iff [f^m] \in J$. Then, for any $\vec{a} \in \mathbf{V}_V(J)$, we must have $[f^m](\vec{a}) = 0$ by definition. This is true iff $f^m(\vec{a}) = 0 \iff f(\vec{a}) = 0 \iff f \in \mathbf{I}_V(\mathbf{V}_V(J))$, which completes this proof.

(iv) ($\subseteq$) Let $\vec{a} \in W$. Consider, any $[f] \in \mathbf{I}_V(W)$. Then, $[f](\vec{a}) = 0$ by definition. But $[f]$ was arbitrary, and it follows that $\vec{a} \in \mathbf{V}_V(\mathbf{I}_V(W))$.
($\supseteq$) If W is a subvariety, then from (i) of Definition 3.1, we can write $W = \mathbf{V}_V(J)$ for some ideal $J \subseteq k[V]$. Pick some $[f] \in J$. Now, by definition, $[f](\vec{a}) = 0, \forall \vec{a} \in W$, which means $[f] \in \mathbf{I}_V(W)$. Now, let $\vec{b} \in \mathbf{V}_V(\mathbf{I}_V(W))$. Then, $[f](\vec{b}) = 0$ by definition of $\mathbf{V}_V$. Since $[f] \in J$ is arbitrary, we can say that $[f](\vec{b}) = 0, \forall \vec{b} \in \mathbf{V}_V(J) = W$. In conclusion, $\vec{b} \in \mathbf{V}_V(\mathbf{I}_V(W)) \implies \vec{b} \in W$.

$\square$

Example 3.3 (Modified Exercise §5.4 11)**.** Consider the variety $V = \mathbf{V}(z - x^2 - y^2) \subseteq \mathbb{R}^3$, and the ideal $J \subseteq \mathbb{R}[V]$ given by $\langle [x] \rangle$. Suppose we wanted to determine $W = \mathbf{V}_V(J)$. It is not hard to see that every point in V can be parameterized as $(u, v, u^2 + v^2)$. Then, W will consist of points of that form so that $[x](u, v, u^2 + v^2) = 0 \iff u = 0$. Which means points in W are exactly the points which look like $(0, v, v^2)$. Indeed, if we follow the proof given for part (i) of Proposition 3.2, then we can see that the ideal corresponding to J in $k[x, y, z]$ is given by $\widetilde{J} = \langle z - x^2 - y^2, x \rangle$, so that we can also write W as $W = \mathbf{V}(\widetilde{J})$.
Next, let $W = \{1, 1, 2\} \subseteq V$. Consider $\mathbf{V}_V([x-1], [y-1])$. Solving $x - 1 = y - 1 = 0$ gives $x = y = 1$. Since points of V look like $(u, v, u^2 + v^2)$, we can conclude that the only point in $\mathbf{V}_V([x-1], [y-1])$ is $(1, 1, 1^2 + 1^2) = (1, 1, 2)$. So, $W = \mathbf{V}_V([x-1], [y-1])$. Now, let $J = \langle [x-1], [y-1] \rangle \subseteq \mathbb{R}[V]$, so that $\mathbf{V}_V([x-1], [y-1]) = \mathbf{V}_V(J) = W$. We claim that $J = \mathbf{I}_V(W)$. First, from Proposition 3.2 part (ii), we can write $J \subseteq \mathbf{I}_V(\mathbf{V}_V(J)) = \mathbf{I}_V(W)$.
We now show $\mathbf{I}_V(W) \subseteq J$. Suppose $[f] \in \mathbf{I}_V(W)$. Then, $[f](1, 1, 2) = 0 \iff f(1, 1, 2) = 0$. Use the Division Algorithm with lex order $(x > y > z)$ to divide f by $x - 1, y - 1$. Then, $f = A(x-1) + B(y-1) + g(z)$. Evaluate $(1, 1, 2)$ to find that $g(2) = 0$. Since $g(z)$ is a univariate polynomial with 2 as a root, we can write $g(z) = (z - 2) \cdot h(z)$.
So, $f = A(x-1) + B(y-1) + g(z) = A(x-1) + B(y-1) + (z-2) \cdot h(z)$. Then, $[f] \in \langle [x-1], [y-1], [z-2] \rangle$. Now, since we are working in the ring $\mathbb{R}[V] = \mathbb{R}[\mathbf{V}(z - x^2 - y^2)]$, we see that $[z - x^2 - y^2] = [0]$. But then $[0] = [z - x^2 - y^2] = [(z-2) - (x^2 - 1) - (y^2 - 1)] = [(z-2) - (x-1)(x+1) - (y-1)(y+1)] = [z-2] - [x+1][x-1] - [y+1][y-1] \implies [z-2] = [x+1][x-1] + [y+1][y-1] \implies [z-2] \in \langle [x-1], [y-1] \rangle$.
This means we can write $\langle [x-1], [y-1], [z-2] \rangle = \langle [x-1], [y-1] \rangle$. Then, $[f] \in \langle [x-1], [y-1] \rangle$, which shows $\mathbf{I}_V(W) \subseteq J$.

The next theorem is the final result of this section, and gives us the analogous Algebra-Geometry Dictionary for subvarieties of V and ideals in $k[V]$. The proof takes advantage of the bijective correspondence between ideals in $k[x_1, \ldots, x_n]$ containing $\mathbf{I}(V)$ and ideals of $k[V]$, and also the Strong Nullstellensatz over $k[x_1, \ldots, x_n]$.

Theorem 3.4 (Subvariety - Ideal Correspondence)**.** *Let k be an algebraically closed field and let $V \subseteq k^n$ be an affine variety.*

(i))The Strong Nullstellensatz in $k[V]$) *If J is any ideal in $k[V]$, then*

$$\mathbf{I}_V(\mathbf{V}_V(J)) = \sqrt{J} = \{[f] \in k[V] \mid [f]^m \in J\}$$

(ii) The correspondences

$$\left\{\begin{matrix} \textit{affine subvarieties} \\ W \subseteq V \end{matrix}\right\} \underset{\mathbf{V}_V}{\overset{\mathbf{I}_V}{\rightleftarrows}} \left\{\begin{matrix} \textit{radical ideals} \\ J \subseteq k[V] \end{matrix}\right\}$$

are inclusion-reversing bijections and are inverses of each other.

We now do some problems to illustrate the theory we learned above.

Problem 3.5 (Exercise §5.4 1)**.** *Let C be the twisted cubic curve in k^3.*

(a) Show that C is a subvariety of the surface $S = V(xz - y^2)$.
(b) Find an ideal $J \subseteq k[S]$ such that $C = \boldsymbol{V}_S(J)$.

Proof.

(a) The twisted cubic is given by $\mathbf{V}(z - x^3, y - x^2)$, and is parameterized as (t, t^2, t^3). If we evaluate $f(x, y, z) = xz - y^2$ at (t, t^2, t^3) we get $f(t, t^2, t^3) = t \cdot t^3 - (t^2)^2 = 0$. So, clearly, $C \subseteq \mathbf{V}(xz - y^2)$.
(b) It is obvious that $J = \langle [z - x^3], [y - x^2] \rangle$ will suffice.

□

Problem 3.6 (Exercise §5.4 2)**.** *Let $V \subseteq \mathbb{C}^n$ be a nonempty affine variety.*

(a) Let $\phi \in \mathbb{C}[V]$. Show that $\mathbf{V}_V(\phi) = \varnothing$ if and only if ϕ is invertible in $\mathbb{C}[V]$.
(b) Is the statement of part (a) true if we replace $\mathbb{C}$ by $\mathbb{R}$? If so, prove it; if not, give a counterexample.

Proof.

(a) ($\Longrightarrow$) Suppose $\mathbf{V}_V(\phi) = \varnothing$. Then, taking $\mathbf{I}_V$ of both sides gives $\mathbf{I}_V(\mathbf{V}_V(\phi)) = \mathbf{I}_V(\varnothing)$. Now to determine $\mathbf{I}_V(\varnothing)$, we will need some logic; looking carefully at Definition 3.1, we can say that $\phi \in \mathbb{C}[V]$ is an element of $\mathbf{I}_V(\varnothing)$ if and only if, $\vec{a} \in \varnothing \implies \phi(\vec{a}) = 0$. Since the antecedent of the conditional is false (there can't be anything in the empty set), overall the conditional $\vec{a} \in \varnothing \implies \phi(\vec{a}) = 0$ is true for any ϕ, and it follows that $\mathbf{I}_V(\varnothing) = \mathbb{C}[V]$. So, $\mathbf{I}_V(\mathbf{V}_V(\phi)) = \mathbf{I}_V(\varnothing) = \mathbb{C}[V]$. From part (i) of Theorem 3.4 (can use since $\mathbb{C}$ is algebraically closed) we can write $\sqrt{\langle \phi \rangle} = \mathbb{C}[V]$. Now, $[1] \in \mathbb{C}[V] \implies [1] \in \sqrt{\langle \phi \rangle} \implies [1]^m = [1^m] = [1] \in \langle \phi \rangle$. So, there exists $\theta \in \mathbb{C}[V]$ such that $\theta \cdot \phi = [1]$, which means ϕ is invertible.
($\Longleftarrow$) Suppose ϕ is invertible. Then, clearly $[1] \in \langle \phi \rangle$. Write $\mathbf{V}_V(\phi) = \mathbf{V}_V(\langle \phi \rangle)$. So, $\vec{a} \in \mathbf{V}_V(\langle \phi \rangle) \implies [1](\vec{a}) = 0$, by definition. But $[1](\vec{a}) = 0$ is impossible. So, $\mathbf{V}_V(\phi)$ must be empty.
(b) Part (a) is not true over $\mathbb{R}$. A counterexample would be letting $\phi = [x_1^2 + 1]$ and choosing V to be any irreducible variety. First, notice that $x_1^2 + 1$ never vanishes over $\mathbb{R}^n$, so that $\mathbf{V}_V(\phi) = \varnothing$. Since V is irreducible, $\mathbb{C}[V]$ is an integral domain by Proposition 1.9. It is now easy to show ϕ cannot have an inverse (contradicting the hypothesis of part (a)), because multiplication by any non-zero polynomial must necessarily increase (or maintain) the multidegree of ϕ (this need not be the case if $\mathbb{C}[V]$ was not an integral domain).

□

It as an easy consequence of Theorem 3.4 that the Weak Nullstellensatz also holds over $k[V]$. Let us prove this.

Problem 3.7 (Exercise §5.4 16)**.** *Let k be algebraically closed.Prove the Weak Nullstellensatz for $k[V]$, which asserts that for any ideal $J \subseteq k[V]$, $\mathbf{V}_V(J) = \varnothing$ if and only if $J = k[V]$. Also explain how this relates to Problem 3.6 when $J = \langle \phi \rangle$.*

Proof. ($\Longrightarrow$) Suppose that for an ideal $J \subseteq k[V]$, we have $\mathbf{V}_V(J) = \varnothing$. Following the exact reasoning we used in Problem 3.6 part (a), we can show that $[1] \in J$. But this obviously implies $J = k[V]$. Indeed, Problem 3.6 was but a special case of the Weak Nullstellensatz when J had a single generator.
($\Longleftarrow$) Suppose $J = k[V]$. Then, $[1] \in J$. $\mathbf{V}_V(J)$ must be empty because $[1]$ never vanishes. □

4. Isomorphism of Varieties

This section formalizes the ideas of Example 1.11.

Definition 4.1. Let $V \subseteq k^m$ and $W \subseteq k^n$ be affine varieties. We say that V and W are isomorphic if there exist polynomial mappings $\alpha : V \to W$ and $\beta : W \to V$ such that $\alpha \circ \beta = \mathrm{id}_W$ and $\beta \circ \alpha = \mathrm{id}_V$, where $\mathrm{id}_W, \mathrm{id}_V$ are identity maps.

An equivalent definition be just saying the polynomial mapping α is bijective and has a polynomial inverse.

Remark 4.2. Note that isomorphisms preserve properties such as irreducibility (see Problem 4.8) and "dimension" of the variety. The question now is how do we tell if two varieties are isomorphic. Since this seems tricky to do geometrically, we use the same old trick: consider the corresponding algebraic structures, namely, the coordinate rings.
First, we must define "suitable" homomorphisms between the two rings (then later worry about bijectivity). They should be "suitable" in the sense that they should be fundamentally dependent on the geometric maps (the polynomial mappings α). Ideally, we would like to "induce" maps between $k[V]$ and $k[W]$ given a polynomial mappings between V and W, and vice-versa. Indeed, this happens in a very natural way, and is the subject of the following result.

Proposition 4.3. *Let V and V be varieties (possibly in different affine spaces).*

(i) (Pullback mapping) *Let $\alpha : V \to W$ be a polynomial mapping. Then for every polynomial function $\phi : W \to k$, the composition $\phi \circ \alpha : V \to k$ is also a polynomial function. Furthermore, the map $\alpha^* : k[W] \to k[V]$ defined by $\alpha^*(\phi) = \phi \circ \alpha$ is a ring homomorphism which is the identity on the constant functions $k \subseteq k[W]$. We call α^* the pullback mapping on functions.*
(ii) Conversely, let $\Phi : k[W] \to k[V]$ be a ring homomorphism which is the identity on constants. Then there is a unique polynomial mapping $\alpha : V \to W$ such that $\Phi = \alpha^*$.

Unsurprisingly, because we are doing Algebraic Geometry, Proposition 4.3 leads to the following beautiful (and convenient) Theorem.

Theorem 4.4. *Two affine varieties $V \subseteq k^m$ and $W \subseteq k^n$ are isomorphic if and only if there is an isomorphism $k[V] \cong k[W]$ of coordinate rings which is the identity on constant functions.*

We conclude this project with a few problems illustrating the theory we have learnt.

Problem 4.5 (Exercise §5.4 4)**.** *Let $V = \mathbf{V}(y - x^n, z - x^m)$, where m, n are any integers $\geqslant 1$. Show that V is isomorphic as a variety to k by constructing explicit inverse polynomial mappings $\alpha : k \to V$ and $\beta : V \to k$.*

Proof. Take x as the coordinate on k. Then, define $\alpha(x) = (x, x^n, x^m)$. It is easy to verify that $\alpha(x) \in V$ for all $x \in k$. Define $\beta(x, y, z) = x$. Now, consider some $(x, y, z) \in V$. Since $y - x^n = z - x^m = 0$, we can write $(x, y, z) = (x, x^n, x^m)$. So, $\alpha \circ \beta(x, y, z) = \alpha(\beta(x, y, z)) = \alpha(x) = (x, x^n, x^m) = (x, y, z) = \mathrm{id}_V(x, y, z)$. Similarly, it is easy to verify $\beta \circ \alpha = \mathrm{id}_W$. ☐

Problem 4.6 (Exercise §5.4 5)**.** *Show that any surface in k^3 with a defining equation of the form $x - f(y, z) = 0$ or $y - g(x, z) = 0$ is isomorphic as a variety to k^2.*

Proof. We will prove this for a defining equation of the form $x - f(y, z) = 0$. It will then obviously follow for an equation of the form $y - g(x, z) = 0$, for the latter is just a symmetric permutation of the variables as arranged in the former equation.
Define polynomial mappings $\alpha : k^2 \to \mathbf{V}(x - f(y, z))$ s.t. $\alpha(y, z) = (f(y, z), y, z)$, and $\beta : V \to k^2$ s.t. $\beta(x, y, z) = (y, z)$. It is easy to verify these maps define an isomorphism between V and k^2. ☐

Problem 4.7 (Exercise §5.4 9)**.** *Let $\alpha : V \to W$ and $\beta : W \to V$ be inverse polynomial mappings between two isomorphic varieties V and W. Let $U = \mathbf{V}_V(I)$ for some ideal $I \subseteq k[V]$. Show that $\alpha(U)$ is a subvariety of W and explain how to find an ideal $J \subseteq k[W]$ such that $\alpha(U) = \mathbf{V}_W(J)$.*

Proof. Consider the isomorphism given by the pullback mapping on $k[V]$, i.e., $\beta^* : k[V] \to k[W]$ s.t. $\beta^*(\theta) = \theta \circ \beta$. We claim $\beta^*(I)$ will suffice as the ideal J. So, let $J = \beta^*(I)$. First, since β^* is an isomorphism, it is easy to prove $\beta^*(I)$ is an ideal in $k[W]$ given that I is an ideal in $k[V]$. Now, we only have to show $\alpha(U) = \mathbf{V}_W(J)$:
($\subseteq$) Let $\vec{x} \in \alpha(U)$. WTS $\phi(\vec{x}) = 0$ for all $\phi \in J = \beta^*(I)$. Consider any $\phi \in J$. Now, $\exists\, \vec{a} \in U$ such that $\alpha(\vec{a}) = \vec{x} \implies \beta(\vec{x}) = \vec{a}$. Write $\phi = \theta \circ \beta$ for some $\theta \in I$. Evaluate $\phi(\vec{x}) = (\theta \circ \beta)(\vec{x}) = \theta(\beta(\vec{x})) = \theta(\vec{a})$. Now, since $\theta \in I$ and $\vec{a} \in U = \mathbf{V}_V(I)$, we have $\theta(\vec{a}) = 0$ by definition. So, $\phi(\vec{x}) = 0$.
($\subseteq$) Let $\vec{x} \in \mathbf{V}_W(J)$. WTS $\vec{x} \in \alpha(U) \iff \exists\, \vec{a} \in U$ such that $\alpha(\vec{a}) = \vec{x}$. We claim $\beta(\vec{x})$ will suffice as $\vec{a}$. So, let $\vec{a} = \beta(\vec{x})$. It is obvious then that $\alpha(\vec{a}) = \vec{x}$ will hold, so we only need to show that $\vec{a} = \beta(\vec{x})$ is an element of $U = \mathbf{V}_V(I)$. Consider any $\theta \in I$. Since the ideal I and J are isomorphic (via the isomorphisms α^* and β^*), we can write $\theta = \alpha^*(\phi) = \phi \circ \alpha$ for some $\phi \in J$. Then, $\theta(\beta(\vec{x})) = (\phi \circ \alpha)(\beta(\vec{x})) = (\phi \circ (\alpha \circ \beta))(\vec{x}) = (\phi \circ \mathrm{id}_V)(\vec{x}) = \phi(\vec{x})$. But since $\vec{x} \in \mathbf{V}_W(J)$ and $\phi \in J$, we have $\phi(\vec{x}) = 0$ by definition. So, $\theta(\vec{a}) = 0$ for all $\theta \in I$, which means $\vec{a} \in U$.
By double containment we have shown that $\alpha(U) = \mathbf{V}_W(J)$, which proves that $\alpha(U)$ is a subvariety of W. ☐

Problem 4.8 (Exercise §5.4 17)**.**

(a) Let $f : R \to S$ be a ring isomorphism. Prove that R is an integral domain if and only if S is an integral domain.
(b) Let $\phi : V \to W$ be an isomorphism of affine varieties. Prove that V is irreducible if and only if W is irreducible.

Proof.

(a) It is sufficient to show that if R has no zero-divisors, then S will also not have any (the reverse direction of the proof will be symmetric). We do this by contradiction. Suppose $s_1, s_2 \in S$ so that $s_1 \neq 0 \neq s_2$ and $s_1 s_2 = 0$. Since, f is an isomorphism, we can write $s_1 = f(r_1), s_2 = f(r_2)$ for some $r_1, r_2 \in R$. Then, $f(r_1)f(r_2) = 0 \implies f(r_1 r_2) = 0$ (because f is a homomorphism). Since f is one-to-one, f maps only $0 \in R$ to $0 \in S$, which implies $r_1 r_2 = 0$. By the same logic, $r_1, r_2 \in \mathbb{R}$ must be non-zero since they map to non-zero elements s_1, s_2. So, we have found non-zero elements in R with a product of 0, which is a contradiction.
(b) Proposition 1.9 tells us that V is irreducible iff $k[V]$ is an integral domain. Theorem 4.4 tells us that $k[V]$ and $k[W]$ are isomorphic because V and W are isomorphic. So, $k[W]$ is an integral domain. Again, from srefProposition1.9, this is true iff W is irreducible, which concludes the proof.

□

Problem 4.9 (Exercise §5.4 6). *Let V be a variety in k^n that is defined by a single equation of the form $x_n - f(x_1, \ldots, x_{n-1}) = 0$. Show that V is isomorphic as a variety to k^{n-1}.*

Proof. Define $\alpha : V \to k^{n-1}$ by $\alpha(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$. Define $\beta : k^{n-1} \to V$ by $\beta(x_1, \ldots, x_{n-1}) = \beta(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}))$. It is easy to check these are polynomial inverses of each other, and hence define an isomorphism between V and k^{n-1}. □